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Time crystals and space crystals: strongly correlated phases of matter with space-time symmetries

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Physics

by

Dominic Victor Else

Committee in charge:

Professor Chetan Nayak, Chair
Professor Cenke Xu
Professor David Weld

June 2018

The Dissertation of Dominic Victor Else is approved.

Professor Cenke Xu

Professor David Weld

Professor Chetan Nayak, Committee Chair

June 2018

Time crystals and space crystals: strongly correlated phases of matter with space-time symmetries

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Dominic Victor Else

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Curriculum Vitæ

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Publications

Classifying symmetry-protected topological phases through the anomalous action of the symmetry on the edge
D. V. Else and C. Nayak
Phys. Rev. B **90**, 235137 (2014)

Classification of topological phases in periodically driven interacting systems
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Phys. Rev. B **93**, 201103 (2016)

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D. V. Else, B. Bauer and C. Nayak
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D. V. Else, B. Bauer and C. Nayak
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R. Thorngren and **D. V. Else**
Phys. Rev. X **8**, 011040 (2018)

Preprints

Exponentially slow heating in short and long-range interacting Floquet systems
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Fields of study

Major field: Theoretical condensed matter physics
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Abstract

Time crystals and space crystals: strongly correlated phases of matter with space-time symmetries

by

Dominic Victor Else

This thesis is concerned with *phases of matter*, one of the central notions in condensed matter physics. Traditionally, condensed matter physics has been concerned with phases of matter *in thermal equilibrium*, which means it is coupled to a heat bath. The main interest of this thesis, however, is *isolated systems*, in which the system is allowed to reach a steady state on its own, without interacting with a heat bath. In such a context it is possible for the steady state to be non-thermal in character, leading to many new phenomena.

A main interest of this thesis will be Floquet systems, which are systems that are periodically driven, for example by a time-oscillatory electric field. In this thesis, we will identify and charcterize phases of matter occuring in Floquet systems that are entirely new, in the sense that they have no analog in equilibrium.

We introduce a "Floquet equivalence principle", which states that Floquet topological phases with symmetry G are in one-to-one correspondence with *stationary* topological phases with additional symmetry. This allows us to leverage the existing literature on topological phases with symmetries to understand Floquet topological phases. Such phases can be stabilized in driven strongly disordered systems through the phenomenon of "many-body localization" (MBL). We discuss properties of Floquet phases such as the "pumping" of lower-dimensional topological phases onto the boundary at each time cycle.

We then turn to spontaneous symmetry-breaking phases. We show that in Floquet systems, there is a striking new kind of such phase: the *Floquet time crystal*, in which the symmetry that is spontaneously broken is discrete time-translation symmetry. Such systems, though driven at frequency ω, respond at a *fractional* frequency ω/n. We show using analytical arguments and numerical evidence that such phases can be stabilized in driven strongly disordered systems through the phenomenon of "many-body localization" (MBL).

Next, we show that both Floquet time crystals and Floquet topological phases can be stabilized even *without* disorder. We establish a new scenario for "pre-thermalization", a phenomenon where the eventual thermalization of the system takes place at a rate that is exponentially small in a parameter. In the intermediate regime, before pre-thermalization, there is a quasi-stationary pre-thermal regime in which Floquet phases can be stabilized.

In a slight digression, we then develop a systematic theory of stationary topological phases with discrete *spatial* symmetries (as opposed to the discrete *temporal* symmetry characterizing Floquet phases), showing that they also satisfy a "crystalline equivalence principle" relating phases of matter with spatial symmetry to phases of matter with internal symmetry. Our arguments are based on notions of "gauging spatial symmetries" as well as a viewpoint based on topological quantum field theory (TQFT).

Finally, we put the Floquet equivalence principle on a systematic footing, and unify it with the crystalline equivalence principle for stationary topological phases, by invoking a powerful homotopy-theoretic viewpoint on phases of matter. The end result is a general theory of strongly correlated phases of matter with space-time symmetries.

Contents

Chapter 1

Introduction

1.1 Phases of matter

This thesis is concerned with *phases of matter*; this is one of the central notions in condensed matter physics. It has long been observed that as the parameters of a system containing many particles are varied continuously, certain physical properties can jump discontinuously, a phenomenon known as a "phase transition". This is an example of an emergent phenomenon, because the transition only becomes truly only becomes discontinuous in the limit as the number of particles in the system goes to infinity.

A phase of matter is a set of parameter values that can be interpolated between without crossing a phase transition. Two parameter values are said to be in *different* phases of matter if it impossible, even in theory, to connect them without a phase transition. The way to show that two systems are in two phases is to describe some discrete property which distinguishes them. Since the property is discrete, it follows that it can only change via some sharp transition.

The classic example of such a discrete property is the spontaneously broken symmetry. This allows us to distinguish, for example, a liquid and a solid; a liquid has continuous rotation and translational symmetry (on average), whereas a solid retains

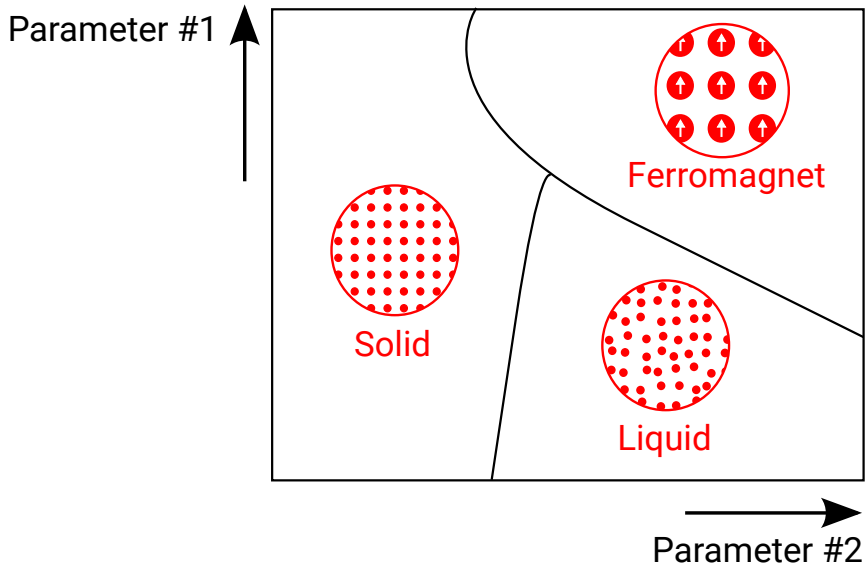

Figure 1.1: Phases of matter

only discrete lattice symmetries. Similarly, in a magnetic system, a ferromagnet has the magnetic moments aligned in a certain direction, breaking rotational symmetry, whereas a paramagnet retains the full rotational symmetry.

More recently, it has been appreciated that there are other, much more subtle discrete properties, that distinguish phases of quantum systems at zero temperature. These are related to the pattern of quantum entanglement in the system, and are referred to as *topological order*.

1.2 In and out of equilibrium

Traditionally, condensed matter physics has been concerned with phases of matter *in thermal equilibrium*. This means that the system of interest is supposed to be coupled to and exchange energy freely with its environment, which acts as a heat bath. Well-known principles of statistical mechanics then imply that the long-time

state of the system should be given by

$$\rho_{\text{thermal}} = \frac{1}{Z} e^{-\beta \hat{H}}, \tag{1.1}$$

where $\hat{H}$ is the Hamiltonian describing the system's evolution. We can then analyze the phases and phase transitions of the state ρ as $\hat{H}$ is varied.

The main interest of this thesis, however, is *isolated systems*, in which the system is allowed to reach a steady state on its own, without interacting with a heat bath. In many cases, this does not actually make any difference, as subsystems of the system can effectively act as heat baths for each other, so that at long times the system still resembles the same thermal state ρ_{thermal}. But there are also many cases in which this does not occur, and instead we obtain a different steady state ρ_{steady}. Specifically, in this thesis we will study phenomena that occur in the context of *many-body-localization* (MBL), in which strong quenched disorder prevents the system from thermalizing, and *prethermalization*, in which a separation of energy scales pushes the thermalization time out to exponentially long times.

Finally, we note that in this dissertation we will mainly be concerned with *Floquet* systems. These are systems which are periodically driven, for example by a time-oscillatory electric field. Specifically, this means that the system evolves under a time-dependent Hamiltonian $\hat{H}(t)$ which is periodic in time; that is, there exists a period T such that $\hat{H}(t+T) = \hat{H}(t)$.

1.3 Floquet phases

A main theme of this dissertation is the identification and classification of phases of matter that occur in the long-time state of Floquet systems that are entirely *new*, in

the sense that they have no analog in equilibrium. Specifically, the discrete properties that distinguish them pertain to the "micro-motion", which is the nature of the evolution that the system undergoes in one drive period T. This micromotion can have non-trivial features (for example quantized charge pumping), corresponding to a genuinely non-equilibrium phase of matter.

However, the simplest and most dramatic example of a new Floquet phase of matter introduced in this dissertation is the *Floquet time crystal*. This is an example of a phase that occurs in the context of MBL. In this phase, the system fails to synchronize with the drive, even at late times. Instead, the system oscillates with period some multiple, for example $2T$, of the drive period T.

The Floquet time crystals, although new to the Floquet context, can still be thought of in terms of the paradigm of spontaneous symmetry breaking. However, the relevant symmetry is the discrete *time translation symmetry* associated with the time-periodicity of the drive, which is spontaneously broken if the system oscillates with a larger period.

This thesis also contains various other results about the classification of Floquet phases and scenarios in which they can occur. In order to make sense of the zoo of potential new Floquet phases, symmetry-breaking and topological, that could exist, it is useful to have an organizing principle. We will argue for an "equivalence principle": Floquet phases are in one-to-one correspondence with stationary phases, but with an augmented symmetry group that takes into account the space-time symmetries of the Floquet phase (for example, time translation symmetry).

Chapter 2

Review: Gapped quantum phases of matter

In this chapter, we are concerned with phases of matter that occur in quantum systems at zero temperature. Thus, they pertain to *ground states* of quantum Hamiltonians. We can further subdivide these phases into two classes, gapped or gapless, depending on whether there is a finite energy gap between the ground state and the first excited state[1]. Here will be concerned only with gapped phases of matter. If a Hamiltonian with local interactions is gapped, then its ground state necessarily has many special properties. For example, correlations always decay exponentially with distance [1], and it is believed (but proven only in one dimension) that the entanglement entropy of any subregion scales like the boundary of that region ("area law") rather than the size of the interior ("volume law") [2, 3]. For brevity, we will refer to zero-temperature phases of matter in gapped local Hamiltonians as GAPGRND phases, and the ground state of a gapped local Hamiltonian as a GAPGRND state.

Let us specify more precisely what we mean by classifying GAPGRND phases on what we mean by classifying phases of matter. The ground states of two gapped local Hamiltonians H_0 and H_1 are then said to be in the same GAPGRND phase

[1]In the definition of gapped, we will allow there to be several degenerate ground states, so long as there is a gap to the other excited states

if there exists a continuous path of Hamiltonians joining them without inducing a ground-state phase transition. More precisely, since ground-state phase transitions are associated with closing of the gap, we say that H_0 and H_1 are in the same phase if there exists $\gamma > 0$ and a continuous path $H(s)$, $0 \leq s \leq 1$, of local Hamiltonians such that $H(0) = H_0$, $H(1) = H_1$, and the gap of $H(s)$ is at least γ for all $0 \leq s \leq 1$. We can also talk about phases of matter in the presence of a symmetry, in which case we require H_0, H_1, and the path $H(s)$ to respect the symmetry.

It is important to realize that, although we have introduced GAPGRND phases and GAPGRND states in the context of Hamiltonians, the classification of GAPGRND phases can be formulated as a statement purely about GAPGRND states [4]. Indeed, later on in this thesis we will be talking about GAPGRND states for which the "parent Hamiltonian" of which they are the ground state (though it, by definition, must exist) has no physical significance. First of all, we note that a GAPGRND state $|\Psi\rangle$ will in general admit several different parent Hamiltonians, say H_0 and H_1, but they always correspond to the same GAPGRND phase, because one can easily show that if H_0 and H_1 are gapped and have the same ground state $|\Psi\rangle$, then the path $(1-s)H_0 + sH_1$ is also gapped (and has ground state $|\Psi\rangle$) for $0 \leq s \leq 1$.

Next, we need to give a definition of "in the same phase" that does not refer to parent Hamiltonians. We can do this by invoking the following (informally stated) theorem [4–6].

Theorem. Two gapped Hamiltonians H_0 and H_1 are in the same phase if and only if there exists a path of quasi-local Hamiltonians $\mathcal{H}(s)$, $0 \leq s \leq 1$ such that $|\Psi_1\rangle = \mathcal{U}\,|\Psi_0\rangle$, where $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are the ground states of H_0 and H_1 respectively, and

$$\mathcal{U} = \mathcal{T} \exp\left(-i \int_0^1 \mathcal{H}(t) dt\right), \tag{2.1}$$

where the symbol $\mathcal{T}$ denotes time-ordering. In other words, by time-evolving with $\mathcal{H}(s)$ we can obtain $|\Psi_1\rangle$ from $|\Psi_0\rangle$. Here by "quasi-local" we mean that $\mathcal{H}(s)$ is a sum of terms supported locally with tails decaying faster than any power law with distance. (For phases in the presence of a symmetry, $\mathcal{H}(s)$ is required to respect the symmetry).

We call a unitary $\mathcal{U}$ of the form Eq. (2.1) a *local unitary*. Thus, a concise statement of the theorem is that two ground states are in the same phase if and only if they are related by a local unitary. Local unitaries have a number of nice properties. For example, they obey a Lieb-Robinson bound [7, 8], which means that the Heisenberg evolution $\mathcal{U}^\dagger \hat{o} \mathcal{U}$ only grows the support of an operator $\hat{o}$ by a constant amount (up to fast-decaying tails).

Chapter 3

Review: Thermalization and Lack Thereof in Isolated Quantum Systems

As we mentioned in the Introduction, in this thesis we are interested in *isolated* systems (not coupled to a heat bath), and in particular isolated systems which do not thermalize. In this chapter, we will first review the properties of isolated systems that do thermalize, and then move on to the other possibilities.

3.1 Thermalization and the eigenstate thermalization hypothesis

If an isolated quantum system, under the time evolution of its Hamiltonian, and for any sufficiently physical initial state, approaches the thermal state ρ_{gibbs} at late times, then we say that it *thermalizes*. Here we must add a caveat, because if the system is initially in a pure quantum state $|\psi\rangle$, then it remains in a pure state for all times, since the time evolution under a Hamiltonian is unitary. Meanwhile, the thermal state ρ_{gibbs} is a mixed state. Therefore, the system can never *precisely* reach the thermal state ρ_{gibbs}. Another way to say this is that since the microscopic time

evolution in quantum mechanics is fundamentally *reversible*, the system cannot reach the Gibbs state because there is only one such state at a given energy density, while there are many possible initial states.

Nevertheless, we say that a system thermalizes if, at late times, it resembles the Gibbs state on any finite subsystem. That is, for any finite subsystem R, we have that

$$\lim_{t\to\infty} \mathrm{Tr}_{R^c} |\Psi(t)\rangle \langle\Psi(t)| = \mathrm{Tr}_{R^c} \rho_{\mathrm{gibbs}} \tag{3.1}$$

where $|\Psi(t)\rangle$ is the state of the system at time t, and Tr_{R^c} denotes the partial trace over the complementary subsystem to R. This avoids the irreversibility problem: the information about the initial state is still in principle present, but it would require the (completely infeasible) measurement of highly non-local observables to recover it.

A very important property that appears (at least empirically) to be true of systems that thermalize is the *eigenstate thermalization hypothesis* (ETH) [9–12], which postulates that a system will thermalize even if the initial state is an eigenstate of the Hamiltonian. Since an eigenstate obviously does not evolve in time, it follows that the eigenstate $|\Psi\rangle$ itself must be thermal, in the sense that for any finite subsystem R,

$$\mathrm{Tr}_{R^c} |\Psi\rangle \langle\Psi| = \mathrm{Tr}_{R^c} \rho_{\mathrm{gibbs}}. \tag{3.2}$$

This implies, in particular, that the entanglement entropy of the eigenstate on a region R, defined by $S_{\mathrm{ent}} = -\mathrm{Tr}(\rho \log \rho)$, where $\rho = \mathrm{Tr}_{R^c} |\Psi\rangle \langle\Psi|$, must scale with the volume of R, since thermodynamic entropy is extensive. This is in contrast to the area law for entanglement entropy for GAPGRND states (see Chapter 2).

3.2 Floquet systems and thermalization

An (isolated) Floquet system is a system that evolves under a time-dependent Hamiltonian $H(t)$ that is periodic, that is there exists a period T such that $H(t+T) = H(t)$. We can define the unitary time-evolution operator

$$U_f = \mathcal{T} \exp\left(-i \int_0^T H(t)dt\right). \tag{3.3}$$

At integer multiples of the driving period $t = nT$, the state of the system $|\Psi(nT)\rangle$ can then be expressed in terms of the initial state $|\Psi(0)\rangle$ as $|\Psi(nT)\rangle = U_f^n |\Psi(0)\rangle$. Thus, a Floquet system can be thought of as the analog of an isolated system with a time-independent Hamiltonian, but where the time evolution happens in discrete steps rather than continuously.

The appropriate analog to the Gibbs state in a Floquet system, and the one to which Floquet systems have been found to thermalize [13–15] is the infinite-temperature, maximally mixed state, $\rho \propto \mathbb{I}$. A way to think about this is that the Gibbs state is the state which maximizes the von Neumann entropy, subject to the constraint of fixed energy (since energy is conserved). Floquet systems do not have conservation of energy due to the time-dependence of the Hamiltonian, and so nothing prevents them from thermalizing to the state of maximum entropy *without* any constraints, which is the infinite temperature state. Of course, the infinite temperature state cannot possibly exhibit any interesting phases of matter, so all the phases of matter we will consider in this thesis will occur in systems for which thermalization is somehow inhibited (for example, by many-body-localization, the subject of the next section).

3.3 Many-body localization

An important class of systems which do not thermalize and violate the ETH are those which are *many-body-localized* (MBL) [16–25]. MBL occurs in systems with strong quenched disorder, which means that the Hamiltonian contains external fields which vary randomly in space. MBL can be considered to be the deformation to interacting systems of Anderson localization, which is a phenomenon that occurs in systems of non-interacting fermions with quenched disorder (similarly to how a Fermi liquid is a deformation to interacting systems of a Fermi gas).

3.3.1 Anderson localization

Before going onto many-body localization, we will briefly review its precursor, Anderson localization [26]. This is a phenomenon that occurs in systems of non-interacting electrons subjected to a random potential. If one places an electron in such a system and then evolves according to Schrödinger's equation, then we say that the electron is localized if the wavefunction remains concentrated in the vicinity of its intial position even at infinite times. That is, the electron does not diffuse. An alternative way to think about localization is in terms of the eigenstates of the Schrödinger equation. These are said to be localized if the probability amplitude is concentrated near a given point, rather than being spread out over the whole system as they would be for a periodic potential (by Bloch's theorem).

An intuitive way to think about localization is by appealing to the atomic limit. Consider, for example, a tight-binding model (the *Anderson model*), which we write

in first-quantized notation as

$$H = \sum_i u_i |i\rangle\langle i| + t \sum_{\langle i,j \rangle} (|i\rangle \langle j| + |j\rangle \langle i|). \tag{3.4}$$

The states $\{|i\rangle\}$ are the atomic orbitals, and u_i are random potentials drawn from some distribution. In the limit $t = 0$, there is no amplitude for electrons to hop between atoms, so naturally the electrons are localized on the atoms. Indeed, this is reflected in the fact that the eigenstates of H are the atomic orbitals $\{|i\rangle\}$. The question is, what happens when we re-introduce nonzero hopping amplitude t? Naively, we might expect to be able to treat this question in perturbation theory. By applying the standard time-independent perturbation theory, one finds that the first-order correction to the eigenstates is

$$|i\rangle' = |i\rangle + t \sum_{j, j\sim i} \frac{1}{u_i - u_j} |j\rangle \,, \tag{3.5}$$

where the sum is over nearest neighbors j to the orbital i. This is still localized, albeit now with some amplitude on the neighboring atoms. Indeed, continuing to higher orders, we find that the orbitals remain localized at any fixed order.

Therefore, we see that if a small hopping does cause the electrons to delocalize, then it must happen non-perturbatively. The obvious culprit for a non-perturbative localization would be *resonances*; that is, nearby sites i and j which have nearly the same energy, $u_i \approx u_j$. Since denominators such as $u_i - u_j$ appear in the perturbation expansion, this is problematic from the point of view of convergence.

Nevertheless, there is reason to believe that resonances might not lead to delocalization for small hopping t. For example, at first order the problematic resonances occur when $|u_i - u_j| \lesssim t$. For t small the spots where this occurs will be rare. An

isolated resonant spot must be dealt with nonperturbatively. However, this does not necessarily ruin localization, because the effect will simply be a reorganization of the orbitals in the vicinity of the location in space where the resonance occurs. Therefore, heuristically, to prove localization one "just" has to prove that the resonances remain sufficiently dilute through each order of perturbation theory, preventing them from mediating any nonperturbative transport. This was attempted by Anderson [26] through a statistical treatment. Rigorous mathematical proofs were given much later [27–29].

3.3.2 Introduction to many-body localization from perturbative analysis

Many-body localization (MBL) is an analog of Anderson localization that occurs in interacting quantum systems. It is sometimes defined as the persistence of signatures of localization when electron-electron interactions are turned on in a system with Anderson localization. However, the phenomena characteristic of MBL are more general than this and can occur, for example, in spin systems which do not have any analog of Anderson localization. Therefore, we will instead introduce MBL in a different way, as follows. A main lesson from Anderson localization is that behavior which might appear to be very fine-tuned (electrons pinned to particular atoms) turn out not to be in the presence of disorder, because perturbing the Hamiltonian merely "dresses" the original orbitals. Therefore, we will now consider an *interacting* Hamiltonian that appears similarly fine-tuned, and again we will find that perturbing the Hamiltonian merely leads to a form of "dressing".

Specifically, we consider the following Hamiltonian for a lattice of spin-1/2 parti-

cles:

$$H = H_0 + \lambda V, \tag{3.6}$$

$$H_0 = \sum_j h_j \sigma_j^z, \tag{3.7}$$

where the h_j are quenched random fields drawn from some distribution. We will not place any restrictions on V, except that it should be a sum of terms acting locally on the lattice.

H_0 appears to be a very fine-tuned Hamiltonian, because its eigenstates are product states, specifically S_i^z eigenstates of the form $|\cdots\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\cdots\rangle$ and so forth. This means that the entanglement entropy of an eigenstate is zero, in sharp contrast to the prediction of the ETH of a volume-law entanglement entropy (see Section 3.1). However, we should now determine to what extent these properties are robust to perturbation theory. As in the case of Anderson localization, we will proceed with a perturbative treatment in λ, and then return to the question of its validity. Specifically, we will aim to perturbatively construct a local unitary $\mathcal{U}$ (see section 2) such that $H_{\text{eff}} := \mathcal{U} H \mathcal{U}^\dagger$ is diagonal in the eigenstates of H_0.

There are many different schemes to construct such a unitary $\mathcal{U}$ perturbatively, but one which ensures that $\mathcal{U}$ is manifestly a *local* unitary at each order in perturbation theory is called "van Vleck perturbation theory" [30]. In this scheme, we attempt to write $\mathcal{U} = e^{iS} := e^{i(\lambda S_1 + \lambda^2 S_2 + \cdots)}$, for Hermitian operators $S_1, S_2, \cdots$, and compute order by order, imposing that at each order the diagonal matrix elements $\langle\Psi_i| S_k |\Psi_i\rangle = 0$ (here $\{|\Psi_i\rangle\}$ is a basis for the eigenstates of H_0.) For example, at first order we have

$$e^{iS}(H_0 + \lambda V)e^{-iS} = H_0 + \lambda V + i\lambda[S, H_0] + O(\lambda^2), \tag{3.8}$$

and imposing that the off-diagonal matrix elements of Eq. (3.8) and the diagonal matrix elements of S are zero immediately gives the first-order contribution to S_1:

$$S_1 = \sum_{i \neq j} \frac{|\Psi_i\rangle \langle \Psi_i| V |\Psi_j\rangle \langle \Psi_j|}{E_i - E_j} \tag{3.9}$$

$$= \sum_s \mathcal{F}(V_s), \tag{3.10}$$

where we write V as a sum of local operators $V = \sum_s V_s$, and we define the map on operators

$$\mathcal{F}(\hat{o}) = \sum_{i \neq j} \frac{|\Psi_i\rangle \langle \Psi_i| \hat{o} |\Psi_j\rangle \langle \Psi_j|}{E_i - E_j} \tag{3.11}$$

where E_i is the energy eigenvalue of $|\Psi_i\rangle$ for H_0.

One can show that if $\hat{o}$ acts on some local set of spins, then so does $\mathcal{F}(\hat{o})$. Hence, S_1 is indeed a sum of local terms. Indeed, one can show that if we continue the van Vleck perturbation to all orders then S remains local at any fixed order.

Hence, if we believe the perturbative approach is accurate, we find that there exists a local unitary $\mathcal{U}$ which relates the eigenstates of $H_0 + \lambda V$. As in the case of Anderson localization, the trouble for convergence arises due to "resonant spots". In the many-body context, a resonant spot is a location in physical space where there exist local operators $\hat{v}$ which have nonzero matrix element $\langle \Psi_i| \hat{v} |\Psi_j\rangle \neq 0$ between states with $E_i - E_j \approx 0$, since then Eq. (3.11) blows up. Nevertheless, when the perturbation is small one hopes that such resonant spots are dilute. The reason is for an H_0 as written in Eq. (3.7) (and, it turns out, for any Hamiltonian exhibiting MBL), the local spectrum – the possible energy differences in eigenstates connected by local operators – is *discrete* and thus, at *typical* point in space, will not include zero. Moreover, if a resonant spot can be treated in a non-perturbative fashion separately from other resonant spots, then the result will be simply be some reorganization

of energy levels described by a unitary operator supported near the resonant spot. Therefore, dilute resonant spots do not prevent the existence of the local unitary $\mathcal{U}$ mentioned above.

Giving a rigorous proof of the existence of MBL based on the ideas above is a challenging problem, due to the difficulty of treating resonances rigorously. Nevertheless, such a proof has been given (albeit with an additional but very reasonable assumption) in one dimension by Imbrie [23]. There is also substantial numerical evidence for MBL in one dimension [17, 18], although one is restricted to only considering relatively small system sizes.

3.3.3 Characteristics of MBL

The key feature of MBL as captured by the model of the previous section is that there exists a single local unitary $\mathcal{U}$ which relates the eigenstates of H to the eigenstates of a Hamiltonian H_0 of the form

$$H_0 = \sum_j h_j \sigma_j^z. \tag{3.12}$$

Indeed, one could well take this as the *definition* of MBL [22]. (This definition can be expanded slightly by allowing different forms of H_0; the important thing is that it is a sum of commuting terms). From this property one can deduce a number of other interesting features of MBL. These features can be grouped into two (related) groups: *eigenstate properties* and *dynamical properties*.

Eigenstate properties. Recall that the eigenstates of H_0 were product states. By assumption, if H is MBL this implies that the eigenstates of H are obtained from those of H_0 by a local unitary $\mathcal{U}$. One can show that $\mathcal{U}$ can only create entanglement in a region proportional to the boundary of that region, so we immediately conclude that

the eigenstates of H_0 obey an area law for the entanglement entropy [22]. This means, in particular, that they do not satisfy the eigenstate thermalization hypothesis. Thus, MBL systems constitute a counter-example to the conjecture that quantum systems generically obey ETH.

Rather than appearing thermal, MBL eigenstates have properties characteristic of gapped ground states of local Hamiltonians; in fact, in the language of Section 2, they are GAPGRND states. To see this, observe that any eigenstate of H_0 can be expressed as the gapped ground state of the Hamiltonian

$$\sum_i \alpha_i \sigma_i^z \tag{3.13}$$

for some set of $\alpha_i = \pm 1$. Therefore, any eigenstate of H is the gapped ground state of the Hamiltonian

$$\sum_i \alpha_i (\mathcal{U} \sigma_i^z \mathcal{U}^\dagger). \tag{3.14}$$

Dynamical properties. A crucial feature of MBL systems is that they have extensively many (in fact, a complete set) of quasi-local integrals of motion [21, 24]. To see this, observe that every eigenstate of H_0 is also a simultaneous eigenstate of all the operators σ_i^z. Therefore, every eigenstate of H is also a simultaneous eigenstate of the operators $\tau_i^z := \mathcal{U} \sigma_i^z \mathcal{U}^\dagger$ (often called "l-bits"). The existence of these local integrals of motion means that the system cannot self-thermalize without a bath (as one might have expected given the breakdown of ETH discussed above). Indeed, since the τ_i^z are integrals of motion their expectation values cannot change, so the system will always retain a memory of its initial state even at infinite times.

3.3.4 Localization protected quantum order

Above, we mentioned that in an MBL system, all the eigenstates are GAPGRND states. It follows that the classification of zero temperature phases matter discussed in Chapter 2 can be extended to MBL systems as well, but it now applies to *all* the eigenstates, not just the ground state [31, 32]. Generally, if all the eigenstates are in a given topological or symmetry-breaking phase, this will have interesting signatures in the dynamics. The most straightforward way to see this is in terms of the local integrals of motion discussed above.

Let us discuss the simplest example, the Ising spin glass. This model has a Hamiltonian of the form

$$H = -\sum_{\langle i,j \rangle} J_{i,j}\sigma_i^z\sigma_j^z + V, \tag{3.15}$$

where V is some local perturbation that respects the Ising symmetry $X := \prod_i \sigma_i^x$. As long as the perturbation does not delocalize the system, then as before there will be a local unitary $\mathcal{U}$ relating the perturbed eigenstates to the unperturbed eigenstates which occur for $V = 0$. Moreover, the unperturbed eigenstates come in degenerate pairs, where the symmetry-respecting states are "cat state" superpositions of symmetry-breaking states, just like in a zero-temperature phase with spontaneous symmetry breaking (now, however, it is true throughout the whole spectrum). One can show that the existence of a (symmetry-respecting) local unitary relating the perturbed eigenstates to the unperturbed eigenstates ensures that these properties also survive in the perturbed eigenstates.

We can use the local unitary $\mathcal{U}$ to define l-bits $\tau_i^z = \mathcal{U}\sigma_i^z\mathcal{U}^\dagger$. The important thing about these l-bits is they anti-commute with X, that is $X\tau_i^z X = -\tau_i^z$. The presence of l-bits not commuting with a symmetry can be taken to be the definition of spontaneous symmetry-breaking in the MBL context. It has interesting consequences

for the dynamics, as follows. Suppose we evolve the system with some generic initial state. It will presumably have $\langle \tau_i^z \rangle \neq 0$, and therefore, under time evolution we will find that $\langle \tau_i^z \rangle \neq 0$ at all times. Therefore, even at late times we find that the state of the system does not respect the symmetry (because any symmetry-respecting state has $\langle \tau_i^z \rangle = 0$). By contrast, if the l-bits had commuted with X, one can show [33] that at late times the state of the system would always respect the symmetry (at least when looking at local observables), even if the initial state did not.

3.3.5 Floquet-MBL

MBL can also occur in Floquet systems [15, 34–37] Recall that these are systems which evolve under a time-periodic Hamiltonian $H(t)$, with $H(t+T) = H(t)$, so that the time evolution at integer multiples of the driving period can be computed by taking powers of the Floquet evolution operator

$$U_f = \mathcal{T} \exp\left(-i \int_0^T H(t) dt\right) \tag{3.16}$$

Suppose that the Hamiltonian is of the form

$$H(t) = H_0 + V(t), \tag{3.17}$$

where H_0 is a time-independent Hamiltonian which is MBL, and $V(t)$ is a weak driving term. Without going into the details, let us note that one can perform a form of perturbation theory [37], formally quite similar to the time-independent perturbation theory of Section 3.3.2, in order to construct a time-periodic local unitary $P(t)$ such

that the transformed Hamiltonian

$$P^\dagger(t)H(t)P(t) - iP^\dagger(t)\partial_t P(t) = D \tag{3.18}$$

is time-independent. As usual, resonant spots cause difficulty. Here, however, resonances can also result from energy levels differing in energy by integer multiples of Ω, the (angular) drive frequency. (Physically, this is because the drive can induce transitions between such energy levels.) Nevertheless, if H_0 is MBL, then we still expect that the local spectrum is discrete, in which case a typical location in space will not be a resonant spot.

For weak enough driving, D is close to H_0. Hence, if H_0 is MBL then we expect that so is D. Thus, there are a complete set of local integrals of motion τ_i^z for D. The existence of these local integrals of motion means that Floquet-MBL systems cannot thermalize. Recall that, for a Floquet system, "thermalization" means heating to infinite temperature. Thus, Floquet-MBL seems to be the only way to get a non-trivial steady state at late times.

3.4 Prethermalization

MBL, discussed in the previous section, takes place only in strongly disordered systems. As far as we know, it seems essential to have disorder to prevent a quantum system from thermalizing, even in the infinite-time limit. On the other hand, in a clean system it is possible to inhibit thermalization such that the thermalization time becomes very large, which will be the subject of this section. There are two different scenarios for prethermalization discussed in Refs. [38–42], and a new variant will be described in Chapter 7. Here we will focus on prethermalization in a Floquet system

at high frequency.

The basic idea is that heating in a weakly driven Floquet system, as in Section 3.3.5, occurs when there are resonances, that is, when the local spectrum of H_0 contains energy differences close to an integer multiple of the driving frequency Ω. In an MBL system, the local spectrum is discrete and thus there are no resonances at a typical location. By contrast, in a non-MBL system the local spectrum is continuous and there are always resonances. That is, there exist $|\Psi_i\rangle$ and $|\Psi_j\rangle$ connected by local operators such that $E_i - E_j \approx n\Omega$ for some n. The case $n = 0$ acts to inhibit localization, but does not lead to heating since the corresponding transition conserves energy. On the other hand, resonances with $n \neq 0$ lead to heating at a rate proportional to the matrix element $\langle\Psi_i| V_n |\Psi_j\rangle$, where $V(t) = \sum_{k=-\infty}^{\infty} V_k e^{ik\Omega t}$ is the Fourier series of $V(t)$. If this matrix element can be made very small, then the effect of heating does not become apparent until correspondingly large time.

A case where this occurs is when the frequency Ω is very large compared to the local energy scale J of the unperturbed Hamiltonian H_0. In that case, any energy levels with $E_i - E_j = n\Omega$ for $n \neq 0$ must be substantially different on at least Ω/J sites, which means that the matrix element $\langle\Psi_i| V_n |\Psi_j\rangle$ is very small. Indeed, a careful analysis shows that, in linear response in the strength λ of the perturbation V, the heating rate scales like $\lambda e^{-\Omega/J}$ [38]. Moreover, going beyond linear response, and following a similar procedure to the MBL case, one can find a time-dependent local unitary change of basis such that the transformed Hamiltonian $H(t)$ is time-independent (and hence corresponds to a conserved "energy"), up to corrections that are suppressed by a factor of $e^{-\Omega/J}$, that is, they do not become important for the dynamics until the heating time $t_* \sim e^{\Omega/J}$.

Finally, let us note that for the above considerations, it turns out not to be required that we are weakly driving a time-independent Hamiltonian H_0. Indeed, suppose we

have any time-periodic Hamiltonian $H(t)$ with local energy scale J satisfying $J \ll \Omega$. Then one can show that there is a time-periodic change of basis such the transformed Hamiltonian takes the form

$$\overline{H} + O(J^2/\Omega), \tag{3.19}$$

where $\overline{H}$ is the time-averaged Hamiltonian $\overline{H} = \frac{1}{T}\int_0^T H(t)dt$. Then we can apply the discussion of the previous paragraphs as before, with $\overline{H}$ playing the role of the "unperturbed" Hamiltonian.

Chapter 4

Review: Spontaneous Symmetry Breaking and Time Crystals

Spontaneous symmetry breaking, the phenomenon where the steady state of a system has less symmetry than its Hamiltonian, is one of the most fundamental notions in physics. Time translation symmetry is one of the most fundamental symmetries in physics, since its generator is the Hamiltonian itself. Nevertheless, when it was first introduced, the idea of a *time crystal*, a system which spontaneously breaks time-translation symmetry (and therefore displays spontaneous oscillations even though the Hamiltonian is time-independent) seemed radical and controversial [43–51]. In this section, we will briefly review the idea of spontaneous symmetry breaking, and what it would mean for a system in thermal equilibrium to be a time crystal. Sadly, such ideas had not been around for very long before a no-go theorem was proven [52]. Nevertheless, they provided the starting point for the ultimately more successful investigation of spontaneously broken time-translation symmetry in non-equilibrium systems, which is presented in later chapters.

4.1 Spontaneous symmetry breaking

Let us discuss what it means for a system in thermal equilibrium to exhibit spontaneous symmetry breaking. Normally, by standard arguments, we expect that in thermal equilibrium the state of a quantum system is given by the canonical ensemble

$$\rho_{\text{gibbs}} = \frac{1}{Z} e^{-\beta \mathcal{H}}, \tag{4.1}$$

where $\mathcal{H}$ is the Hamiltonian. However, this cannot *quite* be right for a system exhibiting spontaneous symmetry breaking. Indeed, if the Hamiltonian $\mathcal{H}$ has a symmetry, which means that there is a unitary operator U such that $U\mathcal{H}U^\dagger = \mathcal{H}$, then it immediately follows from Eq. (4.1) that $U\rho U^\dagger = \rho$, which is to say that ρ is invariant under the symmetry.

The correction to Eq. (4.1) for the case of spontaneous symmetry breaking is also well known, and is related to breaking of ergodicity. Let us first consider for concreteness an Ising symmetry generated by the spin-flip operator X. Then, in the ferromagnetic phase in which the Ising symmetry is spontaneously broken, there are two *different* "thermal states" $\rho_\uparrow$ and $\rho_\downarrow$ in which the system can end up. These states are not themselves invariant under the symmetry; rather, the symmetry interchanges them. They can be distinguished by a macroscopic order parameter, for example the net magnetization. They represent distinct ergodic sectors, because the time taken for a system in one sector to reach the other sector by thermal fluctuations is exponentially large in the system size. The thermal state ρ_{gibbs} is the state corresponding to maximal uncertainty as to which sector the system is in:

$$\rho_{\text{gibbs}} = \frac{1}{2}(\rho_\uparrow + \rho_\downarrow). \tag{4.2}$$

This state of uncertainty is, however, not very sustainable, since the order parameter of the system is very easy to observe. Moreover, once the order parameter has been observed we know that (for sufficiently large systems), it will not change. Hence, the state of the system is best described by the symmetry-breaking states $\rho_\uparrow$ or $\rho_\downarrow$, not ρ_{gibbs}.

These considerations can be extended to a general symmetry group G and unbroken subgroup H. For simplicity, and because it will become relevant shortly, here we will just consider the case $G = \mathbb{R}$ and $H = \mathbb{Z}$. For example, this would describe (leaving aside any Mermin-Wagner objections) a one-dimensional system in continuous space forming a periodic charge density wave (CDW). In general, there should be a Hermitian operator A generating a continuous symmetry $e^{i\alpha A}$ (for any real α), but the physical states are only invariant under the discrete symmetry generated by $e^{i(2\pi)A}$. The physical states are parameterized by a circle[1], so we write them in terms of an angular coordinate as ρ_θ, where $\rho_{\theta+2\pi} = \rho_\theta$. In the CDW example, θ is the displacement of the CDW. The symmetry permutes the physical states according to $e^{i\alpha A}\rho_\theta e^{-i\alpha A} = \rho_{\theta+\alpha}$. The Gibbs state is the state corresponding to maximal uncertainty as to the sector,

$$\rho_{\text{gibbs}} = \frac{1}{2\pi}\int_0^{2\pi} \rho_\theta d\theta \tag{4.3}$$

but as before this uncertainty is quickly dispelled by observing the order parameter, and we will observe a single ρ_θ.

[1]This is because the coset space $\mathbb{R}/\mathbb{Z}$ is a circle.

4.2 Time crystals

Let us now extend these considerations to the case of spontaneously broken time-translation symmetry. Naively, one might expect that the notion of spontaneously broken time-translation symmetry in thermal equilibrium is an oxymoron, because the word "equilibrium" suggests a stationary state. Going beyond mere semantics, one could formulate this argument as follows. In thermal equilibrium, the state of the system should be given by the Gibbs state

$$\rho_{\text{gibbs}} = \frac{1}{Z} e^{-\beta \mathcal{H}}. \tag{4.4}$$

However, if we now consider the time-evolution of this state under evolution by H according to the Schrödinger equation, we find

$$d\frac{\rho_{\text{gibbs}}}{dt} = i[\rho_{\text{gibbs}}, \mathcal{H}] = 0. \tag{4.5}$$

Of course, this is actually only a slight variant of the argument in the previous section that the Gibbs state always respects the symmetries, and it should be resolved in the same way.

Specifically, for a time-translation symmetry the relevant symmetry group is $\mathbb{R}$, and for a time crystal we will want to break this down to time translations by multiples of a discrete period T (that is, the unbroken subgroup is $\mathbb{Z}$), so the structure we expect is the one described in the last paragraph of the previous section. That is, there is a family of symmetry-breaking states ρ_θ indexed by an angular variable θ, and the Gibbs state is a superposition

$$\rho_{\text{gibbs}} = \frac{1}{2\pi} \int_0^{2\pi} \rho_\theta d\theta. \tag{4.6}$$

Moreover, the symmetry-breaking states should be interchanged by the symmetry. In this case the generator of the symmetry is the Hamiltonian $\mathcal{H}$, so we have

$$e^{it\mathcal{H}}\rho_\theta e^{-it\mathcal{H}} = \rho_{\theta+\omega t} \tag{4.7}$$

(for some fixed angular velocity ω). Assuming that the different states ρ_θ are distinguished by some observable order parameter, any uncertainty as to the sector the system is in will quickly be resolved and at any instant we will observe the system to be in a symmetry-breaking state ρ_θ. Crucially, however, Eq. (4.7) says that if we observe the system again at a later time, the observed value of order parameter will oscillate with angular frequency ω. This is the signature of a time crystal.

