University of California
Los Angeles

On the Negative K-theory of Singular Varieties

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Justin Shih

2012

Abstract of the Dissertation

On the Negative K-theory of Singular Varieties

by

Justin Shih

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2012

Professor Christian Haesemeyer, Chair

Let X be an integral n-dimensional variety over a field k of characteristic zero, regular in codimension 1 and with singular locus Z. We establish a right exact sequence, coming from the Brown-Gersten spectral sequence, that computes $K_{1-n}(X)$ from $KH_{1-n}(X)$ and $NK_{1-n}(X)$. We then compute each of these pieces separately, and then analyze the map $NK_{1-n}(X) \longrightarrow K_{1-n}(X)$.

We show that the $KH_{1-n}(X)$ contribution almost has a geometric structure. When k is algebraically closed, X is projective, and Z is either smooth over k or of codimension greater than 2, we prove that there is a 1-motive $M = [L \longrightarrow G]$ over k, and a map $G(k) \longrightarrow KH_{1-n}(X)$ whose kernel and cokernel are finitely generated. Thus the k-points $G(k)$ of the group scheme G approximates $KH_{1-n}(X)$ up to some finitely generated abelian groups. Furthermore, when $n = 3$, the sequence $L(k) \longrightarrow G(k) \longrightarrow KH_{-2}(X)$ is exact. In addition, M is computable, as under Deligne's equivalence between torsion-free 1-motives and torsion-free mixed Hodge structures of type $\{(0,0),(0,1),(1,0),(1,1)\}$ such that $\mathrm{Gr}_1^W H$ is polarizable, the free complex 1-motive $(M \times_k \mathbb{C})_{\mathrm{fr}}$ is the 1-motive that corresponds to the unique largest such H coming from the weight 2 part $W_2 H^n(X(\mathbb{C}),\mathbb{Z})$ of the n^{th} cohomology group $H^n(X(\mathbb{C}),\mathbb{Z})$.

When X is not projective, the result still holds, except that the 1-motive M comes from $W_2 H^n(\overline{X}(\mathbb{C}),\mathbb{Z})$, where $\overline{X}$ is an algebraic compactification of X. Furthermore, the non-lattice parts of the M we get, and hence the map α, are independent of the choice of

compactification.

For the $NK_{1-n}(X)$ contribution, when Z is an isolated singularity, we show that $K_{1-n}(X)$ is an extension of $KH_{1-n}(X)$ by the *cdh*-cohomology group $H^{n-1}_{\mathrm{cdh}}(U, \mathcal{O})$, where U is any open affine neighborhood of Z. Furthermore, $H^{n-1}_{\mathrm{cdh}}(U, \mathcal{O})$ is a finite-dimensional k-vector space, whose dimension is the Du Bois invariant $b^{0,n-1}$ of the isolated singularity Z.

All in all, we have a full computation of $K_{-2}(X)$ when X is three-dimensional over an algebraically closed field and has only isolated singularities.

The dissertation of Justin Shih is approved.

Vasilios Manousiouthakis

Paul Balmer

Alexander Merkurjev

Christian Haesemeyer, Committee Chair

University of California, Los Angeles

2012

To Mom and Dad, for all of your sacrifices, and for 28 years of support and guidance.

To Debbie, for all of your gentle patience and love.

This dissertation would not be possible without any of you.

Table of Contents

Acknowledgments

First and foremost, I owe an immense debt of gratitude to my advisor, Christian Haesemeyer, for guiding me through this winding and sometimes rocky journey. Without your mentorship, patience, and deep mathematical foresight, I would not be typing these words today. Thank you for believing in me.

I would also like to thank Wanshun Wong and Adam Massey for many helpful mathematical discussions, and for answering countless stupid questions I've had. I have benefited greatly from having you two accompany me on this journey.

I am also deeply indebted to other good friends that have kept me sane during my time in graduate school.

Chris Tape – Thank you for all of your advice, companionship, and countless late night online chats. It is rare to find someone that you get along with on so many levels. The circumstances under which we met were certainly suboptimal, but after all these years I can say that it was definitely worth it.

Lindsay Riley – Being friends with you has, luckily for me, increased the average coolness of my LA friends by quite a bit. It's amazing to me that I get to be friends with you. Don't you have better things to do? In conclusion, the combination of beer, cheese sticks, bad flan, and the game is basically one of the worst ever conceived of in the history of mankind.

Siwei Zhu – Thank you for being my roommate during my tenure at UCLA, for many interesting mathematical discussions, and for teaching me how to be greedy! I have stayed somewhat sharp during my time here, in no small part due to being around you.

Jukka Virtanen – Thank you for your friendship over the years, and for giving me a smooth transition in moving out to LA. I really am lucky that I spawned at our office my first

year, and that I have had the good fortunate to stay in close touch with you. MS 6617F OP.

Charlie Cha – Thank you for your loyal friendship, for keeping in touch over the years, and for reminding me how to be more laid back. Why are you so awesome?

Debbie Cheung – Thank you, for everything. Just being around you has undoubtedly made me a better person over the years. I consider myself extremely blessed that I have the privilege of having you in my life at all.

Vita

2001–2006	Undergraduate student, The University of Texas at Austin
2005	Bachelor of Science with Honors, Chemical Engineering, The University of Texas at Austin
2006	Bachelor of Science with Honors, Mathematics, The University of Texas at Austin
2006	Dean's Honored Graduates, College of Natural Sciences, The University of Texas at Austin
2006-2012	Graduate student, UCLA
2006-2012	Teaching Assistant/Associate/Fellow, UCLA
2006-2009	NSF VIGRE Fellowship, UCLA
2007	Master of Arts, Mathematics, UCLA
2009-2010	NSF RTG in Algebra Fellowship, UCLA
2010-2012	Research Assistant, UCLA
2010	Candidate of Philosophy, Mathematics, UCLA

Publications

Alicia Dickenstein, J. Maurice Rojas, Korben Rusek, and Justin Shih. "Extremal real algebraic geometry and $\mathcal{A}$-discriminants." *Mosc. Math. J.*, **7**(3):425452, 574, 2007.

CHAPTER 1

Introduction and History

The origins of algebraic K-theory can be traced back to the late 1950s when Alexander Grothendieck defined $K_0(X)$ of a scheme X as the group of isomorphism classes of locally-free coherent sheaves on X modulo exact sequences [BS58]. Topologists, taking notice of Grothendieck's construction, soon produced an analogous construction for vector bundles over compact Hausdorff spaces. A few years later in 1961, Michael Atiyah and Friedrich Hirzebruch published a landmark paper in which they extended this idea by defining *higher* topological K-groups $K^n(X)$ for all integers n. It led directly to a second, alternative proof of the celebrated Atiyah-Singer index theorem a year later in 1962. On the algebraic side of things, things proceeded more slowly. Progress in the topological case suggested that higher *algebraic* K-theory should exist, and satisfactory definitions for lower K-groups (K_1, K_2) were found, but these were ad-hoc and did not generalize well to higher n. The search for a general definition for higher K-theory went on for the next decade or so until the early 1970s, when Daniel Quillen, in a series of papers, settled things once and for all. He defined higher algebraic K-theory in two different ways, via the Q- and $+$-constructions [Qui10, Qui75]. These equivalent constructions [Qui74] yielded groups $K_n(X)$ for all integers $n \geq 0$, as in the topological case. Quillen would win the Fields Medal a few years later in 1976, in large part due to this foundational work.

Quillen's landmark papers established groups K_n for a general exact category $\mathcal{C}$ (of which the category $\mathcal{VB}(X)$ of vector bundles over a scheme X is one class of examples); in the years since, Quillen's work has been refined by many others, including Waldhausen [Wal85], who defined K-groups for more general categories called *Waldhausen* categories, and Thomason, who extended algebraic K-theory by using the derived category of perfect complexes [TT90]. In addition, many variants of K-theory have appeared across many different fields. Besides topology and algebraic geometry, K-theory has appeared in number theory and functional analysis. In algebraic geometry alone, there are several important variants: K'-theory, which considers all coherent sheaves on a scheme X (rather than just locally-free coherent sheaves, as in K-theory); Karoubi-Villamayor's KV-theory [KV69, KV71], and Weibel's KH-theory [Wei89]. Of these variants, KH-theory will be the most useful to us.

1.1 Overview of Dissertation

The rest of this chapter is dedicated to introducing algebraic K-theory, related variants, and their basic properties. To compute K-groups, we will compute the *homotopy* K-theory KH (Chapter 6) and the *nil* K-theory NK (Chapter 7) pieces, and then put them together. To do this, we will need to use Hironaka's resolution of singularities (Chapter 2), Voevodsky's *cdh*-topology (Chapter 3), simplicial and semisimplicial spaces (Chapter 4), and mixed Hodge structures and 1-motives (Chapter 5). Finally, applications and other consequences of our work are considered in chapter 8.

1.2 Definitions and basic properties

In order to get a basic intuition for algebraic K-theory, we briefly review some of its constructions and properties. We do not claim any originality in this section, and will mostly just quote useful theorems. In *Higher Algebraic K-Theory: I* [Qui10] and *II* [Gra76], Quillen defined the higher algebraic K-theory of exact categories, i.e. those categories that have a distinguished collection of exact sequences.

Quillen's definitions of K-theory are very technical and not particularly well suited for computations. More generally, computing the K-theory of a scheme X is very difficult. The computation of $K_n(\mathbb{Z})$ is still open (see [Wei05] for the current status), and the K-theory

$K_n(\mathbb{C})$ of the complex numbers is still not completely understood [Sus84]. We would like to calculate some of the K-groups of specific varieties, so we are interested in the tools that have been developed to compute K-groups. Despite the difficulties, a good amount of progress has been made over the past few decades in the direction of computing K-groups. This introduction would not be complete without the following standard tools, given below: the projective bundle formula, the localization sequence for K-theory, and the fundamental theorem of K-theory.

Theorem 1.1 (Projective Bundle Formula). *Let X be quasi-compact and quasi-separated, and let $\mathscr{E}$ be a vector bundle of rank n on X with $\pi\colon \mathbb{P}\mathscr{E} \longrightarrow X$ the associated projective space bundle. Then we have, for all $n \in \mathbb{Z}$, an isomorphism*

$$\bigoplus_{i=0}^{r-1} K_n(X) \longrightarrow K_n(\mathbb{P}\mathscr{E}) \tag{1.1}$$

given by $(x_0, \ldots, x_{r-1}) \mapsto \sum_i \pi^*(x_i) \otimes [\mathscr{O}_{\mathbb{P}\mathscr{E}}(-1)]$.

Proof. [TT90, Theorem 4.1] □

Theorem 1.2 (Localization sequence for K-theory). *Let X be quasi-compact and quasi-separated, $U \subset X$ a quasi-compact open subscheme, and let $Y = X - U$ be the reduced induced complement. Then on the spectrum level, we have a homotopy fibration sequence*

$$K(X \text{ on } Y) \longrightarrow K(X) \longrightarrow K(U) \tag{1.2}$$

inducing a long exact sequence on K-groups

$$\cdots \longrightarrow K_n(X \text{ on } Y) \longrightarrow K_n(X) \longrightarrow K_n(U) \longrightarrow K_{n-1}(X \text{ on } Y) \longrightarrow \cdots \tag{1.3}$$

Proof. [TT90, Theorem 7.4] □

In particular, we have the *fundamental theorem of K-theory*:

Corollary 1.3 (Fundamental theorem of K-theory)**.** *We have an exact sequence for any quasi-compact, quasi-separated scheme X, and any $n \in \mathbb{Z}$:*

$$0 \longrightarrow K_n(X) \longrightarrow K_n(X[t]) \oplus K_n(X[t^{-1}]) \longrightarrow K_n(X[t, t^{-1}]) \longrightarrow K_{n-1}(X) \longrightarrow 0. \tag{1.4}$$

Proof. [TT90, Theorem 6.6] □

1.3 Variants of K-theory and related functors

When defining the K-theory of a noetherian scheme, there are two natural choices for the exact category that we can plug into the machine: the category $\mathcal{VB}(X)$ of vector bundles on X, and the larger category $\mathfrak{Coh}(X)$ of coherent sheaves on X. These two categories lead to different K-theories. The first, $\mathcal{VB}(X)$, yields the standard Quillen K-theory, and the second, $\mathfrak{Coh}(X)$, yields K'-theory (also sometimes called G-theory).

It turns out that when X is smooth over k, it satisfies many other additional nice K-theoretic properties. For example, the Bass Fundamental Theorem 1.3 simply becomes $K_n(X \times \mathbb{G}_m) \cong K_n(X) \oplus K_{n-1}(X)$. In addition, the K-theory of X agrees with its K'-theory [Qui10, §6]. Furthermore, its K-theory is zero in negative degrees [TT90, Proposition 6.8]. They also satisfy the important relation $K_n(X) \cong K_n(X \times \mathbb{A}^1)$ [Qui75]. This last property motivates the following definition:

Definition 1.4. Let $F\colon (Sch/k)^{op} \longrightarrow \mathcal{A}$, where $\mathcal{A}$ is additive. We say that F is *homotopy invariant* if for any $X \in Sch/k$, the projection $X \times \mathbb{A}^1 \longrightarrow X$ induces an isomorphism $F(X) \cong F(X \times \mathbb{A}^1)$.

In general, however, K-theory is not homotopy invariant, as the next example shows.

Example 1.5. Let F be a field, and consider $R = F[\epsilon] = F[x]/(x^2)$, the ring of dual numbers over F. We will show that the inclusion $R \longrightarrow R[t]$ does not induce an isomorphism on K_1 groups. The determinant gives a canonical surjection [Wei, Chapter 3]

$$K_1(A) \xrightarrow{\det} A^\times \tag{1.5}$$

for any commutative ring A. For $A = R$, this map is an isomorphism because R is local [Wei, Chapter 3, Lemma 1.4]. On the other hand, $K_1(R[t])$ surjects onto $R[t]^\times \cong R^\times \oplus \epsilon R[t]$. So we have a diagram

$$\begin{tikzcd} K_1(R) \arrow[r, "\cong"] \arrow[d] & R^\times \arrow[d] \\ K_1(R[t]) \arrow[r, two heads] & R[t]^\times \end{tikzcd} \tag{1.6}$$

In particular, the vertical map on the right is a strict inclusion. If the left vertical map were an isomorphism, the left-bottom composite would be surjective. But then the vertical map on the right would have to be surjective, a contradiction.

One attempt at defining higher K-theory came in the form of *Karoubi-Villamayor* K-theory, denoted KV, in 1969 [KV69]. It is defined for a ring R as follows: let $R[\Delta^\bullet]$ be the simplicial ring associated to R (see Example 4.4 for details). Then we define

$$KV_n(R) = \pi_{n-1}(GL(R[\Delta^\bullet])), \tag{1.7}$$

where GL is the direct limit of the groups GL_q [Wei, IV, Definition 11.4]. This definition was not Karoubi and Villamayor's original definition, but was discovered later on by Rector [Wei99]. We mention KV-theory for historical reasons, and because it is closely related to KH-theory, which we now discuss. KH-theory was introduced by Charles Weibel in [Wei89], some number of years after Karoubi and Villamayor discovered KV-theory. KH

is a homotopy invariant K-theory, as the groups KH_n are homotopy invariant for all n (in contrast with the groups KV_n, which are only homotopy invariant for $n \geq 1$). Weibel originally defined KH (for rings) to be $KH_n(R) = \pi_n K^B(R[\Delta^\bullet])$, and then extended it to schemes using Jouanalou's device. We present the more compact definition of KH found in [TT90].

Definition 1.6. Let X be quasicompact and quasiseparated. We then define the homotopy K-groups $KH_n(X)$ to be

$$KH_n(X) = \pi_n \left(\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} K(X \times \Delta^p) \right). \tag{1.8}$$

We will make extensive use of KH, dedicating an entire chapter (see Chapter 6) to the computation of KH groups. A functor closely related to K and KH is that of *nil-K-theory*, denoted by NK, which is the obstruction to homotopy invariance of K-theory.

Definition 1.7. Let X be quasicompact and quasiseparated. We define the *nil-K-groups* $NK_q(X)$ to be

$$NK_q(X) = \ker(K_q(X \times \mathbb{A}^1) \xrightarrow{t=0} K_q(X)). \tag{1.9}$$

NK theory is related to KH theory more directly via the spectral sequence (1.10). This spectral sequence will be the basis for our calculations, and we have an entire chapter (chapter 7) dedicated to computing $NK_{1-n}(X)$ when X is an irreducible n-dimensional variety over k with isolated singularity.

More generally, for a contravariant functor F from Sch/k into an additive category, we may define

$$NF(X) = \ker(F(X \times \mathbb{A}^1) \xrightarrow{t=0} F(X)). \tag{1.10}$$

In particular, we may iterate this construction and define $N^pF := N(N^{p-1}F)$. Furthermore, since the projection $X \times \mathbb{A}^1 \longrightarrow X$ is a section of the closed immersion $\{t = 0\}$, on applying F we see that $NF(X)$ is a direct summand of $F(X \times \mathbb{A}^1)$. Iterating this shows that $N^pF(X)$ is a direct summand of $F(X \times \mathbb{A}^p)$.

1.4 Spectral Sequences

Spectral sequences form a large class of tools by which to compute K-groups. They are, in some sense, generalizations of long exact sequences. Weibel [Wei94, Chapter 5] gives a good introduction to spectral sequences. In the following, we discuss a few spectral sequences which will be useful in our calculations.

Theorem 1.8 (Grothendieck spectral sequence)**.** *Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories, and we have additive functors $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $G\colon \mathcal{B} \longrightarrow \mathcal{C}$ so that G is left exact and F takes injective objects of $\mathcal{A}$ to G-acyclic objects of $\mathcal{B}$. Then there is a spectral sequence*

$$E_2^{p,q} = R^pG(R^qF(A)) \Longrightarrow (R^{p+q}(GF))(A). \tag{1.11}$$

Proof. [Wei94, Theorem 5.8.3] □

The Grothendieck spectral sequence is one of the foundations of homological algebra, and many spectral sequences (including the famous Leray spectral sequence) are specific instances of the Grothendieck spectral sequence.

Theorem 1.9 (Simplicial spectral sequence abutting to KH)**.** *Let X be a scheme over a field k, and let $\Delta^n = \mathrm{Spec}(k[x_0, \ldots, x_n]/(1 - \sum_i x_i))$. Then there is a right half-plane spectral sequence*

$$E^1_{p,q} = K_q(X \times \Delta^p) \Longrightarrow KH_{p+q}(X). \tag{1.12}$$

Proof. [Jar97, Corollary 4.22] □

Applying the Dold-Kan correspondence [Wei94, Theorem 8.4.1] to the E^1 terms for each fixed horizontal line, we obtain the normalized simplicial spectral sequence:

Corollary 1.10 (Normalized simplicial spectral sequence abutting to KH)**.** *Let X be a scheme over a field k, and let $\Delta^n = \operatorname{Spec}(k[x_0, \ldots, x_n]/(1 - \sum_i x_i))$. Then there is a spectral sequence*

$$E^1_{p,q} = N^p K_q(X) \Longrightarrow KH_{p+q}(X). \tag{1.13}$$

This spectral sequence also appears in [Wei89, Theorem 1.3]. Haesemeyer showed that KH satisfies *cdh*-descent in [Hae04] (see Chapter 3 for details). In particular, we have the following spectral sequence:

Theorem 1.11 (*cdh*-descent spectral sequence)**.** *Let X be a scheme, essentially of finite type over a field k of characteristic zero. Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H^p_{\mathrm{cdh}}(X, a_{\mathrm{cdh}} K_{-q}) \Longrightarrow KH_{-p-q}(X), \tag{1.14}$$

where a_{cdh} is the cdh-*sheafification functor.*

Proof. [Hae04, Theorem 1.1]. See Chapter 3 for background on the *cdh*-topology. □

If E is a normal crossing variety (Definition 2.4), the simplicial scheme $\Delta_\bullet E$ associated to E may be used to compute *cdh*-cohomology groups (see Chapter 4 for details):

Theorem 1.12 (Simplicial spectral sequence for *cdh*-cohomology)**.** *For any sheaf $\mathscr{F}$, there is a strongly convergent first quadrant spectral sequence*

$$E_1^{p,q} = H^q_{\mathrm{cdh}}(\Delta_p E, \mathscr{F}) \Longrightarrow H^{p+q}_{\mathrm{cdh}}(E, \mathscr{F}). \tag{1.15}$$

Proof. This is the so-called Čech-to-derived spectral sequence [God73, Theorem 5.4.1] (see also [Con03] for details), applied to the *cdh*-cover $\{\coprod_i E_i \longrightarrow E\}$ (see also Lemma 3.4). □

The inclusion of the nondegenerate components of $\Delta_\bullet E$ into $\Delta_\bullet E$ induces a quasi-isomorphism of spectral sequences as we prove below, so we may use either the simplicial scheme or the semisimplicial scheme $\Delta_\bullet^{alt} E$ associated to E to compute its *cdh*-cohomology groups (again, see Chapter 4 for details).

Theorem 1.13 (Nondegenerate simplicial spectral sequence for *cdh*-cohomology)**.** *For any sheaf $\mathscr{F}$, there is a strongly convergent first quadrant spectral sequence*

$$E_1^{p,q} = H^q_{\mathrm{cdh}}(\Delta_p^{alt} E, \mathscr{F}) \Longrightarrow H^{p+q}_{\mathrm{cdh}}(E, \mathscr{F}). \tag{1.16}$$

Proof. We apply [Wei94, Lemma 8.3.7], to the rows of the E_1 page of the spectral sequence. We consider $H^q_{\mathrm{cdh}}(\Delta_\bullet E, \mathscr{F})$ as a simplicial abelian group, and let $C_\bullet(H^q_{\mathrm{cdh}}(\Delta_\bullet E, \mathscr{F}))$ be the associated chain complex. If $D_\bullet(H^q_{\mathrm{cdh}}(\Delta_\bullet E, \mathscr{F}))$ denotes the subcomplex generated by the image of the degeneracy maps, then $C_\bullet(H^q_{\mathrm{cdh}}(\Delta_\bullet E, \mathscr{F}))$ is quasi-isomorphic to

$$C_\bullet(H^q_{\mathrm{cdh}}(\Delta_\bullet E, \mathscr{F}))/D_\bullet(H^q_{\mathrm{cdh}}(\Delta_\bullet E, \mathscr{F})) \cong C_\bullet H^q_{\mathrm{cdh}}(\Delta_\bullet^{alt} E, \mathscr{F}), \tag{1.17}$$

which was what we wanted to show. □

None of the spectral sequences listed so far abut to the K-theory of X, but instead to the KH-theory of X. Given the limitations of our current tools, it is natural that we attack the problem of computing K by first computing KH. The leftover piece will come from the NK-theory of X, so we will also use a spectral sequence to compute to the NK-theory of X.

