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Pontrjagin forms on certain string homogeneous spaces

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by

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Abstract

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In this thesis, we study the topology and geometry of homogeneous spaces of the form G/T^k, where G is a compact semisimple Lie group, T^k is an embedded torus in G. We say that a principal $Spin(n)$-bundle $P \to M$ admits a string structure if the structure group lifts to $String(n)$. In particular, a spin manifold is string if the principal $Spin(n)$-bundle associated to the tangent bundle has a string structure. It's known that G/T^k are string manifolds and we prove the uniqueness of string structures on G/T^k when G is simply connected . There is a canonical metric on G/T^k, which has positive Ricci curvature. We deform this metric to a 1-parameter family of invariant metrics on G/T^k. We prove that the first Pontrjagin forms associated to the Levi-Civita connection of these metrics do not vanish. This verifies a conjecture by Redden-Stolz on the TMF-Witten genus of Ricci positive string manifolds.

To my parents

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Chapter 1

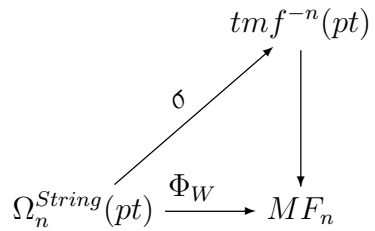
Introduction

1.1 Original motivation

The motivation of this paper arose from the following conjecture, which remains open

Conjecture 1.1.1. (Höhn and Stolz [13]). Let M be a closed oriented n-manifold admitting a string structure. If M admits a metric of positive Ricci curvature, then the Witten genus $\phi_W(M)$ vanishes.

The heuristic argument of Stolz is the combination of two non-rigorous facts: one is that the Witten genus should be the S^1-equivariant index of the Dirac operator on the free loop space LM, and the other is a Weizenboeck-type formula involving $Ric(M)$ such that if $Ric(M) > 0$, then $Ker(D_{LM}) = 0$. The most known examples of this conjecture are homogeneous spaces and complete intersections. Just as the KO-valued α-invariant refines the $\hat{A}$-genus, there is a cohomology theory tmf, or topological modular forms, with string-orientation refining the Witten genus.



The map $tmf^{-*}(pt) \to MF_*$ is a rational isomorphism, with very interesting kernels and cokernels.

A natural question is whether a string manifold M with a positive Ricci curvature metric has vanishing tmf-valued Witten genus. The answer is no! There are several

compact Lie groups whose tmf-valued Witten genus are nonzero torsion classes. This is different from the case of α-invariants, since Stolz showed in 1992 that for simply connected spin manifolds of dimension $\geqslant 5$, all the α-invariants vanish if and only if M admits a metric of positive scalar curvature.

Redden constructed in his thesis a canonical 3-form $H_{g,\mathcal{S}}$ for each string manifold M, with a Riemannian metric g, where $\mathcal{S}$ is a string structure on M. This canonical 3-form satisfies the condition that $dH_{g,\mathcal{S}} = p_1(M), d^*H = 0$ and it's $H^3(M,\mathbb{Z})$-equivariant. (The set of string structures on a string manifold M is a $H^3(M,\mathbb{Z})$-torsor). It's known that for a compact simple Lie group G, there exists a unique (up to a positive scalar) bi-invariant metric, and it has positive Ricci curvature. There is a canonical string structure on G, which is induced from the left-invariant framing on G, which is denoted by $\mathcal{L}$. There is the following fact:

$$\Omega_3^{string} \to tmf^{-3} \cong \mathbb{Z}/24, \quad [SU(2), \mathcal{L}] \mapsto \frac{1}{24}, \quad [SU(2), \partial D^4] \mapsto 0$$

.

Redden showed that $SU(2)$ with the bi-invariant metric g and the bounding string structure $\mathcal{S}$ has $H_{g,\mathcal{S}} = 0$. Redden studied the Berger metrics on $SU(2)$, which are in general only left-invariant, instead of bi-invariant. These Berger metrics depend on a positive real parameter α_1. Ricci curvature is positive if and only if $\alpha_1 > \frac{1}{\sqrt{2}}$. Redden showed that only when $\mathcal{S} = \partial D^4$ and $\alpha_1 = 1$ can we have both $Ric(g_{\alpha_1}) > 0$ and $H_{g,\mathcal{S}} = 0$. Based on this fact, he and Stolz made the following refined conjecture:

Conjecture 1.1.2. Let M be a closed n-dimensional string manifold with a specified structure $\mathcal{S}$. Suppose there exists a metric g such that

$$Ric(g) > 0, \quad and \quad H_{g,\mathcal{S}} = 0 \in \Omega^3(M)$$

Then

$$\sigma([M, \mathcal{S}]) = 0 \in tmf^{-n}(pt)$$

1.2 Summary of results

The primary goal is to study more examples of manifolds with positive Ricci curvature and verify Conjecture 1.1.2. The easiest examples which are close to Lie groups are homogeneous spaces. In an unpublished paper, Hopkins showed by transgression techniques that the tmf-valued Witten genus of certain homogeneous space $Sp(3)/S^1$ is nonzero. It's known that on homogeneous spaces G/H, where G is a compact simple

Lie group, there is a canonical metric which has positive Ricci curvature. To verify the conjecture, we need to show the non-vanishing of the canonical three form H. We study in this thesis all homogeneous spaces of the form G/T^k, where G is a compact semisimple Lie group, and T^k is an embedded torus in G. Since every embedded torus T^k in G is contained in a maximal torus of G, and any two maximal tori of G are conjugate to each other, we can fix a maximal torus T^k, for instance, for unitary groups $SU(n)$, we pick the diagonal matrices as the maximal torus. The first result we have is

Theorem 1.2.1. *Let G be a semisimple, simply connected compact Lie group, and T^k any k-dimensional embedded torus of G, then $H^3(G/T^k, \mathbb{Z}) = 0$. In particular, there is a unique string structure on G/T^k.*

Recall that the canonical 3-form $H_{g,\mathcal{S}}$ has the property that $dH_{g,\mathcal{S}} = p_1(M), d^*H_{g,\mathcal{S}} = 0$. It's not difficult to see that the vanishing of $H_{g,\mathcal{S}}$ is equivalent to the vanishing of $p_1(G/T^k)$. (Since 0 is the only harmonic 3-form by the theorem). Hence we only need to calculate the first Pontryagin form $p_1(G/T^k)$ associated to the Levi-Civita connection. We don't just calculate p_1 with the canonical metric but deform the metric in a similar way as Redden did. We have the following theorem:

Theorem 1.2.2. *Let G be simple, simply connected Lie groups, and T^k any embedded torus in G. Let g_λ be the one-parameter family of Riemannian metrics on G/T^k, which we will define in section 2.1, if $G/T^k \neq SU(3)/S^1$, then the first Pontrjagin form associated to the corresponding Levi-Civita connection is nontrivial.*

This verifies the conjecture of Redden and Stolz in these cases. It would be more interesting to find more of such homogeneous spaces with nonvanishing tmf-valued Witten genus.

The paper is organized as follows:

In Chapter 1, we review the preliminaries, which consists of three parts. In section 1, we review the basic facts about compact Lie groups, including root systems of semi-simple Lie algebras and Killing forms on Lie algebras. In section 2, we recall some known results on differential geometry of homogeneous spaces, including invairant Riemannian metrics, Levi-Civita connections and the associated curvature on homogeneous spaces G/H. In section 3, we briefly review Chern-Weil Theory and define Pontrjagin forms. This part, together with the description of curvature tensor on G/H, will serve for the proof of non-vanishing of the first Pontrjagin form p_1 in Chapter 3.

In Chapter 2, we prove the uniqueness of string structures on G/T^k for semisimple G. We first briefly recall the definition of string structures on spin manifolds. An

important fact is that string structures on a spin manifold M form a $H^3(M,\mathbb{Z})$-torsor. Hence the uniqueness of string structures is equivalent to the vanishing of $H^3(M,\mathbb{Z})$. It's well known that G/T^{max} has a cell decomposition which has only even dimensional cells. Hence it's obvious that $H^3(G/T^{max};\mathbb{Z}) = 0$. In section 2, we generalize this to all G/T^k, for $1 \leqslant k \leqslant rank(G)$.

In Chapter 3, we prove the nonvanishing of the first Pontrjagin form p_1 of homogeneous space G/T^k for G simple of classical types. The proof will mainly be technical computation. The procedure of the proof consists of two main steps: first, assuming that p_1 vanishes, we can get an estimate of the parameter of metrics λ in section 1. Next, with the estimate of λ, we can get contradictions, which is done for two cases: the generic cases and non-generic cases.

Chapter 2

Preliminaries

We describe some basic differential geometry on homogeneous spaces G/H in this chapter. In 2.1, we describe the root system of a compact Lie group, and the Killing form on the Lie algebra $\mathfrak{g}$, which will be important to describe the tangent space at the origin of G/H and invariant metrics on homogeneous spaces. We also review the basics of Chern-Weil theory in section 2.3.

2.1 Preliminaries on compact Lie groups and semi-simple Lie algebras

In this thesis, we will always assume that the Lie group G is compact, connected and simply connected. We will review the basic concepts of such Lie groups in this section. In the first part we will review maximal torus and roots of a Lie group G. In the second part, we will compute the maximal tori and root systems of classical matrix Lie groups. We will refer the details to [4] and [6].

2.1.1 Maximal tori and root systems of compact Lie groups

Definition 2.1.1. A subgroup $T \subset G$ is a **maximal torus** if T is a torus and there is no other torus T' with $T \subsetneq T' \subset G$. By a torus we mean a Lie group that is isomorphic to $\mathbb{R}^k/\mathbb{Z}^k$ for some k.

The existence of maximal tori is clear: since tori are compact and connected, if $T \subsetneq T'$, then $\dim T < \dim T'$. A maximal torus is the same as a maximal connected abelian subgroup, since the closure of such a subgroup is also connected and abelian

and hence is a torus. Maximal torus of a group G is not unique, but close to unique in the following weak sense:

Theorem 2.1.2. *Any two maximal tori in a compact connected Lie group G are conjugate, and every element of G is contained in a maximal torus.*

By restricting the adjoint representation of G on $\mathfrak{g}$ to a maximal torus T, we can study $\mathfrak{g}$ as a T-module. It's known that a T-module is determined by its weights and weight spaces. We define the roots of G by this representation of T:

Definition 2.1.3. The non-trivial weights of the adjoint representation are called **roots** of G. More precisely, we call $\theta : T \to U(1)$ **global** roots of G, the corresponding linear functional $\theta : \mathfrak{t} \to i\mathbb{R}$ **infinitesimal** roots, the induced functional $\Phi : \mathfrak{t} \otimes \mathbb{C} \to \mathbb{C}$ the **complex** roots of G.

By theorem 2.1.2, for two different tori T and T', they are conjugate to each other, the above definition of roots doesn't depend on the choice of a maximal torus T. From now on, we will always fix a maximal torus T.

The adjoint representation decomposes into the direct sum of root spaces:

$$\mathfrak{g} = M_0 \oplus \bigoplus_{\alpha \in R^+} M_\alpha,$$

$$\mathfrak{g}_{\mathbb{C}} = L_0 \oplus \bigoplus_{\alpha \in R} L_\alpha,$$

Here $M_\alpha = (L_\alpha \oplus L_{-\alpha}) \cap \mathfrak{g}$, and R^+ is supposed to contain exactly one element from each pair $\{\alpha, -\alpha\}$, which is called **positive** roots of G.

Consider the complexification $\mathfrak{g}_{\mathbb{C}}$. For each positive root α, we can choose a nonzero vector $e_\alpha \in \mathfrak{g}_\alpha$, and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$. It's not difficult to see that $[e_\alpha, e_{-\alpha}] \in \mathfrak{t}$. We can normalize e_α and $e_{-\alpha}$, such that

$$[e_\alpha, e_{-\alpha}] = t_\alpha,$$

$$[t_\alpha, e_\alpha] = 2e_{-\alpha},$$

$$[t_\alpha, e_{-\alpha}] = -2e_\alpha$$

In other words, $\{t_\alpha, e_\alpha, e_{-\alpha}\}$ spans a sub algebra of $\mathfrak{g}_{\mathbb{C}}$, which is isomorphic to $sl_2(\mathbb{C})$. h_α is called the coroot corresponding to α.

The Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is generated additively by the root vectors e_α, together with the elements in $\mathfrak{t}_{\mathbb{C}}$. The relations are necessarily of the form

$$[e_\alpha, e_\beta] = \begin{cases} n_{\alpha\beta} e_{\alpha+\beta} & \text{if } e_\alpha + e_\beta \text{ is a root} \\ 2t_\alpha & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1.1}$$

$$[t, e_\alpha] = \alpha(t) e_\alpha$$

So far the elements e_α and $e_{-\alpha}$ have been fixed only up to a multiplication by complex numbers of modulus one. They can be chosen so the the numbers $n_{\alpha\beta}$ are integers. We will describe explicitly such e_α's for classical Lie groups.

2.1.2 Examples: root systems of classical Lie groups

In this part, we give the root systems of classical matrix groups. We will give the proof for $G = SU(n)$, and the proof for the other classical Lie groups are similar, which we omit.

a) $G = SU(n)$. The Lie algebra of special unitary groups consists of traceless skew-Hermitian matrices,

$$\mathfrak{su}(n) = \{A \in End(\mathbb{C}^n) | Tr(A) = 0, A +^* A = 0\}$$

The maximal torus of $SU(n)$ consists of the diagonal matrices:

$$D = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}, z_k = exp(2\pi i \theta_k)$$

We use the n-tuple $(\theta_1, \cdots, \theta_n), \theta_1 + \cdots + \theta_n = 0$ to denote D, and have corresponding coordinates for the Lie algebra $\mathfrak{t} \cong \mathbb{R}^{n-1}$. It's clear that the subgroup of diagonal matrices is abelian.

Proposition 2.1.4. The subgroup of diagonal matrices is a maximal torus in $SU(n)$.

Proof. Let $T \subset SU(n)$ be any torus of $SU(n)$, the inclusion gives rise to a unitary representation of T on $\mathbb{C}^n$. Since any complex representation of a torus is a direct sum of one dimensional representations, T can be conjugated, such that it lives in the subgroup of diagonal matrices. □

We now want to determine all the roots of $SU(n)$. In other words, we are looking for all linear functionals $\alpha : \mathfrak{t} \to \mathbb{R}$, such that there is an element x in $\mathfrak{su}(n) \otimes \mathbb{C} = \mathfrak{sl}_n(\mathbb{C})$,

$$[t, x] = \alpha(t) \cdot x, \text{for all } t \in \mathfrak{t}$$

Let $E_{i,j} \in \mathfrak{sl}_n(\mathbb{C}), i \neq j$ be the matrix whose only nonzero entry is a 1 in the (i, j) position. We have

$$[t, E_{i,j}] = (l_i - l_j)(t) \cdot E_{i,j}$$

where l_i is the functional which takes the value of the ith element on the diagonal. Since such $E_{i,j}$'s, together with those traceless diagonal matrices span $\mathfrak{sl}_n(\mathbb{C})$. We have that the roots must be of the form $l_i - l_j$, for $i \neq j, 1 \leqslant, i, j \geqslant n$. To conclude, we have

Proposition 2.1.5. (i)The subgroup of diagonal matrices T of $SU(n)$ is a maximal torus

(ii)Roots: $l_i - l_j, i \neq j, 1 \leqslant, i, j \leqslant n$.

(iii) Positive roots: $l_i - l_j, 1 \leqslant i < j \leqslant n$, for $\alpha = l_i - l_j, e_\alpha = E_{i,j}, t_\alpha = E_{i,i} - E_{j,j}$

We also have the following propositions for the other classical groups:

b) $G = SO(2n + 1)$

To describe the maximal torus of $SO(2n + 1)$, notice that we may decompose

$$\mathbb{R}^{2n+1} = \mathbb{R}^2 \oplus \cdots \mathbb{R}^2 \oplus \mathbb{R}$$

to obtain inclusions

$$T(n) = SO(2n) \times \cdots \times SO(2n) \subset SO(2n) \subset SO(2n + 1)$$

Proposition 2.1.6. (i)The subgroup $T(n)$ is a maximal torus of $SO(n)$

(ii)Roots: $\pm l_i \pm l_j, i \neq j, 1 \leqslant i < j \geqslant n$ and $\pm l_i, 1 \leqslant i \leqslant n$

(iii) Positive roots: $l_i \pm l_j, 1 \leqslant, i < j \geqslant n$ and $l_i, 1 \leqslant i \leqslant n$.

