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Stabilization of a Tower of Universal Deformation Rings

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ABSTRACT OF THE DISSERTATION

Stabilization of a Tower of
Universal Deformation Rings

by

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Given a p-adic absolutely irreducible residual representation of a Galois group with Mazur's finiteness condition Φ_p, we get a universal deformation ring in Mazur's sense. Consider a tower of intermediate field extensions from the base field and restrictions of the given representation to each intermediate field. If all the restrictions are absolutely irreducible, we get a universal deformation ring associated to each restriction. In this way, we get a tower of universal deformation rings and the morphisms between them that are provided by universality. A natural question to ask here is that whether the tower of universal deformation rings stabilize or not, that is, whether the size of the universal deformation rings stops growing or grows indefinitely over the tower. In this thesis, we answer this question in the case of cyclotomic extensions of a number field.

The dissertation of Geunho Gim is approved.

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PUBLICATIONS

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CHAPTER 1

Introduction

After the study of big Hecke algebras by Hida in the 1980s, Mazur conceived an idea of studying universal deformation of a given Galois representation in his foundational work [Maz89]. The theory turned out to be hugely successful as it became a key ingredient of the proof of Fermat's last theorem by A. Wiles in [TW95] and [Wil95]. Recently in [BK], G. Böckle and C. Khare have proved that for a smooth projective curve X over a finite field $\mathbb{F}_q$ and its base change $\overline{X}$ to the algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$, and for a given m-dimensional continuous representation of $\pi_1(X)$ that is absolutely irreducible when restricted to $\pi_1(\overline{X})$, the tower of universal deformation rings given by restrictions to $\pi_1(X \times_{\mathbb{F}_q} \mathbb{F}_{q^n})$ for $n \geq 1$ has stabilization for certain family in it. (See Theorem 1.0.3 for details.) We explore if the same phenomena occur in prime-to-p cyclotomic extensions of a number field with Galois group isomorphic to $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$.

Let p be an odd prime. Let G be a profinite group satisfying Mazur's finiteness condition Φ_p defined as the following.

(Φ_p) For any finite index open subgroup H of G, $H/\langle [H,H], H^p\rangle$ is finite.

Equivalently, for each finite index open subgroup H of G, the maximal pro-p quotient of H is topologically finitely generated. Let $\mathbb{F}$ be a finite field of characteristic p. We consider a continuous, absolutely irreducible representation

$$\overline{\rho} : G \to \mathrm{GL}_m(\mathbb{F})$$

for some $m \geq 1$. We will assume continuity for representations appearing in this thesis even if it is not mentioned specifically. We fix a discrete valuation ring W which is $\mathbb{Z}_p$-free of finite type with maximal ideal $\mathfrak{m}_W$ and $\mathbb{F} = W/\mathfrak{m}_W$. Denote CNL_W for the category of complete noetherian local W-algebras with residue field $\mathbb{F}$. The morphisms in CNL_W are defined to be local homomorphisms. For $A \in CNL_W$, two representations $\rho_1, \rho_2 : G \to \mathrm{GL}_m(A)$ are called strictly equivalent if there is an element $x \in 1 + M_m(\mathfrak{m}_A) = \ker(\mathrm{GL}_m(A) \to \mathrm{GL}_m(\mathbb{F}))$ such that $\rho_2(g) = x\rho_1(g)x^{-1}$ for all $g \in G$. In this case, we denote $\rho_1 \approx \rho_2$.

Theorem 1.0.1 (Mazur)**.** *Let G be a profinite group with Φ_p and $\overline{\rho} : G \to \mathrm{GL}_m(\mathbb{F})$ be a continuous, absolutely irreducible representation. Then, there exists a universal couple $(R, \boldsymbol{\rho})$ for $R \in CNL_W$ and $\boldsymbol{\rho} : G \to \mathrm{GL}_m(R)$ in the following sense. For any $A \in CNL_W$ and a representation $\rho_0 : G \to \mathrm{GL}_m(A)$ with $\rho_0 \mod \mathfrak{m}_A = \overline{\rho}$, there is a unique morphism $\phi : R \to A$ in CNL_W such that $\phi \circ \boldsymbol{\rho} \approx \rho_0$.*

Proof. See [Maz89, Proposition 1]. □

We call $\boldsymbol{\rho}$ the universal deformation of $\overline{\rho}$ and $R = R(\overline{\rho}, W)$ the universal deformation ring. We can consider this as the ring R representing the functor $\mathcal{F}_{\overline{\rho}} : CNL_W \to Sets$ defined by

$$\mathcal{F}_{\overline{\rho}}(A) = \{\rho : G \to \mathrm{GL}_m(A) \mid \rho \mod \mathfrak{m}_A = \overline{\rho}\}/\approx$$

for $A \in CNL_W$ so that $\mathcal{F}_{\overline{\rho}}(-) \cong \mathrm{Hom}_{CNL_W}(R, -)$. We call an element in $\mathcal{F}_{\overline{\rho}}(A)$ an A-deformation of $\overline{\rho}$.

Suppose we have a profinite group G with Φ_p, an absolutely irreducible representation

$$\overline{\rho} : G \to \mathrm{GL}_m(\mathbb{F}),$$

and a tower of subgroups $\{G_n\}_{n=1}^{\infty}$ of G such that $G \supseteq G_1 \supseteq G_2 \supseteq \ldots \supseteq G_\infty = \cap_n G_n$. Note that each subgroup G_n satisfies Φ_p if G does. If we assume that $\overline{\rho}|_{G_n}$ is absolutely irreducible for all n, then we get a family of universal couples $\{(R_n, \boldsymbol{\rho}_n)\}_{n=1}^{\infty}$ coming from $\{\overline{\rho}|_{G_n}\}_{n=1}^{\infty}$ by Mazur's theorem above. For $N \geq n$, we have $\boldsymbol{\rho}_n|_{G_N}$ mod $\mathfrak{m}_{R_n} = \overline{\rho}|_{G_N}$. By universality of R_N, we get a morphism $\theta_{N,n} : R_N \to R_n$ in CNL_W, thus a tower of universal deformation rings $\{(R_n, \theta_{N,n})\}$. A natural question to ask here is the following.

Question 1.0.2. Does the tower of universal deformation rings stabilize? In other words, is there N_0 such that $R_N \cong R_{N_0}$ for all $N \geq N_0$?

Recently, this question has been studied for the function field case by Böckle-Khare in [BK].

Theorem 1.0.3 (Böckle-Khare)**.** *Let X be a smooth projective curve of genus g over a finite field $\mathbb{F}_q$ of characteristic p. Let ℓ be a prime distinct from p, and let $\overline{X}$ (resp. X_n) be the base change of X to $\overline{\mathbb{F}_q}$ (resp. $\mathbb{F}_{q^n}$). Let $\overline{\rho} : \pi_1(X) \to \mathrm{GL}_m(\mathbb{F})$ be a continuous representation that is absolutely irreducible when restricted to $\pi_1(\overline{X})$. For each $n \geq 1$, denote the universal deformation in the sense of Mazur of $\overline{\rho}|_{\pi_1(X_n)}$ by $\rho_n : \pi_1(X_n) \to \mathrm{GL}_m(R_n)$. Then, there exists an n_0 prime to ℓ such that for any multiple of n of n_0 with n prime to ℓ, the canonical homomorphism $R_n \to R_{n_0}$ is an isomorphism.*

Proof. See [BK, Thereom 2.3]. □

In this thesis, we study the above problem in cyclotomic extensions of a number field case, which was never studied before.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Let F be an abelian number field and S be a finite set of primes of F containing infinite places. Let F^S be the maximal extension of F in $\overline{\mathbb{Q}}$

unramified outside S. Then, $G := \mathrm{Gal}(F^S/F)$ is a profinite group satisfying Φ_p. We start from a continuous, absolutely irreducible representation

$$\overline{\rho} : G \to \mathrm{GL}_m(\mathbb{F})$$

with a finite field $\mathbb{F}$. Let F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension with intermediate fields F_n, for each $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0}$ satisfying $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/\ell_1^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_r^{n_r}\mathbb{Z}$. We define $G_n := \mathrm{Gal}(F^S/F_n)$ and suppose the restrictions $\overline{\rho}|_{G_n}$ are all absolutely irreducible. Then, we get a tower of universal deformation rings $\{R_n\}$ from $\{\overline{\rho}|_{G_n}\}$ by Mazur's theorem above. We define $N \geq n$ for $N = (N_1, \ldots, N_r)$, $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0}$ if $N_i \geq n_i$ for all $i = 1, \ldots, r$. In this case, we can ask the same question.

Question 1.0.4. Is there $N_0 \in \mathbb{Z}^r_{\geq 0}$ such that $R_N \cong R_{N_0}$ for all $N \geq N_0$?

We will prove the following result.

Theorem 1.0.5. *Let the notations be as above and suppose that S is the exact set of primes of F above $\ell_1 \cdots \ell_r \infty$.*

1. *$(m = 1)$ The tower $\{R_n\}$ defined above stabilizes.*
2. *$(m \geq 2)$ Let $ad(\overline{\rho})$ be the (m^2-1)-dimensional adjoint representation of $\overline{\rho}$ and suppose that*
$$ad(\overline{\rho}) = \bigoplus_{i=1}^{m^2-1} \xi_i$$
for some characters $\xi_i : G \to \mathbb{F}^\times$. Also suppose that $(F^S)^{\ker \xi_i}/\mathbb{Q}$ is abelian for all i. Then, the tower $\{R_n\}$ stabilizes.

Note 1.0.6. There are a plenty of instances that $ad(\overline{\rho})$ is of the form above and yet $\overline{\rho}$ is absolutely irreducible. See Corollary 6.1.3.

This thesis is organized as follows.

In Chapter 2, we review the basic properties of universal deformation rings and study the tower of universal deformation rings.

In Chapter 3, we give the explicit form of universal deformation rings in 1-dimensional case, and prove the stabilization of the tower.

In Chapter 4, we prove a key lemma (Lemma 4.2.5) which gives an equivalence condition with stabilization of the tower of universal deformation rings.