The scenario just described is very appealing, but we are left to wonder whether it can ever occur for any physical Hamiltonian $\mathcal{H}$. In fact, there is a simple example. Suppose that the system has a U(1) particle number conservation symmetry generated by an operator $\hat{N}$. In that case, ρ_{gibbs} should be replaced by the grand-canonical ensemble state

$$\rho_{gc} = e^{-\beta\mathcal{H}+\mu\hat{N}}, \tag{4.8}$$

where μ is the chemical potential, but the above discussion otherwise carries through unchanged. In particular, if the system condenses into a superfluid, then the superfluid order parameter $\langle b^\dagger \rangle$, where $b^\dagger$ is the particle creation operator, acquires an expectation value (this corresponds to the spontaneous breaking of the U(1) symmetry). Moreover, it is a well-known fact about superfluids that the phase of this order parameter rotates at angular frequency μ. Thus, the structure described above is exactly realized in any superfluid.

On the other hand, this example is unsatisfying for several reasons. First, the

time-translation symmetry breaking is just "piggy-backing" on another spontaneously broken symmetry, which seems too trivial. Second, the superfluid order parameter is difficult to observe – only phase *differences* between two different superfluids are observable, through the Josephson effect. Actually, it is precisely the fact that U(1) is a very good symmetry that makes its order parameter difficult to observe, since measuring it requires breaking the symmetry[2].

For this reason, we really would like to observe a time crystal in a system *without* any symmetries other than time-translation itself. Unfortunately, it turns out that this is impossible in thermal equilbrium. Specifically, it follows from the results of Ref. [52] that the scenario contemplated above, in which Eq. (4.6) satisfied with $\rho_{\text{gibbs}} = \frac{1}{Z}e^{-\beta\mathcal{H}}$, can never occur, at least assuming that $\mathcal{H}$ is local, that the symmetry-breaking states ρ_θ obey a cluster decomposition (correlations decay at large spatial separation), and are distinguished by the expectation value of a local observable[3].

One might ask why time-translation symmetry is the only symmetry that can never be spontaneously broken in thermal equilibrium (without piggy-backing), when, for example, spatial translation symmetry is spontaneously broken in any crystal. The clue lies in the fact that $\mathcal{H}$ appears twice in the above discussion: firstly in Eq. (4.7), as the symmetry generator, and secondly in the definition of $\rho_{\text{gibbs}} \propto e^{-\beta\mathcal{H}}$. For any other symmetry, $\mathcal{H}$ would be replaced by the appropriate symmetry generator in Eq. (4.7), but it would still appear in the definition of ρ_{gibbs}. This is related to the fact that the time direction indeed is a privileged one in statistical mechanics, because it defines what we mean by "equilibrium".

[2]To illustrate this, imagine replacing the U(1) number conservation symmetry with a U(1) spin rotation symmetry. Then the order parameter is a magnetization, which is easy to measure, but the symmetry will not be respected very precisely in any real system because of the presence of stray magnetic fields. These fields will tend to pin the magnetization to a fixed value, preventing the observation of time crystal oscillations.

[3]The assumptions of Ref. [52] are stated in a rather different form, but they can be shown to be consequences of the assumptions stated here.

This therefore concludes our consideration of time crystals in thermal equilibrium: they do not exist, except in a trivial sense. Nevertheless, the ideas of this chapter will reappear in later chapters when we discuss systems out of equilibrium.

Chapter 5

The Floquet Equivalence Principle

In this chapter, we introduce a powerful way to think about Floquet phases of matter. The main result, which we will call the *Floquet Equivalence Principle*, is that Floquet phases with symmetry G are in one-to-one correspondence with *stationary* topological phases with additional symmetry. This allows us to leverage the substantial existing literature on topological phases with symmetries to understand Floquet topological phases. In this chapter, we just give some suggestive arguments in favor of the Floquet Equivalence Principle, prove it in certain cases, and discuss physical consequences. For a more systematic approach, see Chapter 9.

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5.1 Introduction

There are now many known examples of phases of matter which are distinguished not by the symmetries they break spontaneously but through more subtle "topological" orders [53]. Most such phases are not robust to thermal excitations and

therefore were thought to exist only at zero temperature [54, 55]. However, recently it has been appreciated that, in the presence of strong disorder, it is possible for highly excited eigenstates of a many-body system to be *many-body localized (MBL)* [16–25]. Such MBL states are not thermal, and indeed more closely resemble gapped ground states; for example, they obey an area law for the entanglement entropy. This means that they can exhibfit topological phases previously thought to be restricted to zero temperature[31, 32, 56–58].

The lifting of the restriction to ground states also allows us to consider more general "Floquet" systems [59–68], in which the Hamiltonian $H(t)$ is allowed to vary in time, but with periodicity T. The "eigenstates" of such a system are the eigenstates of the Floquet operator $U = U(T)$ which describes the unitary evolution of the system over one time period. Such eigenstates can also be MBL in the presence of strong disorder [15, 34–38], and hence can exhibit topological phases. However, the classification of topological phases in such "Floquet-MBL" systems is in general richer than in the stationary case.

Recently, progress has begun to be made in understanding the classification of topological phases in Floquet-MBL systems with interactions[69, 70]. In particular, Ref. [70] classified phases with a symmetry G and no intrinsic topological order (i.e. *symmetry-protected topological (SPT) phases* [71–89]) in (1+1)-D. The purpose of this chapter is to re-express the classification of Ref. [70] in a concise way, which we feel clarifies the issues involved and streamlines the derivation. We then consider natural extensions, building up to a (conjectured) general correspondence between topological phases in Floquet-MBL systems with symmetry group G, and topological phases in stationary systems with symmetry group $\mathbb{Z} \rtimes G$, where the extra $\mathbb{Z}$ accounts for the discrete time-translation symmetry.

5.2 Assumptions

We will assume that the Floquet operator U can be expressed as a time evolution of a local time-dependent Hamiltonian $H(t)$, with $H(t+T) = H(t)$. Thus,

$$U = \mathcal{T} \exp\left(-i \int_0^T H(t)dt\right), \quad \mathcal{T} = \text{time-ordering}. \tag{5.1}$$

where we assume that the Hamiltonian $H(t)$ is invariant under a representation $V(g)$ of a symmetry group G, where G can contain anti-unitary elements corresponding to a time reversal symmetry. For anti-unitary $g \in G$, what we mean by the Hamiltonian "being invariant" is that $V(g)H(t)V(g)^{-1} = H(T-t)$. This ensures that, in general,

$$V(g)UV(g)^{-1} = U^{\alpha(g)}, \tag{5.2}$$

where $\alpha(g) = -1$ if g is anti-unitary and $+1$ otherwise.

5.3 The SPT classification

The classification of Ref. [70] can be re-expressed in the following way. We define an enlarged symmetry group $\widetilde{G}$ to be the *full* symmetry group of the system, *including* the discrete time translation symmetry inherent in the Floquet setup. Thus, if all of the symmetries of G are unitary, we have $\widetilde{G} = G \times \mathbb{Z}$. More generally, for anti-unitary elements $g \in G$, we have $g\mathbb{T}g^{-1} = \mathbb{T}^{-1}$, where $\mathbb{T}$ is the generator of time translations. Thus, in general $\widetilde{G}$ is a semi-direct product $\widetilde{G} = \mathbb{Z} \rtimes G$. Then in the bosonic case, the classification of Ref. [70] can be reformulated as follows (see Appendix A.1 for a proof):

Result 1. The symmetry-protected topological phases in a periodically driven (1+1) bosonic system exhibiting MBL are classified by the second cohomology group $H^2(\widetilde{G}, U(1))$.

(Here, and later, we will take it to be implicit that $U(1)$ is to be interpreted as a non-trivial $\widetilde{G}$-module, with anti-unitary elements of $\widetilde{G}$ acting as inversion, as in the original classification of SPT phases with anti-unitary symmetries, e.g. see Ref. [80]).

Recall that the bosonic topological phases in a stationary system are classified by $H^2(G, \mathrm{U}(1))$; to obtain the classification in a driven system one simply replaces G by $\widetilde{G}$. In retrospect, this result should be quite natural. Indeed, the classification of stationary SPT phases in (1+1)-D [72, 73, 76, 77], though sometimes expressed in terms of Hamiltonians, is really at its core a classification of *short-range entangled* states (states which are equivalent to a product state by a local unitary) invariant under some local (anti-)unitary representation of a symmetry group (see Appendix A.2 for more details). The gapped ground states of a Hamiltonian are examples of such states, but so are MBL eigenstates of a Floquet operator. (We could even consider eigenstates of the Floquet operator which are not MBL but are separated from all other eigenstates by a quasienergy gap). Thus, the standard classification of (1+1)-D SPT phases can be applied to any such states. However, there is one difference in the Floquet case: as well as the representation of the symmetry G, a Floquet eigenstate is, by definition, also invariant (up to a phase factor) under the Floquet operator U, which is a local unitary since it is the time evolution of a local Hamiltonian. Therefore, we should really include U in the symmetry group to obtain the full classification. [Eq. (5.2) ensures that we then have a representation of the enlarged symmetry group $\widetilde{G} = \mathbb{Z} \rtimes G$].

It is true that, when classifying SPT phases, one normally assumes that the action of the symmetry is "on-site", that is, that each symmetry operator $V(g)$ is a tensor product of its action on each site of the lattice, $V(g) = [v(g)]^{\otimes N}$, which would not be true of the Floquet unitary U. However, all we actually need is that all the symmetry operators (including the Floquet unitary U) can be *restricted* to a region A with boundary while still remaining a representation of $\widetilde{G}$, where by "restriction" of a local unitary U we mean[86] a unitary U_A acting only on the region A which acts the same as U in the interior of A, well away from the boundary. See Appendix A.2 for the derivation of the classification, given such an assumption.

To see that such a restriction is possible, consider for simplicity the case of unitary symmetries. Then if the Hamiltonian $H(t)$ can be written as a sum $H(t) = \sum_X h_X(t)$ of terms supported on local regions X [each of which commutes with the symmetry $V(g)$], then we can define the restriction of the Floquet operator by simply retaining only the terms which act within A, or in other words:

$$U_A = \mathcal{T} \exp\left(-i \int_0^T dt \sum_{X \subseteq A} h_X(t)\right). \tag{5.3}$$

Meanwhile, we define the restriction of $V_A(g)$ in the obvious way, by only acting with the on-site action on sites contained within A. It is easily seen that $V_A(g)$ is still a representation of G, and U_A commutes with $V_A(g)$, so together they form a representation of $\widetilde{G} = \mathbb{Z} \times G$. Similar arguments can be made for anti-unitary symmetries.

We emphasize that our derivation of Result 1 is actually more general than that of Ref. [70]. Firstly, in Ref. [70] the result for non-Abelian G was only stated as a conjecture. Our derivation clearly applies to such G as well. Secondly, we did not need to assume, as did Ref. [70] that *all* the eigenstates of the Floquet operator are

MBL; our classification result applies to any of the eigenstates that happen to be MBL, or separated from the rest of the quasienergy spectrum by a gap. Finally, since our derivation was based on individual eigenstates, it allows for the possibility of different SPT phases coexisting as eigenstates of a single Floquet operator, separated by an eigenstate transition [31, 56].

5.4 Higher dimensional results

When stated in the form given here, classification result of Ref. [70] has obvious generalizations to higher dimensions. In particular, in Ref. [86] we derived the classification of (2+1)-D SPT phases in ground states by considering how the symmetry acts on the boundary. In Ref. [86], we did use the Hamiltonian to argue that the symmetry action on the boundary is well-defined; however, Appendix A.2 shows how to formulate this concept for a single short-range entangled state without reference to a Hamiltonian (and without assuming that the symmetry in the bulk is on-site). Therefore, we can repeat the analysis of Ref. [86] (but taking care to include the Floquet unitary U in the symmetry group), and one finds that

Result 2. The symmetry-protected topological phases in a periodically driven (2+1)-D bosonic system exhibiting MBL are classified by the third cohomology group $H^3(\widetilde{G}, U(1))$.

Again, we simply replace $G \to \widetilde{G}$ compared to the usual stationary case. The anti-unitary case was not explicitly treated in Ref. [86], but it is a straightforward generalization[90]. One can also prove a similar result for fermionic systems.

5.5 General correspondence between stationary and Floquet-MBL topological phases

The above results relied on the method of Ref. [86], which did not consider (at least, not in full generality) SPT phases in higher dimensions, or topological phases beyond SPT. Nevertheless, they motivate us to formulate the following conjecture.

Conjecture 1. The topological phases in a (bosonic/fermionic) periodically driven MBL system in d spatial dimensions with on-site symmetry group G are in one-to-one correspondence with the topological phases in a (bosonic/fermionic) *stationary* MBL system in d spatial dimensions with symmetry group $\widetilde{G} = \mathbb{Z} \rtimes G$ (as defined above).

Here by "topological phases", we mean both symmetry-protected topological (SPT) phases and symmetry-*enriched* topological (SET) phases [91–97]. The rationale for this conjecture is as follows. The classification of gapped ground states is known to depend only on the ground states themselves, not on their parent Hamiltonians [77]. Furthermore, since eigenstates in an MBL system look, roughly speaking, like gapped ground states, one expects to obtain the same classification for such eigenstates. However, in a periodically driven system there is an extra local unitary, beyond the symmetries in the group G, under which these eigenstates are invariant (up to a phase factor) – namely, the Floquet unitary U. Thus, one should treat U as a symmetry for the purpose of obtaining the classification.

The only way we could envision this conjecture failing would be if the non-on-site nature of the Floquet unitary U turned out to be important, in a way that it was not in the case of (1+1)-D and (2+1)-D SPT's. This seems to us unlikely. In fact, we

expect that *any* derivation of the classification of SPT/SET phases – or at least, any derivation which can be formulated in terms of short-range entangled states without reference to Hamiltonians – could probably be applied just as well in the Floquet context, which would prove the conjecture.

We note, however, that probably not all topological phases which can exist at zero temperature can be stabilized in MBL excited states [57]; for this reason, we have been careful to formulate Conjecture 1 in terms of a correspondence with stationary MBL systems, not with zero-temperature states.

5.6 Interpretation of the classification in terms of pumping

Results 1 and 2, and Conjecture 1 in higher dimensions, imply that the classification of SPT phases in bosonic Floquet-MBL systems in d spatial dimensions is $H^{d+1}(\widetilde{G}, U(1))$. In the case of a unitary symmetry, such that $\widetilde{G}$ is just a direct product $\mathbb{Z} \times G$, we can give a simple physical interpretation of this result. From the Künneth formula for group cohomology [97], one finds that

$$H^{d+1}(\mathbb{Z} \times G, \mathrm{U}(1)) = H^{d+1}(G, \mathrm{U}(1)) \times H^{d}(G, \mathrm{U}(1)). \tag{5.4}$$

Thus, the classification is just the usual classification for ground states, plus an extra piece of data given by an element of $H^d(G, \mathrm{U}(1))$. We expect that this extra piece of data can be interpreted as characterizing the fact that each application of the Floquet unitary U "pumps" an additional $(d-1)$-dimensional SPT phase onto the boundary. This is a generalization of the observation in Ref. [70] that in (1+1)-D the extra data is the charge pumped onto each component of the boundary by the Floquet unitary.

A rough physical justification for this interpretation in (2+1)-D (which readily generalizes also to higher dimensions) is as follows. For simplicity we assume that G is Abelian. One can then show that the $H^2(G, \mathrm{U}(1))$ piece of Eq. (5.4) can be extracted from a 3-cocycle $\omega(\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3)$ of the full symmetry group $\widetilde{G}$ by calculating a 2-cocycle of G according to

$$\omega(g_1, g_2) = \frac{\omega(\mathbb{T}, g_1, g_2)\omega(g_1, g_2, \mathbb{T})}{\omega(g_1, \mathbb{T}, g_2)}. \tag{5.5}$$

(where $\mathbb{T}$ is the generator of discrete time translations.) The object Eq. (5.5) has a familiar interpretation [98]. Indeed, suppose we *gauge* the full symmetry group $\widetilde{G} = \mathbb{Z} \times G$. Then the point excitations in the resulting twisted (2+1)-D gauge theory can be classified by the flux $\widetilde{g} \in \widetilde{G}$ they carry. In general, a particle carrying non-trivial flux also carries a *projective* representation of the gauge group. In particular, Eq. (5.5) describes the projective representation of the subgroup G on a particle carrying flux $\mathbb{T}$. Now, in the original ungauged SPT phase, the analog of a flux is a "symmetry twist defect" [96, 99–101] which (since fluxes are confined) must occur at the endpoint of a symmetry twist line. The fact that the endpoints of such symmetry twist lines carry projective representations of G (which can also be derived directly, using the theory of twist defects developed in Ref. [96]) shows that the lines themselves must be in a (1+1)-D SPT phase with respect to G. On the other hand, a *closed* symmetry twist line (with no endpoints) on the boundary ∂A of a region A can be interpreted as the result of applying to the original MBL eigenstate the Floquet unitary U, restricted to the region A. The fact that such a state carries a (1+1)-D SPT on the boundary ∂A indeed shows that the effect of U is to pump a (1+1)-D SPT to the boundary.

On the other hand, we do not expect there to be any similarly simple physical

picture in the anti-unitary case; in Ref. [70] it was found that the extra data for (1+1)-D systems is a somewhat strange "twisted" representation of the symmetry with no obvious physical interpretation.

5.7 Topological phases without symmetry

The above considerations allow us to the establish the existence of topological phases in driven MBL systems that are distinct in the Floquet context, even in the absence of any additional symmetry, but not in the stationary case. Indeed, imagine we take a Floquet system in (2+1) dimensions or higher, with symmetry group $\widetilde{G} = G \times \mathbb{Z}$, and then gauge just the symmetry G. In general, gauging a subgroup of the full symmetry group relates SPT phases to symmetry-enriched topological (SET) phases protected by the remaining global symmetry [94–96]; which, in this case, is simply the discrete time translation symmetry.

5.8 Explicit realization

We have already argued above that the invariants which classify Floquet-MBL topological phases with symmetry G should be the same as in the case of stationary topological phases with symmetry $\mathbb{Z} \rtimes G$. However, one might ask whether there might be an obstruction to realizing any of these "potential" Floquet-MBL topological phases in an explicit model. We argue that this is not the case, *provided* that the corresponding stationary topological phase with symmetry $\mathbb{Z} \rtimes G$ can be realized in a stationary MBL system with symmetry $\widetilde{G}_n = \mathbb{Z}_n \rtimes G$ for some sufficiently large n. Such a system, by definition, consists of a Hamiltonian H which commutes with an on-site representation $V(\widetilde{g})$ of $\widetilde{G}_n$. (A faithful on-site representation of $\mathbb{Z}$ does not

make sense in a lattice system with finite-dimensional Hilbert space per site, hence why we consider $\mathbb{Z}_n$ instead. A system acted on by $\mathbb{Z}_n$ can always be thought of as being acted on by $\mathbb{Z}$ non-faithfully). Then we claim that the Floquet system with Floquet operator $U = e^{iHT}V(\alpha)$ (where α is the generator of $\mathbb{Z}_n$) indeed realizes the desired Floquet-MBL topological phase.

To see this, note that the eigenstates of H are also eigenstates of $V(\alpha)$ [since H commutes with $V(\alpha)$ by assumption] and therefore of U. We can analyze the SPT order of these states by thinking of them either as eigenstates of a stationary system with symmetry $\widetilde{G}$, or as eigenstates of a Floquet system with symmetry G. In fact, the analysis proceeds identically in both cases, with only one difference: in the stationary context, the $\mathbb{Z}$ part of the symmetry is taken to be generated by $V(\alpha)$, whereas in the Floquet context, it is generated by U. However, we can make $U = V(\alpha)$ by sending $T \to 0$ continuously. Since the classification of topological phases is *discrete*, we do not expect that this can change the diagnosed phase. This can be checked explicitly in the (2+1)-D SPT case.

5.9 Conclusion

The perspective on topological phases in Floquet-MBL systems detailed in this Communication opens up many intriguing questions for future study. Indeed, *every* phenomenon that has been studied in the usual stationary case – for example, symmetry fractionalization on topological excitations in symmetry-enriched topological (SET) phases [89, 92, 96] – ought to have analogs in the Floquet-MBL case, but in many cases the possibilities will be richer due to the extra $\mathbb{Z}$ symmetry. We leave further exploration of these phases and their physical properties for future work.

Note added.– Soon after we posted this work on the arXiv, two more preprints

appeared[102, 103] whose results overlap with ours.

Chapter 6

Floquet Time Crystals

The previous chapter exposed the crucial role of the discrete time translation symmetry in understanding Floquet phases of matter. In this chapter, we discuss an even more dramatic consequence: the discrete time-translation symmetry can be *spontaneously broken*. The "Floquet time crystals" in which this occurs are striking examples of entirely new dynamical phases of matter in the non-equilibrium setting.

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6.1 Introduction

Spontaneous symmetry-breaking (SSB) is a pivotal concept in physics, with implications for condensed matter and high-energy physics. It occurs when the ground state or low-temperature states of a system fail to be invariant under symmetries of the Hamiltonian. The Ising model is a prototypical example for this behavior: Here, the symmetry is a simultaneous flip of all the spins, which leaves the energy of a state unchanged. In the ferromagnetic phase, low-energy states are formed with a

non-zero magnetization. For almost every symmetry imaginable, there is a model whose ground state breaks it: crystals break the continuous translational and rotational symmetries of Coulomb interactions; magnetically ordered materials break time-reversal symmetry and spin symmetry, and superfluids break global gauge symmetry. The lone holdout, thus far, has been time-translation symmetry. In this paper, we give a definition of time-translation symmetry breaking, and construct an example of this behavior in a driven many-body localized system.

6.2 Definition of Time Translation Symmetry Breaking

Systems that spontaneously break time-translation symmetry (TTS) have been dubbed "time crystals," in analogy with ordinary crystals, which break spatial translational symmetries [43, 44]. Even defining this notion correctly requires considerable care, and putative models have proven inconsistent [45–51]. The most obvious definition of time-translation symmetry breaking (TTSB) would be that the expectation values of observables are time-dependent in thermal equilibrium. However, this is clearly impossible, since a thermal equilibrium state $\rho = \frac{1}{Z}e^{-\beta H}$ is time-independent by construction (because $[\rho, H] = 0$). A more sophisticated definition of TTSB in terms of correlation functions in the state ρ has been proposed – and ruled out by a no-go theorem – in Ref. [52].

Therefore, we must look beyond strict thermal equilibrium. This should not be too surprising, as the state ρ preserves *all* the symmetries of H, which would suggest that *no* symmetry can be spontaneously broken. For symmetries other than time translation, the resolution to this paradox is well-known: in a system with a

spontaneously broken symmetry, there is ergodicity-breaking and the lifetime of a symmetry-breaking state diverges as the system size grows. Thus, in the thermodynamic limit, the state ρ is unphysical and is never reached. This suggests that an analogous phenomenon should be possible for time translation symmetry, where the time taken to reach a time-independent steady state (such as the thermal state ρ) diverges exponentially with system size.

To turn these considerations into a more useful definition, we observe that, in a quantum system, the ergodicity-breaking in a phase with a spontaneously broken symmetry can be seen at the level of eigenstates. For example, the *symmetry-respecting* ground states of an Ising ferromagnet are $|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\cdots\uparrow\rangle \pm |\downarrow\cdots\downarrow\rangle$. Such long-range correlated "cat states" are unphysical, will immediately decohere given any coupling to the environment, and can never be reached in finite time by any unitary time evolution starting from a short-range correlated starting state. On the other hand, the "physical" combinations $|\uparrow\cdots\uparrow\rangle$ and $|\downarrow\cdots\downarrow\rangle$ break the Ising symmetry.

In the TTSB case, we also need to invoke the intuition that oscillation under time evolution requires the superposition of states whose phases wind at different rates. That is, whereas in the Ising ferromagnet the two cat states $|\pm\rangle$ are degenerate in the thermodynamic limit, in a time-crystal they would need to have *different* eigenvalues under the time-evolution operator. Indeed, consider for simplicity a discrete time evolution operator U_f (which describes periodically driven "Floquet" systems as we discuss further below.) Suppose that the states $|\pm\rangle$ have eigenvalues $e^{i\omega_\pm}$ under U_f. Then, although the unphysical cat states $|\pm\rangle$ are time-invariant (up to a phase), a *physical* state such as $|\uparrow\cdots\uparrow\rangle$ will evolve according to $(U_f)^n|\uparrow\rangle \propto \cos(\omega n)|\uparrow\cdots\uparrow\rangle + i\sin(\omega n)|\downarrow\cdots\downarrow\rangle$, where $\omega = (\omega_+ - \omega_-)/2$.

The above considerations motivate two equivalent definitions of TTSB, using the following terminology/notation. We will say that a state $|\psi\rangle$ has *short-ranged corre-*

lations if, for any local operator $\Phi(x)$, $\langle\psi|\Phi(x)\Phi(x')|\psi\rangle - \langle\psi|\Phi(x)|\psi\rangle\langle\psi|\Phi(x')|\psi\rangle \to 0$ as $|x-x'| \to \infty$, i.e. if cluster decomposition holds. Note that the superpositions defined above are not short-range correlated under this definition, while a state such as $|\uparrow\uparrow\ldots\uparrow\rangle$ is. We assume that time-evolution is described by a time-dependent Hamiltonian $H(t)$, with a discrete time translation symmetry such that $H(t) = H(t+T)$ for some T. Note that we have not assumed a *continuous* time translation symmetry, which will allow us to consider "Floquet" systems driven at a frequency $\Omega = 2\pi/T$. Let $U(t_1, t_2)$ be the corresponding time evolution operator from time t_1 to t_2. We now define (in the thermodynamic limit):

TTSB-1: TTSB occurs if for each t_1, and for every state $|\psi(t_1)\rangle$ with short-ranged correlations, there exists an operator Φ such that $\langle\psi(t_1+T)|\,\Phi\,|\psi(t_1+T)\rangle \neq \langle\psi(t_1)|\,\Phi\,|\psi(t_1)\rangle$, where $|\psi(t_1+T)\rangle = U(t_1+T, t_1)\,|\psi(t_1)\rangle$.

TTSB-2: TTSB occurs if the eigenstates of the Floquet operator $U_f \equiv U(T,0)$ cannot be short-range correlated.

In what follows, we will show how to construct a time-dependent Hamiltonian $H(t)$ which satisfies the conditions for TTSB given above. In such a system, even though the time-evolution is invariant under the discrete TTS generated by time translation by T, the expectation value of some observables is only invariant under translations by nT for some $n > 1$. In other words, the system responds at a *fraction* Ω/n of the original driving frequency.

The first definition puts the time-dependence front and center and is directly connected to how TTSB would be observed experimentally: prepare a system in a short-range correlated state and observe its subsequent time-evolution, which will not be invariant under the TTS of the time evolution operator. But since, in a Floquet eigenstate, observables would necessarily be invariant under the discrete TTS generated by time translation by T, definition TTSB-1 implies that Floquet eigen-

states cannot be short-range correlated, thereby implying TTSB-2. Conversely, if it is impossible to find Floquet eigenstates that are short-range correlated (which is TTSB-2), then it means that short-range correlated states can only be formed by taking superpositions of Floquet eigenstates with different eigenvalues. In such states, observables will not be invariant under the discrete TTS generated by time translation by T, thereby implying TTSB-1. Hence, the two definitions are equivalent. The second definition will prove to be particularly useful for analyzing the results of numerical exact diagonalization of the Floquet operator. When discrete TTS by T is broken down to TTS by nT, the eigenstates of U_f must be superpositions of n different short-range-ordered states.[1] Then, in any Floquet eigenstate, the mutual information $I(A,B) \equiv S_A + S_B - S_{AB}$, where A and B are spatially separated regions of the system and S_X is the von Neumann entropy of the reduced density matrix for region X, satisfies $I(A,B) \to \ln n$ as the system size as well as the sizes of the regions A and B and their separation is taken to infinity [104, 105].

6.3 Floquet-Many-Body-Localization

Generic translationally invariant many-body Floquet systems likely cannot have TTSB, as their eigenstates resemble infinite temperature states and hence are short-range correlated [13–15].[2] This is analogous to the fact (which follows from the results of Ref. [52]) that for continuous time-translation symmetry, TTSB is impossible so long as the eigenstate thermalization hypothesis (ETH)[9–12] is satisfied. However, we can build upon recent developments in the study of Floquet-many-body-localized

[1] To see this, note that we can choose a basis of short-range correlated eigenstates for $(U_f)^n$. By assumption, such states cannot be eigenstates of $(U_f)^k$ for $0 < k < n$. Therefore, U_f generates an orbit of n *different* short-range correlated states. An eigenstate of U_f is an equal-weight superposition over such an orbit.

[2] Nevertheless, an initial state that is not an eigenstate could potentially heat very slowly, leading to non-trivial intermediate-time dynamics [38, 40–42, 106].

(Floquet-MBL) systems [15, 35–37, 67, 69, 70, 102, 103, 107, 108], for which the eigenstates do not resemble infinite temperature states. Instead, the Floquet states of such systems exhibit the characteristics of the energy eigenstates of static MBL [16–21, 24, 109, 110] systems: the eigenstates are local product states, up to finite-depth unitary quantum circuits [22].

In MBL systems, all eigenstates (of the Hamiltonian in the static case or of the Floquet operator in the driven case) behave as ground states and, therefore, SSB or topological order can occur in all eigenstates [22, 31, 58]. In the SSB case, simultaneous eigenstates of the Floquet operator and of the Cartan subalgebra of the symmetry generators cannot be short range correlated. TTSB-2 can then be viewed as a special case of this in which there are no other symmetry generators besides U_f.

In the next paragraph, we construct a Floquet operator and show that it exhibits discrete TTSB. In subsequent paragraphs, we show that this soluble Floquet operator sits in a finite window in parameter space over which TTSB occurs – i.e. that there is a TTSB *phase*. Models which exhibit TTSB (though not identified as such) have previously been considered in Refs. [69, 108]. These models also break another symmetry spontaneously, but this is not essential to achieve TTSB. Our model will be a generalization of that of Refs. [69, 108], with the extra symmetry explicitly broken. By contrast, the models of Refs. [111, 112] rely crucially on an additional symmetry.

6.4 Model and Soluble Point

We consider one-dimensional spin-1/2 systems with Floquet unitaries of the form:

$$U_f = \exp\left(-it_0 H_{\mathrm{MBL}}\right) \exp\left(it_1 \sum_i \sigma_i^x\right) \tag{6.1}$$

We choose $t_1 \approx \pi/2$, such that the application of $\sum_i \sigma_i^x$ in this stroboscopic time evolution has the effect of approximately flipping all of the spins since $\exp(i\frac{\pi}{2}\sum_i \sigma_i^x) = \prod_i i\sigma_i^x$. This is followed by time evolution for an interval t_0 under the Hamiltonian

$$H_{\text{MBL}} = \sum_i \left(J_i \sigma_i^z \sigma_{i+1}^z + h_i^z \sigma_i^z + h_i^x \sigma_i^x \right) \tag{6.2}$$

where J_i, h_i^z, and h_i^x are uniformly chosen from $J_i \in [\frac{J}{2}, \frac{3J}{2}]$, $h_i^z \in [0, h^z]$, $h_i^x \in [0, h]$ where $h \ll J$ is the regime of interest. The period of the drive is $T = t_0 + t_1$. For $h = 0$ and $t_1 = \pi/2$, the eigenstates of H_{MBL} are eigenstates of the individual σ_i^z. Call such an eigenstate $|\{s_i\}\rangle$ with $s_i = \pm 1$ so that $\sigma_k^z|\{s_i\}\rangle = s_k|\{s_i\}\rangle$. Then $H|\{s_i\}\rangle = (E^+(\{s_i\}) + E^-(\{s_i\}))|\{s_i\}\rangle$ where $E^+(\{s_i\}) = \sum_i (J_i s_i s_{i+1})$ and $E^-(\{s_i\}) = \sum_i (h_i^z s_i)$. The Floquet eigenstates are $e^{it_0 E^-(\{s_i\})/2}|\{s_i\}\rangle \pm e^{-it_0 E^-(\{s_i\})/2}|\{-s_i\}\rangle)$, and the corresponding Floquet eigenvalues are $\pm \exp(it_0 E^+(\{s_i\}))$. Hence, TTSB-2 is satisfied for $h = 0$ and $t_1 = \pi/2$.

6.5 Stability of TTSB

We now argue that the preceding conclusions are no fluke: arbitrary weak local T-periodic perturbations of the Floquet operator, such as non-zero h or deviations of the length of the second time-interval from $\frac{\pi}{2}$, do not destroy TTSB, so long as a reasonable but non-trivial assumption about resonances holds. Ordinarily, there would be little doubt that SSB of a discrete symmetry is stable to weak perturbations at zero-temperature in 1D. But since the symmetry in question is TTS, more care seems necessary.

To build confidence in the stability of TTSB, we can exploit the discrete local connectivity of fully MBL systems: that is, for any eigenstate $|i\rangle$, and point x, there is

only a finite number of eigenstates $|j\rangle$ such that the matrix elements $\langle i| \Phi(x) |j\rangle \neq 0$ for some operator $\Phi(x)$ acting locally at x. In particular, generically the (quasi-)energy difference $\omega_j - \omega_i$ for eigenstates connected in this way will not be close to zero. In systems with such a *local spectral gap*, one expects that *local perturbations perturb locally* [5, 113–115], or more precisely, that there exists a single local unitary $\mathcal{U}$ (that is, a unitary which can be expressed as the time evolution of a local Hamiltonian S) which relates perturbed eigenstates to unperturbed eigenstates [22]. Such a local unitary $\mathcal{U}$ cannot possibly connect short-range correlated states with the long-range correlated eigenstates found above. Therefore, the eigenstates of the perturbed Floquet operator still satisfy TTSB-2.

We make these ideas more precise in the Supplementary Material. There, we construct the unitary $\mathcal{U}$ order-by-order in perturbation theory and show that it remains local at all orders, provided that the local spectral gap condition holds. The skeptic might argue, however, that there will always be rare regions (known as "resonances") in which the local spectral gap is arbitrarily small, and that this will spoil the convergence of the perturbation theory. A rigorous treatment of resonances is a difficult problem; however, the principle of "local perturbations perturb locally" has in fact been proven (given certain reasonable assumptions), at least for a particular model of stationary MBL [23].

On the other hand, for sufficiently *large* perturbations, resonances will proliferate and TTSB (and possibly MBL) will be destroyed. As we argue in the Supplementary Material, we expect this to occur when $\lambda \gtrsim \min\{T^{-1}, J\}$, where λ measures the strength of the deviation of the Hamiltonian from the exactly-solvable point.

6.6 Numerical Analysis of U_f

In order to confirm the stability of TTSB, we will simulate the time evolution for one class of perturbations, namely nonzero h in Eq. (1). In the Supplementary Material, we also numerically demonstrate stability with respect to variations of t_1 (see also Ref. [116]). Throughout, we will take $J = h^z = 1$. First, we use the time-evolving block decimation (TEBD) scheme [117] to compute the time evolution of the short-range correlated initial state $[\cos(\pi/8)|\uparrow\rangle + \sin(\pi/8)|\downarrow\rangle]^{\otimes L}$ for system size $L = 200$ and $h = 0.3$ and $t_0 = 1$. The top panel of Figure 6.1 shows the expectation values of the Pauli spin operators, averaged over 146 disorder configurations and over the spatial interval $i \in [50, 150]$. The TEBD calculations were done with Trotter step $0.01T$ and bond dimension $\chi = 50$. The spin-flip part of the Floquet operator is applied instantaneously, which explains why the oscillation appears to be step-like. After an initial transient, the expectation values oscillate at frequency π/T, half the drive frequency.

Lest a skeptic wonder whether such oscillations continue to much later times or decay just beyond the times accessible by TEBD, we analyze smaller systems by numerical exact diagonalization (ED) of the Floquet operator. To extract the time on which the magnetization decays, we consider the time evolution of the magnetization starting from random initial product states that are polarized in the z direction, and compute the average $Z(t) = \overline{(-1)^t \langle \sigma_i^z(t) \rangle \mathrm{sign}(\langle \sigma_i^z(0) \rangle)}$ over 500 disorder realizations and for a fixed position i. As shown in the bottom panel of Fig. 6.1, there is an initial decay of this quantity, which for the parameters chosen here occurs around $t/T = 10$, and then a plateau that extends up to a time that diverges exponentially in the system size, and even for these small system sizes reaches times comparable to the inverse floating point precision. In the Supplementary Material, we explore

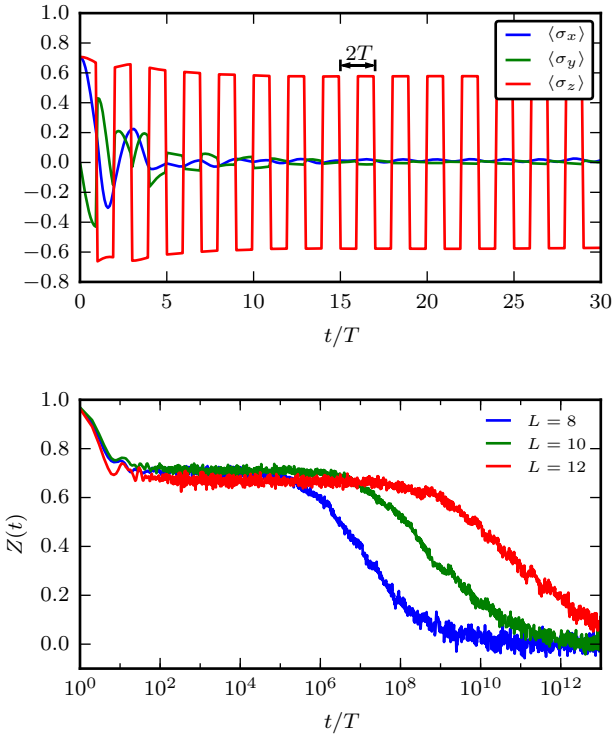

Figure 6.1: The time evolution of a short-range correlated initial state satisfies TTSB-1 for $h = 0.3$. Top Panel: the time-dependence of the disorder-averaged $\langle\sigma_i^x\rangle$, $\langle\sigma_i^y\rangle$, and $\langle\sigma_i^z\rangle$ show that the former two decay rapidly while the latter displays persistent oscillations. (The spin-flip part of sthe Floquet operator is here taken to be applied instantaneously.) Bottom Panel: The decay of the disorder-averaged magnetization, $Z(t)$, as defined in the main text, is found to decay zero on a timescale that diverges exponentially in the system size.

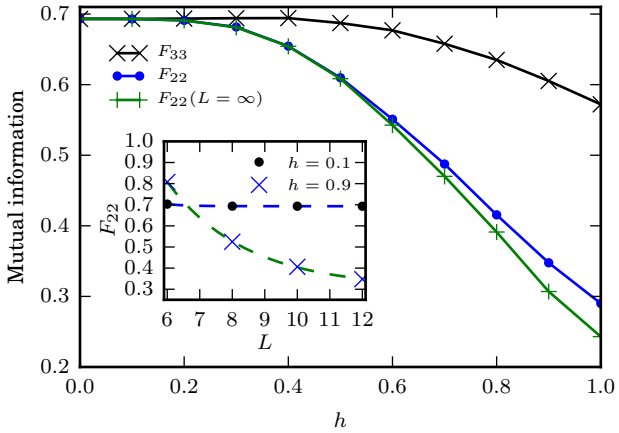

Figure 6.2: The mutual information between the n left- and rightmost sites, F_{nn}, for $n = 2$ and $n = 3$. The main panel shows results for $L = 12$, as well as the extrapolated value of F_{22} for $L \to \infty$. To extrapolate, we fit $F_{22}(L) = F_{22}(\infty) + ce^{-L/\xi}$, with $F_{22}(\infty)$, c and ξ fit parameters. Example fits for $h = 0.1$ and $h = 0.9$ are shown in the inset.

these timescales in more detail and describe ways in which signatures of TTSB can be observed for individual disorder configurations (without disorder averaging).

We now turn to ED of the Floquet operator to verify that TTSB-2 holds. We diagonalize U_f for $L = 6, 8, 10, 12$ sites and 3200 disorder realizations and compute the mutual information between the left- and rightmost n sites, labelled F_{nn}. We find that the mutual information obeys the scaling form: $F_{nn}(h, L) = F_{nn}(g, \infty) + c_n \exp(-L/\xi(h))$. We expect that $F_{nn}(h, \infty) = 0$ in the TTS-invariant phase, $h > h_c$; and $F_{nn}(g, \infty) > 0$ in the TTSB phase, $h < h_c$, with $F_{nn}(g, \infty) \to \ln 2$ as $n \to \infty$. The results in Fig. 6.2 are consistent with this form, with $h_c \gtrsim 1$. It is remarkable that scaling holds even for such small systems, and that $F_{22} \approx F_{33} \approx \log 2$ for $h < 0.3$; evidently, $L = 12$ and $n = 2, 3$ are not so far from the thermodynamic limit.

6.7 Implications of TTSB

In systems exhibiting MBL, it is commonly thought that there exists a *complete* set of local integrals of motion (LIOMs): that is, there is a set of quasi-local operators τ_i^z which commute with each other and with the Floquet operator U_f (or the Hamiltonian in the static case), and such that the eigenvalues of τ_i^z uniquely specify a state in the Hilbert space [21, 24]. Systems with TTSB violate this principle. Indeed, in our model at its soluble point at $h = 0$, the locally indistinguishable states $e^{it_0 E^-(\{s_i\})/2}|\{s_i\}\rangle \pm e^{-it_0 E^-(\{s_i\})/2}|\{-s_i\}\rangle$) No LIOM can distinguish between these two states, so no set of LIOMs can be complete. (Though the existence of a complete set of LIOMs is sometimes taken as the *definition* of MBL, the TTSB phase is still MBL in the sense of, for example, long-time dynamics, since $(U_f)^2$ does have a complete set of LIOMs). By a similar argument, one can show that there does not exist a quasi-local effective Hamiltonian H_{eff} such that $U_f = \exp(-iTH_{\text{eff}})$, whereas for Floquet-MBL systems without TTSB this is likely to be the case [36, 37].

As noted earlier, the oscillations arise from the occurrence of multiplets of states separated in Floquet eigenvalue by Ω/n, where $\Omega = 2\pi/T$ is the drive frequency. We don't use this to identify the TTSB phase in ED because the states are too closely spaced in energy to pick out such multiplets. However, their existence suggests that the system can radiate at frequency Ω/n. The fact that systems oscillating in time can radiate has been cited as an argument against the existence of TTSB [46, 48], since a system maintaining persistent oscillations while simultaneously radiating would be inconsistent with conservation of energy. However, in the Floquet case, this is not an issue since energy is being continually supplied by the drive. Nor does such persistent radiation violate conservation of quasienergy, due the fact that physical (i.e. short-range correlated) states are not quasienergy eigenstates. (For details, see

the Supplementary Material.) On the other hand, in a system that breaks continuous TTS, radiation would cause the system to decay to the ground state, which is reason to doubt that continuous TTSB can occur.

