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# UNIVERSITY OF CALIFORNIA, SAN DIEGO 

Surface Quasigeostrophic Vortex Dynamics and Resulting Transport with Weak Vertical Motion

Dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in

Engineering Sciences (Engineering Physics) by

Cecily Keppel Taylor

Committee in charge:

Professor Stefan Llewellyn Smith, Chair<br>Professor Paola Cessi<br>Professor Melvin Leok<br>Professor James Rottman<br>Professor David Saintillan<br>Professor William Young

The Dissertation of Cecily Keppel Taylor is approved and is acceptable in quality and form for publication on microfilm and electronically.
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Chair

University of California, San Diego

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## Abstract of Dissertation

Surface Quasigeostrophic Vortex Dynamics and Resulting Transport with Weak Vertical Motion<br>by<br>Cecily Keppel Taylor<br>Doctor of Philosophy in Engineering Science (Engineering Physics)<br>University of California, San Diego, 2016<br>Professor Stefan Llewellyn Smith, Chair

The surface quasi-geostrophic (SQG) equations are a model for lowRossby number geophysical flows in which the overall dynamics are governed by buoyancy evolution on the boundary. The model can be used to explore the transition from two-dimensional to three-dimensional mesoscale geophysical flows. We examine SQG vortices and the resulting flow to first order in Rossby number, $O(R o)$. This requires solving an extension to the usual QG equation to compute the velocity corrections, and we demonstrate this mathematical procedure. As we show, it is simple to obtain the vertical velocity, but difficult to find the $O(R o)$ horizontal corrections. Chaotic transport due to three SQG point vortices is studied with Poincaré sections and the Finite

Time Braiding Exponent (FTBE). This chaotic transport is representative of the mixing in the flow, and these terms are used interchangeably in this work. Changes in transport from $O(R o)$ vertical velocity terms are also examined, though without $O(R o)$ horizontal velocities this is not a true solution to the governing equations. We then consider the SQG elliptic vortex solution developed by Dritschel (2011), in which all $O(R o)$ velocities can be calculated. Results show that SQG point vortices exhibit greater mixing at the surface than classical point vortices. There appears to be a minimum FTBE near flow regime boundaries, and generally mixing is greater when the energy of the system is greater. There is also a critical depth below which the FTBE decreases sharply. Finally, including vertical velocity in the point vortex solution increases the observed mixing.

## 1 Introduction

Many fields within the fluid dynamics community require a good understanding of mixing in order to advance in problems such as miniaturizing biomedical devices [48], improving combustion engines [1, 53], and predicting how pollutants carried in the ocean or atmosphere will disperse [15, 43, 34]. While mixing is a familiar concept, the particulars of how to define and measure the extent of mixing in a given flow remains challenging. Mixing is generally decomposed into two stages, starting with the process of stirring, where diffusion is negligible but advection will stretch material lines into filaments (see Figure 1.1 from [52]). The second stage begins when the scales of the filaments are small enough that the advection effects are of the same order as diffusion, and from there the mixed region will homogenize. The initial step of stirring was studied by Aref \& Pomphrey (1980) under the name "chaotic motion" in the simplified case of point vortices in twodimensional flow [5]. In this case, chaos refers to aperiodic behavior in which two points in the flow that are initially very near will exhibit very different trajectories over finite time [4]. The trajectories are a Lagrangian description of how material will be transported in the flow. Physically, a particle in the flow that undergoes chaotic advection will, over time, be carried through the entire region and interact with all of the fluid particles that are bounded by


Figure 1.1: Time series of a set of particles that initially fill a square region being advected by a flow, reproduced from Welander 1955 [52]. Each particle follows a different trajectory in time, and the shape is stretched and twisted into filaments. This is called stirring.
the same material lines. Thus, if dye is introduced into the fluid anywhere in that region, the dye will eventually spread over the entire region, satisfying the typical conceptual definition of mixing. This chaotic advection is what we consider to be mixing in this thesis, where we analyze exact solutions to equations relevant to flow in the ocean.

In the ocean, flows at different scales will have different characteristics.
Figure 1.2 displays some of these scales, along with approximate horizontal lengths of geophysical flows to which they apply and relevant equations of motion. At the largest "basin" width scales, $O(1000 \mathrm{~km})$, the equations are purely planetary geostrophic, where the pressure gradients are balanced by the Coriolis forces. These scales apply to weather systems. At slightly smaller "mesoscales," $O(10-100 \mathrm{~km}$ ), the Quasigeostrophic (QG) equations of motion apply, where inertial terms are small but not neglected. This is relevant to the typical scale of gyres in the ocean. At scales smaller yet, called "submesoscale," the balance of forces is more complicated and the flows are not well understood. Finally, small-scale flows do not experience significant Coriolis forces, and solutions are fully three-dimensional.

At large scales, the ocean can be modeled as a thin body of fluid

## Ocean Scales



Figure 1.2: Various ocean scales, in decreasing order, with approximate horizontal scales and relevant equations of motion.
spread over a rotating sphere. The fluid is stratified and effectively inviscid except at very small length scales. The Coriolis forces are strong relative to local forces, so the Earth's rotation has a significant effect on dynamics. The stratification can be expressed in terms of deviations from a stable state

$$
\rho=\rho_{0}(z)+\rho^{\prime}
$$

The Boussinesq approximation assumes that the deviations are small, $\rho^{\prime} / \rho_{0} \ll$ 1 , and in the ocean where $\rho^{\prime} / \rho_{0}=O\left(10^{-3}\right)$ this is valid [33].

The equations of motion can be nondimensionalized using characteristic scales - velocity $U$, horizontal length $L$, vertical length (i.e. height) $H$, kinematic viscosity $\nu$, Coriolis frequency $f$, and buoyancy frequency $N$ - to find various dimensionless numbers that characterize a regime of motion. For example, inviscid flow is characterized by high Reynold's number $R e=U L / \nu$, and in this regime the viscous stress is negligible away from solid boundaries. In this thesis we are not concerned with flow at boundaries, and so the viscous stresses are neglected. There is also the Rossby number, $R o=U / f L$, comparing the local velocity to the velocity of the rotating coordinates. It is assumed in the work here that $R o \ll 1$. For ocean circulation, $U=0.1 \mathrm{~m} / \mathrm{s}$ and $f_{0}=10^{-4} \mathrm{~s}^{-1}$ at mid-latitudes, so for
the lengths 10-100 km, the Rossby number will be $R o=10^{-3}-10^{-2}$ [33]. Another dimensionless number is the Burger number, $B=N H / f L$, which describes the relative strength of buoyancy forces and Coriolis forces, and provides a ratio of the effective vertical length scale to the horizontal one. For the model of interest here, the Burger number is assumed to be not much smaller than the small Rossby number, specifically $\frac{R o}{B^{2}} \ll 1$. Typically $N=2$ x $10^{-3} \mathrm{~s}^{-1}$ and $H=4 \mathrm{~km}$, so $B=0.8-8$ for the $f_{0}$ and length scales given previously [33]. In the model of interest here, the Burger number is taken to be unity.

Ocean mesoscale flows are characterized by small Rossby number, so that planetary motion dominates the flow and dynamics are mostly horizontal: the rapid rotation suppresses vertical motion by the Taylor-Proudman theorem [7, Section 7.6], and the stratification further minimizes the vertical velocities. Mathematical tools have been developed for studying particle transport for two-dimensional flows. Some researchers include vertical dependence by allowing $z$ to vary as a parameter in the equations [49]. These horizontal flows can be described by a streamfunction, $\psi$, with the velocities described by a Hamiltonian relation, defined here in the geophysical convention,

$$
\begin{equation*}
\dot{x}=-\frac{\partial \psi}{\partial y}, \quad \dot{y}=\frac{\partial \psi}{\partial x} \tag{1.1}
\end{equation*}
$$

One can thus utilize properties of Hamiltonian systems to determine properties of two-dimensional fluid flow [54].

In reality, ocean flow is three-dimensional and cannot be described by (1.1). However, the vertical velocity is relatively weak compared to the horizontal velocity, of order Rossby number. As such, the flow still has a

Hamiltonian relation to leading order, but also has a non-Hamiltonian $O(R o)$ perturbation. Figure 1.3 depicts a chart that demonstrates the added complication going from two-dimensional models to three-dimensional models. Of interest in this thesis are ocean flows that bridge two of these categories, modeled by QG flow. The two-dimensional $O(1)$ QG model is accurate for capturing the two-dimensional inherent dynamics of the system, but other properties of the three-dimensional flow field, such as passive scalar transport, could be significantly altered by this perturbation. For instance, the weak vertical flow could interact with vertical shear, and then the small vertical motion could transport particles into different regimes of motion.

This thesis will examine the effect of this weak vertical velocity on passive scalar transport by considering the Surface Quasigeostrophic (SQG) model [27]. This model is a version of the quasigeostrophic equations, and thus the model lends itself to studying weak vertical velocity effects. By expanding the variables in this small parameter [41], the motion can be examined separately at zeroth and first order. The solution is asymptotically dynamically consistent and allows for the examination of how small vertical velocity can affect transport or other flow properties. In order to determine the fundamental effect of vertical velocity, simple exact solutions are desired; this thesis examines point vortices and elliptic vortices. Vortices are common in the ocean, and idealized point vortices are well-understood in two-dimensional flow and straightforward to model at $O(1)$ in SQG. The elliptic vortex is more complicated, but has the benefit of finite velocities everywhere in the flow. By using these model problems, the effect of the $O(R o)$ corrections on resulting chaotic transport can be isolated.

In order to examine the chaotic motion induced by these vortex solu-

Figure 1.3: A chart depicting the additional complexities between two-dimensional and three-dimensional systems, and how the Quasigeostrophic (QG) model bridges these. All systems may also include time dependence. N-S refers to the Navier-Stokes equations, PE the primitive equations, $\nabla_{H}$ the horizontal gradient, and Ro the Rossby number. Note that QG flow has Hamiltonian dynamics at $O(1)$ as well as three-dimensional flow and variation.
tions, there must be enough free parameters to allow chaos in the flow. In their study of two-dimensional point vortices, Aref \& Pomphrey found that three interacting vortices would follow regular trajectories, and four vortices could follow chaotic trajectories. The four-vortex result also applies to the analysis of flow surrounding three vortices, because a point in the fluid can be modeled as a particle that is passively carried by the fluid, equivalent to a point vortex of zero strength. Thus, by Aref \& Pomphrey's findings, three point vortices will follow periodic paths but can produce chaotic flow in the surrounding fluid. In this thesis, then, three SQG point vortex solutions are considered and their resulting chaotic mixing is examined. In the elliptic vortex case, the additional parameters are expected to induce mixing even for two vortices. While the interaction of point vortices is straightforward, the interaction of elliptic vortices requires integrating over all ellipses to include all contributions to motion. Two methods that approximate this interaction are summarized in Chapter 5. It is found that two elliptic vortices do not follow regular trajectories as in the point vortex case, and instead are quasi-periodic.

By examining mixing for simple exact solutions in the SQG model, this thesis provides insight into a possible driving force of mixing in the ocean as well as a comparison between these dynamics and those found in classical two-dimensional chaotic flows. Since vortices are characteristic structures in the ocean [10, 6], simplified vortices provide an ideal model problem for studying mixing at the mesoscale.

This thesis is structured as follows. First the SQG model is derived in Section 2. Computing the $O(R o)$ corrections to the flow requires inverting Poisson equations, and this analysis is presented in Section 2.1. Then a short
review on tools for quantifying mixing is presented in Chapter 3. For the purposes of comparing different flows, we seek a global measure, and though there is no ideal choice, we choose the Finite Time Braiding Exponent for our analysis. The model problems are then presented: the exact solution of the point vortex flows is derived in Chapter 4 with transport results using a global mixing diagnostic given in Section 4.4. The exact solution of elliptic vortex is presented in Chapter 5 with transport results in Section 5.5. Finally, conclusions are given in Chapter 6.

Chapters 1, 2, 3, 4, and 6, in part, have been submitted for publication of the material as it may appear in Chaos, 2016, Taylor, C. K. and Stefan G. Llewellyn Smith. The dissertation author was the primary investigator and author of this paper.

## 2 Derivation of SQG

In the SQG model, flow is driven by the buoyancy specified at these boundaries. Both semi-infinite space $z<0$ and a finite layer $-D<z<0$ are of interest as relevant to ocean models, and the boundary $z=0$ is considered the surface (see Figure 2.1). The SQG equations are valid in the low-Rossby number limit. The governing equations with the Boussinesq and hydrostatic approximations, given in Vallis (2006) [51], are

$$
\begin{align*}
\frac{D}{D t}\binom{u}{v}+f\binom{-v}{u} & =-\binom{\phi_{x}}{\phi_{y}}  \tag{2.1}\\
\theta & =\phi_{z}  \tag{2.3}\\
u_{x}+v_{y}+w_{z} & =0 \\
\frac{D \theta}{D t}+N^{2} w & =0
\end{align*}
$$

with the conventional material derivative

$$
\frac{D}{D t}=\frac{\partial}{\partial t}+\vec{u} \cdot \nabla
$$

and where the variables $(u, v, w)$ are the velocities in the $(x, y, z)$ directions, respectively; $\phi$ is the geopotential height; and $\theta$ is the buoyancy. Geopotential height refers to the pressure surface compared to a reference height, usually


Figure 2.1: A schematic illustrating the Surface Quasigeostrophic (SQG) model, where flow is governed by potential vorticity which is specified at the boundary and zero elsewhere.
sea level, given by

$$
\phi=\frac{p-p_{0}}{\rho_{0}} .
$$

Buoyancy is the force experienced by a fluid parcel due to differences between its density and the surrounding density,

$$
\theta=-\frac{g\left(\rho^{\prime}-\rho_{0}\right)}{\rho_{0}} .
$$

The physical constants in the equations of motion are the Coriolis parameter, $f$, and the buoyancy frequency, $N$. In addition, the potential vorticity, $q$, is related to the flow by

$$
\begin{equation*}
q=(f \hat{k}+\nabla \times \vec{u}) \cdot \nabla \theta \tag{2.5}
\end{equation*}
$$

For SQG flow, $q=0$ in the interior of the fluid [27]. Thus, the flow is driven only by the buoyancy distribution at the surface.

