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Vertex Operator Algebras and Jacobi Forms

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VERTEX OPERATOR ALGEBRAS AND JACOBI FORMS

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Abstract

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Matthew Thomas Krauel

This thesis develops a theory relating the Jacobi group with n-point functions associated with strongly regular vertex operator algebras. The n-point functions considered here have additional complex variables and generalize n-point functions studied in other works in the mathematics and physics literature. Recursion formulas are discussed which reduce the study of n-point functions to the study of 1-point and 0-point functions.

We consider the space of 1-point functions associated to inequivalent irreducible admissible modules for a strongly regular vertex operator algebra. We develop transformation laws for this space of functions under the Jacobi group. With additional assumptions, we show that 1-point functions are sums of products of 1-point functions of modules for the commutant subVOA of the vertex operator algebra together with a type of Jacobi theta series. Conditions will be given where these functions are vector-valued weak Jacobi forms. A number of corollaries to these results are developed, including a sharper result in the case of holomorphic vertex operator algebras.

Other results contained in this thesis include transformation laws for Jacobi theta functions with spherical harmonics, and a generalization of a result of Miyamoto to include zero modes of elements.

To My Parents,

Tom and Nancy.

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Part I

Background and Preliminaries

Chapter 1

Introduction

The theory of vertex operator algebras (VOAs) is a relatively new area of mathematics related to the physics of conformal field theory (CFT). In recent years, the study of VOAs has also become a mathematically rich subject with connections to group theory, combinatorics, number theory, algebraic geometry, elliptic genera, elliptic cohomology and topology, among other fields. One such connection, that between VOAs and elliptic modular forms, has been a source of inspiration in both physics and mathematics. The presence of elliptic modular forms in the theory of VOAs, however, now appears to be but one strand in a larger web of relations between VOAs and automorphic forms. This thesis strengthens this connection between VOAs and automorphic forms by describing the occurrence of Jacobi forms in the theory of VOAs.

A *vertex operator algebra* V is a vector space over $\mathbb{C}$ with two elements $\mathbf{1}$ and ω called the *vacuum* and *Virasoro* elements, respectively, and which is equipped with a linear map $Y(\cdot, z)\colon V \to (\operatorname{End} V)[[z, z^{-1}]]$ defined by $Y(v, z) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}$, such that a number of axioms are satisfied. Among these axioms is the requirement of a natural action of the Virasoro algebra (of *central charge* c) on V that provides a grading $V = \bigoplus_{n\in\mathbb{Z}} V_n$, where each V_n is finite-dimensional, $V_n = 0$ for n sufficiently small, and V_n is the eigenspace for an operator $L(0)$ that will be defined in Section 3.1. The precise definition of a vertex operator algebra, as well as of a strongly regular vertex operator algebra, their modules, and useful consequences of their definitions, are given in Chapter 3.

The interest in strongly regular vertex operator algebras is a result of the additional structure they possess. It is known that for a strongly regular vertex operator algebra V,

1. V has a finitely many inequivalent irreducible admissible modules, $M^1, \ldots, M^r$ [6],
2. V has a unique symmetric invariant bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that $\langle \mathbf{1}, \mathbf{1} \rangle = -1$ [31],
3. V is C_2-cofinite (defined in Chapter 3),
4. V has another grading (see Subsection 3.1.1 below where $L[0]$ will be defined) such that
$$V = \bigoplus_{n\geq 0} V_{[n]},$$
where $V_{[n]} = \{v \in V \mid L[0]v = nv\}$ and $V_{[0]} = \mathbb{C}\mathbf{1}$, and
5. V_1 and $V_{[1]}$ are reductive Lie algebras [11].

Beyond considering strongly regular vertex operator algebras, this thesis is concerned with VOAs that posses elements $h_1, \ldots, h_n \in V_1$ such that $h_i(m)h_j = \delta_{m,1} \langle h_i, h_j \rangle \mathbf{1}$. Such elements may be found in any strongly regular VOA so long as $V_1 \neq 0$. For such

$h_1, \ldots, h_n$, let G denote the Gram matrix $G = (\langle h_i, h_j \rangle)$ associated with $h_1, \ldots, h_n$. Elements $h_1, \ldots, h_n \in V_1$ are said to satisfy *Condition* $\boldsymbol{H}$ if

1. $h_1(0), \ldots, h_n(0)$ are semisimple operators with integral eigenvalues, and

2. $[h_i, h_j] = 0$ for all i, j.

Similar conditions for $h_1, \ldots, h_n$ have also been considered in works such as [12, 37].

Let $\mathbb{H}$ denote the complex upper-half plane. For an integer k, an *elliptic modular form of weight k* is a holomorphic function $\phi \colon \mathbb{H} \to \mathbb{C}$ that satisfies a certain invariance property with respect to the *modular group* $\mathrm{SL}_2(\mathbb{Z})$ and a growth condition at the infinite cusp. Modular forms have long been a subject of study by mathematicians due to their importance in fields such as number theory. They are also of interest to physicists, as the presence of trace functions invariant under $\mathrm{SL}_2(\mathbb{Z})$ are required in a CFT.

Automorphic forms can loosely be described as functions on higher dimensional spaces which replace $\mathbb{H}$. They generalize elliptic modular forms in the sense that they are invariant under an arithmetic group of higher rank that replaces $\mathrm{SL}_2(\mathbb{Z})$. *Jacobi forms* are such generalizations in which $\mathrm{SL}_2(\mathbb{Z})$ is replaced by the *Jacobi group*, $J_n = \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)$ (the Jacobi group often refers to only the $n = 1$ case, for example in [14]). *Quasi-Jacobi forms* satisfy weaker transformation properties than Jacobi forms. More details for modular and (quasi) Jacobi forms will be provided in Chapter 4.

The original connection between VOAs and modular forms arose via what is called the (genus 1) *partition function* or 0-*point function* defined by

$$Z(\tau) := q^{-c/24} \sum_{n \in \mathbb{Z}} (\dim V_n) q^n,$$

where $q = e^{2\pi i\tau}$ (τ in $\mathbb{H}$) and c is the central charge of V. In 1978, John McKay observed that the first few coefficients of the modular form $j(\tau)$ were "natural" sums of dimensions of irreducible representations of the monster group $\mathbb{M}$. This mysterious connection was labelled "Monstrous Moonshine" and created a lot of interest among mathematicians. VOAs were developed largely as a way to understand McKay's observation [19].

Partition functions are a special case of more general functions called 1-*point functions*, which are of interest in both CFT and the algebraic study of VOAs. For each $v \in V_k$, the endomorphism $v(k-1)$ preserves the graded subspaces V_n of V. We denote $v(k-1)$ by $o(v)$ and call it the *zero mode* of v. We define the (genus 1) 1-*point function* by

$$Z(v,\tau) := q^{-c/24} \sum_{n\in\mathbb{Z}} \mathrm{Tr}_{V_n}\, o(v) q^n. \tag{1.0.0.1}$$

A *module* M for a VOA V is a vector space on which V acts in ways that preserve relevant algebraic structures, including an appropriate grading. The 1-point function above can be defined analogously for any module of a VOA. A vertex operator algebra is called *rational* if every *admissible* V-module is completely reducible. The definition of an admissible module, along with a number of other details pertaining to modules of VOAs will be given in Chapter 3.

For elements $v_1, \ldots, v_m \in V$, an m-point function on a module M^j of V is defined as

$$\mathrm{Tr}_{M^j}\, Y(q_1^{L(0)} v_1, q_1) \cdots Y(q_m^{L(0)} v_m, q_m) q^{L(0)-c/24},$$

where $q_k = e^{2\pi i w_k}$ with $w_k \in \mathbb{C}$, and $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$. In the case $m = 1$, the definition of a 1-point function here is the same as in (1.0.0.1). It is conjectured that 1-point functions associated to irreducible admissible modules of rational VOAs are modular forms on a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The proofs of special cases of this conjecture follow from a result of Zhu [46]. Zhu essentially shows that for elements $v \in V_{[k]}$ of a rational VOA V, the vector consisting of 1-point functions associated to irreducible admissible modules for V is a vector-valued modular form of weight k—a generalization of modular forms that is considered in Section 4.1.

Zhu also establishes a recursion formula in which n-point functions may be written as a sum of $(n-1)$-point functions with modular coefficients [46]. Along with the results pertaining to 1-point functions, this establishes that the space of n-point functions is a vector-valued modular form (see Theorem 5.1.1.1 below for a precise statement of Zhu's Theorem). A number of important extensions of this theory have been developed, such as to orbifold modules [7], intertwining operators [39], and $\mathbb{R}$-graded super VOAs [37].

This thesis considers trace functions of the form

$$\mathrm{Tr}_{M^j}\, Y(q_1^{L(0)}v_1, q_1)\cdots Y(q_m^{L(0)}v_m, q_m)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}, \tag{1.0.0.2}$$

where $\zeta_k = e^{2\pi i z_k}$, $z_k \in \mathbb{C}$ and $h_1, \ldots, h_n \in V_1$ satisfy Condition **H**. In the 1-point case, the function (1.0.0.2) reduces to

$$J_{j,\underline{h}}(v;\tau,\underline{z}) := \mathrm{Tr}_{M^j}\, o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}, \tag{1.0.0.3}$$

for homogeneous $v \in V$.

Establishing a recursion formula which expresses n-point functions of the form (1.0.0.2) as a sum of $(n-1)$-point functions with coefficients that carry an invariance with respect to the Jacobi group will be the purpose of Chapter 5. Such a recursion formula for functions in (1.0.0.2) will follow from results in [37] developed for n-point functions without the additional z variables. In this manner, the study of the Jacobi group invariance for the n-point functions (1.0.0.2) is reduced to establishing an invariance for the 1-point (1.0.0.3) and 0-point functions. The case of the 0-point functions will follow from the 1-point case by taking $v = \mathbf{1}$.

To prove the main transformation laws for the functions of the form (1.0.0.3) we will need a 1-point analogue of a result due to Miyamoto [38]. For a V-module M^j, $u, v \in V_1$ and $w \in V$, we define the function $\Phi_j(v; u, w, \tau)$ by

$$\Phi_j(v; u, w, \tau) := \mathrm{Tr}_{M^j}\, o(v) e^{2\pi i (w(0) + \langle u, w \rangle /2)} q^{L(0) + u(0) + \langle u, u \rangle /2 - c/24}. \tag{1.0.0.4}$$

We extend Miyamoto's proof to include zero modes under certain conditions. This establishes the following theorem. (See Theorem A in [38].)

Theorem 1.0.0.1 *Let V be a rational, C_2-cofinite vertex operator algebra and $M^1, \ldots, M^r$ its finitely many inequivalent irreducible admissible modules. Suppose $v \in V_1$ and $w \in V_{[k]}$ are such that $v(m)w = 0$ for $m \geq 0$. Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,*

$$\Phi_j(v; 0, w, \gamma\tau) = (c\tau + d)^k \sum_{i=1}^{r} A^i_{j,\gamma} \Phi_i(v; cw, dw, \tau), \tag{1.0.0.5}$$

where $A^i_{j,\gamma}$ are scalars that appear in Theorem 5.1.1.1 below.

Proving Theorem 1.0.0.1 is the focus of Chapter 6.

The development of transformation laws for 1-point functions in (1.0.0.3) must be broken into two cases, both of which are considered in Chapter 7. The convergence of such functions will be proved in Section 7.1. Also in this section, transformation laws with respect to the Jacobi group are developed for the functions (1.0.0.3) when $v \in V_{[k]}$ satisfies

$h_i(m)v = 0$ for $m \geq 0$ and $1 \leq i \leq n$. In particular, we will establish the following theorem.

Theorem 1.0.0.2 *Let V be a strongly regular VOA and $M^1, \ldots, M^r$ its finitely many inequivalent irreducible admissible modules. Let $h_1, \ldots, h_n$ satisfy Condition **H**. For any $v \in V$ the function $J_{j,\underline{h}}(v;\tau,\underline{z})$ converges on $\mathbb{H} \times \mathbb{C}^n$ and has a Fourier expansion of the form*

$$J_{j,\underline{h}}(v;\tau,\underline{z}) = q^{\lambda_j - c/24} \sum_{s \geq 0} \sum_{t_1,\ldots,t_n \in \mathbb{Z}} c(s,t_1,\ldots,t_n)\zeta_1^{t_1} \cdots \zeta_n^{t_n} q^s, \tag{1.0.0.6}$$

where $c(s,t_1,\ldots,t_n) \in \mathbb{C}$, λ_j is the conformal weight of M^j, and $\lambda_j = 0$ when $M^j = V$.

Suppose $v \in V_{[k]}$ is such that $h_i(m)v = 0$ for all $1 \leq i \leq n$ and $m \geq 0$. Then $J_{j,\underline{h}}(v;\tau,\underline{z})$ satisfies the following properties:

1. *for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,*

$$J_{j,\underline{h}}\left(v;\gamma\tau,\frac{\underline{z}}{c\tau+d}\right) = (c\tau+d)^k \exp\left(\pi i \frac{cG[\underline{z}]}{c\tau+d}\right) \sum_{\ell=1}^{r} A^{\gamma}_{j,\ell} J_{\ell,\underline{h}}(v;\tau,\underline{z}), \tag{1.0.0.7}$$

   and

2. *for all $[\underline{\lambda},\underline{\mu}] \in \mathbb{Z}^n \times \mathbb{Z}^n$ there is a $j' \in \{1,\ldots,r\}$ such that*

$$J_{j,\underline{h}}\left(v;\tau,\underline{z}+\underline{\lambda}\tau+\underline{\mu}\right) = \exp\left(-\pi i(G[\underline{\lambda}]\tau + 2\underline{z}^t G\underline{\lambda})\right) J_{j',\underline{h}}(v;\tau,\underline{z}). \tag{1.0.0.8}$$

The definition of a Jacobi form (and vector-valued Jacobi form) is given in Section 4.2. It is possible to choose $h_1, \ldots, h_n$ in Theorem 1.0.0.2 so that the corresponding Gram matrix is half-integral. Theorem 1.0.0.2 then implies that for $v \in V_{[k]}$ satisfying the assumptions of the previous theorem, the vector whose components are the functions

$\{J_{j,\underline{h}}(v;\tau,\underline{z}) \mid 1 \le j \le r\}$ is a vector-valued (weak) Jacobi form of weight k and index $G/2$. When V has only one irreducible module (itself), the function $J_{j,\underline{h}}$ is a weak Jacobi form of weight k, index $G/2$, and some character χ. These corollaries will be discussed in Section 7.3.

When $v \in V_{[k]}$ fails to satisfy $h_i(m)v = 0$ for $1 \le i \le n$ and $m \ge 0$, the functions (1.0.0.3) do not necessarily satisfy the transformation laws established in Theorem 1.0.0.2. Section 7.2 addresses this situation. In Subsection 7.2.1 we will show that any element of V can be written as a sum of elements with respect to a convenient decomposition. Taking the trace of any element in V then reduces to taking the trace of each of these individual elements. Among these elements we will find that the only ones that give a nonzero trace are of the form $v = u_1^{i_1}[-m_1]\cdots u_n^{i_n}[-m_n]w$ $(i_j \ge 0)$. Here, the set $\{u_j\}$ is an orthogonal basis for a natural Cartan subalgebra H contained in V_1, and w is in a subVOA $\Omega(0)$ of V that will be discussed later.

This decomposition of elements in V will be useful in Subsection 7.2.2 where we will prove the following theorem.

Theorem 1.0.0.3 *Let V be a strongly regular vertex operator algebra and M^j an irreducible V-module. Let $v \in V$ have the decomposition $v = u_1^{i_1}[-m_1]\cdots u_d^{i_k}[-m_d]w$, $m_1,\ldots,m_d \in \mathbb{N}$, where $w \in \Omega(0)$ and $\{u_t\}_{t=1}^{d}$ is an orthogonal basis for H. Then*

$$J_{j,\underline{h}}(v,\tau,\underline{z}) = \frac{\sum_{s=1}^{\delta} J_{\Omega_{M^j}(\gamma_s),\underline{h}}(w;\tau,\underline{z}) \sum_r f_r(\tau)\theta_{s,\underline{h}}(Q,a_r,k_r,\tau,\underline{z})}{\eta(\tau)^d}, \qquad (1.0.0.9)$$

where $f_r(\tau)$ is a quasi-modular form, $a_r \in H$, and $\theta_{s,\underline{h}}$ is defined as

$$\theta_{s,\underline{h}}(Q,a_r,k_r,\tau,\underline{z}) = \sum_{\alpha\in\Lambda+\gamma_s} \langle a_r,\alpha\rangle^{k_r}\, q^{\langle\alpha,\alpha\rangle/2}\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}. \qquad (1.0.0.10)$$

Here Λ is a positive-definite even lattice and the γ_s are certain states in V that will be discussed along with $\Omega_{M^j}(\gamma_s)$ in Subsection 7.2.1. Q is the quadratic form corresponding to Λ.

Theorem 1.0.0.3 generalizes Theorem 4.2.5 in [13] by including the z-variable and covering a larger class of VOAs. The functions (1.0.0.10) are discussed in Section 4.3 and satisfy certain transformation laws with respect to the Jacobi group. They are essentially Jacobi theta series with spherical harmonics. Conditions for when these functions are Jacobi forms are also established in Section 4.3. This is an extension of results in [43] and [10], and leads to the following refinement of Theorem 1.0.0.3.

Theorem 1.0.0.4 *Let $h_1, \ldots, h_n \in V_1$ be an orthogonal basis of the Cartan subalgebra H of V_1 and which satisfies Condition **H**. Let the rest of the assumptions be as in the previous theorem. Then for each s, $\sum_r f_r(\tau)\theta_{s,\underline{h}}(Q, a_r, k_r, \tau, \underline{z})$ is a sum of quasi-Jacobi forms on $\Gamma_0(N)$. Here N is the level of the integral quadratic form Q.*

The subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ is defined in Section 4.1. The proof of Theorem 1.0.0.4 is found in Subsection 7.2.3 and will rest on results obtained in Section 4.3 dealing with transformation laws of the functions (1.0.0.10).

It is conjectured that the VOA $\Omega(0)$ is strongly regular when V is strongly regular. Assuming this conjecture, combining Theorem 1.0.0.2 and Theorem 1.0.0.4 provide explicit transformation laws for the functions $J_{j,\underline{h}}(v; \tau, \underline{z})$ for any $v \in V$.

Some other work can be found in the literature which discusses Jacobi forms in the theory of vertex operator algebras. For example, in [8] the theory is developed for lattice VOAs, and in [24] it is essentially developed for VOAs associated to the highest weight integrable representations for affine Kac-Moody Lie algebras. However, Theorems 1.0.0.2, 1.0.0.3, and 1.0.0.4 establish a general theory to VOAs which extends beyond these results.

Finally, a few examples illustrating the applications of these theorems will be discussed in Section 7.4. We begin with Chapter 2 where we will establish notational convention that will be used in the following chapters.

Chapter 2

Notation

Throughout this thesis, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the integers, real numbers, and complex numbers, respectively. The set of non-negative integers is written $\mathbb{N}$, while the set of positive integers is labelled $\mathbb{Z}_+$. The complex upper half-plane, defined as the set of complex numbers with positive imaginary part, is denoted $\mathbb{H}$. For variables $z_1, \ldots, z_n$ we will denote the vector $(z_1, \ldots, z_n)$ by $\underline{z}$.