In Chapter 5, we use the key lemma to prove (Theorem 5.1.6) a criterion for stabilization in the case $m = 2$. This is used (Theorem 5.2.4) to show stabilization of certain towers of p-ordinary universal deformation rings.

In Chapter 6, we return to the general m-dimensional case. A stabilization result (Theorem 6.2.1) is obtained by an argument similar to the one used for Theorem 5.1.6.

CHAPTER 2

Universal Deformation Rings

2.1 Notations

We will use the following notations throughout the thesis unless otherwise stated.

For a number field F, let $\mathcal{O}_F$ be the ring of integers of F. For a prime v of F, let F_v (resp. $\mathcal{O}_{F,v}$) be the v-completion of F (resp. $\mathcal{O}_F$) and let $\widehat{\mathcal{O}_F} = \prod_v \mathcal{O}_{F,v}$. Let $\mathbb{A}_F$ be the adele ring of F and $C_F = \mathbb{A}_F^\times / F^\times$ be the idele class group of F. Let $F_\infty = \prod_{v|\infty} F_v$ and $F_{\infty,+}^\times = \prod_{v|\infty} F_{v,+}^\times$ where $F_v = \mathbb{R}$ or $\mathbb{C}$ and $F_{v,+}^\times = \mathbb{R}_+ = (0, \infty)$ or $\mathbb{C}^\times$ depending on v.

Let Cl_F be the class group of F and for an ideal $\mathfrak{A}$ of F, let

$$Cl_F(\mathfrak{A}) = \frac{\{\text{ideals of } F \text{ prime to } \mathfrak{A}\}}{\{(\alpha) \mid \alpha \in F,\ \alpha_v \equiv 1 \bmod \mathfrak{A}\mathcal{O}_{F,v} \text{ for all } v \mid \mathfrak{A}\}}$$

be the ray class group of F modulo $\mathfrak{A}$. In the above definition, $\mathfrak{A}$ could formally contain an archimedean place v, and in that case $\alpha_v \equiv 1$ means α_v is positive. For a principal ideal (a) with $a \in F$, we write $Cl_F(a)$ for short instead of $Cl_F((a))$.

For a local ring R, we denote $\mathfrak{m}_R$ for the maximal ideal of R. We also denote CNL_R for the category of complete noetherian local R-algebras.

For two r-tuples $n = (n_1, \ldots, n_r)$ and $N = (N_1, \ldots, N_r)$ of non-negative integers, we say

$n \leq N$ if $n_i \leq N_i$ for all $i = 1, \ldots, r$.

2.2 A Tower of Universal Deformation Rings

Let F be a number field. We fix distinct odd prime ℓ and p. Let F_∞/F be the cyclotomic $\mathbb{Z}_\ell$-extension such that $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/\ell^n\mathbb{Z}$ for all $n \geq 1$ and $F_\infty = \bigcup_n F_n$. Let S be the finite set of primes of F above ℓ and infinite places, and F^S/F be the maximal unramified extension outside S. Note that $G := \mathrm{Gal}(F^S/F)$ satisfies Φ_p, and so does the subgroup $G_n := \mathrm{Gal}(F^S/F_n)$ for all n.

Theorem 2.2.1 (Frobenius reciprocity)**.** *Let G be a profinite group and H be an open subgroup of finite index. Let K be an algebraically closed field and suppose $(G : H)$ is prime to char(K). If χ and ρ are representations of H and G on K, then*

$$\mathrm{Ind}_H^G(\chi) \supseteq \rho$$

as a direct summand with multiplicity m if and only if

$$\rho|_H \supseteq \chi$$

with multiplicity m.

Proof. See [Mac51, Theorem 4']. □

Theorem 2.2.2 (Mackey)**.** *Let G be a profinite group and H be a normal open subgroup of finite index. Let ρ be an absolutely irreducible representation of H. For $\sigma \in G$, we define ρ^σ by $\rho^\sigma(h) = \rho(\sigma h \sigma^{-1})$ for $h \in H$. Then, $\mathrm{Ind}_H^G \rho$ is absolutely irreducible if and only if ρ^σ's are disjoint with no common irreducible factors for all $\sigma \in G$.*

Proof. See [Mac51, Theorem 6']. □

Lemma 2.2.3. *Let G be a profinite group and H be a normal open subgroup of finite index. Suppose $\Delta = G/H$ is cyclic of order d. We take a representation $\pi : H \to \mathrm{GL}_m(A)$ for a complete noetherian local algebra A and assume the following two conditions.*

1. *$\pi \bmod \mathfrak{m}_A$ is absolutely irreducible.*

2. *$\mathrm{Tr}(\pi) = \mathrm{Tr}(\pi^\sigma)$ for all $\sigma \in G$.*

Then, π can be extended to a representation of G into $\mathrm{GL}_m(B)$ for a local A-algebra B which is A-free of rank at most d.

Proof. See [H2, Corollary 4.37]. □

Lemma 2.2.4. *Let G be a profinite group with a normal subgroup H of finite index and K be a field. Let $\rho : G \to \mathrm{GL}_m(K)$ be a representation for some $m \geq 1$. If G/H is solvable of order prime to m, then the absolute irreducibility of ρ is equivalent to that of $\rho|_H$.*

Proof. It suffices to assume that G/H is cyclic of prime power order ℓ^n for $n \geq 1$ and $\ell \nmid m$. The general case can be deduced from induction.

Suppose ρ is absolutely irreducible but $\rho|_H$ is not. Let ρ_1 be one of the absolutely irreducible direct summand of $\rho|_H$ and let $H' \subseteq G$ be the stabilizer of ρ_1. Then, ρ_1 extends to H' possibly onto a finite extension K'/K by the lemma above. Note that $\rho|_{H'}$ contains the extension $\widetilde{\rho_1}$ of ρ_1, so $H' \subsetneq G$. Since G acts on the irreducible factors of $\rho|_H$ transitively, those of $\rho|_{H'}$ are all distinct by the definition of H'. By Mackey's theorem, $\mathrm{Ind}_{H'}^{G} \widetilde{\rho_1}$ is irreducible.

Since $\widetilde{\rho}_1 \subseteq \rho|_{H'}$, we have $\rho \subseteq \mathrm{Ind}_{H'}^{G} \widetilde{\rho}_1$ by Frobenius reciprocity. This implies $\mathrm{Ind}_{H'}^{G} \widetilde{\rho}_1 = \rho$ by irreducibility of ρ, thus

$$\ell \mid (G : H') \mid \dim(\mathrm{Ind}_{H'}^{G} \widetilde{\rho}_1) = \dim \rho = m,$$

which is a contradiction. $\square$

From the above lemma, if $\overline{\rho}$ is absolutely irreducible, then so is $\overline{\rho}|_{G_n}$ for all n. By Mazur's theorem, we have universal couples $(R_n, \boldsymbol{\rho}_n)$ for all n where $R_n = R_n(\overline{\rho}|_{G_n}, W) \in CNL_W$ and $\boldsymbol{\rho}_n : G_n \to \mathrm{GL}_m(R_n)$. For $n \leq N$, note that $(\rho_n|_{G_N}) \mod \mathfrak{m}_{R_n} = \overline{\rho}|_{G_N}$. By universality of R_N, we get a morphism $R_N \to R_n$ in CNL_W, thus the following tower of universal deformation rings.

$$\begin{tikzcd}
F_\infty = \cup F_n \arrow[r, leftrightarrow] \arrow[d, dash] & R_\infty = \varprojlim_n R_n \arrow[d] \\
\vdots \arrow[d, dash] & \vdots \arrow[d] \\
F_2 \arrow[r, leftrightarrow] \arrow[d, dash] & R_2 \arrow[d] \\
F_1 \arrow[r, leftrightarrow] \arrow[d, dash] & R_1 \arrow[d] \\
F_0 = F \arrow[r, leftrightarrow] & R_0
\end{tikzcd}$$

Question 2.2.5. Does the tower of universal deformation rings stabilize, i.e., is there N_0 such that $R_N \cong R_{N_0}$ for all $N \geq N_0$?

We consider a slightly more general situation. Let $\ell_1, \ldots, \ell_r$ be distinct odd primes different from p and F be a number field.

Let F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension. For every r-tuple $n = (n_1, \ldots, n_r)$ of non-negative integers, there is a unique field F_n such that $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/\ell_1^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_r^{n_r}\mathbb{Z}$ and $F_\infty = \bigcup_n F_n$. Let S be a finite set of primes of F containing primes above $\ell_1, \ldots, \ell_r$ and infinite places, and F^S/F be the maximal unramified extension outside S. Let $\overline{\rho} : G \to \mathrm{GL}_m(\mathbb{F})$ be an absolutely irreducible representation, then similarly by the lemma above and Mazur's theorem we get universal couples $(R_n, \boldsymbol{\rho}_n)$ for all r-tuple $n = (n_1, \ldots, n_r)$ of nonnegative integers. For two r-tuples $n = (n_1, \ldots, n_r)$ and $N = (N_1, \ldots, N_r)$, we define $n \leq N$ if $n_i \leq N_i$ for all $i = 1, \ldots, r$. By universality of R_N, we get a morphism $R_N \to R_n$ in CNL_W and a tower of universal deformation rings like above. We can ask the same question in this case.

Question 2.2.6. Does the tower of universal deformation rings stabilize, i.e., is there an r-tuple N_0 such that $R_N \cong R_{N_0}$ for all $N \geq N_0$?

Note 2.2.7. The above question has never been studied in the case where the base field is a number field. In the function field case, however, some general result has been proven by Böckle-Khare in [BK].