For $\alpha = l_i - l_j$, we have $e_\alpha = E_{i,j} - E_{j+n,i+n}, t_\alpha = (E_{i,i} - E_{n+i,n+i}) - (E_{j,j} - E_{n+j,n+j})$

For $\alpha = l_i + l_j$, we have $e_\alpha = E_{i,n+j} - E_{j,n+i}, t_\alpha = (E_{i,i} - E_{n+i,n+i}) + (E_{j,j} - E_{n+j,n+j})$

For $\alpha = l_i$, we have $e_\alpha = E_{i,i} - E_{n+i,n+i}$

d) $G = Sp(2n)$ The diagonal matrices $T(n)$ of the symplectic groups $Sp(n)$ is a maximal torus, and the root systems of symplectic groups $Sp(2n)$ is given by the following proposition.

Proposition 2.1.7. (i)The subgroup $T(n)$ is a maximal torus of $Sp(n)$

(ii)Roots: $\pm l_i \pm l_j, i \neq j, 1 \leqslant i < j \geqslant n$ and $\pm 2l_i, 1 \leqslant i \leqslant n$

(iii) Positive roots: $l_i \pm l_j, 1 \leqslant, i < j \geqslant n$ and $2l_i, 1 \leqslant i \leqslant n$.

For $\alpha = l_i - l_j$, we have $e_\alpha = E_{i,j} - E_{j+n,i+n}, t_\alpha = (E_{i,i} - E_{n+i,n+i}) - (E_{j,j} - E_{n+j,n+j})$

For $\alpha = l_i + l_j$, we have $e_\alpha = E_{i,n+j} - E_{j,n+i}, t_\alpha = (E_{i,i} - E_{n+i,n+i}) + (E_{j,j} - E_{n+j,n+j})$

For $\alpha = 2l_i$, we have $e_\alpha = E_{i,n+i}, t_\alpha = 2(E_{i,i} - E_{n+i,n+i})$

c) $G = SO(2n)$

The maximal torus of $SO(2n)$ is similar to the $SO(2n+1)$ case. And the root system is described by the following proposition.

Proposition 2.1.8. (i)The subgroup $T(n)$ is a maximal torus of $SO(2n)$

(ii)Roots: $\pm l_i \pm l_j, i \neq j, 1 \leqslant i < j \geqslant n$

(iii) Positive roots: $l_i \pm l_j, 1 \leqslant, i < j \geqslant n$

For $\alpha = l_i - l_j$, we have $e_\alpha = E_{i,j} - E_{j+n,i+n}, t_\alpha = (E_{i,i} - E_{n+i,n+i}) - (E_{j,j} - E_{n+j,n+j})$

For $\alpha = l_i + l_j$, we have $e_\alpha = E_{i,n+j} - E_{n,n+i}, t_\alpha = (E_{i,i} - E_{n+i,n+i}) + (E_{j,j} - E_{n+j,n+j})$

Remark 2.1.9. It's clear that for these classical Lie algebras, if $\alpha + \beta = \gamma$ for three roots, then we have $t_\alpha + t_\beta = t_\gamma$

Complex and Real Lie algebras, Invariant bilinear forms

For the computation in the next section, we will need to focus on the Lie algebra of the compact groups G. So we would prefer the real decomposition:

$$\mathfrak{g} = M_0 \oplus \bigoplus_{\alpha \in R^+} M_\alpha,$$

where each M_α is a real plane with $M_\alpha \otimes_{\mathbb{R}} \mathbb{C} = L_\alpha \oplus L_{-\alpha}$. In the previous section, we have determined the basis e_α for L_α in classical Lie algebras. Now we want to determine the corresponding real basis of M_α. It's not difficult to see that $x_\alpha = e_\alpha - e_{-\alpha}, x_{-\alpha} = \sqrt{-1}(e_\alpha - e_{-\alpha})$ is a basis for M_α, and that $M_0 = \sqrt{-1} \cdot L_0$. Let $h_\alpha = it_\alpha$ We have

$$[h_\alpha, x_\alpha] = [it_\alpha, e_\alpha - e_{-\alpha}] = i(2e_\alpha + 2e_{-\alpha}) = 2x_{-\alpha}$$

$$[h_\alpha, x_\alpha] = [it_\alpha, i(e_\alpha + e_{-\alpha})] = -1(2e_\alpha - 2e_{-\alpha}) = -2x_\alpha$$

$$[x_\alpha, x_{-\alpha}] = [e_\alpha - e_{-\alpha}, i(e_\alpha + e_{-\alpha})] = i[e_\alpha, e_{-\alpha}] - i[e_{-\alpha}, e_\alpha] = 2h_\alpha$$

More generally, we have

$$[h_\beta, x_\alpha] = [it_\beta, e_\alpha - e_{-\alpha}] = i[t_\beta, e_\alpha] - i[t_\beta, e_{-\alpha}] = i\alpha(h_\beta)e_\alpha + i\alpha(h_\beta)e_{-\alpha} = \alpha(h_\beta)x_{-\alpha}$$

$$[h_\beta, x_{-\alpha}] = [it_\beta, i(e_\alpha + e_{-\alpha})] = -[t_\beta, e_\alpha] - [t_\beta, e_{-\alpha}] = -\alpha(h_\beta)e_\alpha + \alpha(h_\beta)e_{-\alpha} = -\alpha(h_\beta)x_\alpha$$

Definition 2.1.10. A symmetric bilinear form $\langle\ ,\ \rangle$ on a Lie algebra $\mathfrak{g}$ is **invariant** if

$$\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$$

Example 2.1.11. The Killing form

$$B(X, Y) = Tr(adX \circ adY), \text{for } X, Y \in \mathfrak{g}$$

is an invariant form

It's not difficult to see that if $\mathfrak{g}$ is a simple Lie algebra, then invariant symmetric bilinear forms on $\mathfrak{g}$ are unique up to a real constant. For the classical Lie algebras: we want to take the following bilinear forms :

(i) $G = SU(n)$,

$$\langle A, B\rangle = -\frac{1}{2}Tr(AB)$$

(i) $G = SO(2n)$,

$$\langle A, B\rangle = -\frac{1}{4}Tr(AB)$$

(i) $G = SU(2n+1)$,

$$\langle A, B\rangle = -\frac{1}{4}Tr(AB)$$

(i) $G = Sp(n)$,

$$\langle A, B\rangle = -\frac{1}{4}Tr(AB)$$

Lemma 2.1.12. $\mathfrak{g} = M_0 \oplus \bigoplus_{\alpha \in R^+} M_\alpha$ is an orthogonal decomposition

Remark 2.1.13. The reason we have different coefficients is the following: notice that all these classical Lie algebras have roots of the form $\alpha = l_i - l_j$, we want to normalize the bilinear forms, so that $\|x_\alpha\| = 1$.

2.2 Differential geometry on homogeneous spaces

We review the basic differential geometry of homogeneous spaces in this section. The main reference for this part is [7]. Let $M = G/H$ be a homogeneous space, where G is a Lie group and H is a closed subgroup of G. The coset H is called the origin of M and will be denoted by o. The group G acts transitively on M in a natural manner; an element $f \in G$ maps a coset $f'H$ into the coset $ff'H$. In particular the subgroup H has the origin o as a fixed point and the linear isotropy representation is by definition the homomorphism of H into the group of linear transformations of $T_o(M)$ which assigns to each $h \in H$ the differential of h at o. A homogeneous space G/H is called **reductive** if the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{h}$ of H and an $ad(H)$-invariant subspace $\mathfrak{m}$, that is, if

(1)$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$

(2)$ad(H)\mathfrak{m} \subset \mathfrak{m}$ We define the notations $[,]_{\mathfrak{h}}$, and $[,]_{\mathfrak{m}}$ by

$$[X, Y] = [X, Y]_{\mathfrak{h}} + [X, Y]_{\mathfrak{m}},\ [X, Y]_{\mathfrak{h}} \subset \mathfrak{h},\ [X, Y]_{\mathfrak{m}} \subset \mathfrak{m}$$

.

Remark 2.2.1. In general the decomposition is not unique. For instance, if the Lie group G is abelian, then any decomposition of $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ as a vector space satisfies the conditions. If $\mathfrak{g}$ is semi-simple, we can define a canonical decomposition in later sections.

Remark 2.2.2. If G is compact, then every homogeneous spaces G/H must be reductive, since any finite dimensional representation of a compact Lie group is completely reducible.

We first give the definition of isotropy groups and isotropy representations.

Definition 2.2.3. Let G be a group of smooth transformations on a manifold M. Let $x \in M$, the isotropy group of the point x G_x is the subgroup of G that leaves x fixed. The isotropy representation

$$Is_x : G_x \to (T_xM)$$

associates to each $h \in G_x$ the differential of h at x.

We have the following lemma, which describes the isotropy representation explicitly :

Lemma 2.2.4. Let $M = G/H$ be a reductive homogeneous spaces, the isotropy representation Is_o can be identified with the representation

$$h \mapsto Ad_G(G_x)|_{\mathfrak{m}}$$

The isotropy representation and linear isotropy group play an important role in the study of invariant objects on homogeneous spaces. The invariant tensor fields on a homogeneous space are in one-to-one correspondence with the tensors on that are invariant with respect to the isotropy representation. In particular, invariant Riemannian metrics, connections and curvatures on homogeneous spaces can be described explicitly by the following propositions.

Proposition 2.2.5. If $M = G/H$ is reductive with an $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, then there is a natural one-to-one correspondence between the G-invariant Riemannian metrics g on $M = G/H$ and the $Ad(H)$-invariant positive definite symmetric bilinear forms $\langle\ ,\ \rangle$ on $\mathfrak{m}$. The correspondence is given by

$$\langle X, Y\rangle = g(X, Y)_o, \text{ for } X, Y \in \mathfrak{m}$$

The following theorem describes all metric compatible connections on the tangent space of the origin o.

Theorem 2.2.6. *Let g be a G-invariant Riemannian metric on a reductive homogeneous space $M = G/H$ with decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then there is a one-to-one correspondence between the set of G-invariant connections compatible with g and the set of linear mappings $\Lambda : \mathfrak{m} \to so(\mathfrak{m})$ such that*

$$\Lambda_{\mathfrak{m}}(adh(Z)) = ad(\lambda(h))(\Lambda_{\mathfrak{m}}(Z)) \text{ for } X \in \mathfrak{m} \text{ and } h \in H$$

Where λ is the isotropy representation $H \to SO(\mathfrak{m})$

In particular, we can give the Levi-Civita connection explicitly.

Theorem 2.2.7. *Let $M = G/H$ be a reductive homogeneous space with an $ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h}+\mathfrak{m}$ and an $Ad(H)$-invariant positive definite symmetric bilinear form $\langle, \rangle$ on $\mathfrak{m}$. Let g be the G-invariant metric corresponding to $\langle\ ,\ \rangle$. Then the Levi-Civita connection for g is given by*

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y)$$

where $U(X, Y)$ is the symmetric bilinear mapping from $\mathfrak{m} \times \mathfrak{m}$ into $\mathfrak{m}$ defined by

$$2\langle U(X, Y), Z\rangle = \langle X, [Z, Y]_{\mathfrak{m}}\rangle + \langle [Z, X]_{\mathfrak{m}}, Y\rangle \text{ for } X, Y, Z \in \mathfrak{m}$$

The following theorem gives the main tool for our calculation of the first Pontrjagin form

Theorem 2.2.8. *The torsion tensor T and the curvature tensor R of the invariant connection corresponding to $\Lambda_{\mathfrak{m}}$ can be expressed at the origin as follows:*

$$T(X,Y)_o = \Lambda(X)Y - \Lambda(Y)X - [X,Y]_{\mathfrak{m}}$$

$$R(X,Y)_o = [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \lambda([X,Y]_{\mathfrak{h}})$$

2.2.1 Examples of computation of the curvature tensor on homogeneous spaces

In this subsection, we will apply the theory in the previous two sections to a special family of homogeneous spaces, namely, we will consider homogeneous spaces of the form G/T^k, where T^k is a torus of dimension k. It's known that all maximal tori of G are conjugate to each other. Hence without loss of generality, we can assume that T^k is contained in a fixed maximal torus T^{max} of G. In this case, we will choose the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{t}$ in the following way: let

$$\mathfrak{m} = \mathfrak{t}' \bigoplus_{\alpha} (x_\alpha \oplus x_{-\alpha})$$

where α runs over all positive roots and $\mathfrak{t}' \oplus \mathfrak{t}$ is the Lie algebra of the fixed maximal torus and $\mathfrak{t}'$ is the orthogonal complement to $\mathfrak{t}$ with respect to the Killing form. The above decomposition of $\mathfrak{m}$ is an orthogonal decomposition of $\mathfrak{m}$. For later calculation of curvature tensors, we pick an orthonomal basis $\{t_1, \cdots, t_{n-k}\}$ of $\mathfrak{t}'$, where n is the rank of G. Hence $\{x_\alpha, x_{-\alpha}\} \bigcup \{t_1, \cdots, t_{n-k}\}$ completes $\{x_\alpha, x_{-\alpha}\}$ to an orthonormal basis of $\mathfrak{m}$.

Each pair $\{x_\alpha, x_{-\alpha}\}$ spans a two-dimensional irreducible (real) representation of T^k, and $\mathfrak{t}'$ is a trivial representation of T^k, since the isotropy representation is just the restriction of the adjoint representation of G to T^k. Now we define a positive definite bilinear form on $\mathfrak{m}$ in the following way: we know that $\mathfrak{m} = \mathfrak{t}' \bigoplus_\alpha (x_\alpha \oplus x_{-\alpha})$ is an orthogonal decomposition with respect to the Killing form. Let

$$\begin{cases} \langle x, y \rangle_\lambda = \langle x, y \rangle & x, y \in \bigoplus_\alpha (x_\alpha \oplus x_{-\alpha}) \\ \langle t_1, t_2 \rangle_\lambda = \lambda \langle t_1, t_2 \rangle & t_1, t_2 \in \mathfrak{t}' \\ \langle x, t \rangle_\lambda = 0 & x \in \bigoplus_\alpha (x_\alpha \oplus x_{-\alpha}), t \in \mathfrak{t}' \end{cases} \tag{2.2.1}$$

In other words, we are only deforming the metric on $\mathfrak{t}'$ with a positive parameter λ. In order to define a corresponding Riemannian metric on G/T^k, by Proposition 2.2.5,

we have to show that this bi-linear form is invariant with respect to the adjoint action of T^k. This is obvious: we only have to check the invariance on $\mathfrak{t}'$ and $\bigoplus_\alpha(x_\alpha \oplus x_{-\alpha})$ respectively, since they are still orthogonal to each other with respect to this new bi-linear form. For $\mathfrak{t}'$, this is trivial since T^k acts trivially. The metric on $\bigoplus_\alpha(x_\alpha \oplus x_{-\alpha})$ is the same as the Killing form, which is even invariant with respect to the adjoint action of G. Hence we can define:

Definition 2.2.9. Let g_λ be the invariant Riemannian metric on G/T^k corresponding to the positive definite bi-linear form $\langle\ ,\ \rangle_\lambda$ on $\mathfrak{m}$. In particular, g_1 is just the metric corresponding to the Killing form, which is bi-invariant.

In 3.2, we defined the symmetric bi-linear map $\mathfrak{m} \times \mathfrak{m}$ into $\mathfrak{m}$ given by

$$2\langle U(x,y),z\rangle_\lambda = \langle x,[z,y]_\mathfrak{m}\rangle_\lambda + \langle [z,x]_\mathfrak{m}, y\rangle_\lambda \text{ for } x,y,z \in \mathfrak{m}$$

The following lemma describes this bilinear mapping

Lemma 2.2.10. For $M = G/T^k$ with the Riemannian metric in Definition 2.2.9, we have that

$$\begin{cases} U(x_\alpha, x_\beta) = 0 & \alpha,\beta \text{ two roots} \\ U(x_\alpha, t) \in \mathrm{Span}\{x_{-\alpha}\} & \alpha \text{ a root}, t \in \mathfrak{t}' \\ U(t_1,t_2) = 0 & t_1, t_2 \in \mathfrak{t}' \end{cases} \tag{2.2.2}$$

Proof. Let α and β be two roots. Then to determine $U(x_\alpha, x_\beta)$. We need to consider $\langle U(x_\alpha,x_\beta)\rangle_\lambda x$ for all basis vectors of $\mathfrak{m}$. For $t \in \mathfrak{t}$, we have $\langle U(x_\alpha,x_\beta),t\rangle_\lambda = \frac{1}{2}(\langle [t,x_\alpha]_\mathfrak{m}, x_\beta\rangle_\lambda + \langle\langle [t,x_\beta]_\mathfrak{m}, x_\alpha\rangle_\lambda)$. And it's obvious that $[t,x_\alpha]_\mathfrak{m} = [t,x_\alpha]$, $[t,x_\beta]_\mathfrak{m} = [t,x_\beta]$. By the invariance of the metric , we have the inner product must be 0.