Of particular importance in the theory of vertex operator algebras is the space of *(doubly-infinite) formal Laurent series* in z with coefficients in a linear space V. This (linear) space is denoted $V[[x, x^{-1}]]$ and defined as

$$V[[x, x^{-1}]] := \left\{ \sum_{n \in \mathbb{Z}} v_n x^n \mid v_n \in V \right\}.$$

Particular subspaces of $V[[x, x^{-1}]]$ which occur in this work include the set of formal power series in x with coefficients in V,

$$V[[x]] := \left\{ \sum_{n \in \mathbb{N}} v_n x^n \mid v_n \in V \right\};$$

its subspace of V-valued polynomials,

$$V[x] := \left\{ \sum_{n\in\mathbb{N}} v_n x^n \mid v_n \in V \text{ and } v_n = 0 \text{ for all but finite many } n \right\};$$

the space of formal Laurent polynomials,

$$V[x, x^{-1}] := \left\{ \sum_{n\in\mathbb{Z}} v_n x^n \mid v_n \in V \text{ and } v_n = 0 \text{ for all but finite many } n \right\};$$

and the space of truncated formal Laurent series,

$$V((x)) := \left\{ \sum_{n\in\mathbb{Z}} v_n x^n \mid v_n \in V \text{ and } v_n = 0 \text{ for } n << 0 \right\}.$$

Each of these sets of formal series may be defined to include additional formal variables. For example, we set

$$V[[x, x^{-1}, y, y^{-1}]] = \left\{ \sum_{m,n\in\mathbb{Z}} v_{nm} x^m y^n \mid v_{mn} \in V \right\}.$$

For a series $f(x, y) \in V[[x, x^{-1}, y, y^{-1}]]$, the *residue* of $f(x, y)$ with respect to x is the coefficient of its x^{-1} term and is denoted $\operatorname{Res}_x f(x, y)$.

The *delta function*, defined by

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n \in \mathbb{C}[[x, x^{-1}]], \tag{2.0.0.1}$$

is used in the definition of a vertex algebra presented in this thesis. It can play a large role in the theory of vertex operator algebras (see [29]). The formal *exponential function* e^x is defined in the usual way as

$$\exp(x) = e^x := \sum_{n\in\mathbb{N}} \frac{1}{n!} x^n.$$

We will use the common convention of setting $q := e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$ throughout this work. For $e^{2\pi iz}$, $z \in \mathbb{C}$, we set $\zeta = \zeta_z := e^{2\pi iz}$. When we have indexed complex variables such as

$z_j \in \mathbb{C}$, we simply write ζ_j to denote ζ_{z_j}.

All binomial expansions are assumed to be expanded in non-negative powers of the second variable. That is, for $n \in \mathbb{Z}$,

$$(x+y)^n = \sum_{i \in \mathbb{N}} \binom{n}{i} x^{n-i} y^i,$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

Caution should be taken. While $(x+y)^n$ equals $(y+x)^n$ for $n \geq 0$, $(x+y)^n$ does not equal $(y+x)^n$ when $n < 0$.

The space of endomorphisms and automorphisms of a linear space V will be denoted by $\operatorname{End} V$ and $\operatorname{Aut} V$, respectively.

Chapter 3

Vertex Operator Algebras

In this chapter we discuss the relevant definitions and properties of vertex operator algebras and their modules that will be used in this thesis. More information can be found in a number of texts, including [29] and [23].

3.1 Vertex algebras and vertex operator algebras

A vector space V is a *vertex algebra* if it contains an element, denoted $\mathbf{1}$ and called the *vacuum vector* of V, and is equipped with a linear map

$$\begin{aligned} Y(\cdot, z)\colon V &\to (\operatorname{End} V)[[z, z^{-1}]], \\ v &\mapsto Y(v, z) := \sum_{n\in\mathbb{Z}} v(n) z^{-n-1}, \end{aligned} \tag{3.1.0.1}$$

such that the following axioms are satisfied for any $u, v \in V$;

1. the *truncation condition*: $u(n)v = 0$ for n sufficiently large,

2. the *creation property*: $Y(u, z)\mathbf{1} = u + \{\text{terms involving positive powers of } z\}$,

3. the *vacuum property*: $Y(\mathbf{1}, z) = 1$, where 1 is the identity operator, and

4. the *Jacobi identity*:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1) \\ = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2). \qquad (3.1.0.2)$$

The vector space V is called the *Fock space* of the vertex algebra, and for each $u \in V$, the infinite sum $Y(u, z)$ is called the *vertex operator (associated with u)*. The importance of the vacuum element and map $Y(\cdot, z)$ in the definition of a vertex algebra is paramount. To emphasize the dependence of the definition on these components, a vertex algebra is often denoted by the triple $(V, Y, \mathbf{1})$ rather than just V.

The endomorphisms $u(n)$ defined in 3.1.0.1 are called the *modes* of u. From the the creation property, it follows that $u(-1)\mathbf{1} = u$ for all $u \in V$. It is useful to express the Jacobi identity in terms of modes. Expanding the vertex operators and delta functions for $\ell, m, n \in \mathbb{Z}$ in 3.1.0.2 and equating the coefficients of the $z_0^{-\ell-1}z_1^{-m-1}z_2^{-n-1}$ terms establishes

$$\sum_{i\geq 0}(-1)^i\binom{l}{i}u(m + \ell - i)v(n + i) - (-1)^\ell\sum_{i\geq 0}(-1)^i\binom{\ell}{i}v(n + \ell - i)u(m + i) \\ = \sum_{i\geq 0}\binom{m}{i}(u(\ell + i)v)(m + n - i). \qquad (3.1.0.3)$$

The identity (3.1.0.3) is found in [15] and is a generalization of the original incarnation of a vertex algebra developed by Richard Borcherds in [3].

Taking $\ell = 0$ in (3.1.0.3) leads to the *commutator formula*,

$$[u(m), v(n)] = \sum_{i\geq 0}\binom{m}{i}(u(i)v)(m + n - i), \qquad (3.1.0.4)$$

for $m, n \in \mathbb{Z}$. Applying (3.1.0.4) to a vertex operator gives

$$[u(m), Y(v, z)] = \sum_{n \in \mathbb{Z}} \sum_{i \geq 0} \binom{m}{i} (u(i)v)(m+n-i) z^{-n-1}, \tag{3.1.0.5}$$

and more generally, for $v_1, \ldots, v_n \in V$,

$$\begin{aligned}
&[u(m), Y(v_1, z_1) \cdots Y(v_n, z_n)] \\
&\qquad = \sum_{j=1}^{n} \sum_{i \geq 0} \binom{m}{i} z_j^{m-i} Y(v_1, z_1) \cdots Y(u(i)v_j, z_j) \cdots Y(v_n, z_n).
\end{aligned} \tag{3.1.0.6}$$

On the other hand, taking $m = 0$ in (3.1.0.3) and relabelling indices establishes the *associator formula*,

$$(u(m)v)(n) = \sum_{i \geq 0} (-1)^i \binom{m}{i} (u(m-i)v(n+i) - (-1)^m v(m+n-i)u(i)), \tag{3.1.0.7}$$

for $m, n \in \mathbb{Z}$. The commutator and associator formulas are useful tools in computations involving vertex operators and their modes.

Also derived from the Jacobi identity is the existence of non-negative integers k and ℓ dependent on the vertex algebra such that

$$(z_1 - z_2)^k [Y(u, z_1), Y(v, z_2)] = 0, \tag{3.1.0.8}$$

and

$$(z_1 + z_2)^\ell Y(Y(u, z_0)v, z_2)w = (z_0 + z_2)^\ell Y(u, z_0 + z_2) Y(v, z_2)w, \tag{3.1.0.9}$$

for all $u, v, w \in V$. These formulas are known as *locallity* and *(weak) associativity*, respectively. It is known (for example [29]) that these two conditions are equivalent to the Jacobi identity in the definition of a vertex algebra.

A space V is a *vertex operator algebra (VOA) of central charge c* if in addition to being a vertex algebra, V has a $\mathbb{Z}$-grading

$$V = \bigoplus_{n\in\mathbb{Z}} V_n,$$

where

$$\dim V_n < \infty,$$

$$V_n = 0 \text{ for } n \text{ sufficiently negative,}$$

and V contains an element $\omega \in V_2$ (called the *conformal* or Virasoro element) such that after defining the operators $L(n)$ by

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega(n) z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},$$

the following conditions hold:

(i) for all $n \in \mathbb{Z}$,

$$V_n = \{v \in V \mid L(0)v = nv\},$$

(ii) for all $v \in V$,

$$Y(L(-1)v, z) = \frac{d}{dz} Y(v, z), \tag{3.1.0.10}$$

and

(iii) for any $m, n \in \mathbb{Z}$, the *Virasoro relation*,

$$[L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0} \frac{m^3 - m}{12} c, \tag{3.1.0.11}$$

holds, where $c \in \mathbb{C}$.

Similar to the case of vertex algebras, the vertex operator algebra V is often denoted by the quadruple $(V, Y, \mathbf{1}, \omega)$.

A *subVOA* of $(V, Y, \mathbf{1}, \omega)$ is a subspace W of V such that $(W, Y, \mathbf{1}, \omega')$ is again a vertex operator algebra. While the map $Y(\cdot, z)$ and vacuum vector $\mathbf{1}$ are shared between W and V, the conformal vectors may be different. In the case that $\omega' = \omega$, $(W, Y, \mathbf{1}, \omega)$ is said to be a *conformal* subVOA.

A subspace I of a vertex algebra V is an *ideal* of V if for all $v \in V$, $u \in I$, and $n \in \mathbb{Z}$, $v(n)u \in I$ and $u(n)v \in I$. The subspaces V and 0 are always ideals of V. For a subspace I of a vertex operator algebra V, conditions (i) and (ii) are equivalent and the definition of an ideal may be slightly simplified. V is called *simple* in the case $V \neq 0$ and V has no other ideals besides itself and 0.

A homogeneous element v in V_k is said to be of *weight* k. When the weight of a homogeneous element v is not specified, we will denote it $\operatorname{wt} v$. Condition (3.1.0.10) is called the $L(-1)$-*derivative property*. Recalling that $L(n) = \omega(n+1)$ for $n \in \mathbb{Z}$ and using (3.1.0.4), it follows that

$$\begin{aligned}
[L(-1), Y(v,z)] &= \sum_{m\in\mathbb{Z}} [\omega(0), v(m)] z^{-m-1} = \sum_{m\in\mathbb{Z}} \sum_{i\geq 0} \binom{0}{i} (\omega(i)v)(m-i) z^{-m-1} \\
&= \sum_{m\in\mathbb{Z}} (L(-1)v)(m) z^{-m-1} = Y(L(-1)v, z).
\end{aligned}$$

Along with the $L(-1)$-derivative property, this establishes the equality

$$[L(-1), Y(v,z)] = Y(L(-1)v, z) = \frac{d}{dz} Y(v,z),$$

for any $v \in V$. In terms of the modes of v this becomes

$$[L(-1), v(n)] = (L(-1)v)(n) = -nv(n-1). \tag{3.1.0.12}$$

Similarly, we find

$$[L(0), v(n)] = [\omega(1), v(n)] = \sum_{i\geq 0} \binom{1}{i} (\omega(i)v)(n+1-i) = (L(-1)v)(n+1) + (L(0)v)(n)$$

$$= -(n+1)v(n) + (\mathrm{wt}\, v)v(n) = (\mathrm{wt}\, v - n - 1)v(n).$$

Let u and v be homogeneous elements in V of weight k and $\mathrm{wt}\, v$, respectively. Then for any $n \in \mathbb{Z}$,

$$L(0)v(n)u = v(n)L(0)u + [L(0), v(n)]u = kv(n)u + (\mathrm{wt}\, v - n - 1)v(n)u$$

$$= (k + \mathrm{wt}\, v - n - 1)v(n)u.$$

It follows that as an operator,

$$v(n)\colon V_k \to V_{k+\mathrm{wt}\, v-n-1}. \tag{3.1.0.13}$$

For a homogeneous element $v \in V$, the unique endomorphism $v(\mathrm{wt}\, v - 1)$ is called the *zero mode* of v and is denoted by $o(v)$. The zero mode of v preserves each graded weight space, so that $o(v)\colon V_n \to V_n$ for all $n \in \mathbb{Z}$.

We finish this subsection with a computation of another result which will be of use later. Let $v \in V_k$ and $w \in V_\ell$. Then $L(0)v(n)w = (\ell + k - n - 1)v(n)w$ and for a variable x we find that

$$e^{xL(0)}Y(v,z)e^{-xL(0)}w = e^{xL(0)} \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}e^{-xL(0)}w = \sum_{n\in\mathbb{Z}} e^{xL(0)}v(n)z^{-n-1}e^{-x\ell}w$$

$$= \sum_{n\in\mathbb{Z}} e^{xL(0)}v(n)wz^{-n-1}e^{-x\ell} = \sum_{n\in\mathbb{Z}} e^{x(\ell+k-n-1)}v(n)wz^{-n-1}e^{-x\ell}$$

$$= \sum_{n\in\mathbb{Z}} e^{x\ell-x\ell+xk+x(-n-1)}v(n)wz^{-n-1} = \sum_{n\in\mathbb{Z}} e^{xk}v(n)we^{x(-n-1)}z^{-n-1}$$

$$= \sum_{n\in\mathbb{Z}} (e^{xL(0)}v)(n)(e^x z)^{-n-1}w = Y(e^{xL(0)}v, e^x z)w.$$

Extending by linearity, this establishes for any $v, w \in V$ that

$$e^{xL(0)}Y(v,z)e^{-xL(0)} = Y(e^{xL(0)}v, e^x z). \tag{3.1.0.14}$$

3.1.1 The "square-bracket" vertex operator algebra

The goal of this subsection is to provide the Fock space V of a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ with an alternate, yet isomorphic, vertex operator algebra structure. This different "square-bracket" vertex operator algebra structure will play a significant role in this thesis. It is a special case of a more general theory (see [22, 2]).

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. Define the "square-bracket" vertex operators on V by

$$\begin{aligned} Y[\cdot, z]\colon V &\to (\operatorname{End} V)[[z, z^{-1}]], \\ v &\mapsto Y[v, z] := Y(q_z^{L(0)}v, q_z - 1). \end{aligned} \tag{3.1.1.1}$$

The resulting modes of $Y[v, z]$ are defined by

$$Y[v, z] = \sum_{n\in\mathbb{Z}} v[n]z^{-n-1}.$$

Typically, $v(n) \neq v[n]$. However, for an integer k,

$$\sum_{i\geq 0} \binom{k}{i} v(i) = \sum_{m\geq 0} \frac{(k+1-\operatorname{wt} v)^m}{m!} v[m]. \tag{3.1.1.2}$$

In particular, taking $k = 0$ shows

$$v(0) = \sum_{m\geq 0} \frac{(1-\operatorname{wt} v)^m}{m!} v[m].$$

It follows that for all $v \in V_1$,

$$v(0) = v[0]. \tag{3.1.1.3}$$

Set

$$\tilde{\omega} = \omega - c/24,$$

and define the operators $L[n]$ by

$$Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} \tilde{\omega}[n] z^{-n-1} = \sum_{n\in\mathbb{Z}} L[n] z^{-n-2}.$$

The operators $L[n]$ satisfy the Virasoro relations (3.1.0.11) and the $L[-1]$-derivative property, (3.1.0.10). Moreover, $L[0]$ provides a grading on V,

$$V = \bigoplus_{n\in\mathbb{Z}} V_{[n]},$$

where $V_{[n]} = \{v \in V \mid L[0]v = nv\}$. It is known (see [22, 46]) that the quadruple $(V, Y[\,], \mathbf{1}, \tilde{\omega})$ is a vertex operator algebra that is isomorphic to $(V, Y, \mathbf{1}, \omega)$. While it is typically not true that $V_n = V_{[n]}$, it is known that for any N,

$$\bigoplus_{n\leq N} V_n = \bigoplus_{n\leq N} V_{[n]}.$$

We let $[\mathrm{wt}]v$ denote the weight of a homogeneous element v in the vertex operator algebra $(V, Y[\,], \mathbf{1}, \tilde{\omega})$. Note that for $v \in V_{[k]}$, the operator $q_z^{L[0]} \colon V \to V[[z]]$ takes $v \mapsto q_z^k v$.

The difference between the two vertex operator algebras $(V, Y, \mathbf{1}, \omega)$ and $(V, Y[\,], \mathbf{1}, \tilde{\omega})$ stems from a difference in the geometric objects for which the respective VOAs are defined on. The VOA structure given by the $(V, Y, \mathbf{1}, \omega)$ defines a VOA on a Riemann sphere. The change of variable $z \mapsto e^z - 1$ defines the VOA $(V, Y[\,], \mathbf{1}, \tilde{\omega})$ on a torus.

3.2 Modules

3.2.1 Definitions

Let V be a vertex operator algebra. A vector space M is said to be a *weak V-module* if it is equipped with a linear map

$$Y^M(\cdot, z)\colon V \to (\operatorname{End} M)[[z, z^{-1}]]$$
$$v \mapsto Y^M(v, z) := \sum_{n\in\mathbb{Z}} v^M(n) z^{-n-1},$$

such that the following axioms are satisfied for all $u, v \in V$ and $w \in M$:

1. the *truncation condition*: $u^M(n)w = 0$ for n sufficiently large,

2. the *vacuum property*: $Y^M(\mathbf{1}, z) = 1$, where 1 is the identity operator of M, and

3. the *Jacobi identity*:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y^M(u,z_1)Y^M(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 + z_1}{-z_0}\right) Y^M(v, z_2)Y^M(u, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y^M(Y(u, z_0)v, z_2). \qquad (3.2.1.1)$$

A V-module M can also be denoted (M, Y^M) to emphasis the map Y^M. When a module is written with an index, i.e. M^j, the notation Y^j is used in place of Y^{M^j}. While the use of M in $v^M(n)$ conveniently emphasizes the action of the mode on M, it will often be omitted.

An *admissible V-module M* is a weak V-module which also carries an $\mathbb{N}$-grading

$$M = \bigoplus_{n\geq 0} M_n$$

such that for any $v \in V_n$,

$$v^M(m)\colon M_k \to M_{k+m-n-1}$$

for all $m, n \in \mathbb{Z}$.

If $M \neq 0$, the grading may be shifted so that $M_0 \neq 0$. We take this to be the case throughout this thesis and refer to M_0 as the *top level* of M.

A weak V-module M is called an *ordinary V-module* (or simply a *V-module*) if M has a $\mathbb{C}$-grading

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda,$$

such that

1. $\dim M_\lambda < \infty$,

2. $M_{\lambda+n} = 0$ for n sufficiently negative, and

3. $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$.

By definition, an admissible module is a weak module. Additionally, it can be shown that an ordinary module is always an admissible module. This provides an inclusion,

$$\text{Ordinary modules} \subseteq \text{Admissible modules} \subseteq \text{Weak modules}.$$

A *submodule* of a V-module M is a subspace U such that (U, Y^M) is also a V-module. If a (weak, admissible, ordinary) module does not contain any proper submodules, it is called *irreducible*. Of central importance to this thesis are the following definitions:

Definition 3.2.1.1 *Let V be a vertex operator algebra.*

1. V is ***rational*** *if its admissible module category is semisimple. That is, if every admissible module is a direct sum of irreducible admissible modules.*

2. *V is C_2-**cofinite** if the subspace $C_2(V) := \langle u(-2)v \mid u, v \in V\rangle$ has finite codimension in V.*

3. *V is of **CFT-type** if it has no negative weight spaces and its weight zero space is one-dimensional. That is,*

$$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus V_2 \oplus \cdots .$$

An element v of V is *quasi-primary* if $L(1)v = 0$, and *primary* if $L(n)v = 0$ for all $n \geq 1$.