Theorem 1.14. *There is a strongly convergent right half plane spectral sequence*

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, a_{\mathrm{Zar}} N^t K_{-q}) \Longrightarrow N^t K_{-p-q}(X), \tag{1.18}$$

where a_{Zar} denotes Zariski sheafification.

Proof. The Brown-Gersten spectral sequence [TT90, Theorem 10.3] splits into a direct sum, as in Definition 1.10. In this way, the above spectral sequence is a direct summand of the Brown-Gersten spectral sequence. □

1.5 Setup and Problem

Let k be a field of characteristic zero. The higher algebraic K-groups of a scheme tend to be, in general, significantly more inaccessible than the lower groups. The following results, known for any n-dimensional noetherian scheme X essentially of finite type over k, exemplify this phenomenon:

1. When $q > n$, $K_{-q}(X) = 0$. [CHS08, Conjecture 0.1]
2. $K_{-n}(X) \cong H^n_{\mathrm{cdh}}(X, \mathbb{Z})$. [Hae04, Theorem 7.1]
3. If, in addition, X is smooth, X has no negative K-groups. [TT90, Proposition 6.8]

where the second isomorphism is with the cohomology of the constant sheaf $\mathbb{Z}$ on the *cdh*-site over X (see Chapter 3). From the above, we see that one of the first unknown cases beyond the boundary of current literature is $K_{1-n}(X)$, where X is singular. But K_0 of curves is well known, and Weibel has computed K_{-1} of normal surfaces [Wei01]. Let X be an normal, irreducible n-dimensional variety (a separated reduced scheme of finite type) over an algebraically closed field k whose singular locus $Z = \mathrm{Sing}(X)$ is either smooth or has codimension greater than 2; we will compute the various pieces that contribute to the group $K_{1-n}(X)$. When $n = 3$, we will have a full computation of $K_{-2}(X)$. Application of the normalized simplicial spectral sequence (1.10) abutting to KH yields the following exact sequence which computes $K_{1-n}(X)$ (see Lemma 7.1 for the proof):

$$NK_{1-n}(X) \longrightarrow K_{1-n}(X) \longrightarrow KH_{1-n}(X) \longrightarrow 0. \tag{1.19}$$

We will compute the pieces $NK_{1-n}(X)$ and $KH_{1-n}(X)$ separately, and then analyze the map $NK_{1-n}(X) \longrightarrow K_{1-n}(X)$.

1.5.1 Motivational Example

When X is a threefold (i.e. $n = 3$) whose singular set $Z = \mathrm{Sing}(X)$ has is smooth and has dimension at most 1, we begin the computation of $KH_{1-n}(X) = KH_{-2}(X)$ here, mostly to motivate the background material found in Chapters 2, 3, 4, and 5. To do this, consider a *good* resolution of singularities for X:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ Z \arrow[r] & X \end{tikzcd} \tag{1.20}$$

where p is a proper birational morphism which is an isomorphism outside of the exceptional divisor E. We note that E must be connected, by Zariski's main theorem (see Lemma 2.3 for details). That this resolution is a *good* resolution means that E has smooth (irreducible) components that intersect transversally in smooth subschemes. More technically, we require E to have *normal crossings* (Definition 2.4). Intuitively, normal crossing divisors locally look like the intersection of coordinate planes. Then *cdh*-descent for KH [Hae04, Theorem 2.4], [CHS08, Theorem 3.4] implies a long exact sequence in KH:

$$\cdots \longrightarrow KH_q(X) \longrightarrow KH_q(Z) \oplus KH_q(\widetilde{X}) \longrightarrow KH_q(E) \longrightarrow \cdots \tag{1.21}$$

Since Z and $\widetilde{X}$ are smooth, KH-theory agrees with K-theory and K'-theory which are zero for negative q, so their negative KH groups vanish. In addition, $KH_q(E) = 0$ for $q < -2$ and $KH_q(X) = 0$ for $q < -3$ by [Hae04, Theorem 7.1]. So we obtain isomorphisms

$$KH_{-q}(X) \cong KH_{-q+1}(E), \tag{1.22}$$

for $q = 2, 3$. Consequently, we can compute $KH_{-1}(E)$ instead of $KH_{-2}(X)$, and we will use the simple normal crossings structure of E to do so. We will do this by using the simplicial and semisimplicial schemes associated to the simple normal crossing divisor E (see Chapter 4).

When the base field k is algebraically closed and of sufficiently small cardinality ($|k| \leq \mathfrak{c}$), we will produce a 1-motive (see Chapter 5) $M = [L \longrightarrow G]$ so that

$$L(k) \longrightarrow G(k) \xrightarrow{\alpha} KH_{-2}(X) \tag{1.23}$$

is exact, and such that $\operatorname{coker}(\alpha)$ is finitely generated. Thus $G(k)$ approximates $KH_{-2}(X)$ up to some finitely generated groups.

CHAPTER 2

Resolution of singularities

2.1 Background, history, and current status

Definition 2.1. Let k be a field, and X be a scheme over k with $Z = \mathrm{Sing}(X_{\mathrm{red}})$ the singular locus. A *resolution of singularities* of X is a proper birational map $p\colon \widetilde{X} \longrightarrow X_{\mathrm{red}}$, where $\widetilde{X}$ is smooth over k, and such that p is an isomorphism on $p^{-1}((X - Z)_{\mathrm{red}})$.

Often times, however, we will be interested not in just a resolution of X in the abstract, but a resolution of X considered as a subvariety of a larger ambient scheme W. In addition, for the purposes of this paper, we will work closely with the exceptional divisor, so we will be especially concerned about having the exceptional divisor be "as nice as possible" in a specific sense, which we will get to later in this chapter. The above definition of a resolution of singularities is standard, but beyond this point, terminology and definitions in the literature vary greatly. Therefore, to fix terminology, we will follow [EH02]. Below, we define the related notions of an *embedded resolution* of X and a *strong resolution* of X, which makes precise what we want out out of a resolution.

Definition 2.2. Let X be a reduced scheme over k, and let $X \longrightarrow W$ be a closed embedding of X into a regular scheme W. An *embedded resolution of X in W* is a proper birational morphism $p\colon \widetilde{W} \longrightarrow W$ satisfying the following:

1. p is a composition of blowups in smooth centers, each transverse to the previous exceptional locus.

2. The total transform $\widetilde{X}$ of X is smooth over k, and has normal crossings (see below, Definition 2.4) with the exceptional locus of $\widetilde{W}$.
3. The morphism $\widetilde{X} \longrightarrow X$ induced by p does not depend on the closed embedding $X \longrightarrow W$.

A *strong resolution of singularities* for X is an embedded resolution for any choice of W.

The first question that one might ask is: do resolutions always exist for any X? What about strong resolutions? To put it bluntly, these questions are hard. The problem of resolution of singularities is very subtle, and like other such reasonably accessible problems (Fermat's last theorem, the four color theorem, the Poincaré conjecture), is well known for producing many incorrect proofs. The history of resolution of singularities in characteristic zero is, very briefly, as follows.

Resolutions of curves are easy and have essentially been known since the 1600s [Kol07, Theorem 1.1]. Consequently, there are a large number of algorithms to resolve the singularities of a curve. Blowing up singular points, taking the normalization of the curve, and Albanese's method, among others, will all resolve the singularities of a curve.

Resolutions of surfaces are much harder than of curves, but are still easy relative to the general case. The standard method of resolving surfaces is to repeatedly (alternatingly) normalize and blow up singular points. There are other methods – for example, Albanese's method extends to surfaces. See [Kol07] for details.

Finally, resolution of singularities for schemes in dimension > 3 become increasingly much more complicated than for surfaces, although Zariski established resolution specifically for threefolds (in characteristic zero) in 1944 [Zar44].

We have the following short lemma.

Lemma 2.3. *Suppose X is a variety over k such that its singular locus $Z = \operatorname{Sing}(X)$ is connected, and let $p\colon \widetilde{X} \longrightarrow X$ be a strong resolution. Then $E = p^{-1}(Z)$ is connected, and its irreducible components are each divisors on $\widetilde{X}$.*

Proof. It is known that blowing up has the universal property of replacing a subscheme (defined by a coherent sheaf of ideals) by a divisor (defined by an invertible sheaf) [Har77, II, Theorems 7.13, 7.14]. Write $E = \cup_i^m E_i$ as the union of its irreducible components, and without loss of generality, consider E_1. The resolution map is surjective, so the images $f(E_i)$ cover Z. Consequently, for any other E_j, there is a sequence $a_0, \ldots, a_r$ of distinct numbers between 1 and m so that $a_0 = 1, a_r = j$, and $f(E_{a_i}) \cap f(E_{a_{i+1}}) \neq \varnothing$. These intersections are closed in X, as the resolution is proper. Pick closed points $x_i \in f(E_{a_i}) \cap f(E_{a_{i+1}})$. Zariski's main theorem asserts that $f^{-1}(x_i)$ is connected. But by definition, $f^{-1}(x_i) \cap E_{a_i} \neq \varnothing$ and $f^{-1}(x_i) \cap E_{a_{i+1}} \neq \varnothing$. So E_{a_i} and $E_{a_{i+1}}$ must be on the same connected component of E. Iterating this, we see that E_1 and E_j are in the same component. But as j was arbitrary, we conclude that E is connected. $\square$

A related question is this: given a resolution, how nice can we make the exceptional divisor E? Ideally, we would want the exceptional divisor to be smooth, but it turns out that this is too strong of a condition. We can get something close, however: we can make E a *simple normal crossing divisor*. This assertion is (essentially) part of requirement 2, above, for a resolution to be a strong resolution.

Definition 2.4. Let E be a reduced effective Cartier divisor with irreducible components $E_1, \ldots, E_m$ on a smooth scheme $\widetilde{X}$. Furthermore, for any subset I of $\{1, \ldots, m\}$, let us write $E_I := \cap_{i \in I} E_i$. We say that E has *normal crossings*, or is a *normal crossing divisor*, if each of the E_I are smooth and $\operatorname{codim}_E E_I = |I| - 1$. Furthermore, we say that E has *simple normal crossings*, or is a *simple normal crossing divisor*, if, in addition, each of the E_I are irreducible.

In particular, all of the components E_i must have the same dimension.

Example 2.5. Consider the node X defined by $x_1^2 = x_0^3 + x_0^2$ in $\mathbb{A}_k^2$. Blowing up the origin gives a closed subvariety $\mathrm{Bl}_0(X)$ of $\mathbb{A}^2 \times \mathbb{P}^1$, defined by

$$\begin{aligned} x_1^2 &= x_0^3 + x_0^2 \\ x_0 y_1 &= x_1 y_0. \end{aligned} \tag{2.1}$$

where y_0, y_1 are the coordinates of $\mathbb{P}^1$. On the chart $y_0 = 1$, we see that the strict transform of the node is given by $x_0 = y_1^2 - 1$, which intersects the exceptional divisor $x_0 = 0$ in two points at $y_1 = \pm 1$, which correspond to the slopes of two tangents of the node at the origin. It turns out that this blowup resolves the singularities of X. The exceptional divisor is the preimage of the origin $(x_0, x_1) = (0, 0)$, which is a copy of $\mathbb{P}^1$.

Example 2.6. Let C be a smooth projective curve, and let X be the cone over C. Then X is normal because it is regular in codimension 1 (its only singular point, the origin, has codimension 2), so we proceed by blowing up the origin. It turns out that this resolves the singularities of X, i.e. $\mathrm{Bl}_0(X)$ is smooth, and the exceptional divisor turns out to be isomorphic to C.

The question of the existence of resolution of singularities in characteristic 0 was finally resolved in 1964 by Hironaka [Hir64]. As a testament to the difficulty of the problem, Hironaka's proof features a long and complicated multiple induction on the dimension, and is over 200 pages, although his proof has been significantly shortened over the years. On the other hand, resolution in characteristic p is still open when X has dimension ≥ 4. Hauser [Hau00] gives a thorough history of the problem.

2.2 The dual complex associated to a resolution of singularities

Given a strong resolution $p\colon \widetilde{X} \longrightarrow X$ with exceptional divisor E that has simple normal crossings, we may construct a finite cell complex associated to E which we denote $\mathcal{D}(E)$, following [Kol12].

Let E be a scheme over k with simple normal crossings, and write $E = \cup_i E_i$ as the union of its irreducible components. For each non-negative integer m, let C^m be a set of m-cells, each one corresponding to each of the irreducible components of each of the m-fold intersections $E_{i_0} \times_E \cdots \times_E E_{i_m}$. We then define $\mathcal{D}(E)^m$ inductively, by letting $\mathcal{D}(E)^0$ be the disjoint union of the elements (vertices) of C^0, and gluing the m-cells of C^m to $\mathcal{D}(E)^{m-1}$ as follows. Let $D^q_{i_0,\ldots,i_m} \in C^m$ be an m-cell corresponding to the q^{th} irreducible component of the intersection $E_{i_0} \times_E \cdots \times_E E_{i_m}$. Then glue $D^q_{i_0,\ldots,i_m}$ to $\mathcal{D}(E)^{m-1}$ along each of the $m-1$ cells $D^{q'}_{i_0,\ldots,\hat{i}_j,\ldots,i_m} \subset \mathcal{D}(E)^{m-1}$ for each $j = 0,\ldots,m$ and each q', in the usual way, whenever the $(q')^{\text{th}}$ component of $E_{i_0} \times_E \cdots \times_E E_{\hat{i_{q'}}} \times_E \cdots \times_E E_{i_m}$ and the q^{th} component of $E_{i_0} \times_E \times \cdots \times_E \times E_{i_m}$ have a nonempty intersection. As m increases, eventually the C^m must become empty, because E has only a finite number of irreducible components. Thus $\mathcal{D}(E)^N = \mathcal{D}(E)^{N+1}$ for sufficiently large N, and we let $\mathcal{D}(E) = \mathcal{D}(E)^N$ for such an N.

Definition 2.7. For a simple normal crossing divisor E, we refer to $\mathcal{D}(E)$ as the *dual complex* associated to E.

We give a simple example to illustrate $\mathcal{D}(E)$.

Example 2.8. Let $E = E_1 \cup E_2 \cup E_3 \cup E_4$ be the union of four hyperplanes in general position in $\mathbb{P}^3$. Then the double intersections $E_i \cap E_j$ are isomorphic to copies of $\mathbb{P}^1$, and the triple intersections $E_i \cap E_j \cap E_m$ are just points. So $\mathcal{D}(E)^0$ consists of four disjoint points; $\mathcal{D}(E)^1$ has an edge for any two distinct vertices; and $\mathcal{D}(E)^2$ has a 3-cell for any three distinct vertices, so that $\mathcal{D}(E)^2 = \mathcal{D}(E)$ is homeomorphic to the 2-sphere.

Different resolutions of X yield different dual complexes $\mathcal{D}(E)$, but it turns out that the homotopy type of $\mathcal{D}(E)$ is independent of the choice of good resolution [Ste06, Theorem 1.2].

Consequently, we will also sometimes use the notation $\mathcal{DR}(X)$ to refer to the homotopy type of a $\mathcal{D}(E)$ corresponding to some good resolution. This is quite nice, but on the other hand $\mathcal{D}(E)$ is in general only a cell complex and need not be a simplicial complex.

This is problematic because we would like $\mathcal{D}(E)$ to be a simplicial complex. In fact, having $\mathcal{D}(E)$ being a simplicial complex is crucial to our future calculations. At the very least, having a simplicial $\mathcal{D}(E)$ simplifies the computations quite a bit. Luckily, such a resolution exists; we begin with establishing an obvious criterion on E to have $\mathcal{D}(E)$ be a simplicial complex. An intersection $\cap_{i\in I}E_i$ is by definition smooth in a simple normal crossing divisor, so it is the disjoint union of its components. We will call $\cap_{i\in I}E_i$ a *bad intersection* if it is not irreducible. It turns out that bad intersections are the only obstruction for $\mathcal{D}(E)$ to be a simplicial complex.

Lemma 2.9. *Given a good resolution $p\colon \widetilde{X} \longrightarrow X$, the dual complex $\mathcal{D}(E)$ is a simplicial complex if and only if each of the intersections $\bigcap_{i\in I} E_i$ is irreducible.*

Proof. Suppose E has m irreducible components, and that $\cap_{i\in I}E_i$ is a bad intersection. Then in the construction of the dual complex, we will have multiple $|I|$-cells glued in the same place, so $\mathcal{D}(E)$ cannot be simplicial.

Conversely, suppose $\cap_{i\in I}E_i$ is irreducible, and consider the corresponding $|I|$-simplex D_I in $\mathcal{D}(E)$. All of the faces of D_I are in $\mathcal{D}(E)$, because any such face corresponds to a smaller intersection of the E_i, which must be nonempty. Furthermore, for any other subset J of $\{1,\dots,m\}$, we have $D_I \cap D_J = D_{I\cup J}$, which is a face of both. □

The preceding criterion eliminates disconnected intersections (which correspond to multiple cells glued to the same vertices). We now proceed with the proof that a resolution of X always exists with dual complex $\mathcal{D}(E)$ a simplicial complex. The idea behind the proof was communicated to us by János Kollár.

Proposition 2.10. *There exists a resolution of X with exceptional divisor E for which $\mathcal{D}(E)$*

is a simplicial complex. Moreover, such a resolution can be obtained from any good resolution by further blowups.

Proof. Let $p\colon \widetilde{X} \longrightarrow X$ be a strong resolution with exceptional divisor E. We will iteratively blow up enough closed subschemes so that the conditions of Lemma 2.9 are satisfied. We begin blowing up components of bad intersections of the smallest dimension, then move up in dimension.

Write $E = \cup_{i=1}^m E_i$ as the union of its irreducible components, and suppose E has no bad intersections of codimension $\geq r$. We will blow up components of bad intersections of codimension r one by one; we claim that when we have blown them all up, the resulting divisor will not have any bad intersections of codimension r. Write $E_I = \cap_{i\in I} E_i$, and let $B_r(p)$ be the number of connected components $E_I^{(j)}$ that belong to some bad intersection E_I (i.e. E_I has more than one connected component). Fix a bad intersection E_{I_0} of codimension r, and without loss of generality, blow up $\widetilde{X}$ along the smooth irreducible center $Z = E_{I_0}^{(1)}$. We claim that if $p'\colon \operatorname{Bl}_{E_{I_0}^{(1)}} \widetilde{X} \longrightarrow X$, then $B_r(p') = B_r(p) - 1$. Continuing in this manner, we will eventually remove all of the bad codimension r intersections of E. This will finish the proposition.

First, we partition the set $\{1, \ldots, m\}$ into three sets:

$$\begin{aligned} I_0 &= \{i \,|\, Z \subset E_i\} \\ I_1 &= \{i \,|\, E_i \cap E_{I_0}^{(1)} \neq \varnothing, i \notin I_0\} \\ I_2 &= \{i \,|\, E_i \cap E_{I_0}^{(1)} = \varnothing\}. \end{aligned} \tag{2.2}$$

When we blow up $\widetilde{X}$ along $E_{I_0}^{(1)}$, we see that $\operatorname{Bl}_{E_{I_0}^{(1)}} E$, the total transform of E, will have $m+1$ components. We will let E_i' denote the total transform of E_i. If $i \in I_0$, then we will have $E_i' : = \operatorname{Bl}_{E_{I_0}^{(1)}} E_i$; if $i \in I_1$, then we will have $E_i' := \operatorname{Bl}_{E_{I_0}^{(1)} \cap E_i} E_i$; and if $i \in I_2$, then since E_i intersects trivially with the center, E_i' will be isomorphic to a copy of E_i. By abuse of notation, we will also use E_i to refer to these E'_i, when it is notationally convenient to do so. The last irreducible component E'_{m+1} of $\mathrm{Bl}_{E^{(1)}_{I_0}} E$ is the exceptional divisor of this blowup; we will also sometimes call this exceptional divisor E' when it is convenient to do so.

For ease of notation, we will also write $E'_J := \cap_{j \in J} E'_j$.

Now that we have established the various irreducible components of $\mathrm{Bl}_{E^{(1)}_{I_0}} E$, we are interested in how they intersect. We list the various types of intersections below.

$$\alpha \subset I_0, \beta \subset I_0 \colon E'_\alpha \cap E'_\beta = E'_{\alpha \cup \beta}$$

$$\alpha \subset I_0, \beta \subset I_1 \colon E'_\alpha \cap E'_\beta = \mathrm{Bl}_{E^{(1)}_{I_0} \cap E_{\alpha \cup \beta}} E_{\alpha \cup \beta}$$

$$\alpha \subset I_0, \beta \subset I_2 \colon E'_\alpha \cap E'_\beta = E_{\alpha \cup \beta}$$

$$\alpha \subset I_0 \colon E'_\alpha \cap E' = \text{exceptional divisor of } \mathrm{Bl}_{E^{(1)}_{I_0}} E_\alpha \longrightarrow E_\alpha$$

$$\alpha \subset I_1, \beta \subset I_1 \colon E'_\alpha \cap E'_\beta = \mathrm{Bl}_{E^{(1)}_{I_0} \cap E_{\alpha \cup \beta}} E_{\alpha \cup \beta}$$

$$\alpha \subset I_1, \beta \subset I_2 \colon E'_\alpha \cap E_\beta = E_{\alpha \cup \beta}$$

$$\alpha \subset I_1 \colon E'_\alpha \cap E' = \text{exceptional divisor of } \mathrm{Bl}_{E^{(1)}_{I_0} \cap E_\alpha} E_\alpha \longrightarrow E_\alpha$$

$$\alpha \subset I_2, \beta \subset I_2 \colon E_\alpha \cap E_\beta = E_{\alpha \cup \beta}$$

$$\alpha \subset I_2 \colon E_\alpha \cap E' = \varnothing$$

$$\alpha \subset I, \beta \subset I_1, \gamma \subset I_2 \colon E'_\alpha \cap E'_\beta \cap E_\gamma = E_{\alpha \cup \beta \cup \gamma}$$

$$\alpha \subset I, \beta \subset I_1 \colon E'_\alpha \cap E'_\beta \cap E' = \text{exceptional divisor of } \mathrm{Bl}_{E^{(1)}_{I_0} \cap E_{\alpha \cup \beta}} E_{\alpha \cup \beta}$$

To finish the proof, we need to check two things. First, we need to check that the intersections of codimension $\geq r$ remain irreducible; second, we need to check that $B_r(p') = B_r(p) - 1$ for the intersections of codimension r. For ease of notation, let $I_3 = \{m+1\}$. For any arbitrary subset J of $\{1, \ldots, m+1\}$, we can partition J according to its intersections with I_0, I_1, I_2, and I_3; then $E'_J = \cap_{i=0}^{3} (E'_{J \cap I_i})$. Suppose now that $|J| > r$, so that $\mathrm{codim}_E E'_J \geq r$; we would like to show that each E'_J is irreducible. For ease of notation, let $\alpha = J \cap I_0, \beta = J \cap I_1, \gamma = J \cap I_2$. We now check the various possibilities for E'_J, using our intersection formulas from above:

$\underline{J = \alpha}$: Since $|I_0| = r < |J|$, we cannot have $J \subset I$.