For $x = x_\gamma$, if $\gamma \neq -\alpha, -\beta$, we still have that $[x_\gamma,x_\alpha]_\mathfrak{m} = [x_\gamma,x_\alpha]$, $[x_\gamma,x_\beta]_\mathfrak{m} = [x_\gamma,x_\beta]$.

For $x = x_{-\alpha}$, then we have that $[x_\gamma,x_\alpha]_\mathfrak{m} = [x_{-\alpha},x_\alpha]_\mathfrak{m} \in \mathfrak{t}'$. Hence we have $\langle [x_{-\alpha},x_\alpha]_\mathfrak{m}, x_\beta\rangle_\lambda = 0$, similarly $\langle [x_{-\alpha},x_\beta]_\mathfrak{m}, x_\alpha\rangle_\lambda = 0$

To see that $U(x_\alpha,t) \in \mathrm{Span}\{x_{-\alpha}\}$. It's enough to apply the fact that x_α and x_β are orthogonal to each other, if α and β are different roots.

□

Let's compute some curvature transformations of G/T^k, which will be important for the computation of the first Pontrjagin forms of G/T^k. We will be concerned

mainly with curvature transformations of the form $R(x_\alpha, x_{-\alpha})x$ for α a positive root of $\mathfrak{g}$. Recall that we fix a basis of $\mathfrak{m}$ to be $\{x_\gamma, x_{-\gamma}\} \cup \{t_1, \cdots, t_{n-k}\}$. By Theorem 2.2.8 we have

(1). $R(x_\alpha, x_{-\alpha})t_i$

$= \Lambda_{\mathfrak{m}}(x_\alpha)(\Lambda_{\mathfrak{m}}(x_{-\alpha})t_i) - \Lambda_{\mathfrak{m}}(x_{-\alpha})(\Lambda_{\mathfrak{m}}(x_\alpha)t_i) - \Lambda([x_\alpha, x_{-\alpha}]_{\mathfrak{m}})(t_i) - [[x_\alpha, x_{-\alpha}]_{\mathfrak{t}}, t_i]$

$= \Lambda_{\mathfrak{m}}(x_\alpha)(\frac{1}{2}[x_{-\alpha}, t_i]_{\mathfrak{m}} + U((x_{-\alpha}, t_i))) - \Lambda_{\mathfrak{m}}(x_{-\alpha})(\frac{1}{2}[x_\alpha, t_i]_{\mathfrak{m}} + U((x_\alpha, t_i)))$

$= \Lambda_{\mathfrak{m}}(x_\alpha)(c_1 x_\alpha) - \Lambda_{\mathfrak{m}}(x_{-\alpha})(c_2 x_{-\alpha})$

$= 0$

(2). For positive root $\gamma \neq \pm\alpha$, we have

$R(x_\alpha, x_{-\alpha})x_\gamma$

$= \frac{1}{4}[x_\alpha, [x_{-\alpha}, x_\gamma]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{4}[x_{-\alpha}, [x_\alpha, x_\gamma]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{2}[[x_\alpha, x_{-\alpha}]_{\mathfrak{m}}, x_\gamma]_{\mathfrak{m}}$
$-[[x_\alpha, x_{-\alpha}]_{\mathfrak{h}}, x_\gamma] - U([x_\alpha, x_{-\alpha}]_{\mathfrak{m}}, x_\gamma)$

$= \frac{1}{4}[x_\alpha, [x_{-\alpha}, x_\gamma]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{4}[x_{-\alpha}, [x_\alpha, x_\gamma]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{2}[[x_\alpha, x_{-\alpha}], x_\gamma]$
$-\frac{1}{2}[[x_\alpha, x_{-\alpha}]_{\mathfrak{h}}, x_\gamma] - U([x_\alpha, x_{-\alpha}]_{\mathfrak{m}}, x_\gamma)$

$= -\frac{1}{4}[[x_\alpha, x_{-\alpha}], x_\gamma] - \frac{1}{2}[[x_\alpha, x_{-\alpha}]_{\mathfrak{h}}, x_\gamma] - U([x_\alpha, x_{-\alpha}]_{\mathfrak{m}}, x_\gamma)$

We have by Lemma 2.2.10,

$U([x_\alpha, x_{-\alpha}]_{\mathfrak{m}}, x_\gamma)$

$= 2\ U((h_\alpha)_{\mathfrak{m}}, x_\gamma)$

$= 2\ \langle U((h_\alpha)_{\mathfrak{m}}, x_\gamma), \frac{x_{-\gamma}}{\|x_\gamma\|}\rangle_\lambda\ \frac{x_{-\gamma}}{\|x_\gamma\|},$

$= \frac{1}{\|x_\gamma\|^2}(\langle (h_\alpha)_{\mathfrak{m}}, [x_{-\gamma}, x_\gamma]_{\mathfrak{m}}\rangle_\lambda + \langle [x_{-\gamma}, (h_\alpha)_{\mathfrak{m}}], x_\gamma\rangle_\lambda)x_{-\gamma}$

$= (-2\frac{\langle (h_\alpha)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda}{\|x_\gamma\|^2})x_\gamma + [(h_\alpha)_{\mathfrak{m}}, x_\gamma]$

Hence we have $R(x_\alpha, x_{-\alpha})x_\gamma$

$= -\frac{1}{4}[[x_\alpha, x_{-\alpha}], x_\gamma] - \frac{1}{2}[[x_\alpha, x_{-\alpha}], x_\gamma] + 2\frac{\langle (h_\alpha)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda}{\|x_\gamma\|^2}x_{-\gamma}$

$= -\frac{3}{4}[[x_\alpha, x_{-\alpha}], x_\gamma] + 2\frac{\langle (h_\alpha)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda}{\|x_\gamma\|^2}x_{-\gamma}$

(3). $R(x_\alpha, x_{-\alpha})x_\alpha$

$$
\begin{aligned}
&= \Lambda_{\mathfrak{m}}(x_\alpha)(\Lambda_{\mathfrak{m}}(x_{-\alpha})x_\alpha) - \Lambda_{\mathfrak{m}}(x_{-\alpha})(\Lambda_{\mathfrak{m}}(x_\alpha)x_\alpha) - \Lambda([x_\alpha, x_{-\alpha}]_{\mathfrak{m}})(x_\alpha) - [[x_\alpha, x_{-\alpha}]_{\mathfrak{t}}, x_\alpha] \\
&= \Lambda_{\mathfrak{m}}(x_\alpha)(\frac{1}{2}[x_{-\alpha}, x_\alpha]_{\mathfrak{m}}) - \Lambda((2h_\alpha)_{\mathfrak{m}})(x_\alpha) - [(2h_\alpha)_{\mathfrak{t}}, x_\alpha] \\
&= \Lambda_{\mathfrak{m}}(x_\alpha)(-(h_\alpha)_{\mathfrak{m}}) - \Lambda((2h_\alpha)_{\mathfrak{m}})(x_\alpha) - [(2h_\alpha)_{\mathfrak{t}}, x_\alpha] \\
&= \frac{1}{2}[x_\alpha, -(h_\alpha)_{\mathfrak{m}}] + U(x_\alpha, -(h_\alpha)_{\mathfrak{m}}) - (\frac{1}{2}[2(h_\alpha)_{\mathfrak{m}}, x_\alpha] + U(2(h_\alpha)_{\mathfrak{m}}, x_\alpha)) - [(2h_\alpha)_{\mathfrak{t}}, x_\alpha] \\
&= -\frac{1}{2}[(h_\alpha)_{\mathfrak{m}}, x_\alpha] - 3U(x_\alpha, (h_\alpha)_{\mathfrak{m}}) - [(2h_\alpha)_{\mathfrak{t}}, x_\alpha] \\
&= -\frac{1}{2}[h_\alpha, x_\alpha] - \frac{3}{2}[(h_\alpha)_{\mathfrak{t}}, x_\alpha] - 3U(x_\alpha, (h_\alpha)_{\mathfrak{m}})
\end{aligned}
$$

We have that

$$
\begin{aligned}
&\langle 3\ U(x_\alpha, (h_\alpha)_{\mathfrak{m}}), x_{-\alpha}\rangle_\lambda \\
&= \frac{3}{2}(\langle [x_{-\alpha}, (h_\alpha)_{\mathfrak{m}}], x_\alpha\rangle_\lambda + \langle [x_{-\alpha}, x_\alpha]_{\mathfrak{m}}, (h_\alpha)_{\mathfrak{m}}\rangle_\lambda) \\
&= \frac{3}{2}(\langle [x_{-\alpha}, (h_\alpha)_{\mathfrak{m}}], x_\alpha\rangle_\lambda + \langle -2(h_\alpha)_{\mathfrak{m}}, (h_\alpha)_{\mathfrak{m}}\rangle_\lambda)
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&R(x_\alpha, x_{-\alpha})x_\alpha \\
&= -\frac{1}{2}[h_\alpha, x_\alpha] - \frac{3}{2}[(h_\alpha)_{\mathfrak{t}}, x_\alpha] - \frac{3}{2}(\langle [x_{-\alpha}, (h_\alpha)_{\mathfrak{m}}], x_\alpha\rangle_\lambda + \langle -2(h_\alpha)_{\mathfrak{m}}, (h_\alpha)_{\mathfrak{m}}\rangle_\lambda)x_{-\alpha} \\
&= -2[h_\alpha, x_\alpha] + 3\langle (h_\alpha)_{\mathfrak{m}}, (h_\alpha)_{\mathfrak{m}}\rangle_\lambda x_{-\alpha}
\end{aligned}
$$

To generalize, we have

Lemma 2.2.11.

$$
\begin{cases}
R(x_\alpha, x_{-\alpha})t = 0 & \text{for } \ t \in \mathfrak{t}' \\
R(x_\alpha, x_{-\alpha})x_\gamma = -\frac{3}{4}[[x_\alpha, x_{-\alpha}], x_\gamma] + 2\frac{\langle (h_\alpha)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda}{\|x_\gamma\|^2}x_{-\gamma} & \text{if } \ \gamma \neq \pm\alpha \\
R(x_\alpha, x_{-\alpha})x_\alpha = -2[h_\alpha, x_\alpha] + 3\langle (h_\alpha)_{\mathfrak{m}}, (h_\alpha)_{\mathfrak{m}}\rangle_\lambda x_{-\alpha}
\end{cases} \tag{2.2.3}
$$

2.3 Chern-Weil theory: from curvature to Pontrjagin forms

The purpose of this section is to give a brief introduction to geometric aspects of the theory of characteristic classes, which was developed by Shiing-shen Chern and André Weil.

2.3.1 Connections on vectors bundles and their curvature

Let $E \to M$ be a smooth real vector bundle over a smooth compact manifold M. We denote by $\Omega^*(M, E)$ the space of smooth sections of the tensor product vector bundle $\Lambda^*(T^*M) \otimes E$,

$$\Omega^*(M, E) := \Gamma(\Lambda^*(T^*M) \otimes E)$$

Definition 2.3.1. A connection ∇^E on E is an $\mathbb{R}$-linear operator $\nabla^E : \Gamma(E) \to \Omega^1(M, E)$, such that for any $f \in C^\infty(M)$ and $X \in \Gamma(E)$, the Leibniz rule holds:

$$\nabla^E(fX) = df \otimes X + f\nabla^E(X)$$

The existence of a connection on a vector bundle can be proved easily by using the method of partition of unity. All connections on it form an infinite dimensional affine space.

One can extend ∇^E to

$$\nabla^E : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$$

We can hence take the composition of ∇^E with it self, which gives rise to the curvature of ∇^E.

Definition 2.3.2. The curvature R^E of a connection ∇^E is defined by

$$R^E : \Gamma(E) \to \Omega^2(M, E)$$

It's not difficult to check that R^E is $C^\infty(M)$-linear, i.e., for any $f \in C^\infty(M)$, $R^E(fX) = fR^E(X)$. Therefore, we can think of R^E as an element of $\Omega^2(End(E))$. More explicitly, let X, Y be two smooth sections of TM, then $R(X, Y) \in \Gamma(End(E))$ by pairing X, Y with $\Omega^2(M)$. For $Z \in \Gamma(E)$, $R(X, Y)$ is given by

$$R(X, Y)Z = \nabla^E_X \nabla^E_Y Z - \nabla^E_Y \nabla^E_X Z - \nabla^E_{[X,Y]} Z$$

Definition 2.3.3. Let ∇ be a connection on a real vector bundle V of rank k. We set

$$\begin{aligned} p(\Omega) &:= det(I + \frac{1}{2\pi}\Omega) \\ &= 1 + p_1(\Omega) + \cdots + p_r(\Omega), \end{aligned}$$

where $p_i(\Omega) \in \Omega^{4i}(M)$.

In particular, we have

$$p_1(\Omega) = \frac{Tr(\Omega^2)}{4\pi^2} = \frac{1}{4\pi^2} \sum det(\Omega_{I,I})$$

. It's now clear that our object is to show the nonvanishing of $\sum det(\Omega_{I,I})$ for certain Riemannian homogeneous spaces.

2.3.2 From curvature to Pontrjagin forms: an example

In this paper, to show the non-vanishing of p_1 of $M = G/H$, we will only focus on the coefficient of the terms $dx_\alpha dx_{-\alpha} dx_\beta dx_{-\beta}$,for each pair of positive roots α and β of the Lie algebra $\mathfrak{g}$, or equivalently, the pairing of p_1 with $x_\alpha \otimes x_{-\alpha} \otimes x_\beta \otimes x_{-\beta}$. Notice that according to Chern-Weil theory, the first Pontrjagin form p_1 is the sum of determinants of all 2×2 minor of the curvature matrix Ω. Notice that the four form $dx_\alpha dx_{-\alpha} dx_\beta dx_{-\beta}$ can be the wedge product of the following three pairs,

$$dx_\alpha dx_{-\alpha} \wedge dx_\beta dx_{-\beta},\ dx_\alpha dx_\beta \ \wedge\ dx_{-\alpha} dx_{-\beta} \text{ and } dx_\alpha dx_{-\beta} \wedge dx_{-\alpha} dx_\beta \text{ and}$$

Let's call the coefficient of the term we can get from the above 3 ways by $f_{\alpha,\beta}$,

$g_{\alpha,\beta}$ and $h_{\alpha,\beta}$. It's obvious from the definition that

$$f_{\alpha,\beta} + g_{\alpha,\beta} + h_{\alpha,\beta} = p_1(x_\alpha \otimes x_{-\alpha} \otimes x_\beta \otimes x_{-\beta})$$

.

A special case is when the homogeneous space is trivial, i.e. , when $H = \{1\}$, $G/H = G$.

We will fix an orthonormal basis of the tangent space of $M = G/T^k$ as we did in the previous section. We will denote $f_{\alpha,\beta}$, $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$ by $f_{G,\alpha,\beta}$, $g_{G,\alpha,\beta}$ and $h_{G,\alpha,\beta}$. We have an orthogonal decomposition

$$\mathfrak{m} = \bigoplus_\alpha (x_\alpha \oplus x_{-\alpha}) \oplus \mathfrak{t}'$$

We pick any orthonormal basis of $\mathfrak{t}'$, then together with $\{x_\alpha, x_{-\alpha}\}$, we can get an orthonomal basis of $\mathfrak{m}$.