6.8 Discussion

The model Eqs. (6.1) and (6.2) is soluble at $h = 0$ because the operator $\exp(i\frac{\pi}{2}\sum_i \sigma_i^x) = \prod_i i\sigma_i^x$ that is applied at the beginning of each driving cycle maps eigenstates of H_{MBL} to eigenstates of H_{MBL}. Analogous soluble models can be constructed for $\mathbb{Z}_n$ spins in which time translation by T is broken down to nT.

Our model has no symmetries, other than discrete time-translation symmetry. Hence, the $\ln 2$ that we find in the mutual information must be a consequence of TTSB; there is no other symmetry to break. However, TTSB can occur in models with other symmetries. A particularly interesting example is given by symmetry-protected topological (SPT) phases of Floquet-MBL systems [70, 102, 103, 107]. In d-dimensions, such phases are classified by $H^{d+1}(G \times \mathbb{Z}, U(1)) = H^{d+1}(G, U(1)) \times H^d(G, U(1))$ [107]. The second factor on the right-hand-side of this equality is a $(d-1)$-dimensional SPT phase that is 'pumped' to the boundary with each application of the Floquet operator, thereby breaking TTS on the boundary.

The definition TTSB-1 naturally suggests an experiment that could observe the phenomenon predicted here. Signatures of MBL have been observed in trapped systems of neutral atoms [118] and trapped ions [119], and signatures of single-particle localization have been seen in coupled superconducting qubits [120]. In any of these systems, one can prepare an arbitrary initial product state, evolve to late times according to a drive in the class considered here, and measure the "spins" in the desired basis. Our prediction is that persistent oscillations will be observed at a fraction of

the drive frequency.

Chapter 7

Prethermal Floquet phases

So far, we have discussed Floquet phases of matter in the context of Floquet-MBL systems, which requires strong quenched disorder. In this chapter, we show that the signatures of Floquet topological phases and Floquet time crystals can also arise in clean systems without MBL. The difference is that, since a generic Floquet system without MBL must inevitably heat to infinite temperature at late times, the Floquet phases of matter manifest themselves only at intermediate times. Nevertheless, this "prethermal" time window can last until a heating time that is exponentially large in a small parameter.

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7.1 Introduction

Much of condensed matter physics revolves around determining which distinct *phases of matter* can exist as equilibrium states of physical systems. Within a phase, the properties of the system vary continuously as external parameters are varied, while different phases are separated by phase transitions, at which the properties change abruptly. An extremely rich set of observed phases can be characterized by symmetry. The best known example is *spontaneous symmetry-breaking*, as a result of which the equilibrium state of the system is less symmetrical than the Hamiltonian. More recently, a set of uniquely quantum phases—*symmetry-protected topological* (SPT) phases [71–89], including topological insulators [121, 122], and *symmetry-enriched topological* (SET) phases [91–97]—has been discovered. These phases, while symmetric, manifest the symmetry in subtly anomalous ways, and are distinct only as long as the symmetry is preserved. We can collectively refer to these three classes of phases as *symmetry-protected phases of matter*.

Thus far, the concept of symmetry-protected phases of matter has not been as succesful in describing systems away from equilibrium. Recently, however, it was realized that certain periodically-driven "Floquet" systems can exhibit distinct phases, akin to those of equilibrium systems [69]. In this paper, we show that there is, in fact, a very general set of non-equilibrium conditions under which such phases can arise, due to a remarkable phenomenon called "pre-thermalization". In Floquet systems, pre-thermalization occurs when a time-dependent change of basis removes all but a small residual time-dependence from the Hamiltonian, and thus allows the properties of the system to be mapped approximately onto those of a system in thermal equilibrium. The residual time-dependence is nearly exponentially-small in a large parameter α of the original Hamiltonian of the system. One can then talk about a

"pre-thermal regime" in which the system reaches a thermal equilibrium state with respect to the approximate effective time-independent Hamiltonian that results from neglecting the small residual time dependence. In this regime, the system can exhibit phases and phase transitions analogous to those seen in thermal equilibrium, such as symmetry-protected phases. Nevertheless, in the original non-rotating frame, the system remains *very far from thermal equilibrium* with respect to the instantaneous Hamiltonian at any given time. After the characteristic time t_*, which is nearly exponentially-long in the large parameter α, other physics (related the residual time-dependence) takes over.

In this paper, we show that pre-thermal systems can also exhibit phases of matter that cannot exist in thermal equilibrium. These novel phases can also be understood as symmetry-protected phases but of a variety that cannot occur in thermal equilibrium: these phases are protected by *discrete time-translation symmetry*. While these include topological phases protected by time-translation symmetry [70, 102, 103, 107], perhaps the most dramatic of these are "time crystals" that spontaneously break time-translation symmetry. The idea of time crystals that spontaneously break *continuous* time-translation symmetry was first proposed by Wilczek and Shapere [43, 44]f, but finding a satisfactory *equilibrium* model has proven difficult and some no-go theorems exist [45–50, 52]. In this paper, we construct pre-thermal "Floquet time crystals", which spontaneously break the *discrete time-translation symmetry* of periodically-driven systems [123] [1]. Floquet time crystals are the focus of this paper, but as a by-product of our analysis, we also find pre-thermal – i.e. non-equilibrium – time crystals that spontaneously break *continuous* time-translation symmetry. We also construct SPT and SET phases protected by discrete time-translation symmetry.

[1]For an alternative view of such systems that focuses on other symmetries of the discrete time-translation operator, see Refs. [33, 69, 108].

Periodically-driven systems have long been considered an unlikely place to find interesting phases of matter and phase transitions since generic driven closed systems will heat up to infinite temperature [13–15]. It has been known that the heating problem can be avoided [15, 35–37, 67] if the system is integrable or if the system has sufficiently strong quenched disorder that it undergoes many-body localization (MBL) [16–22, 24, 109, 110]. However, integrability relies on fine-tuning, and MBL requires the system to be completely decoupled from the environment [124–132]. Furthermore, the disorder must be sufficiently strong, which may be difficult to realize in an experiment but does not constitute fine-tuning.

The central result of this paper is therefore to show that pre-thermalization makes it possible for non-equilibrium phases protected by time-translation symmetry to occur in more generic non-equilibrium systems without the need for fine-tuning, strong disorder, or complete decoupling from the environment. Remarkably, these non-equilibrium phases and phase transitions, which have have no direct analogues in thermal equilbrium, have a mathematical formulation that is identical to that of equilibrium phases, though with a different physical interpretation. Since MBL is not a requirement, it is conceivable that pre-thermal time-translation protected phases could survive the presence of coupling to an environment. In fact, we will discuss a plausible scenario by which these phases can actually be *stabilized* by coupling to a sufficiently cold thermal bath, such that the system remains in the pre-thermal regime even at infinite time.

The structure of the paper will be as follows. In Section 7.2, we state our main technical result. In Section 7.3, we apply this to construct prethermal Floquet time crystals which spontaneously break discrete time-translation symmetry. In Section 7.4, we show that a *continuous* time-translation symmetry can also also be spontaneously broken in the pre-thermal regime for a system with a time-independent

Hamiltonian. In Section 7.5, we outline how our methods can also be applied to construct SPT and SET phases protected by time-translation symmetry. In Section 7.6, we discuss what we expect to happen for non-isolated systems coupled to a cold thermal bath. Finally, we discuss implications and interpretations in Section 7.7.

7.2 Pre-Thermalization Results

The simplest incarnation of pre-thermalization occurs in periodically-driven systems when the driving frequency ν is much larger than all of the local energy scales of the instantaneous Hamiltonian [38–42] (see also Refs. [133–135] for numerical results). The key technical result of our paper will be a theorem generalizing these results to other regimes in which the driving frequency is not greater than *all* the local scales of the Hamiltonian, but there is nevertheless some separation of energy scales. This will allow us to show that time-translation protected phases can exist in the pre-thermal regime. More precisely, in the models that we construct, one local coupling strength is large and the others are small; the drive frequency is large compared to the small couplings, and the parameter α is the ratio of the drive frequency to the largest of the small local couplings. The term in the Hamiltonian with large coupling must take a special form, essentially that of a symmetry generator, that allows it to avoid heating the system.

Accordingly, we will consider a time-dependent Hamiltonian of the form $H(t) = H_0(t) + V(t)$, where $H_0(t)$ and $V(t)$ are periodic with period T. We assume that $\lambda T \ll 1$, where λ is the local energy scale of V. We further assume that $H_0(t)$ has the property that it generates a trivial time evolution over N time cycles: $U_0(NT, 0) =$

$U_0(T,0)^N = 1$, where

$$U_0(t_2,t_1) = \mathcal{T}\exp\left(-i\int_{t_1}^{t_2} H_0(t)\right)dt, \quad \mathcal{T} = \text{time-ordering}. \tag{7.1}$$

We claim that such a time evolution will exhibit pre-thermalizing behavior for $\lambda T \ll 1/N$ even if the local energy scale of $H_0(t)$ is comparable to $1/T$. In other words, such a system exhibits pre-thermalizing behavior when the frequency is large compared some of the couplings (those in $V(t)$) but not others (those in $H_0(t)$), as promised in the introduction.

An easy way to see that this claim is true is to work in the interaction picture (treating V as the "interaction"). Then we see that the time evolution of the total Hamiltonian $H(t)$ over N time cycles is given by

$$U(NT,0) = \mathcal{T}\exp\left(-i\int_0^{NT} V^{\text{int}}(t)dt\right), \tag{7.2}$$

where $V^{\text{int}}(t) = U_0(0,t)^\dagger V(t) U_0(0,t)$ is the representation of $V(t)$ in the interaction picture, and $U_0(0,NT) = 1$ ensures that the time evolution operator Eq. (7.2) is the same in the interaction and Schrödinger pictures. If we rescale time as $t \to t/\lambda$, then Eq. (7.2) describes a system being driven at the large frequency $\nu = 1/(\lambda NT)$ by a drive of local strength 1, which by the results of Refs. [38–42] will exhibit pre-thermalizing behavior for $\nu \gg 1$.

On the other hand, since the above argument for pre-thermalization required coarse-graining the time period from T to NT, it prevents us from identifying phases of matter, such as time crystals or Floquet SPT phases, that are protected by time translation symmetry. The problem is that the time-translation symmetry by T is what allows different phases of matter to be sharply distinguished. This symmetry is

still present, of course (because the coarse-graining is a feature of our description of the system, not the system itself), but it is no longer manifest. Therefore, it is not at all transparent how to understand the different phases of matter in this picture.

In order to proceed further, we will need a new approach. In this paper, we develop a *new* formalism that analyzes $U(T,0)$ itself rather than $U(NT,0)$, allowing the effects of time-translation symmetry to be seen in a transparent way. Our central tool is a theorem that we will prove, substantially generalizing those of Abanin et al.[39]. A more precise version of our theorem will be given momentarily, and the proof will be given in Appendix C; the theorem essentially states that there exists a time-independent local unitary rotation $\mathcal{U}$ such that $U_{\rm f} \approx \widetilde{U}_f = \mathcal{U}^\dagger(Xe^{-iDT})\mathcal{U}$, where $X = U_0(T,0)$ is the time evolution of H_0 over one time cycle, and D is a quasi-local Hamiltonian that commutes with X. The dynamics at stroboscopic times are well-approximated by $\widetilde{U}_f$ for times $t \ll t_*$, where $t_* = e^{O(1/(\lambda T[\log(1/\lambda T)]^3))}$. This result combines ideas in Ref. [39] about (1) the high-frequency limit of driven systems and (2) approximate symmetries in systems with a large separation of scales. Recall that, in the high-frequency limit of a driven system, the Floquet operator can be approximated by the evolution (at stroboscopic times) due a time-independent Hamiltonian, $U_f \approx \exp(-iTH_{\rm eff})$. Meanwhile, in a static system with a large separation of scales, $H = -uL + D_0$, where u is much larger than the couplings in D_0 but $[L, D_0] \neq 0$, Ref. [39] shows that there is a unitary transformation $\mathcal{U}$ such that $\mathcal{U}H\mathcal{U}^\dagger \approx -uL + D$ where $[L, D] = 0$, i.e. the system has an approximate symmetry generated by $\mathcal{U}^\dagger L\mathcal{U}$. Our theorem states that, after a time-independent local unitary change of basis, a periodic Hamiltonian $H(t) = H_0(t) + V(t)$, with $H_0(t)$ satisfying the condition given above, can be approximated, as far as the evolution at stroboscopic times is concerned, by a binary drive that is composed of two components: (1) the action of $H_0(t)$ over one cycle, namely $U_0(T,0)$ and (2) a static Hamiltonian that is invariant under the

symmetry generated by $U_0(T,0)$.

These results might seem surprising, because they imply that the evolution over one time period commutes with a symmetry $X = U_0(T,0)$ [or $\mathcal{U}X\mathcal{U}^\dagger$ in the original basis], despite the fact that the microscopic time-dependent Hamiltonian $H(t)$ had no such symmetry. We interpret this "hidden" symmetry as a shadow of the discrete time-translation symmetry. (For example, the evolution over N time periods also commutes with $\mathcal{U}X\mathcal{U}^\dagger$, but if we add weak NT-periodic perturbations to break the discrete time-translation symmetry then this is no longer the case.) Thus, our theorem is precisely allowing us to get a handle on the implications of discrete time-translation symmetry. Compare Ref. [33], where a similar "hidden" symmetry was constructed for many-body-localized Floquet time crystals.

The preceding paragraphs summarize the physical meaning of our theorem. A more precise statement of the theorem, although it is a bit more opaque physically, is useful because it makes the underlying assumptions manifest. The statement of the theorem makes use of an operator norm $\|O\|_n$ that measures the average over one Floquet cycle of the size of the local terms whose sum makes up a Hamiltonian; the subscript n parametrizes the extent to which the norm suppresses the weight of operators with larger spatial support. An explicit definition of the norm is given in Appendix C. The theorem states the following.

Theorem 1. *Consider a periodically-driven system with Floquet operator:*

$$U_f = \mathcal{T}\exp\left(-i\int_0^T H(t)dt\right) \tag{7.3}$$

where $H(t) = H_0(t) + V(t)$, and $X \equiv U_0(0,T)$ satisfies $X^N = 1$ for some integer N. We assume that $H_0(t)$ can be written as a sum $H_0(t) = \sum_i h_i(t)$ of terms acting only

on single sites i. Define $\lambda \equiv \|V\|_1$. Assume that

$$\lambda T \leq \frac{\gamma \kappa_1^2}{N+3}, \quad \gamma \approx 0.14. \tag{7.4}$$

Then there exists a (time-independent) unitary $\mathcal{U}$ such that

$$\mathcal{U}\, U_f \mathcal{U}^\dagger = X\, \mathcal{T} \exp\left(-i \int_0^T [D + E + V(t)] dt \right) \tag{7.5}$$

where D is local and $[D, X] = 0$; D, E are independent of time; and

$$\|V\|_{n_*} \leq \lambda \left(\frac{1}{2}\right)^{n_*} \tag{7.6}$$

$$\|E\|_{n_*} \leq \lambda \left(\frac{1}{2}\right)^{n_*} \tag{7.7}$$

The exponent n_* is given by

$$n_* = \frac{\lambda_0/\lambda}{[1 + \log(\lambda_0/\lambda)]^3}, \quad \lambda_0 = \frac{(\kappa_1)^2}{72(N+3)(N+4)T} \tag{7.8}$$

Furthermore,

$$\|D - \overline{V}\|_{n_*} \leq \mu(\lambda^2/\lambda_0), \quad \mu \approx 2.9, \tag{7.9}$$

where

$$\begin{aligned} \overline{V} &= \frac{1}{NT} \int_0^{NT} V^{\text{int}}(t) dt \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X^{-k} \left(\frac{1}{T} \int_0^T V^{\text{int}}(t) dt \right) X^k. \end{aligned} \tag{7.10}$$

The proof is given in Appendix C. The statement of the theorem makes use of a number κ_1. It is chosen so that $\|H\|_1$ is finite; the details are given when the norm is given in Appendix C.

Unpacking the theorem a bit in order to make contact with the discussion above, we see that it states that there is a time-independent unitary operator $\mathcal{U}$ that transforms the Floquet operator into the form Xe^{-iDT} with $[D, X] = 0$ and local D, up to corrections that are exponentially small in $n_* \sim 1/(\lambda T[\ln(1/\lambda T)]^3)$. These "error terms" fall into two categories: time-independent terms that do not commute with X, which are grouped into E; and time-dependent terms, which are grouped into $V(t)$. Both types of corrections are exponentially-small in n_*. Since they are exponentially-small $\|E\|_{n_*}, \|V\|_{n_*} \sim (1/2)^{n_*}$, these terms do not affect the evolution of the system until exponentially-long times, $t_* \sim e^{Cn_*}$ (for some constant C). It is not possible to find a time-independent unitary transformation that exactly transforms the Floquet operator into the form Xe^{-iDT} because the system must, eventually, heat up to infinite temperature and the true Floquet eigenstates are infinite-temperature states, not the eigenstates of an operator of the form Xe^{-iDT} with local D. In the interim, however, the approximate Floquet operator Xe^{-iDT} leads to Floquet time crystal behavior, as we will discuss in the next Section.

The proof of Theorem 1 constructs $\mathcal{U}$ and D through a recursive procedure, which combines elements of the proofs of pre-thermalization in driven and undriven systems given by Abanin et al. [39].

In the case of pre-thermal undriven systems, the theorem we need has essentially already been given in Ref. [39], but we will restate the result in a form analogous with Theorem 1, which entails some slightly different bounds (however, they are easily derivable using the techniques of Ref. [39]).

Theorem 2. *Consider a time-independent Hamiltonian H of the form*

$$H = -uL + V, \tag{7.11}$$

where $e^{2\pi i L} = 1$. We assume that L can be written as a sum $L = \sum_i L_i$ of terms acting only on single sites i. Define $\lambda \equiv \|V\|_1$, and assume that

$$\lambda/u \leq \gamma \kappa_1^2, \quad \gamma \approx 0.14. \tag{7.12}$$

Then there exists a local unitary transformation $\mathcal{U}$ such that

$$\mathcal{U} H \mathcal{U}^\dagger = -uL + D + \hat{V} \tag{7.13}$$

where $[L, D] = 0$ and $\hat{V}$ satisfies

$$\|\hat{V}\|_{n_*} \leq \lambda \left(\frac{1}{2}\right)^{n_*} \tag{7.14}$$

where

$$n_* = \frac{\lambda_0/\lambda}{[1 + \log(\lambda_0/\lambda)]^3}, \quad \lambda_0 = \frac{u\kappa_1^2}{144}. \tag{7.15}$$

Furthermore,

$$\|D - \langle V \rangle\|_{n_*} \leq \mu(\lambda^2/\lambda_0), \quad \mu \approx 2.9, \tag{7.16}$$

Here, we have defined, following Ref. [39], the symmetrized operator $\langle V \rangle$ according to

$$\langle V \rangle \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \, e^{iL\theta} \, V \, e^{-iL\theta} \tag{7.17}$$

which, by construction, satisfies $[L, \langle V \rangle] = 0$.

7.3 Pre-thermalized Floquet time crystals

7.3.1 Basic Picture

The results of the previous section give us the tools that we need to construct a model which is a Floquet time crystal in the pre-thermalized regime. Our approach is reminiscent of Ref. [33], where the Floquet-MBL time crystals of Ref. [123] were reinterpreted in terms of a spontaneously broken "emergent" $\mathbb{Z}_2$ symmetry. Here, "emergent" refers to the fact that the symmetry is in some sense hidden – its form depends on the parameters on the Hamiltonian in a manner that is not *a priori* known. Furthermore, it is not a symmetry of the Hamiltonian, but is a symmetry of the Floquet operator.

In particular, suppose that we have a model where we can set $X = \prod_i \sigma_i^x$. (Thus $N = 2$). We then have $U_f \approx \widetilde{U}_f = \mathcal{U}^\dagger(Xe^{-iDT})\mathcal{U}$, where the quasi-local Hamiltonian D by construction respects the Ising symmetry generated by X. This Ising symmetry corresponds to an *approximate* "emergent" symmetry $\mathcal{U}X\mathcal{U}^\dagger$ of U_f ("emergent" for the reason stated above and approximate because it an exact symmetry of $\widetilde{U}_f$, not U_f, and therefore is approximately conserved for times $t \ll t_*$.) Suppose that D *spontaneously breaks* the symmetry X below some finite critical temperature τ_c. For example, working in two dimensions or higher, we could have $D = -J\sum_{\langle i,j\rangle} \sigma_i^z\sigma_j^z$ plus additional smaller terms of strength which break integrability. We will be interested in the regime where the heating time $t_* \gg t_{\mathrm{pre-thermal}}$, where $t_{\mathrm{pre-thermal}}$ is the thermalization time of D.

Now consider the time evolution $|\psi(t)\rangle$, starting from a given short-range correlated state $|\psi(0)\rangle$. We also define the rotated states $|\widetilde{\psi}(t)\rangle = \mathcal{U}\,|\psi(t)\rangle$. At stroboscopic times $t = nT$, we find that $|\widetilde{\psi}(nT)\rangle = (Xe^{-iDT})^n\,|\widetilde{\psi}(0)\rangle$. Since $(Xe^{-iDT})^2 = e^{-2iDT}$, we see that at even multiples of the period, $t = 2nT$, the time evolution of $|\widetilde{\psi}(t)\rangle$ is

described by the time-independent Hamiltonian D. Thus, we expect that, after the time $t_{\mathrm{pre-thermal}}$, the system appears to be in a thermal state of D at temperature τ. Thus, $|\widetilde{\psi}(2nT)\rangle \langle\widetilde{\psi}(2nT)| \approx \widetilde{\rho}$, where $\widetilde{\rho}$ is a thermal density matrix for D at some temperature τ, and the approximate equality means that the expectation values of local observables are approximately the same. Note that for $\tau < \tau_c$, the Ising symmetry of D is spontaneously broken and $\widetilde{\rho}$ must either select a nonzero value for the order parameter $M_{2n} = \langle\sigma_i^z\rangle_{\widetilde{\rho}}$ or have long-range correlations. The latter case is impossible given our initial state, as long-range correlations cannot be generated in finite time. Then, at odd times $t = (2n+1)T$, we have

$$|\widetilde{\psi}((2n+1)T)\rangle \langle\widetilde{\psi}((2n+1)T)| \approx (Xe^{-iDT})\widetilde{\rho}(e^{iDT}X) \tag{7.18}$$

$$= X\widetilde{\rho}X \tag{7.19}$$

(since $\widetilde{\rho}$ commutes with D.) Therefore, at *odd* times, the order parameter

$$M_{2n+1} = \langle\sigma_i^z\rangle_{X\widetilde{\rho}X} = -M_{2n}. \tag{7.20}$$

Thus, the state of the system at odd times is different from the state at even times, and time translation by T is spontaneously broken to time translation by $2T$.

The above analysis took place in the frame rotated by $\mathcal{U}$. However, we can also consider the expectation values of operators in the original frame, for example $\langle\psi(t)|\,\sigma_i^z\,|\psi(t)\rangle = \langle\widetilde{\psi}(t)|\,\mathcal{U}^\dagger\sigma_i^z\mathcal{U}\,|\widetilde{\psi}(t)\rangle$. The rotation $\mathcal{U}$ is close to the identity in the regime where the heating time is large[2], so σ_i^z has large overlap with $\mathcal{U}^\dagger\sigma_i^z\mathcal{U}$ and therefore will display fractional frequency oscillations. We recall that the condition for fractional frequency oscillations in the pre-thermalized regime is that (a) D must

[2]Specifically, it follows from the construction of $\mathcal{U}$ that $\mathcal{U} = 1 + O(\lambda T)$, and $\lambda T \ll 1$ is the regime where the heating time is large.

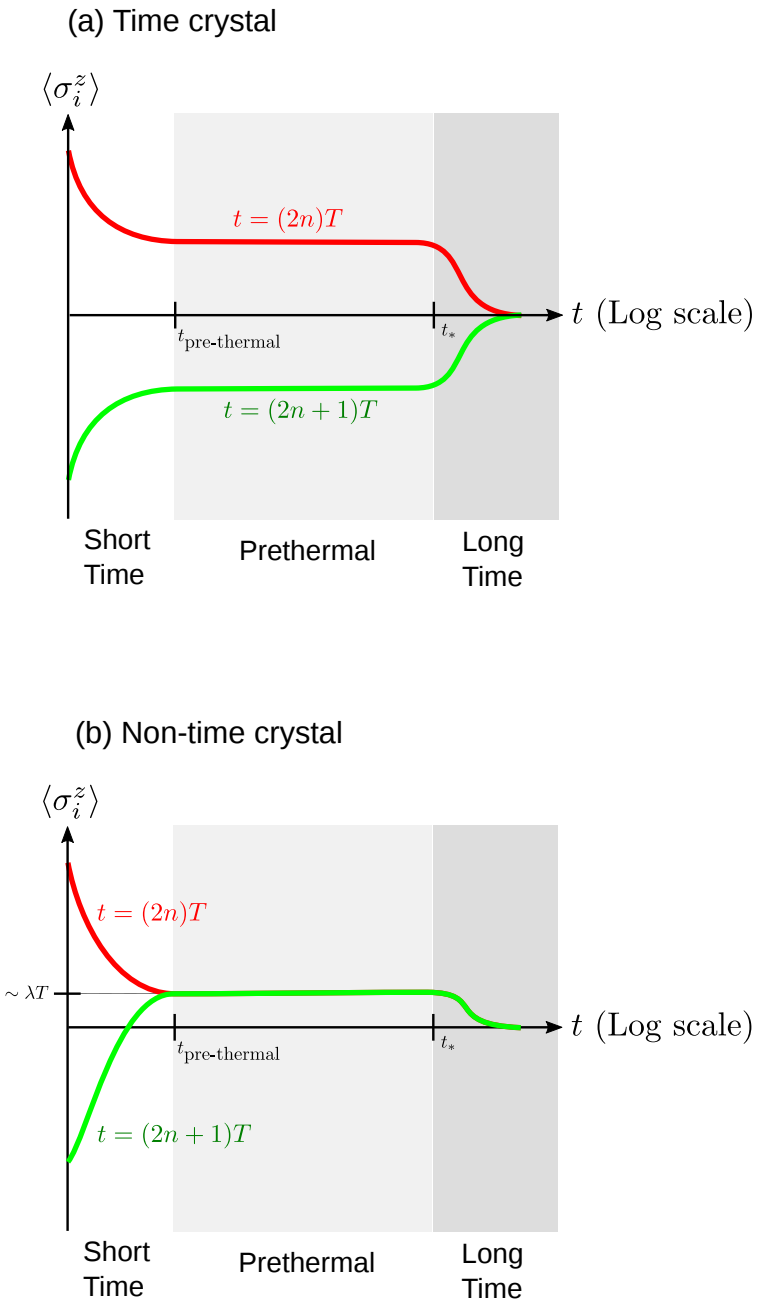

Figure 7.1: The expected time dependence of $\langle \sigma_i^z \rangle$ at stroboscopic times, starting from a state which is low-temperature with respect to $\mathcal{U}D\mathcal{U}^\dagger$ (for example, for a state with all spins polarized in the z direction.), in (a) the pre-thermal time crystal phase, and (b) the non-time crystal pre-thermal phase.

spontaneously break the Ising symmetry X up to a finite critical temperature τ_c; and (b) the energy density with respect to D of $\mathcal{U}\,|\psi(0)\rangle$ must correspond to a temperature $\tau < \tau_c$. In Figure 7.1, we show the expected behavior at low temperatures τ and contrast it with the expected behavior in a system which is not a time crystal in the pre-thermal regime.

7.3.2 Example: periodically-driven Ising spins

Let us now consider a concrete model which realizes the behavior descrived above. We consider an Ising ferromagnet, with a longitudinal field applied to break the Ising symmetry explicitly, and driven at high frequency by a very strong transverse field. Thus, we take

$$H(t) = H_0(t) + V, \tag{7.21}$$

where

$$H_0(t) = -\sum_i h^x(t)\sigma_i^x \tag{7.22}$$

$$V = -J\sum_{\langle i,j\rangle} \sigma_i^z\sigma_j^z - h^z\sum_i \sigma_i^z, \tag{7.23}$$

and we choose the driving profile such that

$$\int_0^T h^x(t)dt = \frac{\pi}{2}, \tag{7.24}$$

ensuring that the "unperturbed" Floquet operator U_0 implements a π pulse, $X = \prod_i \sigma_x^i$, and we can set $N = 2$. (If the driving does not exactly implement a π pulse, this is not a significant problem since we can just incorporate the difference into V.) This implies that $h_x \sim 1/T$, and we assume that $h^z \lesssim J \ll 1/T$.

Then by the results of Section 7.2 (with J playing the role of λ here), we find a quasi-local Hamiltonian $D = \overline{V} + \frac{1}{T}O((JT)^2)$, where

$$\overline{V} = \frac{1}{2T}\int_0^{2T} V_{\text{int}}(t)dt. \tag{7.25}$$

In particular, in the case where the π pulse acts instanteously, so that

$$h^x(t) = \frac{\pi}{2}\sum_{k=-\infty}^{\infty}\delta(t-kT), \tag{7.26}$$

we find that

$$\overline{V} = -J\sum_{\langle i,j\rangle}\sigma_i^z\sigma_j^z \tag{7.27}$$

(this Hamiltonian is integrable, but in general the higher order corrections to D will destroy integrability.) More generally, if the delta function is smeared out so that the π pulse acts over a time window δ, the corrections from Eq. (7.27) will be at most of order $\sim J\delta/T$. Therefore, so long as $\delta \ll T$, then in two dimensions or higher, the Hamiltonian D will indeed spontaneously break the Ising symmetry up to some finite temperature τ_c, and we will observe the time-crystal behavior described above.

7.3.3 Field Theory of the Pre-Thermal Floquet Time Crystal State

The universal behavior of a pre-thermal Floquet time crystal state can be encapsulated in a field theory. For the sake of concreteness, we derive this theory from the model analyzed in the previous section. The Floquet operator can be written, up to

nearly exponential accuracy, as:

$$U_{\rm f} \approx \mathcal{U}(Xe^{-iDT})\mathcal{U}^\dagger \tag{7.28}$$

Consequently, the transition amplitude from an initial state $|\psi_i\rangle$ at time t_0 to a final state $|\psi_f\rangle$ at time t_0+mT can be written in the following form, provided $t_{\rm pre-thermal} < t_0 < t_0 + mT < t_*$:

$$\begin{aligned}\langle\psi_f|\left(U_{\rm f}\right)^m|\psi_i\rangle &= \langle\psi_f|\,\mathcal{U}(Xe^{-iDT})^m\mathcal{U}^\dagger\,|\psi_i\rangle \\ &= \langle\tilde{\psi}_f|\,e^{-iDmT}\,|\tilde{\psi}_i\rangle\end{aligned} \tag{7.29}$$

where $|\tilde{\psi}_i\rangle \equiv \mathcal{U}^\dagger|\psi_i\rangle$ and $|\tilde{\psi}_f\rangle \equiv X^m\,\mathcal{U}^\dagger|\psi_f\rangle$; recall that X^m is 1 or X for, respectively, m even or odd.

The second line of Eq. (7.29) is just the transition amplitude for the quantum transverse field Ising model in $(d+1)$-dimensional spacetime, with $d \geq 2$. The model has nearest-neighbor interaction (7.27) together with higher-order terms that are present in the full expression for D. Hence, it can be represented by the standard functional integral for the continuum limit of the Ising model:

$$\langle\tilde{\psi}_f|\,e^{-iDmT}\,|\tilde{\psi}_i\rangle \;= \int \mathcal{D}\varphi\, e^{i\int d^dx\,dt\,\left[\frac{1}{2}K(\partial_t\varphi)^2-\frac{v^2}{2}K(\nabla\varphi)^2-U(\varphi)\right]} \tag{7.30}$$

where $U(\varphi)$ has minima at $\varphi = \pm\varphi_0$ when the parameters in the Ising model place it in the ordered phase. This functional integral is only valid for wavevectors that are less that a wavevector cutoff: $|q| < \Lambda$, where $\Lambda \ll 1/a$ and a is the spatial lattice spacing. Although the right-hand side of (7.30) has a continuous time variable, it is

only equal to the original peridiodically-driven problem for stroboscopic times $t = mT$ for $m \in \mathbb{Z}$. Note the left-hand side of (7.30) is also well-defined for arbitrary times, i.e. for continuous m, although it, too, only corresponds to the original problem for integer m. Thus the continuous-time effective field theory has a frequency cutoff Λ_ω that we are free to choose. Although the functional integral only corresponds to the original problem for stroboscopic times, the functional integral is well-defined for all times. As a result of the factor of X in $U_{\rm f}$, the field φ is related to the Ising spin according to $\varphi(x, kT) \sim (-1)^k \sigma(x, kT)$. In other words, the field φ in the functional integral has the intepretation of the temporally-staggered magnetization density, just as, in the corresponding description of an Ising anti-ferromagnet, this field would be the spatially-staggered magnetization. Discrete time-translation symmetry, $t \to t+T$ has the following action: $\varphi \to -\varphi$. Thus, the symmetry-breaking phase, in which $\varphi = \pm\varphi_0$, is a pre-thermal Floquet time crystal, in which TTSB occurs, as expected.

The rotated Floquet operator $\mathcal{U}^\dagger U_{\rm f} \mathcal{U}$ has an approximate $\mathbb{Z}_2$ symmetry generated by the operator X since $\mathcal{U}^\dagger U_{\rm f} \mathcal{U} \approx X e^{-iDT}$ and $[D, X] = 0$. Hence, $\mathcal{U}^\dagger X \mathcal{U}$ commutes with the (unrotated) Floquet operator $U_{\rm f}$. It is not a microscopic symmetry in the conventional sense, since $\mathcal{U}^\dagger X \mathcal{U}$ does not commute with the time-dependent Hamiltonian $H(t)$, except for special fine-tuned points in the Floquet time crystal phase. However, since it commutes with the Floquet operator, it is a symmetry of the continuum-limit field theory (7.30). (See Ref. [33] for a discussion of Floquet time crystals in the MBL context that focuses on such symmetries, sometimes called "emergent symmetries".) Within the field theory (7.30), this symmetry acts according to $\varphi \to -\varphi$, i.e. it acts in precisely the same way as time-translation by a single period. Again, this is analogous to the case of an Ising anti-ferromagnet, but with the time-translation taking the place of spatial translation. Thus, it is possible to view the symmetry-breaking pattern as $\mathbb{Z}_{\rm TTS} \times \mathbb{Z}_2 \to \mathbb{Z}$. The unbroken $\mathbb{Z}$ symmetry

is generated by the combination of time-translation by one period and the action of $\mathcal{U}^\dagger X \mathcal{U}$.

However, there is an important difference between a Floquet time crystal and an Ising antiferromagnet. In the latter case, it is possible to explicitly break the the Ising symmetry without breaking translational symmetry (e.g. with a uniform longitudinal magnetic field) and vice versa (e.g. with a spatially-oscillating exchange coupling). In a Floquet time crystal, this is not possible because there is always a $\mathbb{Z}_2$ symmetry $\mathcal{U}^\dagger X \mathcal{U}$ regardless of what small perturbation (compared to the drive frequency) is added to the Hamiltonian. The only way to explicitly prevent the system from having a $\mathbb{Z}_2$ symmetry is to explicitly break the time-translation symmetry. Suppose the Floquet operator is $\mathcal{U} X e^{-iDT} \mathcal{U}^\dagger$. When a weak perturbation with period $2T$ is added, the Floquet operator can be written in the approximate form $\mathcal{U}' e^{-2i(D+Y)T} (\mathcal{U}')^\dagger$ where Y is due to the doubled-period weak perturbation, but it is not possible to guarantee that $[X, Y] = 0$. Thus there is a symmetry generated by an operator of the form $\mathcal{U}^\dagger X \mathcal{U}$ only if time-translation symmetry is present – i.e. it is a consequence of time-translation symmetry and pre-thermalization.

This functional integral is computed with boundary conditions on φ at $t = t_0$ and $t_0 + mT$. Time-ordered correlation functions can be computed by inserting operators between the factors of U_f. However, if we are interested in equal-time correlation functions (at stroboscopic times $t = kT$),

$$\langle\psi| \hat{O}(x, kT) \hat{O}(0, kT) |\psi\rangle \equiv \langle\psi| \left(U_\mathrm{f}\right)^{-k} \hat{O}(x, 0) \hat{O}(0, 0) \left(U_\mathrm{f}\right)^{k} |\psi\rangle \quad (7.31)$$

then we can make use of the fact that the system rapidly pre-thermalizes to replace

$\left(U_{\rm f}\right)^k|\psi\rangle$ by a thermal state:

$$\langle\psi|\left(U_{\rm f}\right)^{-k}\hat{O}(x,0)\hat{O}(0,0)\left(U_{\rm f}\right)^k|\psi\rangle = \mathrm{tr}(e^{-\beta D}\hat{O}(x)\hat{O}(0)) \quad (7.32)$$

where β is determined by $\mathrm{tr}(e^{-\beta D}D) = \langle\psi|D|\psi\rangle$. The latter has an imaginary-time functional integral representation:

$$\mathrm{tr}(e^{-\beta D}\hat{O}(x)\hat{O}(0)) = \int \mathcal{D}\varphi\, e^{-\int d^dx\, d\tau \left[\frac{1}{2}K(\partial_\tau\varphi)^2 + \frac{v^2}{2}K(\nabla\varphi)^2 + U(\varphi)\right]} \quad (7.33)$$

This equation expresses equal-time correlation functions in a pre-thermal Floquet time crystal in terms of the standard imaginary-time functional integral for the Ising model but with the understanding that the field φ in the functional integral is related to the Ising spins in the manner noted above.

In order to compute unequal-time correlation functions, it is convenient to use the Schwinger-Keldysh formalism [136, 137] (see Ref. [138] for a modern review). This can be done by following the logic that led from the first line of Eq. (7.29) to the second and thence to Eq. (7.30). This will be presented in detail elsewhere [139].

We close this subsection by noting that the advantage of the field theory formulation of a pre-thermal Floquet time crystal is the salience of the similarity with the equilibrium Ising model; for instance, it is clear that the transition out of the Floquet time crystal (e.g. as a function of the energy of the initial state) in the pre-thermal regime is an ordinary Ising phase transition. The disadvantage is that it is difficult to connect it to measurable properties in a quantitative way because the field φ has a complicated relationship to the microscopic degrees of freedom.

7.3.4 Relation to formal definitions of time crystals

In the above discussion, we have implicitly been adopting an "operational" definition of time-crystal: it is a system in which, for physically reasonable initial states, the system displays oscillations at a frequency other than the drive frequency forever (or at least, in the pre-thermal case, for a nearly exponentially long time.) This is a perfectly reasonable definition of time crystal, but it has the disadvantage of obscuring the analogies with spontaneous breaking of other symmetries, which tends not to be defined in this way. (Although in fact it could be; for example, an "operational" definition of spontaneously broken Ising symmetry, say, would be a system in which the symmetry-breaking order parameter does not decay with time for physically reasonable initial states[140].) It was for this reason that in Ref. [123] we introduced a formal definition of time-translation symmetry-breaking in MBL systems in terms of eigenstates (two equivalent formulations of which we called TTSB-1 and TTSB-2.)

The definitions TTSB-1 and TTSB-2 of Ref. [123] are natural generalizations of the notion of "eigenstate order" used to define spontaneous breaking of other symmetries in MBL [31, 140]. On the other hand they, like the notion of eigenstate order in general, are not really appropriate outside of the MBL context. In this subsection, we will review the usual formal definitions of spontaneous symmetry breaking in equilibrium. Then we will show how they can be extended in a natural way to time-translation symmetries, and that these extended versions are satisfied by the pre-thermal Floquet time crystals constructed above.

Let us first forget about time-translation symmetry, and consider a time-independent Hamiltonian H with an Ising symmetry generated by X. Let ρ be a steady state of the Hamiltonian; that is, it is invariant under the time evolution generated by H. (Here, we work in the thermodynamic limit, so by ρ we really mean a function which

maps local observables to their expectation values; that is, we define a state in the C^*-algebra sense [141].) Generically, we expect ρ to be essentially a thermal state. If the symmetry is spontaneously broken, then ρ can obey the cluster decomposition (i.e. its correlations can be short-ranged), or it can be invariant under the symmetry X, but *not* both. That is, any state invariant under the symmetry decomposes as $\rho = \frac{1}{2}(\rho_\uparrow + \rho_\downarrow)$, where $\rho_\uparrow$ and $\rho_\downarrow$ have opposite values of the Ising order parameter, and are mapped into each other under X. Thus, a formal definition of spontaneously broken Ising symmetry can be given as follows. We call a symmetry-invariant steady state ρ state an *extremal symmetry-respecting state* if there do not exist states ρ_1 and ρ_2 such that $\rho = p\rho_1 + (1-p)\rho_1$ for some $p \in (0,1)$, where ρ_1 and ρ_2 are symmetry-invariant steady states. We say the Ising symmetry is spontaneously broken if extremal symmetry-invariant steady states do not satisfy the cluster decomposition. Similar statements can be made for Floquet systems, where by "steady state" we fnow mean a state that returns to itself after one time cycle.

We can now state the natural generalization to time-translation symmetry. For time-translation symmetry, "symmetry-invariant" and "steady state" actually mean the same thing. So we say that time-translation symmetry is spontaneously broken if extremal steady states do not satisfy the cluster decomposition. This is similar to our definition TTSB-2 from Ref. [123] (but not exactly the same, since TTSB-2 was expressed in terms of eigenstates, rather than extremal steady states in an infinite system), so we call it TTSB-2′. We note that TTSB-2′ implies that any short-range correlated state ρ, i.e. a state ρ which satisfies the cluster decomposition, must not be an extremal steady state. Non-extremal states never satisfy the cluster decomposition, so we conclude that short-range correlated states must not be steady states at all, so they cannot simply return to themselves after one time cycle. (This is similar to, but again not identical with, TTSB-1 in Ref. [123].)