The governing equations can be non-dimensionalized using character-
istic scales and defining the Rossby number

$$
\epsilon=\frac{U}{f L} \ll 1 .
$$

We then obtain the equations

$$
\begin{align*}
\epsilon \frac{D u}{D t}-v & =-\phi_{x},  \tag{2.6}\\
\epsilon \frac{D v}{D t}+u & =-\phi_{y},  \tag{2.7}\\
\theta & =\phi_{z},  \tag{2.8}\\
u_{x}+v_{y}+\epsilon w_{z} & =0,  \tag{2.9}\\
\frac{D \theta}{D t}+w & =0 \tag{2.10}
\end{align*}
$$

It is convenient to write the variables in terms of three potentials:

$$
\left(\begin{array}{c}
v  \tag{2.11}\\
-u \\
\theta
\end{array}\right)=\nabla \Phi+\nabla \times\left(\begin{array}{c}
F \\
G \\
0
\end{array}\right)
$$

Muraki et al. (1999) show that the variables can be expanded to integer-order in Rossby number ( $u=u^{0}+\epsilon u^{1}+\cdots$ ), and thus the equations above can be further simplified by separating solutions at each order. We must go up to $O(\epsilon)$ to include the effect of vertical velocity, so the equations
are expanded to that order.

$$
\begin{array}{cl}
\nabla^{2} \Phi^{0}=0, & \left(\Phi_{z}^{0}\right)^{s}=\left(\theta^{0}\right)^{s} \\
\nabla^{2} F^{1}=2 J\left(\Phi_{z}^{0}, \Phi_{x}^{0}\right), & \left(F^{1}\right)^{s}=0,  \tag{2.12}\\
\nabla^{2} G^{1}=2 J\left(\Phi_{z}^{0}, \Phi_{y}^{0}\right), & \left(G^{1}\right)^{s}=0, \\
\nabla^{2} \Phi^{1}=\left|\nabla \Phi_{z}^{0}\right|^{2}, & \left(\Phi_{z}^{1}\right)^{s}=\left(\theta^{1}\right)^{s}
\end{array}
$$

The superscript $s$ refers to a condition at the boundary. Note that $F^{1}, G^{1}$ have Dirichlet boundary conditions while $\Phi^{0}, \Phi^{1}$ have Neumann. Once the potentials have been found, the physical variables are calculated from

$$
\begin{array}{rc}
u & \sim-\Phi_{y}^{0}-\epsilon\left(\Phi_{y}^{1}+F_{z}^{1}\right), \\
v & \sim \Phi_{x}^{0}+\epsilon\left(\Phi_{x}^{1}-G_{z}^{1}\right),  \tag{2.13}\\
\theta & \sim \Phi_{z}^{0}+\epsilon\left(\Phi_{z}^{1}+G_{x}^{1}-F_{y}^{1}\right), \\
\epsilon w \sim & \sim\left(F_{x}^{1}+G_{y}^{1}\right) .
\end{array}
$$

### 2.1 Solving Poisson's Equation

In 2002 Hakim et al. [23] determined the particular solutions to the $\mathrm{QG}^{+1}$ potential functions such that their governing equations can be transformed from Poisson equations to Laplace equations:

$$
\begin{align*}
F^{1} & =\Phi_{y}^{0} \Phi_{z}^{0}+\tilde{F}^{1}  \tag{2.14}\\
G^{1} & =-\Phi_{x}^{0} \Phi_{z}^{0}+\tilde{G}^{1}  \tag{2.15}\\
\Phi^{1} & =\frac{1}{2} \Phi_{z}^{0} \Phi_{z}^{0}+\tilde{\Phi}^{1} . \tag{2.16}
\end{align*}
$$

Then

$$
\begin{array}{ll}
\nabla^{2} \tilde{F}^{1}=0, & \tilde{F}^{1 s}=\left(-\Phi_{y}^{0} \Phi_{z}^{0}\right)^{s} \\
\nabla^{2} \tilde{G}^{1}=0, & \tilde{G}^{1 s}=\left(\Phi_{x}^{0} \Phi_{z}^{0}\right)^{s} \\
\nabla^{2} \tilde{\Phi}^{1}=0, & \tilde{\Phi}_{z}^{1 s}=\left(-\Phi_{z}^{0} \Phi_{z z}^{0}\right)^{s} \tag{2.19}
\end{array}
$$

They also proposed that these Laplace equations could be solved by a 2D Fourier Transform in the horizontal variables. For a domain defined by the surface $z=0$ and the interior $z<0$, the solutions are found to be

$$
\begin{align*}
& \hat{\tilde{F}}^{1}=\hat{\tilde{F}}^{1 s} e^{|\vec{k}| z}  \tag{2.20}\\
& \hat{\tilde{G}}^{1}=\hat{\tilde{G}}^{1 s} e^{|\vec{k}| z}  \tag{2.21}\\
& \hat{\tilde{\Phi}}^{1}=\frac{1}{|\vec{k}|} \hat{\tilde{\Phi}}_{z}^{1 s} e^{|\vec{k}| z} \tag{2.22}
\end{align*}
$$

where the hat ^ indicates a Fourier transformed function and $|\vec{k}|$ is the magnitude of the horizontal wavenumber vector. The transform of the boundary condition and the inverse transform to obtain the solution in physical space can be done numerically. In practice, it is the derivatives of these potential functions that are of interest in computing the $O(\epsilon)$ velocities, and these derivatives are also calculated in Fourier space. Derivatives are computed exactly in real space, thus obtaining the $O(1)$ velocities and the particular solutions to the $O(R o)$ potentials. If $O(R o)$ velocities are being calculated, then the derivatives are computed on a grid around the particle positions $(x, y)$ at each $z$ with horizontal resolution of about .01 . The contributions of each vortex are summed, then the surface values are Fourier transformed. From here, $\hat{\tilde{F}}^{1}$, $\hat{\tilde{G}}^{1}$, $\hat{\tilde{\Phi}}^{1}$ and their derivatives are easily calculated in Fourier
space and reverse transformed, thus obtaining the $O(R o)$ velocities.

### 2.1.1 Validation

Because the calculation of $O(R o)$ velocities requires numerical Fourier Transforms and derivative calculations in that space, it is necessary to verify that the solutions are believable. In order to verify the method, an $O(1)$ piecewise continuous vortex solution is considered, the circular version of the vortex presented by Dritschel (2011) [20] which is analyzed in detail in Section 5.

In cylindrical coordinates the governing equations for the circular vortex are

$$
\begin{equation*}
\nabla^{2} \Phi^{0}=0, \quad\left(\Phi_{z}^{0}\right)^{s}=\beta=\beta_{m} \sqrt{1-\frac{r^{2}}{a^{2}}} H(a-r) \tag{2.23}
\end{equation*}
$$

for a radius $a$ and vortex strength $\beta_{m}$, where $H(x)$ is the Heaviside function. The solution comes from (5.2) for the case $b=a$, which can be given by

$$
\Phi^{0}=\left\{\begin{array}{cc}
\frac{\beta_{m}}{8}\left(2-\frac{r^{2}}{a^{2}}\right) & z=0, r \leq a  \tag{2.24}\\
-\frac{\beta_{m}}{4}\left[2 \tan ^{-1}\left(\frac{a}{\sqrt{\sigma}}\right)\left(\frac{r^{2} / 2-z^{2}}{a^{2}}-1\right)\right. & \text { else } \\
\left.-\frac{r^{2}}{a^{2}} \frac{\sqrt{\sigma} / a}{1+\sigma / a^{2}}+\frac{2 z^{2}}{a^{2}} \frac{a}{\sqrt{\sigma}}\right] &
\end{array}\right.
$$

where

$$
\sigma=\frac{a^{2}}{2}\left[\frac{r^{2}+z^{2}}{a^{2}}-1+\sqrt{\frac{4 z^{2}}{a^{2}}+\left(\frac{r^{2}+z^{2}}{a^{2}}-1\right)^{2}}\right] .
$$

For parameters $a=1, \Gamma=\frac{2}{3} \beta_{m} \pi a^{2}=1$ and a domain $-10 \leq(x, y) \leq 10$
with 201 points in each direction, $\psi$ is calculated numerically by two methods, described here. Buoyancy can be given exactly in Fourier space as

$$
\hat{\beta}=\frac{3 \Gamma}{a^{3}|\vec{k}|^{3}}(\sin |\vec{k}| a-|\vec{k}| a \cos |\vec{k}| a)
$$

for a horizontal wavenumber vector of magnitude $|\vec{k}|$. The inverse Fourier Transform of this function is compared to the exact buoyancy in real space in Figure 2.2(a,c). The streamfunction is calculated both from buoyancy given exactly in Fourier Space, $\psi_{1}$, and from the buoyancy given exactly in real space, $\psi_{2}$, and both of these are compared to the exact streamfunction in Figure 2.2(b,d).

Although Gibbs' phenomenon is observed in the inverse transform of $\hat{\beta}$ given in Fourier Space, there are no such problems in the resulting streamfunction $\psi_{1}$. The convergence of both streamfunctions to the exact result for higher resolution and a larger domain is explored for square grids with

$$
N=\{101,201,401,801,1201,1601,2001,2401,2801,3201\}
$$

points in each direction, and a domain length of $\sqrt{N}$. Thus resolution and domain size increase simultaneously. Results are shown in Figure 2.3. Based on the observed convergence, the algorithm is considered reliable and can be applied to the more complicated problem of $O(R o)$ corrections.

The same method is applied to find the potential functions from the equations given by Hakim et al. (2002) [23] in equations (2.17)-(2.19). For $a=1, \Gamma=1$ and the domain $-10 \leq(x, y) \leq 10$ with 401 points in each


Figure 2.2: Comparison of the numerical and analytical solutions for a circular vortex at the surface $z=0$. (a) A comparison of the analytic buoyancy function in real space, $\beta$, (blue) and the inverse transform of the buoyancy function given exactly in Fourier space, $\hat{\beta}$, (red) for parameters $a=1, \Gamma=\frac{2}{3} \beta_{m} \pi a^{2}=1$ and a domain $-10 \leq(x, y) \leq 10$ with 201 points in each direction. (c) The difference between the two buoyancy solutions. (b) A comparison of the exact streamfunction (black), the solution calculated from the buoyancy function given exactly in Fourier Space (blue), and the solution calculated from the buoyancy function given exactly in real space (red). (d) The differences between the two numerical solutions and the exact one in blue and red, respectively.

(a)

Figure 2.3: Residue norms of the numerical solutions for the circular vortex. As the resolution and domain size increase simultaneously, the various difference norms decrease. The square domain length is $\sqrt{N}$ discretized with $N$ points. In the legend, $\psi$ refers to the exact streamfunction, $\psi_{1}$ to the streamfunction numerically calculated from buoyancy given exactly in Fourier space, $\hat{\beta}$, and $\psi_{2}$ to the streamfunction numerically calculated from buoyancy given in real space, $\beta$. The labels "max" and "mean" respectively refer to the maximum and mean of the absolute difference between the numerical streamfunctions and the analytic streamfunction. All are computed at $z=0$.
direction, the $O(1)$ and $O(\epsilon)$ horizontal velocities are compared on planes $z=0$ and $z=-\Delta x=-.05$, one meshgrid from the surface, in Figures 2.42.7. All of the expected symmetries for a circular vortex are exhibited and the singularity at the surface has been eliminated at a depth equal to one horizontal gridpoint length. Note that the computation method is completely independent of the resolution in $z$, so this can be chosen arbitrarily. In Figure 2.8 the maximum $u_{1}$ velocity is compared to the maximum $u_{0}$ velocity with depth, and it is observed that the two velocities decrease at nearly the same rate, with $u_{1}$ slightly slower. Finally, the $O(\epsilon)$ potentials depend quadratically on the vortex strength $\beta_{m}$ while $O(1)$ velocities are linear with this parameter, so this must be chosen carefully so as not to violate the assumptions of the Ro expansion.

Note that $w$ should be zero everywhere due to symmetry, shown here. Converting (2.14)-(2.18) to polar coordinates, we can define

$$
\begin{equation*}
F^{1}=\sin \theta E^{1}, \quad G^{1}=-\cos \theta E^{1} \tag{2.25}
\end{equation*}
$$

where

$$
\begin{equation*}
E^{1}=\Phi_{r}^{0} \Phi_{z}^{0}+\tilde{E}^{1} \tag{2.26}
\end{equation*}
$$

and

$$
\begin{equation*}
\nabla^{2} \tilde{E}^{1}=0, \quad \tilde{E}^{1 s}=\left(-\Phi_{r}^{0} \Phi_{z}^{0}\right)^{s} \tag{2.27}
\end{equation*}
$$

Then

$$
w=F_{x}^{1}+G_{y}^{1}=\cos \theta \sin \theta E_{r}^{1}-\cos \theta \sin \theta E_{r}^{1}=0
$$

This symmetry will be broken when multiple vortices are present.
When $w$ is calculated by taking derivatives of $F$ and $G$ in Fourier


Figure 2.4: The $O(1)$ and $O(\epsilon)$ horizontal velocities for the circular vortex with $a=1, \Gamma=1$ on the $z=0$ plane.


Figure 2.5: The $O(1)$ and $O(\epsilon)$ horizontal velocities for the circular vortex with $a=1, \Gamma=1$ on the $z=0$ plane through $y=0$ for $u$ and $x=0$ for $v$.


Figure 2.6: The $O(1)$ and $O(\epsilon)$ horizontal velocities for the circular vortex with $a=1, \Gamma=1$ on the $z=-\Delta x=-0.05$ plane.


Figure 2.7: The $O(1)$ and $O(\epsilon)$ horizontal velocities for the circular vortex with $a=1, \Gamma=1$ on the $z=-\Delta x=-0.05$ plane through $y=0$ for $u$ and $x=0$ for $v$.


Figure 2.8: The ratio of the maximum $u_{1}$ to the maximum $u_{0}$ with depth.
space, the Gibbs' phenomenon is apparent, as seen at the surface and $z=$ $-\Delta x$ in Figures 2.9 and 2.10, respectively. Alternatively, we can instead compute $F$ and $G$ in Fourier space, then take the derivatives in real space by the MATLAB function gradient. With this analysis, the Gibbs' phenomenon does not appear, and $w$ is of $O\left(10^{-15}\right)$ at the surface. However, below the surface, even this method is inaccurate near the edge of the vortex, as seen in Figure 2.10. In practice, we calculate $w$ by this second method, and look for chaotic advection in the flow between ellipses, therefore avoiding this singularity.

### 2.2 Summary

The Surface Quasigeostrophic (SQG) equations provide an approximation to large-scale ocean flow. The model has two-dimensional dynamics due to buoyancy evolving on the surface, but three-dimensional flow. The governing equations have been expanded about $R o$ to separate the solution at each order, and a numerical method has been presented and validated for calculating $O(1)$ and $O(R o)$ velocities.

Given model problems to the SQG approximation, some of which will be presented in Chapters 4 and 5, the next question is how best to study the effect of vertical velocity on fluid transport. In this system without diffusion, stirring is the relevant property of interest, but tools to measure this have mostly been developed for two-dimensional systems. Thus, these measures must be examined to determine whether they can be applied to three-dimensional flows and how they can be used to understand the underlying geometry of transport. Using an appropriate measure of stirring, the


Figure 2.9: Two calculations of the $O(\epsilon)$ vertical velocity for the circular vortex with $a=1, \Gamma=1$ (a) on the $z=0$ plane, and (b) through $x=$ $\Delta x=0.05$ on this plane. The line $x=0$ was not chosen because both functions are computationally zero along that line. On the left the velocities have been calculated by derivatives of $F, G$ in Fourier space, and on the right the derivatives have been taken in real space using the MATLAB function gradient. Note the severe Gibbs' phenomenon on the left. The velocities on the right are $O\left(10^{-15}\right)$.


Figure 2.10: Two calculations of the $O(\epsilon)$ vertical velocity for the circular vortex with $a=1, \Gamma=1$ (a) on the $z=-\Delta x=-0.05$ plane, and (b) through $x=\Delta x=0.05$ on this plane. The line $x=0$ was not chosen because both functions are computationally zero along that line. On the left the velocities have been calculated by derivatives of $F, G$ in Fourier space, and on the right the derivatives have been taken in real space using the MATLAB function gradient. The Gibbs' phenomenon has lessened on the left as compared to Figure 2.9, but the solution on the right is smoother.
proposed research will examine the fundamental effect of vertical velocity on fluid transport in simple exact solutions to the Surface Quasigeostrophic (SQG) model.

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## 3 Tools for Measuring Transport

The purpose of this work is to understand how transport is affected by a model being perturbed from a two-dimensional Hamiltonian system to a nearly two-dimensional system with weak vertical motion. The project is part of the Multidisciplinary University Research Initiative known as Ocean $3 \mathrm{D}+1$, working to understand and predict ocean flow (funded by the Office of Naval Research). This thesis will contribute to the understanding of the fundamental effect of adding a third dimension to flow in order to gain insight into how current prediction tools can be adapted from two to three dimensions.