Note 2.2.8. In the above, we started from the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension F_∞/F and the representation $\overline{\rho}$ unramified outside S, which is the exact set of primes of F above $\ell_1, \ldots, \ell_r$ and infinite places. But we can allow $\overline{\rho}$ to have ramification outside S by the following argument. Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_m(\mathbb{F})$ be an absolutely irreducible representation. Let $Ram(\overline{\rho})$ be the set of primes of F where $\overline{\rho}$ is ramified. Let $q_1, \ldots, q_s$ be rational primes below $Ram(\overline{\rho}) \setminus S$ and S' be the set of primes of F above $\ell_1, \ldots, \ell_r, q_1, \ldots, q_s$ and infinite places. Let F'_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r} \times \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_s}$-extension, then we have $F_\infty \subseteq F'_\infty$. Let $G' = \mathrm{Gal}(F^{S'}/F)$, then we can regard $\overline{\rho} : G' \to \mathrm{GL}_m(\mathbb{F})$ and ask the same stabilization question for F'_∞/F and the tower $\{(R_n, \boldsymbol{\rho}_n)\}$ where n covers $(r+s)$-tuples of non-negative integers. If the stabilization of universal deformation rings occurs in F'_∞/F, we would get the stabilization for the smaller tower F_∞/F as well. This argument shows that to prove stabilization of universal deformation rings for any representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_m(\mathbb{F})$ unramified outside a finite set of primes over any cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension F_∞/F, it is enough to assume that $\overline{\rho}$ is unramified outside primes above $\ell_1, \ldots, \ell_r$ and infinite places.

CHAPTER 3

1-dimensional Cases

In this chapter we consider 1-dimensional representations and their universal deformation rings. We will use the explicit forms of universal deformation rings to prove stabilization of a tower of universal deformation rings.

3.1 Explicit Forms of Universal Deformation Rings

Let F be a number field and suppose that $F/\mathbb{Q}$ is abelian. Let $p, \ell_1, \ldots, \ell_r$ be distinct odd rational primes and let S be the set of primes of F above $\ell_1, \ldots, \ell_r$ and infinite places. Let $\mathbb{F}$ be a finite field of characteristic p. We fix a discrete valuation ring W which is $\mathbb{Z}_p$-free of finite type with maximal ideal $\mathfrak{m}_W$ and $\mathbb{F} = W/\mathfrak{m}_W$. Denote CNL_W for the category of complete noetherian local W-algebras with residue field $\mathbb{F}$. The morphisms in CNL_W are defined to be local homomorphisms.

We start from a character

$$\overline{\rho} : G \to \mathbb{F}^\times$$

for $G := \mathrm{Gal}(F^S/F)$. Let F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension with $F_\infty = \bigcup_n F_n$ where $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/\ell_1^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_r^{n_r}\mathbb{Z}$ for each $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0}$. Let $G_n := \mathrm{Gal}(F^S/F_n) \subseteq G$ for each $n \in \mathbb{Z}^r_{\geq 0}$. Unlike the general m-dimensional case, we can get the explicit forms of universal deformation rings in this case.

Theorem 3.1.1. *For each r-tuple $n = (n_1, \ldots, n_r)$ of non-negative integers, let $F_{n,p}$ be the maximal p-abelian extension of F_n inside F^S. We define $\widetilde{\overline{\rho}} : G \to W^\times$ by the composition of $\overline{\rho}$ and the Teichmuller lift $\mathbb{F}^\times \to W^\times$. We also define $R_n = W[[\mathrm{Gal}(F_{n,p}/F_n)]]$ and $\boldsymbol{\rho}_n : G_n \to R_n^\times$ by $\boldsymbol{\rho}_n(g) = \widetilde{\overline{\rho}}(g)\overline{g}$ where $\overline{g}$ is the class of $g \in G_n$ in the quotient group $\mathrm{Gal}(F_{n,p}/F_n) = G_n / \mathrm{Gal}(F^S/F_{n,p})$. Then, $(R_n, \boldsymbol{\rho}_n)$ is a universal couple for $\overline{\rho}|_{G_n}$.*

Proof. Suppose we have $A \in CNL_W$ and a character $\chi : G_n \to A^\times$ with $\chi \bmod \mathfrak{m}_A = \overline{\rho}|_{G_n}$. We can regard $\widetilde{\overline{\rho}} : G \to A^\times$ by using the structure morphism $W^\times \to A^\times$. Consider $\widetilde{\overline{\rho}}^{-1}\chi : G_n \to A^\times$. Since $\widetilde{\overline{\rho}}^{-1}\chi \bmod \mathfrak{m}_A = 1$, it assumes values in $1 + \mathfrak{m}_A \subseteq A^\times$. Since $1 + \mathfrak{m}_A$ is p-abelian, $\widetilde{\overline{\rho}}^{-1}\chi$ factors through the maximal p-abelian quotient $\mathrm{Gal}(F_{n,p}/F_n)$.

$$\begin{tikzcd}
G_n = \mathrm{Gal}(F^S/F_n) \arrow[r, "\widetilde{\overline{\rho}}^{-1}\chi"] \arrow[d, two heads] \arrow[dr] & A^\times \\
\mathrm{Gal}(F_{n,p}/F_n) \arrow[r] & 1 + \mathfrak{m}_A \arrow[u, hook]
\end{tikzcd}$$

We define $\phi : R_n = W[[\mathrm{Gal}(F_{n,p}/F_n)]] \to A$ as follows. For $\alpha \in \mathrm{Gal}(F_{n,p}/F_n)$, choose $h \in G_n = \mathrm{Gal}(F^S/F_n)$ with $\overline{h} = \alpha$. Let $\phi(\alpha) = (\widetilde{\overline{\rho}}^{-1}\chi)(h)$, then this is well-defined by the previous argument. We can extend this linearly to R_n. Note that for $g \in G_n$,

$$(\phi \circ \boldsymbol{\rho}_n)(g) = \phi(\widetilde{\overline{\rho}}(g)\overline{g}) = \widetilde{\overline{\rho}}(g)\phi(\overline{g}) = \widetilde{\overline{\rho}}(g)\widetilde{\overline{\rho}}^{-1}(g)\chi(g) = \chi(g)$$

by definition. This shows that $\phi \circ \boldsymbol{\rho}_n = \chi$ and $(R_n, \boldsymbol{\rho}_n)$ is a universal couple. $\square$

3.2 Stabilization of a Tower of Universal Deformation Rings

Let n, N be r-tuples of non-negative integers with $n \leq N$. By universality, we get a morphism $\theta_{N,n} : R_N \to R_n$. Note that since $F_N F_{n,p}/F_N$ is p-abelian, we have $F_N F_{n,p} \subseteq F_{N,p}$.

$$\begin{tikzcd}[column sep=small]
F^S \arrow[ddd, dash] \arrow[drr, dash] & & \\
& & F_{N,p} \arrow[dl, dash] \\
& F_N F_{n,p} \arrow[dl, dash] \arrow[dd, dash] & \\
F_N \arrow[dd, dash] & & \\
& F_{n,p} \arrow[dl, dash] & \\
F_n & &
\end{tikzcd}$$

Since F_N/F_n is a prime-to-p extension, we have $\mathrm{Gal}(F_N F_{n,p}/F_N) \cong \mathrm{Gal}(F_{n,p}/F_n)$. Thus we have the map $\mathrm{Gal}(F_{N,p}/F_N) \twoheadrightarrow \mathrm{Gal}(F_N F_{n,p}/F_N) \xrightarrow{\sim} \mathrm{Gal}(F_{n,p}/F_n)$ and this induces the surjective morphism $\theta_{N,n} : R_N = W[[\mathrm{Gal}(F_{N,p}/F_N)]] \to W[[\mathrm{Gal}(F_{n,p}/F_n)]] = R_n$. Therefore, in 1-dimensional cases, we get stabilization of a tower of universal deformation rings if and only if the size of $\mathrm{Gal}(F_{n,p}/F_n)$ is bounded for $n \geq n_0$ with fixed n_0.

Note that the size of $\mathrm{Gal}(F_{n,p}/F_n)$ is equal to that of p-part of the ray class group of F_n defined by $Cl_{F_n}(\ell_1^\infty \cdots \ell_r^\infty \infty)$. We will use the following lemma to measure the size of p-part of the class group.

Lemma 3.2.1. *The following sequence is exact*

$$\prod_{v|\ell_1\cdots\ell_r} \mathcal{O}_{F_n,v}^\times \to Cl_{F_n}(\ell_1^\infty \cdots \ell_r^\infty \infty) \to Cl_{F_n}(\infty) \to 1$$

for each $n \geq 1$ where v covers primes in F_n dividing $\ell_1 \cdots \ell_r$ in the first term.

Proof. We define $U(\ell_1^m \cdots \ell_r^m) = \{x \in \widehat{\mathcal{O}_{F_n}}^\times \mid x \equiv 1 \pmod{\ell_1^m \cdots \ell_r^m}\}$ for $n \geq 1$ and

$$U := \bigcap_m U(\ell_1^m \cdots \ell_r^m) = \prod_{v \nmid \ell_1, \ldots, \ell_r} \mathcal{O}_{F_n,v}^\times$$

with primes v in F_n. Note that we have a surjection

$$\mathbb{A}_{F_n}^\times / \overline{F_n^\times U (F_n)_{\infty,+}^\times} \twoheadrightarrow \mathbb{A}_{F_n}^\times / F_n^\times \widehat{\mathcal{O}_{F_n}}^\times (F_n)_{\infty,+}^\times = Cl_{F_n}(\infty)$$

with kernel $\displaystyle\prod_{v \mid \ell_1, \ldots, \ell_r} \mathcal{O}_{F_n,v}^\times$. Since

$$\begin{aligned}
\mathbb{A}_{F_n}^\times / \overline{F_n^\times U (F_n)_{\infty,+}^\times} &= \varprojlim_m \mathbb{A}_{F_n}^\times / \overline{F_n^\times U(\ell_1^m \cdots \ell_r^m)(F_n)_{\infty,+}^\times} \\
&= \varprojlim_m Cl_{F_n}(\ell_1^m \cdots \ell_r^m \infty) \\
&= Cl_{F_n}(\ell_1^\infty \cdots \ell_r^\infty \infty),
\end{aligned}$$

we get the desired exact sequence. □

Let's consider the p-part of the exact sequence in the lemma above.