We try to calculate $f_G(x_\alpha, x_{-\alpha}, x_\beta, x_{-\beta})$ for the group $G = SU(n)$ with the bi-invariant metric. We have seen that $R_G(x,y)z = -\frac{1}{4}[[x,y],z]$. We have

$$\begin{cases} R(x_\alpha, x_{-\alpha})t_i = 0 \\ R(x_\alpha, x_{-\alpha})x_\gamma \in \text{Span}\{x_{-\gamma}\} \end{cases} \tag{2.3.1}$$

So we only need to focus on those 2×2 submatrices corresponding to the pair of basis of vectors $\{x_\gamma, x_{-\gamma}\}$. Let's take the group $SU(n)$ as an example, all the roots of $\mathfrak{su}(n)$ are of the form $l_i - l_j$ for $i, j \leqslant n, i \neq j$. Let $\alpha = l_1 - l_2, \beta = l_1 - l_3$. Then $-\frac{1}{4}[[x_\alpha, x_{-\alpha}], x_{-\gamma}] \neq 0$ if and only if $\langle h_\alpha, h_\gamma \rangle_\lambda \neq 0$, hence $\gamma = \pm(l_1 - l_k)$, or $\pm(l_2 - l_k)$.

Hence $-\frac{1}{4}[[x_\alpha, x_{-\alpha}], x_{-\gamma}]$ and $-\frac{1}{4}[[x_\beta, x_{-\beta}], x_{-\gamma}]$ can be simultaneously nonzero only when $\gamma = \pm(l_1 - l_k)$. We have

$$-\frac{1}{4}[[x_\alpha, x_{-\alpha}], x_{(l_1-l_k)}] = -\frac{1}{2}x_{-(l_1-l_k)}, -\frac{1}{4}[[x_\alpha, x_{-\alpha}], x_{-(l_1-l_k)}] = \frac{1}{2}x_{(l_1-l_k)}$$

$$-\frac{1}{4}[[x_\beta, x_{-\beta}], x_{(l_1-l_k)}] = -\frac{1}{2}x_{-(l_1-l_k)}, -\frac{1}{4}[[x_\beta, x_{-\beta}], x_{-(l_1-l_k)}] = \frac{1}{2}x_{(l_1-l_k)}$$

The corresponding 2×2 submatrix is

$$\Omega_{x_\gamma, x_{-\gamma}} = \begin{pmatrix} 0 & -\frac{1}{2}dx_\alpha dx_{-\alpha} - \frac{1}{2}dx_\beta dx_{-\beta} \\ \frac{1}{2}dx_\alpha dx_{-\alpha} + \frac{1}{2}dx_\beta dx_{-\beta} & 0 \end{pmatrix}$$

And the determinant is $\frac{1}{2}dx_\alpha dx_{-\alpha} dx_\beta dx_{-\beta}$. Notice that $k = 2, \cdots, n$, It's not difficult to see that

$f_{G,\alpha,\beta} = \frac{1}{2}(n+1)$

Similarly, we have that $g_G = h_G = -\frac{1}{4}(n+1)$.

We have seen in the previous section that all Lie algebras of classical type have roots $l_i - l_j$. It's straightforward to generalize the above computation to all classical Lie groups

Lemma 2.3.4. (1) For $G = SU(n)$, $f_{G,\alpha,\beta} = \frac{1}{2}(n+1)$

(2) For $G = Sp(n)$, $f_{G,\alpha,\beta} = n + 3$

(3) For $G = SO(2n)$, $f_{G,\alpha,\beta} = n + 1$

(4) For $G = SO(2n+1)$, $f_{G,\alpha,\beta} = n + \frac{3}{2}$

Chapter 3

String structure on G/T^k

In section 3.1, we give the definition of string structures on principal $Spin(n)$-bundles. In particular, a $Spin$-manifold M is called string if the principal $Spin$-bundle associated to the tangent bundle TM has a string structure. In section 2.2, we describe string structures on homogeneous spaces G/T^k. We first give the proof of the known fact that homogeneous spaces G/T^k are stably framed, which gives the existence of string structures on such spaces. We then prove Theorem 3.1.5, which is our first main result.

3.1 String structures on $Spin(n)$-bundles

To define the string group, we first recall the Whitehead tower of a space X. The Whitehead tower of a space X consists of a sequence of spaces $X\langle n+1\rangle \to X\langle n\rangle \to \cdots \to X$, such that the arrows induce isomorphisms $\pi_i(X\langle n\rangle) \cong \pi_i(X)$ for $i \geqslant n$, and $\pi_i(X\langle n\rangle) = 0$ for $i < n$. The first spaces in the Whitehead tower of $O(n)$ are the Lie groups $SO(n)$ and $Spin(n)$.

Definition 3.1.1. $String(n)$ is the topological group that has the homotopy type of $O(n)\langle 4\rangle$. It is the homotopy fiber of the canonical map $Spin(n) \to K(\mathbb{Z}, 3)$.

Since $\pi_i(SO(n)) = 0$ for $i = 4, 5, 6$, $String(n)$ is homotopy equivalent to $O(n)\langle 7\rangle$. We have the following sequences

$$O(n) \leftarrow SO(n) \leftarrow Spin(n) \leftarrow String(n)$$

$$BO(n) \leftarrow BSO(n) \leftarrow BSpin(n) \leftarrow BString(n)$$

$String(n)$ is not a finite dimensional Lie group since $H^3(String(n), \mathbb{Z}) = 0$. There are by now several constructions of $String(n)$, including those based on Von Neumann algebras [14], 2-groups, and by conformal field theory.

Definition 3.1.2. A String structure on a principal $O(n)$-bundle $P \to M$ is a homotopy lifting $\tilde{f}$ of the classifying map $f : M \to BO(n)$ to $BString(n)$.

In particular, a manifold M together with a lifting of the classifying map $\tau : M \to BO(n)$ of the tangent bundle to $BString(n)$ is called a "string manifold", or we may say M is string for the existence of a string structure of the tangent bundle.

Proposition 3.1.3. A string structure on a principal spin bundle exists if and only the characteristic class $\frac{p_1}{2} = 0 \in H^4(M; \mathbb{Z})$, and the space of string structures is a $H^3(M; \mathbb{Z})$-torsor.

Proof. The statements following from the fact that in the universal case, the generator of $H^3(Spin(n); \mathbb{Z})$ transgresses to $\frac{p_1}{2} \in H^4(BSpin(n); \mathbb{Z})$ in the Leray-Serre spectral sequence.

□

3.1.1 Existence and uniqueness of string structures on G/T^k

We first show the existence of a string structure on G/T^k by showing that the tangent bundle of G/T^k is actually stably trivial, which is well-known, see [3],[2] and [10]

Proposition 3.1.4. The tangent bundle of G/T^k is stably trivial, in other words, there exists a trivial bundle E on $M = G/T^k$, such that $E \oplus TM$ is isomorphic to a trivial vector bundle. In particular, $M = G/T^k$ is a string manifold.

Theorem 3.1.5. *Let G be a simply connected, compact and simple Lie group. Then for any embedded torus T^k, $H^2(G/T^k; \mathbb{Z}) = \mathbb{Z}^k$,$H^3(G/T^k; \mathbb{Z}) = 0$, in particular, there is a unique string structure on G/T^k*

Proof. G is simple and simply connected, and hence $H^3(G; \mathbb{Z}) = \mathbb{Z}$,$H^4(G; \mathbb{Q}) = 0$. We can consider the following tower of S^1-principle bundles.

$S^1 \to G/T^{k-1} \to G/T^k$, where $1 \leqslant k \leqslant n = \text{rank } G$.

From the long exact sequence of homotopy groups

$... \to \pi_2(S^1) \to \pi_2(G/T^{k-1}) \to \pi_2(G/T^k) \to \pi_1(S^1) \to \pi_1(G/T^{k-1}) \to \pi_1(G/T^k) \to \pi_0(S^1) \to ...$

we can see that G/T^k is always simply connected. From the middle short exact sequence $0 = \pi_2(S^1) \to \pi_2(G/T^{k-1}) \to \pi_2(G/T^k) \to \pi_1(S^1) \to 0$ we can show inductively that $\pi_2(G/T^k) = \mathbb{Z}^k$. By Hurewicz Theorem, $H^2(G/T^k;\mathbb{Z}) = \mathbb{Z}^k$. Now we need the following

Lemma 3.1.6. $\dim(H^4(G/T^n;\mathbb{Q})) = (n^2+n-2)/2$

Proof. This proof is a simple application of the Bruhat decomposition of the flag variety G^c/B. For more details about Bruhat decomposition, we refer to [6]. It's known that $G/T^n \cong G^c/B$, where G^c is the complexified Lie group of G and B is the Borel subgroup. It's known that G^c/B has a cell structure, which has only even dimensional cells. So the dimension of $H^4(G/T^n;\mathbb{Q})$ is the same as the number of 4 dimensional cells of G^c/B. G^c/B is a disjoint union of the *Bruhat cells* X_W, with W varying over the Weyl group. X_W is isomorphic to the affine space $\mathbb{C}^{l(W)}$, where $l(W)$ is the minimum number of reflections in simple roots whose product is W, i.e., the *length* of W. We only need to find all elements in the Weyl group of length 2. The Weyl group has a presentation $r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1$, where r_i's are the reflection of simple roots and $m_{ij} = 2,3,4,6$,depending on whether roots i,j are unconnected, connected by a simple edge, double edge,triple edge. So actually $r_i r_j = r_j r_i$ if and only if roots i,j are unconnected. So the number of elements of length 2 is equal to $\binom{n}{2}$+ number of edges in the Dynkin diagram). It's easy to see that the number of edges in the Dynkin diagram of any simple Lie algebra of rank n is equal to $n-1$. So we have $\dim(H^4(G/T^n;\mathbb{Q})) = \frac{n(n-1)}{2} + n - 1 = (n^2+n-2)/2$ □

We first show inductively that $H^3(G/T^k;\mathbb{Z})$ is always torsion free. For k =rank(G),this is obvious since G/T^n has no 3 dimensional cells. By looking at the Gysin sequence associated to the S^1 bundle

$S^1 \to G/T^{k-1} \to G/T^k$,

$$H^1(G/T^k;\mathbb{Z}) \rightarrowtail H^3(G/T^k;\mathbb{Z}) \rightarrowtail H^3(G/T^{k-1};\mathbb{Z}) \rightarrowtail H^2(G/T^k;\mathbb{Z}) \xrightarrow{\cup c_1} H^4(G/T^k;\mathbb{Z})$$

The arrow d in the exact sequence is given by cup product with the Euler class, which is just c_1 of the circle bundle. Notice that $H^1(G/T^k;\mathbb{Z}) = 0$ since G/T^k is simply connected, and we have a short exact sequence

$$0 \to H^3(G/T^k;\mathbb{Z}) \to H^3(G/T^{k-1};\mathbb{Z}) \to ker(\cup c_1) \to 0$$

$ker(\cup c_1)$ is a subgroup of $H^2(G/T^k;\mathbb{Z}) = \mathbb{Z}^k$, so $ker(\cup c_1)$ is a free abelian group. So the middle term $H^3(G/T^{k-1};\mathbb{Z})$ is also free.

Hence to show that $H^3(G/T^k;\mathbb{Z}) = 0$, we can just consider rational cohomology and show that $\dim(H^3(G/T^k;\mathbb{Q}))= 0$.

Now we look at the Gysin sequence of cohomology with rational coefficients

$$\cdots \longrightarrow H^3(G/T^{k-1};\mathbb{Q}) \longrightarrow H^2(G/T^k;\mathbb{Q}) \xrightarrow{\cup c_1} H^4(G/T^k;\mathbb{Q}) \longrightarrow H^4(G/T^{k-1};\mathbb{Q}) \longrightarrow \cdots$$

we can see that

$$\begin{aligned}\dim(H^4(G/T^{k-1})) &\geqslant \dim(\text{Coker}(\cup c_1))\\ &= \dim(H^4(G/T^k) - \dim(Im(\cup c_1))\\ &\geqslant \dim(H^4(G/T^k) - \dim(H^2(G/T^k))\\ &= \dim(H^4(G/T^k) - k.\end{aligned}$$

Adding all these inequalities together, we have

$$\begin{aligned}\dim(H^4(G/S^1) &\geqslant \dim(H^4(G/T^n) - (n + (n-1) + (n-2) + \cdots + 2)\\ &= \dim(H^4(G/T^n) - (\frac{n^2+n}{2} - 1)\\ &= (n^2+n-2)/2 - (\frac{n^2+n}{2} - 1)\\ &= 0.\end{aligned}$$

If we can show that $\dim(H^4(G/S^1;\mathbb{Q})$ is actually 0, then we are already done, since it means that all the inequalities above are actually equalities, so $\dim(Im(\cup c_1)) = k$, or in other words $\cup c_1$ is an injection for every k and inductively we have

$$\begin{aligned}&\dim(H^3(G/T^{k-1};\mathbb{Q}))\\ &= \dim(H^3(G/T^k;\mathbb{Q})) + \dim(ker(\cup c_1))\\ &= \dim(H^3(G/T^k;\mathbb{Q})\\ &= \cdots\\ &= \dim(H^3(G/T^n;\mathbb{Q}) = 0\end{aligned}$$

Let's look at the Gysin sequence of the fibre bundle $S^1 \to G \to G/S^1$

$$\cdots \longrightarrow H^3(G;\mathbb{Q}) \longrightarrow H^2(G/S^1;\mathbb{Q}) \longrightarrow H^4(G/S^1;\mathbb{Q}) \xrightarrow{d} H^4(G;\mathbb{Q}) \longrightarrow \cdots$$

It's easy to see that $\dim(H^4(G/S^1;\mathbb{Q}))$ can be at most 1, since $H^4(G;\mathbb{Q}) = 0$ and $H^2(G/S^1;\mathbb{Q})$ is 1 dimensional. Let's consider the complex line bundle $L = G \times_{S^1} \mathbb{C}$, where the S^1 acts on $\mathbb{C}$ by the natural action of S^1 on $\mathbb{C}$, it's obvious from the Gysin sequence that $H^4(G/S^1;\mathbb{Q})$ is generated by $c_1(L)^2$.

Hence to finish the proof, we only need to show that $c_1(L)^2 = 0$. $T(G/S^1) = G \times_{AdS^1} (\mathfrak{g}/\mathfrak{h})$, where $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{h}$ is the Lie algebra of the embedded S^1. So the complexification of the tangent bundle $T_{\mathbb{C}}(G/S^1) = G \times_{AdS^1} (\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}})$,where $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ are the corresponding complification of the Lie algebras. Since all nontrivial irreducible complex representations of S^1 are 1-dimensional, the complex vector bundle $T_{\mathbb{C}}(G/S^1)$ actually splits into line bundles, which we will analyze in detail. Let the Lie algebra of the embedded S^1 be spanned by the element $t = \sum n_i h_i$,where n_i's are integers and h_i consists of a basis of the Cartan subalgebra of $\mathfrak{g}$(also a basis of $\mathfrak{g}_{\mathbb{C}}$ over $\mathbb{C}$). Notice that $T_{\mathbb{C}}(G/S^1) = G \times_{Ad(S^1)} ((\mathfrak{g}/\mathfrak{h}) \otimes \mathbb{C})$. Recall the following decomposition

$$\mathfrak{g} \otimes \mathbb{C} = L_0 \oplus \bigoplus_{\alpha} L_\alpha$$

From this decomposition, we can conclude that $T_{\mathbb{C}}(G/S^1)$ is the direct sum of the following vector bundles:

(1) A trivial vector bundle of dimension $n-1$, where n is the rank of G, i.e., the (complex) dimension of L_0, since S^1 acts trivially on L_0/t

(2) A pair of complex line bundle $L_\alpha, L_{-\alpha}$ for each pair of roots $\alpha, -\alpha$.

Since $T(G/S^1)$ is stably trivial, we have that $c_2(T_{\mathbb{C}}(G/S^1)) = 0$. On the other hand,

$$\begin{aligned} & c_2(T_{\mathbb{C}}(G/S^1)) \\ &= c_2(\bigoplus(L_\alpha \oplus L_{-\alpha})) \\ &= -\sum c_1^2(L_\alpha) \\ &= (-\sum(\alpha(t))^2) c_1^2(L) \end{aligned}$$

The last equality follows from the fact that $c_1(L_\alpha) = \alpha(t) c_1(L)$, which can be explained in the following way: L_α is the complex line bundle associated to the circle bundle $S^1 \to G \to G/S^1$ and the representation $S^1 \to S^1 : e^{i\theta} \mapsto e^{\alpha(t)\cdot i\theta}$, and hence is isomorphic to the $L^{\otimes \alpha(t)}$. It's also clear then $L_{-\alpha} \cong \overline{L}_\alpha$. Notice that when $\mathfrak{g}$ is semisimple, all the roots α span the dual of the Cartan subalgebra of $\mathfrak{g}$, and there exists a root α, such that $\alpha(t) \neq 0$. And hence the conclusion. □

Chapter 4

Non-vanishing of p_1

In this chapter we prove the main result, the non-vanishing of the first Pontrjagin form p_1 on homogeneous spaces G/T^k for G a classical compact Lie group. We will show by contradiction: in section 1, we assume that p_1 vanishes, and we can get a contradiction in section 2.