$\underline{J = \beta}$: E_J has codimension $\geq r$, so is irreducible by assumption. Furthermore, $E_{I_0}^{(1)} \cap E_J \subset E_{I_0 \cup J}$, which is irreducible by assumption and has the same dimension, so the intersection $E_{I_0}^{(1)} \cap E_J$ must either be empty or all of $E_{I_0 \cup J}$. Consequently, $E'_J = \mathrm{Bl}_{E_{I_0}^{(1)} \cap E_J} E_J$ is the blowup of a smooth irreducible scheme along a smooth irreducible subscheme, hence is irreducible.

$\underline{J = \gamma}$: $E'_\gamma = E_\gamma$, which is irreducible by assumption.

$\underline{J = \alpha \cup \beta}$: $E'_\alpha \cap E'_\beta$ is the blowup of $E_{\alpha\cup\beta}$ along $E_{I_0}^{(1)} \cap E_{\alpha\cup\beta}$. As before, $|J| > r$, so $E_{\alpha\cup\beta}$ is irreducible. Also, $E_{I_0}^{(1)} \cap E_J \subset E_{I_0 \cup J}$, which is irreducible by assumption and has the same dimension, so the intersection $E_{I_0}^{(1)} \cap E_J$ must either be empty or all of $E_{I_0 \cup J}$. So E'_J is the blowup of a smooth irreducible scheme along a smooth irreducible subscheme, and in particular must be irreducible.

$\underline{J = \alpha \cup \gamma}$: $E'_\alpha \cap E_\gamma = E_{\alpha\cup\gamma}$, which is irreducible by assumption.

$\underline{J = \alpha \cup I_3}$: $E'_\alpha \cap E'$ is the exceptional divisor of the blowup $E'_\alpha \longrightarrow E_\alpha$. Since $J = \alpha \cup I_3$ and $|J| > r$, $|\alpha| \geq r$, so that in fact, $J = I_0$. Then the blowup of $E_{I_0} = \coprod_i E_{I_0}^{(i)}$ along $E_{I_0}^{(1)}$ is just $\coprod_{i \neq 1} E_I^{(i)}$, so that the component $E_{I_0}^{(1)}$ has disappeared. In this case, the exceptional divisor is empty, i.e. $E'_{I_0} \cap E' = \varnothing$.

$\underline{J = \beta \cup \gamma}$: $E'_\beta \cap E_\gamma = E_{\beta\cup\gamma}$, which is irreducible by assumption.

$\underline{J = \beta \cup I_3}$: $E'_\beta \cap E'$ is the exceptional divisor of the blow up $E'_\beta \longrightarrow E_\beta$. We are blowing up along the intersection $E_{I_0}^{(1)} \cap E_\beta \subset E_{I_0\cup\beta}$, which is irreducible by assumption and has the same dimension, so the intersection $E_{I_0}^{(1)} \cap E_\beta$ is either empty or all of $E_{I_0\cup\beta}$. We are either leaving E_β unchanged, or blowing it up along a smooth irreducible subscheme, so the exceptional divisor will be irreducible.

$\underline{J = \gamma \cup I_3}$: E_γ and $E_{I_0}^{(1)}$, by definition, do not intersect, so E'_J is empty.

$\underline{J = \alpha \cup \beta \cup \gamma}$: $E'_\alpha \cap E'_\beta \cap E_\gamma = E_{\alpha\cup\beta\cup\gamma}$, which is irreducible by assumption.

$\underline{J = \alpha \cup \beta \cup I_3}$: $E'_\alpha \cap E'_\beta \cap E'$ is the exceptional divisor of the blow up $E'_{\alpha\cup\beta} \longrightarrow E'_{\alpha\cup\beta}$. We are blowing up along the intersection $E_{I_0}^{(1)} \cap E_{\alpha\cup\beta} \subset E_{I_0\cup\alpha\cup\beta}$, which is irreducible by assumption and has the same dimension, so the intersection $E_{I_0}^{(1)} \cap E_{\alpha\cup\beta}$ is either empty or all of $E_{I_0\cup\alpha\cup\beta}$. We are either leaving $E_{\alpha\cup\beta}$ unchanged, or blowing it up along a smooth irreducible subscheme, so the exceptional divisor will be irreducible.

It remains to check that this blowup actually reduces the number of bad intersection components. We do this directly by checking the various cases. In particular, the codimension r intersections with E' should all be good intersections.

$\underline{J = \alpha}$: Since $|I_0| = r = |J|$ we must have $J = I_0$, hence $E'_J = \coprod_{i\neq 1} E_i$ has one less irreducible component than the corresponding E_{I_0}. This will be what causes B_r to decrease.

$\underline{J = \beta}$: Write $E_J = \coprod_j E_J^{(j)}$. Then we have $E_{I_0}^{(1)} \cap E_J \subset E_{I_0\cup J}$, which is irreducible. So $E_{I_0}^{(1)} \cap E_J$ is either empty or all of $E_{I_0\cup J}$. In particular, this intersection is contained in one of the $E_J^{(j)}$. Since $E'_J = \mathrm{Bl}_{E_{I_0}^{(1)}\cap E_J} E'_J$, one connected component of E_J is being blown up in a smooth center, so E'_J will be smooth, and will have the same number of connected components.

$\underline{J = \gamma}$: $E'_\gamma = E_\gamma$, which does not change under our blowup.

$\underline{J = \alpha \cup \beta}$: $E'_\alpha \cap E'_\beta$ is the blowup of $E_{\alpha\cup\beta}$ along $E_{I_0}^{(i)} \cap E_{\alpha\cup\beta}$. As before, $|I_0 \cup \beta| > r$, so this intersection is contained in $E_{I_0\cup\beta}$, which is irreducible by assumption. So our intersection is either empty or all of $E_{I_0\cup\beta}$, which will be contained in one irreducible component of $E_{\alpha\cup\beta}$. Then $E'_{\alpha\cup\beta}$ is the blowup of $E_{\alpha\cup\beta}$ in one of its connected components along a smooth connected center, so it will be smooth. In addition, the number of components stays the same.

$\underline{J = \alpha \cup \gamma}$: $E'_\alpha \cap E_\gamma = E_{\alpha\cup\gamma}$, which does not change under blowup.

$\underline{J = \alpha \cup I_3}$: $E'_\alpha \cap E'$ is the exceptional divisor of the blowup $E'_\alpha \longrightarrow E_\alpha$. Since $|J| = r$, we must have $|\alpha| = r - 1$. If we write $E_\alpha = \coprod_j E_\alpha^{(j)}$ as the union of its connected components, then the irreducible $E_{I_0}^{(1)}$ must be contained in one of them, which we will call $E_\alpha^{(1)}$ without loss of generality. Then $E'_J \cap E'$ is the exceptional divisor of $\mathrm{Bl}_{E_{I_0}^{(1)}} E_\alpha^{(1)}$, which will just be a projective bundle over $E_{I_0}^{(1)}$, hence smooth and irreducible. The intersection E'_J is a good intersection.

$\underline{J = \beta \cup \gamma}$: $E'_\beta \cap E_\gamma = E_{\beta \cup \gamma}$, which does not change under blowup.

$\underline{J = \beta \cup I_3}$: $E'_\beta \cap E'$ is the exceptional divisor of the blowup $E'_\beta \longrightarrow E_\beta$. If we write $E_\beta = \coprod_j E_\beta^{(j)}$ as the union of its connected components, then the irreducible $E_{I_0}^{(1)}$ must be contained in one of them, which we will call $E_\beta^{(1)}$ without loss of generality. Then $E'_J \cap E'$ is the exceptional divisor of $\mathrm{Bl}_{E_{I_0}^{(1)}} E_\beta^{(1)}$, which will just be a projective bundle over $E_{I_0}^{(1)}$, hence smooth and irreducible. The intersection E'_J is a good intersection.

$\underline{J = \gamma \cup I_3}$: E_γ and $E_{I_0}^{(1)}$, by definition, do not intersect, so E'_J is empty.

$\underline{J = \alpha \cup \beta \cup \gamma}$: $E'_\alpha \cap E'_\beta \cap E_\gamma = E_{\alpha \cup \beta \cup \gamma}$, which does not change under blowup.

$\underline{J = \alpha \cup \beta \cup I_3}$: $E'_\alpha \cap E'_\beta \cap E'$ is the exceptional divisor of the blowup $E'_\alpha \cap E_\beta = E'_{\alpha \cup \beta} \longrightarrow E_{\alpha \cup \beta}$. If we write $E_{\alpha \cup \beta} = \coprod_j E_{\alpha \cup \beta}^{(j)}$ as the union of its connected components, then the irreducible $E_{I_0}^{(1)}$ must be contained in one of them, which we will call $E_{\alpha \cup \beta}^{(1)}$ without loss of generality. Then $E'_J \cap E'$ is the exceptional divisor of $\mathrm{Bl}_{E_{I_0}^{(1)}} E_{\alpha \cup \beta}^{(1)}$, which will just be a projective bundle over $E_{I_0}^{(1)}$, hence smooth and irreducible. The intersection E'_J is a good intersection.

Looking at the various cases, we see that in each case, we either get a new good intersection (in the case when we intersect with E'), the number of bad components of an intersection stays the same, or in the case of E'_{I_0}, the number of bad components goes down by one. We conclude that $B_r(p') = B_r(p) - 1$, which proves the second claim.

Since blowing up along $E_{I_0}^{(1)}$ preserves the goodness of the intersections of codimension $\geq r$

and lowers the number of bad codimension r intersection components by one, we conclude that this procedure will eventually produce a resolution with all good intersections, i.e. a simplicial $\mathcal{D}(E)$. By construction, the desired resolution is obtained from a good resolution from further blowups. $\square$

Definition 2.11. We call a strong resolution $p\colon \widetilde{X} \longrightarrow X$ an *excellent* resolution if the exceptional divisor E is a simple normal crossing divisor and $\mathcal{D}(E)$ is a simplicial complex.

CHAPTER 3

The *cdh*-topology

Let k be a field of characteristic zero and X a scheme over k. As mentioned in Chapter 1, the *cdh*-cohomology groups with coefficients in the K-theory sheaves can be used to compute KH-groups of X (via the descent spectral sequence, Theorem 1.11). To define the *cdh*-topology on the category Sch/k of schemes over k, we consider two classes of covers: those obtained from Nisnevich squares and those obtained from abstract blow-up squares. A *Nisnevich square* is a pullback diagram

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r, hook] & X \end{tikzcd} \tag{3.1}$$

such that $U \longrightarrow X$ is an open embedding and $V \longrightarrow X$ is an étale morphism that is an isomorphism over $X - U$. Given such a square, $\{U \longrightarrow X, V \longrightarrow X\}$ is a *Nisnevich cover* of X. Such covers generate the Nisnevich topology on Sch/k.

An *abstract blow-up square* is a pullback diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ Z \arrow[r, hook] & X \end{tikzcd} \tag{3.2}$$

such that $Z \longrightarrow X$ is an *closed* embedding, and $p\colon \widetilde{X} \longrightarrow X$ is a proper map that is an isomorphism when restricted to $\widetilde{X} - E$. The *cdh*-topology on Sch/k is the smallest Grothendieck topology that contains Nisnevich covers and covers of the form $\{Z \longrightarrow X, \widetilde{X} \longrightarrow X\}$ obtained from abstract blow-up squares. The "*cdh*-topology" stands for "completely decomposed h-topology," and Voevodsky and Suslin introduced it in 1994 [SV00b] in order to study sheaves of relative cycles. The Nisnevich topology is sometimes called the completely decomposed topology, and the fact that Nisnevich covers are *cdh*-covers is reflected in this name. Evidently, the *cdh*-topology is finer than both the Nisnevich and Zariski topologies. It is also incomparable with the étale topology. Blowups are not flat, so a cover obtained from an actual blowup square will not be étale. Conversely, if $U \longrightarrow X$ is a *cdh*-cover and F is a field that contains k, then $U(F) \longrightarrow X(F)$ will be surjective. But this is not necessarily true for étale morphisms, i.e. $\mathrm{Hom}(\mathbb{C}, \mathbb{R}) \longrightarrow \mathrm{Hom}(\mathbb{R}, \mathbb{R})$ is not surjective. So $\operatorname{Spec}\mathbb{C} \longrightarrow \operatorname{Spec}\mathbb{R}$ is not a *cdh*-cover.

A Grothendieck topology T is *subcanonical* if every representable functor is a sheaf. The *canonical* topology on a category $\mathcal{C}$ is the finest topology such that every representable functor is a sheaf. Almost of the well known Grothendieck topologies are subcanonical. The fpqc topology is subcanonical [FGI05, Theorem 2.55], so all topologies coarser than the fpqc topology are also subcanonical. However, the *cdh*-topology is not subcanonical, and is one of the first examples of a non-subcanonical topology that has seen widespread use. We begin with a lemma that elucidates some of the structure of the *cdh*-topology on Sch/k.

Lemma 3.1. *Schemes over k are locally smooth in the* cdh-*topology.*

Proof. Let X be a scheme over k with singular set Z_0. As we are in characteristic zero, take a resolution $p\colon X_0 \longrightarrow X$. Then $Z_0 \coprod X_0 \longrightarrow X$ is a *cdh*-cover of X. Note that Z_0 necessarily has smaller dimension than X. If Z_0 is singular, we may iterate this process by resolving Z_0, and this process must terminate because the dimension of the singular set decreases each time. In the end, we will produce a smooth *cdh*-cover of X. □

In other words, every scheme has a smooth *cdh*-cover. Notice that the existence of resolution of singularities is crucial here – if k has characteristic p, then schemes over k might not necessarily be locally smooth (although they might be!). Haesemeyer proved in [Hae04] that

KH satisfies *cdh*-descent in characteristic zero. This slightly technical condition means that an elementary Nisnevich square or an abstract blowup square

$$\begin{tikzcd} \widetilde{Y} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} \tag{3.3}$$

yields a homotopy cartesian square of spectra when we apply KH:

$$\begin{tikzcd} KH(\widetilde{Y}) & KH(\widetilde{X}) \arrow[l] \\ KH(Y) \arrow[u] & KH(X) \arrow[l] \arrow[u] \end{tikzcd} \tag{3.4}$$

In particular, we have a spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{cdh}}(X, a_{\mathrm{cdh}}KH_{-q}) \Longrightarrow KH_{-p-q}(X), \tag{3.5}$$

where $a_{\mathrm{cdh}}KH_{-q}$ is the sheafification of the presheaf KH_{-q} in the *cdh*-topology. Since schemes are locally smooth in the *cdh*-topology, and the groups K_{-q} and KH_{-q} agree on smooth schemes, it follows that the sheaves $a_{\mathrm{cdh}}KH_{-q}$ and $a_{\mathrm{cdh}}K_{-q}$ are the same, so that the above spectral sequence is in fact the descent spectral sequence (1.14).

We also have the following lemma.

Lemma 3.2. *Let X be smooth over k, and $\mathscr{F}$ a homotopy invariant sheaf with transfers on the* cdh*-site over X. Then the change of topology morphism induces an isomorphism $H^p_{\mathrm{cdh}}(X, \mathscr{F}) \cong H^p_{Zar}(X, \mathscr{F})$.*

Proof. [VSF00] □

This result is quite useful, because on smooth schemes, all of the sheaves $a_{\mathrm{Zar}}K_n$ are homotopy invariant sheaves with transfers [Voe00, Section 3.4], and computing Zariski cohomology groups is often easier than computing *cdh*-cohomology groups. We will use this fact often in our computations.

Theorem 3.3 (*cdh*-cohomological dimension)**.** *Let X be a separated scheme of finite type over a field k of characteristic zero, and let $\mathscr{F}$ be a* cdh*-sheaf over X. Then $H^p_{\mathrm{cdh}}(X, \mathscr{F}) = 0$ whenever $p > \dim X$.*

Proof. [SV00a, Theorem 5.13] □

In order to compute the KH-groups of X, we will need to know how to compute the *cdh*-cohomology groups of simple normal crossing divisors E.

Lemma 3.4. *Let E be a simple normal crossing scheme, and write $E = \cup_i E_i$ as the union of its irreducible components E_i. Then $\{\coprod_i E_i \longrightarrow E\}$ is a* cdh*-cover of E.*

Proof. We proceed by induction on the number of irreducible components of E. When there is only one irreducible component, it's clear that E is a cover of itself. Next, let $E' = \bigcup_{i \neq 1} E_i$, and observe that the square

$$\begin{tikzcd} E_1 \cap E' \arrow[r] \arrow[d] & E' \arrow[d] \\ E_1 \arrow[r] & E \end{tikzcd} \tag{3.6}$$

is an abstract blowup square. Indeed, $E_1 \longrightarrow E$ is a closed embedding, $E' \longrightarrow E$ is a closed embedding, hence proper, and is an isomorphism outside of E_1. We also note that the diagram (3.6) is also an *actual* blowup of E with center E_1, although this fact is not needed. So $\{E_1 \longrightarrow E, E' \longrightarrow E\}$ is a *cdh*-cover of E. But $\{\coprod_{i \neq 1} E_i \longrightarrow E'\}$ is a cover of E' by induction, and putting these two covers together yields the lemma. □

Taking $\coprod_i E_i$ as a cover of E, we apply the usual Čech cohomology spectral sequence abutting to the derived functor cohomology, which yields spectral sequence (1.12). It also turns out that we can throw away the degenerate parts of the first page – see Theorem 1.13 for a proof. We will also elaborate on this in the next chapter.

CHAPTER 4

Simplicial and semisimplicial schemes

In this chapter, we will define simplicial and semisimplicial schemes, and give some of their basic properties.

Definition 4.1. A *simplicial scheme* over k, denoted $X_\bullet$, is a simplicial object in the category of schemes, i.e. a functor $X\colon \Delta^{op} \longrightarrow Sch/k$ from the simplicial category Δ into the category of schemes over k. Equivalently, it's a sequence of schemes $\{X_p\}_{p\in\mathbb{N}}$ with face maps $d_{i,p}\colon X_p \longrightarrow X_{p-1}$ for $i = 0, \dots, p$ and all $p \in \mathbb{N}$ (except that X_0 has no face maps), and degeneracy maps $s_{i,p}\colon X_p \longrightarrow X_{p+1}$ for $i = 0, \dots, p-1$ and for all $p \in \mathbb{N}$, satisfying the usual simplicial relations.

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i & i &< j \\
d_i s_j &= s_{j-1} d_i & i &< j \\
d_j s_j &= s_{j+1} d_j = \mathrm{id} & & \\
d_i s_j &= s_j d_{i-1} & i &> j+1 \\
s_i s_j &= s_{j+1} s_i & i &\leq j
\end{aligned} \tag{4.1}$$

Definition 4.2. A *semisimplicial scheme* over k, denoted $Y_\bullet$, is a semisimplicial object in the category of schemes over k, i.e. a sequence $\{Y_p\}_{p\in\mathbb{N}}$ of schemes over k, together with face maps $d_{i,p}\colon Y_p \longrightarrow Y_{p-1}$ for $i = 0, \dots, p$ and for all $p \in \mathbb{N}$ (except that Y_0 has no face maps), satisfying the usual relation for degeneracies, namely

$$d_i d_j = d_{j-1} d_i \qquad i < j. \tag{4.2}$$

The notations for a simplicial scheme and a semisimplicial scheme are identical, so there is room for a lot of confusion. If it's not clear from context whether $X_\bullet$ denotes a simplicial or a semisimplicial scheme, we will sometimes denote semisimplicial objects with a superscript "*alt*," for "alternating," e.g. $X_\bullet^{alt}$.

Remark 4.3. Note that from a given simplicial scheme $X_\bullet$, we can produce a semisimplicial scheme $X_\bullet^{alt}$ by simply forgetting the degeneracy maps. A semisimplicial scheme is "half" of a simplicial scheme in the sense that it only has face maps, but not degeneracy maps.

Example 4.4. The standard simplicial ring $\mathbb{Z}[\Delta^\bullet]$ is defined by

$$\mathbb{Z}[\Delta^n] = \mathbb{Z}[t_0, \ldots, t_n]/(t_0 + \cdots + t_n - 1), \tag{4.3}$$

with the face maps $d_{i,n} \colon \mathbb{Z}[\Delta^n] \longrightarrow \mathbb{Z}[\Delta^{n-1}]$ given by

$$d_{i,n}(t_j) = \begin{cases} t_j, & j < i \\ 0, & j = i \\ t_{j-1}, & j > i \end{cases} \tag{4.4}$$

and the degeneracy maps $s_{i,n} \colon \mathbb{Z}[\Delta^n] \longrightarrow \mathbb{Z}[\Delta^{n+1}]$ given by

$$s_{i,n}(t_j) = \begin{cases} t_j, & j < i \\ t_i + t_{i+1}, & j = i \\ t_{j+1}, & j > i. \end{cases} \tag{4.5}$$

For any scheme X, we can construct a simplicial scheme $X[\Delta^\bullet]$ by taking $X[\Delta^n] = X \times_{\mathbb{Z}} \mathbb{Z}[\Delta^n]$, and the face and degeneracy maps induced by those for $\mathbb{Z}[\Delta^\bullet]$. These simplicial schemes are the ones used to define KV theory and KH theory (Section 1.3).