Theorem 3.2.2 (Friedman)**.** *Let $p_1, \ldots, p_s$ be distinct prime numbers, k a finite abelian number field, and K the cyclotomic $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}$-extension of k. For $N = (n_1, \ldots, n_s) \in \mathbb{Z}_{\geq 0}^s$ there is a unique field k_N such that $k \subseteq k_N \subseteq K$ and $[k_N : k] = \displaystyle\prod_s p_i^{n_i}$. For a prime ℓ, we* *define* e_N^ℓ *by* $\ell^{e_N^\ell} \| h_{k_N}$ *where* h_{k_N} *is the class number of* k_N*. Then,*

1. *for* $\ell \nmid p_1 \cdots p_s$*,* e_N^ℓ *is constant for* $N \gg 0$*,*
2. *for* $\ell = p_i$*, there exist integers* $\lambda_i \geq 0$ *and* v_i*, independent of* N*, such that* $e_N^{p_i} = \lambda_i n_i + v_i$ *for* $N \gg 0$*,*
3. *the* p_i*-part of the class group* Cl_{k_N} *is isomorphic to* $(\mathbb{Q}_{p_i}/\mathbb{Z}_{p_i})^{\lambda_i}$*.*

Proof. See [Fri82]. □

Since $F/\mathbb{Q}$ is abelian, the size of p-part of Cl_{F_n} is constant for $n \gg 0$ by the theorem above. Note that since the sizes of

$$Cl_{F_n}(\infty) = \mathbb{A}_{F_n}^\times / F_n^\times \widehat{\mathcal{O}}^\times (F_n)_{\infty,+}^\times$$

and

$$Cl_{F_n} = \mathbb{A}_{F_n}^\times / F_n^\times \widehat{\mathcal{O}}^\times (F_n)_{\infty}^\times$$

differ by a power of 2, they have the same p-part.

Now we will prove that the size of p-part of the first term in Lemma 3.2.1 is also bounded for $n \gg 0$. We need to check two things. First, we will show that there are only finitely many primes in F_∞ dividing $\ell_1, \ldots, \ell_r$ so that the number of terms in the product is bounded for all r-tuples n. Second, we will also prove that the size of p-part of $\mathcal{O}_{F_n,v}^\times$ is bounded for all n and v. Both claims are proved as follows.

Lemma 3.2.3. *Let F be a number field and $\ell_1, \ldots, \ell_r$ be rational primes. Let F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension. Then for any prime $\mathfrak{l}$ in F, there are only finitely many primes in F_∞ that divides $\mathfrak{l}$.*

Proof. Let $F' = F(\mu_{\ell_1^\infty}, \mu_{\ell_2^\infty}, \ldots, \mu_{\ell_r^\infty}) \supseteq F_\infty$. We will prove a stronger result that there are only finitely many primes in F' that divides $\mathfrak{l}$. Let $D_\mathfrak{l} \subseteq \mathrm{Gal}(F'/F)$ be the decomposition group of $\mathfrak{l}$. Since the number of primes in F' over $\mathfrak{l}$ equals the index $(\mathrm{Gal}(F'/F) : D_\mathfrak{l})$, we need to show that this index is finite.

Suppose $\mathfrak{l}$ is above a rational prime $\ell \notin \{\ell_1, \ldots, \ell_r\}$. Then, $D_\mathfrak{l} \subseteq \mathrm{Gal}(F'/F) \subseteq \mathbb{Z}_{\ell_1}^\times \times \cdots \times \mathbb{Z}_{\ell_r}^\times$ is generated by ℓ^f for some $f \geq 1$. Since the projection of $\langle \ell^f \rangle$ to each component $\mathbb{Z}_{\ell_i}^\times$ has infinite order, the index $(\mathrm{Gal}(F'/F) : D_\mathfrak{l})$ is finite.

Suppose $\mathfrak{l}$ is above a rational prime $\ell \in \{\ell_1, \ldots, \ell_r\}$, say $\ell = \ell_1$. In this case, the inertia at ℓ is isomorphic to an open subgroup of $\mathbb{Z}_\ell^\times$ and after taking quotient by the inertia of $\mathrm{Gal}(F'/F)$, $D_\mathfrak{l}$ is generated by a power of ℓ. Thus, the same argument works. $\square$

Lemma 3.2.4. *The size of p-part of $\mathcal{O}_{F_n,v}^\times$ is bounded for all n and v.*

Proof. Suppose $r = 1$ and let $\ell = \ell_1$. We can regard the p-part of $\mathcal{O}_{F_n,v}^\times$ as the intersection $\mathcal{O}_{F_n,v} \cap \mu_{p^\infty}$ inside the fixed algebraic closure $\overline{\mathbb{Q}_\ell}$. Note that the residue field of $\mathcal{O}_{F_n,v}$ is $\mathbb{F}_\ell$ for all n and v. Since there's a one-to-one correspondence between $\mathcal{O}_{F_n,v} \cap \mu_{p^\infty}$ and $\mathbb{F}_\ell \cap \mu_{p^\infty} \subseteq \overline{\mathbb{F}_\ell}$, the size of p-part of $\mathcal{O}_{F_n,v}^\times$ is bounded.

Now let $r \geq 2$ and choose a prime v in F_n such that $v \mid \ell_1$. Then the size of p-part of $\mathcal{O}_{F_n,v}^\times$ is bounded by that of $\mathbb{F}_{\ell_1}(\mu_{\ell_2^\infty}, \ldots, \mu_{\ell_r^\infty}) \cap \mu_{p^\infty} \subseteq \overline{\mathbb{F}_{\ell_1}}$ because the residue field of $\mathcal{O}_{F_n,v}$ is contained in $\mathbb{F}_{\ell_1}(\mu_{\ell_2^\infty}, \ldots, \mu_{\ell_r^\infty})$. We will show that $\mathbb{F}_{\ell_1}(\mu_{\ell_2^\infty}, \ldots, \mu_{\ell_r^\infty}) \cap \mathbb{F}_{\ell_1}(\mu_{p^\infty}) \subseteq \overline{\mathbb{F}_{\ell_1}}$ is finite.

$$
\begin{tikzcd}[column sep=small]
 & \overline{\mathbb{F}_{\ell_1}} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{F}_{\ell_1}(\mu_{\ell_2^\infty}, \ldots, \mu_{\ell_r^\infty}) \arrow[dr, dash] & & \mathbb{F}_{\ell_1}(\mu_{p^\infty}) \arrow[dl, dash] \\
 & \mathbb{F}_{\ell_1}(\mu_{\ell_2^\infty}, \ldots, \mu_{\ell_r^\infty}) \cap \mathbb{F}_{\ell_1}(\mu_{p^\infty}) \arrow[d, dash] & \\
 & \mathbb{F}_{\ell_1} &
\end{tikzcd}
$$

Note that $\mathrm{Gal}(\mathbb{F}_{\ell_1}(\mu_{\ell_2^\infty}, \ldots, \mu_{\ell_r^\infty}) \cap \mathbb{F}_{\ell_1}(\mu_{p^\infty})/\mathbb{F}_{\ell_1})$ is a subquotient of

$$\mathrm{Gal}(\mathbb{F}_{\ell_1}(\mu_{\ell_2^\infty}, \ldots, \mu_{\ell_r^\infty})/\mathbb{F}_{\ell_1}) \hookrightarrow \mathbb{Z}_{\ell_2}^\times \times \cdots \times \mathbb{Z}_{\ell_r}^\times$$

and is also that of

$$\mathrm{Gal}(\mathbb{F}_{\ell_1}(\mu_{p^\infty})/\mathbb{F}_{\ell_1}) \hookrightarrow \mathbb{Z}_p^\times.$$

Therefore, $\mathrm{Gal}(\mathbb{F}_{\ell_1}(\mu_{\ell_2^\infty}, \ldots, \mu_{\ell_r^\infty}) \cap \mathbb{F}_{\ell_1}(\mu_{p^\infty})/\mathbb{F}_{\ell_1})$ has to be finite. This shows that the p-part of $\mathcal{O}_{F_n,v}^\times$ is bounded for all $n = (n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r$ and all primes $v \mid \ell_1$ in F_n. The same argument works for primes v dividing $\ell_2, \ldots, \ell_r$. $\square$

The above two lemmas combined with Lemma 3.2.1 prove the followings.

Proposition 3.2.5. *Let F be an abelian number field and F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension. For each $n = (n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r$, we have an intermediate field F_n such that $\mathrm{Gal}(F_n/F) = \mathbb{Z}/\ell_1^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_r^{n_r}\mathbb{Z}$. Then, the size of p-part of $Cl_{F_n}(\ell_1^\infty \cdots \ell_r^\infty \infty)$ is* *bounded for $n \gg 0$.*

Theorem 3.2.6. *Let F be an abelian number field and F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1}\times\cdots\times\mathbb{Z}_{\ell_r}$-extension. Le S be the set of primes of F above $\ell_1\cdots\ell_r\infty$. Let $\mathbb{F}$ be a finite field of characteristic p which is different from $\ell_1,\ldots,\ell_r$. Let $G = \mathrm{Gal}(F^S/F)$ and*

$$\overline{\rho} : G \to \mathbb{F}^\times$$

be a character. We fix a discrete valuation ring W which is $\mathbb{Z}_p$-free of finite type with residue field $\mathbb{F}$. Let $(R_n, \boldsymbol{\rho}_n)$ be the universal couple corresponding to $\overline{\rho}|_{G_n}$ for all $n = (n_1,\ldots,n_r) \in \mathbb{Z}^r_{\geq 0}$ where $G_n = \mathrm{Gal}(F^S/F_n)$ with $\mathrm{Gal}(F_n/F) = \mathbb{Z}/\ell_1^{n_1}\mathbb{Z}\times\cdots\times\mathbb{Z}/\ell_r^{n_r}\mathbb{Z}$. Then the tower of universal deformation rings $\{R_n\}$ corresponding to $\{\overline{\rho}|_{G_n}\}$ stabilizes, i.e., there is $N_0 \in \mathbb{Z}^r_{\geq 0}$ such that for all $N \geq N_0$ we have $R_N \cong R_{N_0}$.

Note 3.2.7. Note that the stabilization may not occur in non-cyclotomic $\mathbb{Z}_{\ell_1}\times\cdots\times\mathbb{Z}_{\ell_r}$-extensions since we used Friedman's theorem which works only for cyclotomic extensions.

CHAPTER 4

The Key Lemma

In this chapter, we prove a theorem that provides an equivalent condition for the stabilization of a tower of universal deformation rings.