Poincaré maps are a useful visualization tool that provide qualitative insight into mixing. For SQG three point vortices (details presented in Chapter 4), the positions of flow particles are strobed at every period of vortex motion in a co-rotating frome. When a particle samples a large area, chaotic mixing is occuring. Sample Poincaré maps are shown in Figure 3.1.

It can be seen in Figure 3.1 that, while the dynamics are purely two-dimensional, the flow has vertical dependence. Even when particles follow regular trajectories (a,b,c), the shapes of those trajectories change with depth. In the case where mixing is exhibited, the barriers of chaotic motion seem to be three-dimensional surfaces, and islands are observed at a


Figure 3.1: Poincaré maps for SQG point vortices for two distinct vortex configurations sampled at three depths. Vortex positions are shown as crosses at the surface in $(\mathrm{a}, \mathrm{d}, \mathrm{g})$. The upper row shows a nonmixing case, $H=0.48$, and it is observed that the paths change with depth. The middle row is a case where three vortices orbit one another, $H=0.54$, and this induces mixing, even at depth, though the chaotic region changes with $z$. The lowest row is a case where two vortices orbit while the third stays apart, $H=0.58$, and this also induces mixing, with very little mixing observed at $z=-0.5$. Compare the $z=0$ plots (a,d,g) to classical solutions in Kuznetsov \& Zaslavsky (1998) [32].


Figure 3.2: A 3D visualization of the chaotic mixing region of a flow, demonstrating that the islands in the Poincaré maps extend as surfaces in three dimensions. This figure was produced by using a set of Poincaré sections to approximate the three-dimensional chaotic region, then using the MATLAB function alphavol [35] to produce the surface containing this region.
range of depths. A three-dimensional visualization of the chaotic mixing is shown in Figure 3.2, and the islands are shown to extend in depth. This figure was produced by using a set of Poincaré maps over a range of depth to approximate the three-dimensional chaotic region, and the surface of this volume was generated using the MATLAB function alphavol [35]. Two islands near $(x, y)=( \pm 0.2,-0.5)$ descend only partway, whereas islands near $(x, y)=( \pm 1,0)$ descend at least as far as was sampled here.

In order to analyze quantitatively the effect of changing flow parameters, it is necessary to use a single measure that represents the global complexity of a flow. Thus far, tools to quantify stirring have largely been applied to two-dimensional systems, taking advantage of the Hamiltonian relation

$$
\dot{x}=-\frac{\partial H}{\partial y}, \quad \dot{y}=\frac{\partial H}{\partial x} .
$$



Figure 3.3: Examples of flow visualization from other Ocean 3D+1 work studying contra-rotating swirling rolls [13]. On the left is a map of Lyapunov exponents, and on the right are horizontal Poincaré sections with three-dimensional trajectories indicated in color.

In a quasi-two-dimensional system with quasi-Hamiltonian dynamics, how can these tools be modified? Other stirring quantifiers, such as those related to Lyapunov exponents, are valid for any number of dimensions, and produce a scalar field. For a complex three-dimensional field, it is not obvious how best to visualize the results in order to understand the transport properties of the flow, and even less straightforward is how to compare the fields of qualitatively similar but distinct flows. Figure 3.3 shows examples of flow visualization from work studying two contra-rotating swirling rolls within the Ocean 3D+1 project [13].

KAM theory [46] examines the stirring that occurs from perturbing a two-dimensional periodic Hamiltonian system to

$$
H=H_{0}(x, y)+\epsilon H_{1}(x, y, t, \epsilon),
$$

Table 3.1: Summary of stirring measures.

| Measure | Global/Local | Output | Notes |
| :---: | :---: | :---: | :--- |
| FTLE | Local | 3D Field | Most commonly used to find LCSs |
| FSLE | Local | 3D Field | Does not find LCSs as well as FTLE |
| Complexity Measures | Local | Scalar of one trajectory | Perhaps better resolution than FTLEs |
| M | Local | Scalar of one trajectory | Computationally cheaper than FTLE |
| Minimal Stretching | Local | Set of Material Lines | Perhaps does not provide a quantification |
| Hyperbolic Trajectories | Local | Locates Barriers | Also does not provide a quantification |
| Mix-Norm | Local | Single value | Dependent on initial state of tracers |
| ESM | Local | 3D Field | Developed for viscous flow |
| Almost-Invariant Sets | Local | 3D Vector Field | Only identifies most coherent structures |
| FTBE | Global | Single value | Independent of initial state of tracers |

and states that most trajectories will remain close to their original periodic orbit, called "quasi-periodic." This work has been extended to three dimensions [39, 40], but the equations retain the Hamiltonian relation in the horizontal. In SQG, the equations themselves are perturbed such that the Hamiltonian relation does not hold at $O(R o)$. Is there a similar analysis for this sort of perturbation? A brief review of current measures and their applicability to this work is provided here, with a summary in Table 3.1.

### 3.1 Lagrangian Coherent Structures

Lagrangian Coherent Structures (LCSs) refer to "special surfaces of fluid trajectories that organize the rest of the flow into ordered patterns" [25]. As such, they are structures that are barriers to transport. By studying how these barriers change with time, it is possible to see how far chaotic stirring regions extend. For example, Figure 3.1(a) indicates that there is a small LCS surrounding each vortex as well as a larger LCS surrounding all three, and as such a fluid particle that begins between these LCSs must always remain there.

### 3.1.1 Finite-Time Lyapunov Exponents

One classical approach to identifying LCSs is calculating the FiniteTime Lyapunov Exponent (FTLE) [44]. This essentially involves integrating a matrix in time using the Jacobian of the velocity field and finding the largest eigenvalues. This corresponds to how much an infinitesimal area will stretch in time. Surfaces of local maxima, known as "ridges," will follow the LCSs. The main limitations of FTLEs are that they are computation-


Figure 3.4: (a-c) FTLEs corresponding to the chaotic Poincaré maps of Figure 3.1(d-f), respectively. Computed using the software package GAIO [17]. All plots use the same color axes shown on the right.
ally intensive and dependent on initial conditions, so that for a unique flow, the FTLE would change based on the initial position of the tracer particle. Furthermore, the convergence of the FTLE (used to approximate the true Lyapunov exponent) is point-dependent.

Current available packages use crude finite difference methods to calculate the FTLE by the trajectories resulting from advecting a grid of particles [47]. Another calculation that may converge faster uses Gram-Schmidt orthonormalization, calculating the Jacobian of the flow field at each time step [14]. An example of FTLE calculated by the GAIO package [17] is shown in Figure 3.4 for a case of three equal strength point vortices. Such a calculation is computationally intensive, and does not provide a good global measure of mixing.

Finally, recent work by other Ocean 3D+1 investigators has explored the approximation of three-dimensional FTLE fields with calculations using only the horizontal velocities (though including vertical shear), thus reducing computational cost (Sulman et al. 2013) [49]. They examined a quadrupole flow with velocities and velocity gradients consistent with ocean conditions and were able to identify LCS ridges accurately. The FTLEs themselves had
magnitudes "comparable to exact values." The approximation is relevant when the FTLEs between the two-dimensional and three-dimensional flow differ by several magnitudes. However, for the purpose of this work, we seek a global measure of mixing, not a local measure such as the FTLE.

### 3.1.2 Finite-Size Lyapunov Exponents

Another similar calculation is the Finite-Size Lyapunov Exponent (FSLE), where some separation factor $r$ is specified, and the flow is integrated until two points initially separated by $\delta$ are separated by $r \delta$. Karrasch \& Haller (2013) [30] determined that the FSLE does not correspond to LCSs as well as the FTLE, and this still does not provide a global measure, so FSLEs are not considered here.

### 3.1.3 Complexity Measures

Within the Ocean 3D+1 project, Rypina et al. (2011) have developed complexity measures and examined their predictive capabilities of LCSs [45]. Two measures are defined - the correlation dimension and the ergodicity defect - which are both measures of the volume covered by a particle trajectory. They show that for a Duffing oscillator, their measures resolve the LCS better than the conventional FTLE computation (by finite difference methods) for the same distribution of points. This method is able to quantify the stirring of a single trajectory, while the conventional FTLE calculation requires a mesh of trajectories. However, this is still a local rather than a global measure of mixing.

### 3.1.4 Measure $M$

Jiminéz Madrid \& Mancho (2009) [29] formulated a function $M$ that calculates the arc length of the trajectory in a given integration time. In 2010, Mendoza \& Mancho [38] demonstrated that these arc length measures matched FTLE computations. Furthermore, calculating $M$ is cheaper than calculating the FTLE because only one point is advected rather than a matrix or mesh. Again, this is trajectory dependent and does not provide a global measure.

### 3.1.5 Minimally Stretching Material Lines

Haller \& Beron-Vera (2012) [24] considered an alternative theory for identifying coherent structures. They seek material lines (a line of particles advected through the flow) that have minimal stretching in the desired time interval. They define minimal stretching using the Cauchy-Green strain tensor (identical to the Jacobian of the vector field used to find FTLEs) and defining a geodesic deviation to find material lines that lie closest to strainlines and shearlines. Although the method identifies transport barriers, it does not seem to provide a measure for the extent of stirring in the flow. The method has also not yet been extended to three-dimensional flow.

### 3.1.6 Almost-Invariant Sets

Froyland (2005) [21] has developed a probabilistic approach to studying structures within the flow. By approximating the flow advection by a transfer operator, he uses its singular vectors to identify regions of the flow that do not disperse, known as almost-invariant sets. In 2008, Froyland \&

Padberg [22] examined the relationship between LCSs and almost-invariant sets. It appears that the almost-invariant sets relate to the most important barriers in the flow, but do not identify all of the structures present.

### 3.2 Hyperbolic Trajectories

"Hyperbolicity" seems to be the same as the FTLE in Wiggins (2005) [54]. Hyperbolic regions can be estimated using "instantaneous stagnation points," and these ISPs are used to initialize an iterative method that converges on the hyperbolic trajectory. Then, the stable and unstable manifolds of the trajectory are calculated. Again, the method locates transport barriers, but does not seem to provide a quantification of global stirring.

### 3.3 Mix-Norm

Mathew et al. (2005) [36] propose a measure of stirring called the Mix-Norm. They claim that, unlike the Lyapunov exponent, "the Mix-Norm helps to address the mixing efficacy of a flow with respect to an initial scalar field or fluid configuration." The Mix-Norm is calculated by integrating $n-1$ mapping functions on some $n$-dimensional torus. While code for computing the mix-norm is available, this measure requires defining an initial state for the tracers, thus introducing a parameter independent of the flow.

### 3.4 Eulerian Symmetry Measures

King et al. (2001) [31] developed Eulerian Symmetry Measures (ESMs) by analyzing how the Bernoulli function changes throughout the flow. This
seems to predict FSLEs, but if these are not useful for predicting flow features then their approximations may not be worth exploring. The ESM is essentially a measure of how some "dynamical" symmetry of the problem changes in time.

### 3.5 Topological Entropy

Topological entropy represents the exponential growth of the number of distinguishable orbits under the repeated iteration of the flow map; the higher the topological entropy, the more chaotic mixing is present. While this provides a measure of global complexity, it is difficult to compute given only a velocity field. However, an approximation of topological entropy exists called the Finite Time Braiding Exponent (FTBE), and code is readily available $[42,9]$.

### 3.5.1 Finite Time Braiding Exponent

Thiffeault \& Budišić have recently developed a tool called Braidlab that, among other functions, calculates the (FTBE), which approximates topological entropy from particle trajectories [50, 11]. The term "braiding" comes from visualizing two-dimensional trajectories on the $x-y-t$ axes, as shown in Figure 3.5 which is reproduced from the Braidlab guide [50]. Over time, the trajectories will twist around one another, forming a "braid," and the complexity of the braid is used as an approximation of the complexity of the flow, i.e. topological entropy.

The benefit of FTBE over the more commonly used FTLE [44] is that the FTBE provides a global measure of complexity as opposed to a local one


Figure 3.5: A schematic of the braid formed from two-dimensional trajectories traced with time as the vertical axis, reproduced from the Braidlab guide [50]. The number of twists increases the complexity of the braid and results in higher FTBE, reflecting chaotic flow.
[2], allowing us to compare quantitatively the extent of stirring exhibited by different flows and thus explore the parameter space. The FTBE is also independent of initial conditions, and so a single measure of complexity will apply to each unique flow, provided the trajectories are all within the same flow regime, i.e. the regime of chaotic mixing [12]. However, Braidlab can only calculate FTBE for two-dimensional flows, because braiding does not have an analogue for three-dimensional trajectories. Despite this constraint, the FTBE still appears to be the best option for obtaining a global measure of mixing that independent of initial conditions. In this thesis, the FTBE is applied to three-dimensional trajectories projected onto the $x-y$ axis. Because vertical flow is relatively weak, this projection does not result in any unphysical trajectory crossings.

### 3.6 Summary

While Poincaré maps and tools that measure local mixing provide visualizations of chaotic regions and flow boundaries, a global measure of complexity that is independent of initial conditions is needed to compare quantitatively flows with different parameters. The Finite Time Braiding Exponent (FTBE) is chosen for this thesis, though trajectories must be projected onto the $x-y$ plane. With this tool, simple model problems will be presented, and the effect of $O(R o)$ velocities will be examined.

Acknowledgements Chapters 1, 2, 3, 4, and 6, in part, have been submitted for publication of the material as it may appear in Chaos, 2016, Taylor, C. K. and Stefan G. Llewellyn Smith. The dissertation author was the primary investigator and author of this paper.

## 4 Point Vortex Model Problem

Due to Coriolis forces, large-scale geophysical flows commonly form swirling structures, where the fluid exhibits strong circular motion in the horizontal. These rotating structures are called vortices. In the ocean, these vortices have physical scales consistent with the Quasigeostrophic assumptions. By analyzing simplified versions of vortex flow, the effect of the $O(R o)$ vertical velocity on transport can be explored. The simplest form of a vortex is a point vortex, constructed from a mathematical singularity. Classical (2D Euler) point vortices have been studied extensively in the examination of regular and chaotic trajectories [4]. The equations of motion are twodimensional and inviscid,

$$
\begin{equation*}
\frac{D \vec{u}}{D t}=0, \quad \nabla \cdot \vec{u}=0 \tag{4.1}
\end{equation*}
$$

Written instead in terms of a streamfunction and vorticity (using the geophysical convention), the equations become

$$
\begin{equation*}
\nabla^{2} \psi_{2 D}=\omega \tag{4.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega=\hat{z} \cdot(\nabla \times \vec{u}), \quad \nabla \psi_{2 D}=(v,-u) \tag{4.3}
\end{equation*}
$$

In the two-dimensional case, the point vortex is defined by a singular point of vorticity, and thus the streamfunction is the free-space solution to

$$
\begin{equation*}
\nabla^{2} \psi_{2 D}=-\kappa \delta\left(x-x_{0}\right) \delta\left(y-y_{0}\right) \tag{4.4}
\end{equation*}
$$

for a point vortex of strength (or circulation) $\kappa$ and position $\left(x_{0}, y_{0}\right)$. The solution is the classical point vortex

$$
\begin{equation*}
\psi_{2 D}=-\frac{\kappa}{2 \pi} \log \left|\vec{x}-\vec{x}_{0}\right| \tag{4.5}
\end{equation*}
$$

For multiple point vortices of strengths $\kappa_{j}$ and positions $\vec{x}_{j}$, the streamfunctions will combine linearly and the vortices themselves will evolve according to

$$
\begin{equation*}
\left(\dot{x}_{i}, \dot{y}_{i}\right)=-\frac{1}{2 \pi} \sum_{j}^{\prime} \frac{\kappa_{j}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{2}}\left(y_{i}-y_{j},-x_{i}+x_{j}\right) \tag{4.6}
\end{equation*}
$$

Moving now to three-dimensional equations, the $O(1)$ equations for SQG from (2.12) are

$$
\begin{equation*}
\nabla^{2} \Phi^{0}=0, \quad\left(\Phi_{z}^{0}\right)^{s}=\left(\theta^{0}\right)^{s} \tag{4.7}
\end{equation*}
$$

where the superscript $s$ indicates that the variable is evaluated at the surface of the domain, conventionally $z=0$, and the subscript $z$ indicates the $z$ derivative.