Example 4.5. The only other case we will be concerned with is the case of a connected simple normal crossings scheme $E = \cup E_i$, where E_i are the irreducible components of E. The simplicial scheme $\Delta_\bullet E$ and the semisimplicial scheme $\Delta_\bullet^{alt} E$ associated to E are defined as follows. For $\Delta_\bullet E$, we have

$$\Delta_p E = \coprod_{i_0,\dots,i_p} (E_{i_0} \times_E \cdots \times_E E_{i_p}). \tag{4.6}$$

Similarly, for $\Delta_\bullet^{alt}(E)$, we have

$$\Delta_p^{alt} E = \coprod_{i_0<\dots<i_p} (E_{i_0} \times_E \cdots \times_E E_{i_p}), \tag{4.7}$$

with the face maps given by the natural projections from the fiber products, and the degeneracy maps for $\Delta_\bullet E$ induced by the diagonal maps $E_i \longrightarrow E_i \times_E E_i \cong E_i$. Notice that $\Delta_\bullet E$ contains degenerate intersections (e.g. $E_1 \times_E E_1$), where $\Delta_\bullet^{alt} E$ does not. In particular, $\Delta_n E$ is never empty, while $\Delta_n^{alt} E$ is empty when n is larger than the number of irreducible components of E, or when n is larger than the dimension of E.

Remark 4.6. As in Remark 4.3, we can produce a semisimplicial scheme $(\Delta_\bullet E)^{alt}$ from the simplicial scheme $\Delta_\bullet E$ associated to E by forgetting the face maps – but this is different from the semisimplicial scheme $\Delta_\bullet^{alt} E$ associated to E, as it contains degenerate self-intersections (e.g. $E_1 \times_E E_1$). Of these two semisimplicial schemes, we will only use the latter, $\Delta_\bullet^{alt} E$.

Remark 4.7. In [BRS03], the authors consider the category of varieties over k as a subcategory of the larger additive category where morphisms are now formal $\mathbb{Z}$-linear combinations of actual morphisms of varieties. In this additive category, we may take a semisimplicial scheme $X_\bullet$ and produce a chain complex $C_\bullet(X_\bullet)$ where the differentials are the usual alternating sum of face maps. We will need this later in Chapter 6 to produce a 1-motive.

Somewhat related to this discussion is the spectral sequence (1.12) which computes the *cdh*-cohomology of a simple normal crossing divisor. Replacing the "simplicial" rows in the E_1 page with the corresponding nondegenerate (semisimplicial) rows yields a quasi-isomorphism (i.e. the E_2 page is still the same), so we can use either to compute cohomology groups. Technically, we are replacing the rows themselves, but it is convenient to think of replacing the simplicial scheme $\Delta_\bullet E$ by the semisimplicial scheme $\Delta_\bullet^{alt} E$. In practice, using $\Delta_\bullet^{alt} E$ is easier to use since 1) it doesn't have degenerate intersections, and 2) it is finite, i.e. $\Delta_p^{alt} E$ is empty for sufficiently large p. On the other hand, the degenerate intersections of $\Delta_\bullet E$ ensure that $\Delta_p E$ is nonempty for all E.

CHAPTER 5

Mixed Hodge structures and 1-Motives

5.1 Mixed Hodge structures

We motivate this chapter with an example.

Example 5.1. Let X be a compact Kähler manifold. Then the complex cohomology groups $H^n(X, \mathbb{C})$ have a vector space direct sum decomposition

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X), \tag{5.1}$$

such that $H^{p,q} = \overline{H^{q,p}}$ [PS08, Corollary 1.13]. This decomposition is functorial for many operations on cohomology, e.g. if Y is a compact Kähler manifold and $f\colon X \longrightarrow Y$ is a holomorphic map, then the map $H^n(Y, \mathbb{C}) \longrightarrow H^n(X, \mathbb{C})$ induced by pullback is compatible with the direct sum decomposition, i.e. $f(H^{p,q}(Y)) \subset H^{p,q}(X)$. It is also compatible with other natural operations on cohomology, such as cup product and the Künneth formula [PS08, Theorem 5.44, Corollary 5.45]. We will see that $H^n(X, \mathbb{C})$ has the structure of a *pure Hodge structure*.

5.1.1 Filtered vector spaces

The material in this section comes is taken mostly from [Nic05]. We begin with some definitions.

Definition 5.2. V be a vector space over a field k. An *increasing filtration* $F_\bullet$ (resp.

decreasing filtration $F^\bullet$) on V is a chain $\{F_p(V)\}_{p\in\mathbb{Z}}$ (resp. $\{F^p(V)\}_{p\in\mathbb{Z}}$) of subspaces of V satisfying $F_p(V) \subset F_{p+1}(V)$ (resp. $F^{p+1}(V) \subset F^p(V)$) for all p.

If, in addition, there exist $p < q$ so that $F_p(V) = 0$ and $F_q(V) = V$ (resp. $F^p(V) = V$ and $F^q(V) = 0$), then we say that $F_\bullet$ (resp. $F^\bullet$) is *finite*.

We will always denote decreasing filtrations by a superscript, e.g. $F^\bullet$, and increasing filtrations by a subscript, e.g. $F_\bullet$. Given an increasing filtration $F_\bullet$ on V, we can make a decreasing filtration on V via $F^p(V) = F_{-p}(V)$. Henceforth all filtrations will assumed to be decreasing unless otherwise specified. In addition, we will often refer to vector spaces equipped with a given filtration as *filtered vector spaces*, and denote them by the pair $(V, F^\bullet)$.

Given a filtered vector space $(V, F^\bullet)$ and a subspace $W \subset V$, $F^\bullet$ induces a filtration on both W and V/W, via

$$\begin{aligned} F^p(W) &= W \cap F^p(V) \\ F^p(V/W) &= (F^p(V) + W)/W, \end{aligned} \tag{5.2}$$

as we would expect. These two induced filtrations commute: if $W \subset U \subset V$, then the two induced filtrations on U/W (via $U \mapsto U/W$ and via $U/W \subset V/W$) agree.

Now that we have defined filtered vector spaces, we would like to define morphisms between them.

Definition 5.3. A *morphism* $f\colon (V, F^\bullet) \longrightarrow (W, G^\bullet)$ between filtered vector spaces is a k-linear transformation that is compatible with the filtrations, i.e. $f(F^p(V)) \subset G^p(W)$.

The category of filtered vector spaces is additive, but not abelian, as the next example shows.

Example 5.4. Let $V = k^2$ with basis $\{e_1, e_2\}$ and filtration $F^1(V) = V, F^2(V) = ke_2, F^3(V) = 0$, and let $f\colon V \longrightarrow V$ be defined by $f(e_1) = e_2, f(e_2) = 0$. Then f is a morphism of filtered vector spaces with $\ker f = ke_2$, so that the filtration induced on $V/\ker f$ is:

$$
\begin{aligned}
F^1(V/\ker f) &= V/\ker f \\
F^2(V/\ker f) &= 0 \\
F^3(V/\ker f) &= 0,
\end{aligned}
\tag{5.3}
$$

whereas $\operatorname{im} f = ke_2$, and the filtration induced on $\operatorname{im} f$ is

$$
\begin{aligned}
F^1(\operatorname{im} f) &= ke_2 \\
F^2(\operatorname{im} f) &= ke_2 \\
F^3(\operatorname{im} f) &= 0,
\end{aligned}
\tag{5.4}
$$

so that the canonical map $V/\ker f \longrightarrow \operatorname{im} f$ is not an isomorphism of filtered vector spaces.

We would like to know under what conditions the first isomorphism theorem holds. The above example motivates the following definition:

Definition 5.5. A morphism $f\colon (V, F^\bullet) \longrightarrow (W, G^\bullet)$ of filtered vector spaces is called *strict* if $f(F^p(V)) = G^p(W) \cap \operatorname{im} f$.

Given a strict morphism of filtered vector spaces, we then have the first isomorphism theorem: $V/\ker f \cong \operatorname{im} f$, so that the category of filtered vector spaces with strict morphisms indeed is abelian. Keeping the example at the beginning of the chapter in mind, we have the following definition.

Definition 5.6. Let $F^\bullet$ and $G^\bullet$ be two finite filtrations on V. We say that $F^\bullet$ and $G^\bullet$ are n-complementary if $\operatorname{Gr}_F^p \operatorname{Gr}_G^q(V) = 0$ whenever $p + q \neq n$.

It turns out that a pair of n-complementary filtrations gives us the direct sum decomposition we are looking for. We omit the proof of the following lemma.

Lemma 5.7. *Let $F^\bullet$ and $G^\bullet$ be two filtrations on V. Then the following are equivalent.*

1. *$F^\bullet$ and $G^\bullet$ are n-complementary.*
2. *$F^p(V) = \bigoplus_{i \geq p}(F^i(V) \cap G^{n-i}(V))$, and $G^q(V) = \bigoplus_{j \geq q}(G^j(V) \cap F^{n-j}(V))$.*
3. *$F^p(V) \cap G^q(V) = 0, F^p(V) + G^q(V) = V$ whenever $p + q \neq n$.*

We mentioned in Example 5.1 that $H^i(X, \mathbb{C})$ is a pure Hodge structure. We will generalize this to *mixed Hodge structures*, and in order to do so, we need to add a third filtration to the mix.

Definition 5.8. If $F^\bullet, G^\bullet, W_\bullet$ are filtrations on a vector space V, then we say that the triple $(F^\bullet, G^\bullet, W_\bullet)$ is a *complementary triple filtration*, if $\operatorname{Gr}_F^p \operatorname{Gr}_G^q \operatorname{Gr}_n^W(V) = 0$ whenever $p+q \neq n$.

Our next result shows that we also get a direct sum decomposition out of a complementary triple filtration, as in the previous lemma, and as in the example at the beginning of the chapter. The proof is long and technical, so we omit it.

Proposition 5.9. *Let $(F^\bullet, G^\bullet, W_\bullet)$ be a complementary triple filtration on a vector space V. Then we have the following direct sum decomposition for the filtered pieces:*

$$\begin{aligned} W_n(V) &= \bigoplus_{p+q \leq n} I^{p,q} \\ &= \bigoplus_{p+q \leq n} J^{p,q} \\ F^p(V) &= \bigoplus_{k \geq p} \bigoplus_{q} I^{k,q} \\ G^q(V) &= \bigoplus_{k \geq q} \bigoplus_{p} J^{p,k}, \end{aligned}$$

where

$$
\begin{aligned}
I^{p,q} &= (F^p(V) \cap W_{p+q}(V)) \cap \left((G^p(V) \cap W_{p+q}(V)) + \sum_{i=1}^{\infty} (G^{q-i} \cap W_{p+q-i}) \right) \\
J^{p,q} &= (G^q(V) \cap W_{p+q}(V)) \cap \left((F^q(V) \cap W_{p+q}(V)) + \sum_{i=1}^{\infty} (F^{p-i} \cap W_{p+q-i}) \right).
\end{aligned}
\tag{5.5}
$$

Although the $I^{p,q}$ and $J^{p,q}$ are complicated, the proposition is very explicit about how the direct sum decomposition comes about. The takeaway is that a complementary triple filtration yields a direct sum decomposition.

5.1.2 Mixed Hodge structures

Definition 5.10. A *mixed Hodge structure* $H = (H_{\mathbb{Z}}, F^{\bullet}, W_{\bullet})$ is a triple, where $H_{\mathbb{Z}}$ is a free abelian group, $W_{\bullet}$ is an increasing filtration on the $\mathbb{Q}$-vector space $H_{\mathbb{Q}} = \mathbb{Q} \otimes H_{\mathbb{Z}}$, and $F^{\bullet}$ is a decreasing filtration on $H_{\mathbb{C}} = \mathbb{C} \otimes H_{\mathbb{Z}}$, such that the triple $(F^{\bullet}, \overline{F}^{\bullet}, W_{\bullet})$ is a complementary triple filtration, where $\overline{F}^{\bullet}$ is the filtration on $H_{\mathbb{C}}$ obtained by complex conjugation of $F^{\bullet}$.

We refer to $F^{\bullet}$ as the *Hodge filtration* and $W_{\bullet}$ as the *weight filtration*. If there is some n so that $W_{n-1}H_{\mathbb{Q}} = 0, W_n H_{\mathbb{Q}} = H_{\mathbb{Q}}$, then we say that H is *pure of weight* n.

Furthermore, it will be convenient to refer to the *type* of a mixed Hodge structure H: it is the set of all pairs (p, q) such that the $H^{p,q}$ appearing in the direct sum decomposition $H_{\mathbb{C}} = \oplus_{p,q} H^{p,q}$ are nonzero.

The content of Example 5.14 is that when X is a compact Kähler manifold, $H^n(X, \mathbb{Z})$, modulo torsion, has the structure of a pure Hodge structure of weight n. Note that, in the notation of Proposition 5.9, when $G^{\bullet} = \overline{F}^{\bullet}$, we have $J^{p,q} = \overline{I^{q,p}}$, which agrees with the condition $H^{p,q} = \overline{H^{q,p}}$. In fact, we can use Proposition 5.9 to determine the filtrations $F^{\bullet}$ and $W_{\bullet}$ on $H^n(X, \mathbb{Z})$. If we do, we see that

$$
\begin{aligned}
F^p H^n(X,\mathbb{C}) &= \bigoplus_{k\geq p} H^{k,q} \\
W^p H^n(X,\mathbb{C}) &= \begin{cases} H^n(X,\mathbb{C}) & p \geq n \\ 0 & p < n. \end{cases}
\end{aligned}
\tag{5.6}
$$

We would like to build a category of mixed Hodge structures, so we specify morphisms between them.

Definition 5.11. A morphism $f\colon H \longrightarrow H'$ of mixed Hodge structures is a group homomorphism $f_{\mathbb{Z}}\colon H_{\mathbb{Z}} \longrightarrow H'_{\mathbb{Z}}$ that is compatible with the filtrations in the following sense. Specifically, we require f to be such that $f_{\mathbb{Q}}\colon (H_{\mathbb{Q}}, W_\bullet) \longrightarrow (H'_{\mathbb{Q}}, W'_\bullet)$ and $f_{\mathbb{C}}\colon (H_{\mathbb{C}}, F^\bullet) \longrightarrow (H'_{\mathbb{C}}, F'^\bullet)$ are strict morphisms of filtered vector spaces.

Definition 5.12. A *polarization* of a pure Hodge structure H of weight m is a bilinear form $Q\colon H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ such that

1. $Q(\alpha,\beta) = (-1)^m Q(\beta,\alpha)$;
2. $(F^p H_{\mathbb{Q}})^\perp = F^{m-p+1} H_{\mathbb{Q}}$; and
3. The form $(\alpha,\beta) \mapsto Q(C\alpha,\overline{\beta})$ on $H_{\mathbb{C}}$ is positive-definite,

where $C(\alpha) = i^{p-q}\alpha$ whenever $\alpha \in H^{p,q}$. A pure Hodge structure of weight m is called *polarizable* if it admits a polarization. Furthermore, we say that a mixed Hodge structure H is *graded-polarizable*, or often just *polarizable*, if each of the graded pieces $\mathrm{Gr}_m^W H$ of pure weight m are polarizable.

Polarizable Hodge structures often have nice properties that non-polarizable Hodge structures don't have. For example, in the following discussion, we will construct an equivalence between mixed Hodge structures of type $\{(0,0),(0,1),(1,0),(1,1)\}$ such that $\mathrm{Gr}_1^W H$ is polarizable, and the category of 1-motives. The existence of a polarization on $\mathrm{Gr}_1^W H$ guarantees that the objects that comprise the 1-motive, which a priori are just complex-analytic, are actually algebraizable. Some other nice results that come from the existence of polarizations can be found in [PS08], e.g. Corollary 2.11, Theorem 10.13.

Let MHS denote the category of mixed Hodge structures. It turns out that MHS is abelian; see [Del71, Theorem 2.3.5] for the details. Since morphisms respect all of the filtrations, we can also consider the full subcategory $MHS_{\leq p}$ of mixed Hodge structures of weight $\leq p$ for any p. Another subcategory of MHS that will be important to us is the category of mixed Hodge structures of type $\{(0,0),(0,1),(1,0),(1,1)\}$ such that $\mathrm{Gr}_1^W H$ is polarizable; we denote this category by MHS_1.

Deligne proved that the cohomology groups of any complex variety have mixed Hodge structures, by first extending Hodge theory to open smooth varieties, then complete singular varieties, then to all varieties.

Theorem 5.13 (Deligne)**.** *Let X be a complex variety. Then the cohomology groups $H^n(X,\mathbb{Z})$ have Hodge and weight filtrations that give them a mixed Hodge structure. Moreover, given a morphism $f\colon X \longrightarrow Y$, the induced map $H^n(Y,\mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z})$ is a morphism of mixed Hodge structures.*

Proof. See the seminal papers [Del71, Del74]. □

Example 5.14. Consider two elliptic curves $C_1, C_2 \subset \mathbb{P}^2$ in general position, and let $X = C_1 \cup C_2$. Since they are cubic curves, by Bezout's theorem, they will intersect transversally in nine points $P_1, \ldots, P_9$. We then have the diagram

$$\begin{tikzcd} \coprod_i P_i \arrow[r] \arrow[d] & C_2 \arrow[d] \\ C_1 \arrow[r] & X \end{tikzcd} \tag{5.7}$$

Taking cohomology groups yields an exact sequence

$$0 \longrightarrow H^0(X,\mathbb{Z}) \longrightarrow \bigoplus_{i=1}^{2} H^0(C_i,\mathbb{Z}) \longrightarrow H^0(\textstyle\coprod_i P_i,\mathbb{Z}) \tag{5.8}$$

$$\longrightarrow H^1(X,\mathbb{Z}) \longrightarrow \bigoplus_{i=1}^{2} H^1(C_i,\mathbb{Z}) \longrightarrow 0.$$

Then $H^1(C_i,\mathbb{Z}) = \mathbb{Z}^2$, as an elliptic curve is just a torus, and H^0 just counts the number of components. So the exact sequence becomes

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^9 \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow \mathbb{Z}^4 \longrightarrow 0. \tag{5.9}$$

Taking the weight 0 part of this sequence yields

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^9 \longrightarrow W_0H^1(X,\mathbb{Z}) \longrightarrow 0. \tag{5.10}$$

Since the first map is just the diagonal map, we see that $W_0H^1(X,\mathbb{Z})$ has rank 8, and in particular is nontrivial. Furthermore, $H^1(X,\mathbb{Z})$ surjects onto $\mathbb{Z}^4$, which is pure of weight 1, so $\mathrm{Gr}_0^W(H^1(X,\mathbb{Z}))$ has rank 4, hence $H^1(X,\mathbb{Z})$ must have a mixed Hodge structure. In particular, it is not pure of weight 0.

5.2 1-motives

The material in this section, including notations, will mostly follow [BRS03]. Let k be a field of characteristic zero, and fix an algebraic closure $\overline{k}$ of k.

Definition 5.15. A 1-motive over k is an exact diagram of commutative group schemes

$$M = \left[\begin{tikzcd} & & L \arrow[d] & & \\ 0 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 0 \end{tikzcd}\right] \tag{5.11}$$

over k, with L locally finite such that $L(\overline{k})$ is a finitely generated abelian group, T is an algebraic torus, and A is an abelian variety.

We will sometimes write $M = [L \longrightarrow G]$ for convenience. When k is algebraically closed, this definition is equivalent to the one that replaces the group scheme morphism $L \longrightarrow G$ by a group homomorphism $L(k) \longrightarrow G(k)$, as in [Del74]. This is because the image of a given basis of $L(k)$ (which is a set of closed points of L) under the group homomorphism $L(k) \longrightarrow G(k)$ generates its image, hence determines the map $L \longrightarrow G$.

If $M = [L \longrightarrow G]$ and $M' = [L' \longrightarrow G']$, an *effective morphism* $u = (u_{lf}, u_{sa})\colon M \longrightarrow M'$ is just a square

$$\begin{tikzcd} L \arrow[r, "u_{lf}"] \arrow[d] & L' \arrow[d] \\ G \arrow[r, "u_{sa}"] & G' \end{tikzcd} \tag{5.12}$$

We call u a *quasi-isomorphism* if u_{sa} is an isogeny (a surjective morphism with finite kernel) and if the square (5.12) is cartesian. *Morphisms* of 1-motives $M \longrightarrow M'$ are obtained by inverting quasi-isomorphisms, in the same way that the calculus of fractions is used to determine morphisms in Verdier localization. This gives us a category of 1-motives, which we will denote $\mathcal{M}_1(k)$.

Theorem 5.16. *The category $\mathcal{M}_1(k)$ is abelian.*

Proof. [BRS03, Theorem 1.3] □

Example 5.17. Trivial examples of 1-motives include locally finite group schemes $[L \longrightarrow 0]$, algebraic tori $[0 \longrightarrow T]$, abelian varieties $[0 \longrightarrow A]$, and semiabelian varieties $[0 \longrightarrow G]$. We will see in the next section that many more examples (in some sense, all of them) come from mixed Hodge structures.

We now look at 1-motives a little more in depth. We will be interested in 1-motives up to torsion, which we define below.

Definition 5.18. Given a 1-motive

$$M = \left[\begin{tikzcd} & & L \arrow[d, "f"] & & \\ 0 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 0 \end{tikzcd}\right] \tag{5.13}$$

let L_{tor} denote the torsion part of L, and consider the 1-motive

$$M_{\text{tor}} = \left[\begin{tikzcd} & L_{\text{tor}} \cap \ker f \arrow[d, "f"] & \\ 0 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}\right] \tag{5.14}$$

We call this 1-motive the *torsion part* of M. We say that M is *torsion-free* if $M_{\text{tor}} = 0$, and *torsion* if $M = M_{\text{tor}}$. Furthermore, we will let $M_{\text{fr}} := M/M_{\text{tor}}$ denote the free part of M.