4.1 Adjoint Representations

We first recall the definition of an adjoint representation.

Definition 4.1.1. Let G be a profinite group and $\rho : G \to \mathrm{GL}_m(K)$ be a representation to a field K. Let G act on $M_m(K)$ by conjugation, i.e., $g \cdot A = \rho(g)A\rho(g)^{-1}$ for $A \in M_m(K)$. We denote $M_m(K)$ by $Ad(\rho)$ as an m^2-dimensional G-module by the G-action defined above. The trace zero subspace of $M_m(K)$, which has dimension $m^2 - 1$, is stable under this action. We call this $m^2 - 1$ dimensional G-module the adjoint representation of ρ and denote it by $ad(\rho)$.

We will write Ad or ad for short when the base representation is obvious.

4.2 The Key Lemma

We start from the following general result regarding the base change of universal deformation rings.

Lemma 4.2.1. *Let G be a profinite group that satisfies Mazur's finiteness condition Φ_p and let $\mathbb{F}$ be a finite field of characteristic p. Let $\overline{\rho} : G \to \mathrm{GL}_m(\mathbb{F})$ be a continuous representation and let $H \le G$ be a normal open subgroup. Suppose that*

1. *$\overline{\rho}|_H$ is absolutely irreducible,*
2. *the restriction map $H^1(G, ad) \to H^1(H, ad)$ is an isomorphism,*
3. *the restriction map $H^2(G, ad) \to H^2(H, ad)$ is injective.*

Let $R_G = R_G(\overline{\rho}, W)$, $R_H = R_H(\overline{\rho}|_H, W)$ be the universal deformation rings. Then the canonical homomorphism $\pi : R_H \to R_G$ is an isomorphism.

Proof. See [BK, Lemma 2.1]. □

Let $\mathbb{F}$ be a number field $p, \ell_1, \ldots, \ell_r$ be distinct odd rational primes. Let F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension. For each $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\ge 0}$, we have $F_n \subseteq F_\infty$ such that $\mathrm{Gal}(F_n/F) = \mathbb{Z}/\ell_1^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_r^{n_r}\mathbb{Z}$ and $F_\infty = \bigcup_n F_n$. Let S be the set of primes of F above $\ell_1 \cdots \ell_r \infty$ and let F^S/F be the maximal unramified extension outside S. We define $G = \mathrm{Gal}(F^S/F)$, $G_\infty = \mathrm{Gal}(F^S/F_\infty)$, and $G_n = \mathrm{Gal}(F^S/F_n) \le G$ for each $n \in \mathbb{Z}^r_{\ge 0}$.

Lemma 4.2.2. *Let $\mathbb{F}$ be a finite field of characteristic p, and $\overline{\rho} : G \to \mathrm{GL}_m(\mathbb{F})$ be an absolutely irreducible representation. Suppose that $\dim_{\mathbb{F}} H^1(G_\infty, ad)$ is finite. Then, there exists $n \in \mathbb{Z}^r_{\ge 0}$ such that*

1. *the restriction map $H^1(G_N, ad) \to H^1(G_n, ad)$ is an isomorphism,*

2. the restriction map $H^2(G_N, ad) \to H^2(G_n, ad)$ is injective.

for all $N \geq n$.

Proof. We first assume that $r = 1$ and let $\ell = \ell_1$. We follow the proof of [BK, Theorem 2.3]. We have the following spectral sequence

$$E_2^{ij} = H^i(G_n/G_\infty, H^j(G_\infty, ad)) \ \Rightarrow \ H^{i+j}(G_n, ad)$$

for all n. Since the quotient G_n/G_∞ has cohomological dimension 1, we get a short exact sequence

$$0 \to E_\infty^{1,j-1} \to H^j \to E_\infty^{0,j} \to 0$$

for all n. Choose a topological generator γ of $\mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_\ell$, then we can rewrite the above exact sequence by

$$0 \to H^{j-1}(G_\infty, ad)/(1-\gamma^{\ell^n}) \to H^j(G_n, ad) \to H^j(G_\infty, ad)^{\gamma^{\ell^n}} \to 0$$

for $j \geq 1$ where $H^{j-1}(G_\infty, ad)/(1-\gamma^{\ell^n}) = H^{j-1}(G_\infty, ad)/(1-\gamma^{\ell^n})H^{j-1}(G_\infty, ad)$ for short, and $H_j(G_\infty, ad)^{\gamma^{\ell^n}}$ is the fixed part of $H^j(G_\infty, ad)$ by the action of $\langle \gamma^{\ell^n} \rangle = \mathrm{Gal}(F_\infty/F_n) \leq \mathrm{Gal}(F_\infty/F)$. For given $N \geq n$, we consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{j-1}(G_\infty, ad)/(1-\gamma^{\ell^n}) \arrow[r] \arrow[d, "\tau"] & H^j(G_n, ad) \arrow[r] \arrow[d, "\mathrm{res}"] & H^j(G_\infty, ad)^{\gamma^{\ell^n}} \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & H^{j-1}(G_\infty, ad)/(1-\gamma^{\ell^N}) \arrow[r] & H^j(G_N, ad) \arrow[r] & H^j(G_\infty, ad)^{\gamma^{\ell^N}} \arrow[r] & 0
\end{tikzcd}$$

where ι is the inclusion and τ is induced from $1 + \gamma^{\ell^n} + \gamma^{2\ell^n} + \cdots + \gamma^{\ell^N - \ell^n}$. Since ad is finite of characteristic $p \neq \ell$ and $H^1(G_\infty, ad)$ is finite dimensional, we can choose big enough n so that γ^{ℓ^n} is unipotent as an action on $H^1(G_\infty, ad)$. Considering that the number of summands in τ is ℓ^{N-n}, which is prime to p, we can conclude that τ is an isomorphism for $j = 1, 2$. Also this tells us that $\ker(1 - \gamma^{\ell^N}) = \ker((1 - \gamma^{\ell^n})\tau) = \ker(1 - \gamma^{\ell^n})$, thus ι is also an isomorphism for $j = 1$.

This proves that the restriction map in the middle is an isomorphism for $j = 1$ and is an injection for $j = 2$.

For the general case of $r \geq 2$, we can take a topological generator γ of $\mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$ and we can construct the same commutative diagram as above. The number of summands in τ is then the product of powers of $\ell_1, \ldots, \ell_r$, which is still prime to p. So the same proof works in general. $\square$

The two lemmas above and Lemma 2.2.4 prove the following.

Proposition 4.2.3. *Let F be a number field and $p, \ell_1, \ldots, \ell_r$ be distinct odd rational primes. Let F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension with $F_\infty = \bigcup_n F_n$ where $\mathrm{Gal}(F_n/F) = \mathbb{Z}/\ell_1^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_r^{n_r}\mathbb{Z}$ for $n = (n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r$. Let S be the set of primes of F above $\ell_1 \cdots \ell_r \infty$. Let $\mathbb{F}$ be a finite field of characteristic p and $\overline{\rho} : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_m(\mathbb{F})$ be an absolutely irreducible representation for $m \geq 1$. Then, we get a tower of universal deformation rings $\{R_n\}$ associated with $\{\overline{\rho}|_{\mathrm{Gal}(F^S/F_n)}\}$. Suppose that $\dim_{\mathbb{F}} H^1(G_\infty, ad)$ is finite. Then, there exists $N_0 \in \mathbb{Z}_{\geq 0}^r$ such that the canonical homomorphism $R_N \to R_{N_0}$ is an isomorphism for all $N \geq N_0$, i.e., the tower of universal deformation rings $\{R_n\}$ stabilizes.*

In fact, we can also prove that the assumption we used is actually an equivalent condition for stabilization of a tower of universal deformation rings. Recall that each universal deformation ring R_n is in CNL_W for fixed W.

Lemma 4.2.4. *For a universal deformation ring R_n, we define*

$$t_{R_n/W} = \mathrm{Hom}_{\mathbb{F}}(t^*_{R_n/W}, \mathbb{F})$$

where $t^*_{R_n/W} = \mathfrak{m}_{R_n}/(\mathfrak{m}^2_{R_n} + \mathfrak{m}_W)$. Then, we have

$$t_{R_n/W} \cong H^1(G_n, ad)$$

for all n. The space $t_{R_n/W}$ is called the tangent space of $\mathrm{Spec}(R_n)_{/W}$ at $\mathfrak{m}_W$.

Proof. See [H2, Lemma 2.29]. □

Lemma 4.2.5. *We use the same notations as above. Let $\mathbb{F}$ be a finite field of characteristic p and $\overline{\rho} : G \to \mathrm{GL}_m(\mathbb{F})$ be an absolutely irreducible representation. Then, the followings are equivalent.*

1. *$\dim_{\mathbb{F}} H^1(G_\infty, ad)$ is finite.*
2. *The tower of universal deformation rings $\{R_n\}$ stabilize.*

Proof. By the proposition above, the first statement implies the second one. Conversely, suppose that the tower of universal deformation rings $\{R_n\}$ stabilize, i.e., there is $N_0 \in \mathbb{Z}^r_{\geq 0}$ such that $R_N \cong R_{N_0}$ for all $N \geq N_0$. Then, we have

$$H^1(G_N, ad) \cong t_{R_N/W} \cong t_{R_n/W} \cong H^1(G_n, ad)$$

by the lemma above. Note that any cocycle $\sigma \in H^1(G_\infty, ad)$ extends to G_n for some n because G_∞ is discrete and ad is finite. Thus we have

$$H^1(G_\infty, ad) = \varinjlim_n H^1(G_n, ad) = H^1(G_{N_0}, ad),$$

by the assumption. Note that R_{N_0} is noetherian because G_{N_0} satisfies Φ_p (See [H2, Proposition 2.30].) Thus, $H^1(G_\infty, ad) \cong H^1(G_{N_0}, ad) \cong t_{R_{N_0}/W}$ is finite dimensional. $\square$

CHAPTER 5

2-dimensional Cases

In this chapter, we will show that we get the stabilization of a tower of universal deformation rings if we start from an induced representation with certain conditions. We use the same notation as in previous chapters unless stated otherwise.