Definition 3.2.1.2 *A vertex operator algebra V is **strongly regular** if it is rational, C_2-cofinite, of CFT-type, and every element $v \in V_1$ is quasi-primary.*

This thesis is concerned with simple strongly regular vertex operator algebras. Such VOAs come equipped with additional structure that is necessary to obtain the results of this thesis. A table categorizing where the different assumptions are required is included in the Appendix.

The rationality of V implies it has finitely many inequivalent irreducible admissible modules [6] (see also [5] for more comments on regular VOAs). Unless otherwise stated, we denote these inequivalent irreducible modules by $M^1, \ldots, M^r$. A vertex operator algebra V may be viewed as a V-module itself, and in this context, it is referred to as the *adjoint* module. A vertex operator algebra is called *holomorphic* if the adjoint module is its only irreducible admissible module.

A consequence of V being C_2-cofinite is that every weak V-module is an admissible module [1]. Meanwhile, if V is C_2-cofinite and rational, it can be shown that every irreducible admissible module is an ordinary module [1]. In other words, when V is strongly regular, the notions of weak, admissible, and ordinary module are equivalent.

A bilinear form $\langle\cdot,\cdot\rangle_M$ on a V-module M is said to be *invariant* if

$$\langle Y(a,z)u,v\rangle_M = \left\langle u, Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})v\right\rangle_M, \tag{3.2.1.2}$$

for $a \in V$ and $u, v \in M$. When $M = V$, we simply write $\langle\cdot,\cdot\rangle$. It is known [16] that any invariant bilinear form on V is symmetric. Haisheng Li has shown [31] that the space of all symmetric invariant bilinear forms on V is isomorphic to the dual space of $V_0/(L(1)V_1)$. In the case $L(1)V_1 = 0$, it follows that the space of symmetric invariant bilinear forms on V is isomorphic to $V_0 = \mathbb{C}\mathbf{1}$. Furthermore, in the situation that V is of CFT-type, $\langle\cdot,\cdot\rangle$ is nondegenerate if, and only if, V is simple.

For a simple vertex operator algebra V, we normalize an invariant bilinear form $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ so that $\langle\mathbf{1},\mathbf{1}\rangle = -1$. So long as V is of CFT-type, then for $u, v \in V_1$,

$$\begin{aligned}\langle u,v\rangle\,\mathbf{1} &= \mathrm{Res}_z\, z^{-1}\,\langle Y(u,z)\mathbf{1},v\rangle\,\mathbf{1} = \mathrm{Res}_z\, z^{-1}\left\langle \mathbf{1}, Y(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})v\right\rangle \mathbf{1}\\ &= \mathrm{Res}_z\, z^{-1}\sum_{n\in\mathbb{Z}}\left\langle \mathbf{1}, -u(n)vz^{n-1}\right\rangle \mathbf{1} = -u(1)v\,\langle\mathbf{1},\mathbf{1}\rangle\,\mathbf{1}\\ &= u(1)v. \end{aligned} \tag{3.2.1.3}$$

Therefore, when V is a simple strongly regular vertex operator algebra, we have a unique symmetric invariant bilinear form normalized as above that is also non-degenerate.

Consider the bracket $[\cdot,\cdot]\colon V_1 \times V_1 \to V_1$ defined by $[u,v] = u(0)v$. If V is of CFT-type, the bracket $[\cdot,\cdot]$ equips V_1 with the structure of a Lie algebra. If V is strongly regular, V_1 is a *reductive* Lie algebra [11, 34]. This will be of importance in Chapter 7.

3.2.2 Automorphisms

An *automorphism* g of a vertex operator algebra V is an invertible linear map from V to V such that $g(\omega) = \omega$ and

$$gY(v, z)g^{-1} = y(g(v), z)$$

for all $v \in V$. This latter condition is equivalent to

$$gv(n)g^{-1} = (g(v))(n), \tag{3.2.2.1}$$

for all $v \in V$ and $n \in \mathbb{Z}$. The set of all automorphisms of V, denoted $\mathrm{Aut}(V)$, is a group under composition. Note that

$$g\omega(n) = (g(\omega))(n)g = \omega(n)g,$$

showing that g commutes with all modes of ω, and in particular $L(0) = \omega(1)$.

Lemma 3.2.2.1 *For any $u, v \in V$, $k \in \mathbb{Z}$, and $n \geq 0$, we have*

$$(u(0)^n v)(k) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} u(0)^{n-i} v(k) u(0)^i.$$

Proof We will prove the lemma by induction on n. For $n = 1$, (3.1.0.7) gives

$$(u(0)v)(k) = u(0)v(k) - v(k)u(0) = \sum_{i=0}^{1} (-1)^i \binom{1}{i} u(0)^{1-i} v(k) u(0)^i.$$

Suppose that

$$(u(0)^{n-1} v)(k) = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} u(0)^{n-1-i} v(k) u(0)^i.$$

Using (3.1.0.7) and our induction hypothesis, we find

$$
\begin{aligned}
(u(0)^n v)(k) &= (u(0)u(0)^{n-1}v)(k) = u(0)(u(0)^{n-1}v)(k) - (u(0)^{n-1}v)(k)u(0) \\
&= \sum_{i=0}^{n-1}(-1)^i \binom{n-1}{i} u(0)^{n-i}v(k)u(0)^i - \sum_{i=0}^{n-1}(-1)^i \binom{n-1}{i} u(0)^{n-1-i}v(k)u(0)^{i+1} \\
&= u(0)^n v(k) + \sum_{i=1}^{n-1}(-1)^i \binom{n-1}{i} u(0)^{n-i}v(k)u(0)^i \\
&\qquad + \sum_{i=1}^{n-1}(-1)^i \binom{n-1}{i-1} u(0)^{n-i}v(k)u(0)^i + (-1)^n v(k)u(0)^n \\
&= u(0)^n v(k) + \sum_{i=1}^{n-1}(-1)^i \left[\binom{n-1}{i} + \binom{n-1}{i-1}\right] u(0)^{n-i}v(k)u(0)^i + (-1)^n v(k)u(0)^n \\
&= \sum_{i=0}^{n}(-1)^i \binom{n}{i} u(0)^{n-i}v(k)u(0)^i,
\end{aligned}
$$

as desired. □

We are now in position to prove the following lemma.

Lemma 3.2.2.2 *Let $u \in V_1$. Then $e^{u(0)}$ is an automorphism of V.*

Proof Using Lemma 3.2.2.1 we find

$$
\begin{aligned}
(e^{u(0)}v)(k) &= \left(\sum_{n\geq 0} \frac{u(0)^n}{n!} v\right)(k) = \sum_{n\geq 0} \frac{1}{n!}(u(0)^n v)(k) \\
&= \sum_{n\geq 0} \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} u(0)^{n-i}v(k)u(0)^i \\
&= \sum_{n\geq 0} \sum_{i=0}^{n} \left(\frac{u(0)^{n-i}}{(n-1)!}\right) v(k) \left(\frac{(-1)^i u(0)^i}{i!}\right) \\
&= e^{u(0)}v(k)e^{-u(0)}.
\end{aligned}
$$

This shows (3.2.2.1) is satisfied. Since $u \in V_1$, we have $u(0)\mathbf{1} = 0$. Using (3.1.0.5) and (3.1.0.12), it follows that

$$\begin{aligned}u(0)\omega &= u(0)\omega(-1)\mathbf{1} = -[\omega(-1), u(0)]\mathbf{1} \\ &= -\sum_{i\geq 0}(\omega(i)u)(-1-i)\mathbf{1} = -(\omega(0)u)(-1)\mathbf{1} - (\omega(1)u)(0)\mathbf{1} \\ &= -(L(-1)u)(-1)\mathbf{1} - (L(0)u)(0)\mathbf{1} = 0.\end{aligned}$$

This completes the proof of the lemma. □

3.2.3 Twisted modules

Let V be a vertex operator algebra and g an automorphism of V with finite order T. For $0 \leq r \leq T-1$, set

$$V^r := \left\{ v \in V \mid gv = e^{-2\pi i r/T} v \right\}.$$

A linear space M equipped with a linear map

$$\begin{aligned}Y^M \colon V &\to (\mathrm{End}\, M)[[z^{-1/T}, z^{1/T}]], \\ v &\mapsto Y^M(v,z) := \sum_{n \in \mathbb{Z}+1/T} v^M(n) z^{-n-1},\end{aligned}$$

is a *weak g-twisted* V-module if for $u \in V^r$, $v \in V$, and $w \in M$; the truncation condition, $v(m)w = 0$ for $m >> 0$; the vacuum property, $Y_M(\mathbf{1}, z) = 1$; and the twisted-Jacobi identity,

$$\begin{aligned}z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1) \\ = z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v, z_2), \qquad (3.2.3.1)\end{aligned}$$

all hold. Analogous to the definitions for a V-module, a weak g-twisted V-module is called *admissible* if it has a $\frac{1}{T}\mathbb{Z}_+$-grading

$$M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M_n$$

such that for homogeneous $v \in V$, $v^M(m)M_n \subseteq M_{n+\mathrm{wt}\, v-m-1}$. A weak g-twisted V-module M is called an *ordinary* g-twisted V-module (or simply a g-twisted V-module) if M has a complex grading

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda,$$

where $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$, $\dim M_\lambda < \infty$, and $M_{\lambda+n} = 0$ for integers n sufficiently small.

In the case $g = 1$, the definitions for weak, admissible, and ordinary g-twisted V-modules reduce to the original definitions of weak, admissible, and ordinary V-module, respectively. In the admissible g-twisted V-module case, M_λ is again assumed to not be 0 when $M \neq 0$. If M is a simple g-twisted V-module, then

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$$

for some $\lambda \in \mathbb{C}$ such that $M_\lambda \neq 0$. (See also [6, 7] for more details.)

3.2.4 Li's "Delta"-function

Again let V be a vertex operator algebra and g an automorphism of V with finite order T. Consider an element $h \in V$ such that

$$L(n)h = \delta_{n,0}h, \tag{3.2.4.1}$$

$$g(h) = h, \tag{3.2.4.2}$$

$$h(0) \text{ is semisimple on } V, \text{ and} \tag{3.2.4.3}$$

$$[h(m), h(n)] = 0 \tag{3.2.4.4}$$

for all $m, n \in \mathbb{Z}_+$. Moreover, we require that $\operatorname{Spec} h(0) \subseteq \frac{1}{T}\mathbb{Z}$. For such an h, define the map $\Delta_h(z) \colon V \to (\operatorname{End} V)[[z^{-1/T}, z^{1/T}]]$ by

$$\Delta_h(z) := z^{h(0)} \exp\left\{ -\sum_{k\geq 1} \frac{h(k)}{k}(-z)^{-k} \right\}. \tag{3.2.4.5}$$

Note that $\Delta_h(z)$ is invertible. Define the map $Y^M_{\Delta_h(z)}(\cdot, z) \colon V \to (\operatorname{End} M)[[z^{-1/T}, z^{1/T}]]$ by

$$Y^M_{\Delta_h(z)}(v, z) := Y^M\left(\Delta_h(z)v, z\right).$$

Recall from the previous subsection that $e^{2\pi i h(0)}$ is an automorphism of V. The following theorem due to Li [32] (proposition 5.4) is very useful.

Theorem 3.2.4.1 (Li) *Let (M, Y^M) be a g-twisted V-module. Then $(M, Y^M_{\Delta_h(z)})$ is a weak $(ge^{2\pi i h(0)})$-twisted V-module.*

In the context of this thesis, which is concerned with strongly regular vertex operator algebras, weak modules are ordinary modules. In particular, Li's Theorem again holds with weak module replaced by ordinary module.

When (M, Y^M) is an irreducible g-twisted V-module, $(M, Y^M_{\Delta_h(z)})$ is also irreducible. This follows from the invertibility of $\Delta_h(z)$. Taking $g = 1$ in Theorem 3.2.4.1 results in the construction of an $e^{2\pi i h(0)}$-twisted V-module $(M, Y^M_{\Delta_h(z)})$ from the V-module (M, Y^M). In this way, any $e^{2\pi i h(0)}$-twisted module can be constructed from *some* weak V-module. In particular, if $h(0)$ has integral eigenvalues, any V-module (viewed as a 1-twisted V-module) may be constructed from some (perhaps the same) V-module.

Let V be a strongly regular vertex operator algebra and $M^1, \ldots, M^r$ its finitely many inequivalent irreducible admissible modules. Take $h(0)$ to satisfy conditions (3.2.4.1)—(3.2.4.4) and have integral eigenvalues. (Note that an element h that satisfies Condition **H** also satisfies (3.2.4.1)—(3.2.4.4) for $g = 1$ and $g = e^{2\pi i h(0)}$.) Then for the (1-twisted) V-module (M^j, Y^j), there is another (possibly the same) V-module $(M^{j'}, Y^{j'})$ such that the (again 1-twisted) V-module $(M^{j'}, Y^{j'}_{\Delta_h(z)})$ is isomorphic to (M^j, Y^j). In other words, for any of the modules (M^j, Y^j), there exists a $j' \in \{1, \ldots, r\}$ such that there is an isomorphism

$$(M^{j'}, Y^{j'}_{\Delta_h(z)}) \cong (M^j, Y^j). \tag{3.2.4.6}$$

This isomorphism plays a large role in Chapter 7.

3.2.5 The Heisenberg VOA

Let H be a d-dimensional Lie algebra with non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Consider the affinization of the Lie algebra of H, $\hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$, with commutator relations

$$[a \otimes t^m, b \otimes t^n] = \langle a, b \rangle \, \delta_{m+n,0} K, \text{ and}$$

$$[K, a \otimes t^m] = 0,$$

for all $a, b \in H$, and $m, n \in \mathbb{Z}$. Let $H \otimes \mathbb{C}[t]$ act trivially on $\mathbb{C}$, and K act as 1. Consider the induced module

$$\begin{aligned} M_H &= \mathcal{U}(\hat{H}) \otimes_{H \otimes \mathbb{C}[t] \oplus \mathbb{C}K} \mathbb{C} \\ &\cong S(H \otimes t^{-1}\mathbb{C}[t^{-1}]), \end{aligned}$$

where $S(H \otimes t^{-1}\mathbb{C}[t^{-1}])$ is the symmetric algebra and the isomorphism is as linear spaces.

Let the action of $h \otimes t^n$ on M_H be denoted by $h(n)$. For an orthonormal basis $\{h_1, \ldots, h_d\}$ of H, set $\omega_{M_H} = \frac{1}{2}\sum_{i=1}^{d} h_i(-1)^2\mathbf{1}$, where $\mathbf{1} = 1 \otimes 1$. Any element $v \in M_H$ can be written as a linear combination of elements of the form

$$v = a_1(-n_1) \cdots a_k(-n_k)\mathbf{1},$$

for $a_1, \ldots, a_k \in H$, $n_1, \ldots, n_k \in \mathbb{Z}_+$. For such an element $v \in M_H$, define the map $Y(\cdot, z) \colon M_H \to (\operatorname{End} M_H)[[z, z^{-1}]]$ by

$$Y(v, z) = {}^{\circ}_{\circ}\partial^{(n_1-1)}a_1(z) \cdots \partial^{(n_k-1)}a_k(z){}^{\circ}_{\circ},$$

where $\partial^{(n)} = \frac{1}{n!}\left(\frac{d}{dz}\right)^n$ and ${}^{\circ}_{\circ} \cdots {}^{\circ}_{\circ}$ signifies normal ordering (see [15]).

It is known (for example [29]) that $(M_H, Y, \mathbf{1}, \omega_{M_H})$ is a simple (though not rational!) vertex operator algebra with $L(0)$-grading

$$M_H = \bigoplus_{n \geq 0}(M_H)_n,$$

where

$$\begin{aligned} (M_H)_n &= \{v \in M_H \mid L(0)v = nv\} \\ &= \left\langle a_1(-n_1) \cdots a_k(-n_k)\mathbf{1} \mid a_1, \ldots, a_k \in H, n_1, \ldots, n_k \in \mathbb{Z}_+, \sum n_i = n \right\rangle. \end{aligned}$$

There is a natural identification between $(M_H)_1$ and H given by $h(-1)\mathbf{1} \mapsto h$. Moreover, by (3.1.1.3), $a[0] = a(0) = 0$ and $a[1]b = a(1)b = \langle a, b\rangle \mathbf{1}$ for $a, b \in H$.

For $\alpha \in \mathbb{C}$, define the space

$$M_H(\alpha) := M_H \otimes e^{\alpha}. \tag{3.2.5.1}$$

For $n \neq 0$, the operators $a(n) \in \operatorname{End} M_H$ act on $M_H(\alpha)$ via its action on $M(1)$. Meanwhile, $a(0)$ acts on e^{α} by $a(0)e^{\alpha} = \langle a, \alpha\rangle e^{\alpha}$. $M_H(\alpha)$ is an M_H-module. For each α in the dual of H, denoted H°, $M_H(\alpha)$ is an irreducible M_H-module with conformal weight $\frac{1}{2}\langle \alpha, \alpha\rangle$. These are all irreducible modules up to equivalence (see [15, 29]).

Chapter 4

Modular and Jacobi Forms

4.1 Modular forms

4.1.1 Definitions

The *(homogeneous) modular group* is defined by

$$\Gamma := \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}, \tag{4.1.1.1}$$

and is generated by the matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. There is a left-action of the modular group Γ on the complex upper-half plane $\mathbb{H}$ given by Möbius transformations,

$$\begin{aligned} &\Gamma \times \mathbb{H} \to \mathbb{H} \\ &(\gamma, \tau) \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d}, \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \end{aligned}$$

Under this action, $T\tau = \tau + 1$ and $S\tau = -1/\tau$ for all $\tau \in \mathbb{H}$.

Definition 4.1.1.1 *Let k be an integer. A* ***meromorphic modular form*** *of* ***weight*** *k on Γ is a meromorphic function f on $\mathbb{H}$ such that*

1. *for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,*

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \tag{4.1.1.2}$$

   and

2. *f has a Fourier series expansion, or* ***q-expansion***,

$$f(\tau) = \sum_{n \geq N} a_n q^n = g(q), \tag{4.1.1.3}$$

   where $q = e^{2\pi i\tau}$ and $N \in \mathbb{Z}$.

The latter condition states that g is meromorphic at $q = 0$. If g is holomorphic at $q = 0$, then f is said to be holomorphic at infinity.

When $k = 0$, f is often referred to as a *modular function* rather than a modular form of weight 0. We say f is a *holomorphic* modular form of weight k when $N \geq 0$. In this thesis, the term *modular form* will be reserved for holomorphic modular forms. In other words, our definition of modular forms includes the conditions that the functions have no poles in $\mathbb{H}$, and are holomorphic at infinity.

There is another way to write the definition of a modular form that will be of use. Let the notation be as above. For $k \in \mathbb{Z}$, there is a right Γ-action on meromorphic functions in $\mathbb{H}$ given by

$$f|_k\gamma(\tau) = (a\tau + b)^{-k} f(\gamma\tau). \tag{4.1.1.4}$$

In the definition of a modular form, condition (4.1.1.2) can be replaced with requiring that f be invariant under the action $|_k$. That is, requiring $f|_k(\tau) = f(\tau)$ for all $\gamma \in \Gamma$.

Let $\mathcal{M}_k$ denote the space of (holomorphic) modular forms of weight k. It can be shown (see [28]) that $\mathcal{M}_0 = \mathbb{C}$ and $\mathcal{M}_k = 0$ if

1. k is odd,
2. $k < 0$, or
3. $k = 2$.