Deligne [Del71, Del74] is the source of the material found in this section, originally constructing torsion-free 1-motives over algebraically closed fields. His work has been generalized over the years, but the core of many of the arguments remain the same.

5.3 Relation between Mixed Hodge structures and 1-motives

The main theorem we wish to state in this section is the following:

Theorem 5.19. *There is an equivalence of categories*

$$MHS_1 \simeq \mathcal{M}_1(\mathbb{C}). \tag{5.15}$$

Proof. Given a 1-motive $M = [L \longrightarrow G]$, we can construct a mixed Hodge structure $H_{\mathbb{Z}}$ defined by the following pullback diagram:

$$\begin{tikzcd} 0 \arrow[r] & \ker(\exp) \arrow[r] \arrow[d, equal] & H_{\mathbb{Z}} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker(\exp) \arrow[r] & \operatorname{Lie} G \arrow[r, "\exp"] & G \arrow[r] & 0 \end{tikzcd} \tag{5.16}$$

Let $r_{\mathcal{H}} \colon \mathcal{M}_1(\mathbb{C}) \longrightarrow MHS_1$ be the functor that takes M to $H_{\mathbb{Z}}$. We still need to specify the Hodge and weight filtrations on $H_{\mathbb{Z}}$.

$$\begin{aligned} F^2 H_{\mathbb{C}} &= 0 \\ F^1 H_{\mathbb{C}} &= \ker(H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \operatorname{Lie} G) \\ F^0 H_{\mathbb{C}} &= H_{\mathbb{C}} \\ \\ W_2 H_{\mathbb{Q}} &= H_{\mathbb{Z}} \\ W_1 H_{\mathbb{Q}} &= \ker(\exp) \\ W_0 H_{\mathbb{Q}} &= \ker(\exp) \cap \operatorname{Lie} T \\ W_{-1} H_{\mathbb{Q}} &= 0 \end{aligned} \tag{5.17}$$

The map $H_{\mathbb{Z}} \longrightarrow \operatorname{Lie} G$ is a map from a finitely generated abelian group to a complex vector space, so it extends to a map $H_{\mathbb{Z}} \otimes \mathbb{C} \longrightarrow \operatorname{Lie} G$. The maximal torus T sitting inside G induces an inclusion $\operatorname{Lie} T \longrightarrow \operatorname{Lie} G$.

Conversely, given a mixed Hodge structure $H_{\mathbb{Z}}$ of the desired type, we set $L = \operatorname{Gr}_2^W H_{\mathbb{Z}}, G = \operatorname{Ext}^1_{MHS}(\mathbb{Z}, W_1 H_{\mathbb{Z}})$, with the map $L \longrightarrow G$ given by the boundary map

$$\operatorname{Hom}_{MHS}(\mathbb{Z}, \operatorname{Gr}_2^W H_{\mathbb{Z}}) \longrightarrow \operatorname{Ext}^1_{MHS}(\mathbb{Z}, W_1 H_{\mathbb{Z}}) \tag{5.18}$$

induced by the short exact sequence

$$0 \longrightarrow W_1 H_{\mathbb{Z}} \longrightarrow W_2 H_{\mathbb{Z}} \longrightarrow \mathrm{Gr}_2^W H_{\mathbb{Z}} \longrightarrow 0. \tag{5.19}$$

In addition, we have $T = \mathrm{Spec}(\mathbb{C}[\mathrm{Gr}_0^W H_{\mathbb{Z}}])$ and the quotient $A = G/T = W_1 H_{\mathbb{C}}/(W_1 H_{\mathbb{Z}} + (F^1 \cap W_1)H_{\mathbb{C}})$.

Let $r_{\mathcal{M}} \colon MHS_1 \longrightarrow \mathcal{M}_1(\mathbb{C})$ be so that $r_{\mathcal{M}}(H_{\mathbb{Z}}) = [L \longrightarrow G]$. The details of the proof that $r_{\mathcal{H}}$ and $r_{\mathcal{M}}$ are quasi-inverses can be found in [BRS03, Theorem 1.5]; here, we are mostly only interested in knowing how to translate between mixed Hodge structures and 1-motives. □

Example 5.20. We continue our previous example 5.14. We saw that $W_0 H^1(X,\mathbb{Z})$ had rank 8, so the torus part is $T = \mathrm{Spec}(\mathbb{C}[\mathrm{Gr}_0^W H^1(X,\mathbb{Z})]) \cong \mathbb{G}_m^8$. In addition, since $H^1(X,\mathbb{Z})$ has no weight 2 part, there will be no lattice. Finally, if we take the end of the long exact sequence 5.8

$$\cdots \longrightarrow H^0(\textstyle\coprod_i P_i, \mathbb{Z}) \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(C_1,\mathbb{Z}) \oplus H^1(C_2,\mathbb{Z}) \longrightarrow 0, \tag{5.20}$$

we see that $\mathrm{Gr}_1^W H^1(X,\mathbb{C}) \cong H^1(C_1,\mathbb{C}) \oplus H^1(C_2,\mathbb{C})$ as pure Hodge structures of weight 1, which shows that the abelian variety parts must be the same. But as C_1 and C_2 are elliptic curves, the abelian variety must indeed be $C_1 \times C_2$. All in all, the 1-motive that corresponds to $H^1(X,\mathbb{Z})$ is just

$$\left[\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathbb{G}_m^8 \arrow[r] & G \arrow[r] & C_1 \times C_2 \arrow[r] & 0 \end{tikzcd}\right] \tag{5.21}$$

We will, in the next chapter, produce a 1-motive $M = [L \longrightarrow G]$ with a map $\alpha \colon G(k) \longrightarrow KH_{1-n}(X)$ when X is a normal, n-dimensional projective variety over an algebraically closed

k, whose singular locus Z is either smooth or of codimension > 2. In fact, $(M_{\mathbb{C}})_{\mathrm{fr}}$ will come from the unique largest mixed Hodge structure H of type $\{(0,0),(0,1),(1,0),(1,1)\}$ in the weight 2 part $W_2H^n(X(\mathbb{C}),\mathbb{Z})$ of the n^{th} cohomology group $H^n(X(\mathbb{C}),\mathbb{Z})$, such that $\mathrm{Gr}_1^W H$ is polarizable.

CHAPTER 6

Calculation of $KH_{1-n}(X)$

In Section 1.5, we started the calculation of $KH_{-2}(X)$ in the case that X is a threefold, in order to motivate the background material. Here we begin more generally, but in a similar fashion. Let X be an normal, integral n-dimensional variety (where $n \geq 3$) such that $Z = \mathrm{Sing}(X)$ is either smooth or of codimension greater than 2. Instead of taking a good resolution of X, we will insist on taking an *excellent* resolution p of X (Definition 2.11), so that the exceptional divisor E is a simple normal crossing divisor such that the dual complex $\mathcal{D}(E)$ is a simplicial complex. We will compute the various pieces that comprise $KH_{1-n}(X)$, with a full computation of $KH_{-2}(X)$ in the case that $n = 3$. KH satisfies *cdh*-descent (Definition 3.3), so we obtain a long exact sequence

$$\cdots \longrightarrow KH_{1-n}(X) \longrightarrow KH_{1-n}(\widetilde{X}) \oplus KH_{1-n}(Z) \longrightarrow KH_{1-n}(E) \longrightarrow \cdots \tag{6.1}$$

$\widetilde{X}$ is smooth, so its negative KH groups vanish. When Z is smooth, its negative K-groups vanish as well, and when Z is of codimension greater than 2, we have $K_{-q}(Z) = 0$ for $q > n - \operatorname{codim} Z$ (see Section 1.5) In any case, we are left with isomorphisms

$$KH_{-q}(E) \cong KH_{-q-1}(X) \tag{6.2}$$

for $q > 0$ in the case that Z is smooth, and for all $q > n - \operatorname{codim} Z = \dim Z$ when $\operatorname{codim} Z > 2$. Recall that we are interested in the case $q = n - 1$, and in either of these cases, obtain an isomorphism $KH_{2-n}(E) \cong KH_{1-n}(X)$. Notice that when $Z = \coprod_i Z_i$ has more than one connected component, we will have $KH_{1-n}(X) \cong \oplus_i KH_{2-n}(E_i)$, where E_i is the strict transform of Z_i. For each exceptional divisor i, we can compute $KH_{2-n}(E_i)$ using its simple normal crossings structure, so henceforth we will assume that Z, and thus E (see Lemma 2.3) is connected.

Before we use the simple normal crossings structure of E, however, we apply the descent spectral sequence (1.14). We begin with a lemma:

Lemma 6.1. *Let X be a scheme over k, and consider the sheaf $a_{\mathrm{cdh}}K_q$ on the* cdh*-site over X. For $q \leq 1$, we have the following:*

$$a_{\mathrm{cdh}}K_q = \begin{cases} a_{\mathrm{cdh}}\mathbb{G}_m, & q = 1 \\ a_{\mathrm{cdh}}\mathbb{Z}, & q = 0 \\ 0, & q < 0. \end{cases} \tag{6.3}$$

Proof. First, we note that X is locally smooth in the *cdh*-topology (Lemma 3.1), so we may assume that X is smooth and irreducible. In addition, $K_q(U) = 0$ vanishes for smooth U and $q < 0$, so $a_{\mathrm{cdh}}K_q$ vanishes for $q < 0$.

We can sheafify both K_0 and $\mathbb{Z}$ in two steps, as follows:

$$\begin{tikzcd} K_0 \arrow[r] \arrow[d, "\mathrm{rank}"] & a_{\mathrm{Zar}}K_0 \arrow[r] \arrow[d] & a_{\mathrm{cdh}}a_{\mathrm{Zar}}K_0 = a_{\mathrm{cdh}}K_0 \arrow[d] \\ \mathbb{Z} \arrow[r] & a_{\mathrm{Zar}}\mathbb{Z} \arrow[r] & a_{\mathrm{cdh}}a_{\mathrm{Zar}}\mathbb{Z} = a_{\mathrm{cdh}}\mathbb{Z} \end{tikzcd} \tag{6.4}$$

The rank map $a_{\mathrm{Zar}}K_0 \longrightarrow \mathbb{Z}$ is an isomorphism on the stalks, since $K_0(R) = \mathbb{Z}$ when R is local, since every projective module over a local ring is free. So $a_{\mathrm{cdh}}K_0 \longrightarrow a_{\mathrm{cdh}}\mathbb{Z}$ is an isomorphism. We have a similar argument for the sheaf $a_{\mathrm{cdh}}K_1$:

$$\begin{tikzcd} K_1 \arrow[r] \arrow[d] & a_{\mathrm{Zar}}K_1 \arrow[r] \arrow[d] & a_{\mathrm{cdh}}a_{\mathrm{Zar}}K_1 = a_{\mathrm{cdh}}K_1 \arrow[d] \\ \mathbb{G}_m \arrow[r] & \mathbb{G}_m \arrow[r] & a_{\mathrm{cdh}}\mathbb{G}_m \end{tikzcd} \tag{6.5}$$

The map $a_{\mathrm{Zar}}K_1 \longrightarrow \mathbb{G}_m$ is an isomorphism on the stalks, since $K_1(R) = R^\times$ when R is local. So $a_{\mathrm{cdh}}K_1 \longrightarrow a_{\mathrm{cdh}}\mathbb{G}_m$ is an isomorphism locally, hence is an isomorphism globally. □

When there is no ambiguity (for example, when we take cohomology groups), we will sometimes write K_q for $a_{\mathrm{cdh}}K_q$, and similarly for $\mathbb{G}_m$ and $\mathbb{Z}$.

By the above lemma, we conclude that the descent spectral sequence (1.14) resides in the fourth quadrant. Moreover, the spectral sequence degenerates at E_n, because the *cdh*-cohomological dimension of E is at most $\dim E = n - 1$ [SV00a]. In particular, the spectral sequence is bounded. The E_2 page of the spectral sequence is shown below.

$$\begin{tikzcd}
\cdots & 0 & 0 & 0 & 0 & \\
\cdots \arrow[r, dash] & E_2^{n-3,0} \arrow[r, dash] \arrow[drr, "d_2^{n-3,0}"] & E_2^{n-2,0} \arrow[r, dash] & E_2^{n-1,0} \arrow[r, dash] & 0 \arrow[r, dash] & {} \\
\cdots & E_2^{n-3,-1} \arrow[drr, "d_2^{n-3,-1}"] & E_2^{n-2,-1} & E_2^{n-1,-1} & 0 & \\
\cdots & E_2^{n-3,-2} & E_2^{n-2,-2} & E_2^{n-1,-2} & 0 &
\end{tikzcd} \tag{6.6}$$

To calculate $KH_{2-n}(E)$, we need to know about the map $d_2^{n-3,0}$. When $n = 3$, it is zero:

Lemma 6.2. *In the case $n = 3$, the differential $d_2^{0,0}$ is the zero map.*

Proof. Let P be a closed point of E, and consider the diagram

$$\begin{tikzcd}
K_0(E) \arrow[r] \arrow[d, two heads, "\mathrm{rank}"'] & KH_0(E) \arrow[r, two heads] \arrow[d] & E_\infty^{0,0}(E) \arrow[r, tail] \arrow[d] & E_2^{0,0}(E) \arrow[d] \\
K_0(P) \arrow[r, equal] & KH_0(P) \arrow[r, equal] & E_\infty^{0,0}(P) \arrow[r, equal] & E_2^{0,0}(P) = \mathbb{Z}
\end{tikzcd} \tag{6.7}$$

obtained from naturality of both the map $K \longrightarrow KH$ and the spectral sequence. Zariski's main theorem implies that E is connected, so that the vertical map on the right, $H^0_{\mathrm{cdh}}(E, \mathbb{Z}) \longrightarrow H^0_{\mathrm{cdh}}(P, \mathbb{Z})$ is an isomorphism. Furthermore, the vertical map on the left, the rank map,

$K_0(E) \longrightarrow K_0(P)$, is surjective since there are vector bundles on E of any rank. A diagram chase shows that the map $E_\infty^{0,0}(E) \longrightarrow E_2^{0,0}(E)$ is an isomorphism, hence $d_2^{0,0} = 0$. $\square$

This lemma shows that when $n = 3$, we have $E_2^{2,-1} = E_\infty^{2,-1}$, so the spectral sequence reduces to a short exact sequence calculating $KH_{-1}(E)$:

$$0 \longrightarrow H^2_{\mathrm{cdh}}(E, \mathbb{G}_m) \longrightarrow KH_{-1}(E) \longrightarrow H^1_{\mathrm{cdh}}(E, \mathbb{Z}) \longrightarrow 0. \tag{6.8}$$

For arbitrary n, the differential $d_2^{n-3,0}$ may be nonzero, so we only have exactness on the right. Furthermore, we we can extend the exact sequence on the left by precomposing with the E_2 differential $d_2^{n-3,0}$:

$$H^{n-3}_{\mathrm{cdh}}(E, \mathbb{Z}) \xrightarrow{d_2^{n-3,0}} H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m) \longrightarrow KH_{2-n}(E) \longrightarrow H^{n-2}_{\mathrm{cdh}}(E, \mathbb{Z}) \longrightarrow 0. \tag{6.9}$$

To use the fact that E has normal crossings, we invoke the machinery developed in Chapter 4. Let $\Delta_\bullet^{alt}E$ be the semisimplicial scheme associated to the normal crossings divisor E, as in Example 4.5. We apply the spectral sequence (1.13) to $\Delta_\bullet^{alt}E$, with $\mathscr{F} = aK_m$, where $m \in \mathbb{Z}$ is arbitrary:

$$E_1^{p,q} = H^q_{\mathrm{cdh}}(\Delta_p^{alt}E, K_m) \Longrightarrow H^{p+q}_{\mathrm{cdh}}(E, K_m). \tag{6.10}$$

For this first quadrant spectral sequence, many terms are zero. First, $\Delta_p^{alt}E = \varnothing$ for $p > \dim E = n-1$, so $E_1^{p,q} = 0$ for $p > n-1$. Additionally, since $\dim \Delta_p^{alt}E = \dim E - p = n-1-p$, we have $E_1^{p,q} = 0$ for $p + q > n - 1$.

Furthermore, the sheaves $a_{\mathrm{Zar}}K_m$ are homotopy invariant sheaves with transfers in the Zariski topology and the components of $\Delta_p^{alt}E$ are smooth for any p, so we may compute cohomology using the Zariski topology (Theorem 3.2).

We would like a better understanding of the exact sequence (6.9), so using this spectral sequence (1.13), we first compute $H^i_{\mathrm{cdh}}(E,\mathbb{Z})$ for arbitrary i.

Lemma 6.3. $H^i_{\mathrm{cdh}}(E,\mathbb{Z}) \cong H^i(\mathcal{D}(E),\mathbb{Z})$. *In particular, these groups are finitely generated.*

Proof. In addition to the observations above about the spectral sequence, we also get $E_1^{p,q} = 0$ for $q > 0$ since $\mathbb{Z}$ is flasque as a Zariski sheaf, so $H^i_{\mathrm{cdh}}(E,\mathbb{Z})$ is just the cohomology of the complex

$$0 \longrightarrow H^0_{\mathrm{Zar}}(\Delta_0^{alt}E,\mathbb{Z}) \longrightarrow \cdots \longrightarrow H^0_{\mathrm{Zar}}(\Delta_{n-2}^{alt}E,\mathbb{Z}) \longrightarrow H^0_{\mathrm{Zar}}(\Delta_{n-1}^{alt}E,\mathbb{Z}) \longrightarrow 0 \quad (6.11)$$

in degree i. The sheaf $\mathbb{Z}$ only carries information about the components of $\Delta_\bullet^{alt}E$, so this sequence depends only on the semisimplicial structure of E, hence we see that $H^i_{\mathrm{cdh}}(E,\mathbb{Z}) = H^i(\mathcal{D}(E),\mathbb{Z})$.

Additionally, the homotopy type of $\mathcal{D}(E)$ is independent of the choice of resolution [Ste06, Theorem 1.2], so we also have $H^i_{\mathrm{cdh}}(E,\mathbb{Z}) = H^i(\mathcal{DR}(X),\mathbb{Z})$. □

Example 6.4. Using this same approach, we can easily calculate $KH_{-n}(X)$. Applying *cdh*-descent for KH for the same resolution of singularities of X (or just equation (6.2)) yields $KH_{1-n}(E) \cong KH_{-n}(X)$; application of the descent spectral sequence then yields

$$KH_{1-n}(E) \cong E_2^{n-1,0} = H^{n-1}_{\mathrm{cdh}}(E,\mathbb{Z}). \quad (6.12)$$

Then the above lemma tells us that $KH_{1-n}(E) \cong H^i(\mathcal{D}(E),\mathbb{Z}) = H^i(\mathcal{DR}(X),\mathbb{Z})$. □

Applying the above lemma to equation (6.9), we see that the kernel of the map $H^{n-1}_{\mathrm{cdh}}(E,\mathbb{G}_m) \longrightarrow KH_{1-n}(E)$ is a quotient of first term $H^{n-3}_{\mathrm{cdh}}(E,\mathbb{Z})$, so the kernel and cokernel of $H^{n-1}_{\mathrm{cdh}}(E,\mathbb{G}_m) \longrightarrow KH_{1-n}(E)$ are finitely generated. Thus the cohomology group $H^{n-1}_{\mathrm{cdh}}(E,\mathbb{G}_m)$ approximates

$KH_{1-n}(E)$, up to some finitely generated groups. We continue, therefore, by computing $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m)$.

6.1 Calculation of $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m)$

The computation of $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m)$ is more complicated (and interesting) than the computation of $H^{n-2}_{\mathrm{cdh}}(E, \mathbb{Z})$. We begin by analyzing the spectral sequence (1.13).

6.1.0.1 Simplifying the simplicial spectral sequence

We begin with a small lemma.

Lemma 6.5. *Let Y be a smooth scheme over k. Then $H^q_{\mathrm{Zar}}(Y, \mathbb{G}_m) = 0$ whenever $q > 1$.*

Proof. There is a short exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathscr{K}^\times \longrightarrow \mathrm{CaDiv} \longrightarrow 0. \tag{6.13}$$

Since Y is smooth, $\mathscr{K}$ is just the constant sheaf $K(Y)$, and CaDiv is the presheaf of Cartier divisors on Y (which, by definition, is already a sheaf). But for such an Y, the presheaves of Cartier divisors CaDiv, Weil divisors Div, and the class group presheaf Cl are all isomorphic [Har77, II, Theorems 6.11, 6.16], and are sheaves since CaDiv is a sheaf. But Cl is a flasque sheaf by [Har77, II, Proposition 6.5]. So the exact sequence (6.13) is a flasque resolution of $\mathbb{G}_m$, from which we see that $H^q_{\mathrm{Zar}}(Y, \mathbb{G}_m) = 0$ whenever $q > 1$. $\square$

Now, since $\Delta^{alt}_\bullet E$ is a smooth semisimplicial scheme, we apply the above to get $E_1^{p,q} = 0$ for $q > 1$. Thus the spectral sequence degenerates at E_3, and we need only compute $E_3^{n-1,0}$ and $E_3^{n-2,1}$. The E_1 page of the corresponding spectral sequence looks like

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
\cdots \arrow[r] & E_1^{n-3,1} \arrow[r, "d_1^{n-3,1}"] \arrow[rrd, dashed, "d_2^{0,1}"] & E_1^{n-2,1} \arrow[r] & 0 \arrow[r] & 0 \\
\cdots \arrow[r] & E_1^{n-3,0} \arrow[r] & E_1^{n-2,0} \arrow[r] & E_1^{n-1,0} \arrow[r] & 0
\end{tikzcd} \tag{6.14}$$

where now $E_1^{n-1,1} = H^1_{\text{Zar}}(\Delta^{alt}_{n-1}E, \mathbb{G}_m) = 0$ since $\dim \Delta^{alt}_{n-1}E = 0$. There is also a possibly nonzero differential $d_2^{n-3,1}$ on the E_2 page, which we have denoted using a dashed arrow in the diagram above. In order to compute $E_\infty^{n-1,0} = E_3^{n-1,0}$, we need to determine the map $d_2^{n-3,1}$. Applying the global sections of the resolution (6.13) for each $\Delta^{alt}_p E$ in each column yields the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & \text{Div}(\Delta^{alt}_{n-3}E) \arrow[r, dashed] & \text{Div}(\Delta^{alt}_{n-2}E) \arrow[r, dashed] & 0 \\
\cdots \arrow[r, dashed] & k(\Delta^{alt}_{n-3}E)^\times \arrow[r, dashed] \arrow[u] & k(\Delta^{alt}_{n-2}E)^\times \arrow[r, dashed] \arrow[u] & k(\Delta^{alt}_{n-1}E)^\times \arrow[u]
\end{tikzcd} \tag{6.15}$$

The E_1 differentials are given by the semisimplicial structure of $\Delta^{alt}_\bullet E$, i.e. the alternating sum of the face maps. They are defined on the E_0 page at $\text{Div}(\Delta^{alt}_m E)$, but only on those divisors on $\Delta^{alt}_m E$ that intersect $\Delta^{alt}_{m+1}E$ transversally. For example, if two irreducible components E_1 and E_2 of E intersect nontrivially, then $E_1 \cap E_2$ is a divisor on E_1, but the image of $E_1 \cap E_2$ is (clearly) not a divisor on $E_1 \cap E_2$. To remedy this, we will find a quasi-isomorphic subcomplex for which the horizontal maps are defined, then use this subcomplex to show that the map $d_2^{n-3,1}$ in (6.14) is the zero map. Our current discussion motivates the following definition.