4.1 Estimate of λ

In this section we give an estimate of the parameter λ, which is necessary for showing the non-vanishing of $p_1 \in \Omega^4(M)$.

4.1.1 A special case

In this subsection, we deal with a special case. Let $\alpha = l_1 - l_2, \beta = l_1 - l_3$ be two positive roots of the Lie algebra $\mathfrak{su}(n)$. We assume in this subsection that in the homogeneous space $SU(n)/T^k$, we have both h_α and h_β live in the Lie algebra of T^k, i.e., $h_\alpha, h_\beta \in \mathfrak{t}$.

Before stating the tedious calculation, we want to explain the basis idea of how this estimate of λ comes about. We need to calculate $f_{\alpha,\beta}, g_{\alpha,\beta}$ and $h_{\alpha,\beta}$. To calculate $f_{\alpha,\beta}$, we have by lemma 2.2.11

$$R(x_\alpha, x_{-\alpha})x_\gamma = -\frac{3}{4}[[x_\alpha, x_{-\alpha}], x_\gamma] + 2\langle (h_\alpha)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}} \rangle_\lambda$$

$= -\frac{3}{4}[[x_\alpha, x_{-\alpha}], x_\gamma]$ since $(h_\alpha)_{\mathfrak{m}} = 0$ by our assumption.

Similarly

$$
\begin{aligned}
&R(x_\beta, x_{-\beta})x_{-\gamma} \\
&= -\frac{3}{4}[[x_\beta, x_{-\beta}], x_\gamma]
\end{aligned}
$$

Hence, roughly, we have that $f_{\alpha,\beta}$ is "almost" 9 times $f_{G,\alpha,\beta}$. (Recall that $R_G(x_\alpha, x_{-\alpha})x_\gamma = -\frac{1}{4}[[x_\alpha, x_{-\alpha}], x_\gamma]$, by R_G we mean the curvature tensor associated to the Levi-Civita connection of the bi-invariant metric on G.) While $h_{\alpha,\beta}$ is almost the same as $h_{G,\alpha,\beta}$, $g_{\alpha,\beta}$ is almost the same as $g_{G,\alpha,\beta}$. And the estimate of λ just come from this difference here.

Now we start to calculate $f_{\alpha,\beta}, g_{\alpha,\beta}$ and $h_{\alpha,\beta}$. To calculate $f_{\alpha,\beta}$, we calculate the following curvature transformations:

1. $R(x_\alpha, x_{-\alpha})t, R(x_\beta, x_{-\beta})t$ for any base vector $t \in \mathfrak{t}'$.

$$
\begin{aligned}
&R(x_\alpha, x_{-\alpha})t \\
&= [\Lambda_{\mathfrak{m}}(x_\alpha), \Lambda_{\mathfrak{m}}(x_{-\alpha})]t - \Lambda_{\mathfrak{m}}([x_\alpha, x_{-\alpha}]_{\mathfrak{m}}) - [([x_\alpha, x_{-\alpha}]_{\mathfrak{t}}), t] \\
&= 0
\end{aligned}
$$

2. $R(x_\alpha, x_{-\alpha})x_\gamma, R(x_\beta, x_{-\beta})x_{-\gamma}$, for any root $\gamma \neq \pm\alpha, \pm\beta$

$$
\begin{aligned}
&R(x_\alpha, x_{-\alpha})x_\gamma \\
&= \Lambda_{\mathfrak{m}}(x_\alpha)\Lambda_{\mathfrak{m}}(x_{-\alpha})x_\gamma - \Lambda_{\mathfrak{m}}(x_{-\alpha})\Lambda_{\mathfrak{m}}(x_\alpha)x_\gamma - \Lambda_{\mathfrak{m}}([x_\alpha, x_{-\alpha}]_{\mathfrak{m}})x_\gamma - ([[x_\alpha, x_{-\alpha}]_{\mathfrak{t}}, x_\gamma]) \\
&= \frac{1}{4}[x_\alpha, [x_{-\alpha}, x_\gamma]] - \frac{1}{4}[x_{-\alpha}, [x_\alpha, x_\gamma]] - \frac{1}{2}[[x_\alpha, x_{-\alpha}]_{\mathfrak{m}}, x_\gamma] - U([x_\alpha, x_{-\alpha}]_{\mathfrak{m}}, x_\gamma) \\
&\quad -\frac{1}{2}([[x_\alpha, x_{-\alpha}]_{\mathfrak{t}}, x_\gamma]) - \frac{1}{2}([[x_\alpha, x_{-\alpha}]_{\mathfrak{t}}, x_\gamma]) \\
&= \frac{1}{4}[x_\alpha, [x_{-\alpha}, x_\gamma]] - \frac{1}{4}[x_{-\alpha}, [x_\alpha, x_\gamma]] - [[x_\alpha, x_{-\alpha}], x_\gamma] \\
&= \frac{3}{4}[[x_\alpha, x_{-\alpha}], x_\gamma]
\end{aligned}
$$

Similarly we have

$$R(x_\alpha, x_{-\alpha})x_\gamma = \frac{3}{4}[[x_\beta, x_{-\beta}], x_\gamma].$$

3. A little more complicated is when $\gamma = \pm\alpha, \pm\beta$

We have

$$
\begin{aligned}
&R(x_\alpha, x_{-\alpha})x_\alpha \\
&\Lambda_{\mathfrak{m}}(x_\alpha)\Lambda_{\mathfrak{m}}(x_{-\alpha})x_\alpha \\
&= \Lambda_{\mathfrak{m}}(x_\alpha)(\frac{1}{2}[x_{-\alpha}, x_\alpha]_{\mathfrak{m}}) \\
&= \Lambda_{\mathfrak{m}}(x_\alpha)(-(h_\alpha)_{\mathfrak{m}})
\end{aligned}
$$

$$= \frac{1}{2}[x_\alpha, -(h_\alpha)_\mathfrak{m}] + U((x_\alpha), -(h_\alpha)_\mathfrak{m})$$
$$= \frac{1}{2}[x_\alpha, -(h_\alpha)_\mathfrak{m}] + \frac{1}{2}(\langle [x_{-\alpha}, x_\alpha]_\mathfrak{m}, -(h_\alpha)_\mathfrak{m})\rangle_\lambda + \langle x_{\alpha_i}, [x_{-\alpha}, -(h_\alpha)_\mathfrak{m}]_\mathfrak{m}\rangle)x_{-\alpha}$$
$$= \frac{1}{2}(\langle [x_{-\alpha}, x_\alpha]_\mathfrak{m}, -(h_\alpha)_\mathfrak{m}\rangle_\lambda)x_{-\alpha}$$
$$= \langle -(h_\alpha)_\mathfrak{m}, -(h_\alpha)_\mathfrak{m}\rangle_\lambda x_{-\alpha}$$
$$= \frac{1}{4}[x_\alpha, [x_{-\alpha}, x_\alpha]] + (\lambda - 1 - \langle (h_\alpha)_\mathfrak{t}, (h_\alpha)_\mathfrak{t}\rangle_\lambda)x_{-\alpha}$$

Hence we have

$$R(x_\alpha, x_{-\alpha})x_\alpha = -\frac{3}{4}[[x_\alpha, x_{-\alpha}], x_\alpha] + (\lambda - 1 - \langle (h_\alpha)_\mathfrak{t}, (h_\alpha)_\mathfrak{t}\rangle_\lambda)x_{-\alpha}$$
$$R(x_\beta, x_{-\beta})x_{-\alpha} = -\frac{3}{4}[[x_\beta, x_{-\beta}], x_{-\alpha}]$$

From the above calculations we have that

$$f(x_\alpha, x_{-\alpha}, x_\beta, x_{-\beta})$$
$$= 9f_G(x_\alpha, x_{-\alpha}, x_\beta, x_{-\beta}) - 3(\lambda - 1 - \langle (h_\alpha)_\mathfrak{t}, (h_\alpha)_\mathfrak{t}\rangle_\lambda) - 3(\lambda - 1 - \langle (h_\beta)_\mathfrak{t}, (h_\beta)_\mathfrak{t}\rangle_\lambda)$$

3. We also need to compute $g(x_\alpha, x_{-\alpha}, x_\beta, x_{-\beta})$ and $h(x_\alpha, x_{-\alpha}, x_\beta, x_{-\beta})$ For any root $\gamma \neq \pm\alpha, \pm\beta, \pm(\alpha + \beta)$. We have

$$R(x_\alpha, x_\beta)x_\gamma$$
$$= \Lambda_\mathfrak{m}(x_\alpha)\Lambda_\mathfrak{m}(x_\beta)x_\gamma - \Lambda_\mathfrak{m}(x_\beta)\Lambda_\mathfrak{m}(x_\alpha)x_\gamma - \Lambda_\mathfrak{m}([x_\alpha, x_\beta]_\mathfrak{m})x_\gamma - ([[x_\alpha, x_\beta]_\mathfrak{t}, x_\gamma])$$
$$= \Lambda_\mathfrak{m}(x_\alpha)\Lambda_\mathfrak{m}(x_\beta)x_\gamma - \Lambda_\mathfrak{m}(x_\beta)\Lambda_\mathfrak{m}(x_\alpha)x_\gamma - \Lambda_\mathfrak{m}([x_\alpha, x_\beta])x_\gamma$$
$$= \frac{1}{4}[x_\alpha, [x_\beta, x_\gamma]] - \frac{1}{4}[x_\beta, [x_\alpha, x_\gamma]] - \frac{1}{2}[[x_\alpha, x_\beta], x_\gamma]$$
$$= -\frac{1}{4}[[x_\alpha, x_\beta], x_\gamma]$$

For $\gamma = \alpha$, we have

$$R(x_\alpha, x_\beta)x_\alpha = -\frac{1}{4}[[x_\alpha, x_\beta], x_\gamma]$$
$$R(x_{-\alpha}, x_{-\beta})x_\alpha$$
$$= \Lambda_\mathfrak{m}(x_{-\alpha})\Lambda_\mathfrak{m}(x_{-\beta})x_\alpha - \Lambda_\mathfrak{m}(x_{-\beta})\Lambda_\mathfrak{m}(x_{-\alpha})x_\alpha - \Lambda_\mathfrak{m}([x_{-\alpha}, x_{-\beta}]_\mathfrak{m})x_\alpha - ([[x_{-\alpha}, x_{-\beta}]_\mathfrak{t}, x_\alpha])$$
$$= \Lambda_\mathfrak{m}(x_{-\alpha})\Lambda_\mathfrak{m}(x_{-\beta})x_\alpha - \Lambda_\mathfrak{m}(x_{-\beta})\Lambda_\mathfrak{m}(x_{-\alpha})x_\alpha - \Lambda_\mathfrak{m}([x_{-\alpha}, x_{-\beta}])x_\alpha$$
$$= \Lambda_\mathfrak{m}(x_{-\alpha})(\frac{1}{2}[x_{-\beta}, x_\alpha]_\mathfrak{m}) - \Lambda_\mathfrak{m}(x_{-\beta})(\frac{1}{2}[x_{-\alpha}, x_\alpha]_\mathfrak{m}) - \frac{1}{2}[[x_{-\alpha}, x_{-\beta}], x_\alpha]$$

Notice that

$$\Lambda(x_{-\beta})(\frac{1}{2}[x_{-\alpha}, x_\alpha]_\mathfrak{m})$$
$$= \Lambda(x_{-\beta})(-(h_\alpha)_\mathfrak{m})$$

$$= -(\frac{1}{2}[x_{-\beta}, (h_\alpha)_{\mathfrak{m}}] + U(x_{-\beta}, (h_\alpha)_{\mathfrak{m}}))$$
$$= -\frac{1}{2}[x_{-\beta}, (h_\alpha)_{\mathfrak{m}}] - \frac{1}{2}(\langle [x_\beta, x_{-\beta}], (h_\alpha)_{\mathfrak{m}}\rangle_\lambda + \langle [x_\beta, (h_\alpha)_{\mathfrak{m}}], x_{-\beta}\rangle_\lambda)x_\beta$$
$$= -\langle (h_\alpha)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda x_\beta$$
$$= \frac{1}{2}[x_{-\beta}, [x_{-\alpha}, x_\alpha]] + (\frac{1}{2} - \langle (h_\alpha)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda)x_\beta$$
$$= \frac{1}{2}[x_{-\beta}, [x_{-\alpha}, x_\alpha]] + (\frac{1}{2} + \langle (h_\alpha)_{\mathfrak{t}}, (h_\beta)_{\mathfrak{t}}\rangle_\lambda - \frac{\lambda}{2})x_\beta$$

We can conclude that $g(x_\alpha, x_{-\alpha}, x_\beta, x_{-\beta}) = g_G + (\frac{\lambda}{2} - \langle (h_\alpha)_{\mathfrak{t}}, (h_\beta)_{\mathfrak{t}}\rangle_\lambda - \frac{1}{2})$

Similarly $h(x_\alpha, x_{-\alpha}, x_\beta, x_{-\beta}) = h_G + (\frac{\lambda}{2} - \langle (h_\alpha)_{\mathfrak{t}}, (h_\beta)_{\mathfrak{t}}\rangle_\lambda - \frac{1}{2})$

Another thing we need to calculate is

$R(x_\alpha, x_\beta)t$ for $t \in \mathfrak{t}'$.

Let $\{t_1, \cdots, t_{n-k}\}$ be an orthonormal basis of $\mathfrak{t}'$. We know that $R(x_\alpha, x_\beta)t$ must live in the line spanned by $x_{-(l_2-l_3)}$, i.e., $R(x_\alpha, x_\beta)t = cx_{-(l_2-l_3)}$.

Since $x_{-(l_2-l_3)}$ is of unit length, we have $c = \langle R(x_\alpha, x_\beta)t, x_{-(l_2-l_3)}\rangle_\lambda$. Now by the basic symmetry property of the curvature tensor associated to Levi-Civita connection, $\langle R(x_\alpha, x_\beta)t, x_{-(l_2-l_3)}\rangle_\lambda = -\langle R(x_\alpha, x_\beta)x_{-(l_2-l_3)}, t\rangle_\lambda$. Hence

$$c = \langle R(x_\alpha, x_\beta)t, x_{-(l_2-l_3)}\rangle_\lambda$$
$$= -\langle R(x_\alpha, x_\beta)x_{-(l_2-l_3)}, t_i\rangle_\lambda$$
$$= -\langle -\frac{1}{4}[[x_\alpha, x_\beta], x_{-(l_2-l_3)}]_{\mathfrak{m}}, t_i\rangle_\lambda$$
$$= \langle -\frac{1}{4}[x_{l_2-l_3}, x_{-(l_2-l_3)}]_{\mathfrak{m}}, t_i\rangle_\lambda$$
$$= \langle -\frac{1}{2}(h_{l_2-l_3})_{\mathfrak{m}}, t_i\rangle_\lambda$$

On the other hand, we have

$$\langle R(x_{-\alpha}, x_{-\beta})x_{-(l_2-l_3)}, t_i\rangle_\lambda$$
$$= -\langle R(x_{-\alpha}, x_\beta)x_{-(l_2-l_3)}, t_i\rangle_\lambda$$
$$= \langle -\frac{1}{4}[[x_{-\alpha}, x_{-\beta}], x_{-(l_2-l_3)}]_{\mathfrak{m}}, t_i\rangle_\lambda$$
$$= \langle -\frac{1}{4}[-x_{l_2-l_3}, x_{-(l_2-l_3)}], t_i\rangle_\lambda$$
$$= \frac{1}{2}\langle (h_{l_2-l_3})_{\mathfrak{m}}, t_i\rangle_\lambda$$

The determinant is just $\frac{1}{4}\langle (h_{l_2-l_3})_{\mathfrak{m}}, t_i\rangle_\lambda^2 dx_\alpha dx_{-\alpha} dx_\beta dx_{-\beta}$

Taking the sum over the basis vectors $t_1, \cdots, t_k$, we have

$$-\frac{1}{4}\sum_i \langle (h_{l_2-l_3})_{\mathfrak{m}}, t_i\rangle_\lambda^2 = -\frac{1}{4}\|(h_{l_2-l_3})_{\mathfrak{m}}\|^2$$

Now we can conclude that

$g_{\alpha,\beta} = g_{G,\alpha,\beta} - \frac{1}{2}(\|(h_{l_2-l_3})_{\mathfrak{m}}\|^2 - 1) - (\langle (h_{l_1-l_2})_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}}\rangle_\lambda - \frac{1}{2})$

Notice that $(h_{l_2-l_3})_{\mathfrak{m}} = (h_{l_1-l_3})_{\mathfrak{m}} - (h_{l_1-l_2})_{\mathfrak{m}}$, hence $\|(h_{l_2-l_3})_{\mathfrak{m}}\|^2 = \|(h_{l_1-l_2})_{\mathfrak{m}}\|^2 + \|(h_{l_1-l_3})_{\mathfrak{m}}\|^2 - 2\langle (h_{l_1-l_2})_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}}\rangle_\lambda$

We can simplify the above as

$g_{\alpha,\beta} = g_{G,\alpha,\beta} - \frac{1}{2}(\|(h_{l_1-l_2})_{\mathfrak{m}}\|^2 - 1 + \|(h_{l_1-l_3})_{\mathfrak{m}}\|^2 - 1)$

Similarly we have

$h_{\alpha,\beta} = h_{G,\alpha,\beta} - \frac{1}{2}(\|(h_{l_1-l_2})_{\mathfrak{m}}\|^2 - 1 + \|(h_{l_1-l_3})_{\mathfrak{m}}\|^2 - 1)$

Remark 4.1.1. The calculation of $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$ will also be used in later sections.