We note that, for clean systems, the only steady state of the Floquet operator U_f is believed to be the infinite temperature state[13–15] which always obeys the cluster property, and hence time translation symmetry is not broken spontaneously. This does not contradict our previous results, since we already saw that time translation symmetry is only spontaneously broken in the pre-thermal regime, not at infinitely long times. Instead, we should examine the steady states of the *approximate* Floquet operator $\widetilde{U}_f$ which describes the dynamics in the pre-thermal regime. We recall that, after a unitary change of basis, $\widetilde{U}_f = Xe^{-iDT}$, where D commutes with X and spontaneously breaks the Ising symmetry generated by X (for temperatures $\tau < \tau_c$). Hence $\widetilde{U}_f^2 = e^{-2iDT}$. Any steady state ρ of $\widetilde{U}_f$ must be a steady state of $\widetilde{U}_f^2$, which implies (if its energy density corresponds to a temperature $\tau < \tau_c$) that it must be of the form $\rho = t\rho_{SB} + (1-t)X\rho_{SB}X$, where ρ_{SB} is an Ising symmetry-breaking state of temperature τ for the Hamiltonian D. Hence, we see (since ρ_{SB} is invariant under e^{-iDT}) that $\widetilde{U}_f\rho\widetilde{U}_f^\dagger = tX\rho_{SB}X + (1-t)\rho_{SB}$. So if ρ is a steady state of $\widetilde{U}_f$ and not just $\widetilde{U}_f^2$, we must have $t = 1/2$. But then the state ρ clearly violates the cluster property. Hence, time translation is spontaneously broken.

7.4 Spontaneously-broken continuous time-translation symmetry in the pre-thermal regime

7.4.1 Basic Picture

The pre-thermalized Floquet time crystals discussed above have a natural analog in undriven systems with *continuous* time translation symmetry. Suppose we have a

time-independent Hamiltonian

$$H = -uL + V, \tag{7.34}$$

where the eigenvalues of L are integers; in other words, for time $T = 2\pi/u$, the condition $e^{inuLT} = 1$ holds for all $n \in \mathbb{Z}$. We also assume that L is a sum of local terms of local strength $O(1)$; and V is a local Hamiltonian of local strength $\lambda \ll u$. Then by Theorem 3.1 of Ref. [39], restated in Theorem 2 in Section 7.2), there exists a local unitary $\mathcal{U}$ such that $\mathcal{U}H\mathcal{U}^\dagger = -uL + D + \hat{V}$ such that $[D, L] = 0$ and the local strength of $\hat{V}$ is $\sim \lambda\, e^{-O([\log \lambda T]^3/[\lambda T])}$. As noted in Theorem 2 in Section 7.2), the first term in the explicit iterative construction of D in Ref. [39] is $D = \langle V \rangle + \frac{1}{T}O(\lambda T)^2$, where

$$\langle V \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta\, e^{iL\theta} V e^{-iL\theta}. \tag{7.35}$$

As a result of this theorem, such a system has an approximate U(1) symmetry generated by $\mathcal{U}^\dagger L \mathcal{U}$ that is explicitly broken only by nearly exponentially-small terms. Consequently, $\mathcal{U}^\dagger L \mathcal{U}$ is conserved by the dynamics of H for times $t \ll t_* = e^{O([-\log \lambda T]^3/[\lambda T])}$. We will call the Hamiltonian $-uL + D$ the "pre-thermal" Hamiltonian, since it governs the dynamics of the system for times short compared to t_*. We will assume that we have added a constant to the Hamiltonian such that L is positive-definite; this will allow us to abuse terminology a little by referring to the expectation value of L as the "particle number", in order to make analogies with well-known properties of Bose gases, in which the generator of the U(1) symmetry is the particle number operator. In this vein, we will call u the electric potential, in analogy with (negatively) charged superfluids.

We will further suppose that D is neither integrable nor many-body localized, so

that the dynamics of D will cause an arbitrary initial state $|\psi_0\rangle$ with non-zero energy density and non-zero $\langle\psi_0|L|\psi_0\rangle$ to rapidly thermalize on some short (compared to t_*) time scale $t_{\mathrm{pre-thermal}} \sim \lambda^{-1}$. The resulting thermalized state can be characterized by the expectation values of D and L, both of which will be the same as in the initial state, since energy and particle number are conserved. Equivalently, the thermalized state can be characterized by its temperature β (defined with respect to D) and effective chemical potential μ. In other words, all local correlation functions of local operators can be computed with respect to the density matrix $\rho = e^{-\beta(D-\mu L)}$. The chemical potential μ has been introduced to enforce the condition $\mathrm{tr}(\rho L) = \langle\psi_0|L|\psi_0\rangle$.

Now suppose that we choose V such that D spontaneously breaks the $U(1)$ symmetry in some range of temperature $1/\beta$ and chemical potential μ. Suppose, further, that we prepare the system in a short-range correlated initial state $|\psi_0\rangle$ such that the energy density (and hence, its temperature) is sufficiently low, and the number density sufficiently high, so that the corresponding thermalized state spontaneously breaks the U(1) symmetry generated by L. Then, the preceding statement must be slightly revised: all local correlation functions of local operators can be computed with respect to the density matrix $\rho = e^{-\beta(D-\mu L-\epsilon X)}$ for some X satisfying $[X, L] \neq 0$. The limit $\epsilon \to 0$ is taken after the thermodynamic limit is taken; the direction of the infinitesimal symmetry-breaking field X is determined by the initial state. To avoid clutter, we will not explicitly write the ϵXin the next paragraph, but it is understood.

Consider an operator Φ that satisfies $[L, \Phi] = \Phi$. (For example, if we interpret L as the particle number, we can take Φ to be the particle creation operator.) Its

expectation value at time t is given by

$$\begin{aligned}\langle\psi_0|e^{-i(-uL+D)t}\Phi e^{i(-uL+D)t}|\psi_0\rangle \\ &= \mathrm{tr}\big(\big[e^{-i(-uL+D)t}\Phi e^{i(-uL+D)t}\big]\, e^{-\beta(D-\mu L)}\big) \\ &= e^{i(\mu-u)t}\,\mathrm{tr}\big(\big[e^{-i(-\mu L+D)t}\,\Phi\, e^{i(-\mu L+D)t}\big]\, e^{-\beta(D-\mu L)}\big)\end{aligned} \tag{7.36}$$

According to the discussion in Appendix D.1, which makes use of the result of Watanabe and Oshikawa [52], the trace on the right-hand-side of the second equality must be independent of time. Hence, so long as $\mathrm{Tr}(\Phi e^{-\beta(D-\mu L)}) \neq 0$ (which we assume to be true for some order parameter Φ in the symmetry-breaking phase), we find that the expectation value of Φ oscillates with frequency given by the "effective electrochemical potential" $\mu - u$ due to the winding of the phase of Φ.

If the dynamics were exactly governed by $-uL + D$, then the system would oscillate with period $2\pi/(u-\mu)$ forever. As it is, these oscillations will be observed until the exponentially late time t_*. At infinitely long times, the system approaches a thermal state of the full Hamiltonian $-uL + D + \hat{V}$. Since $\hat{V}$ is small, this is approximately the same as a thermal state of $-uL + D$. However, because $\hat{V}$ is not exactly zero, the particle number is not conserved and in equilibrium the system chooses the particle number that minimizes its free energy, which corresponds to the "electrochemical potential" being zero, $\mu - u = 0$. Since this corresponds to zero frequency of oscillations, it follows that no oscillations are observed at infinite time.

The above discussion is essentially the logic that was discussed in Refs. [50, 52, 111], where it was pointed out that a superfluid at non-zero chemical potential is a time crystal as a result of the well-known time-dependence of the order parameter

[142]. However, there is an important difference: the U(1) symmetry is not a symmetry of the Hamiltonian of the problem and, therefore, does not require fine-tuning but, instead, emerges in the $u \to \infty$ limit, thereby evading the criticism [50, 52, 143–145] that the phase winds in the ground state only if the U(1) symmetry is exact.

7.4.2 Example: XY Ferromagnet in a Large Perpendicular Field

Consider the concrete example of a spin-1/2 system in three spatial dimensions, with Hamiltonian

$$H = -h^z \sum_i S_i^z - h^x \sum_i S_i^x - \sum_{i,j} \left[J_{ij}^x S_i^x S_j^x + J_{ij}^y S_i^y S_j^y + J_{ij}^z S_i^z S_j^z \right], \quad (7.37)$$

We take $L = S^z \equiv \sum_i S_i^z$, and the longitudinal magnetic field h^z plays the role of u in the preceding section. We take J_{ij} and J_{ij}^z to vanish except for nearest neighbors, for which $J_{ij}^x = J + \delta J$, $J_{ij}^y = J_y + \delta J$, and $J_{ij}^z = J^z$. (We do not assume $\delta J \ll J$.) The local scale of V is given by $\lambda = \max(J + \delta J, h^x)$, so that the condition $\lambda \ll T^{-1} \sim h^z$ is satisfied if $J + \delta J, h^x \ll h^z$. In this case, D is (to first order) the Hamiltonian of an XY ferromagnet:

$$D = -\sum_{\langle i,j \rangle} \left[J(S_i^x S_j^x + S_i^y S_j^y) + J^z S_i^z S_j^z \right] + \frac{1}{T} O(\lambda / h^z)^2. \quad (7.38)$$

Then, starting from a short-range correlated state with appropriate values of energy and $\langle S^z \rangle$, we expect that time evolution governed by D causes the system to "pre-thermalize" into a symmetry-breaking state with some value of the order

parameter $\langle S_i^+ \rangle = n_0 e^{i\phi}$. According to the preceding discussion, the order parameter will then rotate in time with angular frequency $\omega = \mu - h^z$ (where $\mu \lesssim \lambda$ is determined by the initial value of $\langle S^z \rangle$) for times short compared to the thermalization time t_*.

Note, however, that we have assumed that the system is completely isolated. If the system is not isolated, then the periodic rotation of the order parameter will cause the system to emit radiation, and this radiation will cause the system to decay to its true ground state [46, 48].

7.4.3 Field Theory of Pre-Thermal Continuous-TTSB Time Crystal

For simplicity we will give only the imaginary-time field theory for equal-time correlation functions deep within the pre-thermal regime; the Schwinger-Kelysh functional integral for unequal-time correlation functions, with nearly exponentially-small thermalization effects taken into account, will be discussed elsewhere [139]. Introducing the field $\phi \sim (S_x + iS_y)e^{i(\mu-u)t}$, we apply Eq. (7.36) to the XY ferromagnet of the previous section, thereby obtaining the effective action:

$$S_{\text{eff}} = \int d^d x \, d\tau \, \left[\phi^* \partial_\tau \phi - \mu \phi^* \phi + g(\phi^* \phi)^2 + \ldots \right] \tag{7.39}$$

The ... represents higher-order terms. The U(1) symmetry generated by S^z acts according $\phi \to e^{i\theta}\phi$. Time-translation symmetry acts according to $\phi(t) \to e^{i(\mu-u)a}\,\phi(t+a)$ for any a. Thus, when ϕ develops an expectation value, both symmetries are broken and a combination of them is preserved according to the symmetry-breaking pattern $\mathbb{R}_{\text{TTS}} \times U(1) \to \mathbb{R}$, where the unbroken $\mathbb{R}$ is generated by a gauge transformation by θ and a time-translation $t \to t + \frac{\theta}{\mu - u}$.

From the mathematical equivalence of Eq. (7.39) to the effective field theory of a neutral superfluid, we see that (1) in 2D, there is a quasi-long-range-ordered phase – an 'algebraic time crystal' – for initial state energies below a Kosterlitz-Thouless transition; (2) the TTSB phase transition in 3D is in the ordinary XY universality class in 3D; (3) the 3D time crystal phase has Goldstone boson excitations. If we write $\phi(x,t) = \sqrt{\left(\frac{\mu}{2g} + \delta\rho(x,t)\right)}\, e^{i\theta(x,t)}$, and integrate out the gapped field $\delta\rho(x,t)$, then the effective action for the gapless Goldstone boson $\theta(x,t)$ is of the form discussed in Ref. [144].

7.5 Pre-thermalized Floquet topological phases

We can also apply our general results of Section 7.2 to Floquet symmetry-protected (SPT) and symmetry-enriched (SET) topological phases, even those which don't exist in stationary systems. (We will henceforth use the abbreviation SxT to refer to either SPT or SET phases.)

As was argued in Refs. [102, 107], any such phase protected by symmetry G is analogous to a topological phase of a *stationary* system protected by symmetry $\mathbb{Z} \rtimes G$, where the extra $\mathbb{Z}$ corresponds to the time translation symmetry. Here the product is semi-direct for anti-unitary symmetries and direct for unitary symmetries. For simplicity, here we will consider only unitary symmetries. Similar arguments can be made for anti-unitary symmetries.

We will consider the class of phases which can still be realized when the $\mathbb{Z}$ is refined to $\mathbb{Z}_N$. That is, the analogous stationary phase can be protected by a unitary representation $W(\widetilde{g})$ of the group $\widetilde{G} = \mathbb{Z}_N \times G$. Then, in applying the general result of Section 7.2, we will choose $H_0(t)$ such that its time evolution over one time cycle is equal to $X \equiv W(\mathbb{T})$, where $\mathbb{T}$ is the generator of $\mathbb{Z}_N$. Then it follows that, for a

generic perturbation V of small enough local strength λ, there exists a local unitary rotation $\mathcal{U}$ (commuting with all the symmetries of U_f) such that $U_f \approx \widetilde{U}_f$, where $\widetilde{U}_f = \mathcal{U} X e^{-iDT} \mathcal{U}^\dagger$, D is a quasi-local Hamiltonian which commutes with X, and $\widetilde{U}_f$ well describes the dynamics until the almost exponentially large heating time t_*.

Now let us additionally assume (since we want to construct a Floquet-SxT protected by the symmetry G, plus time-translation) that the Floquet operator U_f is chosen such that it has the symmetry G. Specifically, this means that it is generated by a periodic time evolution $H(t)$ such that, for all $g \in G$, $W(g)H(t)W(g)^{-1}$, By inspection of the explicit construction for $\mathcal{U}$ and D (see Appendix C), it is easy to see that in this case $\mathcal{U}$ is a symmetry-respecting local unitary with respect to $W(g)$, and D commutes with $W(g)$. That is, the rotation by $\mathcal{U}$ *preserves* the existing symmetry G as well as revealing a new $\mathbb{Z}_N$ symmetry generated by X (which in the original frame was "hidden").

Therefore, we can choose D to be a Hamiltonian whose ground state is in the *stationary* SxT phase protected by $\mathbb{Z}_N \times G$. It follows (by the same arguments discussed in Ref. [107] for the MBL case) that the ground state D will display the desired Floquet-SxT order under the time evolution generated by $\mathcal{U}^\dagger U_f \mathcal{U} = X e^{-iDT}$. Furthermore, since Floquet-SxT order is invariant under symmetry-respecting local unitaries, the ground state of $\mathcal{U} D \mathcal{U}^\dagger$ will display the desired Floquet-SxT order under U_f.

We note, however, that topological order, in contrast to symmetry-breaking order, does not exist at nonzero temperature (in clean systems, for spatial dimensions $d < 4$). Thus, for initial state mean energies $\langle D \rangle$ that corresponds to temperatures β^{-1} satisfying $0 < \beta^{-1} \ll \Delta$, where Δ is the bulk energy gap, the system will exhibit exponentiall-small corrections $\sim e^{-\beta\Delta}$ to the quantized values that would be observed in the ground state. This is no worse than the situation in thermal equilbirum where,

for instance, the Hall conductance is not precisely quantized in experiments, but has small corrections $\sim e^{-\beta\Delta}$. However, preparing such an initial state will be more involved than for a simple symmetry-breaking phase. For this reason it is more satisfactory to envision cooling the system by coupling to a thermal bath, as discussed in Section 7.6, which is analogous to how topological phases are observed in thermal equilibrium experiments – by refrigeration.

7.6 Open systems

So far, we have considered only isolated systems. In practice, of course, some coupling to the environment will always be present. One can also consider the effect of classical noise, for example some time-dependent randomness in the parameters of the drive, so that successive time steps do not implement exactly the same time evolution. The Floquet-MBL time crystals of Ref. [123] are not expected to remain robust in such setups, since MBL will be destroyed. Since some amount of coupling to the environment is inevitable in realistic setups, this limits the timescales over which one could expect to observe Floquet-MBL time crystals experimentally.

However, the situation could be quite different for the pre-thermal time crystals of this work. A complete treatment is beyond the scope of the present work, so in this section we will confine ourselves to stating one very interesting hypothesis: Floquet case time-crystals can actually be *stabilized* in open systems so that the oscillations actually continue *forever* for *any* initial state (in contrast to the case of isolated systems, in which, as discussed previously, the oscillations continue only up to some very long time, and only for some initial states). We will not attempt to establish this more rigorously, but simply discuss a plausible scenario by which this would occur. The idea, as depicted in Figure 7.2, is that the heating due to the periodic driving,

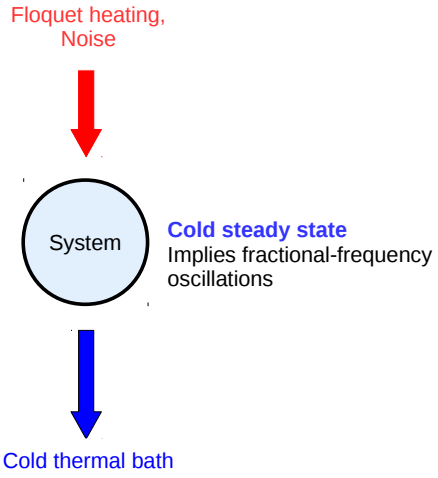

Figure 7.2: So long as the energy inflow due to noise and periodic driving is balanced by the outflow to a cold thermal bath, giving a low-energy steady state, oscillations at a fraction of the drive frequency will be observed.

as well as classical noise sources and other stray couplings to an environment, can be counteracted by cooling from a coupling to a sufficiently cold thermal bath. Provided that the resulting steady-state has sufficiently low "energy", we will argue that that oscillations at a fraction of the drive frequency will be observed in this steady state. Here "energy" means the expectation value of the effective Hamiltonian D which describes the dynamics in the prethermal regime. We discuss this hypothesis further, and show that it indeed implies periodic oscillations, in Appendices D.2 and D.3. We also note that this argument does not apply to the continuous-time time crystals of Section 7.4, since in that case low energy is not a sufficient condition to observe oscillations even in an isolated system; there is also a dependence on the chemical potential μ.

7.7 Discussion

In this paper, we have described how phases protected by time-translation symmetry can be observed in the pre-thermal regime of driven and undriven quantum systems. This greatly increases the set of experimental systems in which such phases can be observed, since, as opposed to previous proposals, we do not require many-body localization to robustly prevent the system from heating to infinite temperature. While many-body localization has been observed in experiments [118, 146, 147], the ideas put forward in this paper significantly reduce experimental requirements as strong disorder is not required.

Our Theorem 1 implies that the time-translation-protected behavior (for example, the fractional-frequency oscillations in the Floquet time crystal) can be observed to nearly exponentially-late times, provided that the drive frequency is sufficently high. However, the rigorous bound given in the theorem – which requires a drive frequency $\sim 10^3$ times larger than the local couplings in the time-dependent Hamiltonian – may not be tight. Therefore, it would be interesting to check numerically whether (in the Floquet time crystal case, say) long-lived oscillations are observed in systems with drive frequency only moderately larger than the local couplings. This may be challenging in small systems, in which there isn't a large separation of energy scales between the local coupling strength and the width of the many-body spectrum (which the frequency should certainly not exceed). In one-dimensional systems, oscillations will not be observed to exponentially-long (in the drive frequency) times, but will have a finite correlation time for any non-zero energy density initial state. However, there will be a universal quantum critical regime in which the correlation time will be the inverse effective temperature.

Although naive application of Theorem 1 suggests that the ideal situation is the

one in which the drive frequency becomes infinitely large, in practice very high-frequency driving will tend to excite high energy modes that were ignored in constructing the model lattice Hamiltonian. For example, if the model Hamiltonian describes electrons moving in a periodic potential in the tight-binding approximation, high frequency driving would excite higher orbitals that were excluded. Thus, the driving frequency Ω needs to be much greater than the local energy scales of the degrees of freedom included in the model Hamiltonian (except for one particular coupling, as discussed in Section 7.3), but also much less than the local energy scales of the degrees of freedom not included. (One cannot simply include *all* degrees of freedom in the model Hamiltonian, because then the norm of local terms would be unbounded, and Theorem 1 would not apply.)

In the case of undriven systems, we have shown that continuous time-translation symmetry breaking can similarly occur on nearly exponentially-long time intervals even without any fine-tuning of the Hamiltonian, provided that there is a large separation of scales in the Hamiltonian. We show how in certain cases this can be described in terms of approximate Goldstone bosons associated with the spontaneously-broken time-translation symmetry.

Our analysis relied on the construction of hidden approximate symmetries that are present in a pre-thermal regime. The analogous symmetries in MBL systems, where they are exact, were elucidated in the interesting work of von Keyserlingk et al. [33]. In the time-translation protected phases discussed here, the symmetry generated by the operator $\mathcal{U}^\dagger X \mathcal{U}$ is enslaved to time-translation symmetry since, in the absence of fine-tuning, such a symmetry exists exists only if time-translation symmetry is present. (That is, if we add fields to the Hamiltonian that are periodic with period nT and not period T, then the hidden symmetry no longer exists.) Moreover, this symmetry is broken if and only if time-translation symmetry is broken. (Similar

statements hold in the MBL case[33].) In the Floquet time crystal case, the hidden symmetry generated by $\mathcal{U}^\dagger X \mathcal{U}$ acts on the order parameter at stroboscopic times in the same way as time-translation by T (a single period of the drive), and therefore it does not constrain correlation functions any more than they already are by time-translation symmetry. The same observation holds for the approximate symmetry generated by L_z in the undriven case.

However, there are systems in which time crystal behavior actually does "piggy-back" off another broken symmetry. This does require fine-tuning, since it is necessary to ensure that the system posseses the "primary" symmetry, but such tuning may be physically natural (e.g. helium atoms have a very long lifetime, leading to a U(1) symmetry). The broken symmetry allows a many-body system to effectively become a few-body system. Thus, time crystal behavior can occur in such systems for the same reason that oscillations can persist in few-body systems. Oscillating Bose condensates (e.g. the AC Josephson effect and the model of Ref. [112]) can, thus, be viewed as fine-tuned time crystals. They are not stable to arbitrary time-translation symmetry-respecting perturbations; a perturbation that breaks the "primary" symmetry will cause the oscillations to decay. Indeed, most few-body systems are actually many-body systems in which a spontaneously-broken symmetry approximately decouples a few degrees of freedom. A pendulum is a system of 10^{23} atoms that can be treated as a single rigid body due to spontaneously-broken spatial translational symmetry: its oscillations owe their persistence to this broken symmetry, which decouples the center-of-mass position from the other degrees of freedom.

With the need for MBL obviated by pre-thermalization, we have opened up the possibility of time-translation protected phases in open systems, in which MBL is impossible [124–132]. In fact, since the results of Appendix D.3 show that TTSB can occur in non-thermal states, it is possible for the coupling to a cold bath to counteract

the heating effect that would otherwise bring an end to the pre-thermal state at time t_*. This raises the possibility of time-translation protected phases that survive to infinite times in non-equilibrium steady states; the construction of such states is an interesting avenue for future work.

Note added: After the submission of this paper, two experimental papers (J. Zhang *et al.*, arXiv:1609.08684 and S. Choi *et al.*, arXiv:1610.08057) have appeared with evidence consistent with the observation of a Floquet time crystal. We note that the J. Zhang *et al.* paper implements disorder by addressing each ion sequentially. A pre-thermal version of this experiment would not need disorder, thereby sidestepping this bottleneck standing in the way of experiments on larger systems. The Choi *et al.* paper occurs in a system that is unlikely to be many-body localize, and therefore occurs during a slow approach to equilibrium. This is unlikely to correspond to a prethermal regime, but the approximate short-time form of the time evolution entailed in our Theorem 1 might still be relevant to understanding the results.

Chapter 8

Classification of phases with spatial symmetries

In this chapter, we develop a systematic framework to understand zero-temperature topological phases with spatial symmetries. This might seem like something of a departure from the rest of the dissertation, which is devoted to non-equilibrium phases in periodically driven systems, but in fact many of the ideas involved here are closely related to the classification of Floquet phases, which are after all protected by a *temporal* symmetry (discrete time-translation symmetry). In fact, in Chapter 9 we will extend the ideas developed here to propose a unified theory of systems with general space-time symmetries, which will include both Floquet phases and stationary crystalline phases as special cases.

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8.1 Introduction

Symmetry is an important feature of many physical systems. Many phases of matter can be characterized in part by the way the symmetry is implemented. For example, liquids and solids are distinguished by whether or not they spontaneously break spatial symmetries. In fact, it was once thought that *all* known phases could be distinguished by their symmetries and that all continuous phase transitions were spontaneous symmetry breaking transitions. The discovery of *topological order*[53] showed that, at zero temperature, there are quantum phases of matter that can be distinguished by patterns of long-range entanglement without the need to invoke symmetry. However, even for topological phases symmetry is important. Any symmetry that is not spontaneously broken in a topological phase must have some action on the topological structure of the phase, and different such patterns can distinguish different phases. Even a phase of matter that is trivial without symmetry can become non-trivial when considering how symmetry is implemented. Topological phases distinguished by symmetry are known as *symmetry-enriched topological (SET)*[91–97] or *symmetry-protected topological (SPT)*[71–80, 82–89] depending on whether they are nontrivial or trivial without symmetry, respectively.

For internal symmetries, which do not move points in space around, very general and powerful ways of understanding SPT and SET phases have been formulated in terms of mathematical notions such as group cohomology[80], category theory[96], and cobordisms[84, 148]. On the other hand, such techniques have not, so far, been extended to the case of *space group* symmetries. We refer to these topological phases enriched by space-group symmetries as *topological crystalline phases*. This is a significant omission because any system which arranges itself into a regular crystal lattice is invariant under one of 230 space groups in three dimensions. Fermionic phases

of matter protected by space-group symmetries are called *topological crystalline insulators* or *topological crystalline superconductors* depending on whether charge is conserved [149–154]. Progress towards a general classification in free-fermion systems has been made [155–161] and some understanding of the effect of interactions been achieved [162–166]. Meanwhile, intrinsically strongly interacting phases protected by spatial symmetries have also been found [92, 96, 97, 167–174]. In particular Ref. [175] gave an approach for deriving the general classification of interacting SPT phases protected by a group of spatial symmetries that leave a given point invariant. However, for SETs and/or general space groups, there is so far no systematic theory analogous to the one that exists for internal symmetries, except in one dimension [76]. Our goal in this paper is to fill this gap.

We will adopt two complementary and related viewpoints to the classification. The first viewpoint is in terms of topological quantum field theories (TQFTs), which are believed to describe the low-energy physics of topological phases. We state and motivate a proposal for how to implement a spatial symmetry in a TQFT.

Our second, more concrete, viewpoint is based on the idea of understanding the SPT or SET order of a system by studying its response to a gauge field. For example, SPTs in (2+1)-D protected by an internal $U(1)$ symmetry can be identified by the topological response to a $U(1)$ gauge field. All such possible responses are described by the Chern-Simons action

$$S = \frac{k}{4\pi} \int A \wedge dA. \tag{8.1}$$

The coefficient k has a physical interpretation as the quantized Hall conductance. Because it is quantized, the only way to get between systems with different values of k is if U(1) symmetry is broken or the gap closes. Further, since this is the

only term that may appear, we learn that the different U(1) SPTs in 2+1D are labelled by this integer. We call this procedure of coupling a G-symmetric system to a background G gauge field "gauging" the G symmetry, though strictly speaking we do not consider making the gauge field dynamical. Stricter terminology would call the dynamical gauge theory the result of gauging and our procedure the first step, called equivariantization, a mouthful, or pregauging. Many of the general approaches to SPT and SET phases can be formulated in terms of gauging[81, 95, 96, 176].

We want to apply similar approaches to the study of systems with spatial symmetry. So we will ask the question

Question 1. What does it mean to gauge a spatial symmetry?

We will give what we believe to be the definitive answer to this question, motivated by the intuition of "gauge fluxes" which for spatial symmetries are crystallographic defects such as dislocations and disclinations. There seems to be a natural generalization of this to symmetries which act on spacetime as well, such as time reversal symmetry or time translation. We will mention briefly this generalization and how the classification extends to these spacetime symmetries, where it agrees with known group cohomology classifications of time reversal-invariant and Floquet SPTs, respectively.

Using the two viewpoints mentioned above, we will elucidate the general theory of crystalline topological phases. Our results are based on a key physical assumption, namely that the phases of matter under consideration are *crystalline topological liquid*, which roughly means that, although crystalline, they preserve a certain degree of "fluidity" in the low-energy limit. The idea is motivated by the notion of "topological liquids" which have an IR limit that is described by a topological quantum field theory (TQFT), i.e. the long-range physics is only sensitive to the topology of the background

manifold. This is in contrast to "fracton" topological phases[177–180] where no such topological IR limit exists. Crystalline topological liquids are a generalization of topological liquids to systems with crystal symmetries.

The main result of this paper is the following.

> **Crystalline Equivalence Principle**: The classification of crystalline topological liquids with spatial symmetry group G is the *same* as the classification of topological phases with *internal* symmetry G.

Compare Ref. [107], where a similar principle was conjectured for symmetry groups containing *time translation symmetry*. This result holds for systems living on a *contractible* space, ie. Euclidean space in d dimensions. On other manifolds, for example Euclidean space with some holes, some new things happen. We note for this correspondence, orientation-reversing symmetries in the space group must correspond to anti-unitary symmetries in the internal group.

We emphasize that the Crystalline Equivalence Principle is expected to hold for both bosonic and fermionic[1] systems, and for both SPT and SET phases. As an example of results that one can deduce from this general principle, we find that bosonic SPT phases protected by orientation-preserving unitary spatial symmetry G are classified by the group cohomology $\mathcal{H}^{d+1}(G, \mathrm{U}(1))$, since that is the classification of internal SPTs with symmetry G. This agrees with a recent classification of a class of tensor networks with spatial symmetries[174]. In (3+1)-D, for space groups containing orientation-reversing transformations, this classification is expected to be incomplete, just as it is for internal symmetry groups containing anti-unitary symmetries[84].

Our results allow for the classification to be explicitly computed in many cases.

[1] There are some caveats for fermionic systems: systems with $R^2 = +1$, where R is a reflection, are in correspondence with systems with $T^2 = (-1)^F$, where T is time-reversal, and vice versa.

Number	Name	Classification
1	p1	0
2	p2	$\mathbb{Z}_2^{\times 4}$
3	pm	$\mathbb{Z}_2^{\times 2}$
4	pg	0
5	cm	$\mathbb{Z}_2$
6	p2mm	$\mathbb{Z}_2^{\times 8}$
7	p2mg	$\mathbb{Z}_2^{\times 3}$
8	p2gg	$\mathbb{Z}_2^{\times 2}$
9	c2mm	$\mathbb{Z}_2^{\times 5}$
10	p4	$\mathbb{Z}_2 \times \mathbb{Z}_4^{\times 2}$
11	p4mm	$\mathbb{Z}_2^{\times 6}$
12	p4gm	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_4$
13	p3	$\mathbb{Z}_3^{\times 3}$
14	p3m1	$\mathbb{Z}_2$
15	p31m	$\mathbb{Z}_2 \times \mathbb{Z}_3$
16	p6	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_3^{\times 2}$
17	p6mm	$\mathbb{Z}_2^{\times 4}$

Table 8.1: The classification of bosonic SPT phases in (2+1)-D protected by space group symmetries, for each of the 17 2-D space groups (sometimes known as "wallpaper groups").

For example, Table 8.1 shows the classification of bosonic SPT phases protected by space-group symmetry in (2+1)-D as obtained from group cohomology. For more details of how Table 8.1 was computed, and the (3+1)-D version of the table, see Appendix E.1.

At a precise level, our results our based on a conjecture about the form of the TQFT which describes the low-energy physics of a crystalline topological phase. We conjecture that, at least for some such phases (i.e. the crystalline topological liquids), the low-energy physics is described by a "spatially dependent TQFT", or in other words a map from the physical space X in which the system lives (usually we would take $X = \mathbb{R}^d$) into the space Θ of TQFTs. For such spatially dependent TQFTs the Crystalline Equivalence Principle is a mathematical theorem which can be rigorously proven.

The outline of our paper is as follows. In Section 8.2, we introduce the notion of a crystalline topological liquid. Then, in Section 8.3 we introduce the key ideas involved in gauging a spatial symmetry. Specifically, in section 8.3.1 we discuss our definition of crystalline gauge field. Then in 8.3.2 we argue that crystalline topological liquids naturally couple to such crystalline gauge fields. In 8.3.3 we use the gauging picture to derive the Crystalline Equivalence Principle, which applies to the physically relevant case of phases of matter in contractible space $\mathbb{R}^d$. In 8.3.4 we discuss extensions to non-contractible spaces and a general classification result for crystalline gauge fields.

In Section 8.4 we give a construction of many crystalline topological liquids from ordinary topological liquids by considering systems which carry both a spatial G symmetry and an internal G symmetry.

In Section 8.5 we describe our approach towards classifying crystalline topological liquids using topological response. In 8.5.1, this is defined in terms of fusion and braiding of symmetry fluxes. In 8.5.2 it is described in terms of effective topological

actions.

In Section 8.6, we describe how our methods can be placed into a general context of position-dependent topological limit, and discuss implications of emergent Lorentz invariance or lack thereof.

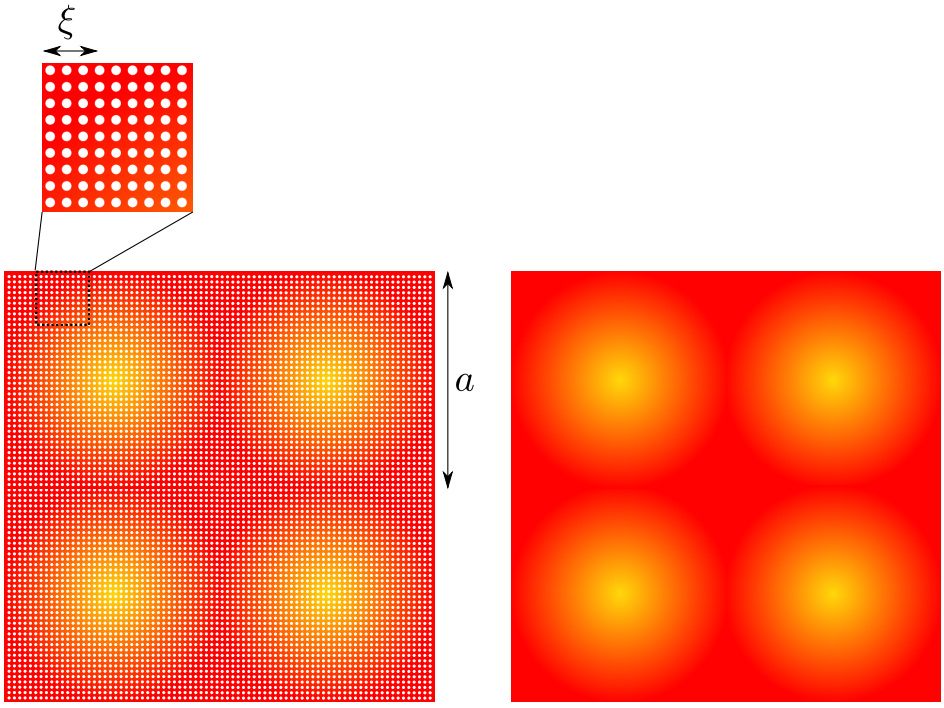
In Section 8.7 we discuss questions for future work.

We hope this paper will inspire the discovery of many curious quantum crystals.

8.2 The topological limit of a crystalline topological phase

In this section, we will briefly outline the arguments based on topological quantum field theory (TQFT) which lead to the Crystalline Equivalence Principle. The mathematical details are left to Section 8.6. The underlying physical concept is that of a *smooth state*. A smooth state is a ground state of a lattice Hamiltonian that is defined on a lattice which is *much* finer than the unit cell with respect to the translation symmetry, such that the lattice spacing l and the correlation length ξ are much smaller than the minimum radius R of spatial variation within the unit cell. The condition $\xi, l \ll a$ (where a is the unit cell size) was discussed as an assumption for classifying crystalline phases in Ref. [181]; our "smooth state" assumption is slightly stronger since we require $\xi, l \ll R$. This implies the condition of Ref. [181] since $R < a$, but the converse need not be true if there are regions in the unit cell where spatial variation happens rapidly (so that $R \ll a$).

A smooth state might not seem like the kind of system one would normally consider; a physical example would be a graphene heterostructure in which a lattice mismatch between two layers results in a Moire pattern with very large unit cell



(a) Smooth state (b) Topological limit

Figure 8.1: (a) In a smooth state, the lattice spacing and the correlation length ξ are much less than the unit cell size a and the radius of spatial variation. (b) The topological response of a crystalline topological liquid is captured by a spatially-dependent TQFT that captures the spatial dependence within each unit cell but "forgets" about the lattice.

[182]. Nevertheless, it is reasonable to expect that the *classification* of smooth states would be the same as the classification of states in general. We will leave a rigorous proof for future work; presently, we merely state it as a conjecture and examine the consequences.

A very important property of a smooth state is that it can be coarse-grained while preserving the spatial symmetries. This is allowed only so long as the coarse-grained lattice is still small compared to the unit cell size, but given the assumption $\xi \ll a$ this still allows us to reach a "topological limit", by which we mean that ξ becomes much smaller than the coarse-grained lattice spacing. Importantly, since the RG can take place in the neighborhood of any given point in the unit cell, the effective field theory that we obtain in this topological limit will still be spatially-dependent. (For this reason, we will avoid referring to the topological limit as an "IR limit", which would be misleading since the unit cell size – but not the lattice spacing! – is still an important length scale).

We expect that this topological limit will, as in the case of systems without spatial symmetries, be described by a topological quantum field theory (TQFT). In fact, given the afore-mentioned spatial dependence, it should be described by a *spatially-dependent TQFT*. We give the precise mathematical definition of this concept in Section 8.6.

Hence, we can define

Definition 1. A crystalline topological liquid is a phase of matter that is characterized by a spatially-dependent TQFT acted upon by spatial symmetries.

We expect that this class of systems is quite large. Certainly, it includes ordinary topological liquids (which, by definition, have no explicit spatial symmetries and can be characterized by a spatially-*constant* TQFT). Moreover, spatially-dependent

TQFTs can capture a wide range of other topological crystalline phenomena, as we shall see.

In Section 8.6, we sketch a proof that on contractible spaces, spatially-dependent TQFTs with spatial symmetries are in one to one correspondence with spatially *constant* TQFTs with internal symmetries. Since the latter are expected to characterize topological phases with internal symmetries, the Crystalline Equivalence Principle follows. In the following sections, we we will discuss how to understand this result in more concrete ways without resorting to the highly abstract formalism of TQFTs.

8.3 Crystalline gauge fields

8.3.1 Gauge fluxes and crystal defects

In order to understand crystalline topological phases, we want to study what it might mean to couple to a background gauge field for a symmetry group G involving some transformation of space itself. More generally, we believe a framework exists where one can also consider symmetries that transform space-*time*. However, for simplicity and to maintain contact with Hamiltonian models we will focus on purely spatial symmetries. We call our object of study the crystalline gauge field.

A special case of a background gauge field is an isolated *gauge flux*. Isolated gauge fluxes are familiar objects for *internal* symmetries. They are objects in space of codimension 2 (i.e. points in 2-D, curves in 3-D) which are labelled by a group element $g \in G$, and a particle moving all the way around one is acted upon by g. Actually, for a non-Abelian group only the conjugacy class of g is gauge-invariant.

Gauge fluxes for spatial symmetries are also labelled by conjugacy classes of G. They are also well-known, but not under that name; they are more commonly referred

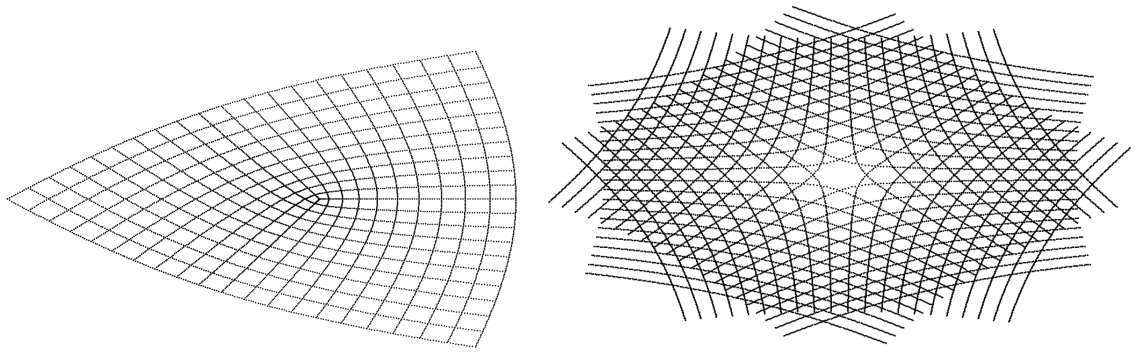


Figure 8.2: An angular defect of 90 degrees in a vertex-centered square lattice and an angular excess of 120 degrees in a face-centered kagome lattice.

to as *crystal defects*. For example, a gauge flux for translational symmetry is a *dislocation* and a gauge flux for a rotational symmetry is a *disclination* (Fig 8.2). In 3d, the direction of dislocation does not have to be in the plane perpendicular to the defect, as in a screw dislocation. A defect for reflection symmetry is like the Möbius band (a cross cap). For a glide reflection we also insert a shift in the lattice as we go around the band. We will see how this zoo of defect configurations is tamed by topology.