Definition 6.6. For each p, we define the *group of good divisors*

$$\text{Div}_g(\Delta^{alt}_p E) = \{D \in \text{Div}(\Delta^{alt}_p E) \mid D \text{ intersects } \Delta^{alt}_m E \text{ transversally for all } m > p\}. \tag{6.16}$$

Remark 6.7. By applying Bertini's theorem, we see that this definition is equivalent to the one that requires the image of D under any composition of any of the face maps d_j to be defined. In addition, while the notation Div_g comes from Carlson [Car85], our definitions are slightly different. Carlson does not require Div_g to be defined so that d_i lands in (our) $\mathrm{Div}_g(\Delta^{alt}_{p+1}E)$ – only that d_i lands in $\mathrm{Div}(\Delta^{alt}_{p+1}E)$. Furthermore, Carlson's definition applies to a more general class of semisimplicial schemes, as we only define Div_g for semisimplicial schemes associated to a special class of simple normal crossing schemes.

Now, we will prove the following:

Lemma 6.8. *For each p, let A_p be the pullback*

$$\begin{tikzcd} A_p \arrow[r] \arrow[d] & \mathrm{Div}_g(\Delta^{alt}_p E) \arrow[d] \\ k(\Delta^{alt}_p E)^\times \arrow[r, "\beta_p"] & \mathrm{Div}(\Delta^{alt}_p E) \end{tikzcd} \tag{6.17}$$

where β_p is the rational function-to-divisor map. Then the vertical maps are a quasi-isomorphism of complexes.

Proof. We add in the horizontal kernels and cokernels to the diagram above, and label the vertical maps:

$$\begin{tikzcd} \ker(\alpha_p) \arrow[r] \arrow[d, equal] & A_p \arrow[r, "\alpha_p"] \arrow[d, "v_k"] & \mathrm{Div}_g(\Delta^{alt}_p E) \arrow[r] \arrow[d, "v_{\mathrm{Div}}"] & \mathrm{coker}(\alpha_p) \arrow[d, "v_{\mathrm{coker}}"] \\ \ker(\beta_p) \arrow[r] & k(\Delta^{alt}_p E)^\times \arrow[r, "\beta_p"] & \mathrm{Div}(\Delta^{alt}_p E) \arrow[r] & \mathrm{Pic}(\Delta^{alt}_p E) \end{tikzcd} \tag{6.18}$$

We now check that v_{coker} is injective. This is a result of the fact that the middle square is a pullback square. Given a section $\overline{s} \in \mathrm{coker}(\alpha_p)$ with $v_{\mathrm{coker}}(\overline{s}) = 0$, we first lift $\overline{s}$ to $s \in \mathrm{Div}_g(\Delta^{alt}_p E)$. Then $v_{\mathrm{Div}}(s)$ maps to zero in $\mathrm{coker}(\beta_p)$, so it lifts to $t \in k(\Delta^{alt}_p E)^\times$. Since $\beta_p(t) = v_{\mathrm{Div}}(s)$, there is some element $r \in A_p$ mapping to s. This shows that $\overline{s} = 0$, establishing injectivity of v_{coker}.

Finally, we show that v_{coker} is surjective. Let $t \in \mathrm{Div}(\Delta_p^{alt} E)$. We would like to lift t to a good divisor on $\Delta_p^{alt} E$. In order to do so, we would like to wiggle t by a principal divisor so that it meets $\Delta_p^{alt} E$ transversally for $q > p$. But since $\Delta_q^{alt} E = \varnothing$ for sufficiently large q and each $\Delta_p^{alt} E$ has a a finite number of components, we may apply Bertini's theorem to find a lift $s \in \mathrm{Div}_g(\Delta_p^{alt} E)$ of t. This shows that v_{coker} is surjective. $\square$

By replacing each column $k(\Delta_p^{alt} E)^\times \longrightarrow \mathrm{Div}(\Delta_p^{alt} E)$ with the quasi-isomorphic complex obtained from $\mathrm{Div}_g(\Delta_p^{alt} E)$ as in the lemma above, we can replace the diagram (6.15) with the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Div}_g(\Delta_{n-3}^{alt} E) \arrow[r] & \mathrm{Div}_g(\Delta_{n-2}^{alt} E) \arrow[r] & 0 \arrow[r] & 0 \\
\cdots \arrow[r] & A_{n-3} \arrow[r] \arrow[u] & A_{n-2} \arrow[r] \arrow[u] & k(\Delta_{n-1}^{alt} E)^\times \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{6.19}$$

so that all of the horizontal maps are indeed defined. We may then use this diagram to calculate the differential $d_2^{n-3,1}$ that appears in the spectral sequence (1.13). We claim this map is zero.

Lemma 6.9. $d_2^{n-3,1}$ *is the zero map.*

Proof. For the purposes of bookkeeping, suppose E has m irreducible components. Let $\mathbf{m} = \{1 < \cdots < m\}$, and let us also set

$$I = \{\{i_0 < \cdots < i_{n-1}\} \,|\, i_1, \ldots, i_{n-1} \in \mathbf{m}\}\,. \tag{6.20}$$

I denotes all ordered subsets of $\{1, \ldots, m\}$ of length n. We will also let $\mathbf{i}$ and $\mathbf{j}$ denote ordered subsets of $\mathbf{m}$ of length $n-2$ and $n-1$, respectively. Keeping tight track of the indices would be a notational burden and detracts from the main thrust of the proof, so there will be some looseness in our usage of $\mathbf{i}$ and $\mathbf{j}$.

Consider a divisor $D \in \mathrm{Div}_g(\Delta^{alt}_{n-3}E)$ that represents an element of $E_2^{n-3,1} = \ker(d_1^{n-3,1})$ (see the diagram (6.14)). The image of D in $\mathrm{Pic}(\Delta^{alt}_{n-2}E)$ is zero, so it pulls back to a rational function $g = (g_{\mathbf{j}}) \in A_{n-2}$ on $\Delta^{alt}_{n-2}E$.

Write $D = (D_{\mathbf{i}})$, and write $D_{\mathbf{i}} = D'_{\mathbf{i}} - D''_{\mathbf{i}}$ such that $D'_{\mathbf{i}}$ and $D''_{\mathbf{i}}$ are effective divisors whose supports intersect in codimension at least two. In the basis of prime divisors on $E_{\mathbf{i}} = E_{i_0} \times_E \cdots \times_E E_{i_{n-3}}$, $D'_{\mathbf{i}}$ and $D''_{\mathbf{i}}$ correspond to the divisors with positive and negative coefficients, respectively. The $D'_{\mathbf{i}}, D''_{\mathbf{i}}$ also correspond to locally principal, closed, codimension one subschemes $Y''_{\mathbf{i}}, Y''_{\mathbf{i}}$, respectively, of $\Delta^{alt}_{n-3}E$. Locally, $Y'_{\mathbf{i}}$ and $Y''_{\mathbf{i}}$ are defined on some open U by the vanishing of sections $f'_{\mathbf{i}}, f''_{\mathbf{i}} \in \Gamma(U, \mathcal{O}_U)$, respectively.

If $\mathbf{i} = \{i_0, \ldots, i_{n-2}\}$ and $\mathbf{i} \subsetneq \mathbf{j} = \{i_0 \ldots, i_r, a, i_{r+1}, \ldots, i_{n-2}\}$ so that $\mathbf{j}$ is obtained from $\mathbf{i}$ by inserting a after the r^{th} element of $\mathbf{i}$, then we define $\mathrm{sign}(\mathbf{i}, \mathbf{j}) = (-1)^r$.

Then on $E_{\mathbf{j}}$, the divisor $\sum_{\mathbf{i} \subsetneq \mathbf{j}} (-1)^{\mathrm{sign}(\mathbf{i},\mathbf{j})}(D_{\mathbf{i}} \cap E_{\mathbf{j}})$ has degree zero, and is locally defined by $f_{\mathbf{j}} := \prod_{\mathbf{i} \subsetneq \mathbf{j}} (f'_{\mathbf{i}}/f''_{\mathbf{i}})^{(-1)^{\mathrm{sign}(\mathbf{i},\mathbf{j})}}$. Then the divisor defined locally by $g_{\mathbf{j}}/f_{\mathbf{j}}$ has no zeroes or poles hence is constant on $E_{\mathbf{j}}$. This shows that the function defined locally by the $f_{\mathbf{j}}$ is in fact actually a rational function, and gives the same divisor class as g. Since D is a good divisor, it meets $\Delta^{alt}_{n-1}E$ transversally, i.e. the support of D does not intersect $\Delta^{alt}_{n-1}E$. Then the zeros of the $f'_{\mathbf{i}}, f''_{\mathbf{i}}$ do not intersect $\Delta^{alt}_{n-2}E$, so we can use the $f_{\mathbf{j}}$ to evaluate $d_2^{n-3,1}(D)$.

Let $P \in \Delta^{alt}_{n-1}E$ be a closed point. We claim that when we push g forward to $k^\times(\Delta^{alt}_{n-1}E)$ and then evaluate at P, we will get 1. By the preceding discussion we can use $f_{\mathbf{j}}$ in place of the $g_{\mathbf{j}}$, and because the differentials are gotten by taking an alternating sum of face maps, we see that $d_2^{n-3,1}(g)(P)$ is is the image of the $f_{\mathbf{i}}$ under the composition of two differentials, hence is trivial. Since g pulls back to something already trivial in $k(\Delta^{alt}_{n-1}E)^\times$, it will be zero in the quotient $E_2^{n-1,0}$. $\square$

Consequently, $E_\infty^{n-1,0} = E_2^{n-1,0} = H^{n-1}(\mathcal{D}(E), k^\times)$, and we have the following corollary.

Corollary 6.10. *Writing* coker(Pic) *for the cokernel of* $\mathrm{Pic}(\Delta^{alt}_{n-3}E) \longrightarrow \mathrm{Pic}(\Delta^{alt}_{n-2}E)$, *we have a short exact sequence that computes* $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m)$:

$$0 \longrightarrow H^{n-1}(\mathcal{D}(E), k^\times) \longrightarrow H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m) \longrightarrow \operatorname{coker}(\operatorname{Pic}) \longrightarrow 0. \tag{6.21}$$

Example 6.11. If $H_{n-2}(\mathcal{D}(E), \mathbb{Z})$ is torsion-free, or if k contains all roots of unity (for example, when k is algebraically closed), then we also have $H^{n-1}(\mathcal{D}(E), k^\times) = H^{n-1}(\mathcal{D}(E), \mathbb{Z}) \otimes k^\times$, via the universal coefficient theorem, which gives a split short exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{n-2}(\mathcal{D}(E), \mathbb{Z}), k^\times) \longrightarrow H^{n-1}(\mathcal{D}(E), k^\times) \longrightarrow \operatorname{Hom}(H_{n-1}(\mathcal{D}(E), \mathbb{Z}), k^\times) \longrightarrow 0. \tag{6.22}$$

The first term is zero, since in the first case $k^\times$ is a divisible group, and in the second case $H_{n-2}(\mathcal{D}(E), \mathbb{Z})$ is free. In addition, since $\dim \mathcal{D}(E) = n-1$, the top homology group of $\mathcal{D}(E)$ is free, which proves the claim. In particular, $H^{n-1}(\mathcal{D}(E), k^\times) \cong (k^\times)^r = T_E(k)$, for some r, is the k-points of some split torus T_E.

The thrust of the next subsection is to show that a 1-motive naturally arises out of the spectral sequence (1.13).

6.1.0.2 Computation of Picard groups

Since all of our schemes are projective over k, the Picard functor is representable [FGI05, Chapter 9]; in particular, Pic^0, the connected component of the Picard scheme, exists. Let the Néron-Severi group, NS, be the presheaf cokernel of the group of components $\operatorname{Pic} / \operatorname{Pic}^0$.

Writing Pic as an extension of NS by Pic^0, we obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\mathrm{Pic}^0) \arrow[r] \arrow[d] & \ker(\mathrm{Pic}) \arrow[r] \arrow[d] & \ker(\mathrm{NS}) \arrow[d] & \\
0 \arrow[r] & \mathrm{Pic}^0(\Delta^{alt}_{n-3}E) \arrow[r] \arrow[d] & \mathrm{Pic}(\Delta^{alt}_{n-3}E) \arrow[r] \arrow[d] & \mathrm{NS}(\Delta^{alt}_{n-3}E) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(\Delta^{alt}_{n-2}E) \arrow[r] \arrow[d] & \mathrm{Pic}(\Delta^{alt}_{n-2}E) \arrow[r] \arrow[d] & \mathrm{NS}(\Delta^{alt}_{n-2}E) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{coker}(\mathrm{Pic}^0) \arrow[r] \arrow[d] & \mathrm{coker}(\mathrm{Pic}) \arrow[r] \arrow[d] & \mathrm{coker}(\mathrm{NS}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{6.23}$$

Taking the second map from the short exact sequence (6.22) and pulling back along the map $\mathrm{coker}(\mathrm{Pic}^0) \longrightarrow \mathrm{coker}(\mathrm{Pic})$ gives us

$$\begin{tikzcd}
0 \arrow[r] & H^{n-1}(\mathcal{D}(E), k^\times) \arrow[r] \arrow[d, equal] & G_E(k) \arrow[r] \arrow[d] & \mathrm{coker}(\mathrm{Pic}^0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{n-1}(\mathcal{D}(E), k^\times) \arrow[r] & H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m) \arrow[r] & \mathrm{coker}(\mathrm{Pic}) \arrow[r] & 0
\end{tikzcd} \tag{6.24}$$

Applying the snake lemma to the above diagram, we see that the two vertical maps on the right have the same kernel and cokernel, and that $\ker(\mathrm{NS})$ surjects onto $\ker(\beta)$:

$$
\begin{tikzcd}
 & & \ker(\mathrm{NS}) \arrow[r, equal] \arrow[d, two heads] & \ker(\mathrm{NS}) \arrow[d, two heads] & \\
 & & \ker(\beta) \arrow[r, equal] \arrow[d, tail] & \ker(\beta) \arrow[d, tail] & \\
0 \arrow[r] & H^{n-1}(\mathcal{D}(E), k^{\times}) \arrow[r] \arrow[d, equal] & G_E(k) \arrow[r] \arrow[d] & \mathrm{coker}(\mathrm{Pic}^0) \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & H^{n-1}(\mathcal{D}(E), k^{\times}) \arrow[r] & H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m) \arrow[r] \arrow[d, two heads] & \mathrm{coker}(\mathrm{Pic}) \arrow[r] \arrow[d, two heads] & 0 \\
 & & \mathrm{coker}(\mathrm{NS}) \arrow[r, equal] & \mathrm{coker}(\mathrm{NS}) &
\end{tikzcd}
\tag{6.25}
$$

Furthermore, since $\mathrm{NS}(\Delta^{alt}_{\bullet} E)$ is a complex, we have a map $\mathrm{NS}(\Delta^{alt}_{n-4} E) \longrightarrow \ker(\mathrm{NS})$. We claim that the composite

$$
\mathrm{NS}(\Delta^{alt}_{n-4} E) \longrightarrow \ker(\mathrm{NS}) \longrightarrow \mathrm{coker}(\mathrm{Pic}^0)
\tag{6.26}
$$

is zero. We can see this from the square

$$
\begin{tikzcd}
\mathrm{Pic}(\Delta^{alt}_{n-4} E) \arrow[r] \arrow[d] & \mathrm{NS}(\Delta^{alt}_{n-4} E) \arrow[d] \\
\ker(\mathrm{Pic}) \arrow[r] & \ker(\mathrm{NS})
\end{tikzcd}
\tag{6.27}
$$

The top horizontal map is surjective, so we may pull back any element of $\mathrm{NS}(\Delta^{alt}_{n-4} E)$ back to $\mathrm{Pic}(\Delta^{alt}_{n-4} E)$. But then we can evaluate the composite (6.26) by pushing down to $\ker(\mathrm{Pic})$ and then forward to $\mathrm{coker}(\mathrm{Pic}^0)$.

For the rest of this chapter, let k be algebraically closed and of sufficiently small cardinality so that there is an embedding $k \longrightarrow \mathbb{C}$. We will show that the diagram

$$\left[\begin{tikzcd} & & \ker(\mathrm{NS}) \arrow[d] & & \\ 0 \arrow[r] & H^{n-1}(\mathcal{D}(E), k^\times) \arrow[r] & G_E(k) \arrow[r] & \operatorname{coker}(\mathrm{Pic}^0) \arrow[r] & 0 \end{tikzcd}\right] \tag{6.28}$$

is the k-points of a 1-motive M'_E, and that

$$\left[\begin{tikzcd} & & H^{n-3}(\mathrm{NS}(\Delta^{alt}_\bullet E)) \arrow[d] & & \\ 0 \arrow[r] & H^{n-1}(\mathcal{D}(E), k^\times) \arrow[r] & G_E(k) \arrow[r] & \operatorname{coker}(\mathrm{Pic}^0) \arrow[r] & 0 \end{tikzcd}\right] \tag{6.29}$$

is the k-points of a 1-motive M_E, where $H_{n-3}(\mathrm{NS}(\Delta^{alt}_\bullet E)) = \operatorname{coker}(\mathrm{NS}(\Delta^{alt}_{n-4}E) \longrightarrow \ker(\mathrm{NS}))$ is the $(n-3)^{\mathrm{rd}}$ cohomology group of the complex $\mathrm{NS}(\Delta^{alt}_\bullet E)$.

Pic^0 is representable, and since k is algebraically closed, the functor of taking k-points is exact. In particular, the k-points of the cokernel of the map between Pic^0 schemes of $\Delta^{alt}_0 E$ and $\Delta^{alt}_1 E$ is the cokernel of the Pic^0 groups, that is, $\operatorname{coker}(\mathrm{Pic}^0)$. In other words, $\operatorname{coker}(\mathrm{Pic}^0)$ is the k-points of the corresponding abelian variety.

Similarly, the group $H^{n-1}(\mathcal{D}(E), k^\times) \cong (k^\times)^r$, are the k-points of some torus, as in Example 6.11. Therefore, for ease of notation and for suggestiveness, let T_E be the (split) torus so that $T_E(k) = H^{n-1}(\mathcal{D}(E), k^\times)$.

We may compose the map $G_E(k) \longrightarrow H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m)$ with the descent spectral sequence edge map $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m) \longrightarrow KH_{2-n}(E)$ to get a map $\alpha := G_E(k) \longrightarrow H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m) \longrightarrow KH_{1-n}(E)$; the cokernel of this composite is an extension of the cokernel of the latter map by the cokernel of the former map:

$$0 \longrightarrow \operatorname{coker}(\mathrm{NS}) \longrightarrow \operatorname{coker}(\alpha) \longrightarrow H^{n-2}(\mathcal{D}(E), \mathbb{Z}) \longrightarrow 0. \tag{6.30}$$

Similarly, the kernel of α is the extension of the kernel of the latter map by the kernel of the former map:

$$0 \longrightarrow \ker(\beta) \longrightarrow \ker(\alpha) \longrightarrow \operatorname{im}(d_2^{n-3,0}) \longrightarrow 0, \tag{6.31}$$

where $d_2^{n-3,0}$ is the E_2 differential $H^{n-3}_{\mathrm{cdh}}(E,\mathbb{Z}) \longrightarrow H^{n-1}_{\mathrm{cdh}}(E,\mathbb{G}_m)$ in the descent spectral sequence (1.14). Since the Néron-Severi groups are finitely generated, ker(NS) is finitely generated, so the quotient $\ker(\beta)$ is finitely generated as well. Furthermore, Lemma 6.3 tells us that the term $\operatorname{im}(d_2^{n-3,0})$ is also finitely generated, so $\ker(\alpha)$, the extension of two finitely generated groups, is always finitely generated. When $n = 3$, Lemma 6.2 implies that the edge map $H^2_{\mathrm{cdh}}(E,\mathbb{G}_m) \longrightarrow KH_{-1}(E)$ is an injection (see the exact sequence (6.8)), so $\ker(\alpha) = \ker(\beta)$. In particular, the sequence

$$\ker(\mathrm{NS}) \longrightarrow G_E(k) \longrightarrow KH_{-2}(X) \tag{6.32}$$

is exact. All in all, for general n, both $\ker(\alpha)$ and $\operatorname{coker}(\alpha)$ are finitely generated, so that $KH_{-1}(E) \cong KH_{-2}(X)$ is the k points of some group scheme, up to some finitely generated groups.

We now verify our earlier claim that the diagrams (6.28) and (6.29) are the k-points of 1-motives.