Hence we have

$$\begin{aligned} f+g+h &= 9f_G + g_G + h_G - 3(\lambda - 1 - \langle (h_\alpha)_{\mathfrak{t}}, (h_\alpha)_{\mathfrak{t}}\rangle_\lambda) - 3(\lambda - 1 - \langle (h_\alpha)_{\mathfrak{t}}, (h_\beta)_{\mathfrak{t}}\rangle_\lambda) \\ &\quad (\|(h_{l_1-l_2})_{\mathfrak{m}}\|^2 - 1 + \|(h_{l_1-l_3})_{\mathfrak{m}}\|^2 - 1) \\ &= 8f_G - 5\lambda + 5 + 3\langle (h_\alpha)_{\mathfrak{t}}, (h_\alpha)_{\mathfrak{t}}\rangle_\lambda + 3\langle (h_\alpha)_{\mathfrak{t}}, (h_\alpha)_{\mathfrak{t}}\rangle_\lambda - \langle (h_\alpha)_{\mathfrak{t}}, (h_\beta)_{\mathfrak{t}}\rangle_\lambda \end{aligned}$$

Since $3\langle (h_\alpha)_{\mathfrak{t}}, (h_\alpha)_{\mathfrak{t}}\rangle_\lambda + 3\langle (h_\alpha)_{\mathfrak{t}}, (h_\alpha)_{\mathfrak{t}}\rangle_\lambda - \langle (h_\alpha)_{\mathfrak{t}}, (h_\beta)_{\mathfrak{t}}\rangle_\lambda \geqslant 0$

we have the estimate

$$8\lambda - 8 \geqslant 8f_G = 8\frac{1}{2}(n+1) = 4n+4$$

Hence

$$\lambda \geqslant \frac{4n+4}{8} + 1 = \frac{n+3}{2}$$

We can get a similar estimate for classical Lie groups of type B, C and D.

Remark 4.1.2. The estimate is only necessary λ. For the case we just considered, actually now matter how big λ is , the coefficient of $dx_\alpha dx_{-\alpha} dx_\beta dx_{-\beta}$ can't vanish, since $(h_\alpha)_{\mathfrak{m}} = (h_\alpha)_{\mathfrak{m}} = 0$. And hence we have actually proved the nonvanishing of p_1 when T is a maximal torus.

Proposition 4.1.3. If T is the maximal torus of G, then the first Pontrjagin form p_1 doesn't vanish.

4.1.2 General case

In the previous subsection, we assumed that we can find two positive roots α and β, such that $h_\alpha, h_\beta \in \mathfrak{t}$, which is a very special case. Let's look at an example, which is very close to the special case.

Example 4.1.4. Let $G = SU(n)$, and $U(1)$ an embedded circle in $SU(n)$, which is generated by $h_{l_1-l_3}$. Let $\alpha = l_1 - l_2, \beta = l_2 - l_3, \gamma = l_1 - l_3$, we have $h_\alpha + h_\beta = h_\gamma$. The idea is to look at the coefficient of the terms $dx_\alpha dx_{-\alpha} dx_\gamma dx_{-\gamma}$ and $dx_\beta dx_{-\beta} dx_\gamma dx_{-\gamma}$. Now let $\zeta \neq \pm\alpha, \pm\beta, \pm\gamma$, then we know that as in the previous part, since $h_\gamma \in \mathfrak{t}$, We have

$$R(x_\gamma, x_{-\gamma})x_\zeta = -\frac{3}{4}[[x_\gamma, x_{-\gamma}], x_\zeta]$$

But $R(x_\alpha, x_{-\alpha})x_\zeta \neq -\frac{3}{4}[[x_\alpha, x_{-\alpha}], x_\zeta], R(x_\beta, x_{-\beta})x_\zeta \neq -\frac{3}{4}[[x_\beta, x_{-\beta}], x_\zeta]$,since $h_\alpha, h_\beta \notin \mathfrak{t}$. However, when we take the sum

$$\begin{aligned}
&R(x_\alpha, x_{-\alpha})x_\zeta + R(x_\beta, x_{-\beta})x_\zeta \\
&= [\Lambda_{\mathfrak{m}}(x_\alpha), \Lambda_{\mathfrak{m}}(x_{-\alpha})]x_\zeta + \Lambda_{\mathfrak{m}}([x_\alpha, x_{-\alpha}]_{\mathfrak{m}})x_\zeta + [[x_\alpha, x_{-\alpha}]_{\mathfrak{t}}, x_\zeta] \\
&+[\Lambda_{\mathfrak{m}}(x_\beta), \Lambda_{\mathfrak{m}}(x_{-\beta})]x_\zeta + \Lambda_{\mathfrak{m}}([x_\beta, x_{-\beta}]_{\mathfrak{m}})x_\zeta + [[x_\beta, x_{-\beta}]_{\mathfrak{t}}, x_\zeta]
\end{aligned}$$

Notice that $\Lambda_{\mathfrak{m}}([x_\alpha, x_{-\alpha}]_{\mathfrak{m}})x_\zeta + \Lambda_{\mathfrak{m}}([x_\beta, x_{-\beta}]_{\mathfrak{m}})x_\zeta$

$= \Lambda_{\mathfrak{m}}(2h_\alpha + 2h_\beta)_{\mathfrak{m}}x_\zeta = \Lambda_{\mathfrak{m}}(2h_\gamma)_{\mathfrak{m}}x_\zeta = 0$

And $[[x_\alpha, x_{-\alpha}]_{\mathfrak{t}}, x_\zeta] + [[x_\beta, x_{-\beta}]_{\mathfrak{t}}, x_\zeta]$

$= [2(h_\alpha)_{\mathfrak{t}}, x_\zeta] + [2(h_\beta)_{\mathfrak{t}}, x_\zeta]$

$= 2[(h_\gamma)_{\mathfrak{t}}, x_\zeta]$

$= 2[h_\gamma, x_\zeta]$

$= [2h_\alpha, x_\zeta] + [2h_\beta, x_\zeta]$

Hence we have $R(x_\alpha, x_{-\alpha})x_\zeta + R(x_\beta, x_{-\beta})x_\zeta$

$= [\Lambda_{\mathfrak{m}}(x_\alpha), \Lambda_{\mathfrak{m}}(x_{-\alpha})]x_\zeta + [[x_\alpha, x_{-\alpha}], x_\zeta]$

$+[\Lambda_{\mathfrak{m}}(x_\beta), \Lambda_{\mathfrak{m}}(x_{-\beta})]x_\zeta + [[x_\beta, x_{-\beta}], x_\zeta]$

$= -\frac{3}{4}[[x_\alpha, x_{-\alpha}], x_\zeta] + -\frac{3}{4}[[x_\beta, x_{-\beta}], x_\zeta]$.

From the above computation, we can assume without loss of generality that both h_α and h_β are in $\mathfrak{t}$. And we can get the same estimate in the previous section.

To conclude, we have the following proposition:

Proposition 4.1.5. (1) For $G = SU(n)$, we have $\lambda \geqslant \frac{4n+4}{8} + 1$

(2) For $G = SO(2n)$, we have $\lambda \geqslant \frac{8(n+1)}{8} + 1$

(3) For $G = SO(2n+1)$, we have $\lambda \geqslant \frac{8n+12}{8} + 1$

(4) For $G = Sp(n)$, we have $\lambda \geqslant \frac{8(n+3)}{8} + 1$

4.2 Contradiction

With the estimate of λ, we can show the non-vanishing of certain terms in $p_1(M)$.

4.2.1 Generic case

Type A_n

We first deal with the generic case. Let $\alpha = l_1 - l_2, \beta = l_3 - l_4$. It's not difficult to see that $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$ are both zero. To see this, we only need to show that $R(x_\alpha, x_\beta)x = 0$ for any $x \in \mathfrak{m}$. We can now focus on $R(x_\alpha, x_\beta)x_\gamma$ and $R(x_{-\alpha}, x_{-\beta})x_{-\gamma}$ for all *positive* roots γ.

Recall that all positive roots of $SU(n)$ are of the form $l_i - l_j, 1 \leqslant i < j \leqslant$, to make the procedure of the calculation clear, we divide the positive roots to the following groups:

(1) First we look at positive roots $\gamma = l_1 - l_k$ and $l_2 - l_k$ for $k > 4$. By lemma 2.2.11 We have

$$\begin{aligned}
&R(x_\alpha, x_{-\alpha})x_{l_1-l_k} \\
&= (-\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda)x_{-(l_1-l_k)} \\
&R(x_\beta, x_{-\beta})x_{-(l_1-l_k)} \\
&= 2\langle (h_\beta)_\mathfrak{m}, (h_{-(l_1-l_k)})_\mathfrak{m}\rangle_\lambda \\
&= -2\langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda \ (h_{-\alpha} = -h_\alpha)
\end{aligned}$$

The corresponding determinant is

$$-3\langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda$$

Symmetrically we consider $\gamma = l_2 - l_k$,

$$R(x_\alpha, x_{-\alpha})x_{l_2-l_k}$$

$= (\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda) x_{-(l_1-l_k)}$

$R(x_\beta, x_{-\beta}) x_{-(l_2-l_k)}$

$= 2\langle (h_\beta)_\mathfrak{m}, (h_{-(l_2-l_k)})_\mathfrak{m}\rangle_\lambda$

$= -2\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda$

The corresponding determinant is

$$+3\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda$$

Let's take the sum of the above determinants.

$-3\langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda$

$+3\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda$

$= -3\langle (h_\beta)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda + 4\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle \langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda$

The identity follows from the fact that $(h_{l_1-l_k})_\mathfrak{m} - (h_{l_2-l_k})_\mathfrak{m} = (h_{l_1-l_2})_\mathfrak{m}$

(2) For positive roots $\gamma = l_3 - l_k, l_4 - l_k, k > 4$. The calculation is similar to (1), and we list the sum of determinants here:

$= -3\langle (h_\beta)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda +$

$4\langle (h_\alpha)_\mathfrak{m}, (h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\beta)_\mathfrak{m}, (h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\alpha)_\mathfrak{m}, (h_{l_4-l_k})_\mathfrak{m}\rangle \langle (h_\beta)_\mathfrak{m}, (h_{l_4-l_k})_\mathfrak{m}\rangle_\lambda$

We notice that in all the sum of the determinants, we have a "linear term" and a "quadratic term". And all the linear terms are the same, $-3\langle (h_\beta)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda$, which doesn't depend on k. Notice that the number of such k is $n-4$. Hence the sum of the linear terms is $-6(n-4)\langle (h_\beta)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda$

(3) We now consider the positive roots $\gamma = l_1 - l_3, l_1 - l_4, l_2 - l_3, l_2 - l_4$

$R(x_\alpha, x_{-\alpha}) x_{l_1-l_3} = (-\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_3})_\mathfrak{m})) x_{-(l_1-l_3)}$

$R(x_\beta, x_{-\beta}) x_{-(l_1-l_3)} = (-\frac{3}{2} + 2\langle (h_\beta)_\mathfrak{m}, (h_{-(l_1-l_3)})_\mathfrak{m})) x_{l_1-l_3}$ so we have

$-\frac{9}{4} + 3\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_3})_\mathfrak{m}\rangle_\lambda - 3\langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_3})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_3})_\mathfrak{m}\rangle_\lambda \langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_3})_\mathfrak{m}\rangle_\lambda$

$R(x_\alpha, x_{-\alpha}) x_{l_1-l_4} = (-\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_4})_\mathfrak{m})\rangle_\lambda x_{-(l_1-l_4)}$

$R(x_\beta, X_{-\beta}) x_{-(l_1-l_4)} = (\frac{3}{2} + 2\langle (h_\beta)_\mathfrak{m}, (h_{-(l_1-l_4)})_\mathfrak{m})) x_{l_1-l_4}$

so we have

$\frac{9}{4} - 3\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_4})_\mathfrak{m}\rangle_\lambda - 3\langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_4})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_4})_\mathfrak{m}\rangle_\lambda \langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_4})_\mathfrak{m}\rangle_\lambda$

$R(x_\alpha, x_{-\alpha})x_{l_2-l_3} = (\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m})\rangle_\lambda x_{-(l_2-l_3)}$

$R(x_\beta, x_{-\beta})x_{-(l_2-l_3)} = (-\frac{3}{2} + 2\langle (h_\beta)_\mathfrak{m}, (h_{-(l_2-l_3)})_\mathfrak{m})\rangle_\lambda x_{l_2-l_3}$

so we have $\frac{9}{4}+3\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m}\rangle_\lambda+3\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m}\rangle_\lambda+4\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m}\rangle_\lambda\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m}\rangle_\lambda$

$R(x_\alpha, x_{-\alpha})x_{l_2-l_4} = (\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_4})_\mathfrak{m})\rangle_\lambda x_{-(l_2-l_4)}$

$R(x_\beta, x_{-\beta})x_{-(l_2-l_4)} = (\frac{3}{2} + 2\langle (h_\beta)_\mathfrak{m}, (h_{-(l_2-l_4)})_\mathfrak{m})\rangle_\lambda x_{l_2-l_4}$

so we have

$-\frac{9}{4}-3\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_4})_\mathfrak{m}\rangle_\lambda+3\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_4})_\mathfrak{m}\rangle_\lambda+4\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_4})_\mathfrak{m}\rangle_\lambda\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_4})_\mathfrak{m}\rangle_\lambda$

Adding these terms together, we get the constant term cancel out, the "linear term" is $-12\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda$. (We will add quadratic terms later).

(3) For $\gamma = l_i - l_j, 3 < i < j \leqslant n,$ $\quad R(x_\alpha, x_{-\alpha})x_{l_i-l_j}$

$= 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_i-l_j})_\mathfrak{m}\rangle_\lambda x_{-(l_i-l_j)}$

$R(x_\beta, x_{-\beta})x_{-(l_i-l_j)}$

$= 2\langle (h_\beta)_\mathfrak{m}, (h_{-(l_i-l_j)})_\mathfrak{m}\rangle_\lambda x_{l_i-l_j}$

And the corresponding determinant is

$$4\langle (h_\alpha)_\mathfrak{m}, (h_{l_i-l_j})_\mathfrak{m}\rangle_\lambda\langle (h_\beta)_\mathfrak{m}, (h_{l_i-l_j})_\mathfrak{m}\rangle_\lambda$$

(5) Now the only positive roots left are just α and β.