While the 2D Euler system is governed by the specified vorticity, in SQG the system is governed instead by the buoyancy at the surface. Thus,
the analogous point vortex flow in SQG is found from the definition

$$
\begin{equation*}
\left(\theta^{0}\right)^{s}=\kappa \delta\left(x-x_{0}\right) \delta\left(y-y_{0}\right) . \tag{4.8}
\end{equation*}
$$

The solution is

$$
\begin{equation*}
\Phi^{0}=-\frac{\kappa}{2 \pi} \frac{1}{\left|\vec{x}-\vec{x}_{0}\right|}, \tag{4.9}
\end{equation*}
$$

given in Held et al. [27]. Note that, unlike for classical point vortices, this solution has vertical dependence. For an arbitrary number of vortices with strengths $\kappa_{j}$ and positions $\vec{x}_{j}$, i.e.

$$
\left(\theta^{0}\right)^{s}=\sum_{j} \kappa_{j} \delta\left(x-x_{j}\right) \delta\left(y-y_{y}\right),
$$

the solution is the linear combination

$$
\begin{equation*}
\Phi^{0}=-\frac{1}{2 \pi} \sum_{j} \frac{\kappa_{j}}{\left|\vec{x}-\vec{x}_{j}\right|} . \tag{4.10}
\end{equation*}
$$

The point vortices themselves will be advected by the flow, neglecting the singular velocity contribution of each point vortex at its own location. From conservation of energy in (2.10), w is of order $\epsilon$ and as such is neglected in (4.7). Each vortex then induces only horizontal motion, so the vertical position of each vortex will remain constant. In this work, we consider one active surface at $z=0$, so the vortices are all constrained to that plane. The horizontal evolution of a given point vortex is determined by the sum of contributions of every other vortex in the system, given by

$$
\begin{equation*}
\left(\dot{x}_{i}, \dot{y}_{i}\right)=-\frac{1}{2 \pi} \sum_{j}^{\prime} \frac{\kappa_{j}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{3}}\left(y_{i}-y_{j},-x_{i}+x_{j}\right) \tag{4.11}
\end{equation*}
$$

where the prime indicates that the self-interaction $i=j$ is ignored.

### 4.1 Three Equal Strength Vortex Case

As detailed in Aref's 1983 review [4], for two-dimensional flow, three point vortices are the minimum number that produces chaotic flow. The three vortices will evolve in a regular pattern, but a passive particle (which can be considered as a vortex of strength 0 ) will follow a chaotic trajectory for certain vortex and initial particle positions. Aref's analysis is based solely on Hamiltonian mechanics, and thus applies to SQG as well. Thus, to study chaos in SQG, we consider three SQG point vortices.

To analyze the vortex interaction, we follow the analysis of Kuznetsov \& Zaslavsky (1998) [32] for classical point vortices of equal strength. If all vortices have equal strength, the strength can be absorbed into the length scales and eliminated from the nondimensional equations, i.e. $\kappa_{i}=1$. Kuznetsov \& Zaslavksy consider the position of each vortex as a complex number, $z_{j}=x_{j}+i y_{j}$, and relocate the origin to the center of vorticity

$$
Z=\frac{\sum_{j} \kappa_{j} z_{j}}{\sum_{j} \kappa_{j}}
$$

They then write the vortex positions in terms of action variables $J_{n}, \theta_{n}$.

$$
\begin{equation*}
z_{j}=\frac{1}{\sqrt{3}} \sum_{n=1}^{2} \sqrt{2 J_{n}} \mathrm{e}^{i \theta_{n}} \mathrm{e}^{-2 i \pi n(j-1) / 3} \quad j=1, \ldots, 3 . \tag{4.12}
\end{equation*}
$$

From here, make another change of variables

$$
\begin{array}{ll}
I_{1}=J_{2}-J_{1}, & I_{2}=J_{2}+J_{1}  \tag{4.13}\\
\phi_{1}=\theta_{2}-\theta_{1}, & \phi_{2}=\theta_{2}+\theta_{1}
\end{array}
$$

These new variables $I_{1}$ and $I_{2}$ have geometric significance, with

$$
\begin{equation*}
I_{1}=A / \sqrt{3}, \quad I_{2}=L^{2} / 4 \tag{4.14}
\end{equation*}
$$

where $A$ represents the signed area of the triangle formed by the vortex positions and $L^{2}=\sum \kappa_{j}\left|z_{j}\right|^{2}$ is the angular momentum, a constant.

The vortex dynamics can then be analyzed by taking advantage of the Hamiltonian relations

$$
\begin{equation*}
\dot{I}_{1}=\frac{\partial H}{\partial \phi_{1}}, \quad \dot{I}_{2}=-\frac{\partial H}{\partial \phi_{2}} \tag{4.15}
\end{equation*}
$$

where the Hamiltonian $H$ is the energy of the system, a constant of motion. Here the derivation for SQG deviates from that for classical point vortices as we use the SQG Hamiltonian,

$$
\begin{equation*}
H=\frac{1}{4 \pi} \sum_{i} \sum_{j}^{\prime} \frac{\kappa_{i} \kappa_{j}}{\left|z_{i}-z_{j}\right|}, \tag{4.16}
\end{equation*}
$$

where the prime indicates that $i=j$ is neglected.
Finally, define the "area variable" as

$$
\begin{equation*}
I=\left(\frac{I_{1}}{I_{2}}\right)^{2}=16 A^{2} / 3 L^{4} \tag{4.17}
\end{equation*}
$$

Because $H$ and $I_{2}$ are constants, the evolution of $I$ depends only on $I$. A


Figure 4.1: Potential function $-\dot{I}^{2}(I)$ showing the evolution of equal strength vortex motion for various energies, $H . H_{c}=0.5623$ is the critical energy that divides the two regimes of motion.
potential function is defined as $-\dot{I}^{2}$ such that where this potential is negative the solution is real, and thus the curves of $\dot{I}^{2}(I)$ can be interpreted as potential wells. These potential curves are shown in Figure 4.1 for various energies. There are two regimes of motion visible, separated by the critical energy $H_{c}=0.5623$. In the higher energy regime, two of the vortices are close enough that they will orbit one another, while the vortex further away remains separate. In the other regime of motion, all three vortices will orbit one another.

These potential wells can be used to determine the period of motion from the integral

$$
\begin{equation*}
T=2 \int_{I_{\min }}^{I_{\max }} \frac{d I}{|\dot{I}|} \tag{4.18}
\end{equation*}
$$

where $I_{\min }$ and $I_{\max }$ refer to the intersections of the well with the $-\dot{I}^{2}=0$ axis, and the factor of 2 is needed to account for the return from $I_{\max }$ to $I_{\text {min }}$ over one cycle. After one cycle $I$ will return to its original value, meaning the vortex triangle has its original area, but the vortices will be permuted
among the triangle vertices. This can be seen clearly in the co-rotating frame (explained below) in Figure 4.3, where in (a) all three vortices orbit and in (b) only two vortices are orbiting. At time $T$, the area has returned to its original value, but the orbiting vortices have changed positions. A higher energy results from two of the vortices orbiting at close range, so the period of vortex motion at higher values of $H$ will be the area variable period $T$ multiplied by an additional factor of 2 . Then for lower energies the the three vortices will permute with one another; thus the period of vortex motion will be $T$ multiplied by an additional factor of 3 .

$$
T_{v}= \begin{cases}3 T & \text { if } H<H_{c}  \tag{4.19}\\ 2 T & \text { if } H>H_{c}\end{cases}
$$

The dependence of $T_{v}$ on $H$ across both regimes of motion is shown in Figure 4.4. Note the singularity at the boundary between the two regimes motion, $H=H_{c}=0.5623$. Referring to Figure 4.1, it can be seen that the $H_{c}$ potential well has a decaying approach to $I_{\min }=0$, implying that the solution will take infinitely long to reach this turning point in $I$ and thus resulting in the singularity in $T_{v}$ exhibited in Figure 4.4.

Figure 4.2 also reveals that the vortices slowly rotate about their center of vorticity. The presence of a rotation becomes clear when $z_{j}$ is written in terms of the new variables.

$$
\begin{align*}
z_{j}(t)=\frac{L}{\sqrt{6}} e^{i \phi_{2}(t) / 2}[ & \left(1-I^{1 / 2}(t)\right)^{1 / 2} e^{-2 \pi i(j-1) / 3} e^{-i \phi_{1}(t) / 2} \\
& \left.+\left(1+I^{1 / 2}(t)\right)^{1 / 2} e^{-4 \pi i(j-1) / 3} e^{i \phi_{1}(t) / 2}\right] . \tag{4.20}
\end{align*}
$$



Figure 4.2: Two regimes of motion are observed for equal strength vortices, (b) lower-energy three-vortex orbits and (a) higher-energy two-vortex orbits. The points indicate initial positions of vortices and the lines indicate trajectories in time. It is observed that the vortices permute and also rotate in time, resulting in braid-like trajectories.


Figure 4.3: Equal strength vortex trajectories in the co-rotating frame (a) up to and (b) just before time $T$, demonstrating the necessary factors for computing $T_{v}$ in (4.19). For (a) the lower-energy three-vortex orbits, a triangle connecting the vortices at $t=0$ (solid) and $t=T$ (dot-dashed) is also shown. The two triangles are equivalent, but the vortex associated with each vertex has changed, therefore requiring a factor of 3 to return each vortex to its original position. For (b) the higher-energy two-vortex orbit, the trajectories are shown for a time just less than $T$ to more clearly show the dynamics. The vortex that remains separate is about to return to its original position, but the two orbiting vortices will have changed positions. Therefore this regime requires a factor of 2 to return each vortex to its original position.


Figure 4.4: Period of motion for equal strength vortices $T_{v}$ vs. $H$. The vertical dotted line shows the boundary between the two regimes of motion, $H_{c}=0.5623$.

Note that the term in the brackets is identical at times $t=0$ and $t=T_{v}$. Therefore, at $t=T_{v}$, each vortex has returned to its original position with a rotation about the center of vorticity of $\phi_{2}\left(T_{v}\right) / 2$. This rotation can be calculated in a similar manner to the period of motion. Using

$$
\begin{equation*}
\dot{\phi}_{2}=-\frac{\partial H}{\partial I_{2}} \tag{4.21}
\end{equation*}
$$

we find

$$
\begin{equation*}
\phi_{2}\left(T_{v}\right)=2 \int_{I_{\min }}^{I_{\max }} \frac{\dot{\phi}_{2}}{|\dot{I}|} d I . \tag{4.22}
\end{equation*}
$$

With this shift, the motion of the vortices is described in its entirety. Vortex trajectories in the co-rotating frame are shown in Figure 4.5. For these plots and all equal strength calculations, we have normalized the horizontal coordinates by $L$.


Figure 4.5: Equal strength vortex trajectories for $H=$ (a) 0.54 , (b) 0.56 , (c) 0.58 in the co-rotating frame. These demonstrate the two regimes of motion from (4.19) and Figure 4.1.

### 4.1.1 Computing Integrals

The integrals of (4.18) and (4.22) are not trivial to calculate. From the change to action variables $I_{n}, \phi_{n}$, we obtain

$$
\begin{align*}
\left|z_{1}-z_{2}\right| & =\sqrt{2 I_{2}+2 \sqrt{I_{2}^{2}-I_{1}^{2}} \sin \left(\phi_{1}+\pi / 6\right)}  \tag{4.23}\\
\left|z_{1}-z_{3}\right| & =\sqrt{2 I_{2}-2 \sqrt{I_{2}^{2}-I_{1}^{2}} \sin \left(\phi_{1}-\pi / 6\right)}  \tag{4.24}\\
\left|z_{2}-z_{3}\right| & =\sqrt{2 I_{2}-2 \sqrt{I_{2}^{2}-I_{1}^{2}} \cos \left(\phi_{1}\right)}  \tag{4.25}\\
H & =\frac{1}{2 \sqrt{2} \pi L}\left[\frac{1}{\left|z_{1}-z_{2}\right|}+\frac{1}{\left|z_{2}-z_{3}\right|}+\frac{1}{\left|z_{1}-z_{3}\right|}\right] \tag{4.26}
\end{align*}
$$

In Kuznetsov and Zaslavsky (1998) [32], the Hamiltonian for classical point vortices was written explicitly in terms of $\cos 3 \phi_{1}$ Further analysis shows that the SQG Hamiltonian is also dependent on $\cos 3 \phi_{1}$ by the more complicated relation

$$
\begin{equation*}
\left(8 H^{2} \pi^{2} L^{2} P_{123}-9 I_{2}^{2}-3 I_{1}^{2}\right)^{2}=4 P_{123}\left(6 I_{2}+4 \sqrt{2} H \pi L \sqrt{P_{123}}\right) \tag{4.27}
\end{equation*}
$$

where

$$
P_{123}=\left|z_{1}-z_{2}\right|\left|z_{1}-z_{3}\right|\left|z_{2}-z_{3}\right|=\sqrt{2 I_{2}\left(I_{2}^{2}+3 I_{1}^{2}\right)-2\left(I_{2}^{2}-I_{1}^{2}\right)^{3 / 2} \cos 3 \phi_{1}} .
$$

Thus, for a given $\left(I_{1}, I_{2}\right)$ and $H>0$, there is a unique $\phi_{1}$ bounded by $0<\phi_{1}<\pi / 3$.

We then calculate

$$
\begin{align*}
\dot{I}_{1}=\frac{\partial H}{\partial \phi_{1}}= & \frac{1}{2 \sqrt{2} \pi L} \sqrt{I_{2}^{2}-I_{1}^{2}}\left[\frac{-\cos \left(\phi_{1}+\pi / 6\right)}{\left|z_{1}-z_{2}\right|^{3}}\right.  \tag{4.28}\\
& \left.+\frac{\cos \left(\phi_{1}-\pi / 6\right)}{\left|z_{1}-z_{3}\right|^{3}}-\frac{\sin \phi_{1}}{\left|z_{2}-z_{3}\right|^{3}}\right]
\end{align*}
$$

and from here easily find $\dot{I}=2 I_{1} \dot{I}_{1} / I_{2}^{2}$.
From the initial conditions of the vortices, the constants of motion $I_{2}=L^{2} / 4$ and $H$ are obtained. The Hamiltonian determines the regime of motion, which determines whether the origin is a zero of $\dot{I}$. The other zeros of $\dot{I}$ are found numerically with MATLAB's fzero function. Finally, to calculate (4.18), we must make a final change of variables to avoid the singularities of calculating $1 /|\dot{I}|$ at its zeros. After the change $I=I_{\min }+\left(I_{\max }-I_{\min }\right) \sin ^{2} \alpha$, the integral becomes

$$
T=2 \int_{0}^{\pi / 2} \frac{2\left(I_{\max }-I_{\min }\right) \sin \alpha \cos \alpha}{|\dot{I}(I)|} d \alpha
$$

with the singularity eliminated. In practice, we evaluate the integral from $\left[10^{-6}, \pi / 2-10^{-6}\right]$.