Let $\mathcal{M}$ denote the space consisting of modular forms with $\mathbb{Z}$-grading

$$\mathcal{M} = \bigoplus_{k \geq 0} \mathcal{M}_{2k} = \mathbb{C} \oplus \mathcal{M}_4 \oplus \mathcal{M}_6 \oplus \cdots . \qquad (4.1.1.5)$$

A typical element of $\mathcal{M}$ is a finite sum of modular forms of various weights, but not in general a modular form itself.

Suppose $f(\tau) \in \mathcal{M}_{2k}$ and $g(\tau) \in \mathcal{M}_{2\ell}$ for $k, \ell \geq 0$. Then $f(\tau)g(\tau)$ is again holomorphic in $\mathbb{H} \cup \{\infty\}$. Moreover, $(fg)|_{2k+2\ell}\gamma(\tau) = f(\tau)g(\tau)$ for all $\gamma \in \Gamma$. It follows that $f(\tau)g(\tau)$ is a modular form of weight $2(k+\ell)$. In other words, pointwise multiplication of modular forms defines a product such that $\mathcal{M}$ is a $\mathbb{Z}$-graded algebra.

Set

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

The set $\Gamma(N)$ is a subgroup of Γ. By definition, $\Gamma(1) = \Gamma$. We are also interested in the subgroup $\Gamma_0(N)$ of Γ defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \bmod N \right\}. \qquad (4.1.1.6)$$

A subgroup of Γ is called a *congruence subgroup* of Γ if it contains $\Gamma(N)$ for some N. The definition of a modular form can be extended to any congruence subgroup of Γ.

Let Hol_q denote the space of holomorphic functions $f(\tau)$ on $\mathbb{H}$ such that as $\tau = x + iy \to i\infty$ (meaning $y \to \infty$, $x, y \in \mathbb{R}$), there exists a scalar K and $N \in \mathbb{N}$ such that $|f(\tau)| < Ky^N$. We define a *(holomorphic) vector-valued modular form* of weight k to be a tuple of functions

$$\mathbf{F}(\tau) = (f_1(\tau), \ldots, f_r(\tau))$$

together with a representation $\rho\colon \Gamma \to \mathrm{GL}_r(\mathbb{C})$ such that each $f_j(\tau) \in \mathrm{Hol}_q$ and

$$\mathbf{F}^t|_k\gamma(\tau) = \rho(\gamma)\mathbf{F}^t(\tau)$$

for all $\gamma \in \Gamma$. Here t denotes the transpose of the vector and $\mathbf{F}^t|_k\gamma(\tau)$ is defined as

$$\mathbf{F}^t|_k\gamma(\tau) = (f_1|_k\gamma(\tau), \ldots, f_r|_k\gamma(\tau))^t\,.$$

Moreover, we require that each function $f_j(\tau)$ $(1 \le j \le r)$ have a convergent q-expansion holomorphic at infinity:

$$f_j(\tau) = \sum_{n\ge 0} a_n(j) q^{n/N_j}$$

for positive integers N_j.

4.1.2 Eisenstein series

Consider the sum

$$G_{2k}(\tau) = \sum_{0\ne(a,b)\in\mathbb{Z}^2} \frac{1}{(a\tau+b)^{2k}}. \qquad (4.1.2.1)$$

In the case that $k \ge 2$, $G_{2k}(\tau)$ satisfies the invariance

$$G|_{2k}\gamma(\tau) = G(\tau)$$

for all $\gamma \in \Gamma$. Moreover, for all $k \geq 2$, G_{2k} is holomorphic on $\mathbb{H}$ and has the q-expansion

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n, \tag{4.1.2.2}$$

where $\zeta(2k) = \sum_{n=1}^{\infty}\frac{1}{n^{2k}}$ and $\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$ (see [28]). Therefore G_{2k} is also holomorphic at infinity, and it follows that for $k \geq 2$, each $G_{2k}(\tau)$ is a modular form of weight $2k$. Such functions are called Eisenstein series. They may also be written

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}. \tag{4.1.2.3}$$

The case $k = 1$ in (4.1.2.1) results in a series which does not converge. Instead, define a function $G_2(\tau)$ on $\mathbb{H}$ by

$$G_2(\tau) = \frac{\pi^2}{3} + \sum_{a\in\mathbb{Z}\setminus\{0\}}\sum_{b\in\mathbb{Z}}\frac{1}{(a\tau+b)^2}. \tag{4.1.2.4}$$

The function $G_2(\tau)$ is holomorphic in $\mathbb{H}$ and at infinity, and also has the q-expansion (4.1.2.2) for $k = 1$. However,

$$G_2(\gamma\tau) = (c\tau+d)^2 G_2(\tau) - 2\pi i c(c\tau+d) \tag{4.1.2.5}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ (see [28] for details). Therefore, G_2 does not satisfy (4.1.1.2) and is not a modular form. It is, however, also called an Eisenstein series and is of great importance to us.

There are several normalizations of the functions $G_{2k}(\tau)$ that are convenient. We define the functions $E_{2k}(\tau)$ by

$$E_{2k}(\tau) = \frac{1}{(2\pi i)^{2k}}G_{2k}(\tau). \tag{4.1.2.6}$$

A word of caution should be made as there are competing and inconsistent notations in the literature. The notation E_{2k} chosen here is the same as E_{2k}, in say [7], but different than the E_{2k} normalization in other works. The series E_{2k} are also called *Eisenstein series*, and from now on the term Eisenstein series will refer to these functions rather than the functions G_{2k}. With this notation, transformation (4.1.2.5) for E_2 reads

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{2\pi i}. \tag{4.1.2.7}$$

One importance of Eisenstein series is their use as generators of the graded algebra $\mathcal{M} = \bigoplus_{k\geq 0} \mathcal{M}_{2k}$. Namely, $\mathcal{M}$ is the weighted polynomial algebra

$$\mathcal{M} = \mathbb{C}[E_4, E_6]. \tag{4.1.2.8}$$

If the function E_2 is also included, the weighted polynomial $\mathbb{Z}$-graded algebra of *quasi-modular forms*,

$$\mathcal{Q} = \mathbb{C}[E_2, E_4, E_6], \tag{4.1.2.9}$$

is obtained. The space $\mathcal{Q}$ is graded in the same way as $\mathcal{M}$.

The Eisenstein series $E_2(\tau)$ plays a role in the *modular derivative* ∂ defined for $f(\tau) \in \mathcal{M}_k$ by

$$\partial f(\tau) = \partial_k f(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau} f(\tau) + kE_2(\tau) f(\tau). \tag{4.1.2.10}$$

It can be verified that

$$(\partial_k f)|_{k+2}\gamma(\tau) = \partial_k f|_k \gamma(\tau),$$

showing that ∂ maps modular forms of weight k to modular forms of weight $k + 2$ (see Lemma 4.2.2.1 below for a similar calculation).

4.1.3 Elliptic functions

For $(z,\tau) \in \mathbb{C} \times \mathbb{H}$, we define the Weierstrass $\wp$-function by

$$\wp(z,\tau) := \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \left(\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2} \right), \tag{4.1.3.1}$$

where $\omega_{m,n} = 2\pi i(m\tau + n)$. The function $\wp(z,\tau)$ is periodic in the variable z with periods $2\pi i$ and $2\pi i\tau$ [28]. $\wp(z,\tau)$ also has the expansion

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{n\geq 4, n\in 2\mathbb{Z}} (n-1)E_n(\tau)z^{n-2}, \tag{4.1.3.2}$$

where the functions E_n are the Eisenstein series defined above. For $(w,\tau) \in \mathbb{C} \times \mathbb{H}$, define the functions $P_k(w,\tau)$ for $k \geq 1$ by

$$P_k(w,\tau) := \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \left(\frac{n^{k-1}q_w^n}{1-q^n} + \frac{(-1)^k n^{k-1} q_w^{-n} q^n}{1-q^n} \right), \tag{4.1.3.3}$$

where $q_w = e^{2\pi i w}$ and $q = e^{2\pi i\tau}$. The functions $P_k(w,\tau)$ for $k \geq 1$ could also be defined recursively by

$$P_k(w,\tau) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} P_1(w,\tau) = \frac{1}{z^k} + (-1)^k \sum_{n\geq k} \binom{n-1}{k-1} E_n(\tau) z^{n-k}. \tag{4.1.3.4}$$

These functions are often called *Weierstrass functions* and are related to the classical Weierstrass $\wp$-function by the equalities $P_2(w,\tau) = \wp(w,\tau) + E_2(\tau)$ and $\wp(w,\tau) = P_1(w,\tau) - zE_2(\tau)$ (see [7, 28]). For $k \geq 2$, $P_k(w,\tau)$ is periodic in w with period $2\pi i$ and $2\pi i\tau$. $P_1(w,\tau)$ is still periodic in w with period $2\pi i$, however, $P_1(w + 2\pi i\tau, \tau) = P_1(w,\tau) - 1$.

4.1.4 Other useful functions on $\mathbb{H}$

Define the *Dedekind η-function* by

$$\eta(\tau) := q^{1/24} \prod_{n\geq 1} (1-q^n). \tag{4.1.4.1}$$

The η-function is holomorphic on $\mathbb{H}$ and is closely related to the *unrestricted partition function* $p(n)$ by

$$\eta(\tau)^{-1} = q^{-1/24} \prod_{n\geq 1} (1-q^n)^{-1} = q^{-1/24} \sum_{n\geq 0} p(n) q^n$$
$$= q^{-1/24}(1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots).$$

Though the η-function is not a modular form on Γ, it will play a role in describing the modular invariance of n-point functions discussed later in this thesis.

4.2 Jacobi forms

4.2.1 Definitions

Let F^t denote the transpose of a square matrix F. A matrix F is symmetric if $F^t = F$. When every off-diagonal entry of F is half-integral and the diagonal entries are integral, we call F half-integral. For a vector $\underline{z}$ we let $F[\underline{z}]$ denote $\underline{z}^t F \underline{z}$, and for a function ϕ defined on $\mathbb{C}^n$ we write $\phi(\underline{z})$ for $\phi(z_1, \ldots, z_n)$.

Definition 4.2.1.1 *Let k be an integer and F be a real symmetric, positive-definite, half-integral $n \times n$ matrix. A holomorphic function $\phi\colon \mathbb{H} \times \mathbb{C}^n \to \mathbb{C}$ is a Jacobi form on Γ of weight k and index F if*

1. *for all* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{\underline{z}}{c\tau + d}\right) = (c\tau + d)^k \exp\left(2\pi i \frac{cF[\underline{z}]}{c\tau + d}\right) \phi(\tau, \underline{z}), \tag{4.2.1.1}$$

2. for any $(\underline{\lambda}, \underline{\mu}) \in \mathbb{Z}^n \times \mathbb{Z}^n$*,*

$$\phi(\tau, \underline{z} + \underline{\lambda}\tau + \underline{\mu}) = \exp\left(-2\pi i(\tau F[\underline{\lambda}] + 2\underline{z}^t F\underline{\lambda})\right)\phi(\tau, \underline{z}), \tag{4.2.1.2}$$

and

3. ϕ *has a Fourier-Jacobi expansion*

$$\phi(\tau, \underline{z}) = \sum_{\substack{\underline{r}\in\mathbb{Z}^n, \ell\in\mathbb{Q}, \\ 4\ell - F^{-1}[\underline{r}]\geq 0}} c(\ell, \underline{r}) q^\ell \exp\left(2\pi i(\underline{z}^t \underline{r})\right), \tag{4.2.1.3}$$

where $\ell \geq \ell_0$ *for some* ℓ_0*. In the case that condition 3 is replaced with the weaker condition,*

3. ϕ *has a Fourier-Jacobi expansion*

$$\phi(\tau, \underline{z}) = \sum_{\substack{\ell\in\mathbb{Q} \\ \ell\geq\ell_0}} \sum_{\underline{r}\in\mathbb{Z}^n} c(\ell, \underline{r}) q^\ell \exp\left(2\pi i(\underline{z}^t \underline{r})\right), \tag{4.2.1.4}$$

then $\phi(\tau, \underline{z})$ *is called a weak Jacobi form. In the case* $\ell_0 \geq 0$*,* ϕ *is a holomorphic (weak) Jacobi form, otherwise it is meromorphic.*

We will often use $\gamma\underline{z}$ to denote $\frac{\underline{z}}{c\tau+d}$ for γ labelled as in the previous definition. If $\underline{z} = \underline{0}$ the definition of a Jacobi form above reduces to the definition of a modular form. When $n = 1$ and $F[\underline{z}]$ is the quadratic form z^2, the definition is the same as presented in [14]. We use a more general definition here which coincides with that used in [44]. Sometimes we will refer to such functions as *matrix* Jacobi forms.

Another approach to the definition of Jacobi forms is to focus on the actions of Γ and $\mathbb{Z}^n \times \mathbb{Z}^n$ on holomorphic functions $\phi\colon \mathbb{H} \times \mathbb{C}^n \to \mathbb{C}$. For an integer k and matrix F as above, define the operators $|_{k,F}$ and $|_F$ by

$$(\phi|_{k,F}\gamma)(\tau, \underline{z}) := (c\tau + d)^{-k} \exp\left(-2\pi i \frac{cF[\underline{z}]}{c\tau + d}\right) \phi\left(\gamma\tau, \frac{\underline{z}}{c\tau + d}\right), \tag{4.2.1.5}$$

and

$$(\phi|_F[\underline{\lambda},\underline{\mu}])(\tau,\underline{z}) := \exp\left(2\pi i(F[\underline{\lambda}]\tau + 2\underline{z}^t F\underline{\lambda})\right)\phi(\tau,\underline{z}+\underline{\lambda}\tau+\underline{\mu}), \tag{4.2.1.6}$$

respectively. In this way, the definition of a Jacobi form reads as: a holomorphic function $\phi\colon \mathbb{H}\times\mathbb{C}^n \to \mathbb{C}$ such that

1. $(\phi|_{k,F}\gamma)(\tau,\underline{z}) = \phi(\tau,\underline{z})$ for any $\gamma\in\Gamma$,
2. $(\phi|_F[\underline{\lambda},\underline{\mu}])(\tau,z) = \phi(\tau,\underline{z})$ for all $[\underline{\lambda},\underline{\mu}]\in\mathbb{Z}^n\times\mathbb{Z}^n$, and
3. ϕ has a Fourier expansion (4.2.1.3) as before.

The space of all weak Jacobi forms on $\mathbb{H}\times\mathbb{C}^n$ will be denoted $\mathcal{J}^n$ and the space of weak Jacobi forms of weight k and index F is denoted $\mathcal{J}^n_{k,F}$.

An example of a Jacobi form is the Weierstrass $\wp$-function (see [14]). This function has the required Fourier expansion and satisfies $(\wp|_{2,0}\gamma)(\tau,z) = \wp(\tau,z)$ and $(\wp|_0[\lambda,\mu])(\tau,z) = \wp(\tau,z)$ for all $\gamma\in\Gamma$ and $[\lambda,\mu]\in\mathbb{Z}\times\mathbb{Z}$. Therefore, $\wp(\tau,z)$ is a Jacobi form of weight 2 and index 0. That is, $\wp(\tau,z)\in\mathcal{J}^1_{2,0}$.

Let $\mathrm{Hol}_{\mathbb{H}\times\mathbb{C}^n}$ denote the space of holomorphic functions on $\mathbb{H}\times\mathbb{C}^n$. We define a *vector-valued weak Jacobi form* of weight k and index F to be a tuple of functions

$$\Phi(\tau,\underline{z}) = (\phi_1(\tau,\underline{z}),\ldots,\phi_r(\tau,\underline{z}))$$

together with a representation $\rho\colon \Gamma\ltimes(\mathbb{Z}^n\times\mathbb{Z}^n)\to \mathrm{GL}_r(\mathbb{C})$, such that

1. $\Phi^t|_{k,F}\gamma(\tau,\underline{z}) = \rho(\gamma)\Phi^t(\tau,\underline{z})$ for all $\gamma\in\Gamma$,
2. $\Phi^t|_F[\underline{\lambda},\underline{\mu}](\tau,\underline{z}) = \rho([\underline{\lambda},\underline{\mu}])\Phi^t(\tau,\underline{z})$ for all $[\underline{\lambda},\underline{\mu}]\in\mathbb{Z}^n\times\mathbb{Z}^n$, and
3. each $\phi_j(\tau,\underline{z})\in\mathrm{Hol}_{\mathbb{H}\times\mathbb{C}^n}$ and has an expansion of the form (4.2.1.4).

Here the slash operators $\Phi^t|_* A(\tau, \underline{z})$ are defined as

$$\Phi^t|_* A(\tau) = \left(f_1|_* A(\tau, \underline{z}), \ldots, f_r|_* A(\tau, \underline{z})\right)^t,$$

where $|_* A$ is either $|_{k,F}\gamma$ or $|_F[\underline{\lambda}, \underline{\mu}]$. In the case each ϕ_j satisfies an expansion of the form (4.2.1.3), then Φ is a vector-valued Jacobi form.

4.2.2 Differential operators

Let k and $m \neq 0$ be integers. Define the differential operator $\mathcal{L}_k$ by

$$\mathcal{L}_k = \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{1}{4m}\left(\frac{1}{2\pi i}\frac{d}{dz}\right)^2 + \frac{2k-1}{2}E_2(\tau). \tag{4.2.2.1}$$

We have the following lemma.