$R(x_\alpha, x_{-\alpha})x_\alpha = (-4 + 3\langle (h_\alpha)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda)x_{-\alpha}$

$R(x_\beta, x_{-\beta})x_{-\alpha} = 2\langle (h_\beta)_\mathfrak{m}, (h_{-\alpha})_\mathfrak{m}\rangle_\lambda x_\alpha$

$R(x_\alpha, x_{-\alpha})x_\beta = 2\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda x_{-\beta}$

$R(x_\beta, x_{-\beta})x_{-\beta} = (4 + 3\langle (h_\beta)_\mathfrak{m}, (h_{-\beta})_\mathfrak{m}\rangle_\lambda)x_\beta$

The sum of the determinants is

$-8\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda+6\langle (h_\alpha)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda\langle (h_\beta)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda+6\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda\langle (h_\beta)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda$

Adding all these determinants together, we have

$4\sum_{\gamma\in R^+}\langle (h_\alpha)_\mathfrak{m}, (h_\gamma)_\mathfrak{m}\rangle_\lambda\langle (h_\beta)_\mathfrak{m}, (h_\gamma)_\mathfrak{m}\rangle_\lambda+(2\|(h_\alpha)_\mathfrak{m}\|^2+2\|(h_\beta)_\mathfrak{m}\|^2-6n-4)\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda$

It turns out the "quadratic term"

$$\sum_{\gamma} 4\langle (h_\alpha)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda \langle (h_\beta)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda$$

can be expressed in a simple form. Let

$$(h_\alpha)_{\mathfrak{m}} = a_1 e_1 + \cdots + a_n e_n$$

$$(h_\beta)_{\mathfrak{m}} = b_1 e_1 + \cdots + b_n e_n$$

$\langle (h_\alpha)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda = \langle (h_\alpha)_{\mathfrak{m}}, h_\gamma\rangle_\lambda = \dfrac{\lambda}{2}(a_i - a_j)$

$\langle (h_\beta)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda = \langle (h_\beta)_{\mathfrak{m}}, h_\gamma\rangle_\lambda = \dfrac{\lambda}{2}(b_i - b_j)$ for $\gamma = l_i - l_j$

Hence

$$\sum_{\gamma} 4\langle (h_\alpha)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda \langle (h_\beta)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda = \sum_{1\leqslant i<j\leqslant n} \lambda^2 (a_i - a_j)(b_i - b_j)$$

The coefficient of the a_1 in the above expression is $\lambda^2(b_1 - b_2 + b_1 - b_3 + \cdots + b_1 - b_n) = \lambda^2((n-1)b_1 - (b_2 + b_3 + \cdots + b_n)) = n\lambda^2 b_1$.

So the above expression is $n\lambda^2(a_1 b_1 + \cdots + a_n b_n) = 2n\lambda\langle (h_\alpha)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda$

Hence

$4\sum\langle (h_\alpha)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda \langle (h_\beta)_{\mathfrak{m}}, (h_\gamma)_{\mathfrak{m}}\rangle_\lambda + (2\|(h_\alpha)_{\mathfrak{m}}\|^2 + 2\|(h_\beta)_{\mathfrak{m}}\|^2 - 6n - 4)\langle (h_\alpha)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda$

$= (2n\lambda + 2\|(h_\alpha)_{\mathfrak{m}}\|^2 + 2\|(h_\beta)_{\mathfrak{m}}\|^2 - 6n + 8)\langle (h_\alpha)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda$

By lemma 4.1.5, we have

$2n\lambda - 6n - 4 \geqslant 2n(\dfrac{n+3}{2} - 6n - 4) = n^2 - 3n - 4 > 0$ for $n \geqslant 4$. Combined with the condition that $\langle (h_\alpha)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda \neq 0$, we get the nonvanishing of the coefficient of $dx_\alpha dx_{-\alpha} dx_\beta dx_{-\beta}$

Type B_n, C_n and D_n

In this subsection, we prove the nonvanishing of p_1 for G of type B_n, C_n and D_n. The root system of type B, C and D are very similar. We will only prove for type B, the proof for type C and type D are almost the same.

For type B_n, we only consider the case when $n \geqslant 3$, since when type B_2 is the same as type C_2. We can hence find $i < j$, such that $\langle (h_{l_i})_{\mathfrak{m}}, (h_{l_j})_{\mathfrak{m}}\rangle_\lambda \neq 0$. Hence for $l \neq i, j$(since $n \geqslant 3$), we have either $\langle (h_{l_i})_{\mathfrak{m}} + (h_{l_l})_{\mathfrak{m}}, (h_{l_j})_{\mathfrak{m}}\rangle_\lambda \neq 0$ or

$\langle (h_{l_i})_{\mathfrak{m}} - (h_{l_l})_{\mathfrak{m}}, (h_{l_j})_{\mathfrak{m}} \rangle_\lambda \neq 0$. We can assume without loss of generality that $\langle (h_{l_i})_{\mathfrak{m}} - (h_{l_l})_{\mathfrak{m}}, (h_{l_j})_{\mathfrak{m}} \rangle_\lambda \neq 0$, and assume that $i = 1, j = 2, l = 3$

Let $\alpha = l_1 - l_2, \beta = l_3$. Since neither $\alpha + \beta$ or $\alpha - \beta$ is a root of $\mathfrak{g}$, we have that $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$, by an argument similar to type A_n.

So we only need to calculate $f_{\alpha,\beta}$. We have the following:

(1) $R(x_\alpha, x_{-\alpha})t = 0$ for any $t \in \mathfrak{t}'$

(2) $R(x_\alpha, x_{-\alpha})x_{l_1-l_3} = (-\frac{3}{2} + 2\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}} \rangle_\lambda) x_{-(l_1-l_3)}$

$R(x_\beta, x_{-\beta})x_{-(l_1-l_3)} = (-\frac{3}{2} - 2\langle (h_\beta)_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}} \rangle_\lambda) x_{l_1-l_3}$

The corresponding determinant is

$-\frac{9}{4} + 3\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}} \rangle_\lambda - 3\langle (h_\beta)_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}} \rangle_\lambda + 4\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}} \rangle_\lambda \langle (h_\beta)_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}} \rangle_\lambda$

Similarly,

$R(x_\alpha, x_{-\alpha})x_{l_1+l_3} = (-\frac{3}{2} + 2\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_1+l_3})_{\mathfrak{m}} \rangle_\lambda) x_{-(l_1+l_3)}$

$R(x_\beta, x_{-\beta})x_{-(l_1+l_3)} = (\frac{3}{2} - 2\langle (h_\beta)_{\mathfrak{m}}, (h_{l_1+l_3})_{\mathfrak{m}} \rangle_\lambda) x_{l_1+l_3}$

The corresponding determinant is

$\frac{9}{4} - 3\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_1+l_3})_{\mathfrak{m}} \rangle_\lambda - 3\langle (h_\beta)_{\mathfrak{m}}, (h_{l_1+l_3})_{\mathfrak{m}} \rangle_\lambda + 4\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_1+l_3})_{\mathfrak{m}} \rangle_\lambda \langle (h_\beta)_{\mathfrak{m}}, (h_{l_1+l_3})_{\mathfrak{m}} \rangle_\lambda$

$R(x_\alpha, x_{-\alpha})x_{l_2-l_3} = (\frac{3}{2} + 2\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_2-l_3})_{\mathfrak{m}} \rangle_\lambda) x_{-(l_2-l_3)}$

$R(x_\beta, x_{-\beta})x_{-(l_2-l_3)} = (-\frac{3}{2} - 2\langle (h_\beta)_{\mathfrak{m}}, (h_{l_2-l_3})_{\mathfrak{m}} \rangle_\lambda) x_{l_2-l_3}$

The corresponding determinant is

$\frac{9}{4} + 3\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_2-l_3})_{\mathfrak{m}} \rangle_\lambda + 3\langle (h_\beta)_{\mathfrak{m}}, (h_{l_2-l_3})_{\mathfrak{m}} \rangle_\lambda + 4\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_2-l_3})_{\mathfrak{m}} \rangle_\lambda \langle (h_\beta)_{\mathfrak{m}}, (h_{l_2-l_3})_{\mathfrak{m}} \rangle_\lambda$

$R(x_\alpha, x_{-\alpha})x_{l_2+l_3} = (\frac{3}{2} + 2\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_2+l_3})_{\mathfrak{m}} \rangle_\lambda) x_{-(l_2+l_3)}$

$R(x_\beta, x_{-\beta})x_{-(l_2+l_3)} = (\frac{3}{2} - 2\langle (h_\beta)_{\mathfrak{m}}, (h_{l_2+l_3})_{\mathfrak{m}} \rangle_\lambda) x_{l_2+l_3}$

The corresponding determinant is

$-\frac{9}{4} - 3\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_2+l_3})_{\mathfrak{m}} \rangle_\lambda + 3\langle (h_\beta)_{\mathfrak{m}}, (h_{l_2+l_3})_{\mathfrak{m}} \rangle_\lambda + 4\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_2+l_3})_{\mathfrak{m}} \rangle_\lambda \langle (h_\beta)_{\mathfrak{m}}, (h_{l_2+l_3})_{\mathfrak{m}} \rangle_\lambda$

Taking the sum of the above determinants, we have

$-18\langle (h_\alpha)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}} \rangle_\lambda +$ "quadratic terms"

For $k > 3$. Let $\gamma = l_1 - l_k$, then

$R(x_\alpha, x_{-\alpha})x_{l_1-l_k} = (-\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda)x_{-(l_1-l_k)}$

$R(x_\beta, x_{-\beta})x_{-(l_1-l_k)} = -2\langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda x_{l_1-l_k}$

$R(x_\alpha, x_{-\alpha})x_{l_2-l_k} = (-\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda)x_{-(l_2-l_k)}$

$R(x_\beta, x_{-\beta})x_{-(l_2-l_k)} = -2\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda x_{l_2-l_k}$

The sum of the corresponding determinant is

$-3\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda + 4\langle (h_\beta)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\alpha)_\mathfrak{m}, (h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\alpha)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda$

$R(x_\alpha, x_{-\alpha})x_{l_1+l_k} = (-\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_1+l_k})_\mathfrak{m}\rangle_\lambda)x_{-(l_1+l_k)}$

$R(x_\beta, x_{-\beta})x_{-(l_1+l_k)} = -2\langle (h_\beta)_\mathfrak{m}, (h_{l_1+l_k})_\mathfrak{m}\rangle_\lambda x_{l_1+l_k}$

$R(x_\alpha, x_{-\alpha})x_{l_2+l_k} = (-\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_2+l_k})_\mathfrak{m}\rangle_\lambda)x_{-(l_2+l_k)}$

$R(x_\beta, x_{-\beta})x_{-(l_2+l_k)} = -2\langle (h_\beta)_\mathfrak{m}, (h_{l_2+l_k})_\mathfrak{m}\rangle_\lambda x_{l_2+l_k}$

The sum of the corresponding determinant is

$-3\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda + 4\langle (h_\beta)_\mathfrak{m}, (h_{l_1+l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\alpha)_\mathfrak{m}, (h_{l_1+l_k})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\beta)_\mathfrak{m}, (h_{l_2+l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\alpha)_\mathfrak{m}, (h_{l_2+l_k})_\mathfrak{m}\rangle_\lambda$

$R(x_\alpha, x_{-\alpha})x_{l_3-l_k} = 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda x_{-(l_3-l_k)}$

$R(x_\beta, x_{-\beta})x_{-(l_3-l_k)} = (\frac{3}{2} - 2\langle (h_\beta)_\mathfrak{m}, (h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda x_{l_3-l_k}$

$R(x_\alpha, x_{-\alpha})x_{l_3+l_k} = 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_3+l_k})_\mathfrak{m}\rangle_\lambda x_{-(l_3+l_k)}$

$R(x_\beta, x_{-\beta})x_{-(l_3+l_k)} = (\frac{3}{2} - 2\langle (h_\beta)_\mathfrak{m}, (h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda)x_{l_2-l_k}$

The sum of the corresponding determinant is

$-6\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda + 4\langle (h_\beta)_\mathfrak{m}, (h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\alpha)_\mathfrak{m}, (h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda + 4\langle (h_\beta)_\mathfrak{m}, (h_{l_3+l_k})_\mathfrak{m}\rangle_\lambda \langle (h_\alpha)_\mathfrak{m}, (h_{l_3+l_k})_\mathfrak{m}\rangle_\lambda$

For $l > k > 3$, we have

$R(x_\alpha, x_{-\alpha})x_{l_k-l_l} = 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_k-l_l})_\mathfrak{m}\rangle_\lambda x_{-(l_k-l_l)}$

$R(x_\beta, x_{-\beta})x_{-(l_k-l_l)} = -2\langle (h_\beta)_\mathfrak{m}, (h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda x_{l_k-l_l}$

The corresponding determinant is $4\langle (h_\beta)_\mathfrak{m}, (h_{l_k-l_l})_\mathfrak{m}\rangle_\lambda \langle (h_\alpha)_\mathfrak{m}, (h_{l_k-l_l})_\mathfrak{m}\rangle_\lambda$

We can thus take the sum and conclude that the coefficient is

$$(-9n-9)\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda + 4\sum_{\gamma\in R^+} \langle (h_\beta)_\mathfrak{m}, (h_\gamma)_\mathfrak{m}\rangle_\lambda \langle (h_\alpha)_\mathfrak{m}, (h_\gamma)_\mathfrak{m}\rangle_\lambda + 6\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda (\|h_\alpha\|^2 + \|h_\beta\|^2)$$

Now let $(h_\alpha)_\mathfrak{m} = a_1 h_{l_1} + \cdots + a_n h_{l_n}, (h_\beta)_\mathfrak{m} = b_1 h_{l_1} + \cdots + b_n h_{l_n}$

$$\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m} \rangle_\lambda = \frac{\lambda}{2} \sum_i a_i b_i$$

Then

$$4 \sum_{\gamma \in R^+} \langle (h_\beta)_\mathfrak{m}, (h_\gamma)_\mathfrak{m} \rangle_\lambda \langle (h_\alpha)_\mathfrak{m}, (h_\gamma)_\mathfrak{m} \rangle_\lambda = 4(\sum_i \frac{\lambda^2}{4} a_i b_i + \sum_{i<j} \frac{\lambda^2}{4}(a_i - a_j)(b_i - b_j) + \sum_{i<j} \frac{\lambda^2}{4}(a_i + a_j)(b_i + b_j))$$

$= (2n-1)\lambda^2 \sum a_i b_i$

$= (4n-2)\lambda \langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m} \rangle_\lambda$

By lemma 4.1.5 $\lambda \geqslant \frac{8n+12}{8} + 1$, we have

$(4n-2)\lambda - 9n - 9 \geqslant (4n-2)(n + \frac{5}{2}) - 9n - 9 = 4n^2 + 8n - 5 - 9n - 9 = 4n^2 - n - 14 > 0$ for $n \geqslant 2$. We then get the nonvanishing by the genericity.

4.2.2 Non-generic case

We first show that for groups G of type $B_n (n \geqslant 4)$, C_n and D_n, G/T^k can be non-generic only when T^k is the maximal torus.

Type B_n, C_n and D_n

Proposition 4.2.1. If G of type $B_n (n \geqslant 4)$, C_n and D_n, G/T^k can be non-generic only when T^k is the maximal torus.

Proof. For type D_n, we have roots h_i, for $1 \leqslant i \leqslant n$, and $\langle h_{l_i}, h_{l_j} \rangle_\lambda = 0$ if $i \neq j$. If G/T^k is generic, by definition we have $\langle (h_{l_i})_\mathfrak{m}, (h_{l_j})_\mathfrak{m} \rangle_\lambda = \langle (h_{l_i})_\mathfrak{m}, h_{l_j} \rangle_\lambda = 0$. Then $(h_{l_i})_\mathfrak{m}$ can only be 0 or h_{l_i}. If T^k is not maximal, we can find $1 \leqslant i, j \leqslant n$, such that $(h_{l_i})_\mathfrak{m} = h_{l_i}, (h_{l_j})_\mathfrak{m} = 0$. Then $\langle (h_{l_i + l_j})_\mathfrak{m}, (h_{l_i + l_j})_\mathfrak{m} \rangle_\lambda = \langle h_{l_i}, h_{l_i} \rangle_\lambda \neq 0$, which contradicts the assumption that G/T^k is non-generic. Similarly we can proof for type B_n.