The same procedure is followed to evaluate (4.22) using

$$
\begin{align*}
\dot{\phi}_{2}=-\frac{\partial H}{\partial I_{2}}=\frac{1}{2 \sqrt{2} \pi L} & {\left[\frac{1}{\left|z_{1}-z_{2}\right|^{3}}\left(1+\frac{I_{2}}{\sqrt{I_{2}^{2}-I_{1}^{2}}} \sin (\phi+\pi / 6)\right)\right.} \\
& +\frac{1}{\left|z_{1}-z_{3}\right|^{3}}\left(1-\frac{I_{2}}{\sqrt{I_{2}^{2}-I_{1}^{2}}} \sin (\phi-\pi / 6)\right)  \tag{4.29}\\
& \left.+\frac{1}{\left|z_{2}-z_{3}\right|^{3}}\left(1-\frac{I_{2}}{\sqrt{I_{2}^{2}-I_{1}^{2}}} \cos \phi\right)\right]
\end{align*}
$$

From these integrals, the equal strength vortex period and rotation are calculated, and the flow simulations are transformed into the co-rotating frame.

### 4.2 Three Arbitrary Strength Vortex Case

Aref (1979) described the motion of three classical vortices of arbitrary strength by noting that these sums are constants of motion

$$
\sum_{i} \kappa_{i} x_{i}, \quad \sum_{i} \kappa_{i} y_{i}, \quad \sum_{i} \kappa_{i}\left(x_{i}^{2}+y_{i}^{2}\right)
$$

and therefore the sum

$$
\frac{1}{2} \sum_{i, j} \kappa_{i} \kappa_{j} l_{i j}^{2}, \quad l_{i j}=\left|z_{i}-z_{j}\right|
$$

is constant and independent of the choice of coordinates. Additionally, the Hamiltonian is a constant. With the convention $\kappa_{1} \geq \kappa_{2}>0$, the above can be used to define a constant parameter $C$ such that

$$
\kappa_{1} \kappa_{2} l_{12}^{2}+\kappa_{2} \kappa_{3} l_{23}^{2}+\kappa_{3} \kappa_{1} l_{31}^{2}=3 \kappa_{1} \kappa_{2} \kappa_{3} C .
$$

$C$ is essentially a time scale for relative motion. Note that $L=\sum_{i} \kappa_{i}\left(x_{i}^{2}+y_{i}^{2}\right)$, and with the origin defined as the center of vorticity,

$$
L=3 \frac{\prod_{j} \kappa_{j}}{\sum_{j} \kappa_{j}} C
$$

so fixing $L=1$ in the equal strength case where all $\kappa_{i}=1$ is equivalent to fixing $C=1$.

For nonzero $C$ we can define trilinear coordinates

$$
b_{1}=\frac{l_{23}^{2}}{\kappa_{1} C}, \quad b_{2}=\frac{l_{13}^{2}}{\kappa_{2} C}, \quad b_{3}=\frac{l_{12}^{2}}{\kappa_{3} C},
$$

with

$$
b_{1}+b_{2}+b_{3}=3 .
$$

Additionally, the physical regime is where the vortex positions can form a triangle, which in trilinear coordinates is expressed as

$$
\left(\kappa_{1} b_{1}\right)^{2}+\left(\kappa_{2} b_{2}\right)^{2}+\left(\kappa_{3} b_{3}\right)^{2} \leq 2\left(\kappa_{1} \kappa_{2} b_{1} b_{2}+\kappa_{2} \kappa_{3} b_{2} b_{3}+\kappa_{1} \kappa_{3} b_{1} b_{3}\right)
$$

From here, the SQG analysis differs from Aref's because we introduce the SQG Hamiltonian

$$
H=\frac{1}{4 \pi} \sum_{\alpha, \beta}^{\prime} \frac{\kappa_{\alpha} \kappa_{\beta}}{l_{\alpha \beta}} .
$$

Rewriting this in terms of trilinear coordinates we find

$$
\begin{aligned}
& H=\frac{\kappa_{1} \kappa_{2} \kappa_{3}}{2 \pi|C|^{1 / 2}}\left(\frac{1}{\left|b_{1} \kappa_{1}\right|^{1 / 2} \kappa_{1}}+\frac{1}{\left|b_{2} \kappa_{2}\right|^{1 / 2} \kappa_{2}}+\frac{1}{\left|b_{3} \kappa_{3}\right|^{1 / 2} \kappa_{3}}\right), \\
& \frac{1}{\left|b_{1} \kappa_{1}\right|^{1 / 2} \kappa_{1}}+\frac{1}{\left|b_{2} \kappa_{2}\right|^{1 / 2} \kappa_{2}}+\frac{1}{\left|b_{3} \kappa_{3}\right|^{1 / 2} \kappa_{3}}=\frac{2 \pi H|C|^{1 / 2}}{g^{3}}=\theta
\end{aligned}
$$

where $g=\left(\kappa_{1} \kappa_{2} \kappa_{3}\right)^{1 / 3}$ is the geometric mean. The second line defines a new constant of motion $\theta$ that specifies the phase trajectory in trilinear coordinates.

A particular set of vortex properties will correspond to a point in the plane of trilinear coordinates within the physical regime (see Figure 4.6). As time evolves, the vortices will trace the phase trajectory curve. If the trajectory goes off to infinity, as in plot (c), the vortices scatter. Intersections of the trajectories with the physical regime boundary are points where the vortices are collinear. At the fixed points of the trajectories, which are at the center of the concentric curves in plots (a) and (b) and at trajectory intersections in plots (c) and (d), the vortices exhibit rigid motion. As for classical two-dimensional point vortices, this fixed point is

$$
\begin{equation*}
\left(b_{1}^{\star}, b_{2}^{\star}, b_{3}^{\star}\right)=\frac{1}{h}\left(\frac{1}{\kappa_{1}}, \frac{1}{\kappa_{2}}, \frac{1}{\kappa_{3}}\right), \tag{4.30}
\end{equation*}
$$

where

$$
h=\frac{1}{3}\left(\frac{1}{\kappa_{1}}+\frac{1}{\kappa_{2}}+\frac{1}{\kappa_{3}}\right)
$$

is the harmonic mean.
If $C$ is zero, the analysis is much the same with the change

$$
\begin{gathered}
b_{1}=\frac{l_{23}^{2}}{\kappa_{1}}, \quad b_{2}=\frac{l_{13}^{2}}{\kappa_{2}}, \quad b_{3}=\frac{l_{12}^{2}}{\kappa_{3}}, \\
b_{1}+b_{2}+b_{3}=0 .
\end{gathered}
$$

Expressed in terms of only $\left(b_{1}, b_{2}\right)$, the trajectories will be given by

$$
\frac{1}{\left|b_{1} \kappa_{1}\right|^{1 / 2} \kappa_{1}}+\frac{1}{\left|b_{2} \kappa_{2}\right|^{1 / 2} \kappa_{2}}+\frac{1}{\left|\left(b_{1}+b_{2}\right) \kappa_{3}\right|^{1 / 2} \kappa_{3}}=\theta
$$



Figure 4.6: Phase trajectories for various vortex strengths, with the value of $\theta$ for each trajectory indicated. The three axes represent trilinear coordinates and the black curve shows the physical regime boundary. Compare to Figures 2, 3, and 4 in Aref (1979) [3] .


Figure 4.7: Phase trajectories for the special case $C=0$ and two vortex strength combinations, with the value of $\theta$ for each trajectory indicated. The axes represent $b_{1}, b_{2}$ with $b_{3}=-b_{1}-b_{2}$, and the black lines show the physical regime boundary. In the case of (a), the physical regime boundaries lie along the $\left(b_{1}, b_{2}\right)$ axes. Compare to Figures 5 and 6 in Aref (1979) [3].
as shown in Figure 4.7.
Unlike the equal strength case, for arbitrary strength vortices the period and shift of vortex motion cannot be written explicitly, but they can be computed from the resulting vortex trajectories. We perform this calculation by finding where $A$ and its derivative return to their initial values via linear interpolation between the two nearest time steps. The regime of motion is determined by observation, and the rotation angle about the center of vorticity is determined by interpolating the second vortex's position (chosen arbitrarily) at time $T$ and comparing this to its initial position. Due to the possibility of interpolation errors, all resulting vortex trajectories are visually confirmed to be periodic. Some examples of resulting vortex trajectories are shown in Figure 4.8. The resulting transport for the $O(1)$ vortex solutions will be presented in Section 4.4.


Figure 4.8: Select examples of vortex trajectories for the three arbitrary strength point vortex solution.

## 4.3 $O($ Ro $)$ Velocities

The small $\epsilon$-order, or Ro-order, corrections to the velocities from Muraki et al. (1999) [41] are given in (2.13). The $O(R o)$ solutions include derivatives of the $O(1)$ solution, and in the point vortex case, where the $O(1)$ solution is singular, these terms are very problematic. However, the vertical velocity can instead be obtained from the energy conservation equation

$$
\frac{D \theta}{D t}+w=0
$$

For the point vortex solution, it is found that

$$
\begin{equation*}
w^{1}=3 z \sum_{j} \frac{\kappa_{j}}{2 \pi} \frac{\left(\vec{u}-\dot{\vec{x}}_{j}\right) \cdot\left(\vec{x}-\vec{x}_{j}\right)}{\left|\vec{x}-\vec{x}_{j}\right|^{5}} \tag{4.31}
\end{equation*}
$$

where

$$
\vec{u}-\dot{\vec{x}}_{i}=\sum_{j}^{\prime} \frac{\kappa_{j}}{2 \pi}\left[\frac{\left(-y+y_{j}, x-x_{j}, 0\right)}{\left|\vec{x}-\vec{x}_{j}\right|^{3}}-\frac{\left(-y_{i}+y_{j}, x_{i}-x_{j}, 0\right)}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{3}}\right]
$$

and the prime on the sum indicates that the $i=j$ term is not considered. It is thus possible to examine the effect of vertical velocity on mixing, though in this case the solution is not dynamically consistent because the velocity is not divergenceless at $O(\epsilon)$.


Figure 4.9: Poincaré maps for three SQG equal strength point vortices for two distinct vortex configurations sampled at three depths. Vortex positions are shown as crosses at the surface in ( $\mathrm{a}, \mathrm{d}, \mathrm{g}$ ). The upper row shows a nonmixing case, $H=0.48$, and it is observed that the paths change with depth. The middle row is a case where three vortices orbit one another, $H=0.54$, and this induces mixing, even at depth, though the chaotic region changes with $z$. The lowest row is a case where two vortices orbit while the third stays apart, $H=0.58$, and this also induces mixing, with very little mixing observed at $z=-0.5$. Compare the $z=0$ plots ( $\mathrm{a}, \mathrm{d}, \mathrm{g}$ ) to classical solutions in Kuznetsov \& Zaslavsky (1998) [32].


Figure 4.10: The power spectrum density (PSD) estimated by MATLAB's pwelch function for $H=0.54$ at $z=0$ for (a) the $x$ and (b) $y$ components of vortex trajectories. The range of energetic frequencies indicates chaotic trajectories.

### 4.4 Point Vortex FTBE Results

### 4.4.1 Three Equal Strength Vortex Case

The presence of mixing can be confirmed by visualizing transport with Poincaré maps. Select maps for $O(1)$ flow at several depths are given in Figure 4.9. Trajectory positions are strobed for every period of vortex motion, and where trajectories cover a large area, there is chaotic mixing. Chaos can also be confirmed by the range of energetic frequencies in the power spectrum density (PSD), estimated by MATLAB's pwelch function and shown in Figure 4.10 for $H=0.54$ at $z=0$.

The FTBE is dependent on the number of trajectories, so we fix our analysis to include 64 trajectories and integrate over 90 periods of vortex motion. As an estimate, the FTBE also varies slightly according the initial conditions of the trajectories. To quantify this variation, 13 different sets of 64 trajectories were used to generate a mean FTBE and standard deviation


Figure 4.11: Sample FTBE calculations over time for both the $O(1)$ and $O(R o)$ equal strength point vortex solutions at $z=-0.25$.


Figure 4.12: The mixing induced by the $O(1)$ solution of equal strength point vortices across a wide range of energies. The vertical line indicates the boundary between the two regimes of motion, given by $H_{c}=0.5623$.
for each flow considered. For the equal strength case, we have fixed $L=1$, so the only varying parameter is $H$. For sample cases of both the $O(1)$ and $O(R o)$ equal strength point vortex solutions at $z=-0.25$, Figure 4.11 shows that both the mean and standard deviation of the calculated FTBEs converge well over time.

FTBE vs. $H$ for the $O(1)$ equal strength solution is shown over a wide range of $H$ in Figure 4.12. There is a minimum $H_{\min } \approx 0.49$ that allows for mixing. Above this minimum, the level of mixing is relatively constant in


Figure 4.13: For three equal strength SQG vortices, the FTBE is calculated for various vortex configurations characterized by the Hamiltonian, $H$, plotted (a) at the surface and (c) with depth. For comparison, the FTBE for three equal strength two-dimensional vortices are shown in (b). Error bars of the FTBE are determined by statistical analysis of several choices of trajectory subsets. The vertical dotted line indicates the boundary between the two regimes of vortex motion, which for SQG is $H_{c}=0.5623$ and for 2D is $H_{c}=\frac{\log 2}{4 \pi}$.
the lower energy regime, then there is a dip in the extent of mixing near the boundary between regimes of motion. In the higher energy regime, mixing increases with $H$ up to a maximum, and then falls to zero as the vortex separation becomes too large to mix effectively. The window around the regime boundary is further examined and compared to the classical twodimensional solution for an analogous regime in Figure 4.13(a,b). For both solutions the boundary between the two regimes of vortex motion is indicated with the vertical dotted line, and it seems that both show qualitatively similar behavior, but the SQG case exhibits higher levels of mixing than the classical case. Finally, in Figure 4.13(c) the FTBE is calculated at increasing depth, where the plane of vortex motion is $z=0$. While the vortices still produce mixing at a depth close to the surface, the FTBE falls steeply at depths below a critical depth of approximately $z=-0.25$. This is consistent with qualitatively simpler Poincaré maps generated at depth $z=-0.5$ shown in Figure 4.9(f).

## Including $w$

As described in Section 4.3, obtaining the full $O(R o)$ corrections is not easily done. However, the vertical velocity is easily found and given in (4.31). By adding only $w$ to the solution, we do not have a dynamically consistent flow, but can still obtain some insights into the effect of weak vertical flow on mixing. FTBE results comparing the $O(R o)$ and $O(1)$ solutions is shown in Figure 4.14 for $R o=0.01,0.1$. At the surface, the solutions are identical, which follows from the constraint that $w=0$ at the surface. At depths below the surface, including $w$ seems to increase mixing, with larger $w$ (resulting from larger $R o$ ) producing more mixing. This difference in mixing is also


Figure 4.14: Comparison between the $O(1)$ equal strength point vortex solutions and those including $w$ at height (a) $z=0$, (b) $z=-0.25$, and (c) $z=-0.5$. Since $w=0$ at the surface, it is expected that the solutions overlap as seen in (a).


Figure 4.15: Poincaré maps of the $O(1)$ equal strength point vortex solutions and those including $w$, all at $z=-0.25$. Representative values $H=0.54$ and $H=0.58$ are shown for $(\mathrm{a}, \mathrm{d}) R o=0,(\mathrm{~b}, \mathrm{e}) R o=0.01$, and $(\mathrm{c}, \mathrm{f}) R o=0.1$.
visualized by Poincaré maps in Figure 4.15. In the case with $w$, the particle positions are projected into the $x-y$ plane to obtain the maps. Here it appears that the area of the islands around each vortex increases with $R o$, but the edge of the outer mixing boundary seems slightly larger in (c,d) for $R o=0.1$.