Lemma 4.2.2.1 *Let k and m be as above. The operator $\mathcal{L}_k$ maps weak Jacobi forms of weight k and index m to weak Jacobi forms of weight $k+2$ and index m.*

Proof Let ϕ be a weak Jacobi form. We will prove the following two properties:

1. for any $\gamma \in \Gamma$,

$$(\mathcal{L}_k\phi)|_{k+2,m}\gamma = \mathcal{L}(\phi|_{k,m}\gamma), \tag{4.2.2.2}$$

   and

2. for any $[\lambda, \mu] \in \mathbb{Z} \times \mathbb{Z}$,

$$(\mathcal{L}_k\phi)|_m[\lambda, \mu] = \mathcal{L}_k(\phi|_m[\lambda, \mu]). \tag{4.2.2.3}$$

First, we find

$$
\begin{aligned}
&\mathcal{L}_k(\phi|_{k,m}\gamma) = \left(\frac{1}{2\pi i}\frac{d}{d\tau} - \frac{1}{4m(2\pi i)^2}\frac{d^2}{dz^2} + \frac{2k-1}{2}E_2(\tau)\right)\left((c\tau+d)^{-k}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z)\right) \\
&= -\frac{kc}{2\pi i}(c\tau+d)^{-k-1}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z) + mc^2z^2(c\tau+d)^{-k-2}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z) \\
&\quad + (c\tau+d)^{-k}e^{-2\pi im\frac{cz^2}{c\tau+d}}\left(\frac{1}{2\pi i}\frac{d}{d\tau}\phi(\gamma\tau,\gamma z)\right) \\
&\quad - \frac{1}{4m}\frac{1}{2\pi i}\frac{d}{dz}\Bigg[-2mcz(c\tau+d)^{-k-1}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma,\tau,\gamma z) \\
&\qquad + \frac{1}{2\pi i}(c\tau+d)^{-k}e^{-2\pi im\frac{cz^2}{c\tau+d}}\left(\frac{d}{dz}\phi(\gamma\tau,\gamma z)\right)\Bigg] \\
&\quad + \frac{2k-1}{2}E_2(\tau)(c\tau+d)^{-k}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z) \\
&= -\frac{kc}{2\pi i}(c\tau+d)^{-k-1}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z) \\
&\quad + mc^2z^2(c\tau+d)^{-k-2}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z) \qquad (4.2.2.4) \\
&\quad + (c\tau+d)^{-k}e^{-2\pi im\frac{cz^2}{c\tau+d}}\left(\frac{1}{2\pi i}\frac{d}{d\tau}\phi(\gamma\tau,\gamma z)\right) \\
&\quad + \frac{c}{2(2\pi i)}(c\tau+d)^{-k-1}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z) \\
&\quad - mc^2z^2(c\tau+d)^{-k-2}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z) \qquad (4.2.2.5) \\
&\quad + \frac{cz}{2(2\pi i)}(c\tau+d)^{-k-1}e^{-2\pi im\frac{cz^2}{c\tau+d}}\left(\frac{d}{dz}\phi(\gamma\tau,\gamma z)\right) \qquad (4.2.2.6) \\
&\quad + \frac{cz}{2(2\pi i)}(c\tau+d)^{-k-1}e^{-2\pi im\frac{cz^2}{c\tau+d}}\left(\frac{d}{dz}\phi(\gamma\tau,\gamma z)\right) \qquad (4.2.2.7) \\
&\quad - \frac{1}{4m}(c\tau+d)^{-k}e^{-2\pi im\frac{cz^2}{c\tau+d}}\left(\frac{d^2}{dz^2}\phi(\gamma\tau,\gamma z)\right) \\
&\quad + \frac{2k-1}{2}E_2(\tau)(c\tau+d)^{-k}e^{-2\pi im\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z).
\end{aligned}
$$

Cancelling out (4.2.2.4) and (4.2.2.5), and adding (4.2.2.6) and (4.2.2.7) together, we have the above equals

$$
\begin{aligned}
&-\frac{kc}{2\pi i}(c\tau+d)^{-k-1}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z) && (4.2.2.8)\\
&\quad+(c\tau+d)^{-k}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(\frac{1}{2\pi i}\frac{d}{d\tau}\phi(\gamma\tau,\gamma z)\right) && (4.2.2.9)\\
&\quad+\frac{c}{2(2\pi i)}(c\tau+d)^{-k-1}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z) && (4.2.2.10)\\
&\quad+\frac{cz}{2\pi i}(c\tau+d)^{-k-1}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(\frac{d}{dz}\phi(\gamma\tau,\gamma z)\right) && (4.2.2.11)\\
&\quad-\frac{1}{4m}(c\tau+d)^{-k}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(\frac{d^2}{dz^2}\phi(\gamma\tau,\gamma z)\right) && (4.2.2.12)\\
&\quad+\frac{2k-1}{2}E_2(\tau)(c\tau+d)^{-k}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z). && (4.2.2.13)
\end{aligned}
$$

We now consider the terms (4.2.2.8), (4.2.2.10), and (4.2.2.13). We find that adding these together they become

$$
\begin{aligned}
&k((c\tau+d)^2E_2(\tau)-\frac{c}{2\pi i}(c\tau+d))(c\tau+d)^{-k-2}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z)\\
&\;-\frac{1}{2}E_2(\tau)(c\tau+d)^{-k}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z)\\
&\quad+\frac{1}{2}\frac{c}{2\pi i}(c\tau+d)^{-k-1}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z)\\
&\;=(c\tau+d)^{-k-2}e^{-2\pi i m\frac{cz^2}{c\tau+d}}[kE_2(\gamma\tau)]\phi(\gamma\tau,\gamma z)\\
&\quad-\frac{1}{2}\left((c\tau+d)^2E_2(\tau)-\frac{1}{2\pi i}c(c\tau+d)\right)(c\tau+d)^{-k-2}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\phi(\gamma\tau,\gamma z)\\
&\;=(c\tau+d)^{-k-2}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left[\frac{2k-2}{2}E_2(\gamma\tau)\right]\phi(\gamma\tau,\gamma z).
\end{aligned}
$$

Meanwhile, combining the terms (4.2.2.9) and (4.2.2.11) gives

$$\begin{aligned}
&(c\tau+d)^{-k}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(\frac{1}{2\pi i}\frac{d}{d\tau}\phi(\gamma\tau,\gamma z)\right)\\
&\quad+(c\tau+d)^{-k-1}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(\frac{cz}{2\pi i}\frac{d}{dz}\phi(\gamma\tau,\gamma z)\right)\\
&=(c\tau+d)^{-k}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(\frac{1}{2\pi i}\frac{d}{d\tau}\phi(\gamma\tau,\gamma z)\right)\\
&\quad+(c\tau+d)^{-k-1}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(+\frac{1}{2\pi i}\frac{cz}{(c\tau+d)}\frac{d}{d(\gamma z)}\phi(\gamma\tau,\gamma z)\right)\\
&\quad(c\tau+d)^{-k}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\frac{1}{2\pi i}\left(\frac{d}{d\tau}+\frac{cz}{(c\tau+d)^2}\frac{d}{d(\gamma z)}\right)\phi(\gamma\tau,\gamma z)\\
&=(c\tau+d)^{-k-2}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(\frac{1}{2\pi i}\frac{d}{d(\gamma\tau)}\phi(\gamma\tau,\gamma z)\right),
\end{aligned}$$

where we used

$$\begin{aligned}
\frac{d}{d\tau}\phi(\gamma\tau,\gamma z)&=\left[\frac{d(\gamma\tau)}{d\tau}\frac{d}{d(\gamma\tau)}+\frac{d(\gamma z)}{d\tau}\frac{d}{d(\gamma z)}\right]\phi(\gamma\tau,\gamma z)\\
&=\left[\frac{1}{(c\tau+d)^2}\frac{d}{d(\gamma\tau)}-\frac{cz}{(c\tau+d)^2}\frac{d}{d(\gamma z)}\right]\phi(\gamma\tau,\gamma z).
\end{aligned}$$

Finally, we note that the term (4.2.2.12) equals

$$(c\tau+d)^{-k-2}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(-\frac{1}{4m}\frac{d^2}{d(\gamma z)^2}\phi(\gamma\tau,\gamma z)\right).$$

Making these substitutions we find

$$\begin{aligned}
&\mathcal{L}_k(\phi|_{k,m}\gamma)\\
&\quad=(c\tau+d)^{-k-2}e^{-2\pi i m\frac{cz^2}{c\tau+d}}\left(\frac{1}{2\pi i}\frac{d}{d(\gamma\tau)}-\frac{1}{4m}\frac{d^2}{d(\gamma z)^2}+\frac{2k-1}{2}E_2(\gamma\tau)\right)\phi(\gamma\tau,\gamma z)\\
&\quad=(\mathcal{L}_k\phi)|_{k+2,m}\gamma,
\end{aligned}$$

as desired. This proves (4.2.2.2).

We now prove (4.2.2.3). We have

$$
\begin{aligned}
&\mathcal{L}_k(\phi|_m[\lambda,\mu]) \\
&= \left(\frac{1}{2\pi i}\frac{d}{d\tau} - \frac{1}{4m(2\pi i)^2}\frac{d^2}{dz^2} + \frac{2k-1}{2}E_2(\tau)\right)\left(e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z+\lambda\tau+\mu)\right) \\
&= m\lambda^2 e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z+\lambda\tau+\mu) + \frac{1}{2\pi i}e^{2\pi i m(\lambda^2\tau+2\lambda z)}\left(\frac{d}{d\tau}\phi(\tau, z+\lambda\tau+\mu)\right) \\
&\quad - \frac{1}{4m}\frac{1}{2\pi i}\frac{d}{dz}\Bigg[2m\lambda e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z+\lambda\tau+\mu) \\
&\qquad + \frac{1}{2\pi i}e^{2\pi i m(\lambda^2\tau+2\lambda z)}\left(\frac{d}{dz}\phi(\tau, z+\lambda\tau+\mu)\right)\Bigg] \\
&\quad + \frac{2k-1}{2}E_2(\tau)e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z+\lambda\tau+\mu) \\
&= m\lambda^2 e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z+\lambda\tau+\mu) + \frac{1}{2\pi i}e^{2\pi i m(\lambda^2\tau+2\lambda z)}\left(\frac{d}{d\tau}\phi(\tau, z+\lambda\tau+\mu)\right) \\
&\quad - \frac{1}{4m}4m^2\lambda^2 e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z+\lambda\tau+\mu) \\
&\quad - \frac{1}{4m}\frac{1}{2\pi i}2m\lambda e^{2\pi i m(\lambda^2\tau+2\lambda z)}\left(\frac{d}{dz}\phi(\tau, z+\lambda\tau+\mu)\right) \\
&\quad - \frac{1}{4m}\frac{1}{2\pi i}2m\lambda e^{2\pi i m(\lambda^2\tau+2\lambda z)}\left(\frac{d}{dz}\phi(\tau, z+\lambda\tau+\mu)\right) \\
&\quad - e^{2\pi i m(\lambda^2\tau+2\lambda z)}\left(\frac{1}{4m}\frac{1}{(2\pi i)^2}\frac{d^2}{dz^2}\phi(\tau, z+\lambda\tau+\mu)\right) \\
&\quad + \frac{2k-1}{2}E_2(\tau)e^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z+\lambda\tau+\mu)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi i} e^{2\pi i m(\lambda^2\tau+2\lambda z)} \left(\frac{d}{d\tau}\phi(\tau, z+\lambda\tau+\mu) \right) \\
&\quad - \frac{\lambda}{2\pi i} e^{2\pi i m(\lambda^2\tau+2\lambda z)} \left(\frac{d}{dz}\phi(\tau, z+\lambda\tau+\mu) \right) \\
&\quad - \frac{1}{4m}\frac{1}{2\pi i} 2m\lambda e^{2\pi i m(\lambda^2\tau+2\lambda z)} \left(\frac{d}{dz}\phi(\tau, z+\lambda\tau+\mu) \right) \\
&\quad - e^{2\pi i m(\lambda^2\tau+2\lambda z)} \left(\frac{1}{4m}\frac{1}{(2\pi i)^2}\frac{d^2}{dz^2}\phi(\tau, z+\lambda\tau+\mu) \right) \\
&\quad + \frac{2k-1}{2} E_2(\tau) e^{2\pi i m(\lambda^2\tau+2\lambda z)} \phi(\tau, z+\lambda\tau+\mu). \qquad (4.2.2.14)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{d}{d\tau}\phi(\tau, z+\lambda\tau+\mu) &= \left[\frac{d\tau}{d\tau}\frac{d}{d\tau} + \frac{d(z+\lambda\tau+\mu)}{d\tau}\frac{d}{d(z+\lambda\tau+\mu)} \right] \phi(\tau, z+\lambda\mu) \\
&= \left[\frac{d}{d\tau} + \lambda\frac{d}{d(z+\lambda\tau+\mu)} \right] \phi(\tau, z+\lambda\tau+\mu),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dz}\phi(\tau, z+\lambda\tau+\mu) &= \left[\frac{d(z+\lambda\tau+\mu)}{dz}\frac{d}{d(z+\lambda\tau+\mu)} \right] \phi(\tau, z+\lambda\mu) \\
&= \frac{d}{d(z+\lambda\tau+\mu)}\phi(\tau, z+\lambda\tau+\mu).
\end{aligned}
$$

Plugging these into (4.2.2.14) we find

$$
\begin{aligned}
\mathcal{L}_k(\phi|_m[\lambda,\mu]) &= \left(\frac{1}{2\pi i}\frac{d}{d\tau} - \frac{1}{4m}\left(\frac{1}{2\pi i}\frac{d}{d(z+\lambda\tau+\mu)} \right)^2 + \frac{2k-1}{2}E_2(\tau) \right) \phi(\tau, z+\lambda\tau+\mu) \\
&= (\mathcal{L}_k\phi)|_m[\lambda,\mu].
\end{aligned}
$$

This completes the proof of (4.2.2.3), and with it Lemma 4.2.2.1. □

4.2.3 Quasi-Jacobi forms

A holomorphic function $\phi(\tau, \underline{z})$ on $\mathbb{H} \times \mathbb{C}^n$ is a *(weak) quasi-Jacobi form* of weight k and index F (F as in Definition 4.2.1.1) if for fixed $\tau \in \mathbb{H}$, $\underline{z} \in \mathbb{C}^n$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and $[\underline{\lambda}, \underline{\mu}] \in \mathbb{Z}^n \times \mathbb{Z}^n$, we have

1. $\phi|_{k,F}\gamma(\tau, \underline{z}) \in \mathrm{Hol}_{\mathbb{H}\times\mathbb{C}^n}\left[\frac{cz_1}{c\tau+d}, \ldots, \frac{cz_n}{c\tau+d}, \frac{c}{c\tau+d}\right]$ with coefficients dependent only on ϕ, and

2. $\phi|_F[\underline{\lambda}, \underline{\mu}](\tau, \underline{z}) \in \mathrm{Hol}_{\mathbb{H}\times\mathbb{C}^n}[\lambda_1, \ldots, \lambda_n]$ with coefficients dependent only on ϕ.

In other words, there are holomorphic functions $S_{i_1,\ldots,i_n,j}(\phi)$ and $T_{i_1,\ldots,i_n}(\phi)$ on $\mathbb{H} \times \mathbb{C}^n$ determined only by ϕ such that

$$\begin{aligned}&(c\tau + d)^{-k} e^{-2\pi i \frac{cF[\underline{z}]}{c\tau+d}} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{\underline{z}}{c\tau + d}\right) \\ &\quad = \sum_{\substack{i_1 \leq s_1, \ldots, i_n \leq s_n \\ j \leq t}} S_{i_1,\ldots,i_n,j}(\phi)(\tau, \underline{z}) \left(\frac{cz_1}{c\tau + d}\right)^{i_1} \cdots \left(\frac{cz_n}{c\tau + d}\right)^{i_n} \left(\frac{c}{c\tau + d}\right)^{j}, \end{aligned} \tag{4.2.3.1}$$

and

$$e^{2\pi i(\tau F[\underline{\lambda}] + 2\underline{z}^t F \underline{\lambda})} \phi(\tau, \underline{z} + \underline{\lambda}\tau + \underline{\mu}) = \sum_{i_1 \leq s_1, \ldots, i_n \leq s_n} T_{i_1,\ldots,i_n}(\phi)(\tau, \underline{z}) \lambda_1^{i_1} \cdots \lambda_n^{i_n}. \tag{4.2.3.2}$$

If $\phi \neq 0$, we can take $S_{s_1,\ldots,s_n,t}(\phi) \neq 0$ and $T_{s_1,\ldots,s_n}(\phi) \neq 0$, and we say ϕ is a quasi-Jacobi form of *depth* $(s_1, \ldots, s_n, t)$. In the case $\underline{z} = z$ and $F = 0$, this definition of a quasi-Jacobi form reduces to that in [33]. When $n > 1$ in the definition above, we will sometimes refer to these functions as matrix quasi-Jacobi forms. (See also [26] for another definition of quasi-Jacobi form.)

For $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$, define the functions

$$E_n(\tau, z) := \sum_{(a,b)\in\mathbb{Z}^2} \frac{1}{(z + a\tau + b)^n}. \tag{4.2.3.3}$$

Such functions converge absolutely for $n \geq 3$ on $\mathbb{H} \times \mathbb{C}$. The cases $n = 1, 2$ also converge utilizing 'Eisenstein summation' (see page 10 of [33]). Moreover, for $n \geq 3$ the functions $E_n(\tau, z)$ are weak Jacobi forms of weight n and index 0 on $\mathbb{H}\times\mathbb{C}$. (See [33] for more details.) The function E_1 has the transformation laws

$$E_1\left(\gamma\tau, \frac{z}{c\tau + d}\right) = (c\tau + d)E_1(\tau, z) + \frac{\pi i c}{2} z \qquad \text{and} \tag{4.2.3.4}$$

$$E_1(\tau, z + \lambda\tau + \mu) = E_1(\tau, z) - 2\pi i\lambda, \tag{4.2.3.5}$$

while E_2 satisfies

$$E_2\left(\gamma\tau, \frac{z}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau, z) + \frac{\pi i c(c\tau + d)}{2} \qquad \text{and} \tag{4.2.3.6}$$

$$E_2(\tau, z + \lambda\tau + \mu) = E_2(\tau, z), \tag{4.2.3.7}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $[\lambda, \mu] \in \mathbb{Z} \times \mathbb{Z}$ [33]. In other words, $E_1(\tau, z)$ is a quasi-Jacobi form of index 0, weight 1, and depth $(1, 0)$, while $E_2(\tau, z)$ is a quasi-Jacobi form of index 0, weight 2, and depth $(0, 1)$. $E_2(\tau, z) - E_2(\tau)$ is a weak Jacobi form of index 0 and weight 2.

The algebra $\mathcal{Q}^{\mathcal{J}^1} := \mathbb{C}[E_2(\tau), E_n(\tau, z), n \geq 1]$ is the space of quasi-Jacobi forms on $\mathbb{H} \times \mathbb{C}$ (see Proposition 2.9 in [33]). That is, any quasi-Jacobi form on $\mathbb{H} \times \mathbb{C}$ can be generated by $E_2(\tau)$ and $E_n(\tau, z)$, for $n \geq 1$.

The functions $E_m(\tau, z)$ may be generalized as follows. For $m \geq 1$, define the functions $E_m(\tau, \underline{z})$ by

$$E_m(\tau, \underline{z}) := \sum_{(a,b)\in\mathbb{Z}^2} \frac{1}{(z_1 + \cdots + z_n + a\tau + b)^m}. \tag{4.2.3.8}$$

Noticing that $E_m(\tau, \underline{z}) = E_m(\tau, z_1 + \cdots + z_n)$ and applying the transformation properties known for $E_m(\tau, z)$, it follows that $E_m(\tau, \underline{z})$ is a weak matrix Jacobi form of index zero and weight m on $\mathbb{H} \times \mathbb{C}^n$ for $m \geq 3$. $E_1(\tau, \underline{z})$ is again a matrix quasi-Jacobi form of index zero, weight 1, and depth $(1, 0)$ on $\mathbb{H} \times \mathbb{C}^n$, while $E_2(\tau, \underline{z})$ is a matrix quasi-Jacobi form of index zero, weight 2, and depth $(0, 1)$ on $\mathbb{H} \times \mathbb{C}^n$.

Let $\mathcal{Q}^{\mathcal{J}^n}$ denote the space of quasi-Jacobi forms on $\mathbb{H} \times \mathbb{C}^n$. It will not be discussed here whether the functions $E_m(\tau, \underline{z})$ generate $\mathcal{Q}^{\mathcal{J}^n}$, however, by the discussion above, it is clear that they are contained in $\mathcal{Q}^{\mathcal{J}^n}$.