5.1 Stabilization of a Tower of Universal Deformation Rings

Let K, F be number fields with $[K : F] = 2$ and $p, \ell_1, \ldots, \ell_r$ be distinct rational primes. Let $\mathbb{F}$ be a finite field of characteristic p and $\overline{\phi} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{F}^\times$ be a continuous character. Then, we consider the representation

$$\overline{\rho} = \mathrm{Ind}_K^F \overline{\phi} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\mathbb{F})$$

induced from $\overline{\phi}$. We assume that $\overline{\rho}$ is absolutely irreducible. Let S be the set of primes of F over $\ell_1 \cdots \ell_r \infty$ and suppose that $\overline{\rho}$ is unramified outside S. Thus we can regard $\overline{\rho} : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_2(\mathbb{F})$.

Let F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension, then for each $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0}$ we have an intermediate field F_n with $\mathrm{Gal}(F_n/F) = \mathbb{Z}/\ell_1^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell_r^{n_r}\mathbb{Z}$. Define $G_n = \mathrm{Gal}(F^S/F_n)$, then we get universal couples $(R_n, \boldsymbol{\rho}_n)$ from $\overline{\rho}|_{G_n}$ for each n by Lemma 2.2.4 and Mazur's theorem.

Definition 5.1.1. We define $\overline{\phi}^-$ by

$$\overline{\phi}^-(g) = \overline{\phi}(g)\overline{\phi}(cgc^{-1})^{-1}$$

for $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ inducing a nontrivial automorphism on K.

Lemma 5.1.2. *If the character $\overline{\phi}^-$ defined above has order 2, then it has an extension $\widetilde{\overline{\phi}^-} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{F}^\times$ which also has order 2.*

Proof. Since $\overline{\phi}^-$ has order 2, it is $\mathrm{Gal}(K/F)$-invariant. Thus, it extends to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ by [H2, Corollary 4.37]. The coefficient field of the extension is intact as the induced representation is dihedral modulo center. Since $\mathrm{Gal}(\overline{\mathbb{Q}}^{\ker(ad(\overline{\rho}))}/F)$ is dihedral of order 4, there is no element of order 4. This proves that $\widetilde{\overline{\phi}^-}$ has to be quadratic. □

Lemma 5.1.3. *For $\overline{\rho}$ and $\overline{\phi}$ defined above, suppose that $\overline{\phi}^-$ has order 2. Then we have*

$$ad(\overline{\rho}) = ad(\mathrm{Ind}_K^F \overline{\phi}) = \chi \oplus \widetilde{\overline{\phi}^-} \oplus \widetilde{\overline{\phi}^-}\chi$$

where χ is the quadratic character defined by the symbol $\left(\frac{K/F}{\cdot}\right)$.

Proof. Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and $H = \mathrm{Gal}(\overline{\mathbb{Q}}/K) \leq G$. We realize $\overline{\rho} = \mathrm{Ind}_K^F(\overline{\phi})$ as a 2×2 diagonal matrix with diagonal entries $\overline{\phi}$ and $\overline{\phi}'$ when restricted to H. We choose an element $s \in G \setminus H$ inducing a non-trivial automorphism of F as an anti-diagonal matrix. This follows from the matrix form of induced representation, i.e., an element in $\overline{\rho}(G)$ is diagonal and an element in $\overline{\rho}(G \setminus H)$ is anti-diagonal. Let G act on $M_2(\mathbb{F})$ by conjugation from $\overline{\rho}$. Let D (resp. A) be the subspace of diagonal (resp. anti-diagonal) matrices in $M_2(\mathbb{F})$. By the above shape, we can see that D and A are stable under the conjugate action of $\overline{\rho}$ on $M_2(\mathbb{F})$. We can further decompose D as a direct sum of S and S' where S is the subspace of scalar matrices and S' is that of matrices of the form $\mathrm{diag}(a, -a)$ for $a \in \mathbb{F}$.

$$M_2(\mathbb{F}) = D \oplus A = S \oplus S' \oplus A.$$

It is easy to see that the conjugation by $\overline{\rho}$ acts as trivial on S and as χ on S'. Let N^+ (resp. N^-) be the upper-nilpotent subspace (resp. the lower-nilpotent subspace) in A. The conjugation by $\overline{\rho}$ interchanges these two subspaces. The subgroup H acts on N^+ by $\overline{\phi}^-$ while it acts on N^- by $(\overline{\phi}^-)^{-1}$. This shows that the action of $\overline{\rho}$ on A is isomorphic to $\mathrm{Ind}_K^F \overline{\phi}^-$.

This proves that we have

$$Ad(\overline{\rho}) = Ad(\mathrm{Ind}_K^F \overline{\phi}) = \mathbf{1} \oplus \chi \oplus \mathrm{Ind}_K^F \overline{\phi}^-,$$

and

$$ad(\overline{\rho}) = \chi \oplus \mathrm{Ind}_K^F \overline{\phi}^-.$$

Also by the previous lemma, $\overline{\phi}^-$ has an extension $\widetilde{\overline{\phi}^-}$ and we get

$$\mathrm{Ind}_K^F \overline{\phi}^- = \widetilde{\overline{\phi}^-} \oplus \widetilde{\overline{\phi}^-}\chi,$$

which completes the proof. □

We will show that $\dim_{\mathbb{F}} H^1(G_n, ad(\overline{\rho}))$ is bounded for $n \gg 0$, thus the tower $\{R_n\}$ stabilize by Lemma 4.2.5. We have

$$H^1(G_n, ad(\overline{\rho})) = H^1(G_n, \chi) \oplus H^1(G_n, \widetilde{\overline{\phi}^-}) \oplus H^1(G_n, \widetilde{\overline{\phi}^-}\chi)$$

by the above lemma. Let $\xi \in \{\chi, \widetilde{\overline{\phi}^-}, \widetilde{\overline{\phi}^-}\chi\}$. For each $\xi : G_n \to \mathbb{F}^\times$, we define a finite extension F_n^{ξ}/F_n so that $\ker \xi = \mathrm{Gal}(F^S/F_n^{\xi})$.

Theorem 5.1.4 (Inflation-restriction sequence)**.** *Let G be a group, H a normal subgroup of G and A an abelian G-group. Then we have the following exact sequence.*

$$0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A)^{G/H} \to H^2(G/H, A^H) \to H^2(G, A)$$

where $A^H = H^0(H, A)$ for an H-group A.

We will apply the theorem above to $G = \mathrm{Gal}(F^S/F_n)$, $H = \mathrm{Gal}(F^S/F_n^{\xi})$ and a G-group $A = \mathbb{F} = \mathbb{F}(\xi)$ defined by the action of G_n on $\mathbb{F}$ by ξ. We get the following exact sequence

$$\begin{aligned} 0 \to H^1(\operatorname{im}\xi, \mathbb{F}^{\mathrm{Gal}(F^S/F_n^{\xi})}) \to H^1(\mathrm{Gal}(F^S/F_n), \mathbb{F}) \to H^1(\mathrm{Gal}(F^S/F_n^{\xi}), \mathbb{F})^{\operatorname{im}\xi} \\ \to H^2(\operatorname{im}\xi, \mathbb{F}^{\mathrm{Gal}(F^S/F_n^{\xi})}) \end{aligned}$$

from the theorem where $\operatorname{im}\xi = \mathrm{Gal}(F_n^{\xi}/F_n) = G/H$. Since $|\operatorname{im}\xi|$ is prime to p and $|\mathbb{F}|$ is a power of p, we get

$$H^1(\operatorname{im}\xi, \mathbb{F}^{\mathrm{Gal}(F^S/F_n^{\xi})}) = H^2(\operatorname{im}\xi, \mathbb{F}^{\mathrm{Gal}(F^S/F_n^{\xi})}) = 0.$$

This shows

$$H^1(G_n, \xi) = H^1(\mathrm{Gal}(F^S/F_n), \mathbb{F}) \cong H^1(\mathrm{Gal}(F^S/F_n^{\xi}), \mathbb{F})^{\operatorname{im}\xi}$$

by exactness of the sequence. Note that the action of $\mathrm{Gal}(F^S/F_n^{\xi})$ on $\mathbb{F}$ is trivial by definition of F_n^ξ. Thus, we get

$$H^1(\mathrm{Gal}(F^S/F_n^\xi), \mathbb{F}) \cong \mathrm{Hom}(\mathrm{Gal}(F^S/F_n^\xi), \mathbb{F}).$$

Since $F^S = (F_n^\xi)^S$ and $\mathbb{F}$ is a p-group, we get

$$|\mathrm{Hom}(\mathrm{Gal}(F^S/F_n^\xi), \mathbb{F})| = |Cl_{F_n^\xi}(\ell_1^\infty \cdots \ell_r^\infty \infty)_p|$$

where $Cl_{F_n^\xi}(\ell_1^\infty \cdots \ell_r^\infty \infty)_p$ is the maximal p-quotient of $Cl_{F_n^\xi}(\ell_1^\infty \cdots \ell_r^\infty \infty)$. Note that $F_n^\xi = F^\xi F_n$ for all n, thus F_∞^ξ/F^ξ is also a cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension. Suppose that $F^\xi/\mathbb{Q}$ is abelian for all $\xi \in \{\chi, \widetilde{\overline{\phi}^{-}}, \widetilde{\overline{\phi}^{-}}\chi\}$.

Lemma 5.1.5. $\dim_{\mathbb{F}} H^1(G_n, \xi)$ *is bounded for* $n \gg 0$.