Statistics of particle depth for the $O(R o)$ solution are given in Figure 4.16, indicating how particles tend to move between levels. For particles initiated at $z=-0.25$ in (a,b), the particles stay generally around their initial depth, with the statistical spread of particles increasing with Ro. At the lower initial depth of $z=-0.5$ in ( $\mathrm{c}, \mathrm{d}$ ), the particles trend away from this initial depth, increasing slightly in the case $H=0.54$, Ro $=0.1$ and decreasing notably in the case $H=0.58, R o=0.01,0.1$. Note, however, that there is negligible mixing at this depth, from 4.14(c).


Figure 4.16: Statistics of particle height over time from the equal strength point vortex solution including $w$. Representative values $H=0.54$ and $H=0.58$ are shown for particles initially on $(\mathrm{a}, \mathrm{c}) z=-0.25,(\mathrm{~b}, \mathrm{~d}) z=-0.5$.

We can also examine how varying parameters $R o$ as well as $L$ may vary the trend of FTBE with depth. There is no characteristic vertical length, but as seen in Figure 4.17(a,c,d), varying the characteristic horizontal length $L$ will change the critical depth, from approximately $z=-0.075$ in the $L=0.1$ case to approximately $z=-0.75$ for $L=10$. This relation is further explored in Figure 4.18, where it can be seen that the critical depth seems to change according to $L^{1 / 2}$. The comparisons of Figure 4.17(a,c,d) additionally shows that the extent of mixing has decreased, which follows intuitively from the weaker velocities due to larger vortex spacing. As seen in Figure $4.17(\mathrm{~b})$, changing Ro does not affect this critical depth, but does seem to cause a sharper dropoff in FTBE for higher energies. This may be due to an attraction towards periodic trajectories, so that particles below the critical depth tend to drift deeper and no longer experience mixing. Note also that the larger islands with larger $R o$ in Figure 4.15 seem to also indicate an attraction toward periodic trajectories. However, the complexity of the flow still seems to increase in Figure 4.14. More examination is needed to fully understand this trend.

### 4.4.2 Three Arbitrary Strength Vortex Case

From the phase trajectories in Figure 4.6, it is clear that 5 constants must be designated to specify one unique arbitrary strength solution: three vortex strengths define the phase space, $\theta$ specifies a particular phase trajectory, and $C$ defines the time scale of vortex motion evolving along that trajectory. This parameter space is therefore very large, and as a preliminary examination only select slices are considered here. The integration time was also limited to only 60 periods of vortex motion.


Figure 4.17: Comparing the $O(1)$ equal strength point vortex solutions of FTBE vs. depth under changes to (a) $L$ and (b) Ro. These plots show there is a critical depth beyond which the FTBE decreases sharply. For $L=1$ the critical depth appears to be $z=-0.25$, and increasing $R o$ increases the dropoff of FTBE for higher energies. For $L=10$, the critical depth has increased to $z=-0.75$, and mixing has decreased.

(a)

Figure 4.18: How the critical depth, below which the FTBE falls steeply, changes with $L$ in the $O(1)$ equal strength vortex solution. For comparison, the line of $L^{1 / 2}$ is also shown.

In the first case, the three initial distances between vortices are designated as $l_{13}=l_{12}=2 l_{23}=0.2$ and the vortex strengths are constrained by $\kappa_{1}=\kappa_{2}$. With this it is possible to solve for both these strengths as well as the third vortex strength from specifying two constants of motion. The parameter domain was chosen to be extensive, $-100<\theta<100,-1<C<1$, and a grid of 256 samples was examined. For these calculations, horizontal lengths are not normalized by $L$, as this results in changing $C$. All the simulations exhibited mixing, and the resulting FTBE's across this course grid are shown in Figure 4.19. Although only preliminary, the results seem to indicate a line in $\theta-C$ space of maximum FTBE. A more rigorous computational exploration of the parameter space would be needed to fully uncover the trends in FTBE.

As a second case, the constant of motion $C$ is fixed as unity as it was in the equal strength case, and the vortex strengths are constrained by $\kappa_{1}=$ $\kappa_{2}=1$. Then, the system is determined by the $\theta-\kappa_{3}$ parameter space. Based on phase trajectory plots, a representative parameter domain was chosen to


Figure 4.19: Course FTBE results for arbitrary vortex strength cases shown for (a) the $\theta-C$ parameter space where the initial configuration is fixed and $\kappa_{1}=\kappa_{2}$, and (b) the $\theta-\kappa_{3}$ parameter space where $\kappa_{1}=\kappa_{2}=1$ and $C=1$. Negative FTBE represent nonphysical solutions, also separated by a black contour, whereas zero FTBE is physical but not energetic enough to induce mixing.
be $-2<\theta<5,-8<\kappa_{3}<8$. In this case, many parameter gridpoints did not have mixing, either because there was not a physical solution (marked as negative and separated by a black contour in Figure 4.19(b)), or because the solution resulted in two vortices orbiting so energetically as to dominate the dynamics, resulting in periodic flow trajectories (marked as zeros).

### 4.5 Summary

SQG point vortices provide a mathematically simple exact solution for which the surrounding flow can be given explicity. Both equal and arbitrary strength solutions of three interaction vortices exhibit regular vortex motion, and the vortex motion is expressed in terms of a period and slow rotation about the center of vorticity, which are calculated directly for equal strength vortices and from vortex trajectories for arbitrary strength. There are two
regimes of motion, one in which all three vortices orbit one another, and one in which two vortices set up an orbit and the third vortex remains apart. A flow is uniquely determined by 5 constants, such as the three vortex strengths, $H$ or $\theta$, and $L$ or $C$. Of the $O(R o)$ corrections, only vertical velocity can be computed due to the singularities of the $O(1)$ streamfunction.

The three-vortex solution was shown to produce chaotic trajectories in the surrounding flow, visualized by Poincaré maps and quantified by the FTBE. The FTBE exhibits a local minimum at the boundary between flow regimes in the case of equal strength vortices. There is also a critical depth below which the FTBE falls steeply which is dependent on the constant of motion $L=\sum_{i} \kappa_{i}\left|z_{i}\right|^{2}$. Adding $O(R o)$ vertical velocity increases the FTBE and results in larger barriers to chaotic mixing surrounding the vortices. Preliminary results for arbitrary strength vortices indicate a possible ridge of increased mixing in $\theta-C$ space, but additional examination of the multidimensional parameter space is required to make rigorous conclusions.

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## 5 Elliptic Vortex Model Problem

For a solution that is dynamically consistent, even at $O(R o)$, a different model problem must be used. Harvey \& Ambaum (2011) [26] found that discontinuities in surface buoyancy necessarily result in infinite velocities at those locations. Thus, a continuous buoyancy distribution is needed. One such solution was detailed by Dritschel (2011) [19], who considered the exact solution in QG for an ellipsoid containing a region of constant potential vorticity, which will rotate and maintain its shape (see also Dritschel et al. (2004) [20]). Dritschel orients this ellipsoid along the $z=0$ plane so that one of its axes is vertical, then takes the limit as that axis length goes to zero. The potential vorticity is then contained on the boundary, and thus is an exact solution to SQG. The resulting governing equations are

$$
\begin{equation*}
\Delta \Phi^{0}=0, \quad\left(\Phi_{z}^{0}\right)^{s}=\beta(x, y)=\beta_{m} \sqrt{1-\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}} \tag{5.1}
\end{equation*}
$$

where $a, b$ are the semi-major and -minor axis lengths, respectively, aligned along the Cartesian axes, and $\beta_{m}$ is a constant designating the strength of the ellipse. The streamfunction exterior to the ellipsoid was derived by Laplace (1784) [16], and after taking the vertical axis to zero it becomes

$$
\begin{equation*}
\Phi^{0}=-\beta_{m} \frac{a b}{4} \int_{\sigma}^{\infty} \frac{d u}{\sqrt{\left(u+a^{2}\right)\left(u+b^{2}\right) u}}\left(1-\frac{x^{2}}{u+a^{2}}-\frac{y^{2}}{u+b^{2}}-\frac{z^{2}}{u}\right) \tag{5.2}
\end{equation*}
$$



Figure 5.1: An example of a streamfunction induced by a Dritschel elliptic vortex, shown at the surface and in the rotating reference frame of the ellipse, from Dritschel (2011) [19]. The ellipse will maintain its shape as it rotates.
where $\sigma$ is the largest root of

$$
\begin{equation*}
\frac{x^{2}}{\sigma+a^{2}}+\frac{y^{2}}{\sigma+b^{2}}+\frac{z^{2}}{\sigma}=1 . \tag{5.3}
\end{equation*}
$$

Dritschel found that the interior streamfunction of the flattened ellipsoid is also given by (5.2) with $\sigma=0$, as well as $z=0$ since the ellipse is at the surface (note that the $\sigma$ used here is modified from Dritschel's by $1 / a b)$. A single elliptic vortex will rotate about its center and maintain its shape. The surrounding flow is shown in Figure 5.1 in the co-rotating frame on the plane $z=0$ reproduced from Dritschel (2011) [19]. Below the surface, three-dimensional trajectories in the flow will be periodic.

For velocity computations, the streamfunction is evaluated by starting with $\sigma$ at the surface, which can be calculated exactly. This is computed for all $(x, y)$ particle positions. Then, $\sigma$ is calculated for descending particle depths $z$ via Newton's method in order to trace the appropriate root of


Figure 5.2: A schematic from Melander (1986) [37] displaying the relevant variables for calculating the relative motion of two elliptic vortices. This analysis was applied to Dritschel's ellipses by Bersanelli (2010) [8].
(5.3). With $\sigma$ thus evaluated for all particle positions $(x, y, z)$, as well as $(x, y, 0)$, the streamfunction $\Phi^{0}$ and its derivatives are calculated exactly. The integrals in (5.2) are elliptic and evaluated using MATLAB functions rd and rf [28].

If multiple vortices are present, they will interact with one another and lose their elliptic shapes [20]. In this thesis, we are not interested in solving this interaction exactly, but are looking for a manageable buoyancy solution that induces mixing in order to examine the effect of $O(R o)$ velocities on transport. As such, the vortex interaction is approximated such that the ellipses will remain ellipses as the flow evolves. Two approximations that have been developed in the literature are presented here.

### 5.1 Moment Model

One solution for the interaction of two of Dritchel's elliptic vortices was found using a moment model determined by Bersanelli (2010) [8]. He followed the derivation by Melander et al. (1986) [37], using perturbation
analysis in the limit that the vortices are widely spaced relative to their own maximum diameter, and where the vortices are shaped such that their centers of vorticity are within their respective boundaries (see Figure 5.2 reproduced from Melander et al. (1986) [37]).

The moment model begins by considering velocities acting on a point in ellipse $k\left(\vec{x} \in E_{k}\right)$ induced by ellipse $k^{\prime}$

$$
\dot{\vec{x}}=\left.\binom{-\partial_{y}}{\partial_{x}} \psi_{k^{\prime}}\right|_{\vec{x} \in E_{k}} .
$$

For a surface buoyancy distribution (in coordinates aligned with the ellipse axes)

$$
\begin{equation*}
\beta_{k}(\xi, \eta)=\beta_{m, k} \sqrt{1-\frac{\xi^{2}}{a^{2}}-\frac{\eta^{2}}{b^{2}}}, \tag{5.4}
\end{equation*}
$$

the streamfunction for ellipse $k$ can be given by the Green's function solution to

$$
\Delta \psi=0, \quad \psi_{z}^{s}=\beta_{k},
$$

which is

$$
\begin{equation*}
\psi_{k}(x, y, z)=-\frac{1}{2 \pi} \int_{E_{k}} \frac{\beta_{k}(\xi, \eta)}{\sqrt{\left(x_{k}+\xi-x\right)^{2}+\left(y_{k}+\eta-y\right)^{2}+z^{2}}} d \xi d \eta \tag{5.5}
\end{equation*}
$$

where $\vec{x}_{k}$ is the center of the ellipse, defined by its center of vorticity

$$
\begin{equation*}
\beta_{m, k} \vec{x}_{k}=\frac{\int_{E_{k}} \beta_{k}(\xi, \eta) \vec{\xi} d \xi d \eta}{\int_{E_{k}} \beta_{k}(\xi, \eta) d \xi d \eta} . \tag{5.6}
\end{equation*}
$$

For $\left|\vec{x}-\vec{x}_{k}\right| \gg|\vec{\xi}|$, i.e. position $\vec{x}$ far from $E_{k}$, the binomial expan-
sion formula is used to approximate $\psi$ to $O\left(d^{4} / R^{3}\right)$ for a maximum ellipse diameter $d$ [37] as

$$
\begin{align*}
& \psi_{k}(x, y) \approx-\frac{1}{2 \pi}\left[\frac{J_{k}^{(0,0)}}{R}+\frac{1}{2}\left\{\quad\left[\frac{3\left(x-x_{k}\right)^{2}}{R^{5}}-\frac{1}{R^{3}}\right] J_{k}^{(2,0)}+\right.\right. \\
&\left.\left.\frac{6\left(x-x_{k}\right)\left(y-y_{k}\right)}{R^{5}} J_{k}^{(1,1)}+\left[\frac{3\left(y-y_{k}\right)^{2}}{R^{5}}-\frac{1}{R^{3}}\right] J_{k}^{(0,2)}\right\}\right] \tag{5.7}
\end{align*}
$$

where $R=\sqrt{\left(x-x_{k}\right)^{2}+\left(y-y_{k}\right)^{2}}$ and $J_{k}^{(m, n)}$ are geometric moments given by

$$
\begin{equation*}
J_{k}^{(m, n)}=\int_{E_{k}} \beta_{k}(\xi, \eta) \xi^{m} \eta^{n} d \xi d \eta \tag{5.8}
\end{equation*}
$$

The order-zero moment is the circulation

$$
\begin{equation*}
J_{k}^{(0,0)}=\int_{E_{k}} \beta_{k}(\xi, \eta) d \xi d \eta=\Gamma_{k}=\frac{2}{3} A_{k} \beta_{m, k} \tag{5.9}
\end{equation*}
$$

where $A_{k}=\pi a_{k} b_{k}$ is the area of the ellipse.
The ellipse center will evolve in time according to

$$
\begin{align*}
\dot{\vec{x}}_{k} & =\frac{1}{\Gamma_{k}} \frac{d}{d t} \int_{E_{k}} \sqrt{1-\frac{\xi^{2}}{a_{k}^{2}}-\frac{\eta^{2}}{b_{k}^{2}}} \vec{\xi} d \xi d \eta  \tag{5.10}\\
& =\frac{1}{\Gamma_{k}} \int_{E_{k}} \frac{D}{D t} \sqrt{1-\frac{\xi^{2}}{a_{k}^{2}}-\frac{\eta^{2}}{b_{k}^{2}}} \vec{\xi} d \xi d \eta  \tag{5.11}\\
& =\frac{1}{\Gamma_{k}} \int_{E_{k}} \sqrt{1-\frac{\xi^{2}}{a_{k}^{2}}-\frac{\eta^{2}}{b_{k}^{2}}} \dot{\vec{\xi}} d \xi d \eta \tag{5.12}
\end{align*}
$$

where in the last line we have used the fact that by definition

$$
\begin{equation*}
\frac{d}{d t} \Gamma_{k}=\int_{E_{k}} \beta_{m, k} \frac{D}{D t} \sqrt{1-\frac{\xi^{2}}{a_{k}^{2}}-\frac{\eta^{2}}{b_{k}^{2}}} d \xi d \eta=0 \tag{5.13}
\end{equation*}
$$

so therefore

$$
\frac{D}{D t} \sqrt{1-\frac{\xi^{2}}{a_{k}^{2}}-\frac{\eta^{2}}{b_{k}^{2}}}=0
$$

everywhere, such that the buoyancy distribution moves with the ellipse.
The velocities are

$$
\begin{equation*}
\dot{\vec{\xi}}=\left.\binom{-\partial_{y}}{\partial_{x}} \psi_{k}\right|_{\vec{\xi}}+\left.\binom{-\partial_{y}}{\partial_{x}} \psi_{k^{\prime}}\right|_{\vec{\xi}}, \tag{5.14}
\end{equation*}
$$

but noting that $\vec{x}_{k}$, the center of vorticity for ellipse $k$, can only move by the flow induced by ellipse $k^{\prime}$, we neglect the $\psi_{k}$ term above.