Generalizing these examples, we can give a systematic definition of crystalline gauge flux, and more generally of a crystalline gauge field. For motivation, one can look again at Fig 8.2. The original lattice Λ is a regular square or kagome lattice. The crucial property the defect lattice Σ is that away from the singular point in the middle, it looks *locally* the same as Λ, meaning that in a neighborhood of every face except the central one there is an invertible map sending Σ to Λ. However, there is no *global* map sending Σ to Λ. Indeed, if we try to extend the domain of our map, we will eventually create a discontinuity after encircling the singularity. This is shown in Fig 8.3. For the 90 degree angular defect, the discontinuity is a branch cut such that the limits on either side are related by a 90 degree rotation. For a crystal defect,

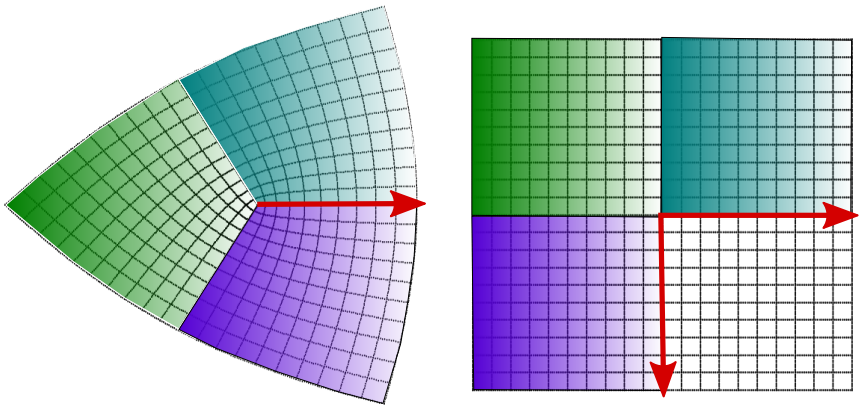


Figure 8.3: A 90 degree disclination maps discontinuously to the square lattice, as indicated with the colored quadrants. The red line is the branch cut across which the image rotates by 90 degrees. Because the discontinuity is by a rotation in G, this map descends to a continuous map from the disclination to the quotient of the square lattice by G.

this discontinuity is always by a G transformation and labels the symmetry flux of the defect.

To further motivate the definition, let us recall the definition of a gauge field for an *internal* (discrete) symmetry. Gauge fields for discrete symmetries are somewhat more esoteric than gauge fields for continuous groups (like the familiar electromagnetic vector potential A_μ). One way to think about them is that they encode "twisted boundary conditions". For example, threading a non-trivial gauge flux for an Ising symmetry through a system living on a circle means that we make a cut and identify spin-up on one side of the cut with spin-down on the other side of the cut ("anti-periodic boundary conditions"). In general, to specify a gauge field on a manifold M we can build M up out of "patches". The boundaries between patches ("domain walls") are "twisted" by an element $g \in G$ of the symmetry group ("transition functions"), which tells us how to identify the patches. A discrete gauge field must be "flat", which is to say there can be no non-trivial holonomy around a vertex where several patches intersect, as shown in Figure 8.4. This is to say there is no G-flux

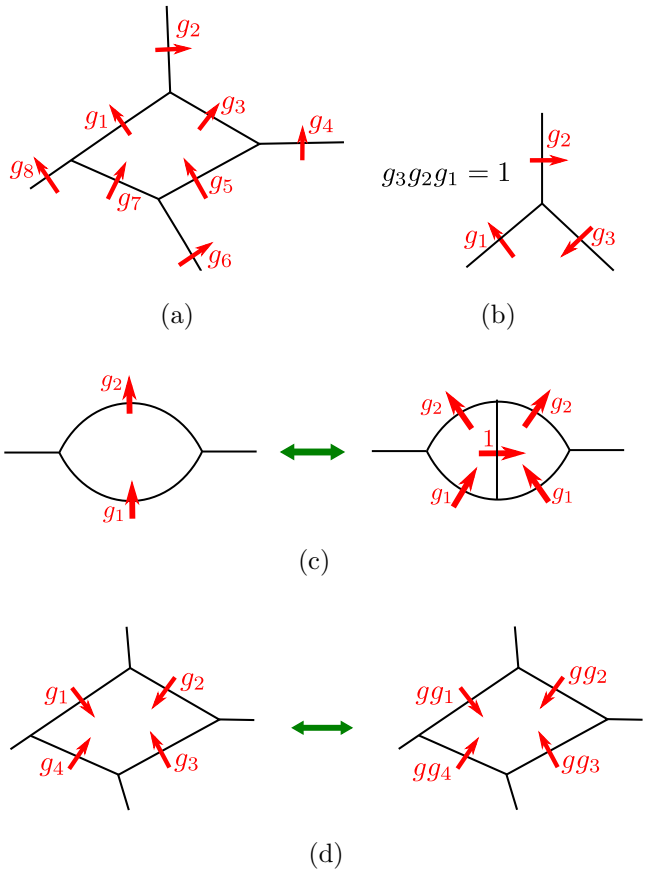


Figure 8.4: The "patches" picture of a gauge field for an internal symmetry. (a): The manifold M is divided up into patches, and the boundaries between patches are twisted by a group element $g \in G$. (b): The flatness constraint implies that the holonomy around a vertex must be trivial. (c) and (d): We identify configurations that differ by dividing patches or by acting on a patch with some $g \in G$.

through the vertices (or along such line-like junctions in a 3d picture). There is some inherent gauge freedom: firstly, we can merge or split patches, provided that the boundaries thus created or destroyed are twisted by the trivial element $1 \in G$; secondly, we can apply an element $g_p \in G$ of the symmetry group to a given patch p, which has the effect of multiplying the twist carried by the boundaries of this patch by g_p. This gauge freedom relates two different representations of the same gauge field. More abstractly (but equivalently), we can define a gauge field as a principal G-bundle over M [183].

As an example, we can consider a g-flux at the origin of the plane. This g-flux is defined as a G gauge field on the plane minus the origin. It may be defined using a single (simply-connected) patch which meets itself along a domain wall extending from the origin to infinity. This domain wall is labeled with the transition function g, indicating that a point charge taking along a path encircling the origin will return to its original position with any internal degrees of freedom transformed by the symmetry g. The similarity between the internal symmetry flux and the crystal defect is striking. It leads us to identify the role of the branch cut in the latter with the domain wall of the former.

With this identification in hand, we are ready to state our definition of crystalline gauge field, by directly generalizing the patches picture of internal symmetry gauge fields. An important novelty will be that the lattice geometry is defined by the crystalline gauge background. That is, we fix our physical space X containing the lattice Λ. X is usually $\mathbb{R}^d$, a torus, or some related spacetime. G acts on X preserving Λ. The lattice with defects Σ will be embedded in a different space M. For example, in the disclination, M is the plane minus the origin.

To specify a crystalline gauge field, we will start with the same data we had before: a collection of patches U_i dividing $M = \bigcup_i U_i$, with domain walls between

intersecting patches $U_i \cap U_j \neq 0$ labelled by elements $g_{ij} = g_{ji}^{-1} \in G$, with the flatness condition $\prod_i g_{i,i+1} = 1$ imposed over all contractible loops. This is the definition of an internal symmetry G gauge field, but it is not the end of the story, because as we saw in the examples above, there is an extra feature of crystalline gauge fluxes which needs to be captured: a *map* $f : M \to X$. This represents the (continuum limit) of the identification between the lattices Σ embedded in M and Λ embedded in X. Inside each patch U_i, this map $f : U_i \to X$ is continuous, but on the boundaries between intersecting patches $U_i \cap U_j \neq 0$ we impose the twisted continuity condition that for any $m \in U_i \cap U_j$, the limit of $f(m')$ as $m' \to m$ in U_i and the limit of $f(m')$ as $m' \to m$ in U_j are related in X from the former to the latter by application of g_{ij}. For example, in Figure 8.3, the different colored quadrants are patches on M (which in this case is the punctured plane $\mathbb{R}^2 \setminus \{0\}$), and the thick red line denotes a boundary between patches which is twisted by a 90 degree clockwise rotation as we pass from the teal patch to the violet patch. We impose the same gauge freedom as before [Figure 8.4(c) and 8.4(d)], except that when we act on a patch by g, as shown in Figure 8.4(d), then inside the patch we replace the function f according to $f(m) \to gf(m)$.

There is a final condition we need to impose, related to the orientation (or lack thereof) of the manifold M. It is standard lore that a topological phase that is not reflection invariant cannot be put on an unorientable manifold, and moreover, that for a reflection invariant system, putting it on a unorientable manifold is essentially threading a "flux" of the reflection symmetry. So in order to enforce compatibility with these notions, we define $\mu(g) = -1$ if g acts in an orientation-reversing way on X, and $\mu(g) = 1$ otherwise. For any closed loop γ in M, we can define the "flux" g_γ, which is the product of the twist over each boundary crossed by γ. We also define $\lambda(\gamma) = \pm 1$ depending on whether going around the loop γ would reverse the

orientation on M. We require that $\lambda(\gamma) = \mu(g_\gamma)$.

For completeness, we will also formulate a more abstract mathematical definition. Basically we are specifying some extra data on top of a principal G-bundle. Formally, we have

Definition 2. A *crystalline gauge field* is a pair $(\pi, \hat{f})$, where $\pi : P \to M$ is a principal G bundle, and $\hat{f} : P \to X$ is a continuous map satisfying satisfying $\hat{f}(gp) = g\hat{f}(p)$ for all $p \in P$, $g \in G$. We require that the homomorphism $\mu : G \to \mathbb{Z}_2$ (where $\mu(g) = -1$ if g has orientation-reversing action on X) reduces π to the orientation bundle of M. We say that two pairs $(\pi, \hat{f})$, $(\pi', \hat{f}')$ represent the same crystalline gauge field if the principal G-bundles $\pi : P \to M$ and $\pi' : P' \to M$ are isomorphic by a map $\sigma : P \to P'$ such that $\hat{f}' \circ \sigma = \hat{f}$.

The map $\hat{f}$ in the definition above always induces a map g from $P/G = M$ into X/G. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \text{mod } G} \\ M & \xrightarrow{g} & X/G \end{array} . \tag{8.2}$$

It should be clear, from the disclination example, that crystalline gauge fields can describe the crystal defects which were our original motivation. However, now that have given a general definition, we had better ask whether *all* crystalline gauge fields admit such a physical interpretation. In particular, there ought to be a well-defined sense of what it means to *couple* to a general crystalline gauge field.

For internal symmetries it is familiar how to couple to a gauge field, at least when that gauge field lives on $M = X$. Given a gauge field A for a (discrete) internal symmetry G, described using patches and transition functions, and given a

Hamiltonian H that commutes with the symmetry, we can define a Hamiltonian $H[A]$ that describes the system coupled to the gauge field. To do this, we assume that H can be written as a sum of local terms. Then, $H[A]$ contains a local term for each local term in H. The terms in H which act only within a patch carry over to $H[A]$ without change, while for terms in H which act in multiple patches, we must first perform a gauge transformation so that the term acts in a single patch, add it to the Hamiltonian, and then reverse that gauge transformation. See, for example [96].

Now suppose that we want to do the same thing for crystalline gauge fields. For crystal defects (for example, the disclination in Figure 8.3) it should be clear how to do this; locally, the defect lattice looks the same as the original lattice, so we just pull local terms in X back into M. On the other hand, this construction doesn't necessarily work for a general crystalline gauge field. We have to impose a condition which we call *rigidity*.

Definition 3. A crystalline gauge field (expressed in terms of patches, twisted boundary conditions, and a map $f : M \to X$) is *rigid* if near any point $m \in M$ that maps into a lattice point in X under f, there exists a local neighborhood U containing m such that, after making a gauge transformation such that U is contained in a single patch, f is injective (one-to-one) when restricted to U; and, moreover, the image of U under f contains all lattice points that are coupled to $f(m)$ by a term in the Hamiltonian.[2]

This somewhat technical definition is best understood by considering examples of crystalline gauge fields which are *not* rigid. An extreme example is the case where $f : M \to X$ is the constant function: there is some $x_* \in X$ such that $f(m) = x_*$ for all $m \in M$. In other words, every point in M gets identified with a single point in X.

[2]For certain applications, this last condition may be relaxed near a boundary of M. Terms in the Hamiltonian which fall of the edge may need to be discarded or modified in some arbitrary manner.

If the Hamiltonian in X has terms coupling x_* with some other nearby point, then there is no way to define corresponding terms acting in M, since the nearby point does not correspond to any point in M. More generally, rigidity fails when there are points at which f is not locally invertible; if f is a smooth map between manifolds, this is equivalent to saying that there are points at which its Jacobian vanishes.

For a rigid crystalline gauge field, on the other hand, there is always a well-defined procedure to couple it to the Hamiltonian. The idea is that rigidity guarantees that the local neighborhood is always sufficiently well-behaved that it makes sense to pull terms in the Hamiltonian from X back into M. This is illustrated in Appendix E.2

Finally, let us remark on a interesting property of the the definition of crystalline gauge field: in the case that the whole symmetry group acts internally (that is, the action of G on X is trivial), we might have expected the definition to reduce to the usual notion of a gauge field for an internal symmetry. However, this is evidently not the case, because there is still the map $f : M \to X$ (which in this case must be globally continuous). We believe that, in fact, this may be a more complete formulation of a gauge field for an internal symmetry.

8.3.2 Crystalline topological liquids

From the discussion in the preceding discussion, it might seem that we should only consider *rigid* crystalline gauge fields. Now, however, we want to argue that this is too restrictive. One indeed should require a crystalline gauge field A to be rigid if one wants to go from a Hamiltonian H to a Hamiltonian $H[A]$ coupled to A. But such a microscopic lattice Hamiltonian is a property of the system *in the ultra-violet (UV)*. On the other hand, when classifying topological phases, what we actually care about is the *low-energy limit*. The central conjecture of this work is that it is well-defined

to discuss the low-energy topological response to *any* crystalline gauge field (not just a rigid one).

One reason for this is that a spatially-dependent TQFT that is invariant under a spatial symmetry can be expressed as a single TQFT coupled to a background field which is precisely our crystalline gauge background of Def 2 (with no rigidiy constraints)! This should be compared with the result for internal G symmetry which says that a G action on a (single) TQFT is equivalent to a TQFT with an ordinary background G gauge field. In other words, topological field theories can be gauged and the resulting topological gauge theory retains all the information of the original theory and its symmetry action[184][3]. We discuss this further in section 8.6.

Such considerations provide the mathematical basis for our conjecture about the gauge response. Nevertheless, since these arguments are very abstract and potentially unappealing to readers not familiar with TQFTs, we will also give a more concrete prescription for coupling smooth states (recall that we introduced this concept in Section 8.2) to a general crystalline gauge field. For simplicity, we will only consider the case where there are no orientation-reversing symmetries, although we expect that this restriction can be lifted.

The idea is that there is a simple set of data which one can use to specify a smooth state. Firstly, in the neighborhood of every point in space, we need to specify the orientation of the fine lattice; this can be specified through a *framing* of the manifold M (i.e. a continuous choice of basis for the tangent space at every point). Moreover, in the neighborhood of every point in space, the state looks like it respects the (orientation-preserving) spatial symmetries of the fine lattice (globally, of course, this is not the case). Hence, there is a map $\psi : M \to \Omega$, where Ω is the space of all ground states invariant under the spatial symmetries of the fine lattice. (For our

[3]In the mathematics literature, this is often stated "equivariantization is an equivalence".

arguments, it won't be important to characterize Ω precisely). For a smooth state, we require this map to be continuous.

As a warm-up, we will first show how to define coupling to a gauge field for an *internal* discrete unitary symmetry G in terms of smooth states. Let Ω be a space of ground states, with G acting on Ω as a tensor product over every site, with the action at a given site described by the representation $u(g)$. Let $\psi \in \Omega$ be a G-invariant state. Now, given a framed manifold M and a G gauge field A (i.e. collection of patches on M with G-twisted boundary condition; alternatively, a principal G-bundle over M), we will show how to define a smooth state $\psi[A] : M \to \Omega$. For each g we define a continuous path $u(g;t), t \in [0,1]$ such that $u(g;0) = \mathbb{I}$ and $u(g;1) = u(g)$. Given that ψ is G-invariant, acting with $[u(g;t)]^{\otimes N}$ on ψ defines a *loop* $\psi_g(t) \in \Omega$, such that $\psi_g(0) = \psi_g(1) = \psi$. Then, inside each patch we just set $\psi[A](m) = \psi$. But we decorate patch boundaries twisted by a group element $g \in G$ by the corresponding loop. That is, we require that, as m crosses such a boundary, $\psi[A](m)$ goes through the loop described by $\psi_g(m;t)$. One might wonder whether this procedure is well-defined at the intersections between patch boundaries. For example, an obstruction would occur if the composition of the paths ψ_{g_1}, ψ_{g_2} and $\psi_{(g_1 g_2)^{-1}}$ defines a non-contractible loop, i.e. a non-trivial element in the fundamental group $\pi_1(\Omega)$. In Appendix E.3, we show that such obstructions can never arise, provided that we sufficiently enlarge the on-site Hilbert space dimension. We also give a more rigorous formulation in terms of the classifying space BG.

Now we return to the case of a crystalline gauge field, but by way of simplification we first consider the case where there is no symmetry. Then a crystalline gauge field A on a manifold M is simply a continuous map $f : M \to X$. In general, there is no way to define the Hamiltonian $H[A]$. But for a smooth state $\psi : X \to \Omega$ there is a well-defined way to define a corresponding smooth state $\psi[A] : M \to \Omega$ which

describes ψ coupled to A. Indeed, we just define $\psi[A](m) = \psi(f(m))$. (To completely specify the state, we also have to choose a framing on M). This should be compared with Kitaev's "weak symmetry breaking" paradigm[185], where our Ω plays the role of Kitaev's Y.

Finally, we can combine the ideas from the previous two paragraphs to give a prescription for coupling a smooth state to a crystalline gauge field for a symmetry G acting on X, living on a manifold M. The crystalline gauge field is specified (according to the discussion in Section 8.3.1) by a collection of patches on M with twisted boundaries, and a function $f : M \to X$ respecting the twisted boundary conditions. We assume the symmetry action takes the form $U(g) = S(g)[u(g)]^{\otimes N}$, where $S(g)$ is a unitary operator that simply permutes lattice sites around according to the spatial action, and $[u(g)]^{\otimes N}$ is an on-site action. Then we define a path $u(g;t)$ for $t \in [0,1]$ such that $u(g,0) = \mathbb{I}$, $u(g,1) = u(g)$. By acting with $[u(g;t)]^{\otimes N}$ we obtain a path $\psi_g(x;t)$ in M. It's not a loop this time, though; instead G-invariance of ψ implies that $\psi_g(x;0) = \psi(x)$, $\psi_g(x;1) = \psi(gx)$. Now we can define the coupled state $\psi[A]$ as follows. Inside each patch, we have $\psi[A](m) = \psi(f(m))$. Then, for patches connected by boundaries twisted by $g \in G$, we connect up the $\psi[A]$ in the respective patches by means of the paths $\psi_g(x;t)$. The previously noted endpoints of these paths are consistent with the fact that $f(m)$ jumps to $gf(m)$ as one crosses the boundary. Again, we defer the proof that this procedure is well-defined at the intersection of boundaries to Appendix E.3.

At this point, the careful reader might raise an objection. In our statement of the conjecture about coupling to a crystalline gauge field, we did *not* require the manifold M to be framed, only orientable (the orientability condition comes from our stipulation that there are no orientation-reversing symmetries, and from the compatibility condition between the orientation bundle of M and the crystalline gauge

field discussed in Section 8.3.1 and again in Section 8.6). But so far, our smooth state arguments only showed how to couple to crystalline gauge fields on framed manifolds. There are two questions that still need to be addressed:

- Question 1. Does the topological response depend on the choice of framing?
- Question 2. Can the topological response be defined on oriented manifolds that do not admit a framing?

These questions need to be addressed in any formulation of continuum limit. For bosonic systems we expect that the continuum limit, if it exists, can be defined on any oriented manifold and doesn't depend on any extra structure. For fermionic systems it also can depend on a spin or spin^c structure. There are of course systems which, while gapped, still exhibit some metric or framing dependence in the IR, eg. Witten's famous framing anomaly of Chern-Simons theory [186]. We will later approach these questions in the TQFT framework of section 8.6. For now let us think about these questions from the perspective of smooth states.

For Question 1, we observe that that changing the framing corresponds to changing the fine lattice, and generally speaking, most topological phases have a "liquidity" property that ensures that the ground states on different lattices can be related by local unitaries. Since the states live on different lattices, this requires bringing in and/or removing additional ancilla spins that are not entangled with anything else, as is standard protocol when defining local equivalence of quantum states. Such a liquidity property will be necessary for the crystalline topological liquid condition to be satisfied. There are some notable exceptions, such as fracton phases [179], of which a simple example is a stack of toric codes. We do not expect such fracton phases to be crystalline topological liquids.

As for Question 2, we believe that the answer is probably yes. To illustrate the

issues at play, consider the 2-sphere. This is an orientable 2-manifold which does not admit a framing. As a consequence, there is no way to put a regular square lattice on a 2-sphere; there must be at least a singular face which is not a square or a singular vertex which is not 4-valent. So one cannot strictly define a smooth state. But we expect that there are ways to "patch up" such singular points so that they don't affect the long-range topological response. For example, the toric code is usually defined on a square lattice, which cannot be placed onto the sphere, but it is easy to put a toric-code-like state on the sphere by allowing a few non-square faces.

We emphasize that coupling to non-rigid crystalline gauge fields is what allows us to establish the crystalline equivalence principle. For example, for internal symmetries one could consider braiding symmetry fluxes around each other. Does this make sense in the case of, for example, disclination defects? If the disclinations were interpreted strictly as lattice defects this would not be possible, since there is no continuous deformation of a lattice containing two disclinations such that the two disclinations move around each other with the lattice returning to its original configuration. But if we interpret disclination defects as special cases of (generally non-rigid) crystalline gauge fields, then this braiding process is allowed. The physical interpretation is that in the course of the braiding process, additional sites get coupled to, and superfluous sites decoupled from, the system by means of local unitaries (as discussed above in the context of the framing dependence). That is, the lattice geometry changes along the path.

In conclusion, this discussion motivates our terminology of "crystalline topological liquid": although such systems are "crystalline" in the sense that they have spatial symmetries, they are also "topological liquids" in the sense that the lattice is not fixed but can be transformed into other geometries by means of local unitaries (with ancillas). This is also consistent with our picture from Section 8.2 that the topological

response of crystalline topological liquids "forgets" about the lattice.

8.3.3 The Crystalline Equivalence Principle

Most of the time, we will be interested in topological crystalline phases in Euclidean space $X = \mathbb{R}^d$. Moreover, the topological response should only depend on the deformation class of the crystalline gauge field. It turns out that for $X = \mathbb{R}^d$ there is a very simple characterization of the collapsible homotopy classes of crystalline gauge fields:

Theorem 3. *If X is contractible (e.g. $X = \mathbb{R}^d$), then the deformation classes of crystalline gauge fields are in one-to-one correspondence with* internal *gauge fields.*

That is, in the "patches" formulation of crystalline gauge fields, the deformation classes remember only the twisted boundary conditions and not the function $f : M \to X$. This theorem is a corollary of the more general classification theorem for crystalline gauge fields. See Thm 7. However, here we remark on an elementary way to see one part of Thm 3: namely, that homotopy classes can only depend on the twisted boundary conditions. (For the moment we will not attempt to prove the other part, namely that *any* configuration of twisted boundary conditions has at least one function f respecting it). Although the proposition holds more generally, for simplicity we consider the case where $X = \mathbb{R}^d$ and where the G action on X is affine linear:

$$gx = A_g x + b_g, \tag{8.3}$$

where A_g is a $(d \times d)$ matrix and b_g is a length d vector. We then observe that given a patch configuration on M with twisted boundary conditions, and two maps $f_0 : M \to X$ and $f_1 : M \to X$ respecting the same twisted boundary conditions, then

there is a continuous interpolation

$$f_s = (1-s)f_0 + sf_1, \tag{8.4}$$

which respects the same twisted boundary conditions all the way along the path.

Thm 3 allows us to deduce the most important result of this paper. Thm 3 shows that deformation classes of crystalline gauge fields are in one-to-one correspondence with principle G-bundles. On the other hand, deformation classes of gauge fields for an *internal* symmetry also correspond to principal G-bundles. Topological phases are distinguished by their response to background gauge fields. Therefore we conclude the

> **Crystalline Equivalence Principle**: The classification of crystalline topological liquids on a contractible space with spatial symmetry group G is the *same* as the classification of topological phases with *internal* symmetry G.

To be precise, the orientation-reversing symmetries on the spatial side are identified with the anti-unitary symmetries on the internal side.

8.3.4 Beyond Euclidean space

Before we delve into the details of how to classify crystalline topological liquids by their topological response to gauge fields, we recall that the above considerations refer to topological phases that exist in Euclidean space $\mathbb{R}^d$. In principle one can consider the more exotic problem of classifying topological phases on non-contractible spaces; for example, the d-sphere, the d-torus, or a Euclidean space with holes[4]. The practical

[4]We emphasize that, in the absence of translation symmetry, it does not make sense to relate a topological phase defined on one compact space to one defined on another space with different

relevance of this problem may be a bit obscure, but from a theoretical point of view we find it more enlightening to formulate the problem we are interested in – Euclidean space – as a special case of the more general problem. It also illustrates an important conceptual point: the Crystalline Equivalence Principle is not something that *a priori* had to be true. Rather, it is a consequence of the fact that systems of physical interest live in Euclidean space.

On contractible spaces, we had the classification Theorem 3 for crystalline gauge fields. This classification theorem is a special case of the more general result (see Appendix E.4 and Theorem 7) that deformation classes of crystalline gauge fields $M \to X$ are classified by homotopy classes of maps from M into the "homotopy quotient" $X//G$, pronounced "X mod mod G". For X contractible, $X//G$ is homotopic to the "classifying space" BG, so we recover Theorem 3 if we invoke the well-known fact that principal G-bundles over M are classified by homotopy classes of maps $M \to BG$.

8.4 Exactly solvable models

It is of course important to show that we can explicitly construct Hamiltonians realizing topological crystalline phases classified in this work. We do this using a "bootstrap" construction. This is really a meta-construction, in the sense that it is a prescription for going from a construction for an SPT or SET phase with *internal* symmetry to a construction for a topological crystalline phase. A similar idea was used by one of us to construct phases of matter protected by time-translation symmetry in Ref. [107].

For simplicity we consider the case where the entire symmetry group G acts spa-

topology. That is, the classification can depend on the background space.

tially, i.e. the internal subgroup is trivial. We will also consider the case where G does not contain any orientation-reversing transformations, and we work in Euclidean space, $X = \mathbb{R}^d$. First of all, let φ be a surjective homomorphism from the symmetry group G to a finite group G_f. We use one of many approaches to construct a topological liquid with an *internal* symmetry G_f. In most of these approaches, there is no obstacle to construct the Hamiltonian to *also* have a spatial symmetry G, which commutes with G_f so that the full symmetry group is $\widetilde{G} = G \times G_f$ (for example, in the case of bosonic SPTs, this can be shown explicitly using the construction of Ref. [80], as detailed in Appendix E.5). We then can imagine deforming Hamiltonian to break the full symmetry group $\widetilde{G}$ down to the *diagonal subgroup*

$$G' = \{(g, \varphi(g)) \in \widetilde{G}\} \cong G. \tag{8.5}$$

We expect that this model will be in the topological crystalline phase that corresponds to the internal symmetry-protected phase we started with via the crystalline equivalence principle. Indeed, we can do this construction on a lattice with lattice spacing much less than the unit cell size (thus giving a smooth state), and verify that, for the original model (without the $\widetilde{G}$-breaking perturbation), following the prescription given in Section 8.3.1 to couple to a crystalline gauge field for the diagonal subgroup G' gives the same result as coupling to an internal gauge field for the internal subgroup G. (A similar argument can be given in the spatially-dependent TQFT picture of Section 8.6).

Let us briefly sketch how to extend the above construction to symmetry groups G containing orientation-reversing transformations. A general topological phase is not reflection-invariant, so the above argument needs to be modified. We expect that a topological liquid can always be made invariant under a spatial symmetry G if

we make the orientation-reversing elements of G act *anti-unitarily*; we can call this suggestively the "CPT princple"[5] We prove this explicitly for bosonic SPT phases in Appendix E.5. We then proceed as before, starting from a $(G \times G_f)$-symmetric topological phase, where the internal symmetry $\varphi(g) \in G_f$ acts anti-unitarily if g was orientation-reversing. Then eventually the symmetry gets broken down to the diagonal subgroup G', which contains spatial symmetries, possibly orientation-reversing, but all acting unitarily (since the orientation-reversing elements of G, which we have taken to act anti-unitarily, get paired with anti-unitary elements of G_f). We expect that this gives the topological crystalline phase corresponding to the original internal symmetry-protected phase via the crystalline equivalence principle, but explicitly determining the topological response would involve explaining what it means to gauge an anti-unitary symmetry, which we will not attempt to do (but see Ref. [187].)

8.5 Topological Response and Classification

In this section, we will discuss how our understanding of what it means to gauge a spatial symmetry allows us to classify topological phases by their topological responses. Basically, any approach to understanding topological phases with *internal* symmetries which relies on gauging the symmetry, can be applied equally well to space-group symmetries by coupling to crystalline gauge fields. Moreover, in Euclidean space, Theorem 7 should imply that we obtain the same classification as for internal symmetries, in accordance with the Crystalline Equivalence Principle. In non-contractible spaces we may obtain a different classification.

There are two main approaches to thinking about topological response. The first

[5]This is related to, but not a consequence of, the CPT *theorem*, because here we are talking about lattice models, not relativistic quantum field theories. The CPT principle doesn't claim that *every* lattice model is CPT invariant, which would be demonstrably false; rather, it posits that in any topological phase there is at least one CPT-invariant point.

is a bottom-up approach where one starts with a Hamiltonian in a lattice model and one attempts to work out all the topological excitations. For example in 2+1D, one has anyons and symmetry fluxes and one can ask about how they interact. This is tabulated mathematically in a G-crossed braided fusion category [96, 188] and one can try to work out a classification of these objects or at least find some interesting examples and then look for lattice realizations.

The second approach is a top-down one where one first assumes the existence of a low energy and large system size ("IR") limit of the gapped system. This is a topological quantum field theory (TQFT) of some sort and one can just try to guess what it is from the microscopic symmetries, entanglement structure (short-range vs. long-range), and so on. One can make a bold statement that all possible IR limits are of a certain type of TQFT and then try to classify all of those. Despite its obvious lack of rigor, this approach has proven successful.

One reason for this is that it is often possible to bridge the two perspectives. For example, a G-SPT can be understood in terms of an effective action $\omega \in H^{D+1}(BG, \mathbb{Z})$[80, 176] leading ultimately to a TQFT. But considering the fusion of symmetry fluxes also leads to an element of $\mathcal{H}^D(G, U(1))$ through a higher associator of symmetry fluxes (in $2+1$-D, it is the F symbol). These are equivalent under the isomorphism $H^{D+1}(BG, \mathbb{Z}) = \mathcal{H}^D(G, U(1))$. In general, defects such as anyons and symmetry fluxes can be described in the TQFT framework through the language of "extended TQFT".

Let us now discuss how these methods can be extended to the case of spatial symmetries.

8.5.1 Flux fusion and braiding for SET phases in (2+1)-D with spatial symmetry

If we want to classify symmetry-enriched phases in (2+1)-D phases we can consider the "bottom-up" approach of Ref. [96]. There, one has a topological phase with an internal symmetry G, and one envisages coupling to a classical background gauge field. In particular, one can consider gauge-field configurations in which the gauge fluxes are localized to a discrete set of points. One can then consider the algebraic structure of braiding and fusion of such gauge fluxes, which is an extension of the braiding and fusion of the intrinsic excitations (anyons) that exist without symmetry. This structure is argued to be described by a mathematical object called a "G-crossed braided tensor category". For a crystalline topological liquid on Euclidean space, we expect that the equivalence between crystalline gauge fields and G-connections allows the arguments to carry over without significant change. (We will leave a detailed derivation for future work.) On non-contractible spaces, presumably a generalization of the arguments of Ref. [96] should be possible, but we will not explore this.

8.5.2 Topological Response as Effective Action

Another way to compute topological response, which does not involve braiding or fusing fluxes is by computing twisted partition functions. That is, given a background gauge field (ordinary or crystalline) A on a spacetime M, we can compute the partition function of $Z(M, A)$ and compare it to the untwisted partition function $Z(M)$. The assumption is that

$$Z(M, A)/Z(M)$$

tends to a complex number of modulus 1 in the limit that M becomes very large compared to the correlation length. In favorable situations, such as a crystalline topological liquid, the limiting phase is a topological invariant of M and its gauge background A. We call this the topological response of our system to A and its log the effective action for the gauge background A. In some cases, like $M = Y \times S^1$, $Z(M, A)$ can be interpreted as some kind of "twisted trace" of symmetry operators, as we soon discuss. In general there is such an interpretation but it involves topology-changing operators [189]. [6]. What is most important for classification of phases is that it is a number that captures some (or all) of the data in a "spatially-dependent TQFT", which we introduce in Section 8.6 as the mathematical way to describe a "crystalline topological liquid" phase of matter.

For internal symmetries of bosonic systems, we know that in this case, the limiting ratio can be written

$$Z(M, A)/Z(M) \to \exp\left(2\pi i \int_M \omega(A)\right), \tag{8.6}$$

where $\omega(A)$ is a gauge-invariant top form made out of the gauge field. In the case of a crystalline gauge field $A = (P, M, \pi, \hat{f})$, we will also assume that the topological response is an exponentiated integral:

$$Z(X, A)/Z(X) \to \exp\left(2\pi i \int_M \omega(\alpha, \hat{f})\right), \tag{8.7}$$

where $\omega(\alpha, \hat{f})$ is a top form on M made of the twisting field $\alpha \in H^1(M, G)$ which

[6] Indeed, on a general spacetime, a generic choice of time direction defines a Morse function and a foliation of spacetime by spatial slices. At critical points of this Morse function, the spatial slice is singular and we have a topology changing operator that gets us from the Hilbert space just before the critical point to the Hilbert space just after. These are all handle attachments and can be thought of as generalized flux fusion processes.

classifies the cover P and the map $\hat{f}$, used to pull back densities from X. In the case that G is purely internal, α plays the role of A in (8.6).

As discussed in Ref [176], responses of the form (8.6) are the same thing as cocycles in group cohomology, defined as cohomology of the classifying space $H^{D+1}(BG, \mathbb{Z})$, where D is the dimension of spacetime X. This reproduces the classification of internal symmetry bosonic SPTs in Ref [80]. To construct the effective action of A, we use the fact that the gauge field A itself is the same as a map $A : X \to BG$, and given a D-cocycle on BG, we can pull it back along this map to get $\omega(A)$ over X.

Analogously, we can think of our crystalline gauge field as a map $A : M \to X//G$ (see Appendix E.4) and take any form in $H^D(X//G, U(1))$, pull it back along this map to M to get a $\omega(\alpha, \hat{f})$ and integrate it. We just need to be a little careful with coefficients. We intend to integrate $\omega(\alpha)$ over M, but if G contains orientation-reversing elements like mirror and glide reflections (or time reversal), then M may likely be unorientable. Integration on an unorientable M is done by choosing a local orientation: orienting M away from some hypersurface N and performing the integration on $M - N$ with its orientation. To ensure the integral does not depend on this local orientation, we need our top form $\omega(\alpha)$ to switch sign with the local orientation is reversed. Mathwise, this means that $\omega(\alpha)$ should live in cohomology $H^D(M, U(1)^{or})$ with twisted coefficients $U(1)^{or}$. Luckily, if X is orientable, then the unorientability of M is entirely due to orientation-reversing elements of G, so if we use twisted cohomology $H^D(X//G, U(1)^{or})$ where orientation-reversing elements of G act on $U(1)$ by $\theta \mapsto -\theta$, then the coefficients will pull back properly. This cohomology group is well known in algebraic topology as the equivariant cohomology of X, and is written

$$H^D_G(X, U(1)^{or}) := H^D(X//G, U(1)^{or}).$$

Another subtlety comes from considering the identity map $M = X \to X$ as a crystalline gauge field. Any non-trivial topological response to the identity cover is equivalent to a shift of all the partition functions by a phase. We may as well consider only the subgroup of all equivariant cohomology classes which pulled back along the identity map are trivial. This is called reduced cohomology and is denoted with a tilde $\tilde{H}$.

Summarizing, we find:

Theorem 4. *Homotopy-invariant effective actions in $D = d+1$ spacetime dimensions for crystalline gauge fields $A : M \to X//G$ which may be written as integrals over M are in correspondence with "twisted reduced equivariant cohomology":*

$$\tilde{H}_G^{D+1}(X, \mathbb{Z}^{or}).$$

8.6 Spatially-dependent TQFTs

Here, we will explain our proposal for the description of the low-energy limit of a crystalline topological phase in terms of a TQFT. In this setting, our results, such as the crystalline equivalence principle, and the fact that the low-energy limit can be coupled to an arbitrary crystalline gauge field, can be proven mathematically. We will focus here on the physical motivations; however, we give enough detail that the full mathematically rigorous treatment should be apparent to TQFT experts.

Recall that the starting point is that a phase of matter should have a spatially-dependent "topological limit", which we expect to be described by a *spatially-dependent TQFT*. Indeed, we define

Definition 4. A $(d+1)$-*dimensional spatially-dependent TQFT* on a space X is a continuous map $\sigma : X \to \Theta$, where Θ is the space of all $(d+1)$-dimensional TQFTs.

Now, what exactly do we mean by "space of all TQFTs"? Familiar notions of TQFTs (at least in 2+1D) look quite rigid, suggesting that any such space would be discrete. However, we want to argue that there is a natural way to think about TQFTs as living in a richer topological space Θ. First of all, we note that for classifying phases of matter it will not be necessary to specify Θ exactly, only up to homotopy equivalence. Let us discuss a physical motivation for the homotopy type of Θ.

Generally, specifying the homotopy type of a topological space involves identifying points, paths between points, deformations between paths, and so on. The idea is that the structure of Θ should represent features of ground states of quantum lattice models. Thus, the points in Θ should correspond to ground states of quantum lattice models; the paths in Θ should correspond to continuous paths of ground states of quantum lattice models; and so on. There is another way to interpret these statements. A path in the space of ground states of quantum lattice models can also be implemented spatially, giving rise to an interface of codimension 1. Similarly, deformations between paths give rise to interfaces of codimension 2 between interfaces of codimension 1, and so on. (See Figure 8.5).

Roughly, therefore, the idea is that Θ should have the homotopy type of a cell complex with vertices v labeled by $(d+1)$-dimensional TQFTs $T(v)$. Edges $e : v \to w$ are labeled by invertible d-dimensional topological defects $D(e)$ between $T(v)$ and $T(w)$. 2-Cells f with $\partial f = v_1 \xrightarrow{e_{12}} \cdots v_n \xrightarrow{e_{n1}} v_1$ are labeled by invertible $d-1$-dimensional junctions between the defects $D(e_{12}) \cdots D(e_{n1})$. This continues all the way down to 0-dimensional defects, which for topological field theories with a unique ground state on a sphere is a copy of the complex numbers. [7] In [188], this space

[7] Note that if two topological theories share an invertible topological defect, it means they are isomorphic, so in a formulation of TQFT up to isomorphism, eg. modular tensor category, each component of Θ will have a single vertex, perhaps with many other cells attached to it. In a state sum or tensor network formulation, on the other hand, there could be lots of state sums giving rise to the same TQFT with invertible MPO defects between them[180].

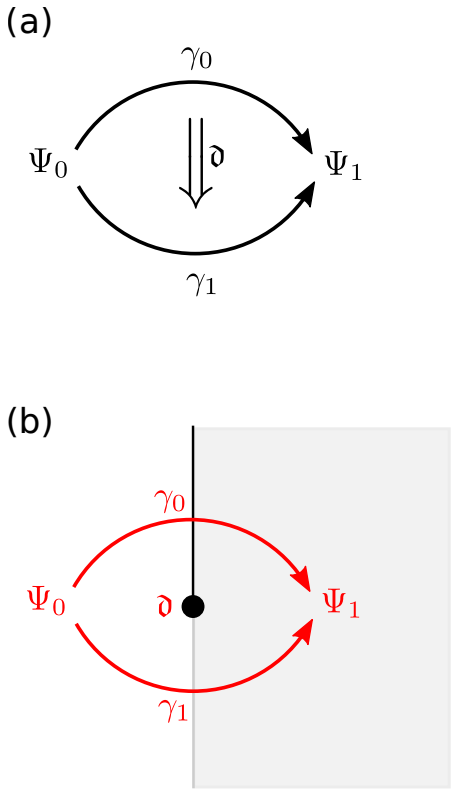


Figure 8.5: (a) Specifying the homotopy type of the space Θ of all TQFTs involves specifying points in this space, paths between arrows (single arrows), deformations between paths (double arrow), and so on. We want these to capture features of the space of quantum ground states. (b) These features can also be interpreted as interfaces. Depicted is a spatial configuration of interfaces in a 2-dimensional system, with two 1-dimensional interfaces separated by a junction of dimension 0. We can imagine that these interfaces are "smoothed out" such that the spatial variation occurs on scales large compared to the lattice spacing (thus, we have a a "smooth state" as discussed in Sections 8.2 and 8.3.2). Traversing a path in $\mathbb{R}^2$ from the left half-plane to the right half-plane, the local quantum state goes through the path γ_0 or γ_1 depending on whether the path in $\mathbb{R}^2$ goes through the upper 1-dimensional interface or the lower one. As one deforms the path in $\mathbb{R}^2$ through the 0-dimensional junction (black dot), the corresponding path in the space of quantum states goes through the deformation described by $\mathfrak{d}$.

was considered for $d = 3$ in the tensor category framework and was referred to as the Brauer-Picard 3-groupoid.

A version of the bulk-boundary correspondence says that the set of boundary conditions and boundary operators determines the bulk topological field theory (see [190] for some perspective on this in general dimensions and [191–193] in 2+1D especially). For theories admitting gapped (therefore topological in the IR) boundary conditions, this is the Baez-Dolan-Lurie cobordism theorem (sometimes "hypothesis") [189, 194], which characterizes possible boundary data as special objects in a $d+1$-category $\mathcal{C}$. This characterization can be used to construct Θ in a mathematically precise way. (Specifically, it is a space whose homotopy type is described by the core of the category $\mathcal{C}$).

Let us now consider the effect of symmetries. There is a natural way to define a G-action on a TQFT. From the Baez-Dolan-Lurie framework, one can show that a TQFT with symmetry G is equivalent to TQFT coupled to a background G gauge field. What we mean by the latter is the following. A $(d+1)$-dimensional TQFT assigns topological invariants to manifolds; for example, it assigns complex numbers (the partition function) to $(d+1)$-dimensional manifolds, and finite-dimensional Hilbert spaces (the state space) to d-dimensional manifolds. A $(d+1)$-dimensional TQFT coupled to a background G gauge field assigns invariants to G-manifolds: manifolds decorated with G gauge fields. Physically, this is supposed to describe response the topological response of the system to background gauge fields. We want to extend this result to systems with spatial symmetries.