Proposition 6.12. *The diagrams (6.28) and (6.29) are the k-points of 1-motives over k.*

Proof. For ease of notation, we will let $A_\bullet \coloneqq C_\bullet(\Delta^{alt}_\bullet E)$ denote be the complex associated to the semisimplicial scheme $\Delta^{alt}_\bullet E$ (see Remark 4.7). We now check that our construction agrees with [BRS03], where we take $X_\bullet = \Delta^{alt}_\bullet E$. In addition, $\Delta^{alt}_\bullet E$ is already projective, so there is no need to take a compactification. So we have, using the notation of [BRS03],

$$
\begin{aligned}
{}^{o}W'^{0}(A_\bullet) &= A_\bullet \\
{}^{o}W'^{1}(A_\bullet) &= E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_2 \longrightarrow E_1 \\
&\vdots \\
{}^{o}W'^{n-1}(A_\bullet) &= E_n \longrightarrow E_{n-1} \\
{}^{o}W'^{n}(A_\bullet) &= E_n \\
{}^{o}W'^{n+1}(A_\bullet) &= \varnothing
\end{aligned}
\tag{6.33}
$$

and

$$
\begin{aligned}
{}^{o}W''^{-1}(A_\bullet) &= A_\bullet \\
{}^{o}W''^{0}(\Delta^{alt}_\bullet E) &= A_\bullet \\
{}^{o}W''^{1}(\Delta^{alt}_\bullet E) &= \varnothing
\end{aligned}
\tag{6.34}
$$

so that

$$
\begin{aligned}
{}^{o}W''^{-1}(A_\bullet) &= \Delta^{alt}_\bullet E \\
{}^{o}W''^{0}(A_\bullet) &= \Delta^{alt}_\bullet E \\
{}^{o}W'^{1}(A_\bullet) &= E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_2 \longrightarrow E_1 \\
&\vdots \\
{}^{o}W'^{n-1}(A_\bullet) &= E_n \longrightarrow E_{n-1} \\
{}^{o}W'^{n}(A_\bullet) &= E_n \\
{}^{o}W'^{n+1}(A_\bullet) &= \varnothing
\end{aligned}
\tag{6.35}
$$

where the chain maps are the alternating sum of the face maps. In addition, in any explicitly written-out complexes, the rightmost term has homological degree zero. Then the spectral sequence [BRS03, 3.1.3] with $r = 0$ is

$$E_1^{p,q} = H^q(\Delta_p^{alt} E, \mathbb{G}_m) \Longrightarrow H^{p+q}(\mathcal{K}'), \tag{6.36}$$

which is the spectral sequence (1.13). Next, with the notation of [BRS03], we claim that the diagrams (6.28) and (6.29) are the k-points of the 1-motives $M'_{n-1}(A_\bullet) = [\Gamma'_{n-1}(A_\bullet) \longrightarrow G_{n-1}(A_\bullet)]$ and $M'_{n-1}(A_\bullet) = [\Gamma'_{n-1}(A_\bullet) \longrightarrow G_{n-1}(A_\bullet)]$, where

$$\begin{aligned} \Gamma'_{n-1}(A_\bullet) &= \ker\left(\partial\colon P_{n-3}(A_\bullet)/P_{n-3}(A_\bullet)^0 \longrightarrow P_{\geq n-2}(A_\bullet)/P_{\geq n-2}(A_\bullet)^0\right) \\ \Gamma_{n-1}(A_\bullet) &= \operatorname{coker}\left(\mathrm{NS}(\Delta_{n-4}^{alt} E) \longrightarrow \Gamma'_{n-1}(A_\bullet)\right) \end{aligned} \tag{6.37}$$

and

$$G_{n-1}(A_\bullet) = \operatorname{coker}\left(\partial\colon P_{n-3}(A_\bullet) \longrightarrow P_{\geq n-2}(A_\bullet)\right), \tag{6.38}$$

where

$$\begin{aligned} P_{\geq i}(A_\bullet) &= H^{i+1}(W^i \mathcal{K}') \\ P_i(A_\bullet) &= H^{i+1}(Gr_W^i \mathcal{K}'). \end{aligned} \tag{6.39}$$

and $\mathcal{K}'$ is as in spectral sequence (6.36). To check this, we first compute the lattices $\Gamma'_{n-1}(A_\bullet)$ and $\Gamma_{n-1}(A_\bullet)$. We can see that $P_{n-3}(A_\bullet) = \mathrm{Pic}(\Delta_{n-3}^{alt} E)$; we still need to know $P_{\geq n-2}(A_\bullet)$. For the latter, the above spectral sequence (6.36) gives a short exact sequence

$$0 \longrightarrow H^{n-1}(\mathcal{D}(E), k^\times) \longrightarrow P_{\geq n-2}(A_\bullet) \longrightarrow \mathrm{Pic}(\Delta_{n-2}^{alt} E) \longrightarrow 0. \tag{6.40}$$

Consider the pullback of the diagram along the inclusion $\mathrm{Pic}^0(\Delta^{alt}_{n-2}E) \longrightarrow \mathrm{Pic}(\Delta^{alt}_{n-2}E)$. We claim that the pullback of this square is $P_{\geq n-2}(A_\bullet)^0$:

$$\begin{tikzcd} 0 \arrow[r] & H^{n-1}(\mathcal{D}(E), k^\times) \arrow[r] \arrow[d, equal] & P_{\geq n-2}(A_\bullet)^0 \arrow[r] \arrow[d] & \mathrm{Pic}^0(\Delta^{alt}_{n-2}E) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^{n-1}(\mathcal{D}(E), k^\times) \arrow[r] & P_{\geq n-2}(A_\bullet) \arrow[r] & \mathrm{Pic}(\Delta^{alt}_{n-2}E) \arrow[r] & 0 \end{tikzcd} \tag{6.41}$$

The middle term in the top row, the pullback, is an extension of two connected group schemes, hence is connected. In addition, the connected component of the identity $P_{\geq n-2}(A_\bullet)^0$, injects into the pullback, so the pullback must be isomorphic to the connected component of the identity, as we have indicated on the diagram above. Now applying the snake lemma to the above diagram, we obtain

$$P_{\geq n-2}(A_\bullet)/P_{\geq n-2}(A_\bullet)^0 \cong \mathrm{NS}(\Delta^{alt}_{n-2}E), \tag{6.42}$$

so that $\Gamma'_{n-1}(A_\bullet) = \ker(\mathrm{NS}(\Delta^{alt}_{n-3}E) \longrightarrow \mathrm{NS}(\Delta^{alt}_{n-2}E))$, which agrees with our lattice $L_E = \ker(\mathrm{NS})$. The related lattice $\Gamma_{n-1}(A_\bullet)$ is just the cokernel

$$\begin{aligned} \Gamma_{n-1}(A_\bullet) &= \mathrm{coker}(\mathrm{NS}(\Delta^{alt}_{n-4}E) \longrightarrow \Gamma'_{n-1}(A_\bullet)) \\ &= H^{n-3}(\mathrm{NS}(A_\bullet)) \end{aligned} \tag{6.43}$$

which agrees with our other lattice term in (6.29). It remains to check that the semiabelian variety $G_{n-1}(A_\bullet)$ agrees with our G_E. Using the short exact sequence above that calculates $P_{\geq n-2}(A_\bullet)$, we get

$$\begin{tikzcd} & & \mathrm{Pic}^0(\Delta^{alt}_{n-3}E) \arrow[d, "g"] & & \\ 0 \arrow[r] & T_E \arrow[r] & P_{\geq n-2}(A_\bullet)^0 \arrow[r] & \mathrm{Pic}^0(\Delta^{alt}_{n-2}E) \arrow[r] & 0 \end{tikzcd} \tag{6.44}$$

where $G_{n-1}(A_\bullet)$ is the cokernel of the vertical map g. We take the pullback of the first horizontal map $T_E \longrightarrow P_{\geq n-2}(A_\bullet)^0$ along g:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \mathrm{Pic}^0(\Delta^{alt}_{n-3}E) \arrow[d] \\ T_E \arrow[r] & P_{\geq n-2}(A_\bullet)^0 \end{tikzcd} \tag{6.45}$$

Because the bottom map is a closed immersion, we see that W is a closed subgroup of $\mathrm{Pic}^0(\Delta^{alt}_{n-3}E)$. Furthermore, because the square is a pullback square, the induced map on cokernels is injective. We add these observations to the diagram (6.44):

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ 0 \arrow[r] & W \arrow[r] \arrow[d, "f"] & \mathrm{Pic}^0(\Delta^{alt}_{n-3}E) \arrow[r] \arrow[d, "g"] & \mathrm{Pic}^0(\Delta^{alt}_{n-3}E)/W \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & T_E \arrow[r] \arrow[d] & P_{\geq n-2}(A_\bullet)^0 \arrow[r] \arrow[d] & \mathrm{Pic}^0(\Delta^{alt}_{n-2}E) \arrow[r] \arrow[d] & 0 \\ & \operatorname{coker} f \arrow[r] & G_{n-1}(A_\bullet) \arrow[r] & \operatorname{coker} h \arrow[r] & 0 \end{tikzcd} \tag{6.46}$$

To finish, we need the following lemma:

Lemma 6.13. *The k-points of the bottom row of the above diagram (6.46) is the same as the short exact sequence in the top row of the diagram (6.24).*

Proof. Let $W' = \operatorname{im} f$ denote the image of W in T_E. Since $\mathrm{Pic}^0(\Delta^{alt}_{n-3}E)$ is an abelian variety, it is proper over k, so that g is also proper [Har77, II, Corollary 4.8]. We have already observed that W is a closed subscheme of $\mathrm{Pic}^0(\Delta^{alt}_{n-3}E)$, hence proper over k. Furthermore, the map $W \longrightarrow T_E$ is also proper, so W' is a closed subvariety of T_E that is proper over k [Har77, II, Exercise 4.4]. On the other hand, T_E is affine, and W', being closed in T_E, is also affine. But then W' is finite over k, as it is affine and proper over k[Har77, II, Exercise 4.6].

In addition, since W' is a finite subgroup of T_E, we claim that coker f is isomorphic to T_E. T_E is a group of multiplicative type, and since all finite subgroups of a group of multiplicative type are also of multiplicative type, W' is of multiplicative type [Wat79, 2.2]. There is an anti-equivalence between group schemes of multiplicative type over k and finite abelian groups [KMR98, Proposition 20.17]. Here, the map $W' \longrightarrow T_E$ corresponds to a surjective map of a lattice onto a finite abelian group. The kernel of this map must also be finitely generated free abelian of the same rank, so that coker f must be isomorphic to a copy of T_E.

Finally, since the top right horizontal map is surjective, coker h is the same as the cokernel of the map $\mathrm{Pic}^0(\Delta^{alt}_{n-3}E) \longrightarrow \mathrm{Pic}^0(\Delta^{alt}_{n-2}E)$, as in our case. □

Applying the snake lemma and making the identification coker $f \cong T_E$ yields a short exact sequence of commutative group schemes

$$0 \longrightarrow T_E \longrightarrow G_{n-1}(A_\bullet) \longrightarrow \mathrm{coker}(\mathrm{Pic}^0) \longrightarrow 0, \tag{6.47}$$

which agrees with our construction. □

The main theorem in the Barbieri-Viale, Rosenschon, Saito paper on Deligne's conjecture on 1-motives [BRS03, Theorem 0.1] tells us that, under the equivalence 5.15, the free part 1-motive $(M_E)_{\mathrm{fr}}$, after base extending to $\mathbb{C}$, comes from a unique mixed Hodge structure H_E in $W_2H^{n-1}(E(\mathbb{C}), \mathbb{Z})$. (More specifically, H_E is the unique largest torsion-free mixed Hodge structure of type $\{(0,0),(0,1),(1,0),(1,1)\}$ in $W_2H^{n-1}(E(\mathbb{C}), \mathbb{Z})$ such that $\mathrm{Gr}_1^W H_E$ is polarizable.)

6.2 Independence of the choice of resolution

Now that we have the 1-motive $M_E = [L_E \longrightarrow G_E]$, we wish to determine to what extent these are independent of the choice of resolution. Under the equivalence (5.15), we get another 1-motive, which we will denote $M = [L \longrightarrow G]$, that comes from a unique mixed Hodge structure H in $W_2H^n(X(\mathbb{C}), \mathbb{Z})$, of the considered type. We will not only establish to what extent M_E is independent of the choice of resolution, but also we will establish a relation between M_E and M. The precise formulation is as follows.

Proposition 6.14. *For each resolution $p\colon \widetilde{X} \longrightarrow X$, there exists an (effective) morphism $M_E \longrightarrow M$, which is an isomorphism on the non-lattice parts. In particular, the 1-motive M_E is independent of the choice of resolution p unless $n = 3$, in which case we have a surjection $L_E \longrightarrow L$.*

Proof. Taking the long exact sequence in singular cohomology (of the $\mathbb{C}$-points) induced by the blowup square (1.20), we obtain

$$\cdots \longrightarrow H^{r-1}(\widetilde{X}, \mathbb{Z}) \oplus H^{r-1}(Z, \mathbb{Z}) \longrightarrow H^{r-1}(E, \mathbb{Z}) \longrightarrow H^r(X, \mathbb{Z}) \longrightarrow \cdots \tag{6.48}$$

From this long exact sequence, we get a map $H_E \longrightarrow H$ of mixed Hodge structures, since the weights are functorial with respect to morphisms. Since the groups $H^i(Z(\mathbb{C}), \mathbb{Z})$ vanish for $i > n-2$, the groups $H^i(\widetilde{X}(\mathbb{C}), \mathbb{Z})$ are pure of weight i, and $n \geq 3$, taking the weight 2 part of the sequence yields an isomorphism $W_2H^{n-1}(E(\mathbb{C}), \mathbb{Z}) \cong W_2H^n(X(\mathbb{C}), \mathbb{Z})$ unless $n = 3$, in which case we only have a surjection. Similarly, taking the weight 1 part of the above sequence yields an isomorphism $W_1H^{n-1}(E(\mathbb{C}), \mathbb{Z}) \cong W_1H^n(X(\mathbb{C}), \mathbb{Z})$. The weight 2 part contains the lattice, and the weight 1 part contains the rest of the 1-motive, proving the claim. □

Remark 6.15. Because the map $L_E \longrightarrow G$ factors through L, we see from the composite

$$L_E(k) \twoheadrightarrow L(k) \longrightarrow G(k) \longrightarrow KH_{1-n}(E) \tag{6.49}$$

that the images of L_E and L in G are the same. So when $n = 3$, the sequence

$$L(k) \longrightarrow G(k) \longrightarrow KH_{-2}(E) \tag{6.50}$$

is still exact.

Another way to see that the torus $H^{n-1}(\mathcal{D}(E), k^\times)$ is independent of the resolution is to see that the homotopy type of $\mathcal{D}(E)$ is independent of the choice of resolution [Ste08]. In particular, all of the cohomology groups $H^i(\mathcal{D}(E), A)$, for any abelian group A, are independent of the choice of resolution. So we see that the groups $H^i(\mathcal{D}(E), \mathbb{Z})$ for $i = n-3, n-2$, coming out of the exact sequence (6.9) that computes $KH_{2-n}(E)$ is also independent of the choice of resolution. Thus $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m)$ is independent of the choice of resolution as well. More directly, we can apply *cdh*-descent to the cohomology groups themselves; we get a long exact sequence

$$\cdots \longrightarrow H^{n-1}_{\mathrm{cdh}}(Z, \mathbb{G}_m) \oplus H^{n-1}_{\mathrm{cdh}}(\widetilde{X}, \mathbb{G}_m) \longrightarrow H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m) \longrightarrow H^{n}_{\mathrm{cdh}}(X, \mathbb{G}_m) \longrightarrow \cdots \tag{6.51}$$

Since Z and $\widetilde{X}$ are smooth, their *cdh*-cohomology groups agree with their Zariski cohomology groups, and by Lemma 3.2, we obtain an isomorphism $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m) \cong H^{n}_{\mathrm{cdh}}(X, \mathbb{G}_m)$, which gives an alternate way to see that $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m)$ is independent of the choice of resolution.

Now that $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m)$ is independent of the choice of resolution, the exact sequence (6.21) shows that the group coker(Pic) is also independent of the choice of resolution. Furthermore, since coker(Pic^0) was independent of the choice of resolution, the cokernel of coker(Pic^0) $\longrightarrow$ coker(Pic), which is coker(NS), is also independent of the choice of resolution, as is the kernel of that map. In summary, all of the various groups appearing in the diagram (6.23) are independent of the choice of resolution *except* possibly the group ker(NS), and only in the case $n = 3$. We give an example to show that indeed this is the case, that ker(NS) is not independent of the choice of resolution when $n = 3$.

Example 6.16. Suppose we have an 3-dimensional integral X with a smooth singular locus Z of dimension ≤ 1. Suppose that we also have an excellent resolution $p\colon \widetilde{X} \longrightarrow X$ with exceptional divisor E that has at least 2 irreducible components E_1, E_2 that have a nonempty intersection E_{12} (which must be a smooth curve). Let the other irreducible components of E be $E_3, \ldots, E_m$. Take a closed point x that lies in E_{12} but does not lie in any of the other E_i. If we take the blowup along x, we obtain a diagram

$$\begin{tikzcd} \mathrm{Bl}_x E \arrow[r] \arrow[d] & \mathrm{Bl}_x \widetilde{X} \arrow[d] \\ E \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ Z \arrow[r] & X \end{tikzcd} \tag{6.52}$$

so that $\mathrm{Bl}_x\widetilde{X} \longrightarrow X$ is also an excellent resolution. $\mathrm{Bl}_x E$ then has $m+1$ irreducible components: $E_1' = \mathrm{Bl}_x E_1, E_2' = \mathrm{Bl}_x E_2, E_3, \ldots, E_m$, and a new component E' that is the exceptional divisor of $\mathrm{Bl}_x\widetilde{X}$. The relationships between the intersections of the various components are given below.

$$\begin{aligned} & E_1' \cap E_2' = \mathrm{Bl}_x E_{12} \cong E_{12} && \\ & E_i' \cap E = \text{exceptional divisor of } \mathrm{Bl}_x E_i, \longrightarrow E_i && i = 1, 2 \\ & E_i \cap E' = \varnothing && i > 2 \\ & E_i \cap E_j' \cong E_i \cap E_j && i > 2, j = 1, 2 \end{aligned} \tag{6.53}$$

In general, for a smooth surface S that contains a point y, then we will have [Har77, V, Theorem 5.8]

$$\mathrm{NS}(\mathrm{Bl}_y S) = \mathrm{NS}(S) \oplus \mathbb{Z}, \tag{6.54}$$

so that from the diagram

$$
\begin{tikzcd}
\mathrm{NS}(\Delta_0^{alt}E) \arrow[r] \arrow[d] & \mathrm{NS}(\Delta_0^{alt}E) \oplus \mathrm{NS}(E') \oplus \mathbb{Z}^2 \arrow[d] \\
\mathrm{NS}(\Delta_1^{alt}E) \arrow[r] & \mathrm{NS}(\Delta_1^{alt}E) \oplus \mathbb{Z}^2
\end{tikzcd}
\tag{6.55}
$$

we see that $\mathrm{NS}(\Delta_0^{alt}E)$ has changed by $\mathrm{NS}(E') \oplus \mathbb{Z}^2$ and that $\mathrm{NS}(\Delta_1^{alt}E)$ has changed by $\mathbb{Z}^2$. Since E' is projective, $\mathrm{NS}(E')$ has rank at least 1, so that $\ker(\mathrm{NS})$ must become strictly bigger, and in particular depends on the choice of resolution of X.

This makes sense, because by Proposition 6.14, we have in general only a surjection $\ker(\mathrm{NS}) \longrightarrow L$ and not an isomorphism.

Remark 6.17. When X is projective, the 1-motive is independent of the choice of good resolution p unless $n = 3$, in which case the non-lattice parts of the 1-motive are independent of the choice of good resolution p. Therefore, to calculate $KH_{1-n}(X)$ when X is not projective, we need only take an algebraic compactification $\overline{X}$ of X, and *then* compute $KH_{1-n}(\overline{X})$, as $KH_{1-n}(X) \cong KH_{2-n}(E) \cong KH_{1-n}(\overline{X})$. This shows that $KH_{1-n}(X)$ is independent of the choice of algebraic compactification $\overline{X}$. This result makes sense in light of the observation that negative KH vanishes for smooth schemes, and we compactify away from the singular locus. In some sense, we are computing, KH_{1-n} of the singularity x_0 locally sitting inside X.

We wrap things up by putting together everything we have proven so far.

Theorem 6.18 (Main Theorem for $KH_{1-n}(X)$)**.** *Let X be an normal, integral n-fold over an algebraically closed field k of characteristic zero, with singular locus $Z = \mathrm{Sing}(X)$ such that Z is smooth or $\operatorname{codim} Z > 2$. Then there exists a 1-motive*

$$
M = \left[\begin{tikzcd}
 & & L \arrow[d] & & \\
0 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 0
\end{tikzcd} \right]
\tag{6.56}
$$

and a map $\alpha\colon G(k) \longrightarrow KH_{1-n}(X)$*, natural in* X*, whose kernel and cokernel are finitely generated. If* $p\colon \widetilde{X} \longrightarrow X$ *is any good resolution of singularities, then* $\ker(\alpha)$ *and* $\operatorname{coker}(\alpha)$ *have the more explicit descriptions* (6.31) *and* (6.30)*, respectively. In particular, the descriptions of* $\ker(\alpha)$ *and* $\ker(\beta)$ *are independent of the choice of resolution of* X*.*

Furthermore, if $X \longrightarrow \overline{X}$ *is an algebraic compactification of* X*, then after base extending to* $\mathbb{C}$*, the (torsion-free)* 1*-motive* $(M_{\mathbb{C}})_{\mathrm{fr}}$ *corresponds, under the equivalence* (5.15)*, to the unique largest torsion-free mixed Hodge structure* H *of type* $\{(0,0),(0,1),(1,0),(1,1)\}$ *in* $W_2H^n(\overline{X}(\mathbb{C}),\mathbb{Z})$ *such that* $\mathrm{Gr}_1^W H$ *is polarizable. Moreover, the non-lattice parts of* M*, and hence* α*, are independent of the choice of algebraic compactification* $X \longrightarrow \overline{X}$*.*

Finally, when $n = 3$*, then we have the additional property that the sequence*

$$L(k) \longrightarrow G(k) \longrightarrow KH_{-2}(X) \tag{6.57}$$

is exact.