Lemma 4.2.3.1 *The space of quasi-Jacobi froms $\mathcal{Q}^{\mathcal{J}^n}$ is closed under the partial derivatives $\frac{d}{d\tau}$ and $\frac{d}{dz_j}$, $1 \leq j \leq n$.*

Proof Suppose $\phi(\tau, \underline{z})$ is a quasi-Jacobi form of weight k, index F, and depth $(s_1, \ldots, s_n, t)$. Let the components of F be denoted by $F = (F_{pq})$. Using the product rule we find for any r, $1 \leq r \leq n$, that

$$
\begin{aligned}
&(c\tau + d)^{-k} e^{-2\pi i \frac{cF[z]}{c\tau+d}} \left(\frac{d}{dz_r} \phi(\gamma\tau, \gamma\underline{z}) \right) \\
&= \frac{d}{dz_r} \left[(c\tau + d)^{-k} e^{-2\pi i \frac{cF[z]}{c\tau+d}} \phi(\gamma\tau, \gamma\underline{z}) \right] - \left(\frac{d}{dz_r} (c\tau + d)^{-k} e^{-2\pi i \frac{cF[z]}{c\tau+d}} \right) \phi(\gamma\tau, \gamma\underline{z}) \\
&= \frac{d}{dz_r} \left[\sum_{\substack{i_1 \leq s_1, \ldots, i_n \leq s_n \\ j \leq t}} S_{i_1, \ldots, i_n, j}(\phi)(\tau, \underline{z}) \left(\frac{cz_1}{c\tau + d} \right)^{i_1} \cdots \left(\frac{cz_n}{c\tau + d} \right)^{i_n} \left(\frac{c}{c\tau + d} \right)^{j} \right] \\
&\quad + 2\pi i \left(\sum_{p=1}^{n} F_{pr} \frac{cz_p}{c\tau + d} + \sum_{q=1}^{n} F_{rq} \frac{cz_q}{c\tau + d} \right) (c\tau + d)^{-k} e^{-2\pi i \frac{cF[z]}{c\tau+d}} \phi(\gamma\tau, \gamma\underline{z}),
\end{aligned}
$$

which then equals

$$= \sum_{\substack{i_1\le s_1,\ldots,i_n\le s_n\\ j\le t}} \left(\frac{d}{dz_r}S_{i_1,\ldots,i_n,j}(\phi)(\tau,\underline{z})\right)\left(\frac{cz_1}{c\tau+d}\right)^{i_1}\cdots\left(\frac{cz_n}{c\tau+d}\right)^{i_n}\left(\frac{c}{c\tau+d}\right)^{j}$$

$$+ \sum_{\substack{i_1\le s_1,\ldots,i_n\le s_n\\ j\le t}} (i_r S_{i_1,\ldots,i_n,j}(\phi)(\tau,\underline{z}))$$

$$\cdot\left(\frac{cz_1}{c\tau+d}\right)^{i_1}\cdots\left(\frac{cz_r}{c\tau+d}\right)^{i_r-1}\cdots\left(\frac{cz_n}{c\tau+d}\right)^{i_n}\left(\frac{c}{c\tau+d}\right)^{j+1}$$

$$+ \sum_{\substack{i_1\le s_1,\ldots,i_n\le s_n\\ j\le t}} \sum_p (2\pi i F_{pr} S_{i_1,\ldots,i_n,j}(\phi)(\tau,\underline{z}))$$

$$\cdot\left(\frac{cz_1}{c\tau+d}\right)^{i_1}\cdots\left(\frac{cz_p}{c\tau+d}\right)^{i_p+1}\cdots\left(\frac{cz_n}{c\tau+d}\right)^{i_n}\left(\frac{c}{c\tau+d}\right)^{j}$$

$$+ \sum_{\substack{i_1\le s_1,\ldots,i_n\le s_n\\ j\le t}} \sum_q (2\pi i F_{rq} S_{i_1,\ldots,i_n,j}(\phi)(\tau,\underline{z}))$$

$$\cdot\left(\frac{cz_1}{c\tau+d}\right)^{i_1}\cdots\left(\frac{cz_q}{c\tau+d}\right)^{i_q+1}\cdots\left(\frac{cz_n}{c\tau+d}\right)^{i_n}\left(\frac{c}{c\tau+d}\right)^{j}.$$

This shows that $(d/dz_r\phi)|_{k,F}\gamma(\tau,\underline{z})$ is in $\mathrm{Hol}_{\mathbb{H}\times\mathbb{C}^n}\left[\frac{cz_1}{c\tau+d},\ldots,\frac{cz_n}{c\tau+d},\frac{c}{c\tau+d}\right]$, as desired. Similar calculations show this is also true for $(d/d\tau\phi)|_{k,F}\gamma(\tau,\underline{z})$. The same can be done for the partial derivatives in (4.2.3.2). □

4.2.4 Twisted elliptic functions

For $w \in \mathbb{C}$, $\underline{z} \in \mathbb{C}^n$, and $\tau \in \mathbb{H}$ such that $|q| < \left|e^{2\pi i w}\right| < 1$ and $\zeta_{z_1+\cdots+z_n} \neq 1$, we define the *'twisted' Weierstrass functions* $\tilde{P}_k(w,\underline{z},\tau)$ by

$$\tilde{P}_k(w,\underline{z},\tau) := \frac{(-1)^k}{(k-1)!}\sideset{}{'}\sum_{\ell\in\mathbb{Z}} \frac{\ell^{k-1}q_w^{\ell}}{1-\zeta_1^{-1}\cdots\zeta_n^{-1}q^{\ell}}, \tag{4.2.4.1}$$

where $q = e^{2\pi i\tau}$, $q_w = e^{2\pi i w}$, $\zeta_j = e^{2\pi i z_j}$, and the notation $\sum'$ signifies that $\ell = 0$ is omitted if $\zeta_1\cdots\zeta_n = 1$. When $z = z_1+\cdots+z_n$ the functions $\tilde{P}_k(w,z,\tau) = \tilde{P}_k(w,\underline{z},\tau)$ are the same as the functions $P_k \begin{bmatrix} \zeta \\ \\ 1 \end{bmatrix} (w,\tau)$ in [37], $P_k(1,\zeta^{-1},w,\tau)$ in [7], and $(-2\pi i)^k \hat{\mathcal{P}}_k(q_w,q,\zeta)$ in [17]. In the case $\zeta_1\cdots\zeta_n = 1$, the functions (4.2.4.1) are simply those of (4.1.3.3).

Similar to the 'untwisted' Weierstrass functions, $\tilde{P}_k(w,\underline{z},\tau)$ can be defined recursively for $k \geq 1$ by

$$\tilde{P}_k(w,\underline{z},\tau) = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dw^{k-1}}\tilde{P}_1(w,\underline{z},\tau). \tag{4.2.4.2}$$

Writing (4.2.4.1) as

$$\tilde{P}_k(w,\underline{z},\tau) = \frac{(-1)^k}{(k-1)!}\sum_{\ell=1}^{\infty}\left(\frac{\ell^{k-1}q_w^{\ell}\zeta_1^{-1}\cdots\zeta_n^{-1}}{1-\zeta_1^{-1}\cdots\zeta_n^{-1}q^{\ell}} + \frac{(-1)^k\ell^{k-1}q_w^{-\ell}q^{\ell}\zeta_1\cdots\zeta_n}{1-\zeta_1\cdots\zeta_n q^{\ell}}\right), \tag{4.2.4.3}$$

we are able to discuss its convergence.

Lemma 4.2.4.1 *The functions $\tilde{P}_k(w,\underline{z},\tau)$ converge for $|q| < |q_w| < 1$ and $\zeta_{z_1+\cdots+z_n} \neq 1$.*

Proof Note that for some ℓ', each term

$$\left|\frac{\ell^{k-1}q_w^{\ell}\zeta_1^{-1}\cdots\zeta_n^{-1}}{1/2}\right|, \qquad \text{and} \qquad \left|\frac{\ell^{k-1}q_w^{-\ell}q^{\ell}\zeta_1\cdots\zeta_n}{1/2}\right|,$$

is larger than the corresponding terms

$$\left|\frac{\ell^{k-1}q_w^{\ell}\zeta_1^{-1}\cdots\zeta_n^{-1}}{1-\zeta_1^{-1}\cdots\zeta_n^{-1}q^{\ell}}\right|, \qquad \text{and} \qquad \left|\frac{\ell^{k-1}q_w^{-\ell}q^{\ell}\zeta_1\cdots\zeta_n}{1-\zeta_1\cdots\zeta_n q^{\ell}}\right|,$$

respectively, for all $\ell > \ell'$. This follows since $q^{\ell} \to 0$ as $\ell \to \infty$.

Let us now consider the series,

$$\sum_{\ell=1}^{\infty}\left(\ell^{k-1}q_w^{\ell}\zeta_1^{-1}\cdots\zeta_n^{-1} + (-1)^k\ell^{k-1}q_w^{-\ell}q^{\ell}\zeta_1\cdots\zeta_n\right),$$

and in particular the two series

$$\zeta_1^{-1}\cdots\zeta_n^{-1}\sum_{\ell=1}^{\infty}\ell^{k-1}q_w^{\ell} \qquad \text{and} \qquad (-1)^k\zeta_1\cdots\zeta_n\sum_{\ell=1}^{\infty}\ell^{k-1}q_w^{-\ell}q^{\ell}. \tag{4.2.4.4}$$

For each term in the first series of (4.2.4.4) we find

$$\lim_{\ell\to\infty}(\ell^{k-1}q_w^\ell)^{1/\ell} = \lim_{\ell\to\infty}(\ell^{1/\ell})^{k-1}q_w = q_w,$$

and therefore the corresponding series converges so long as $|q_w| < 1$ by the root test. Similarly, for the terms in the second series, we find

$$\lim_{\ell\to\infty}(\ell^{k-1}q_w^{-\ell}q^\ell)^{1/\ell} = \lim_{\ell\to\infty}(\ell^{1/\ell})^{k-1}q_w^{-1}q = q_w^{-1}q,$$

so that the series converges when $\left|q_w^{-1}q\right| = |q_w|^{-1}|q| < 1$, or equivalently, $|q| < |q_w|$.

We conclude that

$$\sum_{\ell>\ell'}^{\infty}\left(\frac{\ell^{k-1}q_w^\ell\zeta_1^{-1}\cdots\zeta_n^{-1}}{1-\zeta_1^{-1}\cdots\zeta_n^{-1}q^\ell} + \frac{(-1)^k\ell^{k-1}q_w^{-\ell}q^\ell\zeta_1\cdots\zeta_n}{1-\zeta_1\cdots\zeta_n q^\ell}\right)$$

converges on $|q| < |q_w| < 1$. Since there are only a finite many additional terms in (4.2.4.3), $\tilde{P}_k(w,\underline{z},\tau)$ converges on the same domain. □

Define the functions $\tilde{G}_k(\tau,\underline{z})$ by

$$\tilde{G}_{2k}(\tau,\underline{z}) := 2\xi(2k) + \frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{\ell=1}^{\infty}\left(\frac{\ell^{2k-1}q^\ell\zeta_1^{-1}\cdots\zeta_n^{-1}}{1-q^\ell\zeta_1^{-1}\cdots\zeta_n^{-1}} + \frac{\ell^{2k-1}q^\ell\zeta_1\cdots\zeta_n}{1-q^\ell\zeta_1\cdots\zeta_n}\right), \quad (4.2.4.5)$$

$$\tilde{G}_{2k+1}(\tau,\underline{z}) := \frac{(2\pi i)^{2k+1}}{(2k)!}\sum_{\ell=1}^{\infty}\left(\frac{\ell^{2k}q^\ell\zeta_1^{-1}\cdots\zeta_n^{-1}}{1-q^\ell\zeta_1^{-1}\cdots\zeta_n^{-1}} - \frac{\ell^{2k}q^\ell\zeta_1\cdots\zeta_n}{1-q^\ell\zeta_1\cdots\zeta_n}\right), \quad (4.2.4.6)$$

$$\tilde{G}_1(\tau,\underline{z}) := (2\pi i)\sum_{\ell=1}^{\infty}\left(\frac{q^\ell\zeta_1^{-1}\cdots\zeta_n^{-1}}{1-q^\ell\zeta_1^{-1}\cdots\zeta_n^{-1}} + \frac{q^\ell\zeta_1\cdots\zeta_n}{1-q^\ell\zeta_1\cdots\zeta_n}\right) + \frac{2\pi i}{1-\zeta_1^{-1}\cdots\zeta_n^{-1}} - \pi i, \quad (4.2.4.7)$$

and set

$$\tilde{E}_m(\tau,\underline{z}) := \frac{1}{(2\pi i)^m}\tilde{G}_m(\tau,\underline{z}). \quad (4.2.4.8)$$

The above functions are called 'twisted Eisenstein series' in [7, 17, 37], though here they have additional complex variables. These additional complex variables do not add much difficulty, however, as in most calculations they collapse to the single complex variable case via

$$\tilde{E}_m(\tau, \underline{z}) = \tilde{E}_m(\tau, z_1 + \cdots + z_n).$$

The following lemma follows as in Proposition 2 in [37] with the same proof (see also display (C.14) in [17]).

Lemma 4.2.4.2

$$\tilde{P}_k(w, \underline{z}, \tau) = \frac{1}{w^k} + (-1)^k \sum_{m \geq k} \binom{m-1}{k-1} \tilde{E}_m(\tau, \underline{z}) w^{m-k}.$$

□

We have the following functional equation for $\tilde{E}_k(\tau, \underline{z})$.

Lemma 4.2.4.3 *For weights $m \geq 1$, the functions $\tilde{E}_m(\tau, \underline{z})$ satisfy (4.2.3.1) for the matrices S and T with index $F = 0$.*

Proof We first take $\underline{z} = z$. The result follows from a transformation discussed in [17]. In particular, it is established there that

$$\tilde{E}_m\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \sum_{k=0}^{m} \frac{(-1)^{k-m}}{(m-k)!} \tilde{E}_k(\tau, z) z^{m-k} \tau^k,$$

where we take $\tilde{E}_0(\tau, z)$ to be 1. Therefore,

$$\tau^{-m} \tilde{E}_m\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \sum_{k=0}^{m} \frac{(-1)^{k-m}}{(m-k)!} \tilde{E}_k(\tau, z) z^{m-k} \tau^{k-m},$$

so that

$$\begin{aligned} \tau^{-m} \tilde{E}_m\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \sum_{k=0}^{m} \frac{(-1)^{k-m}}{(m-k)!} \tilde{E}_k(\tau, z) z^{m-k} \tau^{k-m} \\ &= \sum_{k=0}^{m} \frac{(-1)^{k-m}}{(m-k)!} \tilde{E}_k(\tau, z) \left(\frac{z}{\tau}\right)^{m-k}. \end{aligned} \tag{4.2.4.9}$$

This proves the transformation for the matrix S. For the matrix T, we have $T \cdot \tau \mapsto \tau + 1$, and we find $\tilde{E}_m(\tau+1, z) = \tilde{E}_m(\tau, z)$. Therefore (4.2.3.1) holds for all $\gamma \in \Gamma$.

We now consider the general case of $\underline{z}$. Using $\tilde{E}_m(\tau, \underline{z}) = \tilde{E}_m(\tau, z_1 + \cdots + z_n)$ and (4.2.4.9), we find

$$\begin{aligned}
\tau^{-m}\tilde{E}_m\left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}\right) &= \tau^{-m}\tilde{E}_m\left(-\frac{1}{\tau}, \frac{z_1 + \cdots + z_n}{\tau}\right) \\
&= \sum_{k=0}^{m} \frac{(-1)^{m-k}}{(m-k)!}\tilde{E}_k(\tau, z_1 + \cdots + z_n)\left(\frac{z_1 + \cdots + z_n}{\tau}\right)^{m-k} \\
&= \sum_{k=0}^{m} \sum_{\substack{i_1 + \cdots + i_n = m-k \\ i_1, \ldots, i_n \geq 0}} C_{i_1,\ldots,i_n} \frac{(-1)^{m-k}}{(m-k)!}\tilde{E}_k(\tau, \underline{z})\left(\frac{z_1}{\tau}\right)^{i_1} \cdots \left(\frac{z_n}{\tau}\right)^{i_n},
\end{aligned}$$

where $C_{i_1,\ldots,i_n}$ is a scalar given by the expansion of $((z_1 + \cdots + z_n)/\tau)^{m-k}$. This proves the claim for the matrix S. The case of the matrix T is again trivial. □

4.3 Jacobi theta series with spherical harmonics

Let Q be a positive definite quadratic form, and B be the associated bilinear form so that $2Q(x) = B(x, x)$. Let A be the matrix of Q of even rank $f = 2r$. Since A is positive-definite, its determinant $|A| > 0$. Fix $h \in \mathbb{Z}^f$. It is well known (page 81 of [14], for example) that the *Jacobi theta function*

$$\sum_{n \in \mathbb{Z}^f} q^{Q(n)} \zeta^{B(n,h)}$$

is a Jacobi form of weight r and index $Q(h)$.

We now fix a vector $v \in \mathbb{C}^f$. In this section, transformation laws under the Jacobi group for the functions

$$\theta_h(Q, v, k, \tau, z) := \sum_{m \in \mathbb{Z}^f} B(v, m)^k q^{Q(m)} \zeta^{B(m,h)}, \tag{4.3.0.10}$$

and more generally,

$$\theta_{\underline{h}}(Q, v, k, \tau, \underline{z}) := \sum_{m \in \mathbb{Z}^f} B(v, m)^k q^{Q(m)} \zeta_1^{B(m,h_1)} \cdots \zeta_n^{B(m,h_n)}, \tag{4.3.0.11}$$

are developed. As mentioned above, the case $(k, \tau, z) = (0, \tau, z)$ of (4.3.0.10) is found in [14]. The case $(k, \tau, z) = (k, \tau, 0)$ is established in [10] and is a generalization of the work of Schoeneberg [43]. In fact, most of the results developed in this section mimic the work of Schoeneberg [43], Dong and Mason [10], and Ogg [40]. We attempt to maintain similar notation and structure to these works (in particular [10]), and will often make reference to results contained within them.

To establish our desired transformation laws (see Theorems 4.3.0.5 and 4.3.4.2 below), we will utilize the theory of *Jacobi-like forms*. These are holomorphic functions $\phi(\tau, z, X)$ on $\mathbb{H} \times \mathbb{C} \times \mathbb{C}$ given by

$$\phi(\tau, z, X) = \sum_{n \geq 0} \phi^{(n)}(\tau, z)(2\pi i X)^n,$$

and which satisfy for some $\ell, F \in \mathbb{C}$, an integer k, and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$;

$$\phi\left(\gamma\tau, \gamma z, \frac{X}{(c\tau + d)^2}\right) = \chi(d) e^{-2\pi i \frac{cz^2}{c\tau + d} F} (c\tau + d)^k \exp\left(\frac{c\ell X}{c\tau + d}\right) \phi(\tau, z, X), \tag{4.3.0.12}$$

where χ is some character (see [45]). Here, the $\phi^{(n)}(\tau, z)$ denote the coefficients of the $2\pi i X$ terms, and are holomorphic functions on $\mathbb{H} \times \mathbb{C}$. In the case $\ell = 0$, equating the coefficients of each $(2\pi i X)^n$ term shows that the functions $\phi^{(n)}(\tau, z)$ are holomorphic Jacobi forms of weight $k + 2n$ and index F. In the case $\ell = 1$, $\phi(\tau, z, X)$ is a Jacobi-like form on $\Gamma_0(N)$ of weight k, index F, and character χ. The X may be normalized so that ℓ equals either 0 or

1.

While the coefficients of a Jacobi-like form may not be Jacobi forms themselves, often the coefficients of the product of two Jacobi-like forms will be a Jacobi form. This situation arises here, and will be discussed below.

For fixed Q, v, and h as above, we define the function $\Theta_h(Q, v, \tau, z, X)$ on $\mathbb{H}\times\mathbb{C}\times\mathbb{C}$ by

$$\Theta_h(Q, v, \tau, z, X) := \sum_{n\geq 0} \frac{2^n \theta_h(Q, v, 2n, \tau, z)}{(2n)!}(2\pi i X)^n. \tag{4.3.0.13}$$

Moreover, set

$$\gamma(t, k) = \frac{k!}{2^t t!(k-2t)!}, \tag{4.3.0.14}$$

and

$$\Psi_h(Q, v, 2k, \tau, z) = \sum_{t=0}^{k} \gamma(t, 2k) E_2(\tau)^t \theta_h(Q, v, 2k-2t, \tau, z). \tag{4.3.0.15}$$

It is functions of the form (4.3.0.13) and (4.3.0.15) that will be used to prove results about θ_h. For $n > 0$, define $\epsilon(n)$ by

$$\epsilon(n) = \left(\frac{(-1)^r \det(A)}{n}\right),$$

and

$$\epsilon(-n) = (-1)^r \epsilon(n).$$

$\epsilon(\bmod N)$ is a Dirichlet character (see page 216 in [43] for more details). The main theorem we wish to establish in this section is the following:

Theorem 4.3.0.4 *Let the notation be as above. Suppose the matrix that represents the quadratic form Q has rank $f = 2r$. Suppose $B(v, h) = 0$ and let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ *and* $[\lambda, \mu] \in \mathbb{Z}^2$.