Let us first review the case of a TQFT $\theta \in \Theta$ with an internal unitary G-action. Indeed, we define:

Definition 5. A G action on a TQFT is a collection of isomorphisms $\phi_g : \theta \to \theta$ for

each $g \in G$, with consistency data.

In fact, in the Baez-Dolan-Lurie framework discussed above, isomorphisms are just paths in the space Θ. These have the interpretation of defects of codimension 1. In fact, these are just symmetry twist branch cuts (e.g. see Ref. [96]), such that particles moving through them get acted upon by the symmetry G. What we mean by "consistency data" is that the implementation of the relations of G are also data in the G-action (see for instance [195]). This data describes the codimension 2 junctions where domain walls fuse, the codimension 3 singularities where two junctions slide past each other, and so on. In fact, a more succinct way to formulate this definition is that a (anomaly-free, see below) TQFT with G symmetry is a continuous map $\phi : BG \to \Theta$. The statement about equivalence between TQFTs with G-action and TQFTs coupled to background gauge field then follows from the following general consequence of the Baez-Dolan-Lurie framework (see Thm 2.4.18 of [194]):

Lemma 1. *For any space W, a continuous map $f : W \to \Theta$ is equivalent to a TQFT for manifolds equipped with maps into W.*

Indeed, we set $W = BG$ and note that maps into BG are the same as G gauge fields.

Finally, we are ready to consider the general case of a spatially-dependent TQFT with a spatial symmetry G. We define:

Definition 6. A $(d+1)$*-dimensional spatially-dependent TQFT with symmetry G* on a space X is an action of the group G on X along with a G-equivariant map $\sigma : X \to \Theta$, meaning for all x and g we have a choice of isomorphism

$$\phi_{g,x} : \sigma(g \cdot x) \simeq \sigma(x). \tag{8.8}$$

(with consistency data).

Note that the isomorphisms should be taken to be unitary or anti-unitary for orientation-preserving or orientation-reversing symmetries respectively.

Once all the appropriate consistency data has been taken into account, we find that a spatially-dependent TQFT with an orientation-preserving spatial symmetry G corresponds to a map from the homotopy quotient $X//G$ (discussed in section 8.3.4 and appendix E.4) into Θ. (We will not discuss the orientation-reversing case here). Applying Lemma 1, we find

Theorem 5. *A $(d+1)$-dimensional spatially-dependent TQFT on X with symmetry G is equivalent to a TQFT for $(d+1)$-manifolds M equipped with a (homotopy class of) map $M \to X//G$, where $X//G$ is the homotopy quotient we have discussed in section 8.3.4.*

This statement suggests that we can consider any map $M \to X//G$ as a crystalline gauge background, whereas in section 8.3.1 we only showed how to couple a Hamiltonian to a rigid crystalline gauge background. Indeed, spatially-dependent TQFT mathematically formalizes our notion of smooth states in section 8.3.2 and appendix E.3. Further, restricting to the case that X is contractible, $X//G$ is homotopy equivalent to BG, so we find the same classification whether G acts internally or on X.

8.7 Open problems

In this work we have presented a general framework for understanding the classification of interacting topological crystalline phases, for both bosons and fermions.

An important question for future work is to understand the physical signatures of these phases.

The classic signature of an SPT phase is the protected gapless modes on the boundary (though in strongly interacting systems the boundary can also spontaneously break the symmetry or be topologically ordered). One would expect similar statements to hold for crystalline SPT phases, but there are some caveats. Firstly, of course, a boundary will in general explicitly break the spatial symmetry down to a subgroup, and one only expects protected modes when the phase is still non-trivial with respect to this subgroup. But even then there are exceptions. For example, an SPT protected in 1-D by inversion symmetry about $x = 0$ does *not* have a protected degeneracy when placed on the interval $[-L, L]$, even though the entire boundary (comprising two points) is in fact invariant under the symmetry [72]. Another example is a phase in 2-D with a C_4 rotation symmetry, which can be constructed using the techniques of Ref. [175]. A ground state in this phase is equivalent by a local unitary to a product state, with a C_4 charge pinned to the origin; therefore, there will not be any non-trivial edge states for any choice of boundary. Thus, it is still an open question to determine what is the criterion which ensures protected boundary modes. A way to answer this would be to extend our spatial symmetry gauging procedure to systems with boundary. This is, however, beyond the scope of the present work.

Another question is the robustness of the topological crystalline phases that we have found to disorder, which explicitly breaks the spatial symmetries. There are some topological crystalline phases which have been argued to be robust to disorder, so long as the spatial symmetry is respected *on average* [161]. It would be interesting to determine the general circumstances under which this happens.

Chapter 9

Systematic treatment of various "Equivalence Principles"

In this chapter, I will extend and unify some of the ideas touched on in previous chapters into a more coherent framework.

9.1 Homotopy theory viewpoint on the classification

The main tool in this chapter will be a powerful viewpoint on topological phases with and without symmetries, that allows us to reason in very general terms about such phases. It was introduced by Kitaev in Appendix F of Ref. [185], and we review it here.

The central idea is that in each spatial dimension d, the set of all possible GAPGRND states (recall the discussion of GAPGRND states in Section 2) with dimension-k spins at each site should form a *topological space* $\Omega_d^{(k)}$. As we do not want to have any restriction coming from local Hilbert space dimension, we will consider the limit $\Omega_d := \Omega_d^{(\infty)}$. The problem of classifying topological phases amounts to finding the connected components $\pi_0(\Omega_d)$.

To classify topological phases with symmetry, we have to specify an action of a symmetry group G on Ω_d, and then we can find the connected components of Ω_d^G, the subspace of Ω_d left fixed by the action of G. However, generally in classification of phases we want to identify phases in systems transforming under different representations of G as being in some sense the "same phase" (in the sense, for example, that they have the same low energy physics).

The key insight of Kitaev was to identify an invariant that is not sensitive to the microscopic details. Specifically, it can be shown (see Appendix F of Ref. [185]) that for any unitary action of G on $\omega \in \Omega_d^G$, there is a corresponding map from $BG \to \Omega_d^{(\infty)} := \Omega_d$, where BG is the so-called "classifying space" of the group G, which is defined to be $BG = EG/G$, where EG is a contractible space on which G acts freely. (The resulting BG is independent of the choice of EG, up to homotopy equivalence[1]). It is then reasonable to conjecture that topological phases correspond to homotopy classes of maps $BG \to \Omega_d$. The classification problem is then reduced to understanding the structure of the space Ω_d.

Most of the well-known partial classification results for topological phases with symmetry can be interpreted as arising from some partial understanding of or approximation to the space Ω_d. For example, Kitaev (Appendix F of Ref. [185]) derives the classification of symmetry fractionalization on anyons from this point of view. Moreover, all of the proposed partial classifications for invertible phases seem to take the form of generalized cohomology theories, and any such generalized cohomology theory classifies homotopy classes of maps $BG \to \Omega_d$ for some appropriate choice of space Ω_d for each dimension d (the "spectrum") [196, 197].

We will also mention an even more abstract point of view. Generally, we expect

[1]Two spaces X and Y are homotopy equivalent if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \cdot g$ and $g \cdot f$ are homotopic to the identity maps $Y \to Y$ and $X \to X$ respectively.

that the low-energy physics of a topological phase should be described by a topological quantum field theory (TQFT). As discussed in Section 8.6, there is a well-defined sense in which we can define a space of n-dimensional TQFTs, which we call Θ. A TQFT with a symmetry action G turns out to be equivalent to a map $BG \to \Theta$. Therefore, we can conjecture that topological phases with symmetry G should be classified by homotopy classes of maps $BG \to \Theta$. This is equivalent to the previous classification (homotopy classes of maps $BG \to \Omega$) if the spaces Θ and Ω are homotopy equivalent, which we conjecture to be the case. In fact, since the definition of TQFT is not entirely fixed – in mathematical language, we have the freedom to choose the category which the TQFT functors should target – one might even say that we ought to *choose* the target category in order to ensure that the homotopy equivalence $\Theta \simeq \Omega$ holds, since that ensures that the TQFTs are accurately capturing the physics of microscopic ground states.

Finally, let us note that all the statements we have made relate to systems with *unitary* symmetries. In general, we also want to consider systems with anti-unitary symmetries (such as time reversal). In the TQFT formalism, there is a natural extension of the above discussion to anti-unitary symmetries, but we do not know how to extend Kitaev's more microscopic argument. Therefore, in this chapter we will mainly restrict ourself to unitary symmetries. However, we do expect that the results will hold also for anti-unitary symmetries, for reasons that we will mention.

9.2 The Floquet equivalence principle

The Floquet Equivalance Principle conjectured in Chapter 5 can be rigorously proven if we assume the homotopy-theoretic viewpoint discussed above. Recall that this Floquet Equivalence Principle states that

Floquet topological phases with symmetry G are in one-to-one correspondence with stationary topological phases with symmetry $G \times \mathbb{Z}$.

First we need to give a precise definition of what we mean by "Floquet topological phase". Here we will be mainly concerned with the non-trivial topological features that can be observed in a single eigenstate of the Floquet evolution operator, as in the cases considered in Chapter 5. (In Floquet-MBL systems, where every eigenstate of the Floquet evolution operator is localized, there may be non-trivial topological features of the entire Floquet evolution that are not observable in a single eigenstate [198–202]; we are explicitly not considering such features here).

The new ingredient in a Floquet system, compared to a stationary system, is that even an eigenstate of the Floquet evolution undergoes a non-trivial evolution over one driving period. Let us consider a time-periodic Hamiltonian $H(t)$, with $H(t+T) = H(t)$, and we define the Floquet evolution operator

$$U_f = \mathcal{T} \exp\left(-i \int_0^T H(t) dt\right). \tag{9.1}$$

Then for any eigenstate $|\Psi\rangle$ of U_f, we can define a family of states,

$$|\Psi(t)\rangle = \mathcal{T}' \exp\left(-i \int_0^t H(t') dt'\right) |\Psi\rangle \tag{9.2}$$

which describes the micromotion of the eigenstate over a driving period. Now let us assume that $|\Psi\rangle$ is a GAPGRND state (as it is in a Floquet-MBL system, for example). It follows that $|\Psi(t)\rangle$ is a GAPGRND state for all t, because according Eq. (9.2), $|\Psi(t)\rangle$ is related to $|\Psi\rangle$ by a local unitary $U(t)$, and if K is a local Hamiltonian which has $|\Psi\rangle$ as its gapped ground state, then $U(t)KU(t)^{-1}$ has $|\Psi(t)\rangle$ as its gapped ground state.

In particular, since $|\Psi(T)\rangle \propto |\Psi\rangle$, $|\Psi(t)\rangle$ defines a *loop* in the space of GAPGRND states in d dimensions, Ω_d, introduced in Section 9.1, which means a map from the circle $S^1 \to \Omega_d$. Hence, it is reasonable to define a Floquet topological phase (without symmetry) to be a homotopy class of such maps (i.e. an equivalence class under continuous deformations).

But now we can recall that, in the homotopy theoretic framework, a stationary phase with a $\mathbb{Z}$ symmetry is supposed to correspond to a map $B\mathbb{Z} \to \Omega_d$, where $B\mathbb{Z}$ is the classifying space of $\mathbb{Z}$. The Floquet Equivalence Principle for Floquet topological phases without symmetries then immediately follows from the mathematical fact that

$$B\mathbb{Z} \simeq S^1. \tag{9.3}$$

Here the "$\simeq$" symbol denotes homotopy equivalence.

We can easily generalize this argument to systems with symmetries. Suppose that the Hamiltonian $H(t)$ commutes (at all times) with a representation $U(g)$ of an internal symmetry G. Suppose that we have an eigenstate $|\Psi\rangle$ of U_f which is also invariant under G (that is, the symmetry G is not spontaneously broken). It then follows that the path $|\Psi(t)\rangle$ also is invariant under $U(g)$ for all t. Then, following similar arguments again to Appendix F of Ref. [185], one wants to classify maps $S^1 \times BG \to \Omega_d$. A Floquet topological phase with symmetry G corresponds to an equivalence class of this map. The Floquet Equivalence Principle then follows from the equation

$$B(\mathbb{Z} \times G) \simeq S^1 \times BG. \tag{9.4}$$

9.3 Symmetry-breaking phases

The Floquet Equivalence Principle introduced in Chapter 5 was initially stated for Floquet *topological* phases: that is, those which do not spontaneously break any symmetries. However, it is natural to ask whether the correspondence also holds for spontaneous symmetry-breaking phases. For example, a stationary phase with symmetry $\mathbb{Z}$ ought to be able to spontaneously break $\mathbb{Z} \to n\mathbb{Z}$. The corresponding Floquet phases are, of course, the time crystals discussed in Chapter 6.

In fact, it is straightforward to argue that the correspondence does indeed hold for all kinds of phases of matter with symmetry, both spontaneous symmetry breaking and topological. Indeed, we expect that all such phases are describable in terms of the homotopy theoretic framework discussed in the previous sections, provided that one replaces the space Ω_d discussed in the previous section (the space of ground states in d spatial dimensions) with the space $\widetilde{\Omega_d}$ of all possible GAPGRND *multiplets*. A GAPGRND multiplet is just a set of n orthogonal GAPGRND states.

By similar arguments to before, we find that any GAPGRND multiplet that is mapped onto itself by a symmetry G (the individual GAPGRND states may get permuted among themselves) gives rise to a map $BG \to \widetilde{\Omega_d}$. The rest of the arguments from Section 9.2 carry over, and so we obtain a more general form of the Floquet Equivalence Principle that holds for all phases, topological and spontaneous symmetry breaking.

9.4 More general temporal symmetries

Now that we have put the Floquet equivalence principle on general footing, we are able to make a more general statement. Let $|\Psi(t)\rangle$ be any time-dependent family of

states, i.e. a map $\mathbb{R} \to \Omega_d$. It is not important for the argument that the parameter represent time, but in applications to Floquet systems, this will be its interpretation. Now we consider a symmetry G with some continuous action $\alpha : G \times \mathbb{R} \to \mathbb{R}, (g,t) \mapsto \alpha_g(t)$. For example, in the case of discrete time translation generated by $\mathbb{T}$, we have $\alpha_{\mathbb{T}^n}(t) = t + nT$. Moreover, the specification of the symmetry action also involves an on-site unitary representation $U(g)$. We say that the family of states $|\Psi(t)\rangle$ is invariant under the temporal symmetry described by the pair (α, G) if

$$U(g)\,|\Psi(\alpha_g(t))\rangle \propto |\Psi(t)\rangle\,. \tag{9.5}$$

More generally, we can say that a family of multiplets $\mathcal{P}(t)$ is a *representation* of the temporal symmetry (α, G) if

$$U(g)\mathcal{P}(\alpha_g(t)) = \mathcal{P}(t) \tag{9.6}$$

Thus, we are considering a notion of symmetry that generalizes both discrete time-translation symmetry $[\alpha_{\mathbb{T}^n}(t) = t + nT]$, $U(g) = \mathbb{I}$ and the usual equal-time symmetries $[\alpha_g(t) = t]$.

By similar arguments to Appendix F of Ref. [185], we find that for a symmetric family of states $|\Psi(t)\rangle$, there is a map $(\mathbb{R} \times EG)/G \to \Omega_d$, where EG is a contractible space with a free action of G. Similarly, for a family of ground state multiplets that is a representation, there is a map $(\mathbb{R} \times EG)/G \to \widetilde{\Omega}_d$.

Then we can invoke the mathematical fact that

$$(\mathbb{R} \times EG)/G \simeq BG, \tag{9.7}$$

where the left-hand side means the quotient of $\mathbb{R} \times EG$ by the diagonal action, where

G acts freely on EG as mentioned, and G acts on $\mathbb{R}$ through α. To see Eq. (9.7), we just observe that $\mathbb{R} \times EG$ is also a contractible space with a free action of G, and therefore Eq. (9.7) follows from the uniqueness of BG up to homotopy equivalence.

This gives us a more powerful version of the equivalence principle: the classification of symmetric families of states with symmetry group G as described is the *same* as the classification of stationary topological phases with symmetry group G. The Floquet equivalence principle described above is a special case when $G = \mathbb{Z}_{\text{time translation}} \times G_{\text{equal-time}}$. However, we can also consider more general symmetries, for example a time translation followed by a spin-flip.

So far, we have only been talking about symmetric families of states/multiplets. Let us show how they arise from dynamics. We say that a time-dependent Hamiltonian is invariant under the temporal symmetry (α, U) if

$$U(g)H(t)U(g)^\dagger = \alpha_g'(t)H(\alpha_g(t)), \tag{9.8}$$

where $\alpha_g'(t) = \frac{d}{dt}\alpha_g(t)$. In particular, this implies that the unitary propagator $\mathcal{U}(t_2; t_1)$, which is defined by

$$\mathcal{U}(t;t) = \mathbb{I}, \quad i\frac{\partial}{\partial t_2}\mathcal{U}(t_2;t_1) = H(t_2)\mathcal{U}(t_2;t_1) \tag{9.9}$$

satisfies

$$U(g)\mathcal{U}(t_2;t_1)U(g)^{-1} = \mathcal{U}(\alpha_g(t_2);\alpha_g(t_1)). \tag{9.10}$$

We can then define

$$V(g) = \mathcal{U}(0;\alpha_g(0))U(g). \tag{9.11}$$

It can be shown that $V(g)$ is a representation of G. Indeed, we have

$$V(g_1)V(g_2) = \mathcal{U}(0;\alpha_{g_1}(0))U(g_1)\mathcal{U}(0;\alpha_{g_2}(t))U(g_2) \tag{9.12}$$

$$= \mathcal{U}(0;\alpha_{g_1}(0))\Big[U(g_1)\mathcal{U}(0;\alpha_{g_2}(0))U(g_1)^{-1}\Big]\Big[U(g_1)U(g_2)\Big] \tag{9.13}$$

$$= \mathcal{U}(0;\alpha_{g_1}(0))\mathcal{U}(\alpha_{g_1}(0);\alpha_{g_1}(\alpha_{g_2}(0)))U(g_1g_2) \tag{9.14}$$

$$= \mathcal{U}(0;\alpha_{g_1g_2}(0))U(g_1g_2) \tag{9.15}$$

$$= V(g_1g_2). \tag{9.16}$$

Moreover, defining the time evolution of a state as $|\Psi(t)\rangle = \mathcal{U}(t;0)\,|\Psi\rangle$, we see that $|\Psi(t)\rangle$ is invariant under the temporal symmetry action (α, U) if and only if it is invariant under V. Similarly, the time evolution of a subspace gives a representation of (α, U) if and only if the subspace is a representation of V. Hence, if we decompose the Hilbert space as a sum of irreps under $V(g)$, each irrep will lead to an (irreducible) representation of (α, U).

In particular, consider the case of a Floquet system, so that G has a discrete subgroup $\mathbb{Z}$ corresponding to discrete time translations, generated by $\mathbb{T}$. Any irrep of V can be decomposed into irreps of $\mathbb{Z}$, which (since $\mathbb{Z}$ is Abelian) necessarily will be eigenstates of $U_f = U(0,T)^\dagger = V(\mathbb{T})^\dagger$. If the system is Floquet-MBL, then these eigenstates are GAPGRND states, therefore, the whole irrep under V will constitute a GAPGRND multiplet, which can be classified using the general approach described above. We find that the possible phases are in one-to-one correspondence with the phases for a system with the same symmetry group, but leaving points in space invariant and acting at equal times.

It is also possible to argue directly for this result along the same lines as in Chapter 5: namely, that the representation $V(g)$ defined above should be treated the same

as an on-site representation of the same symmetry group, and its non-on-site nature seems unlikely to make any difference. (In particular, this point of view suggests that the same result should hold for anti-unitary symmetries).

9.5 Space-time symmetries

Recall that we argued for a *crystalline* equivalence principle in Chapter 8, for stationary phases of matter with *spatial* symmetries, i.e. symmetries which relate different points in space (at equal times). One of the main ideas of Chapter 8 was that crystalline phases on a space X (for example, $X = \mathbb{R}^d$) should correspond to homotopy classes of maps

$$\sigma : (X \times EG)/G \to \Theta_d, \tag{9.17}$$

and G acts both on EG as usual, and on X corresponding to the spatial action of the symmetry. Here Θ_d is the space of d-dimensional TQFTs, and the formulation as stated is for systems without orientation-reversing symmetries, though it can be generalized. (We did not, however, give a rigorous proof of this result starting from microscopic lattice models).

If X is a contractible space, then $(X \times EG)/G$ is homotopy equivalent to BG, and so there is a one-to-one correspondence between phases with spatial symmetry and phases with the same symmetry group, but acting internally, i.e. without moving points in space around.

By contrast, we showed above that families of states with *temporal* symmetries, which relate different times but at the same point in space, should correspond to

homotopy classes of maps

$$(\mathbb{R} \times EG)/G \to \Omega_d, \tag{9.18}$$

where Ω_d is the space of ground states in d dimensions. As mentioned in Section 9.1, we expect that Ω_d and Θ_d are homotopy equivalent spaces, and so we can also replace Ω_d with Θ_d in Eq. (9.18).

Now we can imagine going further, and consider symmetries which relate different points in *space-time*. An example would be a time-translation followed by a spatial rotation (a "time-screw" [203]) or spatial reflection ("time-glide") [203, 204]. (The latter symmetry is orientation reversing, so we would have to use the appropriate extension of the framework described here). We will not attempt to give a careful justification here, but it is natural to conjecture, given the preceding results, that phases with such symmetries correspond to homotopy classes of maps

$$(\mathbb{R} \times X \times EG)/G \to \Omega_d \tag{9.19}$$

where G has some specified action on space-time $\mathbb{R} \times X$. If X is contractible then $(\mathbb{R} \times X \times EG)/G \simeq BG$, and hence there is a general *spatiotemporal equivalence principle*: phases with space-time symmetry are in one to correspondence with phases with the same symmetry group, but acting internally and at equal times.

Appendix A

Floquet-SPT classifications

A.1 Equivalence of 1-D Floquet classifications

Here we will show that the classification of Ref. [70] is equivalent to $H^2(\mathbb{Z} \rtimes G, \mathrm{U}(1))$, as claimed in Section 5.3. We do this by exploiting the connection between the second cohomology group and projective representations.

Suppose we have a projective representation $V(g\mathbb{T}^n) : g \in G, n \in \mathbb{Z}$ of $\mathbb{Z} \rtimes G$. Then we can define a *new* representation $V'(g\mathbb{T}^n) = V(g)V(\mathbb{T})^n$. Clearly, since V is a projective representation, $V'(g\mathbb{T}^n)$ can differ from $V(g\mathbb{T}^n)$ at most by a phase factor $\chi(g\mathbb{T}^n)$. Thus, defining the corresponding 2-cocycles ω and ω' by

$$V(x_1)V(x_2) = \omega(x_1, x_2)V(x_1x_2), \tag{A.1}$$

where $x_1 = g_1\mathbb{T}^{n_1}$, etc. (and similarly for ω'), we find that they are in the same equivalence class [thus, they correspond to the same element of $H^2(\mathbb{Z} \rtimes G, \mathrm{U}(1))$]. On the other hand, ω' is completely determined once we know its restriction ω'_G to G and the extra data $\chi(g) = V(g)\mathbb{T}V(g)^{-1}\mathbb{T}^{-\alpha(g)}$ (where $\alpha(g) = 1$ for unitary g and -1 for

anti-unitary g). One can verify that χ must satisfy the equation

$$\chi(gh) = \chi(g)^{\alpha(h)}\chi(h)^{\alpha(g)}. \tag{A.2}$$

Thus, up to equivalence, the 2-cocycle ω of $\mathbb{Z} \rtimes G$ is fully determined by a 2-cocycle ω'_G of G, and χ satisfying Eq. (A.2). This is indeed the classification of Ref. [70].]

A.2 Deriving the SPT classification

Here we will briefly recap the argument for the $H^{d+1}(G, \mathrm{U}(1))$ classification of SPT ground states in $d = 1$ and $d = 2$, taking care to formulate it in such a way as to make it clear that it can also be applied to give a $H^{d+1}(\mathbb{Z} \rtimes G, \mathrm{U}(1))$ classification in Floquet systems. Suppose we have some short-range entangled state $|\Psi\rangle$ defined on a system without boundary, such that $|\Psi\rangle$ is invariant under the local unitary (or anti-unitary) representation $V(g)$ of a symmetry. Now imagine some subregion M of the whole system, and consider the *subspace* $\mathcal{P}_{M,|\Psi\rangle}$ of "boundary states" defined in the Hilbert space of M which *complete* to $|\Psi\rangle$, in the sense that they are identical to $|\Psi\rangle$ away from the boundary of M. The restriction $V_M(g)$ of the symmetry operation $V(g)$ to the region M must preserve this subspace (note that this restriction is still well-defined even for anti-unitary symmetries, since we can take it to act only on the Hilbert space of M.) Thus, it is well-defined to talk about the action of the symmetry on the boundary states.

Moreover, if we assume that $|\Psi\rangle$ is short-range entangled, this implies that there exists a local unitary $\mathcal{D}$ which transforms $|\Psi\rangle$ into a product state $|\phi\rangle^{\otimes N}$. The restriction $\mathcal{D}_M$ must then transform the states in $\mathcal{P}_{M,|\Psi\rangle}$ into the states which look like a product of $|\psi\rangle$'s away from the boundary. Thus, if we started with a system

in d spatial dimensions, we can identify the boundary states with the states of a $(d-1)$-dimensional system. In the case $d = 1$, the boundary is just a set of points and we classify the SPT order from the *projective* representation of the symmetry on a boundary point [72, 73, 75, 77]. In $d = 2$, we can classify the SPT order by considering a symmetry restriction procedure as described in Ref. [86].

Appendix B

More on Floquet Time Crystals

B.1 Local structure of Floquet perturbation theory

Consider a soluble Floquet operator

$$U_f^0 = \mathcal{T} e^{-i\int_0^1 H_0(t)dt}, \tag{B.1}$$

and a time-dependent local perturbation $\lambda V(t)$, and define

$$U_{\rm f} = \mathcal{T}\exp\left(-i\int_0^1 [H_0(t)+\lambda V(t)]dt\right), \quad \mathcal{T} = \text{time-ordering}. \tag{B.2}$$

By Trotterizing, we can show that

$$U_{\rm f} = U_f^0 \times \mathcal{T}\exp\left(-i\int_0^1 (U^0)^\dagger(t)\lambda V(t)U^0(t)\right), \tag{B.3}$$

where $U^0(t) = \mathcal{T}e^{-i\int_0^t H_0(t')}$. Hence, without loss of generality we can just consider a perturbed Floquet operator $U_{\rm f} = U_{\rm f}^0\, U'$, where where $U' = \mathcal{T}\exp\left(-i\int_0^1 \lambda V(t)dt\right)$ for some local time-dependent V. We label the eigenstates of $U_{\rm f}^0$ as $|i\rangle$, with quasienergies

ω_i. We will now construct, order-by-order, a local unitary rotation that diagonalizes the perturbation.

First order. At first order we look for a unitary $e^{i\lambda S}$ such that $e^{i\lambda S}U_f e^{-i\lambda S}$ is diagonal (to first order in λ). Expanding $e^{i\lambda S}U_{\mathrm{f}}^0 U' e^{-i\lambda S}$ to first-order in λ and taking the matrix elements with $\langle i|$ and $|j\rangle$, we see that we can make it diagonal to this order by taking:

$$\langle i|\, S\, |j\rangle = \frac{\langle i|\, \overline{V}\, |j\rangle}{e^{i(\omega_i-\omega_j)}-1} \quad (i \neq j) \tag{B.4}$$

where $\overline{V} = \int_0^1 V(s)ds$. We can choose to set $\langle i|\, S\, |i\rangle = 0$.

It might not be clear whether this S is local, given that the eigenstates $|i\rangle$ might be highly non-local "cat states". To see that it is, we adapt an idea originally due to Hastings [205] (as refined in Ref. [6]) to the Floquet case. First write $\overline{V}$ as a sum of local terms $\overline{V} = \sum_X \overline{V}_X$, where $\overline{V}_X$ is supported on a bounded region X. Then we can write $S = \sum_X S_X$, where

$$S_X = \sum_{i\neq j} \frac{|i\rangle\langle i|\, \overline{V}_X\, |j\rangle\langle j|}{e^{i(\omega_i-\omega_j)}-1} \equiv \sum_{i\neq j} f(\omega_i-\omega_j)\, |i\rangle\langle i|\, \overline{V}_X\, |j\rangle\langle j|\,, \tag{B.5}$$

Now suppose that there are no "resonances" near X, by which we mean that $|e^{i(\omega_i-\omega_j)}-1| > \gamma > 0$ for all i, j for which the matrix element $\langle i|\, \overline{V}_X\, |j\rangle$ is nonzero. Then we can replace $f(\omega)$ with $\widetilde{f}(\omega)$ in Eq. (B.5), where $\widetilde{f}(\omega)$ is an infinitely differentiable function with period 2π such that $\widetilde{f}(0) = 0$ and $\widetilde{f}(\omega) = f(\omega)$ for $|e^{i(\omega_i-\omega_j)}-1| > \gamma$. By taking matrix elements one can then verify that

$$S_X = \sum_{n=-\infty}^{\infty} a_n (U_{\mathrm{f}}^0)^{-n} \overline{V}_X (U_{\mathrm{f}}^0)^n, \tag{B.6}$$

where a_n are the Fourier series coefficients of $\widetilde{f}$: $\widetilde{f}(\omega) = \sum_{n=-\infty}^{\infty} e^{in\omega} a_n$. From this,

we can show that S_X is quasi-local provided that $U_{\rm f}^0$ obeys a Lieb-Robinson bound. In particular, however, if we choose $U_{\rm f}^0$ such that $(U_{\rm f}^0)^n$ doesn't increase the support of operators by more than an n-independent constant, we see that S_X is still strictly local on a region of slightly larger size. In particular, this can be shown to be true of the Floquet operator $U_{\rm f}^0$ in Eq. (6.1) in the main text. To see this, note that in $(U_{\rm f}^0)^n$ we can move all the spin flips to the end at the cost of simply changing the sign of the h_i's during the course of the evolution, and the time evolution of a Hamiltonian which is the sum of terms, each of which is a product of Pauli σ_z operators (even if the coefficients vary with time) never increases the support of operators by more than a constant amount.

All orders. Suppose that we have found a unitary rotation which diagonalizes the perturbation to order λ^n, such that $U_{\rm f} = U_{\rm f}^0 U'$, with

$$\mathcal{V} = \exp\left(-i\left\{V_d + \lambda^{n+1} V_{nd} + O(\lambda^{n+2})\right\}\right), \tag{B.7}$$

where V_d is diagonal, $V_d = O(\lambda)$ and V_{nd} is non-diagonal. (At first-order, i.e. $n = 0$, if U' was originally the evolution of a time-dependent Hamiltonian we can still generate such an expression for $\mathcal{V}$ using the Campbell-Baker-Haussdorf formula.) Then we want to find S such that $e^{iS} U_{\rm f} e^{-iS}$ is diagonal to order λ^{n+1}, or equivalently, writing $e^{iS} U_{\rm f} e^{-iS} = U_{\rm f}^0 U''$, that U'' is diagonal. We see that

$$U'' = (U_f^0)^\dagger e^{iS} (U_f^0) U' e^{-iS} = e^{i(U_f^0)^\dagger S U_f^0} U' e^{-iS}. \tag{B.8}$$

From the Campbell-Baker-Haussdorf formula, we see that

$$U'' = \exp\left(i\left\{-V_d - \lambda^{n+1} V_{nd} + (U_{\rm f}^0)^\dagger S U_{\rm f}^0 - S\right\} + O(\lambda^{n+2})\right), \tag{B.9}$$

and hence we set the expression in {...} to be diagonal. Taking off-diagonal matrix elements gives

$$\langle i|\, S\, |j\rangle = \frac{\langle i|\, \lambda^{n+1} V_{nd}\, |j\rangle}{e^{i(\omega_i - \omega_j)} - 1} \quad (i \neq j), \tag{B.10}$$

and we choose to set $\langle i|\, S\, |i\rangle = 0$. We can then repeat the process at next order, with $n' = n+1$, $V'_d = V_d + \lambda^{n+1} V_{nd} - (U^0_{\rm f})^\dagger S U^0_{\rm f} + S$, and V'_{nd} equal to the coefficient of λ^{n+2} in the Campbell-Baker-Haussdorf expansion Eq. (B.9).

We observe that at all orders in the perturbation theory, locality is preserved. The only operations contained in the exponentials are addition, conjugation by $U^0_{\rm f}$, taking nested commutators (through the Campbell-Baker-Haussdorf expansion), and evaluating expressions of the form Eq. (B.10). The first three *manifestly* preserve locality, and the last one preserves locality in the absence of resonances for the same reasons discussed in the first-order section above. Therefore, the unitary rotation that relates the eigenstates of $U_{\rm f}$ to the eigenstates of $U^0_{\rm f}$ is indeed a *local* unitary at all orders.

Effect of resonances. In the above discussion, we have ignored the effect of resonances. At low orders, resonances can be accounted for by treating the dilute resonant spots separately[23]. Therefore, we expect that a modified version of the perturbation theory will remain valid provided that the non-resonant terms in the series converge sufficiently fast. This corresponds to the requirement that, for typical levels i and j which are connected locally, the perturbation denominator

$$\Delta_{i,j} = \left| \frac{\lambda}{e^{i(\omega_j - \omega_i)} - 1} \right| \ll 1. \tag{B.11}$$

For the exactly solvable Floquet operator U^0_f considered in this paper, the typical value of $e^{i(\omega_j - \omega_i)}$ is set by the parameter J. (See the eigenvalues of U^0_f calculated

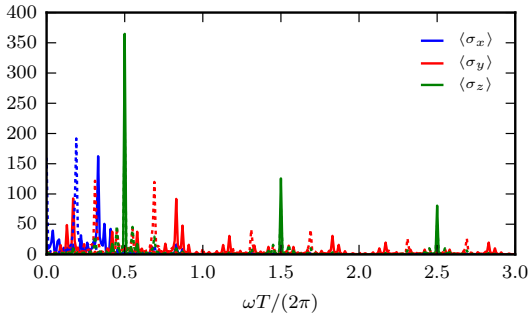


Figure B.1: The Fourier transform of the time evolution at late times (taken over the interval $200 < t < 300$) for two individual disorder realizations (shown as solid and dotted lines respectively), at $h = 0.3$. The dominant peaks at $\omega = (2k+1)\pi/T$ are universal, whereas the smaller peaks at other locations are disorder-dependent.

in the main text). We note that, if $J \ll 1$, then typically $|\omega_j - \omega_i| \sim J$ and thus, $\Delta_{i,j} \sim \lambda/J$. On the other hand, if $J \gtrsim 1$, then typically $|\omega_j - \omega_i| \gtrsim 1$ and hence $\Delta_{i,j} \sim \lambda$. Therefore, we see that the condition for Eq. (B.11) to hold is $\lambda \ll \min\{1, J\}$, or (restoring the driving period T, which was set to 1 in the above discussion):

$$\lambda \ll \min\{T^{-1}, J\}. \tag{B.12}$$

B.2 Numerical Observation of Persistent Oscillations at Very Late Times

In a single disorder realization, we can go to much later times in TEBD. Moreover, experiments might be carried out in a small number of disorder realizations. As noted in the main text, $\langle\sigma_i^x\rangle$ and $\langle\sigma_i^y\rangle$ are noisier in individual disorder realizations. However, one can still observe a clear signature of TTSB by looking at the Fourier transform of the time dependence of a single disorder realization, as shown in Fig. B.1. There are strong peaks at π/T, with subleading peaks at $(2k+1)\pi/T$, indicating the fractional frequency response. (The other peaks in the Fourier transform have their

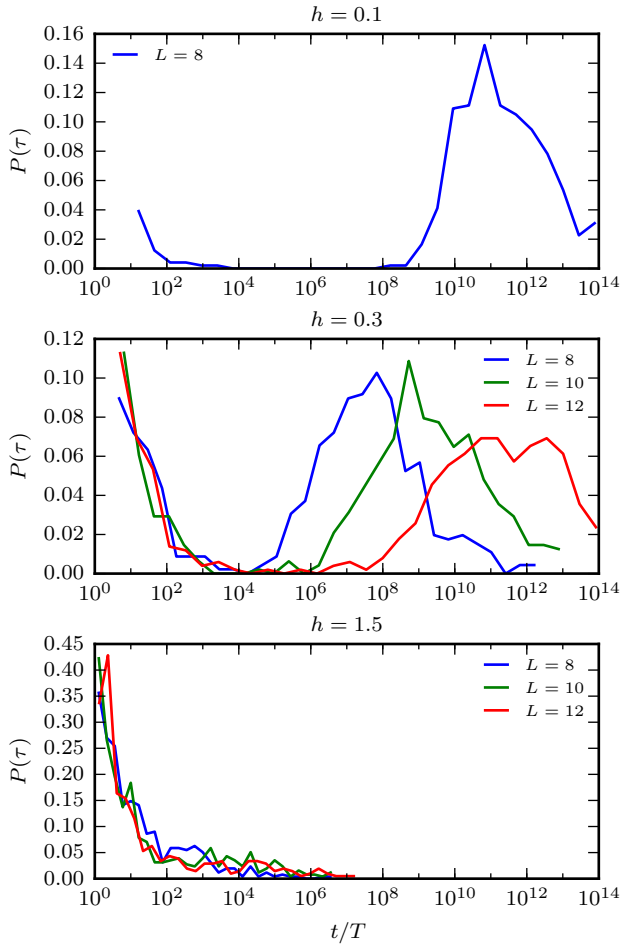


Figure B.2: Histogram of the characteristic timescale τ, as defined in the text, for different values of the magnetic field. From top to bottom: $h = 0.1$ deep in the TTSB phase, $h = 0.3$ in the same regime as discussed in the main text, and $h = 1.5$ beyond the TTSB phase.

origins in the discreteness of the local quasienergy spectrum near a given point, and can be distinguished by the fact that their positions vary depending on the disorder realization.) These results indicate that the oscillations persist to later times than shown in the upper panel of Fig. 6.1 in the main text and that they are visible even in a single disorder realization.

To more carefully examine the decay of the oscillations, we turn again to exact diagonalization. For a given disorder realization and initial state, we can determine a characteristic timescale τ by computing $(-1)^t\langle\sigma_i^z(t)\rangle\text{sign}(\langle\sigma_i^z(0)\rangle)$. This is defined such that it is positive for small times, and we define τ to be the time at which this observable first changes its sign. In Fig. B.2, we show a histogram of these τ for different system sizes and strengths of the magnetic field. We observe a very interesting structure: deep in the TTSB phase, at $h = 0.1$, we find a single large peak at very large times (here, we show only $L = 8$ since for larger systems the τ are too large compared to the floating point precision). In an intermediate range, such as $h = 0.3$, we find two pronounced peaks, where the location of the first peak does not depend on system size while the second peak is centered around a time that diverges exponentially. The relative weight of the two peaks seems unaffected by system size. In this regime, the average of τ is dominated by rare instances with very large τ, while the typical value is dominated by instances with short characteristic times. In the disorder-averaged value of the magnetization $Z(t)$, which was discussed in the main manuscript, the first peak in the distribution of τ manifests in the decay from the initial value to the intermediate plateau, and the second peak corresponds to the decay from this plateau to zero. Finally, in the limit of very large h where the system has been driven out of the TTSB phase, we find the histogram to be dominated by a peak at short times.

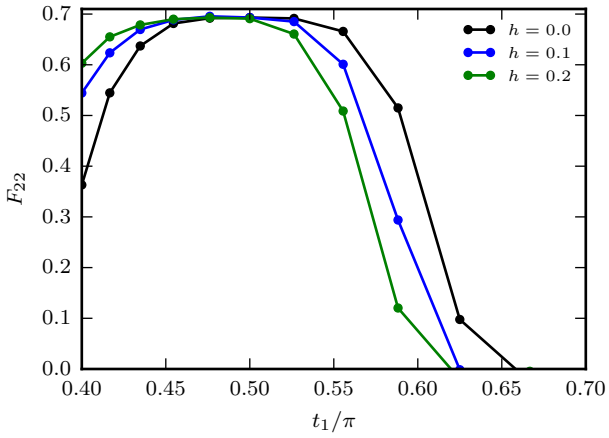


Figure B.3: Mutual information between the two left- and right-most sites, F_{22}, in eigenstates of U_f for $t_1 \neq \pi/2$ (see Eq. (B.13)). The data shown here is extrapolated to the thermodynamic limit using the techniques discussed in the main manuscript and from data for $L = 6 - 12$ and $t_0 = 1$.

B.3 Perturbation of the driving pulse time

We now consider the evolution under the Floquet operator

$$U_f = \exp\left(-it_0 H_{\mathrm{MBL}}\right) \exp\left(it_1 \sum_i \sigma_i^x\right) \tag{B.13}$$

for $t_1 \neq \pi/2$. By the arguments given above, the TTSB phase should be stable against small perturbations of the Floquet drive, which includes changing t_1 away from $\pi/2$. We confirm this numerically by performing exact diagonalization of U_f for a range of t_1 (with $J = h^z = 1$) and measuring the mutual information between well-separated regions in the eigenstates, analogous to the computation performed for Fig. 2 of the main manuscript. Our results are shown in Fig. B.3. We also calculated the real time evolution for $h = 0.1$, $t_1/\pi = 0.475$, and verified that it shows qualitatively similar behavior to that depicted in Figure 1 in the main text.