CHAPTER 7

Calculation of $NK_{1-n}(X)$

Now that we have a good description of $KH_{1-n}(X)$, we turn our attention towards $NK_{1-n}(X)$, the other remaining contribution to $K_{1-n}(X)$. For this chapter, let X be a (not necessarily irreducible) n-dimensional variety over k with isolated singularity x_0. We begin with a lemma justifying the exact sequence (1.19).

Lemma 7.1. *There is an exact sequence*

$$NK_{1-n}(X) \xrightarrow{d_1^{1,1-n}} K_{1-n}(X) \longrightarrow KH_{1-n}(X) \longrightarrow 0. \tag{7.1}$$

Proof. We apply the normalized simplicial spectral sequence (1.10) to X. The K-dimension theorem [CHS08, Conjecture 0.1] implies that the groups $N^pK_{-q}(X)$ are zero whenever $q \geq n$ and $p \geq 1$. The E_1 page of this spectral sequence is shown below.

$$\begin{array}{ccccccccc}
\ddots & & \vdots & & \vdots & & \vdots & & \iddots \\
& & \Big| & & & & & & \\
0 & \longleftarrow & K_{2-n}(X) & \longleftarrow & NK_{2-n}(X) & \longleftarrow & N^2K_{2-n}(X) & \longleftarrow & \cdots \\
& & \Big| & & & & & & \\
0 & \longleftarrow & K_{1-n}(X) & \longleftarrow & NK_{1-n}(X) & \longleftarrow & N^2K_{1-n}(X) & \longleftarrow & \cdots \\
& & \Big| & & & & & & \\
0 & \longleftarrow & K_{-n}(X) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
& & \Big| & & & & & & \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array} \tag{7.2}$$

We can see that there are no nonzero differentials coming into or going out of $(0, 1-n)$

after the first page, so that $E^{\infty}_{0,1-n} = E^2_{0,1-n}$. In addition, all of the groups $E^{\infty}_{p,1-n-p}$ are zero, except when $p = 0$. This gives us the exact sequence we are looking for. $\square$

We now reduce to the case when X is affine.

Lemma 7.2. *$N^tK_{-q}(X) \cong N^tK_{-q}(U)$ for any $q \in \mathbb{Z}$, any $t \geq 1$, and any open $U \subset X$ containing the isolated singularity x_0.*

Proof. We apply the spectral sequence (1.14) to X. Because smooth schemes are K_{-q}-regular, it follows that for any smooth open subscheme $U \subset X$, we have $N^tK_{-q}(U) = 0$ whenever $t \geq 1$, as we have indicated above. Since X has only a singularity at x_0, we have $N^tK_{-q}(U) = 0$ whenever $x_0 \notin U$. It follows that the Zariski sheaf aN^tK_{-q} is a skyscraper sheaf supported at x_0. In particular, aN^tK_{-q} is flasque, so it has no higher cohomologies. Since $E_2^{p,q} = 0$ for $p > 0$ (and also $p < 0$ trivially), all differentials are zero, so we conclude that $E_2^{p,q} = E_{\infty}^{p,q}$, and thus $H^0_{\mathrm{Zar}}(X, aN^tK_{-q}) \cong N^tK_{-q}(X)$. But since aN^tK_{-q} is a skyscraper sheaf, we have $(aN^tK_{-q})(U) = (aN^tK_{-q})(X)$, proving the claim. $\square$

In particular, we may choose $U = \operatorname{Spec} R$ to be an open affine neighborhood of x_0. The intuition here is that since the N^tK_{-q}-groups are zero on smooth schemes, they only detect singularities, and their value depends only on the type of singularity involved.

Recall that we are interested in the case $q = n - 1$. Cortiñas, et al. [CHW10, Example 3.5, Proposition 4.1] elucidates the structure of the N^pK_q groups, which, specializing to $p = 1$ and $q = n - 1$, gives

$$NK_{1-n}(X) \cong NK_{1-n}(U) \cong H^{n-1}_{\mathrm{cdh}}(U, \mathcal{O}) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]. \tag{7.3}$$

The maps in the spectral sequence (1.14) are induced by the maps on the p-simplicial structure of $X \times \mathbb{A}^p$; in particular,

$$NK_{-q}(X) = \ker(\partial_0 : K_{-q}(X \times \mathbb{A}^1) \xrightarrow{t=0} K_{-q}(X)), \tag{7.4}$$

where t is the parameter of $\mathbb{A}^1$ – the same t as in equation (7.3). The decomposition (7.3) found in [CHW10] boils down to applying the Künneth formula for Hochschild homology $HH_n(R[t]) \cong \oplus_{i+j=n} HH_i(R) \otimes_{\mathbb{Q}} HH_j(\mathbb{Q}[t])$ [Wei94, Proposition 9.4.1], from which we see that the t in the $\mathbb{Q}[t]$ is indeed the parameter t in the copy of $\mathbb{A}^1$ when computing the N-functors. The groups $HH_j(\mathbb{Q}[t])$ are given in [Wei94, Exercise 9.1.3], albeit with several typos.

The differential $\partial_0 - \partial_1 \colon K_{1-n}(X \times \mathbb{A}^1) \longrightarrow K_{1-n}(X)$ reduces to just $-\partial_1$ on $NK_{1-n}(U) = \ker(\partial_0)$, and ∂_1 just sets $t = 1$. Therefore, the image of $NK_{1-n}(X)$ in $K_{1-n}(X)$ is isomorphic to $H^{n-1}_{\mathrm{cdh}}(U, \mathcal{O})$. In summary, we have proven that

Proposition 7.3. *There is a short exact sequence*

$$0 \longrightarrow H^{n-1}_{\mathrm{cdh}}(U, \mathcal{O}) \longrightarrow K_{1-n}(X) \longrightarrow KH_{1-n}(X) \longrightarrow 0\,. \tag{7.5}$$

that computes $KH_{1-n}(X)$.

Remark 7.4. The observation here that the maps in the spectral sequence come from the simplicial structure on $X \times \mathbb{A}^p$ can be taken further. For example, we can try to say something about $K_{2-n}(X)$. We have, in a similar way, via [CHW10, Corollary 4.2],

$$\begin{aligned} N^2K_{1-n}(U) &\cong NK_{1-n}(U) \otimes_{\mathbb{Q}} s\mathbb{Q}[s] \\ &\cong H^{n-1}_{\mathrm{cdh}}(U, \mathcal{O}) \otimes_{\mathbb{Q}} r\mathbb{Q}[r] \otimes_{\mathbb{Q}} s\mathbb{Q}[s]. \end{aligned} \tag{7.6}$$

The top face map from $K_{1-n}(X \times \mathbb{A}^2) \longrightarrow K_{1-n}(X \times \mathbb{A}^1)$ sends $1 - r - s$ to zero, so it sends r to t and s to $1 - t$. Therefore, the image of $d_1^{2,1-n}$ in $NK_{1-n}(X)$ is just

$$H^{n-1}_{\mathrm{cdh}}(U, \mathcal{O}) \otimes_{\mathbb{Q}} t(1-t)\mathbb{Q}[t] \subset H^{n-1}_{\mathrm{cdh}}(U, \mathcal{O}) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]. \tag{7.7}$$

which is precisely the kernel of the map $\partial_1 = (t \mapsto 1)$. The E_1 page of the spectral sequence is therefore exact at $(1, 1-n)$, and so $0 = E_2^{1,1-n} = E_\infty^{1,1-n}$. We may make the same argument for the map $d_1^{3,1-n} : N^3K_{1-n}(X) \longrightarrow N^2K_{1-n}(X)$. Let us write

$$\begin{aligned} N^3K_{1-n}(X) &\cong NK_{1-n}(X) \otimes s\mathbb{Q}[s] \otimes t\mathbb{Q}[t] \\ &\cong H^{n-1}_{\text{cdh}}(U, \mathcal{O}) \otimes r\mathbb{Q}[r] \otimes s\mathbb{Q}[s] \otimes t\mathbb{Q}[t] \\ N^2K_{1-n}(X) &\cong H^{n-1}_{\text{cdh}}(U, \mathcal{O}) \otimes u\mathbb{Q}[u] \otimes v\mathbb{Q}[v]. \end{aligned} \tag{7.8}$$

The differential coming out of $N^3K_{1-n}(X)$ is just the one that sends $1 - r - s - t$ to 0, so $r \mapsto u$, $s \mapsto v$, $t \mapsto 1 - u - v$, so the image of this map is just

$$(1 - u - v)(H^{n-1}_{\text{cdh}}(U, \mathcal{O}) \otimes u\mathbb{Q}[u] \otimes v\mathbb{Q}[v]) = H^{n-1}_{\text{cdh}}(U, \mathcal{O}) \otimes uv(1 - u - v)\mathbb{Q}[u, v], \tag{7.9}$$

which is precisely the kernel of the map $\partial_2 = d_1^{2,1-n}$. Thus $0 = E_3^{2,1-n} = E_\infty^{2,1-n}$, and we conclude that we have an exact sequence

$$NK_{2-n}(X) \longrightarrow K_{2-n}(X) \longrightarrow KH_{2-n}(X) \longrightarrow 0. \tag{7.10}$$

In particular, the map $K_{2-n}(X) \longrightarrow KH_{2-n}(X)$ is surjective.

As we have already noted, this group is independent of the choice of open affine neighborhood U of the singularity x_0. The following lemma makes this statement precise.

Lemma 7.5. *Let $V \subset U$ be an open affine neighborhood of x_0. Then the inclusion $V \hookrightarrow U$ induces an isomorphism $H^{n-1}_{\text{cdh}}(U, \mathcal{O}) \cong H^{n-1}_{\text{cdh}}(V, \mathcal{O})$.*

Proof. Take a Nisnevich cover $\{V \longrightarrow U, V' \longrightarrow U\}$, and then cover V' by open affines V_i'. Since V' is smooth, so are all of the V_i', and in particular, they have no higher *cdh*-cohomology groups (Theorem 3.2). A standard Čech spectral sequence argument then shows that the induced map is an isomorphism. $\square$

Alternatively, this isomorphism can be obtained directly from Proposition 7.3, by seeing that the kernel $H^{n-1}_{\mathrm{cdh}}(U,\mathcal{O})$ of the map $K_{1-n}(X) \longrightarrow KH_{1-n}(X)$ is independent of the choice of open affine neighborhood U containing the isolated singularity x_0.

The discussion using the decomposition (7.3) yielding the short exact sequence (7.5) is a reasonable description of $K_{1-n}(X)$, but *cdh*-cohomology groups are often difficult to compute. It turns out that we can be more explicit in our description of $K_{1-n}(X)$ in the exact sequence (1.14) by identifying the term $H^{n-1}_{\mathrm{cdh}}(U,\mathcal{O})$ in terms of known invariants of the singularity x_0.

Definition 7.6. Let R be a finite type k-algebra such that $U = \operatorname{Spec} R$ has only isolated singularities. The *generalized Du Bois invariants* $b^{p,q}$ for $p \geq 0, q \geq 1$ are

$$b^{p,q} = \operatorname{length} H^q_{\mathrm{cdh}}(U,\Omega^p). \tag{7.11}$$

These invariants are finite by [CHW11]. Du Bois invariants were introduced by Steenbrink [Ste97]. By [CHW11, Lemma 2.1, Equation 2.7], we see that $H^{n-1}_{\mathrm{cdh}}(U,\mathcal{O})$ is a k-vector space of dimension $b^{0,n-1}$. In particular, its dimension is finite.

Finally, in the case of $n = 3$, we have a full computation of $K_{-2}(X)$.

Corollary 7.7. *Let X be an integral threefold with an isolated singularity x_0. Then for any open affine U containing x_0, $K_{-2}(X)$ is an extension of $KH_{-2}(X)$ by $H^2_{\mathrm{cdh}}(U,\mathcal{O})$, where $KH_{-2}(X)$ has the description given by Theorem 6.18, and $H^2_{\mathrm{cdh}}(U,\mathcal{O})$ is a k-vector space of finite dimension $b^{0,2}$.*

CHAPTER 8

Applications, examples and related questions

In this chapter, we give some applications and examples, and then discuss possible future research directions.

8.1 Applications and examples

We begin with a few simple examples. Because the dual complex $\mathcal{D}(E)$ appears multiple times in our computation, many of the simplifications in particular instances come from knowledge about $\mathcal{D}(E)$.

Example 8.1. When x_0 is a *Du Bois singularity*, then $H^i_{\mathrm{cdh}}(U, \mathcal{O}) = 0$ for $i > 0$. In this case, the exact sequence (7.5) yields an isomorphism $K_{1-n}(X) \cong KH_{1-n}(X)$. Furthermore, when x_0 is a rational singularity (i.e. for any resolution $p\colon \widetilde{X} \longrightarrow X$, the higher direct images $R^i p_* \mathcal{O}_{\widetilde{X}}$ are zero) , then we use the fact [Ste06] that $H^{n-1}(\mathcal{D}(E), \mathbb{Z}) = 0$ to see that in fact there is no torus part in the 1-motive constructed in chapter 6.

Example 8.2 (Isolated hypersurface singularity)**.** Let $x_0 \in X$ be a isolated hypersurface singularity in the case that X is a complex threefold. Then [Ste08, Theorem 3.3] says that $\mathcal{DR}(X)$ has trivial fundamental group, so that $H^1(E, \mathbb{Z}) = 0$, so that $H^3_{\mathrm{cdh}}(X, \mathbb{G}_m) \cong KH_{-2}(X)$.

Example 8.3 (Isolated toric singularity)**.** When x_0 is an isolated singularity of a toric variety X, then [Ste05, Theorem 3] tells us that $\mathcal{D}(E)$ is homotopy equivalent to a point, so that $H^i(\mathcal{D}(E), A) = 0$ whenever $i > 0$ and for any A. In this case, we have $H^{n-1}_{\mathrm{cdh}}(E, \mathbb{G}_m) \cong KH_{1-n}(E)$, and that there is no torus part, so that G is an abelian variety.

8.2 Related Questions

There are many ways we can try to generalize using our techniques:

1. Resolution of singularities holds for threefolds in characteristic p, so we can try the same techniques. We won't be able to relate it to any Hodge structures, but perhaps there is something analogous.
2. See to what extent our techniques will work for computing $K_{-1}(X)$ for a threefold X with isolated singularities.

For 2., we may take an excellent resolution of X, but then the term $H^2_{\mathrm{cdh}}(E, aK_2)$ shows up when applying the descent spectral sequence – a term that would require further investigation, since we currently do not know of any good descriptions of aK_2.

There are other interesting questions that we don't quite know the answer to.

Question 8.4. Theorem 6.14 asserts that in the computation of the 1-motive that approximates $KH_{-2}(X)$, we get, for any choice of resolution $p\colon \widetilde{X} \longrightarrow X$, a surjection $L_E \longrightarrow L$. It is then natural to ask the following questions:

1. Does there always exist a resolution p of X for which $L_E \longrightarrow L$ is an isomorphism?
2. If there does not exist such a resolution p, then what is the minimum of $\dim L_E - \dim L$, and how can we relate this invariant to other invariants of X? Does it depend only on the singularity, or on the choice of X?
3. We can ask a similar question for L and $\ker(\beta)$: under what conditions on X (especially when $n = 3$) is $L = \ker(\beta)$?

In chapter 7, we were able to determine some of the maps in the spectral sequence (1.10). Because these maps are induced by the simplicial face maps, it may be possible to completely describe all of the differentials involved.

In Chapter 6, we established a full calculation of $KH_{-2}(X)$, when X is a threefold with smooth singular locus $Z = \mathrm{Sing}(X)$ of dimension ≤ 1. The only obstruction to the full calculation of $KH_{1-n}(X)$ for arbitrary n is the differential $d_2^{n-3,0}$ appearing in the descent spectral sequence (1.11). In the $n = 3$ case, we were able to show that this differential was the zero map. Presently, we do not understand this differential, and it is possible that its image is torsion, or possibly even zero.

References

[BRS03] L. Barbieri-Viale, A. Rosenschon, and M. Saito. "Deligne's conjecture on 1-motives." *Ann. of Math. (2)*, **158**(2):593–633, 2003.

[BS58] Armand Borel and Jean-Pierre Serre. "Le théorème de Riemann-Roch." *Bull. Soc. Math. France*, **86**:97–136, 1958.

[Car85] James A. Carlson. "The one-motif of an algebraic surface." *Compositio Math.*, **56**(3):271–314, 1985.

[CHS08] G. Cortiñas, C. Haesemeyer, M. Schlichting, and C. Weibel. "Cyclic homology, *cdh*-cohomology and negative K-theory." *Ann. of Math. (2)*, **167**(2):549–573, 2008.

[CHW10] G. Cortiñas, C. Haesemeyer, Mark E. Walker, and C. Weibel. "Bass' NK groups and cdh-fibrant Hochschild homology." *Invent. Math.*, **181**(2):421–448, 2010.

[CHW11] G. Cortiñas, C. Haesemeyer, Mark E. Walker, and C. Weibel. "A negative answer to a question of Bass." *Proc. Amer. Math. Soc.*, **139**(4):1187–1200, 2011.

[Con03] Brian Conrad. "Cohomological Descent.", 2003.

[Del71] Pierre Deligne. "Théorie de Hodge. II." *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[Del74] Pierre Deligne. "Théorie de Hodge. III." *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[EH02] Santiago Encinas and Herwig Hauser. "Strong resolution of singularities in characteristic zero." *Comment. Math. Helv.*, **77**(4):821–845, 2002.

[FGI05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[God73] Roger Godement. *Topologie algébrique et théorie des faisceaux*. Hermann, Paris, 1973. Troisième édition revue et corrigée, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII, Actualités Scientifiques et Industrielles, No. 1252.

[Gra76] Daniel Grayson. "Higher algebraic K-theory. II (after Daniel Quillen)." In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pp. 217–240. Lecture Notes in Math., Vol. 551. Springer, Berlin, 1976.

[Hae04] Christian Haesemeyer. "Descent properties of homotopy K-theory." *Duke Math. J.*, **125**(3):589–620, 2004.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Hau00] Herwig Hauser. "Resolution of singularities 1860–1999." In *Resolution of singularities (Obergurgl, 1997)*, volume 181 of *Progr. Math.*, pp. 5–36. Birkhäuser, Basel, 2000.

[Hir64] Heisuke Hironaka. "Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II." *Ann. of Math. (2) 79 (1964), 109–203; ibid. (2)*, **79**:205–326, 1964.

[Jar97] J. F. Jardine. *Generalized étale cohomology theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[KMR98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[Kol07] János Kollár. *Lectures on resolution of singularities*, volume 166 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2007.

[Kol12] János Kollár. "Dual graphs of exceptional divisors.", March 2012.

[KV69] Max Karoubi and Orlando Villamayor. "Foncteurs K^n en algèbre et en topologie." *C. R. Acad. Sci. Paris Sér. A-B*, **269**:A416–A419, 1969.

[KV71] Max Karoubi and Orlando Villamayor. "K-théorie algébrique et K-théorie topologique. I." *Math. Scand.*, **28**:265–307 (1972), 1971.

[Nic05] Liviu Nicolaescu. "Mixed Hodge structures.", 2005.

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.

[Qui74] Daniel Quillen. "Higher K-theory for categories with exact sequences." In *New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972)*, pp. 95–103. London Math. Soc. Lecture Note Ser., No. 11. Cambridge Univ. Press, London, 1974.

[Qui75] Daniel Quillen. "Higher algebraic K-theory." In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pp. 171–176. Canad. Math. Congress, Montreal, Que., 1975.

[Qui10] Daniel Quillen. "Higher algebraic K-theory: I [MR0338129]." In *Cohomology of groups and algebraic K-theory*, volume 12 of *Adv. Lect. Math. (ALM)*, pp. 413–478. Int. Press, Somerville, MA, 2010.

[Ste97] Joseph H. M. Steenbrink. "Du Bois invariants of isolated complete intersection singularities." *Ann. Inst. Fourier (Grenoble)*, **47**(5):1367–1377, 1997.

[Ste05] D. A. Stepanov. "A note on the dual complex associated to a resolution of singularities.", September 2005.

[Ste06] D. A. Stepanov. "Combinatorial structure of exceptional sets in resolution of singularities.", November 2006.

[Ste08] D. A. Stepanov. "A note on resolution of rational and hypersurface singularities." *Proc. Amer. Math. Soc.*, **136**(8):2647–2654, 2008.

[Sus84] Andrei A. Suslin. "On the K-theory of local fields." In *Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983)*, volume 34, pp. 301–318, 1984.

[SV00a] Andrei Suslin and Vladimir Voevodsky. "Bloch-Kato conjecture and motivic cohomology with finite coefficients." In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pp. 117–189. Kluwer Acad. Publ., Dordrecht, 2000.

[SV00b] Andrei Suslin and Vladimir Voevodsky. "Relative cycles and Chow sheaves." In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pp. 10–86. Princeton Univ. Press, Princeton, NJ, 2000.

[TT90] R. W. Thomason and Thomas Trobaugh. "Higher algebraic K-theory of schemes and of derived categories." In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pp. 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Voe00] Vladimir Voevodsky. "Cohomological theory of presheaves with transfers." In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pp. 87–137. Princeton Univ. Press, Princeton, NJ, 2000.

[VSF00] Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.

[Wal85] Friedhelm Waldhausen. "Algebraic K-theory of spaces." In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pp. 318–419. Springer, Berlin, 1985.

[Wat79] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979.

[Wei] Charles Weibel. "The K-book: An introduction to K-theory.".

[Wei89] Charles A. Weibel. "Homotopy algebraic K-theory." In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, volume 83 of *Contemp. Math.*, pp. 461–488. Amer. Math. Soc., Providence, RI, 1989.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.

[Wei99] Charles A. Weibel. "The development of algebraic K-theory before 1980." In *Algebra, K-theory, groups, and education (New York, 1997)*, volume 243 of *Contemp. Math.*, pp. 211–238. Amer. Math. Soc., Providence, RI, 1999.

[Wei01] Charles Weibel. "The negative K-theory of normal surfaces." *Duke Math. J.*, **108**(1):1–35, 2001.

[Wei05] Charles Weibel. "Algebraic K-theory of rings of integers in local and global fields." In *Handbook of K-theory. Vol. 1, 2*, pp. 139–190. Springer, Berlin, 2005.

[Zar44] Oscar Zariski. "Reduction of the singularities of algebraic three dimensional varieties." *Ann. of Math. (2)*, **45**:472–542, 1944.