We Taylor expand the $\psi_{k^{\prime}}$ terms to second order about the centroid $\vec{x}_{k}$ and obtain

$$
\begin{equation*}
\left.\dot{\vec{x}}_{k} \approx\left[1+\frac{1}{2 \Gamma_{k}}\left(J_{k}^{(2,0)} \partial_{x}^{2}+2 J_{k}^{(1,1)} \partial_{x} \partial_{y}+J_{k}^{(0,2)} \partial_{y}^{2}\right)\right]\binom{-\partial_{y}}{\partial_{x}} \psi_{k^{\prime}}\right|_{\vec{x}_{k}} \tag{5.15}
\end{equation*}
$$

Using the approximation for $\psi$ from above, rewriting in polar coordinates $\vec{x}_{k}-\vec{x}_{k^{\prime}}=R\left(\cos \theta_{k, k^{\prime}}, \sin \theta_{k, k^{\prime}}\right)$, and using symmetry to define $\theta=\theta_{1,2}=$ $\theta_{2,1}+\pi$, we obtain

$$
\begin{align*}
\dot{x}_{k} \approx & (-1)^{k} \frac{\Gamma_{k^{\prime}}}{2 \pi}\left\{\frac{\sin \theta}{R^{2}}-\sum_{i=k, k^{\prime}} \frac{3}{2 R^{4} \Gamma_{i}}\left[J_{i}^{(2,0)} \sin \theta\left(1-5 \cos ^{2} \theta\right)\right.\right. \\
& \left.\left.+2 J_{i}^{(1,1)} \cos \theta\left(1-5 \sin ^{2} \theta\right)+J_{i}^{(0,2)} \sin \theta\left(3-5 \sin ^{2} \theta\right)\right]\right\}  \tag{5.16}\\
\dot{y}_{k} \approx & (-1)^{k-1} \frac{\Gamma_{k^{\prime}}}{2 \pi}\left\{\frac{\cos \theta}{R^{2}}-\sum_{i=k, k^{\prime}} \frac{3}{2 R^{4} \Gamma_{i}}\left[J_{i}^{(2,0)} \cos \theta\left(3-5 \cos ^{2} \theta\right)\right.\right. \\
& \left.\left.+2 J_{i}^{(1,1)} \sin \theta\left(1-5 \cos ^{2} \theta\right)+J_{i}^{(0,2)} \cos \theta\left(1-5 \sin ^{2} \theta\right)\right]\right\} \tag{5.17}
\end{align*}
$$

Now, using the geometric relations

$$
\begin{align*}
& \left(x_{1}-x_{2}\right)\left(\dot{x}_{1}-\dot{x}_{2}\right)+\left(y_{1}-y_{2}\right)\left(\dot{y}_{1}-\dot{y}_{2}\right)=R \dot{R}  \tag{5.18}\\
& \left(x_{1}-x_{2}\right)\left(\dot{y}_{1}-\dot{y}_{2}\right)-\left(y_{1}-y_{2}\right)\left(\dot{x}_{1}-\dot{x}_{2}\right)=R^{2} \dot{\theta} \tag{5.19}
\end{align*}
$$

we find

$$
\begin{align*}
& \dot{\theta}=\frac{\Gamma_{1}+\Gamma_{2}}{2 \pi}\left\{\frac{1}{R^{3}}+\sum_{i} \frac{3}{4 R^{5} \Gamma_{i}}\left[J_{i}^{(2,0)}(1+3 \cos 2 \theta)+2 J_{i}^{(1,1)}(3 \sin 2 \theta)\right.\right. \\
& \left.\left.\quad \quad+J_{i}^{(0,2)}(1-3 \cos 2 \theta)\right]\right\},  \tag{5.20}\\
& \dot{R}=-\frac{\Gamma_{1}+\Gamma_{2}}{2 \pi} \sum_{i} \frac{3}{2 R^{4} \Gamma_{i}}\left[J_{i}^{(2,0)} \sin 2 \theta-2 J_{i}^{(1,1)} \cos 2 \theta-J_{i}^{(0,2)} \sin 2 \theta\right] . \tag{5.21}
\end{align*}
$$

Now if we evaluate the geometric moments

$$
\begin{align*}
J_{k}^{(2,0)} & =\frac{2 \beta_{m, k} A_{k}^{2}}{15 \pi \lambda_{k}}\left(\lambda_{k}^{2}+\left(1-\lambda_{k}^{2}\right) \sin ^{2} \phi_{k}\right)  \tag{5.22}\\
J_{k}^{(0,2)} & =\frac{2 \beta_{m, k} A_{k}^{2}}{15 \pi \lambda_{k}}\left(\lambda_{k}^{2}+\left(1-\lambda_{k}^{2}\right) \cos ^{2} \phi_{k}\right)  \tag{5.23}\\
J_{k}^{(1,1)} & =-\frac{\beta_{m, k} A_{k}^{2}}{15 \pi \lambda_{k}}\left(1-\lambda_{k}^{2}\right) \sin 2 \phi_{k} \tag{5.24}
\end{align*}
$$

where here $\lambda_{k}$ is the aspect ratio $a_{k} / b_{k}$, the evolution equations become

$$
\begin{align*}
& \dot{\theta}=\frac{\Gamma_{1}+\Gamma_{2}}{2 \pi R^{5}}\left\{R^{2}+\frac{3 A_{1}}{20 \pi \lambda_{1}}\left[\left(1+\lambda_{1}^{2}\right)-3\left(1-\lambda_{1}^{2}\right) \cos 2\left(\phi_{1}-\theta\right)\right]\right. \\
& \left.+\frac{3 A_{2}}{20 \pi \lambda_{2}}\left[\left(1+\lambda_{2}^{2}\right)-3\left(1-\lambda_{2}^{2}\right) \cos 2\left(\phi_{2}-\theta\right)\right]\right\},  \tag{5.25}\\
& \dot{R}=-\frac{\Gamma_{1}+\Gamma_{2}}{2 \pi R^{4}}\left\{\frac{3 A_{1}}{10 \pi \lambda_{1}}\left(1-\lambda_{1}^{2}\right) \sin 2\left(\phi_{1}-\theta\right)\right. \\
& \left.+\frac{3 A_{2}}{10 \pi \lambda_{2}}\left(1-\lambda_{2}^{2}\right) \sin 2\left(\phi_{2}-\theta\right)\right\} . \tag{5.26}
\end{align*}
$$

These equations describe how the vortex centers evolve in time, and
next we must derive how each ellipse evolves in time, which is based on the quantities

$$
\begin{align*}
J_{k} & =J_{k}^{(2,0)}+J_{k}^{(0,2)}=\frac{2 \beta_{m, k} A_{k}^{2}}{15 \pi} \frac{1+\lambda_{k}^{2}}{\lambda_{k}}  \tag{5.27}\\
D_{k} & =J_{k}^{(2,0)}-J_{k}^{(0,2)}=-\frac{2 \beta_{m, k} A_{k}^{2}}{15 \pi} \frac{1-\lambda_{k}^{2}}{\lambda_{k}} \cos 2 \phi_{k} \tag{5.28}
\end{align*}
$$

Similarly to the centroid, we need to compute how the geometric moments evolve. We will explicitly show the analysis for $J_{k}^{(2,0)}$ as representative of the three second-order moments.

$$
\begin{equation*}
\dot{J}_{k}^{(2,0)}=\frac{d}{d t} \int_{E_{k}} \omega \xi^{2} d \xi d \eta=\int_{E_{k}} 2 \omega \xi \dot{\xi} d \xi d \eta \tag{5.29}
\end{equation*}
$$

In this case the velocity given by (5.14) must include both terms, but if we separate the self-interaction terms as $\dot{J}_{\star k}^{(2,0)}$ and Taylor expand the $\psi_{k^{\prime}}$ terms to first order, we obtain

$$
\begin{align*}
\dot{J}_{k}^{(2,0)} & \approx \dot{J}_{\star k}^{(2,0)}-\left.\left[2 J_{k}^{(2,0)} \partial_{x} \partial_{y}+2 J_{k}^{(1,1)} \partial_{y}^{2}\right] \psi_{k^{\prime}}\right|_{\vec{x}_{k}}  \tag{5.30}\\
\dot{J}_{k}^{(0,2)} & \approx \dot{J}_{\star k}^{(0,2)}+\left.\left[2 J_{k}^{(0,2)} \partial_{x} \partial_{y}+2 J_{k}^{(1,1)} \partial_{x}^{2}\right] \psi_{k^{\prime}}\right|_{\vec{x}_{k}}  \tag{5.31}\\
\dot{J}_{k}^{(1,1)} & \approx \dot{J}_{\star k}^{(1,1)}+\left.\left[J_{k}^{(2,0)} \partial_{x}^{2}-J_{k}^{(0,2)} \partial_{y}^{2}\right] \psi_{k^{\prime}}\right|_{\vec{x}_{k}} \tag{5.32}
\end{align*}
$$

Combining, we get

$$
\begin{align*}
\dot{J}_{k} & =\dot{J}_{\star k}-\left.\left[2 D_{k} \partial_{x} \partial_{y}-2 J_{k}^{(1,1)}\left(\partial_{x}^{2}-\partial_{y}^{2}\right)\right] \psi_{k^{\prime}}\right|_{\vec{x}_{k}}  \tag{5.33}\\
\dot{D}_{k} & =\dot{D}_{\star k}-\left.\left[2 J_{k} \partial_{x} \partial_{y}+2 J_{k}^{(1,1)}\left(\partial_{x}^{2}+\partial_{y}^{2}\right)\right] \psi_{k^{\prime}}\right|_{\vec{x}_{k}} \tag{5.34}
\end{align*}
$$

which we match to the time derivatives of the definitions

$$
\begin{align*}
\dot{J}_{k} & =\frac{2 \beta_{m, k} A_{k}^{2}}{15 \pi} \frac{\left(\lambda_{k}^{2}-1\right)}{\lambda_{k}^{2}} \dot{\lambda}_{k}  \tag{5.35}\\
\dot{D}_{k} & =\frac{2 \beta_{m, k} A_{k}^{2}}{15 \pi}\left[\frac{1-\lambda_{k}^{2}}{\lambda_{k}} 2 \sin 2 \phi_{k} \dot{\phi}_{k}+\frac{1+\lambda_{k}^{2}}{\lambda_{k}^{2}} \cos 2 \phi_{k} \dot{\lambda}_{k}\right] \tag{5.36}
\end{align*}
$$

The self-interaction terms come from the flow within the ellipse, given in the frame rotating with the ellipse as a constant rate

$$
\begin{gather*}
\frac{\dot{x}}{a_{k}}=-\Omega_{k} \frac{y}{b_{k}}, \quad \frac{\dot{y}}{b_{k}}=\Omega_{k} \frac{x}{a_{k}},  \tag{5.37}\\
\Omega_{k}=\frac{\beta_{m, k}}{\sqrt{a_{k} b_{k}}} \frac{\lambda_{k} R_{d}\left(0, \lambda_{k}^{-1}, \lambda_{k}\right)-\lambda_{k}^{-1} R_{d}\left(0, \lambda_{k}, \lambda_{k}^{-1}\right)}{3\left(\lambda_{k}-\lambda_{k}^{-1}\right)} \tag{5.38}
\end{gather*}
$$

where

$$
\begin{equation*}
R_{d}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)=\frac{3}{2} \int_{0}^{\infty} \frac{d u}{\sqrt{\left(u+\alpha_{1}\right)\left(u+\alpha_{2}\right)\left(u+\alpha_{3}\right)^{3}}} \tag{5.39}
\end{equation*}
$$

This is calculated using the MATLAB function rd [28].
Therefore the self-interaction terms are

$$
\dot{J}_{\star k}=0, \quad \dot{D}_{\star k}=-4 J_{k}^{(1,1)} \Omega_{k}
$$

Solving for the time derivatives of $\left(\lambda_{k}, \phi_{k}\right)$ we find

$$
\begin{align*}
& \dot{\lambda}_{k}=-\frac{3 \Gamma_{k^{\prime}}}{2 \pi} \frac{\lambda_{k}}{R^{3}} \sin 2\left(\phi_{k}-\theta\right)  \tag{5.40}\\
& \dot{\phi}_{k}=\Omega_{k}-\frac{\Gamma_{k^{\prime}}}{4 \pi R^{3}}+\frac{3 \Gamma_{k^{\prime}}}{4 \pi R^{3}} \frac{1+\lambda_{k}^{2}}{1-\lambda_{k}^{2}} \cos 2\left(\phi_{k}-\theta\right) \tag{5.41}
\end{align*}
$$

### 5.2 Point Vortex Substitution

An alternative to this derivation comes from an approximation by Dritschel et al. (2004) [20]. In their analysis, the ellipse of continuous buoyancy is replaced by a set of point vortices. The strengths and positions of each point vortex are calculated such that the geometric moments of the system of point vortices match those of the original elliptic vortex up to the desired order.

The moments of an ellipse are given in (5.8). Moments of a set of point vortices are

$$
\begin{equation*}
J_{P V}^{(m, n)}=\sum_{j} \kappa_{j} x_{j}^{m} y_{j}^{n} \tag{5.42}
\end{equation*}
$$

As Dritschel et al. (2004) show, order $m=2$ is sufficient for accurately calculating vortex motion [20]. They calculated the strengths and positions of the seven point vortex replacements at this order to be

$$
\begin{align*}
& \kappa_{j=1, \ldots, 6}=\frac{7 \Gamma}{60}, \quad \kappa_{7}=\frac{3 \Gamma}{10} \\
& \vec{x}_{j=1, \ldots, 6}=\sqrt{\frac{4}{7}}(a \cos (j \pi / 3-\pi / 6), b \sin (j \pi / 3-\pi / 6)), \quad \vec{x}_{7}=0 . \tag{5.43}
\end{align*}
$$

This approximation was computed using the code 2DHelm developed by Dritschel et al. (2002) and modified by Poje [18].

### 5.3 Comparison

We can compare the evolution of these two approximations for arbitrary parameters. I have chosen initial conditions

$$
\begin{gathered}
R=10, \quad \theta=0, \quad a_{1}=2, \quad b_{1}=1, \quad a_{2}=1.5, \quad b_{2}=1, \\
\phi_{1}=0, \quad \phi_{2}=1, \quad \beta_{m, 1}=1, \quad \beta_{m, 2}=1,
\end{gathered}
$$

and obtained the following results. Note that the accuracy of the moment model is here $O(0.26)$. Surprisingly, the vortex centers appeared to follow regular trajectories, though the evolution equations do not obviously lead to regular behavior. Defining a period of motion as the time when $\theta$ returns to zero, the period was found to be $T=5317$, and motion over one period was considered. Note that the motion is not truly periodic because the ellipse aspect ratios and orientations do not return to their initial values at this time.