B.4 Radiation Emitted from a TTSB System at Lowest Order in Perturbation Theory

Let us suppose, for illustrative puposes, that our spin system is coupled to the electromagnetic field through the Jaynes-Cummings Hamiltonian

$$H_1 = V(a + a^\dagger), \quad V = g\sum_i \sigma_i^z, \tag{B.14}$$

where $a^\dagger$, a are creation/annihilation operators for photons of frequency $\Omega/2$ (where $\Omega = 2\pi/T$ is the drive frequency.) The most general Hamiltonian will include couplings of the electromagnetic field to σ_i^x, σ_i^y, and σ_i^z. Here, we focus on the last of these three types of couplings, since this is the only one that can cause transitions between Floquet eigenstates separated by quasi-energy $\Omega/2$ in lowest-order perturbation theory about the soluble $h = 0$ point. The transition amplitude between initial and final Floquet eigenstates $|m\rangle$, $|n\rangle$ is given, in the interaction picture, by:

$$A_{m,n} = \langle n, 1|\, \mathcal{T} \exp\left(-i\int_{-\infty}^{\infty} dt H_1(t')\right) |m, 0\rangle \tag{B.15}$$

where $\langle n, 1|$ is the state with the spin system in the state $|n\rangle$ and a single photon (and similarly for $|m, 0\rangle$), and $H_1(t) \equiv U_0^\dagger(t, -\infty)H_1U_0(t, -\infty)$ and $U_0(t, -\infty) \equiv \mathcal{T}\exp\big(-i\int_{-\infty}^{t} dt' H(t')\big)$. The unperturbed Hamiltonian $H(t)$ is the stroboscopic Hamiltonian given in Eq. (6.1) in the main text and the text below it. We write $t = kT + s$, where $s \in [0, T)$. Then we can write $U_0(t, jT) = U_0(s, 0)U_{\rm f}^{k-j}$. To lowest-order, the

transition amplitude can be written in the form:

$$A_{m,n} = -i \sum_{k=-\infty}^{\infty} \int_0^T ds \langle n,1|(U_{\rm f}^k)^\dagger U_0^\dagger(s,0) H_1 U_0(s,0) U_{\rm f}^k|m,0\rangle$$
$$= -i \sum_{k=-\infty}^{\infty} e^{ikT(\omega_n-\omega_m+\Omega/2)} \int_0^1 ds e^{i(\Omega/2)s} \langle n|U_0^\dagger(s,0) V U_0(s,0)|m\rangle \tag{B.16}$$
$$\equiv -if(\omega_n-\omega_m) \int_0^T ds e^{i(\Omega/2)s} \langle n|U_0^\dagger(s,0) V U_0(s,0)|m\rangle, \tag{B.17}$$

where

$$f(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega + \left[k+\frac{1}{2}\right]\Omega\right). \tag{B.18}$$

This matrix element is generally non-zero. For instance, consider the soluble point $h = 0$. We take the spin-flip part of the Floquet operator to act instantaneously such that $\int_0^T ds e^{i(\Omega/2)s} U_0^\dagger(s,0)\sigma_i^z U_0(s,0) \propto \sigma_i^z$. Then the initial and final states are $|\pm\rangle \equiv (\exp(it_0 E^-(\{s_i\})/2)|\{s_i\}\rangle \pm \exp(-it_0 E^-(\{s_i\})/2)|\{-s_i\}\rangle)/\sqrt{2}$, we find that $\langle -|\sigma_i^z|+\rangle = \langle +|\sigma_i^z|-\rangle = 1$ for any i, and hence

$$A_{+-} = A_{-+} \propto -\frac{2\pi i g N}{T}\delta(0). \tag{B.19}$$

Now consider a locally-prepared initial state, such as

$$|\{s_i\}\rangle = (|+\rangle \pm |-\rangle)/\sqrt{2}, \tag{B.20}$$

[Here we have set $h_i^z = 0$ in order to unclutter the equations, so that $E^-(\{s_i\}) = 0$.] Then, in the absence of a coupling to the electromagnetic field, it would not change with time in the interaction picture. (The fractional frequency response in the interaction picture comes from the time evolution of observables.) However, Eq. (B.19) tells us that at lowest-order in perturbation theory, the system can emit a

photon at frequency $\Omega/2$ and transition from $|-\rangle \leftrightarrow |+\rangle$. However, this only changes the superposition Eq. (B.20) by a global phase factor ± 1. One might wonder why this does not violate conservation of quasienergy modulo Ω, given that a photon of frequency $\Omega/2$ has been emitted. However, we observe that the state Eq. (B.20) is not a quasienergy eigenstate; rather, it is a superposition of two eigenstates with quasienergies differing by $\Omega/2$. Therefore, its quasienergy is only well-defined modulo $\Omega/2$. We note that neither $|+\rangle$ nor $|-\rangle$ is "higher" in quasienergy. The system can emit a photon of energy $\Omega/2$ while transitioning from $|+\rangle$ to $|-\rangle$ or from $|-\rangle$ to $|+\rangle$ since since $-\Omega/2 \equiv \Omega/2 \pmod{\Omega}$. [Mathematically, this corresponds to the statement that $f(\Omega/2) = f(-\Omega/2)$ in Eq. (B.17).]

Appendix C

Proof of prethermalization results

C.1 Definition of the norm

Let's suppose, for the sake of concreteness, that we have a spin system with a local time-dependent Hamiltonian of the form:

$$\begin{aligned} H(t) &= \sum_{i,j} J_{i,j}^{\alpha\beta}(t) S_i^\alpha S_j^\beta + \sum_{i,j,k} K_{i,j,k}^{\alpha\beta\gamma}(t) S_i^\alpha S_j^\beta S_k^\gamma + \dots \\ &= \sum_p \sum_{p-\text{tuples}} A_{i_1,\dots,i_p} \end{aligned} \tag{C.1}$$

Here $\alpha = x, y, z$ are the components of the spins, and i, j, k are lattice sites. In the first line, we have explicitly written the 2-site and 3-site terms; the $\dots$ represents terms up to n-site terms, for some finite n. It is assumed that these interactions have finite range $r \geq n$ such that all of the sites in a k-site term are within distance r. In the second line, we have re-expressed the Hamiltonian in a more generic form in terms of p-site terms $A_{i_1,\dots,i_p}$ with $i_1 \neq \dots \neq i_p$. To avoid clutter, we have not explicitly denoted the t-dependence of $A_{i_1,\dots,i_p}$. We define the local instantaneous

norm $\|A_{i_1,\ldots,i_p}\|_n$ according to

$$\|A_{i_1,\ldots,i_p}\|_n^{\text{inst}} \equiv e^{p\kappa_n}\|A_{i_1,\ldots,i_p}\| \tag{C.2}$$

where $\|A_{i_1,\ldots,i_p}\|$ is the operator norm of $A_{i_1,\ldots,i_p}$ at a given instant of time t and

$$\kappa_n \equiv \kappa_1/[1+\ln n]. \tag{C.3}$$

We make this choice of n-dependence of κ_n, following Ref. [39] for reasons that will be clear later. We then average the instantaneous norm over one cycle of the drive:

$$\|A_{i_1,\ldots,i_p}\|_n \equiv \frac{1}{T}\int_0^T dt\, \|A_{i_1,\ldots,i_p}\|_n^{\text{inst}} \tag{C.4}$$

It is only in this step that we differ from Abanin et al. [39], who consider the supremum over t rather than the average. In analyzing the Floquet operator, i.e. the evolution due to H at stroboscopic times, it is the total effect of H, which is determined by its integral over a cycle, that concerns us. Error terms that act over a very short time, even if they are relatively strong, have little effect on the Floquet operator so long as their norm, as defined above, is small. Finally, we define the global time-averaged norm of the Hamiltonian H:

$$\|H\|_n \equiv \sup_j \sum_p \sum_{p-\text{tuples}} \Big[\sum_k \delta_{j,i_k}\Big] \|A_{i_1,\ldots,i_p}\|_n \tag{C.5}$$

The term in square braces restricts the sum to p-tuples that contain the site j.

C.2 More technical statement of Theorem 1

Theorem 1 stated above will follow from the following slightly more technical formulation. For notational simplicity we work in units with $T = 1$. **?**

Theorem 1′. *Consider a periodically-driven system with Floquet operator:*

$$U_f = \mathcal{T}\exp\left(-i\int_0^T [H_0(t) + V(t)]dt\right), \tag{C.6}$$

where $X \equiv \mathcal{T}\exp\left(-i\int_0^T H_0(t)\right)$ *satisfies* $X^N = 1$ *for some integer* N*, and we assume that* H_0 *can be written as a sum* $H_0(t) = \sum_i h_i(t)$ *of terms acting on single sites* i*. Define* $\lambda \equiv \|V\|_1$*. Then there exists a sequence of quasi-local* A_n *such that, defining* $\mathcal{U}_n = e^{-iA_n}\cdots e^{-iA_1}$*, we have*

$$\mathcal{U}_n U_f \mathcal{U}_n^\dagger = X\,\mathcal{T}\exp\left(-i\int_0^1 [D_n + E_n + V_n(t)]dt\right), \tag{C.7}$$

where $[D_n, X] = 0$*;* D_n, E_n *are independent of time; and*

$$\|V_n\|_n, \|E_n\|_n \leq 2K_n\lambda^n, \tag{C.8}$$

$$\|A_n\|_n \leq (N+1)K_n\lambda^n, \tag{C.9}$$

$$\|D_n - D_{n-1}\|_n \leq K_n\lambda^n, \tag{C.10}$$

where we have defined $\lambda \equiv \|V\|_1$*, and*

$$K_n = C^{n-1}\prod_{k=1}^{n-1} m(k), \quad C = 2(N+3)(N+4),$$
$$m(n) = \frac{18}{\kappa_{n+1}(\kappa_n - \kappa_{n+1})}. \tag{C.11}$$

These bounds hold provided that $n \leq n_*$, with

$$n_* = \frac{\lambda_0/\lambda}{[1+\log(\lambda_0/\lambda)]^3}, \quad \lambda_0 = (36C)^{-1} \tag{C.12}$$

and provided that

$$\lambda < \frac{\mu}{N+3}, \quad \mu \approx 0.07. \tag{C.13}$$

Theorem 1 follows from Theorem 1′, because n_* is chosen such that $n \leq n_*$ implies $Cm(n) \leq \frac{1}{2\lambda}$. It then follows that $K_{n+1}\lambda^{n+1}/(K_n\lambda^n) = Cm(n)\lambda \leq \frac{1}{2}$, and hence that $K_n\lambda^n \leq \lambda/2^{n-1}$. Moreover, we obtain Eq. (7.9) by summing Eq. (C.10), from which we see that $\|D_n - D_1\|_n \leq \sum_{k=2}^{\infty} K_k\lambda^k \leq K_2\lambda^2 \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^{k-2} = 2K_2\lambda^2 = 2Cm(1)\lambda^2 \approx 2.9\lambda^2/\lambda_0$. (Here we use the fact that $\|\cdot\|_{n+1} \leq \|\cdot\|_n$.)

In the next sections, we will give a proof of Theorem 1′.

C.3 Iterative construction

The idea is to construct the D_n, V_n, E_n, A_n discussed above iteratively. That is, suppose that at the n-th step, we have

$$\mathcal{U}_n U_f \mathcal{U}_n^\dagger \equiv U_f^{(n)} = X\,\mathcal{T}\exp\left(-i\int_0^1 \mathcal{H}_n(t)dt\right), \tag{C.14}$$

where $\mathcal{H}_n(t) = F_n + V_n(t)$, with $F_n = \int_0^T \mathcal{H}_n(t)dt$ time-independent. We will choose to separate the time-independent piece F_n according to $F_n = D_n + E_n$, where $D_n = \langle F_n\rangle$, and we have defined the symmetrization

$$\langle O\rangle = \frac{1}{N}\sum_{k=0}^{N-1} X^{-k}OX^k. \tag{C.15}$$

In particular, this implies that $[D_n, X] = 0$ and $\langle D_n \rangle = D_n$, and therefore $\langle E_n \rangle = \langle F_n \rangle - \langle D_n \rangle = D_n - D_n = 0$.

We will now introduce a local unitary $\mathcal{A}_n = e^{-iA_n}$, which we use to rotate the Floquet operator $U_f^{(n)}$, giving a new Floquet operator

$$U_f^{(n+1)} \equiv \mathcal{A}_n U_f^{(n)} \mathcal{A}_n^\dagger = X\mathcal{T} \exp\left(-i \int_0^1 \mathcal{H}_{n+1}(t)dt\right). \tag{C.16}$$

The ultimate goal, decomposing $\mathcal{H}_{n+1}(t) = D_{n+1} + E_{n+1} + V_{n+1}(t)$ as before, is to ensure that the residual error terms E_{n+1} and V_{n+1} are much smaller than E_n and V_n. This goal is achieved in two separate steps. The first step ensures that E_{n+1} is small (that is, the time-independent part of $\mathcal{H}_{n+1}(t)$ nearly commutes with X), and the second step ensures that V_{n+1} is small.

Step One.– This step proceeds similarly to the recursion relation of Abanin et al [39] for the *time-independent* case (Section 5.4 of Ref. [39]). There the recursion relation was designed to make the Hamiltonian commute with its zero-th order version. This is analogous to our present goal of making the Floquet operator commute with X. Here, we adapt the analysis of Ref. [39] to the Floquet case.

We observe that

$$\begin{aligned}
U_f^{(n+1)} &= \mathcal{A}_n U_f^{(n)} \mathcal{A}_n^\dagger && \text{(C.17)}\\
&= X\left[X^\dagger \mathcal{A}_n X \times \mathcal{T} \exp\left(-i \int_0^1 \mathcal{H}_n(t)dt\right) \times \mathcal{A}_n^\dagger\right], && \text{(C.18)}\\
&= X\left[e^{-X^\dagger i A_n X} \times \mathcal{T} \exp\left(-i \int_0^1 \mathcal{H}_n(t)dt\right) \times e^{iA_n}\right] && \text{(C.19)}\\
&= X \times \mathcal{T} \exp\left(-i \int_0^1 \mathcal{H}'_n(t)dt\right), && \text{(C.20)}
\end{aligned}$$

where

$$\mathcal{H}'_n(t) = \begin{cases} \frac{1}{a}(-A_n) & 0 \le t \le a \\ \frac{1}{1-2a}\mathcal{H}_n\left(\frac{t-a}{1-2a}\right) & a \le t \le (1-a), \\ \frac{1}{a}(X^\dagger A_n X) & (1-a) \le t \le 1, \end{cases} \tag{C.21}$$

(for some constant $a \in [0, 1/2]$ which can be chosen arbitrarily.) Let us decompose $\mathcal{H}'_n(t) = D'_n + V'_n(t)$, where $D'_n = \frac{1}{T}\int_0^1 \mathcal{H}'_n(t)$. Our goal will be to ensure that the time-independent part D'_n commutes with X. It turns out this can actually be achieved exactly, and in particular we can choose A_n such that $D'_n = D_n$.

To this end, we first observe that

$$D'_n = D_n + E_n + X^\dagger A_n X - A_n. \tag{C.22}$$

We now claim that $D'_n = D_n$ if we choose

$$A_n := \frac{1}{N}\sum_{k=0}^{N-1}\sum_{p=0}^{k} E_n^{(p)}, \quad E_n^{(p)} = X^{-p} E X^p. \tag{C.23}$$

To see this, note that, by construction,

$$X^\dagger A_n X - A_n = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{p=0}^{k}[E_n^{(p+1)} - E_n^{(p)}] \tag{C.24}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}[E_n^{(k+1)} - E_n] \tag{C.25}$$

$$= -E_n + \langle E_n \rangle, \tag{C.26}$$

$$= -E_n, \tag{C.27}$$

since $\langle E_n \rangle = 0$.

Step Two.– The next step is now to find a new time-dependent Hamiltonian

$\mathcal{H}_{n+1}(t)$ which gives the same unitary evolution as $\mathcal{H}'_n(t)$ over the time interval $[0,1]$, while making the time-dependent part smaller. That is, making the decomposition $\mathcal{H}_{n+1}(t) = D_{n+1} + E_{n+1} + V_{n+1}(t)$ as before, the goal is to make V_{n+1} small. In fact, this is precisely the problem already considered by Abanin et al[39], and we can use the procedure described in Section 4.1 of that paper.

One might worry whether Step Two undoes the good work done by Step One. That is, does making V_{n+1} small come at the cost of making E_{n+1} larger again? However, this turns out not to be a problem, as the bounds we derive below will make clear.

C.4 Bounds on Error terms

Now we will derive bounds that quantify the success of the iterative procedure described in the previous subsection at making the residual error terms E_n and V_n small. Analysis proceeds in similar way to Abanin et al[39]. We define

$$d(n) = \|D_n\|_n, \quad v(n) = \|V_n\|_n, \quad v'(n) = \|V'_n\|_n,$$
$$e(n) = \|E_n\|_n, \quad \delta d(n) = \|D_{n+1} - D_n\|_{n+1}, \quad \text{(C.28)}$$

First of all, from Eq. (C.23) we have a bound on A_n:

$$\|A_n\|_n \leq \frac{N+1}{2} e(n) \quad \text{(C.29)}$$

From Eq. (C.21) we observe that

$$V_n'(t) = \begin{cases} \frac{1}{a}(-A_n) - D_n & 0 \leq t \leq a \\ \frac{1}{1-2a}\left[2aD_n + E_n + V_n\left(\frac{t-a}{1-2a}\right)\right] & a \leq t \leq (1-a), \\ \frac{1}{a}(X^\dagger A_n X) - D_n & (1-a) \leq t \leq 1, \end{cases} \tag{C.30}$$

and hence

$$v'(n) \leq 2\|A_n\|_n + \|E_n\|_n + \|V_n\|_n + 4a\|D_n\|_n \tag{C.31}$$

Hence, we can send $a \to 0$ to give (using Eq. (C.29))

$$v'(n) \leq (N+2)e(n) + v(n). \tag{C.32}$$

Then, as our construction of $\mathcal{H}_{n+1}$ from $\mathcal{H}_n'$ is the one described in Section 4.1 of Abanin et al, we can use their bounds

$$\|D_{n+1} + E_{n+1} - D_n\|_{n+1} \leq \epsilon_n/2 \tag{C.33}$$

$$v(n+1) \leq \epsilon_n \tag{C.34}$$

where

$$\epsilon_n = m(n)v'(n)\big(d(n) + 2v'(n)\big), \tag{C.35}$$

$$m(n) = \frac{18}{(\kappa_{n+1} - \kappa_n)\kappa_{n+1}}. \tag{C.36}$$

These bounds hold provided that

$$3v'(n) \leq \kappa_n - \kappa_{n+1} \tag{C.37}$$

Since $D_{n+1} - D_n = \langle D_{n+1} + E_{n+1} - D_n \rangle$, we see that

$$\delta d(n) \leq \|D_{n+1} + E_{n+1} - D_n\|_{n+1} \leq \epsilon_n/2 \tag{C.38}$$

and

$$e(n+1) \leq \|D_{n+1} + E_{n+1} - D_n\|_{n+1} + \|D_{n+1} - D_n\|_{n+1} \leq \epsilon_n \tag{C.39}$$

C.5 Proof of Theorem 1′ by induction

The idea now is to apply the bounds of the previous subsection recursively to give bounds expressed in terms of the original Floquet operator,

$$U_f = U_f^{(1)} = \mathcal{T} \exp\left(-i \int_0^1 [H_0(t) + V(t)]\right) \tag{C.40}$$

$$= X\mathcal{T} \exp\left(-i \int_0^1 V_{\text{int}}(t)dt\right), \tag{C.41}$$

and in particular the quantity $\lambda \equiv \|V_{\text{int}}\|_1 = \|V\|_1$. First of all, we write $\mathcal{H}_1(t) \equiv V_{\text{int}}(t) = F_1 + V_1(t)$, where $F_1 = \int_0^1 V_{\text{int}}(t)dt$, and then separate $F_1 = D_1 + E_1$, where $D_1 = \langle F_1 \rangle$. We note that $\|F_1\|_1 \leq \lambda$, which implies that $v(1) \leq \|V_{\text{int}}\|_1 + \|F_1\|_1 \leq 2\lambda$, and $d(1) \leq \lambda$. In turn this gives $e(1) \leq \|D_1\|_1 + \|F_1\|_1 \leq 2\lambda$.

Now we proceed by induction. Suppose that we have some n such that, for all $1 \leq k \leq n$, we have

$$e(k), v(k) \leq 2K_k \lambda^k, \tag{C.42}$$

and for all $1 \leq k < n$,

$$\delta d(k) \leq K_{k+1}\lambda^{k+1} \tag{C.43}$$

where the coefficients K_k satisfy $K_{k+1}/K_k \leq \frac{1}{2\lambda}$. (The preceding discussion shows that this induction condition is satisfied for $n = 1$ with $K_1 = 1$.)

Then from Eq. (C.32) we find that

$$v'(n) \leq 2c_N K_n \lambda^n, \quad c_N = N + 3, \tag{C.44}$$

and hence

$$\epsilon_n \leq m(n) 2c_N K_n \lambda^n (d(n) + 2c_N K_n \lambda^n). \tag{C.45}$$

We note that the triangle inequality and the fact that $\|\cdot\|_n$ decreases with n ensures that $d(n+1) - d(n) \leq \delta d(n)$. Hence we can bound $d(n)$ by

$$\begin{aligned} d(n) &\leq d(1) + \sum_{k=1}^{n-1} \delta d(k) && \text{(C.46)} \\ &\leq \lambda + \sum_{k=1}^{n-1} K_{k+1}\lambda^{k+1} && \text{(C.47)} \\ &= \sum_{k=1}^{n} K_k \lambda^k && \text{(C.48)} \\ &\leq \sum_{k=1}^{n} \lambda \left(\frac{1}{2}\right)^{k-1} && \text{(C.49)} \\ &\leq 2\lambda && \text{(C.50)} \end{aligned}$$

In Eq. (C.49), we used the inequality $K_{k+1}/K_k \leq 1/(2\lambda)$. This same inequality also

ensures that $K_n\lambda^n \leq \lambda$, so inserting into Eq. (C.45) gives

$$\begin{aligned}\epsilon_n &\leq m(n)2c_N K_n(2+2c_N)\lambda^{n+1} \\ &\equiv 2Cm(n)K_n\lambda^{n+1} \\ &\equiv K_{n+1}\lambda^{n+1}. \end{aligned} \tag{C.51}$$

Here we chose

$$K_{n+1} = Cm(n)K_n, \quad C = 2c_N(1+c_N). \tag{C.52}$$

Next we need to examine the conditions under which Eq. (C.37) holds. Given the bounds on $v'(n)$ and using the inequality $K_n\lambda^n \leq \lambda(1/2)^{n-1}$, it is sufficient to demand that

$$3c_N(1/2)^{n-1}\lambda \leq \kappa_{n+1} - \kappa_n, \tag{C.53}$$

or in other words

$$\lambda \leq \frac{1}{3c_N}\max_{n\in\mathbb{N}}\left[2^{n-1}(\kappa_{n+1}-\kappa_n)\right] = \frac{1}{3c_N}(\kappa_2-\kappa_1) \approx \frac{0.14\kappa_1}{N+3}. \tag{C.54}$$

Provided that Eq. (C.54) holds, we then find that

$$\delta d(n), v(n+1)/2, e(n+1)/2 \leq K_{n+1}\lambda^{n+1}. \tag{C.55}$$

Therefore, we can continue the induction provided that $K_{n+1}/K_n \leq \frac{1}{2\lambda}$. Since $K_{n+1}/K_n = Cm(n)$, this is true provided that $n \leq n_*$. This completes the proof of Theorem 1′.

Appendix D

More details on prethermalization

D.1 Proof of phase-winding when a $\mathrm{U}(1)$ symmetry is spontaneously broken

Here we intend to prove the claim made in Section 7.4.1 above that the expectation value

$$\mathrm{Tr}(\rho_X e^{itK}\Phi e^{-itK}) \equiv g_X(t) \tag{D.1}$$

must be independent of time t, where we have defined $K \equiv D - \mu L$ and $\rho_X \equiv \lim_{\epsilon\to 0^+} \frac{1}{\mathcal{Z}} e^{-\beta(K+\epsilon X)}$. The idea is to make a connection with results of Ref. [52]; however, these were expressed in terms of *two-point* correlation functions, and also did not have the ϵX term in the definition of the density matrix. To make a connection, we assume that the symmetric density matrix $\rho = \frac{1}{\mathcal{Z}} e^{-\beta K}$ can be recovered by symmetrizing a symmetry-breaking state,

$$\rho = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta L}\rho_X e^{i\theta L} d\theta, \tag{D.2}$$

and that the symmetry-breaking state ρ_X is short-range correlated. Now we calculate the two-point correlation function (where $\Phi(x)$ and $\Phi(y)$ are two operators acting at

different spatial locations x and y)

$$f(t) = \mathrm{Tr}[\rho e^{itK}\Phi(x)e^{itK}\Phi^\dagger(y)] \tag{D.3}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta \mathrm{Tr}[e^{-i\theta L}\rho_X e^{i\theta L}e^{itK}\Phi(x)e^{-itK}\Phi(y)] \tag{D.4}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta \mathrm{Tr}[\rho_X e^{itK}\{e^{i\theta L}\Phi(x)e^{-i\theta L}\}e^{-itK}\{e^{i\theta L}\Phi^\dagger(y)e^{-i\theta L}\}] \tag{D.5}$$

$$= \mathrm{Tr}[\rho_X\{e^{-itK}\Phi(x)e^{itK}\}\Phi^\dagger(y)\}] \tag{D.6}$$

$$= g_X(t)[g_X(0)]^*, \tag{D.7}$$

where we used the fact that L and K commute and that $e^{i\theta L}\Phi e^{-i\theta L} = e^{i\theta}\Phi$. In the last line we sent $|x-y| \to \infty$ and used the assumption that ρ_X has short-range correlations.

Now, the theorem of Ref. [52] rigorously proves that the function $f(t)$ must be independent of time. Hence, unless $g_X(0) = 0$, we conclude that $g_X(t)$ must be independent of time. (If $g_X(0) = 0$ but $g_X(t)$ is not independent of time then there must be some t such that $g_X(t) \neq 0$. Then we can just relabel the time-coordinate so that $g_X(0) \neq 0$ and repeat the argument.)

D.2 Open systems

In this section, we will elaborate on our hypothesis for open systems introduced in Section 7.6 above, namely that in a large class of systems the steady state will have low energy. First we need to clarify what we mean by "energy" and "steady state" in the Floquet context. Let $H_S(t)$ be the time-evolution of the system alone (not taking to account the coupling to the environment.) We define the Floquet operator $U_f = \mathcal{T}\exp\left(-i\int_0^T H_S(t)dt\right)$. Recall that in the regime discussed in Section

7.3, where λ as defined there satisfies $\lambda T \ll 1$, we can write $H_S(t) = \widetilde{H}_S(t) + V(t)$. Here $V(t)$ is a very weak residual perturbation, and $\widetilde{H}_S(t)$ is such that, if we define the approximate Floquet operator by $\widetilde{U_f} = \mathcal{T}\exp\left(-i\int_0^T \widetilde{H}_S(t)\right)$, then it can be expressed, following a local unitary time-independent change of basis (which we will here set to 1 for notational simplicity), as $\widetilde{U_f} = Xe^{-iDT}$, where $X^2 = 1$ and D is a quasi-local Hamiltonian D that commutes with X. In particular, we have $\widetilde{U_f}^2 = e^{-2iDT}$. This implies that we can make a time-*dependent* local unitary change of basis $W(t)$, periodic with period $2T$ and satisfying $W(0) = 1$, such that the transformed Hamiltonian, which is related to $\widetilde{H}_S(t)$ according to

$$\widetilde{H}'_S = WH_SW^\dagger + i[\partial_t W]W^\dagger, \tag{D.8}$$

is time-independent and equal to D. Therefore, in this new reference frame, it is clear that we should refer to the expectation value of D as "energy". We emphasize that we have not gotten rid of the time-dependence completely: even in the new reference frame the residual driving term $V(t)$, as well as any couplings to the environment, will still be time-dependent. (Due to the time-dependent change of basis, the latter will gain a time-dependence even if it was originally time-independent.)

The steady state is now determined by some balance between the residual periodic driving $V(t)$, the classical noise, and the coupling to the environment. We leave a detailed analysis of this open system process for future work[1], but we expect that in a suitable regime the energy-density of the steady state will be low. We will now explain why this implies oscillations (which are observed in the *original* reference frame, not the rotating one defined above.)

Consider a short-range correlated steady state ρ whose energy density with respect

[1]For one study of steady states of many-body Floquet systems coupled to a bath, see Ref. [206]

to D is small. Recall that in Section 7.3.1 we argued that if ρ is a thermal state it must spontaneously break the symmetry generated by X, and it follows that under $\widetilde{U_f}$ it oscillates at twice the drive frequency. Of course, for an open system the steady-state need not be thermal, and time evolution of the open system is not exactly given by $\widetilde{U_f}$. However, as we prove in Appendix D.3, even non-thermal states must fail to be invariant under the symmetry X if their energy density with respect to D is sufficiently small, provided that they satisfy a physically reasonable "thermalizability" condition. Moreover, if $\lambda T \ll 1$ (so that we can approximate $\widetilde{U_f} \approx X$), and the coupling to the environment sufficiently weak, then the resulting state after one time period is approximately given by $X\rho X^\dagger$, which by the preceding discussion is *not* the same as ρ. (We make this argument more precise in Section D.3.) Thus, provided that the energy of the steady-state is sufficiently small, it does not return to itself after one time period, and oscillations with period $2T$ will be observed.

Generic baths will destroy continuous-time time crystals. The difference with the discrete-time case is the existence of an extra variable characterizing thermal states of D; namely, the chemical potential μ. This extra variable is needed because of the presence of the hidden $U(1)$ symmetry in the continuous-time regime. (There is no analogous variable when the hidden symmetry is *discrete*). Thus, one certainly cannot make any statement that all low-energy states of D oscillate, because, in particular, a *thermal* state of D in which the electrochemical potential $\mu - u = 0$ does not oscillate. A coupling to a generic bath will not preserve the hidden $U(1)$ symmetry, and thus to the extent that the steady state of an open system process is close to a thermal state of D, we in fact expect it to have $\mu - u = 0$, since this corresponds to minimizing the free energy.

In principle, one could fine-tune the bath so that it repects the symmetry. This would allow the time crystal to survive, but is clearly contrived. One might wonder

whether the bath itself could also pre-thermalize: if we could consider the bath to be included in the Hamiltonian (7.34) then it could have an approximate U(1) symmetry along with the rest of the system. This would require the local terms in the bath Hamiltonian to be much smaller than the coupling u in Eq. (7.34). However, for most of the physically relevant baths that one would want to consider (for example, phonons), the local terms in the bath Hamiltonian are in fact unbounded.

D.3 Spontaneous symmetry breaking for non-thermal states

Let D be a quasi-local Hamiltonian for which the thermal states spontaneously break an on-site $\mathbb{Z}_N$ symmetry generated by X for energy densities $e < e_c$. More precisely, what we mean is the following, where we define the local distance between two states on a region A according to

$$\|\rho_1 - \rho_2\|_A = \|(\rho_1)_A - (\rho_2)_A\|_1 \tag{D.9}$$

where $\|\cdot\|_1$ is the trace norm, and $(\rho)_A = \mathrm{Tr}_{A^c}\rho$ is the reduced state of ρ on A.

Assumption 1 (Spontaneous symmetry-breaking)**.** There exists some finite region A and some $\gamma > 0$, such that, for any short-range correlated thermal state ρ_τ with energy density $e < e_c$, we have $\|\rho_\tau - X^k \rho_\tau X^{-k}\|_A \geq \gamma$ for all $0 < k < N$.

Now let ρ be *any* state (not necessarily thermal) such that the energy density $\epsilon \equiv \langle D \rangle_\rho / V < \epsilon_c$ (with V the volume of the system.) We assume the following *thermalizability* condition, which roughly states that ρ can thermalize when time-evolved under D. More precisely:

Assumption 2 (Thermalizability)**.** There exist a time t_1 and a short-range correlated thermal state ρ_τ with the same energy density as ρ, such that $\|\rho(t_1) - \rho_\tau\|_A \leq \gamma/8$, where $\rho(t) = e^{-iDt_1}\rho e^{iDt_1}$.

From Assumptions 1 and 2 we derive the following lemma, which quantifies the sense in which the state ρ must break the symmetry.

Lemma 1. *There exists a finite region A' such that $\|\rho - X^k \rho X^{-k}\|_{A'} \geq 3\gamma/4$.*

Proof. From the triangle inequality it follows that

$$\|\rho(t_1) - X^k \rho(t_1) X^{-k}\|_A \tag{D.10}$$

$$\geq \|\rho_\tau - X^k \rho_\tau X^{-k}\|_A - \|\rho(t_1) - X^k \rho(t_1) X^{-k} - (\rho_\tau - X^k \rho_\tau X^{-k})\|_A \tag{D.11}$$

$$\geq \gamma - 2\gamma/8 \tag{D.12}$$

$$= 3\gamma/4. \tag{D.13}$$

Using the characterization of the trace norm as

$$\|\rho\|_1 = \sup_{\hat{o}:\|\hat{o}\|=1} |\langle \hat{o} \rangle_\rho|, \tag{D.14}$$

it follows that there exists an operator $\hat{o}_A$ supported on A, with $\|\hat{o}_A\| = 1$, such that $|\langle X^{-k} \hat{o}_A X^k - \hat{o}_A \rangle_{\rho(t_1)}| \geq 3\gamma/4$. Now, since D is quasi-local, it must obey a Lieb-Robinson bound [207, 208], which implies that there exists a local operator $\hat{O}_{A'}$ supported on a finite region A' such that $\|\hat{o}(t_1) - \hat{O}_{A'}\| \leq \gamma/8$, where $\hat{o}(t_1) =$

$e^{iDt_1}\hat{o}e^{-iDt_1}$. Hence we see that

$$|\langle X^{-k}\hat{O}_{A'}X^k - \hat{O}_{A'}\rangle_\rho| \tag{D.15}$$

$$\geq -\gamma/4 + |\langle X^{-k}\hat{o}_A(t_1)X^k - \hat{o}_A(t_1)\rangle_\rho| \tag{D.16}$$

$$= -\gamma/4 + |\langle X^{-k}\hat{o}_A X^k - \hat{o}_A\rangle_{\rho(t_1)}| \tag{D.17}$$

$$\geq -\gamma/4 + 3\gamma/4. \tag{D.18}$$

$$= \gamma/2. \tag{D.19}$$

To get to line Eq. (D.17), we used the fact that X and D commute. The lemma follows. □

Now consider a system which in isolation would evolve under a time-dependent Hamiltonian $H(t)$, which is periodic with period T. We assume that $H(t)$ exhibits the pre-thermalization phenomena discussed in the main text. That is, we assume that the Floquet operator can be approximated according to $U_f \approx \widetilde{U_f} = Xe^{-iDT}$, where D is quasi-local and commutes with X, and where U_f is close to $\widetilde{U_f}$ in the sense that

$$\|U_f^\dagger O_{A'} U_f - \widetilde{U_f}^\dagger O_{A'}\widetilde{U_f}\| \leq \frac{\gamma}{8}\|O_{A'}\| \tag{D.20}$$

for any operator $O_{A'}$ supported on A'.

Let $\rho_{\text{open}}(t)$ be the reduced state of the system (tracing out the bath) at time t, taking into account the system-bath coupling, and we assume that $\rho_{\text{open}}(0) \equiv \rho$ satisfies Assumption 2 above. We assume the coupling to the bath is sufficiently weak, in the following sense:

Assumption 3 (Weak coupling)**.** For any time $0 \leq t \leq T$, we have $\|\rho^{\text{int}}_{\text{open}}(t) - \rho\|_{A'} \leq \gamma/8$.

Here we defined the interaction picture state $\rho^{\text{int}}_{\text{open}}(t) = U(0,t)^{-1}\rho_{\text{open}}(t)U(0,t)$, where $U(0,t)$ is the time evolution generated by $H(t)$. If we were to set the coupling to the bath to zero then the state $\rho^{\text{int}}_{\text{open}}(t)$ would be constant in time, so Assumption 3 corresponds to weak coupling. Finally, we will assume that the strength of DT is small enough so that

Assumption 4. For any observable $O_{A'}$ supported on A', we have

$$\|e^{-iDT}O_{A'}e^{iDT} - O_{A'}\| \leq \frac{\gamma}{8}\|O_{A'}\| \tag{D.21}$$

This will always be true in the regime of interest, $\lambda T \ll 1$ (where λ is as defined in Section 7.2), because $\|D\|_{n_*}$ is $O(\lambda)$ [see Eq. (7.9) in Theorem 1].

From the above assumptions we can now derive our main result:

Theorem 6.

$$\|\rho_{\text{open}}(T) - \rho\|_{A'} \geq \gamma/8. \tag{D.22}$$

Proof.

$$\|\rho_{\text{open}}(T) - \rho\|_{A'} \tag{D.23}$$

$$= \|U_f\rho^{\text{int}}_{\text{open}}(T)U_f^\dagger - \rho\|_{A'} \tag{D.24}$$

$$\geq -\gamma/8 + \|\widetilde{U}_f\rho^{\text{int}}_{\text{open}}(T)\widetilde{U}_f^\dagger - \rho\|_{A'} \tag{D.25}$$

$$= -\gamma/8 + \|e^{-iDT}\rho^{\text{int}}_{\text{open}}(T)e^{iDT} - X^\dagger\rho X\|_{A'} \tag{D.26}$$

$$\geq -\gamma/8 - \gamma/8 + \|\rho^{\text{int}}_{\text{open}}(T) - X^\dagger\rho X\|_{A'} \tag{D.27}$$

$$\geq -\gamma/8 - \gamma/8 - \gamma/8 + \|\rho - X^\dagger\rho X\|_{A'} \tag{D.28}$$

$$\geq -\gamma/8 - \gamma/8 - \gamma/8 + \gamma/2. \tag{D.29}$$

$$= \gamma/8. \tag{D.30}$$

□

In other words, the state of the open system at times $t = T$ and $t = 0$ are locally distinguishable. That is, for the stated assumptions, the state of the system does not synchronize with the drive and time translation symmetry is spontaneously broken.

Appendix E

More on topological phases with spatial symmetries

E.1 Computing the bosonic classification

A nice feature of our results, at least in the case of bosonic crystalline SPTs (in Euclidean space) is that the classification is readily computable. According to the general discussion of Section 8.5.2, we see that the classification in d space dimensions for a given space group G is given by $H^{d+2}(BG, \mathbb{Z}^{\mathrm{or}})$. Computing this object turns out to be within the capabilities of the GAP computer algebra program [209]. We show the results in Table 8.1 (in the introduction) for the (2+1)-D case and in Table E.1 for the (3+1)-D case. There were 3 space groups in (3+1)-D for which the classification took too long to compute and is not shown.

We recall that this classification is expected to be *complete* in (2+1)-D, and for the Sohncke groups (those not containing any orientation-reversing elements) in (3+1)-D. What about explicit constructions of these phases? Let us fix some element $\omega \in H^{d+2}(BG, \mathrm{Z}^{\mathrm{or}})$. Suppose that there exists a finite group G_f and a group homomorphism $\varphi : G \to G_f$ such that ω is in the image of the map $\mathcal{H}^{d+1}(G_f, \mathrm{U}(1)^{\mathrm{or}}) \cong H^{d+2}(BG_f, \mathbb{Z}^{\mathrm{or}}) \to H^{d+2}(BG, \mathbb{Z}^{\mathrm{or}})$ induced by φ. Then indeed we have an explicit

construction of the crystalline SPT correponding to ω, using the bootstrap argument of Section 8.4 (leveraging, for example, the construction of Ref. [80] for the SPT protected by G_f acting internally). We conjecture that there will always be some such G_f for any element of $H^{d+2}(BG, \mathbb{Z}^{\mathrm{or}})$.

E.2 Coupling a Hamiltonian to a rigid crystalline gauge field

In this appendix, we explain how to couple a finite range Hamiltonian to a crystalline gauge field. To fix notation, X will be the physical space with G action, Λ the crystalline lattice therein, M the test space, divided into patches $\bigcup_i U_i = M$ with local homeomorphisms $f : U_i \to X$ and transition functions $g_{ij} \in G$ such that for all $x \in U_i \cap U_j$, $f_i(x) = g_{ij} f_j(x)$. We will use the shorthand A to denote the whole crystalline gauge field.

We begin by defining the Hilbert space on M, assuming that the Hilbert space of X is local to the lattice Λ, that is, there is a space $\mathcal{H}_x$ for every $x \in \Lambda$ and $\mathcal{H}_X = \bigotimes_{x\in\Lambda} \mathcal{H}_x$. We define the pulled-back lattice $\Sigma = \bigcup_j f_j^{-1}\Lambda$ and assign to each $m \in f_j^{-1}\Lambda$ the Hilbert space $\mathcal{H}_m(A) := \mathcal{H}_{f_j(m)}$. The total Hilbert space may be written $\mathcal{H}(A) = \bigotimes_{m\in\Sigma} \mathcal{H}_m(A)$.

Next we discuss (rigid) gauge transformations. These come in three sorts. The first are homotopies of the maps f_j (fixing the boundary). We suppose that the patches are transverse to the lattice (this is generic) so that each $m \in \Sigma$ lies in a unique $U_j =: U_{j(m)}$. In the rigid case, these are simply continuous deformations of the lattice in M[1].

[1] In the non-rigid case, new lattice sites could appear or disappear in conjugate pairs by creating "folds" of f_j.