Proof. By Proposition 3.2.5, the size $|Cl_{F_n^\xi}(\ell_1^\infty \cdots \ell_r^\infty \infty)_p|$ is bounded for $n \gg 0$. Since

$$|H^1(G_n, \xi)| = |H^1(\mathrm{Gal}(F^S/F_n^\xi), \mathbb{F})^{\mathrm{im}\,\xi}| \leq |H^1(\mathrm{Gal}(F^S/F_n^\xi), \mathbb{F})| = |Cl_{F_n^\xi}(\ell_1^\infty \cdots \ell_r^\infty \infty)_p|,$$

we get the desired result. $\square$

Theorem 5.1.6. *Let* K, F, $\overline{\rho} = \mathrm{Ind}_K^F \overline{\phi} : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_2(\mathbb{F})$ *be as defined above. Suppose that* $\overline{\phi}^{-}$ *has order* 2 *and* F^ξ *is abelian over* $\mathbb{Q}$ *for* $\xi = \chi, \widetilde{\overline{\phi}^{-}}, \widetilde{\overline{\phi}^{-}}\chi$. *Then the tower of universal deformation rings* $\{R_n\}$ *defined on* F_∞/F *stabilize.*

Proof. By Lemma 5.1.3 and Lemma 5.1.5,

$$\dim_{\mathbb{F}} H^1(G_n, ad) = \sum_{\xi \in \{\chi, \widetilde{\overline{\phi}^{-}}, \widetilde{\overline{\phi}^{-}}\chi\}} \dim_{\mathbb{F}} H^1(G_n, \xi)$$

is bounded for $n \gg 0$. Now we use the fact that $H^1(G_\infty, ad) = \varinjlim_n H^1(G_n, ad)$ and apply Lemma 4.2.5. $\square$

5.2 p-ordinary Cases

In this section, we will allow our starting representation $\overline{\rho}$ and deformations of it to have ramification at p and consider p-ordinary universal deformation rings instead of full universal deformation rings. We will start by reviewing basic definitions first.

5.2.1 p-ordinary Universal Deformation Rings

Let F be an abelian number field and $p, \ell_1, \ldots, \ell_r$ be distinct odd rational primes. Let S be the set of primes of F above $p\ell_1 \cdots \ell_r \infty$. We start from an absolutely irreducible Galois representation

$$\overline{\rho} : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_2(\mathbb{F})$$

for some finite field $\mathbb{F}$ of characteristic p. Note that unlike the case in the previous section, we allow $\overline{\rho}$ to have ramification at p. We fix a discrete valuation ring W finite free over $\mathbb{Z}_p$ with $W/\mathfrak{m}_W = \mathbb{F}$ and consider the category of complete noetherian local W-algebras CNL_W. For each $A \in CNL_W$, we consider deformations $\rho : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_2(A)$ with $\rho \bmod \mathfrak{m}_A = \overline{\rho}$ satisfying

$$\text{(ord)} \quad \rho|_{D_\mathfrak{p}} \cong \begin{pmatrix} \epsilon_\mathfrak{p} & * \\ 0 & \delta_\mathfrak{p} \end{pmatrix} \text{ with } \delta_\mathfrak{p} \text{ unramified, } \epsilon_\mathfrak{p} \text{ ramified, and } \overline{\epsilon_\mathfrak{p}} \text{ ramified}$$

where $D_\mathfrak{p}$ is the decomposition group at a prime $\mathfrak{p}$ of F above p.

Definition 5.2.1. We call a deformation $\rho : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_2(A)$ of $\overline{\rho}$ p-ordinary if it satisfies (ord) for $\mathfrak{p} \mid p$.

Theorem 5.2.2 (Mazur). *Consider the functor* $\mathcal{F}_{\overline{\rho}}^{\mathrm{ord}} : CNL_W \to Sets$ *defined by*

$$\mathcal{F}_{\overline{\rho}}^{\mathrm{ord}}(A) = \{\rho : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_2(A) \mid \rho \bmod \mathfrak{m}_A = \overline{\rho}, \rho \textit{ satisfies } (\mathrm{ord})\}/\approx$$

for $A \in CNL_W$. *Then, this functor is representable and we have a universal couple* $(R^{\mathrm{ord}}, \boldsymbol{\rho}^{\mathrm{ord}} : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_2(R^{\mathrm{ord}}))$ *in the sense that for* $\rho \in \mathcal{F}_{\overline{\rho}}^{\mathrm{ord}}(A)$ *there exists a unique morphism* $\phi_\rho : R^{\mathrm{ord}} \to A$ *in* CNL_W *such that* $\phi_\rho \circ \boldsymbol{\rho}^{\mathrm{ord}} \approx \rho$.

Proof. See [H2, Theorem 3.30]. □

The ring R^{ord} in the theorem above is called a p-ordinary universal deformation ring. The Galois group $G := \mathrm{Gal}(F^S/F)$ acts on $M_2(\mathbb{F})$ by conjugation and the trace zero subspace is stable under this action. We call this Galois module of dimension 3 the adjoint representation $ad(\overline{\rho})$ of $\overline{\rho}$. We will write $ad(\overline{\rho})$ by ad for short. We fix a prime $\mathfrak{p}$ of F above p.

The underlying space

$$V = V(ad) = \{T \in \mathrm{End}_{\mathbb{F}}(V(\overline{\rho})) \mid tr(T) = 0\}$$

has a three-step filtration $0 \subseteq V_{\mathfrak{p}}^{+} \subseteq V_{\mathfrak{p}}^{-} \subseteq V$ where

$$V_{\mathfrak{p}}^{-}(ad) = \{T \in V(ad) \mid T(V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}})) \subseteq V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}})\}$$

and

$$V_{\mathfrak{p}}^{+}(ad) = \{T \in V_{\mathfrak{p}}^{-}(ad) \mid T(V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}})) = 0\}.$$

We take a base of $V(\overline{\rho})$ so that $\overline{\rho}|_{D_{\mathfrak{p}}} = \begin{pmatrix} \epsilon_{\mathfrak{p}} & * \\ 0 & \delta_{\mathfrak{p}} \end{pmatrix}$, then we have

$$V_{\mathfrak{p}}^{+} = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}, \quad V_{\mathfrak{p}}^{-} = \left\{ \begin{pmatrix} a & * \\ 0 & -a \end{pmatrix} \right\}$$

inside $M_2(\mathbb{F})$. We define the adjoint Selmer group $\mathrm{Sel}_F(ad)$ by

$$\mathrm{Sel}_F(ad) = \ker \left(H^1(G, ad) \xrightarrow{res} \prod_{\mathfrak{p}|p} H^1 \left(D_{\mathfrak{p}}, \frac{V_{\mathfrak{p}}}{V_{\mathfrak{p}}^{+}} \right) \right),$$

and $H^1_{\mathrm{ord}}(G, ad) := \mathrm{Sel}_F(ad) \subseteq H^1(G, ad)$.

Let F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension, then similarly we have universal couples $(R_n^{\mathrm{ord}}, \boldsymbol{\rho}_n^{\mathrm{ord}})$ for each $n \in \mathbb{Z}_{\geq 0}^r$ and a tower of p-ordinary universal deformation rings $\{R_n^{\mathrm{ord}}\}$. We can ask the same question as before.

Question 5.2.3. Does the tower of p-ordinary universal deformation rings $\{R_n^{\mathrm{ord}}\}$ stabilize?

5.2.2 Stabilization of a Tower of p-ordinary Universal Deformation Rings

We follow the same notation as in the previous section.

Let $G = \mathrm{Gal}(F^S/F)$ and define $A = \mathbb{F}[X]/[X^2] \in CNL_W$. We write ϵ for the class $[X] \in A$ so that $A = \mathbb{F}[\epsilon]$ and $\epsilon^2 = 0$. We fix a deformation $\rho : G \to \mathrm{GL}_2(\mathbb{F}[\epsilon])$ where $\rho \bmod \mathfrak{m}_{\mathbb{F}[\epsilon]} = \overline{\rho}$.

Write $\rho = \overline{\rho} \oplus u'_\rho \epsilon$ for $u'_\rho : G \to M_2(\mathbb{F})$. Define $u_\rho = u'_\rho \overline{\rho}^{-1}$ with $u_\rho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$, then we get $tr(u_\rho) = 0$ because

$$1 = \det(\rho\overline{\rho}^{-1}) = \det \begin{pmatrix} 1 + a\epsilon & b\epsilon \\ c\epsilon & 1 + d\epsilon \end{pmatrix} = 1 + (a + d)\epsilon$$

gives $a + d = 0$. Thus, we can regard $u_\rho : G \to ad$.

Let $g, h \in G$, then we have

$$\begin{aligned} \overline{\rho}(gh) + u'_\rho(gh)\epsilon &= \rho(gh) \\ &= \rho(g)\rho(h) \\ &= \overline{\rho}(g)\overline{\rho}(h) + (\overline{\rho}(g)u'_\rho(h) + u'_\rho(g)\overline{\rho}(h))\epsilon, \end{aligned}$$

thus

$$u'_\rho(gh) = \overline{\rho}(g)u'_\rho(h) + u'_\rho(g)\overline{\rho}(h).$$

This shows

$$\begin{aligned} u_\rho(gh) &= u'_\rho(gh)\overline{\rho}(gh)^{-1} \\ &= (\overline{\rho}(g)u'_\rho(h)\overline{\rho}(g)^{-1})\overline{\rho}(h)^{-1} + u'_\rho(g)\overline{\rho}(g)^{-1} \\ &= gu_\rho(h) + u_\rho(g), \end{aligned}$$

i.e., $u_\rho : G \to ad$ is a 1-cocycle.

Conversely, starting from a cocycle $u : G \to ad$, we can reverse the steps to get $\rho : G \to \mathrm{GL}_2(\mathbb{F}[\epsilon])$ such that $u = u_\rho$.

Let K, F be number fields with $[K : F] = 2$. We consider an induced representation

$$\overline{\rho} = \mathrm{Ind}_K^F \overline{\phi} : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_2(\mathbb{F})$$

from a character $\overline{\phi}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Suppose $\overline{\rho}$ is p-ordinary. Note that we have

$$H^1(G, ad) = H^1(G, \chi) \oplus H^1(G, \widetilde{\overline{\phi}^-}) \oplus H^1(G, \widetilde{\overline{\phi}^-}\chi)$$

where $\chi, \widetilde{\overline{\phi}^-}$ are defined as in the previous section.

Note that we have an element $c \in G$ with $\overline{\rho}(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since c acts as 1 on the subspace $\left\{ \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \right\}$ while c acts as -1 on the subspace $\left\{ \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix} \right\}$ as an action of G on ad, we have the following decomposition of trace-zero subspace of $M_2(\mathbb{F})$

$$\left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\} \bigoplus \left\{ \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \right\} \bigoplus \left\{ \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix} \right\}$$

where the action of G on each component on the right-hand side coincides with the characters $\chi, \widetilde{\overline{\phi}^-}, \widetilde{\overline{\phi}^-}\chi$ respectively. (See the proof of Lemma 5.1.3.)