In all comparisons to follow, the moment model is shown in blue and the Dritschel approximation is in orange. First the vortex centers' evolution over time is examined in Figure 5.3(a), and the approximations essentially overlap. The orientation angle between ellipses $\theta$ also matches nearly exactly in Figure 5.3(b). Next the distance between the ellipse centers is shown in Figure 5.4, where the interval $T / 10$ was chosen for illustrative purposes. The solutions still appear to match very well. Over this interval, the aspect ratios also match very well (see Figure 5.5), but looking over the entire period shows a drift away form initial conditions in the Dritschel approximation. Finally, the orientation angle of ellipse one about its center is given in Figure 5.6, and


Figure 5.3: A comparison of the moment model (blue) with the Dritschel approximation (orange) of the interaction of two elliptic vortices. The evolution of vortex centers is shown on the $x-y$ plane in (a), and the evolution of the orientation angle between them is shown in (b).
the solutions here also matches well. Note that only the evolution of ellipse one is shown in Figures 5.5-5.6, but it is representative of both ellipses.

Thus, the approximations appear to agree, with the Dritschel solution exhibiting a long-time trend of $\lambda$ deviating from initial conditions. Therefore, it seems that the moment model is more stable over long time, and this model will be used for transport calculations.


Figure 5.4: A comparison of the distance between the two vortex centers calculated from the moment model (blue) with the Dritschel approximation (orange).


Figure 5.5: The aspect ratio of one ellipse over time for the intervals (a) $T / 10$ and (b) $T$, the period of vortex center motion, calculated from the moment model (blue) with the Dritschel approximation (orange). Although the solutions match well over the shorter interval (a), in (b) the Dritschel solution exhibits a slow trend away from initial conditions.


Figure 5.6: The orientation angle of ellipse one about its center, calculated from the moment model (blue) with the Dritschel approximation (orange).

## 5.4 $O($ Ro $)$ Velocities

Details of the calculation have been presented in Section 2.1. Figure 5.7 shows the velocities resulting from the two elliptic vortices considered above. The vertical velocity should be zero at the surface by (2.12) and (2.13), and the calculations meet this condition. Interestingly, the $O(R o)$ horizontal velocities are of opposite sign to the $O(1)$ velocities, suggesting that the $O(1)$ model might overestimate the strength of SQG flow.

### 5.5 Elliptic Vortex FTBE Results

### 5.5.1 Two Elliptic Vortices

For two elliptic vortices, there appears to be a negligible amount of mixing. The particle trajectories are quasiperiodic, not chaotic, as shown in Figure 5.8 in the frame of motion where the ellipses remain on the $x$-axis. These trajectories were computed for a time interval of 10,000 , over which ellipse one completed 193 completed rotations about its axis and ellipse two completed 243 complete rotations. While the trajectories appear to intersect with the ellipse on the left, this is only the initial position of the ellipse, and the trajectories had buoyancies exactly equal to zero, confirming that there was no vortex intrusion. The power spectral density (PSD) of the trajectories, estimated by pwelch in MATLAB, additionally indicates only one or two dominant frequencies in Figure 5.9, implying there is no chaos.

Trajectories are additionally examined away from the surface and for nonzero Ro in Figure 5.10. Differences in the mean particle buoyancy between

Figure 5.7: The calculated velocities at $O(1)$ and $O(R o)$ for two of Dritschel's elliptic vortices. The colorbar shown for $v_{1}$ is used for all horizontal velocities, while vertical velocity uses a separate colorbar as shown. The vortex on the left has
 separation between them is given by $R=10, \theta=0$. Each velocity field is shown at heights $z=0,-0.5,-0.1$, as labeled.


Figure 5.8: Trajectories for 16 particles initiated on $z=0$ near $(0,0)$ for the ellipses shown in Figure 5.7 and a time interval of 10,000 in the frame of reference where the ellipses remain on the $x$ axis. The initial position of the ellipses is shown along with the streamfunction. Over time, the ellipse rotation is such that the particles do not intersect the ellipse as the trajectories appear to show.


Figure 5.9: The power spectral density (PSD) estimated by MATLAB's pwelch function for the ellipses in Figure 5.7 for (a) the $x$ and (b) $y$ components of particle trajectories.


Figure 5.10: The (a) mean particle depth and (b) mean buoyancy surface for the particles shown in Figure 5.8, but initiated at $z=0$ and -0.2. In (b) the buoyancy remains exactly zero for particles initiated at $z=0$, confirming that particles do not intersect with the ellipse. For particles initiated at $z=$ -0.2 , particle depth for nonzero Ro and buoyancy for all Ro show oscillatory behavior with small amplitude. Buoyancy results are nearly identical for $R o=0,0.01,0.1$.
$R o=0,0.01,0.1$ were negligible, of $O\left(10^{-6}\right)$, so only one plot is shown. For particles initiated at $z=-0.2$, the horizontal trajectories are the same as at the surface, and buoyancy exhibits oscillatory behavior with small amplitude. Oscillatory behavior is also seen in the mean particle depth for nonzero Ro, and the deviation from initial depth scales with $R o$, while for $R o=0$ the depth change is exactly zero.

The results presented here are not meant to provide a rigorous proof that this flow does not result in chaos, but for the purposes of this thesis we seek a flow with more immediate mixing. Because we are looking for the effect of $O(R o)$ velocities interacting with horizontal mixing, and not whether $O($ Ro $)$ velocities alone can induce mixing, the model flow should exhibit mixing at $O(1)$. Based on the mixing results for point vortices, three elliptic vortices are considered.

### 5.5.2 Three Elliptic Vortices

In an attempt to induce mixing, we introduce a third elliptic vortex. The interaction follows from that presented in Section 5, resulting in

$$
\begin{align*}
& \dot{x}_{i} \approx-\sum_{j}^{\prime} \frac{\Gamma_{j}}{2 \pi}\left\{\frac{y_{i}-y_{j}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{3}}-\sum_{\alpha=i, j} \frac{3}{2\left|\vec{x}_{i}-\vec{x}_{j}\right|^{5} \Gamma_{\alpha}}[ \right. \\
& J_{\alpha}^{(2,0)}\left(y_{i}-y_{j}\right)\left[1-5 \frac{\left(x_{i}-x_{j}\right)^{2}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{2}}\right] \\
& +2 J_{\alpha}^{(1,1)}\left(x_{i}-x_{j}\right)\left[1-5 \frac{\left(y_{i}-y_{j}\right)^{2}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{2}}\right] \\
& \left.\left.+J_{\alpha}^{(0,2)}\left(y_{i}-y_{j}\right)\left[3-5 \frac{\left(y_{i}-y_{j}\right)^{2}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{2}}\right]\right]\right\}  \tag{5.44}\\
& \dot{y}_{i} \approx \sum_{j}^{\prime} \frac{\Gamma_{j}}{2 \pi}\left\{\frac{x_{i}-x_{j}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{3}}-\sum_{\alpha=i, j} \frac{3}{2\left|\vec{x}_{i}-\vec{x}_{j}\right|^{5} \Gamma_{\alpha}}[ \right. \\
& J_{\alpha}^{(2,0)}\left(x_{i}-x_{j}\right)\left[3-5 \frac{\left(x_{i}-x_{j}\right)^{2}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{2}}\right] \\
& +2 J_{\alpha}^{(1,1)}\left(y_{i}-y_{j}\right)\left[1-5 \frac{\left(x_{i}-x_{j}\right)^{2}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{2}}\right] \\
& \left.\left.+J_{\alpha}^{(0,2)}\left(x_{i}-x_{j}\right)\left[1-5 \frac{\left(y_{i}-y_{j}\right)^{2}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{2}}\right]\right]\right\}  \tag{5.45}\\
& \dot{\lambda}_{i}=\sum_{j}^{\prime} \frac{3 \Gamma_{j}}{2 \pi} \frac{\lambda_{i}}{\left|\vec{x}_{i}-\vec{x}_{j}\right|^{5}}\left\{2\left(x_{i}-x_{j}\right)\left(y_{i}-y_{j}\right) \cos 2 \phi_{i}\right. \\
& \left.-\left[\left(x_{i}-x_{j}\right)^{2}-\left(y_{i}-y_{j}\right)^{2}\right] \sin 2 \phi_{i}\right\}  \tag{5.46}\\
& \dot{\phi}_{i}=\Omega_{i}-\sum_{j}^{\prime} \frac{\Gamma_{j}}{4 \pi\left|\vec{x}_{i}-\vec{x}_{j}\right|^{3}} \\
& -\frac{3 \Gamma_{j}}{4 \pi\left|\vec{x}_{i}-\vec{x}_{j}\right|^{5}} \frac{1+\lambda_{i}^{2}}{1-\lambda_{i}^{2}}\left\{2\left(x_{i}-x_{j}\right)\left(y_{i}-y_{j}\right) \sin 2 \phi_{i}\right. \\
& \left.+\left[\left(x_{i}-x_{j}\right)^{2}-\left(y_{i}-y_{j}\right)^{2}\right] \cos 2 \phi_{i}\right\} \tag{5.47}
\end{align*}
$$

These three ellipses follow trajectories like those of the point vortices,

and their centers are periodic, though the aspect ratios and orientation angles of the ellipses do not return to their initial positions with the vortex centers. Velocities for three elliptic vortices are shown in Figure 5.11. Because the $O(R o)$ velocities are quadratic with the maximum buoyancy, vortex parameters must be chosen carefully so as not to violate the assumptions of the small Ro expansion. However, choosing parameters that are similar to those examined for point vortices is most likely to generate a flow with mixing. Balancing these two objectives, parameters are chosen to be $\Gamma=1$, $a=0.5, b=0.25, \phi=0$ for all vortices, with positions of $( \pm 2,-\sqrt{18} / 3)$ and $(0,2 \sqrt{18} / 3)$ such that the combined center of vorticity is the origin and $L=\sum \Gamma\left|\vec{x}_{i}\right|^{2}=10$. The streamfunction here is accurate to $O(0.054)$.

A Poincaré map from 16 particles initiated at $z=0$ near $(-2, \sqrt{18} / 3)$ and an integration time of 5,000 is shown in Figure 5.12 and compared to point vortex results. Additionally in (c) the full chaotic mixing region for the point vortex chase is shown from an integration time of 75,000 . The area spanned by the elliptic vortex map is much smaller than that of the point vortices. Additionally, while the point vortices produce islands of chaotic mixing that trajectories cannot penetrate, the elliptic vortex trajectories are able to pass very close to the vortices. In this particular simulation, the allowed tolerance in the MATLAB stiff ODE solver ode23t was not sufficiently small, and particles actually intruded the vortex patch (see Figure 5.13), which is not physically allowed. This error also seems to result in problems creating a braid from the trajectories, and FTBE could not be calculated. This simulation alone took approximately 35 hours to run, and attempts to run simulations at lower error tolerances resulted in software crashes.

The trajectories' power spectral density is given in Figure 5.14. By


Figure 5.12: Poincaré maps from 16 particles initiated at $z=0$ near $(-2, \sqrt{18} / 3)$ for (a) the ellipses shown in Figure 5.11 and (b) point vortices of the same circulation and positions, for a time interval of 5,000 . The period of motion of the ellipse centers is approximately 250 while the period of motion of the point vortices is approximately 255 . Vortex trajectories and initial positions are also shown, with ellipses to scale. The Poincaré map for the point vortices for a longer time interval of 75,000 is given in (c) to show the entire mixing region.


Figure 5.13: Calculations of the buoyancy of tracer particles over time. Solid vertical bars result from rapid oscillations. The buoyancy at the center of the ellipse is also indicated, and clearly particles are penetrating the vortex, which is nonphysical.


Figure 5.14: The power spectral density (PSD) estimated by MATLAB's pwelch function for the ellipses in Figure 5.11 for (a) the $x$ and (b) $y$ components of particle trajectories.
the Poincare map, the three elliptic vortices seem to induce mixing, and this is confirmed by the power spectral density. However, as this flow is unphysical, it is not known whether this mixing would also be observed in accurate trajectories. Physical intuition suggests that, because the elliptic vortices act as point vortices to first order, mixing will be present for these solutions. Confirmation of this is left to future work.

### 5.6 Summary

The elliptic vortex provides an exact solution to the SQG approximation that has no singularities in the $O(1)$ streamfunction, therefore allowing for the calculation of complete $O(R o)$ corrections. A moment model is used to approximate the interaction between multiple vortices, while flow calculations are done exactly. Despite the relatively large number of free parameters, two ellipses do not exhibit mixing. Three ellipses were considered, but the required computation time was very large, so these results were not thoroughly examined. The goal of this work was to find a manageable solution to SQG for which the $O(R o)$ velocities could be calculated and resulting effects on transport could be evaluated, but these vortex solutions did not provide such a model problem and other exact solutions must be considered.

## 6 Conclusions

The Surface Quasigeostrophic (SQG) approximation provides a model with horizontal dynamics with weak vertical flow. Simple exact solutions to this model can provide insight into the effect of this weak vertical velocity on transport. The Finite Time Braiding Exponent (FTBE) seems to provide a good measure of global complexity [50, 11], revealing quantitative trends in mixing based on the flow parameters. While the value of the FTBE is not meant to be construed as a physical value, it can be used to compare relative complexity between flows, where a higher FTBE indicates higher complexity of chaotic mixing. Because the SQG model is applicable for largescale ocean flows, understanding mixing in this model will provide insights into the physical processes behind pollution dispersion or vehicle trajectories. As a two-dimensional model with vertical dependence, such examination also shows how tools for studying mixing in two-dimensional flow can be modified to apply to more complicated, three-dimensional problems.

In order to build model problems for an examination of chaotic transport, this thesis presented a novel analysis of SQG point vortex interactions as well as an exploration of the surrounding flow properties. As in the classical two-dimensional case, three vortices follow regular trajectories themselves and can induce chaos in the surrounding flow [5]. This mixing has
been diagnosed using the classical tool of Poincaré maps and the new FTBE tool. These vortices behave qualitatively in the same way as classical twodimensional point vortices, exhibiting the same two regimes of motion in the equal strength case, and similar phase trajectories in the arbitrary strength cases. At $O(1)$, the SQG flow is two-dimensional but has vertical dependence.

The first trend observed for $O(1)$ SQG point vortex flows is that the FTBE is relatively constant in the regime of three orbiting vortices, then a minimum is found near the regime boundary, and mixing then increases with energy in the regime of two orbiting vortices until the vortex separation becomes too great. Also, the SQG case exhibits an overall higher extent of mixing than the classical two-dimensional case. The second trend is that the FTBE falls steeply below a critical depth that depends on $L$. For $L=1$ this depth is approximately $z=-0.25$, and for $L=10$ it is approximately $z=-0.75$. Larger $L$ also appears to decrease FTBE. A final trend is given in a preliminary investigation of FTBE for arbitrary vortex strength which seems to indicate a line in $\theta-C$ space that corresponds to maximum mixing.

Attempts to calculate $O(R o)$ velocities for point vortices proved to be challenging. However, the vertical velocity can instead be obtained from the energy conservation equation

$$
w+\frac{D \theta}{D t}=0 .
$$

It is thus possible to examine the effect of vertical velocity on mixing, though in this case the solution is not dynamically consistent because the velocity is not divergenceless. Results for the equal strength case show that including vertical velocity will increase FTBE everywhere but the surface, with more
mixing for larger Ro. Including $R o$ also seems to enhance the rate of dropoff of FTBE with depth below the critical depth, which remains the same. The direction in which particles will trend in the vertical may depend on the regime of motion, though more investigation is needed. Visualizations of trajectories seem to indicate an attraction toward periodic trajectories, such that particles below the critical depth trend downward and the islands of mixing around the vortex centers grow with Ro, but more examination is needed to confirm this trend.

For a dynamically consistent solution, a different model problem must be used. This work examined elliptic vortices formed by flattened constant-potential-vorticity ellipsoids, as in Dritschel et al. (2004) [20]. Two vortices did not appear to induce mixing, while three vortices proved computationally intensive beyond the scope of this work, and so other models must be chosen to further examine how the complete $O(R o)$ velocity corrections affect transport in the SQG approximation.

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