Number	Name	Classification
1	P1	0
2	$P\bar{1}$	$\mathbb{Z}_2^{\times 8}$
3	P2	$\mathbb{Z}_2^{\times 4}$
4	$P2_1$	0
5	C2	$\mathbb{Z}_2^{\times 2}$
6	Pm	$\mathbb{Z}_2^{\times 4}$
7	Pc	0
8	Cm	$\mathbb{Z}_2^{\times 2}$
9	Cc	0
10	P2/m	$\mathbb{Z}_2^{\times 18}$
11	$P2_1/m$	$\mathbb{Z}_2^{\times 6}$
12	C2/m	$\mathbb{Z}_2^{\times 11}$
13	P2/c	$\mathbb{Z}_2^{\times 6}$
14	$P2_1/c$	$\mathbb{Z}_2^{\times 4}$
15	C2/c	$\mathbb{Z}_2^{\times 5}$
16	P222	$\mathbb{Z}_2^{\times 16}$
17	$P222_1$	$\mathbb{Z}_2^{\times 4}$
18	$P2_12_12$	$\mathbb{Z}_2^{\times 2}$
19	$P2_12_12_1$	0
20	$C222_1$	$\mathbb{Z}_2^{\times 2}$
21	C222	$\mathbb{Z}_2^{\times 9}$
22	F222	$\mathbb{Z}_2^{\times 8}$
23	I222	$\mathbb{Z}_2^{\times 8}$
24	$I2_12_12_1$	$\mathbb{Z}_2^{\times 3}$
25	Pmm2	$\mathbb{Z}_2^{\times 16}$
26	$Pmc2_1$	$\mathbb{Z}_2^{\times 4}$
27	Pcc2	$\mathbb{Z}_2^{\times 4}$
28	Pma2	$\mathbb{Z}_2^{\times 4}$
29	$Pca2_1$	0
30	Pnc2	$\mathbb{Z}_2^{\times 2}$
31	$Pmn2_1$	$\mathbb{Z}_2^{\times 2}$
32	Pba2	$\mathbb{Z}_2^{\times 2}$
33	$Pna2_1$	0
34	Pnn2	$\mathbb{Z}_2^{\times 2}$
35	Cmm2	$\mathbb{Z}_2^{\times 9}$
36	$Cmc2_1$	$\mathbb{Z}_2^{\times 2}$
37	Ccc2	$\mathbb{Z}_2^{\times 3}$
38	Amm2	$\mathbb{Z}_2^{\times 9}$
39	Aem2	$\mathbb{Z}_2^{\times 4}$
40	Ama2	$\mathbb{Z}_2^{\times 3}$
41	Aea2	$\mathbb{Z}_2$
42	Fmm2	$\mathbb{Z}_2^{\times 6}$
43	Fdd2	$\mathbb{Z}_2$
44	Imm2	$\mathbb{Z}_2^{\times 8}$
45	Iba2	$\mathbb{Z}_2^{\times 2}$
46	Ima2	$\mathbb{Z}_2^{\times 3}$
47	Pmmm	$\mathbb{Z}_2^{\times 42}$
48	Pnnn	$\mathbb{Z}_2^{\times 10}$
49	Pccm	$\mathbb{Z}_2^{\times 17}$
50	Pban	$\mathbb{Z}_2^{\times 10}$
51	Pmma	$\mathbb{Z}_2^{\times 17}$
52	Pnna	$\mathbb{Z}_2^{\times 4}$
53	Pmna	$\mathbb{Z}_2^{\times 10}$
54	Pcca	$\mathbb{Z}_2^{\times 5}$
55	Pbam	$\mathbb{Z}_2^{\times 10}$
56	Pccn	$\mathbb{Z}_2^{\times 4}$
57	Pbcm	$\mathbb{Z}_2^{\times 5}$
58	Pnnm	$\mathbb{Z}_2^{\times 9}$
59	Pmmn	$\mathbb{Z}_2^{\times 10}$
60	Pbcn	$\mathbb{Z}_2^{\times 3}$
61	Pbca	$\mathbb{Z}_2^{\times 2}$
62	Pnma	$\mathbb{Z}_2^{\times 4}$
63	Cmcm	$\mathbb{Z}_2^{\times 10}$
64	Cmce	$\mathbb{Z}_2^{\times 7}$
65	Cmmm	$\mathbb{Z}_2^{\times 26}$
66	Cccm	$\mathbb{Z}_2^{\times 13}$
67	Cmme	$\mathbb{Z}_2^{\times 17}$
68	Ccce	$\mathbb{Z}_2^{\times 7}$
69	Fmmm	$\mathbb{Z}_2^{\times 20}$
70	Fddd	$\mathbb{Z}_2^{\times 6}$
71	Immm	$\mathbb{Z}_2^{\times 22}$
72	Ibam	$\mathbb{Z}_2^{\times 10}$
73	Ibca	$\mathbb{Z}_2^{\times 5}$
74	Imma	$\mathbb{Z}_2^{\times 13}$
75	P4	$\mathbb{Z}_2 \times \mathbb{Z}_4^{\times 2}$
76	$P4_1$	0
77	$P4_2$	$\mathbb{Z}_2^{\times 3}$
78	$P4_3$	0

Table E.1: The "230-fold way". This table shows the classification of bosonic crystalline SPT phases in (3+1)-D for each of the 3-D space groups. For space groups 227, 228 and 230 the classification has not been computed.

Number	Name	Classification
79	I4	$\mathbb{Z}_2 \times \mathbb{Z}_4$
80	$I4_1$	$\mathbb{Z}_2$
81	$P\bar{4}$	$\mathbb{Z}_2^{\times 3} \times \mathbb{Z}_4^{\times 2}$
82	$I\bar{4}$	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_4^{\times 2}$
83	P4/m	$\mathbb{Z}_2^{\times 12} \times \mathbb{Z}_4^{\times 2}$
84	$P4_2/m$	$\mathbb{Z}_2^{\times 11}$
85	P4/n	$\mathbb{Z}_2^{\times 3} \times \mathbb{Z}_4^{\times 2}$
86	$P4_2/n$	$\mathbb{Z}_2^{\times 4} \times \mathbb{Z}_4$
87	I4/m	$\mathbb{Z}_2^{\times 8} \times \mathbb{Z}_4$
88	$I4_1/a$	$\mathbb{Z}_2^{\times 3} \times \mathbb{Z}_4$
89	P422	$\mathbb{Z}_2^{\times 12}$
90	$P42_12$	$\mathbb{Z}_2^{\times 4} \times \mathbb{Z}_4$
91	$P4_122$	$\mathbb{Z}_2^{\times 3}$
92	$P4_12_12$	$\mathbb{Z}_2$
93	$P4_222$	$\mathbb{Z}_2^{\times 12}$
94	$P4_22_12$	$\mathbb{Z}_2^{\times 5}$
95	$P4_322$	$\mathbb{Z}_2^{\times 3}$
96	$P4_32_12$	$\mathbb{Z}_2$
97	I422	$\mathbb{Z}_2^{\times 8}$
98	$I4_122$	$\mathbb{Z}_2^{\times 5}$
99	P4mm	$\mathbb{Z}_2^{\times 12}$
100	P4bm	$\mathbb{Z}_2^{\times 4} \times \mathbb{Z}_4$
101	$P4_2cm$	$\mathbb{Z}_2^{\times 6}$
102	$P4_2nm$	$\mathbb{Z}_2^{\times 5}$
103	P4cc	$\mathbb{Z}_2^{\times 3}$
104	P4nc	$\mathbb{Z}_2 \times \mathbb{Z}_4$
105	$P4_2mc$	$\mathbb{Z}_2^{\times 9}$
106	$P4_2bc$	$\mathbb{Z}_2^{\times 2}$
107	I4mm	$\mathbb{Z}_2^{\times 7}$
108	I4cm	$\mathbb{Z}_2^{\times 4}$
109	$I4_1md$	$\mathbb{Z}_2^{\times 4}$
110	$I4_1cd$	$\mathbb{Z}_2$
111	$P\bar{4}2m$	$\mathbb{Z}_2^{\times 13}$
112	$P\bar{4}2c$	$\mathbb{Z}_2^{\times 10}$
113	$P\bar{4}2_1m$	$\mathbb{Z}_2^{\times 5} \times \mathbb{Z}_4$
114	$P\bar{4}2_1c$	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_4$
115	$P\bar{4}m2$	$\mathbb{Z}_2^{\times 13}$
116	$P\bar{4}c2$	$\mathbb{Z}_2^{\times 7}$
117	$P\bar{4}b2$	$\mathbb{Z}_2^{\times 5} \times \mathbb{Z}_4$
118	$P\bar{4}n2$	$\mathbb{Z}_2^{\times 5} \times \mathbb{Z}_4$
119	$I\bar{4}m2$	$\mathbb{Z}_2^{\times 9}$
120	$I\bar{4}c2$	$\mathbb{Z}_2^{\times 6}$
121	$I\bar{4}2m$	$\mathbb{Z}_2^{\times 8}$
122	$I\bar{4}2d$	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_4$
123	P4/mmm	$\mathbb{Z}_2^{\times 32}$
124	P4/mcc	$\mathbb{Z}_2^{\times 13}$
125	P4/nbm	$\mathbb{Z}_2^{\times 13}$
126	P4/nnc	$\mathbb{Z}_2^{\times 8}$
127	P4/mbm	$\mathbb{Z}_2^{\times 15} \times \mathbb{Z}_4$
128	P4/mnc	$\mathbb{Z}_2^{\times 8} \times \mathbb{Z}_4$
129	P4/nmm	$\mathbb{Z}_2^{\times 13}$
130	P4/ncc	$\mathbb{Z}_2^{\times 5}$
131	$P4_2/mmc$	$\mathbb{Z}_2^{\times 24}$
132	$P4_2/mcm$	$\mathbb{Z}_2^{\times 18}$
133	$P4_2/nbc$	$\mathbb{Z}_2^{\times 8}$
134	$P4_2/nnm$	$\mathbb{Z}_2^{\times 13}$
135	$P4_2/mbc$	$\mathbb{Z}_2^{\times 8}$
136	$P4_2/mnm$	$\mathbb{Z}_2^{\times 14}$
137	$P4_2/nmc$	$\mathbb{Z}_2^{\times 8}$
138	$P4_2/ncm$	$\mathbb{Z}_2^{\times 10}$
139	I4/mmm	$\mathbb{Z}_2^{\times 20}$
140	I4/mcm	$\mathbb{Z}_2^{\times 14}$
141	$I4_1/amd$	$\mathbb{Z}_2^{\times 9}$
142	$I4_1/acd$	$\mathbb{Z}_2^{\times 5}$
143	P3	$\mathbb{Z}_3^{\times 3}$
144	$P3_1$	0
145	$P3_2$	0
146	R3	$\mathbb{Z}_3$
147	$P\bar{3}$	$\mathbb{Z}_2^{\times 4} \times \mathbb{Z}_3^{\times 2}$
148	$R\bar{3}$	$\mathbb{Z}_2^{\times 4} \times \mathbb{Z}_3$
149	P312	$\mathbb{Z}_2^{\times 2}$
150	P321	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_3$
151	$P3_112$	$\mathbb{Z}_2^{\times 2}$
152	$P3_121$	$\mathbb{Z}_2^{\times 2}$
153	$P3_212$	$\mathbb{Z}_2^{\times 2}$
154	$P3_221$	$\mathbb{Z}_2^{\times 2}$
155	R32	$\mathbb{Z}_2^{\times 2}$
156	P3m1	$\mathbb{Z}_2^{\times 2}$

Table E.1: (continued)

Number	Name	Classification
157	P31m	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_3$
158	P3c1	0
159	P31c	$\mathbb{Z}_3$
160	R3m	$\mathbb{Z}_2^{\times 2}$
161	R3c	0
162	P$\bar{3}$1m	$\mathbb{Z}_2^{\times 9}$
163	P$\bar{3}$1c	$\mathbb{Z}_2^{\times 3}$
164	P$\bar{3}$m1	$\mathbb{Z}_2^{\times 9}$
165	P$\bar{3}$c1	$\mathbb{Z}_2^{\times 3}$
166	R$\bar{3}$m	$\mathbb{Z}_2^{\times 9}$
167	R$\bar{3}$c	$\mathbb{Z}_2^{\times 3}$
168	P6	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_3^{\times 2}$
169	P6_1	0
170	P6_5	0
171	P6_2	$\mathbb{Z}_2^{\times 2}$
172	P6_4	$\mathbb{Z}_2^{\times 2}$
173	P6_3	$\mathbb{Z}_3^{\times 2}$
174	P$\bar{6}$	$\mathbb{Z}_2^{\times 4} \times \mathbb{Z}_3^{\times 3}$
175	P6/m	$\mathbb{Z}_2^{\times 10} \times \mathbb{Z}_3^{\times 2}$
176	P6_3/m	$\mathbb{Z}_2^{\times 4} \times \mathbb{Z}_3^{\times 2}$
177	P622	$\mathbb{Z}_2^{\times 8}$
178	P$6_1$22	$\mathbb{Z}_2^{\times 2}$
179	P$6_5$22	$\mathbb{Z}_2^{\times 2}$
180	P$6_2$22	$\mathbb{Z}_2^{\times 8}$
181	P$6_4$22	$\mathbb{Z}_2^{\times 8}$
182	P$6_3$22	$\mathbb{Z}_2^{\times 2}$
183	P6mm	$\mathbb{Z}_2^{\times 8}$
184	P6cc	$\mathbb{Z}_2^{\times 2}$
185	P6_3cm	$\mathbb{Z}_2^{\times 2}$
186	P6_3mc	$\mathbb{Z}_2^{\times 2}$
187	P$\bar{6}$m2	$\mathbb{Z}_2^{\times 9}$
188	P$\bar{6}$c2	$\mathbb{Z}_2^{\times 3}$
189	P$\bar{6}$2m	$\mathbb{Z}_2^{\times 9} \times \mathbb{Z}_3$
190	P$\bar{6}$2c	$\mathbb{Z}_2^{\times 3} \times \mathbb{Z}_3$
191	P6/mmm	$\mathbb{Z}_2^{\times 22}$
192	P6/mcc	$\mathbb{Z}_2^{\times 9}$
193	P6_3/mcm	$\mathbb{Z}_2^{\times 9}$
194	P6_3/mmc	$\mathbb{Z}_2^{\times 9}$
195	P23	$\mathbb{Z}_2^{\times 4} \times \mathbb{Z}_3$
196	F23	$\mathbb{Z}_3$
197	I23	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_3$
198	P$2_1$3	$\mathbb{Z}_3$
199	I$2_1$3	$\mathbb{Z}_2 \times \mathbb{Z}_3$
200	Pm$\bar{3}$	$\mathbb{Z}_2^{\times 14} \times \mathbb{Z}_3$
201	Pn$\bar{3}$	$\mathbb{Z}_2^{\times 4} \times \mathbb{Z}_3$
202	Fm$\bar{3}$	$\mathbb{Z}_2^{\times 6} \times \mathbb{Z}_3$
203	Fd$\bar{3}$	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_3$
204	Im$\bar{3}$	$\mathbb{Z}_2^{\times 8} \times \mathbb{Z}_3$
205	Pa$\bar{3}$	$\mathbb{Z}_2^{\times 2} \times \mathbb{Z}_3$
206	Ia$\bar{3}$	$\mathbb{Z}_2^{\times 3} \times \mathbb{Z}_3$
207	P432	$\mathbb{Z}_2^{\times 6}$
208	P$4_2$32	$\mathbb{Z}_2^{\times 6}$
209	F432	$\mathbb{Z}_2^{\times 4}$
210	F$4_1$32	$\mathbb{Z}_2$
211	I432	$\mathbb{Z}_2^{\times 5}$
212	P$4_3$32	$\mathbb{Z}_2$
213	P$4_1$32	$\mathbb{Z}_2$
214	I$4_1$32	$\mathbb{Z}_2^{\times 4}$
215	P$\bar{4}$3m	$\mathbb{Z}_2^{\times 7}$
216	F$\bar{4}$3m	$\mathbb{Z}_2^{\times 5}$
217	I$\bar{4}$3m	$\mathbb{Z}_2^{\times 5}$
218	P$\bar{4}$3n	$\mathbb{Z}_2^{\times 4}$
219	F$\bar{4}$3c	$\mathbb{Z}_2^{\times 2}$
220	I$\bar{4}$3d	$\mathbb{Z}_2 \times \mathbb{Z}_4$
221	Pm$\bar{3}$m	$\mathbb{Z}_2^{\times 18}$
222	Pn$\bar{3}$n	$\mathbb{Z}_2^{\times 5}$
223	Pm$\bar{3}$n	$\mathbb{Z}_2^{\times 10}$
224	Pn$\bar{3}$m	$\mathbb{Z}_2^{\times 10}$
225	Fm$\bar{3}$m	$\mathbb{Z}_2^{\times 13}$
226	Fm$\bar{3}$c	$\mathbb{Z}_2^{\times 7}$
227	Fd$\bar{3}$m	???
228	Fd$\bar{3}$c	???
229	Im$\bar{3}$m	$\mathbb{Z}_2^{\times 13}$
230	Ia$\bar{3}$d	???

Table E.1: (continued)

The second type are given by the action of a group element $g_j \in G$ on a U_j and are analogous to ordinary gauge transformations. To define these, we need to assume the symmetry action on $\mathcal{H}_X$ is "ultralocal", meaning that it is a tensor product operator $U(g) = \bigotimes_{x\in\Lambda} U(g)_x$ where $U(g) : \mathcal{H}_x \to \mathcal{H}_{gx}$. Then we can isolate the part acting on $f_j(U_j)$, $U(g_j)_j = \bigotimes_{x\in\Lambda\cap f_j(U_j)} U(g)_x$ and apply this to $\mathcal{H}(A)$. This takes us to a different Hilbert space $\mathcal{H}(A^{g_j})$, where A^{g_j} is the crystalline gauge field obtained from A by replacing f_j with $g_j f_j$ and g_{ij} with $g_{ij}g_j^{-1}$ for all adjacent U_i to U_j.

The third type involve moving the patches themselves. This is actually a combination of the previous type of gauge transformation as well as splitting or joining patches. A patch U becomes split into $U_1 \cup U_2$ with f_1, f_2 defined by restricting f and $g_{12} = 1$. Likewise, if there are every any adjacent patches $U_{i,j}$ with $g_{ij} = 1$, then f_i and f_j can be joined to a continuous function across both patches which can then be considered a single patch $U_i \cup U_j$. In both cases the adjacent transition functions do not change. Moving a domain wall can then be achieved by first splitting a patch, applying a G element to the new patch, and joining patches again.

Now we discuss how to couple a Hamiltonian to this crystalline gauge field. For each $m \in \Sigma$ and each term h in the Hamiltonian H acting on $f_j(m)$, we will have a corresponding term in the Hamiltonian $H(A)$ acting on $\mathcal{H}(A)$. If the support of h lies entirely inside $f_j(U_j)$, then it acts on $\bigotimes_{x\in f_j(U_j)\cap\Lambda} \mathcal{H}_x = \bigotimes_{m\in U_j\cap\Sigma} \mathcal{H}_m$, which is a tensor factor of $\mathcal{H}(A)$ so we can include h in $H(A)$ with no issue.

Difficulty comes when the support of h is not contained inside any one $f_j(U_j)$. This is where we have to use the rigidity assumption. We assume that it is possible to move the patch U_j by a gauge transformation so that h is contained in $f_j(U_j)$ (the Hamiltonian built so far comes along for the ride according to our gauge transformation operator). Then we add h to the Hamiltonian and perform the inverse gauge transformation to return to the original gauge field configuration. Compare Appendix

E.4, especially Fig E.4.

As a simple example of this technique, consider a 1+1D spin-1/2 Ising model, focusing on a specific edge 12 with Hamiltonian term X_1X_2 and global $\mathbb{Z}_2$ symmetry $\bigotimes_j Z_j$, where X, Z denote Pauli spin operators. Suppose that 1 and 2 belong to different patches with a non-trivial transition function. Then rather than adding X_1X_2 to the Hamiltonian, we first perform a gauge transformation Z_2, which pushes the domain wall off to the right and we get the term $-X_1X_2$. Note because $\mathbb{Z}_2$ is a symmetry, it doesn't matter which way we push the domain wall off. Using Z_1 would result in the same term.

We end this appendix with a second method for describing the Hamiltonian coupled to a crystalline gauge field, which is equivalent but does not require one to perform gauge transformations to obtain all the terms in the Hamiltonian. In this version, the patches U_j are taken to be an open covering of M and are allowed to overlap. Then a lattice (hence a Hilbert space) is first defined on the disjoint union $\bigsqcup_j U_j$ by $\tilde{\Sigma} := \bigsqcup_j f_j^{-1}\Lambda$. We denote the associated Hilbert space $\mathcal{H}_{\tilde{M}} = \bigotimes_j \mathcal{H}_{U_j}$, where $\mathcal{H}_{U_j} = \bigotimes_{m\in\Sigma\cap U_j} \mathcal{H}_m$. Note that the map $\bigsqcup_j U_j \to \bigcup_j U_j = M$ sends $\tilde{\Sigma}$ to Σ. Then rigidity means that for each $m \in \Sigma$, and for each term h acting on $f(m)$, there is some $U_j \ni m$ such that the support of h is contained in $f_j(U_j)$. We choose h to act on the U_j part of the Hilbert space $\mathcal{H}_{\tilde{M}}$. Then we project everything to $\mathcal{H}_M$ by identifying duplicated vertices $m \in U_j, m' \in U_k$ in the disjoint union by the transition maps $U(g_{ij}) : \mathcal{H}_{U_j} \to \mathcal{H}_{U_k}$. A simple example is shown in Fig E.1.

This method is particularly convenient for describing crystal defects. In the case of a single defect in $\mathbb{R}^d$ supported along ∂H, where H is a $d-1$-dimensional branch cut (which, fixing ∂H, is a choice of gauge), the defect space $M = \mathbb{R}^d - \partial H$ can be covered with a single patch U given by a thickening of $\mathbb{R}^d - H$, which intersects itself in M along a neighborhood of H. In other words, the degrees of freedom near the

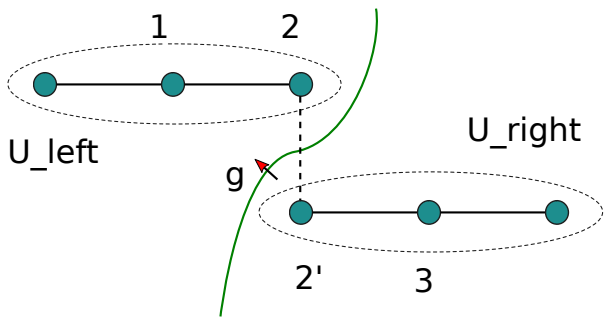


Figure E.1: In this approach to defining the Hamiltonian coupled to crystalline gauge field, patches are allowed to overlap to include some vertices. In this particular example, $U_{left} \cap U_{right}$ includes vertex 2, which gets duplicated. Hamiltonian terms (denoted by solid edges) lying entirely inside U_{left} or U_{right} are taken to act on those Hilbert spaces. Then spurious degrees of freedom are eliminated by applying a projection operator which in a product state basis identifies the state at 2 with g applied to the state at 2'. This is indicated by the green curve labelled by g cutting the dashed vertical line from 2 to 2'.

branch cut are doubled (see Figure 8.3, coupled to either side of the branch cut, and then reglued by a projection map twisted by the crystal symmetry.

E.3 Coupling smooth states to gauge fields

Here we prove the claims made in Section 8.3.2 about the well-definedness of the construction to couple smooth states to gauge fields. We first consider the case of an internal symmetry G. We adapt an argument due to Kitaev (Appendix F of Ref. [185]). We assume that our original ground state ψ lives on a lattice with a spin of Hilbert space dimension d at each site. However, we will define the space Ω which our smooth states target to be the space of states with Hilbert space dimension $m > d$ per site. Of course, given a choice of isometric embedding $e : \mathbb{C}^d \to \mathbb{C}^m$, we could think of our original state ψ as living in Ω too. The resulting state depends on e and we call it $e(\psi)$.

Recall that the symmetry is assumed to act on-site, with the action on each site described by a representation $u(g) \in U(d)$. For each $g \in G$, we also considered a path $u(g;t)$, $t \in [0,1]$ such that $u(g;0) = \mathbb{I}$ and $u(g;1) = u(g)$. Then, (at least locally) we can reformulate the prescription in Section 8.3.2 for defining the smooth state $\psi[A] : M \to \Omega$ as follows in terms of a *spatially-dependent* isometric embedding $e_m : \mathbb{C}^d \to \mathbb{C}^m$, according to $\psi[A](m) = e_m(\Psi)$. We then require that when passing over a patch boundary twisted by $g \in G$, e_m goes through the continuous path obtained by acting with $u(g;t)$. But now we see that there will not be any obstructions to making this process well-defined due to non-contractible loops (or higher non-trivial homotopy groups) at intersections between patch boundaries, provided that we take m sufficiently large. This is because in the limit $m \to \infty$ the space $\mathrm{Emb}(d,m)$ of all isometric embeddings $\mathbb{C}^d \to \mathbb{C}^m$ is contractible, i.e. all its homotopy groups are trivial.

A more rigorous (and succinct) way to think about the above construction is obtained by thinking about the classifying space BG. Indeed, since $EG := \lim_{m\to\infty}\mathrm{Emb}(d,m)$ is a contractible space with a free action of G, it follows that EG/G is a model for BG, and we find that there is a continuous map $BG \to \Omega$. A G gauge field over M is the same as a principal G-bundle over M, which can be represented by a a continuous map $M \to BG$. Hence, composing these two maps gives a smooth state $\psi[A] : M \to \Omega$.

The "patch" version of the argument for a crystalline gauge field proceeds similarly to above and we will not write it out again. Let us simply note that a rigorous version of the construction can be formulated in terms of the homotopy quotient $X//G$. Indeed, given a smooth state $\psi : X \to \Omega_d$ (where Ω_d is the space of ground states with Hilbert space dimension d per site), there is a map from $X \times \mathrm{Emb}(d,m) \to \Omega$ defined by $(x,e) \mapsto e(\psi(x))$. This map is invariant under the diagional action of G.

Therefore, taking the limit $m \to \infty$, we find a map from $(X \times EG)/G = X//G \to \Omega$. A crystalline gauge field on M can be represented by a map from $M \to X//G$. By composing these two maps we obtain a smooth state $\psi[A] : M \to \Omega$.

E.4 Lattice Crystalline Gauge Fields

The cellular description we give in this section is dual to the patch picture we gave in Section 8.3.1, where g elements labelled codimension 1 walls between volumes in the crystal. Here in order to compare with the usual definition of a lattice gauge field, we label edges with g elements.

Recall for a discrete group G a lattice gauge field has a very nice description where each edge e gets a group label $g_e \in G$ and any 2-face τ imposes a flatness constraint

$$\prod_{e \in \partial\tau} g_e = 1, \tag{E.1}$$

where the multiplication is performed in the order the edges are encountered in a circular traversal of the boundary. This conservation law allows us to express these labels as a configuration of domain walls running about our manifold. The conservation law says that a g_1 and a g_2 fuse to a $g_1 g_2$. The domain walls are codimension one so fusion can be non-commutative in this way.

Let's imagine drawing a configuration like this on X where the G elements act non-trivially on X. Does this make sense? Let's look at a particular edge, Fig E.2. It looks like an edge from $x \to y$, but if we push the domain wall out of the way, we see the actual data there is an edge (actually path; see below) from x to gy! This means that while our underlying manifold has points labelled by points in X, it is perhaps a different space M! To see what data is assigned to a face or higher facet,

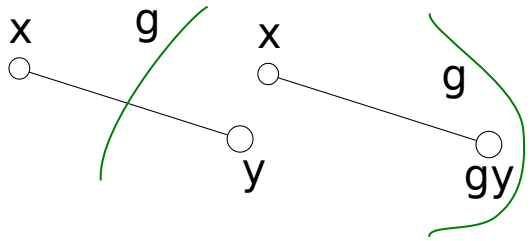


Figure E.2: When the g domain wall is pulled off of this edge, it is revealed to be an edge from $x \to gy$. Note the similarity with the Hamiltonian coupling procedure in Appendix E.2.

one performs a similar procedure, pushing all the domain walls off and collecting g labels. The flatness condition on G implies that this is always unambiguous. At a symmetry defect like the core of a disclination, the flatness condition is violated and it is impossible to unambiguously assign a face of X to the core of the defect. When this happens, the underlying space M may have different topology from X! In fact, we may end up with a space M whose labels don't even close up unto a map to X! In such a case, we end up with only a map $P \to X$, where P is the G-cover corresponding to the g labels (equivalently the G gauge bundle).

Note that if X is contractible the extra information beyond the G gauge field, the X labels, contributes no non-trivial data up to homotopies of this map. Indeed this is basically another proof of the Crystalline Equivalence Principle.

Let us try to be more systematic about the construction. We start with a warm-up, just describing cellular maps $\hat{f} : M \to X$ in a lattice gauge theoryish way. A cellular map means the n-skeleton of M gets sent to the n-skeleton of X for every n. This means every vertex $m \in M$ gets a vertex $\hat{f}(m) \in X$, every edge $e : m_1 \to m_2 \in M$ gets a *path* $\hat{f}(e) : \hat{f}(m_1) \rightsquigarrow \hat{f}(m_2) \in M$, every plaquette τ gets a chain $\hat{f}(\tau)$ with $\partial \hat{f}(\tau) = \hat{f}(\partial\tau)$, every volume gets a 3-chain with prescribed boundary and so on.

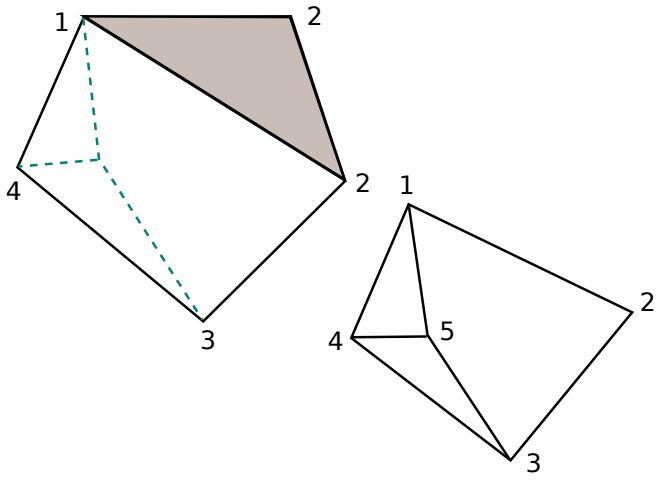


Figure E.3: Here we depict of a piece of M (northwest) mapping to a piece of X (southeast). We have given the vertices of X unique labels and labelled the vertices of M with their image vertices in X. Note that vertex 2 has two adjacent preimages. This edge of M is mapped to a degenerate edge and the triangle it lies on (grey) is mapped to a degenerate face 122 in X. Note also that vertex $5 \in X$ has no preimage and to map faces to faces we must refine the lattice of M, depicted by the dotted blue lines.

This data describes a general partial covering $M \to X$ (i.e. a map which gives a rigid crystalline gauge field with trivial transition functions).

To account for maps which are not locally homeomorphisms, we need to include in this definition the *degenerate facets of* X. For example, if we had the constant map $M \mapsto x \in X$, this definition only makes sense if there is a hidden edge $\mathrm{id} : x \to x$, hidden faces $x \to x \to x$, $x \to x \to y$, and so on. All higher degenerate facets should be included as well.

This means that any map $\hat{f} : M \to X$ is homotopic to one given first by refinement of the lattice in M and then by labelling vertices, edges, faces, ... of the refinement with vertices, edges, faces, ... (possibly degenerate ones) of X. This should be intuitive, since the cell structure in M is not really physical. It's just a way to encode the topology of M combinatorially.

Now let us consider maps with G-twisted continuity conditions. As before we

assign vertices of X to vertices of M. Before to an edge in M we would assign a path $x \rightsquigarrow y$ connecting the X labels x and y of the endpoints. For G-fs, these paths can pass through domain walls, resulting in something we call a G-path:

$$x_1 \rightsquigarrow y_1 \xrightarrow{g_1} x_2 \rightsquigarrow \cdots \xrightarrow{g_k} y_k.$$

Around the boundary of a face $\tau \in M$, we get a G-path by concatenating the G-paths on each edge. Our conservation law

$$\prod_j g_j = 1$$

must be supplemented by the condition that the boundary G-path forms a G-loop:

$$y_k = x_1.$$

If this is the case, then we can push all the g's to the right, acting on the paths as we do to obtain an honest path $x_1 \rightsquigarrow g_k^{-1} \cdots g_1^{-1} x_1$. If the G conservation law holds then this path is a loop in X. This is just like pushing the domain walls off τ towards vertex 1. We ask that τ be assigned a chain with boundary equal to this loop. A picture of this is depicted in Fig E.4.

Now we discuss homotopies of this data (collapsible crystalline gauge transformations). Such a homotopy $A(0) \mapsto A(1)$ is itself a crystalline gauge field $A(t)$ but on the prism $M \times [0,1]$ with boundary conditions equal to $A(0)$ and $A(1)$ on each copy of M.

As a first warm-up, let's just consider ordinary G gauge fields. See Fig E.5. There is a cell complex of $M \times [0,1]$ with one inner $p+1$-cell for every p cell of M. These inner cells are the only ones where the boundary conditions do not fix the data. For

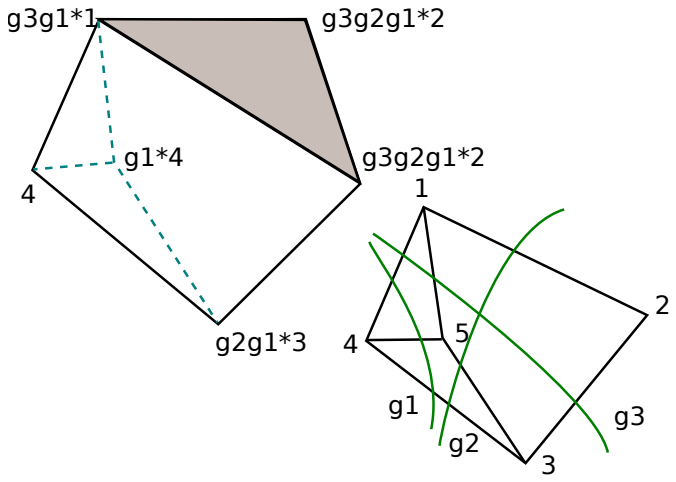


Figure E.4: The conservation law for G labels allows us to draw them as G domain walls in X. Then in any contractible patch of M we can describe our local map $M \to X$ by "pushing off the domain walls". Then we look at the northwest picture of our patch in M. See how the vertices have been transformed; so have the edges. Then we fill in the transformed picture with faces of X as we would in describing an ordinary map $M \to X$. This always requires a choice of basepoint. Here our basepoint is 4 and we have pushed all the domain walls (green) straight to the east. The choice of basepoint is like a local choice of gauge. It should be compared with the construction for coupling to Hamiltonians in Appendix E.2.

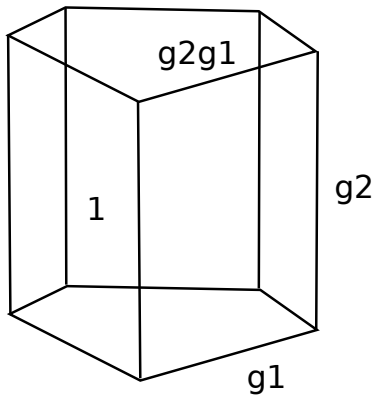


Figure E.5: A prism $M \times [0,1]$ mapping to BG means an assigning of G labels also to the interior edges. These correspond with the vertices of M so we can think of them as a function $g : M^0 \to G$ where M^0 is the set of vertices of M. Then the conservation law on the internal faces of the prism forces a constraint between corresponding edge labels in each M. The constraint reads that the top labels are the gauge transformation of the bottom labels by g. The direction is fixed by an orientation of the internal prism edges. If we reverse all of them, it takes $g \mapsto g^{-1}$ (and locally as well).

an ordinary G gauge field we must specify the G labels on the inner edges. These correspond to vertices of M, so the data is like an element of G for each vertex of M. The flatness condition on the inner faces determines how these must act on the edge variables.

A second warm-up, really getting going this time, is to consider homotopies of a map $M \to X$. This is the case with no symmetry, $G = 1$. This gets quite complicated but it is possible to divide homotopies into elementary pieces, where all the inner p-cells but one are degenerate but one: τ_p, meaning the map $M \to X$ does not change away from τ_p. The map $h : M \times [0,1] \to X$ identifies τ_p with a p-chain $h(\tau_p)$ and because all other inner cells in $M \times [0,1]$ are degenerate, $\partial h(\tau_p)$ is divided into two $p-1$-chains in the image of the boundaries: $\partial h(\tau_p) = h(N_0) \sqcup h(N_1)$, where $N_j \subset M \times \{j\}$ are $p-1$-chains in M. In fact these are the same $p-1$-chains and

$h(\tau_p)$ is telling us how they move inside X during the homotopy h. A general gauge transformation of A is essentially a combination of these two ingredients.

Just as the cellular description of G gauge fields reflects a convenient cellular structure of BG, what we have described above amounts to a cellular structure on the homotopy quotient $X//G$. One can see what we've written as a simultaneous construction of $X//G$ and a proof of

Theorem 7. *A crystalline gauge field is the same as a cellular map $A : M \to X//G$ with the cell structure induced by the action of G on a compatible cell structure of X. Thus, gauge equivalence classes of crystalline gauge fields are the same as homotopy classes of maps $A : M \to X//G$.*

There is a nice way to get a handle on the homotopy type of $X//G$. Recall from, eg. Ref. [210], that BG, the classifying space for ordinary G-gauge fields and the special case of our construction when X is a point, is itself constructed as an ordinary quotient EG/G, where EG is some (usually very large) *contractible* space on which G acts freely. For discrete groups, EG can be constructed as a simplicial complex where vertices are group elements $g \in G$, edges are pairs, triangles are triples, and so on. The gluing maps use the G multiplication. For example, an edge (g_0, g_1) is glued to g_1 and to $g_0 g_1$; a triangle (g_0, g_1, g_2) is glued to (g_1, g_2), $(g_0, g_1 g_2)$; and $(g_0 g_1, g_2)$, and so on. This space has a G action which acts on all the labels simultaneously. It's also contractible. The quotient structure is the usual structure on BG. Likewise, we can invent a cell structure on the space $EG \times X$ so that the quotient structure is the one we've described on $X//G$. This proves $X//G = EG \times X/G$ where G acts diagonally. In fact, to preserve the homotopy type of $X//G$, we just need any space EX which is homotopy equivalent to X and on which G acts freely. $EG \times X$ is an example, but if G already acts freely on X, then X itself is an example and the homotopy quotient

reduces to the ordinary quotient $X//G = X/G$. In the other extreme, which G is a purely internal symmetry, $X//G = BG \times X$.

E.5 Explicit constructions for bosonic SPTs

In Ref. [80], a prescription was given to construct a ground-state wavefunction for an SPT phase protected by a finite internal symmetry group G_{int}. As stated in the main text, we want to leverage this construction in a "bootstrap" procedure to construct a wavefunction for an SPT phase protected by a spatial symmetry, as outlined in Section 8.4. For our current discussion, the important requirement is that we must be able to choose the wavefunction to be invariant under *both* an internal symmetry G_{int} and a spatial symmetry G_{spatial}. Ultimately, the symmetry protecting the crystalline SPT phase will be the diagonal subgroup G_{phys}. Recall that we take orientation-reversing elements of G_{spatial} to also act anti-unitarily, in accordance with the CPT principle. (Thus, the orientation-reversing symmetries in G_{phys} are a composition of two anti-unitary operators, and so end up being unitary.)

Let us briefly review the construction of Ref. [80]. This construction starts from an element of the group cohomology group $\mathcal{H}^{d+1}(G_{\text{int}}, \mathrm{U}(1))$. This cohomology class is represented by a $(d+1)$-cocycle in homogeneous form, which is a function $\nu : G_{\text{int}}^{\times d+1} \to \mathrm{U}(1)$ satisfying

$$g \cdot \nu(g_1, \cdots, g_{d+1}) = \nu(gg_1, \cdots, gg_{d+1}) \quad \forall g \in G_{\text{int}} \tag{E.2}$$

$$\prod_{i=0}^{d+2} \nu^{(-1)^i}(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_{d+2}) = 1, \tag{E.3}$$

where $g \cdot \nu$ denotes the action of G_{int} on $U(1)$, i.e. anti-unitary elements of G_{int} act by inversion.

To construct the wavefunction on some d-dimensional spatial manifold, one first chooses a triangulation of the manifold. The spins will live on the vertices of this triangulation, and they will each carry a Hilbert space with basis $\{|g\rangle : g \in G_{\text{int}}\}$, on which G_{int} acts by left-multiplication: $|h\rangle \xrightarrow{g} |gh\rangle$. Then one chooses a branching structure, which is a choice of direction on the edges of the triangulation, such that there are no directed cycles on any d simplex. A branching structure allows us to define an ordering of the vertices on any d-simplex. The wavefunction of Ref. [80] is then defined as a superposition

$$|\Psi\rangle = \sum_{\{g_i\}} \left(\prod_{\Delta} \alpha_{\Delta} \left(g_{\Delta} \right) \right) |\{g_i\}\rangle \,, \tag{E.4}$$

where the sum is over all configurations $\{g_i\}$ of group elements $g \in G_{\text{int}}$ for every vertex, and the product is over all d-simplices. The phase factor α_{Δ} associated to a d-simplex Δ is defined by

$$\alpha_{\Delta}(g_{\Delta}) = \nu^{s(\Delta)}(g_*, g_1, \cdots, g_d), \tag{E.5}$$

where $g_1, \cdots, g_d$ are the group elements living on the vertices of the simplex (ordered according to the branching structure), $g_* \in G_{\text{int}}$ is some fixed group element which is chosen to be the same for every d-simplex (the resulting wavefunction turns out not to depend on g_* on any closed manifold); and $s(\Delta) = \pm 1$ is the orientation of the d-simplex (see Ref. [80] for further details). It can be verified that the wavefunction $|\Psi\rangle$ so defined is indeed invariant under the action of G_{int}.

Now it remains to show that $|\Psi\rangle$ can also be taken to be invariant under the action of a spatial symmetry G_{spatial}. We take the action of G_{spatial} on the Hilbert space of the spins to be inherited from its action on the space manifold; that is, it simply permutes

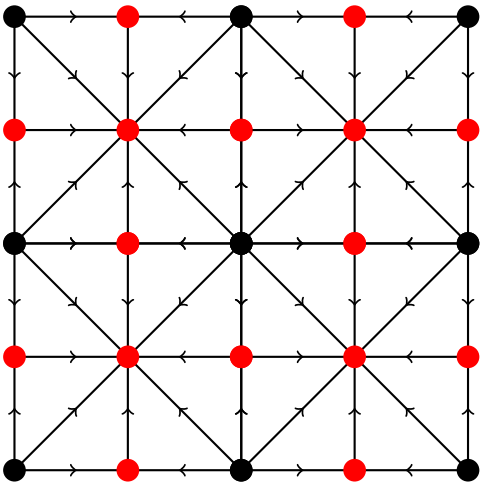


Figure E.6: A $p4m$-invariant triangulation and branching structure. The block dots are the vertices of the original $p4m$-invariant cellulation (the simple square lattice) and the red dots are the vertices that had to be added (through the barycentric subdivision) to get a $p4m$-invariant triangulation and branching structure.

the spins. (For orientation-reversing elements of G_{spatial}, this is followed by complex conjugation, in accordance with our stipulation that orientation-reversing elements of G_{spatial} should act anti-unitarily). This will evidently be the case provided that the locations of the vertices, the triangulation, and the branching structure are all invariant under the action of G_{spatial}. (For orientation-revering elements, note that the effect of the complex conjugation is cancelled by the reversal of the orientation of the simplices). To achieve this, we can start from a G_{spatial}-invariant cellulation of the spatial manifold (which can be obtained, for example, via the Wigner-Seitz construction), then take its barycentric subdivision, which gives a G_{spatial}-invariant triangulation. Moreover, one can show that there is always a G_{spatial}-invariant branching structure on this triangulation. The resulting triangulation and branching structure is illustrated in Figure E.6 for the case $d = 2$ and $G_{\text{spatial}} = p4m$ (the symmetry group of the simple square lattice).

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