For type C_n, we are assuming here $n \geqslant 4$. We have that for any 4-tuples $1 \leqslant i < j < k < l \leqslant n$, we must have

$$\langle (h_{l_i \pm l_j})_\mathfrak{m}, (h_{l_k \pm l_l})_\mathfrak{m} \rangle_\lambda = 0$$

. Hence we also have $\langle (h_{l_i})_\mathfrak{m}, (h_{l_j})_\mathfrak{m} \rangle_\lambda = \langle (h_{l_i})_\mathfrak{m}, h_{l_j} \rangle_\lambda = 0$ for $i < j$. Then the argument is the same as for type C_n.

□

Remark 4.2.2. Here to consider C_n for $n \geqslant 3$ is enough, since when $n = 2$, C_n is only semisimple, when $n = 3$, type C_3 Lie algebra is isomorphic to type A_3 Lie algebra, i.e. the Lie algebra of $SU(4)$.

Hence for type B_n, C_n and D_n, we have proved the non-vanishing of p_1 for all homogeneous space G/T^k.

Type A_n

However, there are more non-generic homogeneous spaces when $G = SU(n)$, which is of type A_n.

Example 4.2.3. 1. Let $G = SU(3)$, then any homogeneous space $SU(3)/T^k$ is non-generic. Positive roots of $\mathfrak{su}_3$ are $l_1 - l_2, l_1 - l_3$ and $l_2 - l_3$. for any two α, β, we have $\langle h_\alpha, h_\beta \rangle_\lambda \neq 0$

2. Let $G = SU(4)$, let $U(1) \subset SU(4)$ be the circle generated by $3e_1 - e_2 - e_3 - e_4$. Then the homogeneous space $SU(4)/U(1)$ is non-generic. For instance, let $\alpha = l_1 - l_2, \beta = l_3 - l_4$, then $\langle h_\alpha, h_\beta \rangle_\lambda = 0$, but we also have $\langle (h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m} \rangle_\lambda = 0$

$SU(n), n \geqslant 4$

For the non-generic case, we will look at the two roots $\alpha = l_1 - l_2$ and $\beta = l_1 - l_3$, instead of $l_1 - l_2$ and $l_3 - l_4$. The calculation of the coefficient of $dx_\alpha dx_{-\alpha} dx_\beta dx_{-\beta}$ is more complicated than the generic case, since now $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$ are no longer zero.

The calculation of $f_{\alpha,\beta}$ is similar to the generic case.

1. $R(x_\alpha, x_{-\alpha})t = 0, R(x_\alpha, x_{-\alpha})t = 0$ for any $t \in \mathfrak{t}'$.

2. If $\gamma = l_1 - l_k, l_2 - l_k, l_3 - l_k, k > 3$.

We have

$$
\begin{aligned}
& R(x_\alpha, x_{-\alpha}) x_{l_1 - l_k} \\
= & (-\frac{3}{2} + 2\langle (h_\alpha)_\mathfrak{m}, (h_{l_1 - l_k})_\mathfrak{m} \rangle_\lambda) x_{-(l_1 - l_k)}, \\
& R(x_\beta, x_{-\beta}) x_{-(l_1 - l_k)} \\
= & (\frac{3}{2} + 2\langle (h_\beta)_\mathfrak{m}, (h_{-(l_1 - l_k)})_\mathfrak{m} \rangle_\lambda) x_{l_1 - l_k}
\end{aligned}
$$

The determinant corresponding to these two transformations is

$$-(-\frac{3}{2}+2\langle(h_\alpha)_\mathfrak{m},(h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda)(\frac{3}{2}+2\langle(h_\beta)_\mathfrak{m},(h_{-(l_1-l_k)})_\mathfrak{m}\rangle_\lambda)$$
$$=\frac{9}{4}+3\langle(h_\beta)_\mathfrak{m},(h_{-(l_1-l_k)})_\mathfrak{m}\rangle_\lambda-3\langle(h_\alpha)_\mathfrak{m},(h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda$$
$$+4\langle(h_\alpha)_\mathfrak{m},(h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda\langle(h_\beta)_\mathfrak{m},(h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda$$

For $\gamma=l_2-l_k$, we have

$$R(x_\alpha,x_{-\alpha})x_{l_2-l_k}$$
$$=(\frac{3}{2}+2\langle(h_\alpha)_\mathfrak{m},(h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda)x_{-(l_2-l_k)},$$
$$R(x_\beta,x_{-\beta})x_{-(l_2-l_k)}$$
$$=2\langle(h_\beta)_\mathfrak{m},(h_{-(l_2-l_k)})_\mathfrak{m}\rangle_\lambda x_{l_2-l_k},$$

The corresponding determinant is

$$(\frac{3}{2}+2\langle(h_\alpha)_\mathfrak{m},(h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda)2\langle(h_\beta)_\mathfrak{m},(h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda$$
$$=3\langle(h_\beta)_\mathfrak{m},(h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda+4\langle(h_\alpha)_\mathfrak{m},(h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda)\langle(h_\beta)_\mathfrak{m},(h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda$$

Similarly, if $\gamma=l_3-l_k$, the corresponding determinant is given by

$$2\langle(h_\alpha)_\mathfrak{m},(h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda(\frac{3}{2}+2\langle(h_\beta)_\mathfrak{m},(h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda)$$

$$=3\langle(h_\alpha)_\mathfrak{m},(h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda+4\langle(h_\alpha)_\mathfrak{m},(h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda\langle(h_\beta)_\mathfrak{m},(h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda$$

Take the above three determinants, we have

$$\frac{9}{4}-6\langle(h_\alpha)_\mathfrak{m},(h_\beta)_\mathfrak{m}\rangle_\lambda+4\langle(h_\alpha)_\mathfrak{m},(h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda)\langle(h_\beta)_\mathfrak{m},(h_{l_2-l_k})_\mathfrak{m}\rangle_\lambda$$
$$+4\langle(h_\alpha)_\mathfrak{m},(h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda\langle(h_\beta)_\mathfrak{m},(h_{l_3-l_k})_\mathfrak{m}\rangle_\lambda$$
$$+4\langle(h_\alpha)_\mathfrak{m},(h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda\langle(h_\beta)_\mathfrak{m},(h_{l_1-l_k})_\mathfrak{m}\rangle_\lambda$$

(3) For $\gamma=l_i-l_j, 3<i<j\leqslant n,\qquad R(x_\alpha,x_{-\alpha})x_{l_i-l_j}$

$$=2\langle(h_\alpha)_\mathfrak{m},(h_{l_i-l_j})_\mathfrak{m}\rangle_\lambda x_{-(l_i-l_j)}$$
$$R(x_\beta,x_{-\beta})x_{-(l_i-l_j)}$$
$$=2\langle(h_\beta)_\mathfrak{m},(h_{-(l_i-l_j)})_\mathfrak{m}\rangle_\lambda x_{l_i-l_j}$$

And the corresponding determinant is

$$4\langle(h_\alpha)_\mathfrak{m},(h_{l_i-l_j})_\mathfrak{m}\rangle_\lambda\langle(h_\beta)_\mathfrak{m},(h_{l_i-l_j})_\mathfrak{m}\rangle_\lambda$$

(4) We consider $\gamma = l_1 - l_3, l_1 - l_2, l_1 - l_3$

$R(x_\alpha, x_{-\alpha})x_{l_1-l_2}$

$= R(x_\alpha, x_{-\alpha})x_\alpha$

$= (-4 + 3\langle(h_\alpha)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda)$

$R(x_\beta, x_{-\beta})x_{-\alpha}$

$= (\frac{3}{2} + 2\langle(h_\alpha)_\mathfrak{m}, (h_{-\alpha})_\mathfrak{m}\rangle_\lambda)x_\alpha$

The determinant is

$$6 - \frac{9}{2}\langle(h_\alpha)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda - 8(h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m} + 6\langle(h_\alpha)_\mathfrak{m}, (h_\alpha)_\mathfrak{m}\rangle_\lambda\langle(h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda$$

For $l_1 - l_3$, we have

$R(x_\alpha, x_{-\alpha})x_\beta$

$= (-\frac{3}{2} + 2\langle(h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda)x_{-\beta}$

$R(x_\beta, x_{-\beta})x_{-(l_1-l_3)}$

$= R(x_\beta, x_{-\beta})x_{-\beta}$

$= (4 + 3\langle(h_\beta)_\mathfrak{m}, (h_{-\beta})_\mathfrak{m}\rangle_\lambda)$

The determinant is

$$6 - \frac{9}{2}\langle(h_\beta)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda - 8(h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m} + 6\langle(h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda\langle(h_\beta)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda$$

$R(x_\alpha, x_{-\alpha})x_{l_2-l_3}$

$= (\frac{3}{2} + 2\langle(h_\alpha)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m}\rangle_\lambda)x_{-(l_2-l_3)}$

$R(x_\beta, x_{-\beta})x_{-(l_2-l_3)}$

$= (\frac{3}{2} + 2\langle(h_\beta)_\mathfrak{m}, (h_{-(l_2-l_3)})_\mathfrak{m}\rangle_\lambda)x_{l_2-l_3}$

The determinant is

$$-\frac{9}{4} - 3\langle(h_\alpha)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m}\rangle_\lambda - 3\langle(h_\beta)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m} + 4\langle(h_\alpha)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m}\rangle_\lambda 2\langle(h_\beta)_\mathfrak{m}, (h_{l_2-l_3})_\mathfrak{m}\rangle_\lambda$$

Taking the sum of the above three determinants, we get

$$12 - \frac{9}{4} - 22\langle(h_\alpha)_\mathfrak{m}, (h_\beta)_\mathfrak{m}\rangle_\lambda - \frac{3}{2}\|(h_\alpha)^2_\mathfrak{m}\| - \frac{3}{2}\|(h_\beta)^2_\mathfrak{m}\| + \text{"}quadratic\ term\text{"}$$

We can conclude that

$$f_{\alpha,\beta} = 9 \times f_{G,\alpha,\beta} + 3 - \frac{3}{2}\langle (h_\alpha)_{\mathfrak{m}}, (h_\alpha)_{\mathfrak{m}}\rangle_\lambda - \frac{3}{2}\langle (h_\beta)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda$$
$$+(2n\lambda + \|(h_\alpha)_{\mathfrak{m}}\|^2 + \|(h_\beta)_{\mathfrak{m}}\|^2 - 6n - 4)\langle (h_\alpha)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda$$

By Remark 4.1.1, we have

$$g_{\alpha,\beta} = g_{G,\alpha,\beta} - \frac{1}{2}(\|(h_\alpha)_{\mathfrak{m}}\|^2 - 1 + \|(h_\beta)_{\mathfrak{m}}\|^2 - 1)$$

$$h_{\alpha,\beta} = h_{G,\alpha,\beta} - \frac{1}{2}(\|(h_\alpha)_{\mathfrak{m}}\|^2 - 1 + \|(h_\beta)_{\mathfrak{m}}\|^2 - 1)$$

We can conclude that the coefficient of the term $dx_\alpha dx_{-\alpha} dx_\beta dx_{-\beta}$

is $9 \times f_{G,\alpha,\beta} + 2(3 - \frac{3}{2}\langle (h_\alpha)_{\mathfrak{m}}, (h_\alpha)_{\mathfrak{m}}\rangle_\lambda - \frac{3}{2}\langle (h_\beta)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda) + g_{G,\alpha,\beta} - \frac{1}{2}(\|(h_\alpha)_{\mathfrak{m}}\|^2 -$
$1 + \|(h_\beta)_{\mathfrak{m}}\|^2 - 1) + h_{G,\alpha,\beta} - \frac{1}{2}(\|(h_\alpha)_{\mathfrak{m}}\|^2 - 1 + \|(h_\beta)_{\mathfrak{m}}\|^2 - 1) + "quadratic\ terms"$

$= 8 \times f_{G,\alpha,\beta} + 2(3 - \frac{3}{2}\langle (h_\alpha)_{\mathfrak{m}}, (h_\alpha)_{\mathfrak{m}}\rangle_\lambda - \frac{3}{2}\langle (h_\beta)_{\mathfrak{m}}, (h_\beta)_{\mathfrak{m}}\rangle_\lambda) - \frac{1}{2}(\|(h_\alpha)_{\mathfrak{m}}\|^2 - 1 +$
$\|(h_\beta)_{\mathfrak{m}}\|^2 - 1) - \frac{1}{2}(\|(h_\alpha)_{\mathfrak{m}}\|^2 - 1 + \|(h_\beta)_{\mathfrak{m}}\|^2 - 1) + "quadratic\ terms"$

$= 8 \times f_{G,\alpha,\beta} + (8 - 4\|(h_\alpha)_{\mathfrak{m}}\|^2 - 4\|(h_\beta)_{\mathfrak{m}}\|^2) + "quadratic\ terms"$

Now let $\gamma = l_1 - l_4$.

$\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}}\rangle_\lambda = \langle (h_\alpha)_{\mathfrak{m}}, (h_{l_1-l_4})_{\mathfrak{m}}\rangle_\lambda$, since $(h_{l_1-l_4})_{\mathfrak{m}} - (h_{l_1-l_3})_{\mathfrak{m}} = (h_{l_3-l_4})_{\mathfrak{m}}$

and $\langle (h_\alpha)_{\mathfrak{m}}, (h_{l_3-l_4})_{\mathfrak{m}}\rangle_\lambda = 0$ by genericity.

The difference of the coefficients of $dx_\alpha dx_{-\alpha} dx_\beta dx_{-\beta}$ and $dx_\alpha dx_{-\alpha} dx_\gamma dx_{-\gamma}$ is $c(||(h_{l_1-l_3})_{\mathfrak{m}}||^2 - ||(h_{l_1-l_4})_{\mathfrak{m}}||^2)$. If they both vanish, we must have

$||(h_{l_1-l_3})_{\mathfrak{m}}|| = ||(h_{l_1-l_4})_{\mathfrak{m}}||$.

$$||(h_{l_1-l_3})_{\mathfrak{m}}||^2$$
$$= \langle (h_{l_1-l_4})_{\mathfrak{m}} - (h_{l_1-l_3})_{\mathfrak{m}}, (h_{l_1-l_4})_{\mathfrak{m}} - (h_{l_1-l_3})_{\mathfrak{m}}\rangle_\lambda$$
$$= ||(h_{l_1-l_4})_{\mathfrak{m}}||^2 - 2\langle (h_{l_1-l_4})_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}}\rangle_\lambda + ||(h_{l_1-l_3})_{\mathfrak{m}}||^2$$

This implies that $\langle (h_{l_1-l_4})_{\mathfrak{m}}, (h_{l_1-l_3})_{\mathfrak{m}}\rangle_\lambda = \frac{1}{2}||(h_{l_1-l_4})_{\mathfrak{m}}||^2$

By the Weyl group action, it's clear that for any triple $1 \leqslant i < j < k \leqslant n$, we have

$$\langle (h_{l_i-l_j})_{\mathfrak{m}}, (h_{l_i-l_k})_{\mathfrak{m}}\rangle_\lambda = \frac{1}{2}||(h_{l_i-l_j})_{\mathfrak{m}}||^2 = \frac{1}{2}||(h_{l_i-l_k})_{\mathfrak{m}}||^2$$

Notice that $\langle (h_{l_i-l_j})_{\mathfrak{m}}, (h_{l_i-l_k})_{\mathfrak{m}}\rangle_\lambda = \langle (h_{l_i-l_j})_{\mathfrak{m}}, h_{l_i-l_k}\rangle_\lambda$ By the definition of $\langle\ ,\ \rangle_\lambda$, we have that $(h_{l_1-l_2})_{\mathfrak{m}}$ must be must of the form

$$(h_{l_1-l_2})_{\mathfrak{m}} = (a+2c)e_1 + ae_2 + (a+c)e_3 + (a+c)e_4 + \cdots + (a+c)e_n$$

And

$$(h_{l_1-l_3})_{\mathfrak{m}} = (a+2c)e_1 + (a+c)e_2 + ae_3 + (a+c)e_4 + \cdots + (a+c)e_n$$

for $a, c \in \mathbb{R}$

Hence $(h_{l_2-l_3})_{\mathfrak{m}} = (h_{l_1-l_3})_{\mathfrak{m}} + (h_{l_1-l_2})_{\mathfrak{m}} = ce_2 - ce_3$, and it must be that $(h_{l_2-l_3})_{\mathfrak{m}} = h_{l_2-l_3}$, and this can only be the trivial homogeneous space.

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