For a 1-cocycle $u \in H^1(G, ad)$, we have

$$u|_{I_{\mathfrak{p}}} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

because p-ordinarity assumption shows $(2,2)$-component of the above matrix is zero and the fact that $tr(u) = 0$ shows $(1,1)$-component is zero as well. By comparing this shape with the decomposition of ad above, we can conclude that each component must be zero, i.e., $u|_{I_{\mathfrak{p}}} = 0$.

Theorem 5.2.4. *Let the notations same as above. Let F_∞/F be the cyclotomic $\mathbb{Z}_{\ell_1}\times\cdots\times\mathbb{Z}_{\ell_r}$-extension and $\{R_n^{\mathrm{ord}}\}$ be the tower of p-ordinary universal deformation rings associated to $\overline{\rho}$ over F_∞/F. Suppose that F^ξ are abelian over $\mathbb{Q}$ for $\xi \in \{\chi, \widetilde{\overline{\phi}^{-}}, \widetilde{\overline{\phi}^{-}}\chi\}$. Then the tower $\{R_n^{\mathrm{ord}}\}$ stabilizes.*

Proof. Let $S' \subseteq S$ be the set of primes of F above $\ell_1\cdots\ell_r\infty$. For each $\xi \in \{\chi, \widetilde{\overline{\phi}^{-}}, \widetilde{\overline{\phi}^{-}}\chi\}$, we have

$$H^1_{\mathrm{ord}}(G, ad) \cong H^1_{\mathrm{ord}}(\mathrm{Gal}(F^{S'}/F), ad)$$

since any cocycle $u \in H^1(G, ad)$ is unramified at p by the above argument. By Theorem 5.1.6, we know that $\dim_{\mathbb{F}} H^1(\mathrm{Gal}(F_n^{S'}/F_n), ad)$ is bounded for $n \gg 0$, thus so is $\dim_{\mathbb{F}} H^1_{\mathrm{ord}}(\mathrm{Gal}(F_n^{S'}/F_n), ad)$. This shows $\dim_{\mathbb{F}} H^1_{\mathrm{ord}}(G_n, ad)$ is also bounded for $n \gg 0$, thus $\dim_{\mathbb{F}} H^1_{\mathrm{ord}}(G_\infty, ad)$ is finite. $\square$

Note 5.2.5. Note that without assuming p-ordinarity, the tower of full universal deformation rings could grow indefinitely when we allow ramification at p.

CHAPTER 6

General m-dimensional Cases

In this chapter, we will start from an m-dimensional representation induced from a character just like in 2-dimensional case. We will need to impose certain conditions on the induced representation to have stabilization of a tower of universal deformation rings. We use the same notation as in previous chapters unless stated otherwise.

6.1 Induced Representations

We start the section by the following lemma.

Lemma 6.1.1. *Let G be a profinite group and H be a normal subgroup with cyclic quotient G/H. Suppose $\Delta := G/H$ is of order m. Let $\sigma \in \Delta$ be a generator. For a character $\phi : H \to \mu_m$ of order m and $\tau \in \Delta$, we define $\phi^\tau : H \to \mu_m$ by*

$$\phi^\tau(h) = \phi(\widetilde{\tau} h \widetilde{\tau}^{-1})$$

for $h \in H$ and $\widetilde{\tau} \in G$ with projected image $\tau \in \Delta$. Then, we have

$$Ad(\operatorname{Ind}_H^G \phi) \cong \bigoplus_{i=0}^{m-1} \operatorname{Ind}_H^G \phi^{\sigma^i - 1}$$

where $\phi^{\sigma^i - 1} = \phi^{\sigma^i} \phi^{-1}$.

Proof. Since $\operatorname{Ind}_H^G \phi|_H = \bigoplus_{i=0}^{m-1} \phi^{\sigma^i}$, we have

$$Ad(\operatorname{Ind}_H^G \phi)|_H = \bigoplus_{i,j=0}^{m-1} \phi^{\sigma^i - \sigma^j}.$$

On the other hand, we have

$$\begin{aligned}\bigoplus_{i=0}^{m-1} \operatorname{Ind}_H^G \phi^{\sigma^i - 1}|_H &= \bigoplus_{i,j=0}^{m-1} (\phi^{\sigma^i - 1})^{\sigma^j} \\ &= \bigoplus_{i,j=0}^{m-1} \phi^{\sigma^i - \sigma^j} = Ad(\operatorname{Ind}_H^G \phi)|_H\end{aligned}$$

since $(i,j) \mapsto (i+j, j)$ gives an automorphism of $(\mathbb{Z}/m\mathbb{Z})^2$. This shows $Ad(\operatorname{Ind}_H^G \phi)$ and $\bigoplus_{i=0}^{m-1} \operatorname{Ind}_H^G \phi^{\sigma^i - 1}$ have the same trace on H, and also on $G \setminus H$ as they are both zero. Since the traces of two representations match on G, they agree on G as well. $\square$

Proposition 6.1.2. *Let M/F be a cyclic extension of degree $m > 1$ with $\langle \sigma \rangle = \mathrm{Gal}(M/F)$. Let $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mu_m(\overline{\mathbb{Q}})$ be a character of order m such that $\overline{\mathbb{Q}}^{\ker \xi} = M' = MF'$ with F'/F cyclic of order m which is disjoint from M/F, and $\mathrm{Gal}(M'/F) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. Then, there exists a finite order character ϕ of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$ such that $\xi = \phi^{\sigma - 1}$. Also $\operatorname{Ind}_M^F \phi$ is absolutely irreducible.*

Proof. Since we have $\xi^\sigma = \xi$, ξ extends to a character $\widetilde{\xi} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mu_{m^2}(\mathbb{Q})$ by [H2, Corollary 4.37]. Note that $\widetilde{\xi}$ has order m because $\mathrm{Gal}(M'/F) \cong (\mathbb{Z}/m\mathbb{Z})^2$. Then for $x \in \mathbb{A}_F^{\times}$, we have

$$\xi(x) = (\widetilde{\xi} \circ N_{M/F})(x) = \widetilde{\xi}(xx^{\sigma} \cdots x^{\sigma^{m-1}}) = \widetilde{\xi}(x)^m = 1$$

if we regard $\widetilde{\xi}$ as a character of $\mathbb{A}_F^{\times}$. This shows $\xi|_{C_F} = 1$ for the idele class group $C_F = \mathbb{A}_F^{\times}/F^{\times}$ of F. Note that we have $C_F = \ker(\sigma - 1 : C_M \to C_M)$ by Hilbert's theorem 90. Since ξ vanishes on C_F, it factors through $\mathrm{im}(\sigma - 1 : C_M \to C_M)$, thus extends to a character ϕ of C_M such that $\phi^{\sigma-1} = \xi$. We can regard ϕ as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$ by class field theory.

Since $\phi^{\sigma} = \phi\xi$ and ξ is fixed by σ, we can show that $\phi^{\sigma^i} = \xi^i\phi$ by induction. This shows that ϕ^{σ^i} are all distinct for $i = 0, \ldots, m-1$ because ξ has order m. By [Mac51, Theorem 6'], $\mathrm{Ind}_M^F \phi$ is absolutely irreducible. $\square$

The lemma and the proposition above prove the following.

Corollary 6.1.3. *Under the same notation in the above, we have*

$$Ad(\mathrm{Ind}_M^F \phi) \cong \bigoplus_{i,j=0}^{m-1} \widetilde{\xi}^i \chi^j$$

for the character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mu_m$ *such that* $\ker \chi = \mathrm{Gal}(M/F)$.

Note 6.1.4. Note that Lemma 5.1.3 is a special case of the above corollary.

6.2 Stabilization of a Tower of Universal Deformation Rings

We use the same notations as in Proposition 6.1.2. Let $\mathbb{F}$ be a finite field of characteristic p with $(p, m) = 1$ which is big enough so that can regard the characters ξ and ϕ defined above to have values in $\mathbb{F}^\times$.

Let $\ell_1, \ldots, \ell_r$ be distinct odd rational primes such that $(\ell_1 \cdots \ell_r, pm) = 1$. Let S be the set of primes of F above $\ell_1 \cdots \ell_r \infty$. We start from the induced representation

$$\overline{\rho} = \mathrm{Ind}_M^F \phi : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_m(\mathbb{F})$$

defined in Proposition 6.1.2. We consider the cyclotomic $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_r}$-extension F_∞/F and the tower of universal deformation rings $\{R_n\}$ for $n \in \mathbb{Z}^r_{\geq 0}$. Note that by Lemma 2.2.4 and Proposition 6.1.2, universal deformation rings R_n exist for all n.

Theorem 6.2.1. *Let $\overline{\rho} : \mathrm{Gal}(F^S/F) \to \mathrm{GL}_m(\mathbb{F})$ be defined as above and assume that $M'/\mathbb{Q}$ is abelian. Then, the tower of universal deformation rings $\{R_n\}$ stabilizes.*

Proof. Since $ad(\overline{\rho}) \subseteq Ad(\overline{\rho})$, we can write ad as a direct sum of $m^2 - 1$ characters by Corollary 6.1.3. Then, the proof is similar to that of Theorem 5.1.6.

Note that you can regard M' as a composite of two $\mathbb{Z}/m\mathbb{Z}$-extensions defined by characters $\widetilde{\xi}$ and χ each. Thus by assuming $M'/\mathbb{Q}$ is abelian, it implies that the splitting fields $F^{\widetilde{\xi}}$ and F^χ of $\widetilde{\xi}$ and χ are abelian over $\mathbb{Q}$. Thus, so is $F^{\widetilde{\xi}^i \chi^j}$ for all i, j. This shows that we can apply the proof of Theorem 5.1.6 similarly. $\square$

Note 6.2.2. Once we have a suitable definition of p-ordinarity in m-dimensional case, we could prove the stabilization of the tower of p-ordinary universal deformation rings from a m-dimensional representation by using similar arguments in Section 5.2 by using decomposition of ad by $m^2 - 1$ components.

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