1. If $\langle v, v\rangle = 0$, *then*

$$\theta_h(Q, v, 2n, \gamma\tau, \gamma z) = \epsilon(d)e^{-2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau + d)^{2n+r}\theta_h(Q, v, 2n, \tau, z), \qquad (4.3.0.16)$$

and

$$\theta_h(Q, v, 2n, \tau, z + \lambda\tau + \mu) = e^{2\pi i Q(h)(\lambda^2\tau+2z\lambda)}\theta_h(Q, v, 2n, \tau, z). \qquad (4.3.0.17)$$

2. If $\langle v, v\rangle = 1$, *then*

$$\Psi_h(Q, v, \gamma\tau, \gamma z) = \epsilon(d)e^{-2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau + d)^{2n+r}\Psi(Q, v, \tau, z), \qquad (4.3.0.18)$$

and

$$\Psi_h(Q, v, \tau, z + \lambda\tau + \mu) = e^{2\pi i Q(h)(\lambda^2\tau+2z\lambda)}\Psi_h(Q, v, \tau, z). \qquad (4.3.0.19)$$

The crux of establishing Theorem 4.3.0.4 is in the proof of the following theorem.

Theorem 4.3.0.5 *Let the notation be as before. Suppose* $B(v, h) = 0$. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, *we have*

$$\Theta_h\left(Q, v, \gamma\tau, \gamma z, \frac{X}{(c\tau + d)^2}\right) = \epsilon(d)e^{2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau + d)^r \exp\left(\frac{2cQ(v)X}{c\tau + d}\right)\Theta_h(Q, v, \tau, z, X). \qquad (4.3.0.20)$$

Later in this section we will be interested in the specific case $2Q(v) = \langle v, v\rangle$. After scaling, Theorem 4.3.0.5 essentially takes two forms of interest: the case when v is a null vector ($\langle v, v\rangle = 0$), and the case v is a unit vector ($\langle v, v\rangle = 1$). In the case v is a null vector, Theorem 4.3.0.5 implies transformation (4.3.0.16) by equating coefficients as discussed above. This is an extension of results developed in the modular theta series case by Schoeneberg [42] and Hecke [20] to the case of Jacobi theta series. In fact, Hecke [20] has proved that the functions $\langle v, m\rangle^k$ are spherical harmonics of degree k and that every spherical harmonic of degree k is a linear combination of other spherical harmonics of degree k. In this context, the functions considered in (4.3.0.10) are Jacobi theta series with spherical harmonics. On the other hand, taking v to be a unit vector will lead to (4.3.0.18) and is a generalization of [10].

Subsection 4.3.1 will establish Theorem 4.3.0.5 and Subsection 4.3.2 will show how this result is used to prove (4.3.0.18). In Subsection 4.3.3 we prove (4.3.0.17) and (4.3.0.19).

We finish this section by developing transformation laws for matrix Jacobi theta series. With the notation already in use, define

$$\Theta_{\underline{h}}(Q, v, \tau, \underline{z}, X) = \sum_{n\geq 0} \frac{2^n \theta_{\underline{h}}(Q, v, 2n, \tau, \underline{z})}{(2n)!} (2\pi i X)^n \tag{4.3.0.21}$$

and

$$\Psi_{\underline{h}}(Q, v, 2k, \tau, \underline{z}) = \sum_{t=0}^{k} \gamma(t, 2k) E_2(\tau)^t \theta_{\underline{h}}(Q, v, 2k-2t, \tau, \underline{z}). \tag{4.3.0.22}$$

In Subsection 4.3.4 we will extend Theorem 4.3.0.5 to the functions $\theta_{\underline{h}}$ and $\Psi_{\underline{h}}$ on $\mathbb{H} \times \mathbb{C}^n$.

4.3.1 Transformation laws of $\mathbf{SL}_2(\mathbb{Z})$ on θ_h

This subsection mimics proofs found in [43] and [10]. Let Q be as above. Let $x = (x_1, \ldots, x_f)$, and let x^t denote the transpose of x. Consider the series

$$\Theta_h(Q, x) := \sum_{m\in\mathbb{Z}^f} e^{-2Q(m+x)} e^{-2B(m+x,h)}. \tag{4.3.1.1}$$

Let A be the matrix corresponding to Q such that $2Q(x) = x^t Ax$ and $B(x, h) = x^t Ay$. Then

$$\Theta_h(Q, x) = \sum_{m \in \mathbb{Z}^f} a_m e^{2\pi i m^t x},$$

with Fourier coefficients

$$\begin{aligned} a_m &= \int_0^1 \cdots \int_0^1 \sum_{n \in \mathbb{Z}^f} e^{-(n+x)^t A(n+x)} e^{-2(n+x)^t Ah} e^{-2\pi i m^t x} dX \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-x^t Ax - 2\pi i m^t x} e^{-2x^t Ah} dX. \end{aligned} \tag{4.3.1.2}$$

Take $x = A^{-1}y$ so that $x^t Ax = y^t A^{-1} y$, $x^t Ah = y^t h$, and $dY = |A|\, dX$. Then (4.3.1.2) becomes

$$a_m = \frac{1}{|A|} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(y^t A^{-1} y + 2\pi i m^t A^{-1} y)} e^{-2y^t h} dY. \tag{4.3.1.3}$$

Using that

$$(y + \pi i m)^t A^{-1} (y + \pi i m) = y^t A^{-1} y + 2\pi i m^t A^{-1} y - \pi^2 m^t A^{-1} m,$$

equation (4.3.1.3) becomes

$$\begin{aligned} a_m &= \frac{1}{|A|} e^{-\pi^2 m^t A^{-1} m} e^{2\pi i m^t h} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(y+\pi i m)^t A^{-1} (y + \pi i m)} e^{-2(y+\pi i m)^t h} dY \\ &= \frac{1}{|A|} e^{-\pi^2 m^t A^{-1} m} e^{2\pi i m^t h} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-y^t A^{-1} y} e^{-2y^t h} dY. \end{aligned} \tag{4.3.1.4}$$

Completing the square in the argument of the integrand in (4.3.1.4) gives

$$-y^t A^{-1} y - 2y^t h = -(y + Ah)^t A^{-1} (y + Ah) + h^t Ah,$$

so that (4.3.1.4) becomes

$$a_m = \frac{1}{|A|} e^{-\pi^2 m^t A^{-1} m} e^{2\pi i m^t h} e^{h^t Ah} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(y+Ah)^t A^{-1} (y+Ah)} dY. \tag{4.3.1.5}$$

Take a vector w such that $|w| = w_1^2 + \cdots + w_f^2$, and consider a real matrix L such that $A = LL^t$. Set $Lw = y + Ah$ so that $dY = |L|\, dW$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(y+A^{-1}h)^t A(y+A^{-1}h)} dY &= |L| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-w_1^2+\cdots+w_f^2} dz_1 \cdots dw_f \\ &= |L| \prod_{j=1}^{f} \int_{-\infty}^{\infty} e^{-w_j^2} dw_j \\ &= |A|^{1/2} \pi^{f/2}.\end{aligned}$$

The Fourier coefficients in (4.3.1.5) are now

$$a_m = \frac{\pi^{f/2}}{|A|^{1/2}} e^{-\pi^2 m^t A^{-1} m} e^{2\pi i m^t h} e^{h^t A h}. \tag{4.3.1.6}$$

Replace A with $\pi\epsilon A$, where $\epsilon > 0$. Equation (4.3.1.6) now becomes

$$a_m = \frac{\pi^{f/2}}{(\sqrt{\pi\epsilon})\, |A|^{1/2}} e^{-\frac{\pi}{\epsilon} m^t A^{-1} m} e^{2\pi i m^t h} e^{\pi\epsilon h^t A h}. \tag{4.3.1.7}$$

Replacing A with $\pi\epsilon A$ also in (4.3.1.1) and using (4.3.1.2) and (4.3.1.3) we have

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^f} e^{-\pi\epsilon(m+x)^t A(m+x)} e^{-\pi\epsilon(m+x)^t A h} \\ &\qquad = \frac{\pi^{f/2}}{(\sqrt{\pi\epsilon})^f\, |A|^{1/2}} \sum_{m\in\mathbb{Z}^f} e^{-\frac{\pi}{\epsilon} m^t A^{-1} m} e^{2\pi i m^t h} e^{\pi\epsilon h^t A h} e^{2\pi i m^t x} \\ &\qquad = \frac{e^{\pi\epsilon h^t A h}}{(\sqrt{\epsilon})^f\, |A|^{1/2}} \sum_{m\in\mathbb{Z}^f} e^{-\frac{\pi}{\epsilon} m^t A^{-1} m + 2\pi i m^t h + 2\pi i m^t x}.\end{aligned} \tag{4.3.1.8}$$

As explained in [43] (page 205), this last equation is valid so long as $\sqrt{\epsilon} > 0$. Setting $\epsilon = -i\tau$ gives

$$\sum_{m\in\mathbb{Z}^f} e^{\pi i\tau(m+x)^t A(m+x)} e^{\pi i\tau(m+x)^t A h} = \frac{e^{-\pi i\tau h^t A h}}{(\sqrt{-i\tau})^f\, |A|^{1/2}} \sum_{m\in\mathbb{Z}^f} e^{-\frac{\pi i}{\tau} m^t A^{-1} m + 2\pi i m^t h + 2\pi i m^t x}, \tag{4.3.1.9}$$

where the imaginary part of $\tau > 0$ and the root is chosen so that it has positive real part. Recalling that $2Q(m) = m^tAm$ and $B(m,h) = m^tAh$, we can write (4.3.1.9) as

$$\sum_{m\in\mathbb{Z}^f} e^{2\pi i\tau Q(m+x)}e^{2\pi i\tau B(m+x,h)} = \frac{e^{-2\pi i\tau Q(h)}}{(\sqrt{-i\tau})^f\,|A|^{1/2}} \sum_{m\in\mathbb{Z}^f} e^{-\frac{\pi i}{\tau}m^tA^{-1}m+2\pi i m^t h+2\pi i m^t x}. \tag{4.3.1.10}$$

Consider the linear operator

$$\mathcal{L} := \sum_{\nu=1}^{f} \ell_\nu \frac{\partial}{\partial x_\nu}, \quad \ell_\nu \in \mathbb{C}$$

and its powers $\mathcal{L}^n$, $n \in \mathbb{N}$. The operator $\mathcal{L}$ has the properties (see [43]),

$$\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v), \qquad \mathcal{L}(uv) = \mathcal{L}(u)v + u\mathcal{L}(v),$$

$$\mathcal{L}(cu) = c\mathcal{L}(u) \quad \text{for } c \in \mathbb{C}, \quad \text{and } \mathcal{L}(e^u) = e^u\mathcal{L}(u).$$

Set

$$\ell^t = (\ell_1, \ldots, \ell_f).$$

Then

$$\mathcal{L}(2Q(m+x)) = \mathcal{L}(m+x)^tA(m+x) = 2\ell^tA(m+x),$$

and applying $\mathcal{L}$ again shows

$$\mathcal{L}^2(2Q(m+x)) = \mathcal{L}^2(m+x)^t(m+x)^tA(m+x) = 2\ell^tA\ell = 2\cdot 2Q(\ell).$$

On the other hand, we find

$$\mathcal{L}(B(m+x,h)) = \mathcal{L}((m+x)^tAh) = \ell^tAh = B(\ell,h),$$

and $\mathcal{L}^2((m+x)^tAh)=0$. Applying $\mathcal{L}$ to the left hand side of (4.3.1.10) gives

$$\mathcal{L}\left(\sum_{m\in\mathbb{Z}^f} e^{2\pi i\tau Q(m+x)}e^{2\pi i\tau B(m+x,h)}\right)=\sum_{m\in\mathbb{Z}^f}\mathcal{L}\left(e^{\pi i\tau[(m+x)^tA(m+x)+2(m+x)^tAh]}\right)$$

$$=\sum_{m\in\mathbb{Z}^f}\mathcal{L}(\pi i\tau[(m+x)^tA(m+x)+2(m+x)^tAh])e^{\pi i\tau[(m+x)^tA(m+x)+2(m+x)^tAh]}$$

$$=\sum_{m\in\mathbb{Z}^f}(\pi i\tau)(2\ell^tA(m+x)+2\ell^tAh)e^{2\pi i\tau[Q(m+x)+B(m+x,h)]}.$$

If we apply $\mathcal{L}$ again we find

$$\mathcal{L}^2\left(\sum_{m\in\mathbb{Z}^f} e^{2\pi i\tau[Q(m+x)+B(m+x,h)]}\right)$$

$$=\sum_{m\in\mathbb{Z}^f}\Big(\mathcal{L}(\pi i\tau[2\ell^tA(m+x)+2\ell^tAh])e^{\pi i\tau[(m+x)^tA(m+x)+2(m+x)^tAh]}$$

$$+(\pi i\tau[2\ell^tA(m+x)+2\ell^tAh])^2e^{\pi i\tau[(m+x)^tA(m+x)+2(m+x)^tAh]}\Big)$$

$$=\sum_{m\in\mathbb{Z}^f}\pi i\tau[2\ell^tA\ell+[\pi i\tau(2\ell^tA(m+x)+2\ell^tAh)]^2]e^{\pi i\tau[(m+x)^tA(m+x)+2(m+x)^tAh]}.$$

More generally, we have

$$\mathcal{L}^k\left(\sum_{m\in\mathbb{Z}^f} e^{2\pi i\tau[Q(m+x)+B(m+x,h)]}\right)$$

$$=\sum_{j=0}^{[k/2]}\sum_{m\in\mathbb{Z}^f}\gamma(j,k)(2\pi i\tau)^{k-j}(2Q(\ell))^j(\ell^tA(m+x)+\ell^tAh)^{k-2j}e^{2\pi i\tau(Q(m+x)+B(m+x,h))},$$

For $\gamma(j,k)$ defined in (4.3.0.14) and $0\le j\le[k/2]$, $k\ge 0$ (see [10]). Here $[k/2]$ denotes the greatest integer less than or equal to $k/2$.

Applying $\mathcal{L}^k$ to both sides of (4.3.1.10) gives

$$\sum_{j=0}^{[k/2]}\sum_{m\in\mathbb{Z}^f}\gamma(j,k)(2\pi i\tau)^{k-j}(2Q(\ell))^j(\ell^tA(m+x)+\ell^tAh)^{k-2j}e^{2\pi i\tau(Q(m+x)+B(m+x,h))}$$

$$=e^{-2\pi i\tau Q(h)}\frac{(2\pi i)^k}{(\sqrt{-i\tau})^f\,|A|^{1/2}}\sum_{m\in\mathbb{Z}^f}(m^t\ell)^ke^{-\frac{\pi i}{\tau}m^tA^{-1}m+2\pi i m^th+2\pi i m^tx}.$$

As in [43] and [10] we now replace τ with $-1/\tau$ and x with p/N (N is the level of A). On the right hand side we also replace m by Am_1/N. Recalling that $f = 2r$ and setting $D = |A|$, we have

$$\sum_{j=0}^{[k/2]} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv p(N)}} N^{2j-k}\left(-\frac{2\pi i}{\tau}\right)^{k-j} \gamma(j,k)(2Q(\ell))^j(\ell^t Am + \ell^t Ah)^{k-2j} e^{-\frac{2\pi i}{\tau}(Q(m)/N^2+B(m,h)/N)}$$
$$= e^{\frac{2\pi i}{\tau}Q(h)} \frac{(2\pi i)^k \tau^r N^{-k}}{i^r\sqrt{D}} \sum_{\substack{m_1\in\mathbb{Z}^f \\ Am_1\equiv 0(N)}} (\ell^t Am_1)^k e^{2\pi i\tau Q(m_1)/N^2+2\pi i B(m_1,h)/N+2\pi i m_1^t Ap/N^2}. \tag{4.3.1.11}$$

Next replace h with zh to find

$$\sum_{j=0}^{[k/2]} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv p(N)}} N^{2j-k}\left(-\frac{2\pi i}{\tau}\right)^{k-j} \gamma(j,k)(2Q(\ell))^j(\ell^t Am + z\ell^t Ah)^{k-2j} e^{-\frac{2\pi i}{\tau}(Q(m)/N^2+zB(m,h)/N)}$$
$$= e^{\frac{2\pi i z^2}{\tau}Q(h)} \frac{(2\pi i)^k \tau^r N^{-k}}{i^r\sqrt{D}} \sum_{\substack{m_1\in\mathbb{Z}^f \\ Am_1\equiv 0(N)}} (\ell^t Am_1)^k e^{2\pi i\tau Q(m_1)/N^2+2\pi i z B(m_1,h)/N+2\pi i m_1^t Ap/N^2}. \tag{4.3.1.12}$$

For now on we assume $\ell^t Ah = 0$ (this is the assumption $B(\ell, h) = 0$). Then (4.3.1.12) becomes

$$\sum_{j=0}^{[k/2]} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv p(N)}} N^{2j-k}\left(-\frac{2\pi i}{\tau}\right)^{k-j} \gamma(j,k)(2Q(\ell))^j(\ell^t Am)^{k-2j} e^{-\frac{2\pi i}{\tau}(Q(m)/N^2+zB(m,h)/N)}$$
$$= e^{\frac{2\pi i z^2}{\tau}Q(h)} \frac{(2\pi i)^k \tau^r N^{-k}}{i^r\sqrt{D}} \sum_{\substack{m_1\in\mathbb{Z}^f \\ Am_1\equiv 0(N)}} (\ell^t Am_1)^k e^{2\pi i\tau Q(m_1)/N^2+2\pi i z B(m_1,h)/N+2\pi i m_1^t Ap/N^2}. \tag{4.3.1.13}$$

As done in [43], taking p so that $Ap \equiv 0 \bmod N$, we have $m_1^t Ap/N^2 \bmod 1$ depends only on

$p \bmod N$ and (4.3.1.13) may be written as

$$\sum_{j=0}^{[k/2]} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv p(N)}} N^{2j-k} \left(-\frac{2\pi i}{\tau}\right)^{k-j} \gamma(j,k)(2Q(\ell))^j (\ell^t Am)^{k-2j} e^{-\frac{2\pi i}{\tau}(Q(m)/N^2+zB(m,h)/N)}$$

$$= e^{\frac{2\pi i z^2}{\tau}Q(h)} \frac{(2\pi i)^k \tau^r}{i^r\sqrt{D}} \sum_{\substack{g \bmod N \\ Ag\equiv 0 \bmod N}} e^{2\pi i g A p/N^2}$$

$$\cdot \frac{1}{N^k} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv g \bmod N}} (\ell^t Am)^k e^{2\pi i\tau Q(m)/N^2+2\pi i z B(m,h)/N}. \tag{4.3.1.14}$$

Since

$$\left(-\frac{2\pi i}{\tau}\right)^{k-j} (2Q(\ell))^j = \left(-\frac{Q(\ell)\tau}{\pi i}\right)^j (-2\pi i)^k,$$

we can rewrite (4.3.1.14) as

$$\sum_{j=0}^{[k/2]} \left(-\frac{Q(\ell)\tau}{\pi i}\right)^j \gamma(j,k)\frac{1}{N^{k-2j}} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv p(N)}} (\ell^t Am)^{k-2j} e^{-\frac{2\pi i}{\tau}(Q(m)/N^2+zB(m,h)/N)}$$

$$= e^{\frac{2\pi i z^2}{\tau}Q(h)} \frac{(-i)^{2k+r}\tau^{r+k}}{\sqrt{D}} \sum_{\substack{g \bmod N \\ Ag\equiv 0 \bmod N}} e^{2\pi i g A p/N^2}$$

$$\cdot \frac{1}{N^k} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv g \bmod N}} (\ell^t Am)^k e^{2\pi i\tau Q(m)/N^2+2\pi i z B(m,h)/N}. \tag{4.3.1.15}$$

In the case $z = 0$, this becomes equation (3.8) in [10]. If we also were to assume $Q(\ell) = 0$ (which corresponds to the case v is a null vector above) we would have equation (12) of [43]. Following the notation in [10] we set

$$\theta(A,p,\ell,k,\tau,z) = \frac{1}{N^k} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv p(N)}} (\ell^t Am)^k e^{2\pi i\tau Q(m)/N^2+2\pi i z B(m,h)/N}. \tag{4.3.1.16}$$

We also set

$$\begin{aligned}
&\Theta(A,p,\ell,k,j,\tau,z)\\
&= e^{\frac{2\pi i z^2}{\tau}Q(h)}\frac{(-i)^{r+2k}\tau^{r+k-2j}}{\sqrt{D}}\sum_{\substack{g \bmod N\\ Ag\equiv 0(N)}} e^{2\pi i g^t Ap/N^2}\theta(A,g,\ell,k-2j,\tau,z). \qquad (4.3.1.17)
\end{aligned}$$

We have the following theorem.

Theorem 4.3.1.1 *With the notation as before, we have*

$$\theta(A,p,\ell,k,-1/\tau,z/\tau)=\sum_{j=0}^{[k/2]}\left(\frac{Q(\ell)\tau}{\pi i}\right)^j\gamma(j,k)\Theta(A,p,\ell,k,j,\tau,z). \qquad (4.3.1.18)$$

Proof This is an extension of Theorem 3.3 in [10] to include the z-variable. The proof remains the same and we maintain the notation of [10]. Set $\Theta_j=\Theta(A,p,\ell,k-2j,0,\tau,z)$, $\theta_j=\theta(A,p,\ell,k-2j,-1/\tau,z/\tau),[k/2]=K,(Q(\ell)\tau)/\pi i=M$. Then (4.3.1.18) becomes

$$\theta_0=\sum_{j=0}^{K}M^j\gamma(j,k)\Theta_j.$$

Equation (4.3.1.15) can now be written as

$$\Theta_0=\sum_{i=0}^{K}(-M)^i\gamma(i,k)\theta_i.$$

We also find by replacing k by $k-2j$ that

$$\Theta_j=\sum_{i=0}^{K-j}(-M)^i\gamma(i,k-2j)\theta_{i+j}.$$

It follows that

$$\begin{aligned}
\sum_{j=0}^{K}M^j\gamma(j,k)\Theta_j &= \sum_{j=0}^{K}M^j\gamma(j,k)\left(\sum_{i=0}^{K-j}(-M)^i\gamma(i,k-2j)\theta_{i+j}\right)\\
&= \sum_{s=0}^{K}M^s\gamma(s,k)\theta_s\sum_{i=0}^{s}(-1)^i\binom{s}{i}\\
&= \theta_0,
\end{aligned}$$

since the terms in $\sum_{i=0}^{s}(-1)^s\binom{s}{i}$ cancel except when $s=0$. Here we set $s=i+j$ and used the identity $\gamma(j,k)\gamma(i,k-2j)=\binom{s}{i}\gamma(s,k)$. $\square$

Continuing to mimic the structure and proofs of [10], we establish the analogue of their Theorem 3.4 (loc. cit.) to include a z-variable.

Theorem 4.3.1.2 *Suppose A has rank $f=2r$. If $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_0(N)$, and if $d>0$, then*

$$e^{-2\pi i\frac{cz^2Q(h)}{c\tau+d}}(c\tau+d)^{-(r+k)}\theta\left(A,p,\ell,k,\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)$$
$$=\exp(2\pi iQ(p)ab/N^2)\epsilon(d)\sum_{j=0}^{[k/2]}\left(\frac{Q(\ell)c}{\pi i(c\tau+d)}\right)^j\gamma(j,k)\theta(A,bp,\ell,k-2j,\tau,z). \quad (4.3.1.19)$$

In particular, if also $d>0$ and we take $p=0$, then

$$e^{-2\pi i\frac{cz^2Q(h)}{c\tau+d}}(c\tau+d)^{-(r+k)}\theta\left(A,\ell,k,\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)$$
$$=\epsilon(d)\sum_{j=0}^{[k/2]}\left(\frac{Q(\ell)c}{\pi i(c\tau+d)}\right)^j\gamma(j,k)\theta(A,\ell,k-2j,\tau,z). \quad (4.3.1.20)$$

Recalling that $Ap=0(\mathrm{mod}\,N)$, then (4.3.1.16) shows that

$$\theta(A,p,\ell,k,\tau+1,z)=\exp(2\pi iQ(p)/N^2)\theta(A,p,\ell,k,\tau,z). \quad (4.3.1.21)$$

Moreover, if $c>0$ then following equation 18 in [43] (page 211) we have

$$\theta(A,p,\ell,k,\tau,z)=\sum_{\substack{g\,\mathrm{mod}\,cN\\ g\equiv p(N)}}\theta(cA,g,\ell,k,c\tau,z). \quad (4.3.1.22)$$

The next couple of displays depend little on the z-variable, and are established just as in [10] (displays (3.17) through (3.20)). We carry out the calculations again here, carrying along the additional z-variable terms and making the necessary changes.

Let γz denote $\frac{z}{c\tau+d}$. Using (4.3.1.21) and (4.3.1.22), along with Theorem 4.3.1.1 we find that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $c > 0$ we have

$$\theta(A,p,\ell,k,\gamma\tau,\gamma z) = \theta(A,p,\ell,k,c^{-1}(a-(c\tau+d)^{-1},\gamma z)$$

$$= \sum_{\substack{g \bmod cN \\ g\equiv p(N)}} \theta(cA,g,\ell,k,a-(c\tau+d)^{-1},\gamma z)$$

$$= \sum_{\substack{g \bmod cN \\ g\equiv p(N)}} \exp(2\pi iacQ(g)//c^2N^2)\theta(cA,g,\ell,k,-(c\tau+d)^{-1},\gamma z)$$

$$= \sum_{\substack{g \bmod cN \\ g\equiv p(N)}} \sum_{j=0}^{[k/2]} \exp(2\pi iacQ(g)//c^2N^2) \left(\frac{Q(\ell)c(c\tau+d)}{\pi i}\right)^j \gamma(j,k)\Theta(cA,g,\ell,k,j,c\tau+d,z)$$

$$= \sum_{\substack{g \bmod cN \\ g\equiv p(N)}} \sum_{j=0}^{[k/2]} \sum_{\substack{q \bmod cN \\ cAq\equiv 0(cN)}} \exp(2\pi iaQ(g)/cN^2) \left(\frac{Q(\ell)c(c\tau+d)}{\pi i}\right)^j \gamma(j,k)$$

$$\cdot e^{2\pi i\frac{cz^2}{c\tau+d}Q(h)} \frac{(-i)^{r+2(k-2j)}}{\sqrt{c^f D}} (c\tau+d)^{r+k-2j} \exp(2\pi i g^t Aq/cN^2)\theta(cA,q,\ell,k-2j,c\tau+d,z)$$

$$= e^{2\pi i\frac{cz^2}{c\tau+d}Q(h)} \frac{(c\tau+d)^{r+k}(-i)^{r+2k}}{c^r\sqrt{D}} {\sum}' \sum_{j=0}^{[k/2]} \exp(2\pi i(aQ(g)+dQ(q)+g^tAq)/N^2)\gamma(j,k)$$

$$\cdot \left(\frac{cQ(\ell)}{(c\tau+d)\pi i}\right)^j \theta(cA,q,\ell,k-2j,c\tau,z) \qquad (4.3.1.23)$$

where ${\sum}'$ denotes that we have taken $g \bmod cN$, $q \bmod cN$, $g \equiv p(N)$, and $Aq \equiv 0(N)$. Note that cN arises as the level of cA.

Just as in Schoeneberg ([43], page 213), we use the functions defined by

$$\phi_{p,q} = \sum_{\substack{g \bmod cN \\ g\equiv p(N)}} \exp(2\pi i(aQ(g)+dQ(q)+g^tAq)/N^2). \qquad (4.3.1.24)$$

Making note that $\phi_{p,q}$ depends only on q only modulo N, (4.3.1.23) may be written as

$$e^{2\pi i \frac{cz^2}{c\tau+d}Q(h)} \frac{(c\tau+d)^{r+k}(-i)^{r+2k}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q \bmod cN \\ Aq\equiv 0(N)}}$$

$$\cdot\, \phi_{p,q}\gamma(j,k) \left(\frac{cQ(\ell)}{(c\tau+d)\pi i}\right)^j \theta(cA,q,\ell,k-2j,c\tau,z)$$

$$= e^{2\pi i \frac{cz^2}{c\tau+d}Q(h)} \frac{(c\tau+d)^{r+k}(-i)^{r+2k}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{q_1 \bmod N} \phi_{p,q_1}\gamma(j,k) \left(\frac{cQ(\ell)}{(c\tau+d)\pi i}\right)^j$$

$$\cdot \sum_{\substack{q \bmod cN \\ Aq\equiv 0(N), q\equiv q_1(N)}} \theta(cA,q,\ell,k-2j,c\tau,z).$$

Applying (4.3.1.22) again establishes the following lemma which is Lemma 3.5 of [10] with a z-variable.

Lemma 4.3.1.3 *We have*

$$e^{-2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau+d)^{-(r+k)}\theta(A,p,\ell,k,\gamma\tau,\gamma z)$$

$$= \frac{(-i)^{r+2k}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N \\ Aq_1\equiv 0(N)}} \phi_{p,q_1}\gamma(j,k) \left(\frac{cQ(\ell)}{(c\tau+d)\pi i}\right)^j \theta(A,q_1,\ell,k-2j,\tau,z), \quad (4.3.1.25)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *with* $c > 0$.

As in [10] we now assume that $d \equiv 0 (\bmod N)$ and use techniques of Schoeneberg ([43], page 214) to rewrite (4.3.1.25) as

$$e^{-2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau+d)^{-(r+k)}\theta(A,p,\ell,k,\gamma\tau,\gamma z)$$

$$= \frac{(-i)^{r+2k}\phi_{p,0}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N \\ Aq_1\equiv 0(N)}} \exp(-2\pi i p^t A q_1 b/N^2)$$

$$\gamma(j,k) \left(\frac{cQ(\ell)}{(c\tau+d)\pi i}\right)^j \theta(A,q_1,\ell,k-2j,\tau,z). \quad (4.3.1.26)$$

We now replace τ with $-1/\tau$ and z with z/τ in (4.3.1.26) so that

$$\frac{a\tau+b}{c\tau+d} \mapsto \frac{b\tau-a}{d\tau-c} \qquad \text{and} \qquad \frac{z}{c\tau+d} \mapsto \frac{z}{d\tau-c}.$$

While this again mimics [10], deviations occur due to the z-variable. Using Theorem 4.3.1.1 we find (4.3.1.26) becomes

$$\begin{aligned}
&e^{-2\pi i z^2 \frac{c}{\tau(d\tau-c)}Q(h)} \left(\frac{d\tau-c}{\tau}\right)^{-(r+k)} \theta\left(A,p,\ell,k,\frac{b\tau-a}{d\tau-c},\frac{z}{d\tau-c}\right)\\
&= \frac{(-i)^{r+2k}\phi_{p,0}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N\\ Aq_1\equiv 0(N)}} \sum_{u=0}^{[(k-2j)/2]} \exp(-2\pi i p^t A q_1 b/N^2)\gamma(j,k)\\
&\quad \cdot \left(\frac{cQ(\ell)\tau}{(d\tau-c)\pi i}\right)^j \left(\frac{Q(\ell)\tau}{\pi i}\right)^u \gamma(u,k-2j)\Theta(A,q_1,\ell,k-2j,u,\tau,z)\\
&= \frac{(-i)^{r+2k}\phi_{p,0}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N\\ Aq_1\equiv 0(N)}} \sum_{u=0}^{[(k-2j)/2]} \sum_{\substack{g \bmod N\\ Ag\equiv 0(N)}} \exp(-2\pi i p^t A q_1 b/N^2)\gamma(j+u,k)\\
&\quad \cdot \binom{j+u}{j} \left(\frac{c}{d\tau-c}\right)^j \left(\frac{cQ(\ell)\tau}{\pi i}\right)^{j+u} e^{2\pi i \frac{z^2}{\tau}} \frac{(-1)^{r+2(k-2j)}}{\sqrt{D}} \tau^{r+k-2j-2u}\\
&\quad \cdot \exp(2\pi i g^t A q_1/N^2)\theta(A,g,\ell,k-2j-2u,\tau,z).
\end{aligned}$$

Since

$$\frac{c}{\tau(d\tau-c)} + \frac{1}{\tau} = \frac{d}{d\tau-c},$$

we find

$$\begin{aligned}
&e^{-2\pi i \frac{dz^2}{d\tau-c}Q(h)} (d\tau-c)^{-(r+k)} \theta\left(A,p,\ell,k,\frac{b\tau-a}{d\tau-c},\frac{z}{d\tau-c}\right)\\
&= \frac{(-1)^r\phi_{p,0}}{c^r D} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N\\ Aq_1\equiv 0(N)}} \sum_{u=0}^{[(k-2j)/2]} \sum_{\substack{g \bmod N\\ Ag\equiv 0(N)}} \exp(2\pi i (g-bp)^t A q_1/N^2)\gamma(j+u,k)\binom{j+u}{j}\\
&\quad \cdot \left(\frac{c}{d\tau-c}\right)^j \left(\frac{Q(\ell)}{\pi i\tau}\right)^{j+u} \theta(A,g,\ell,k-2(j+u),\tau,z). \qquad (4.3.1.27)
\end{aligned}$$

Schoeneberg shows ([43], page 214) that

$$\sum_{\substack{q_1 \bmod N \\ Aq_1 \equiv 0(N)}} \exp(2\pi i(g-bp)^t Aq_1/N^2) = D\delta_{g,bp},$$

where $\delta_{g,bp}$ is the Kronecker delta and g, bp are considered modulo N. Following [10], we find (4.3.1.27) becomes

$$\begin{aligned}
&e^{-2\pi i \frac{dz^2}{d\tau-c}Q(h)}(d\tau-c)^{-(r+k)}\theta\left(A,p,\ell,k,\frac{b\tau-a}{d\tau-c},\frac{z}{d\tau-c}\right) \\
&= \frac{(-1)^r\phi_{p,0}}{c^r}\sum_{j=0}^{[k/2]}\sum_{u=0}^{[(k-2j)/2]}\gamma(j+u,k)\binom{j+u}{j}\left(\frac{c}{d\tau-c}\right)^j\left(\frac{Q(\ell)}{\pi i\tau}\right)^{j+u} \\
&\qquad \cdot\theta(A,bp,\ell,k-2(j+u),\tau,z) \\
&= \frac{(-1)^r\phi_{p,0}}{c^r}\sum_{j=0}^{[k/2]}\sum_{t=0}^{[k/2]}\gamma(t,k)\binom{t}{j}\left(\frac{c}{d\tau-c}\right)^j\left(\frac{Q(\ell)}{\pi i\tau}\right)^t\theta(A,bp,\ell,k-2t,\tau,z) \\
&= \frac{(-1)^r\phi_{p,0}}{c^r}\sum_{j=0}^{[k/2]}\gamma(t,k)\left(\frac{d\tau}{d\tau-c}\right)^t\left(\frac{Q(\ell)}{\pi i\tau}\right)^t\theta(A,bp,\ell,k-2t,\tau,z).
\end{aligned}$$

At this time, we make the change of variables $\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The previous display now reads

$$\begin{aligned}
&e^{-2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau+d)^{-(r+k)}\theta\left(A,p,\ell,k,\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) \\
&= \frac{(-1)^r\phi_{p,0}}{d^r D}\sum_{j=0}^{[k/2]}\gamma(t,k)\left(\frac{Q(\ell)c}{\pi i(c\tau+d)}\right)^t\theta(A,bp,\ell,k-2t,\tau,z).
\end{aligned} \tag{4.3.1.28}$$

The proof of Theorem 4.3.1.2 is complete following the use of the equality

$$\frac{\phi_{p,0}}{d^r} = \exp(2\pi i Q(p)ab/N^2)\epsilon(d)$$

due to Schoeneberg ([43], page 215).

The proof of Theorem 4.3.0.5 is obtained from using the fact that if $p=0$ then

$\theta(A, p, \ell, k, \tau, z)$ is the function $\theta_h(A, \ell, k, \tau, z)$, along with the following calculation (see page 24 of [10]):

$$\Theta_h\left(Q, v, \gamma\tau, \gamma z, \frac{X}{(c\tau+d)^2}\right) = \sum_{n\geq 0} \frac{2^n \theta_h(Q, v, 2n, \gamma\tau, \gamma z)}{(2n)!}\left(\frac{2\pi i X}{(c\tau+d)^2}\right)^n$$

$$= \epsilon(d) e^{2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau+d)^r \sum_{n\geq 0}\sum_{j=0}^{n} \frac{2^n}{(2n)!}\left(\frac{2Q(v)c}{2\pi i(c\tau+d)}\right)^j$$

$$\cdot \gamma(j, 2n)(c\tau+d)^{2n}\theta(Q, v, 2n-2j, \tau, z)\left(\frac{2\pi i X}{(c\tau+d)^2}\right)^n$$

$$= \epsilon(d) e^{2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau+d)^r \sum_{n\geq 0}\sum_{j=0}^{n} \frac{2^{n-j}}{(2n-2j)!}$$

$$\cdot \theta(Q, v, 2n-2j, \tau, z)(2\pi i X)^{n-j}\left(\frac{2Q(v)c}{c\tau+d}\right)^j \frac{X^j}{j!}$$

$$= \epsilon(d) e^{2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau+d)^r \exp\left(\frac{2Q(v)cX}{c\tau+d}\right)\Theta(Q, v, \tau, z, X).$$

This last calculation establishes (4.3.0.16) for the case $d > 0$.

Let $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ be such that $d > 0$, and let $\gamma_2 = -\gamma_1$. Note that $\gamma_2 z = -\gamma_1 z$ and $\gamma_2\tau = \gamma_1\tau$. We find

$$\theta_h(Q, v, 2n, \gamma_2\tau, \gamma_2 z) = \sum_{m\in\mathbb{Z}} B(v,m)^{2n} e^{2\pi i\gamma_2\tau Q(m)} e^{2\pi i \gamma_2 z B(m,h)}$$

$$= \sum_{m\in\mathbb{Z}} B(v,m)^{2n} e^{2\pi i\gamma_1\tau Q(m)} e^{2\pi i \gamma_2 z B(-m,h)} = \sum_{m\in\mathbb{Z}} B(v,-m)^{2n} e^{2\pi i\gamma_1\tau Q(-m)} e^{2\pi i \gamma_2 z B(m,h)}$$

$$= \sum_{m\in\mathbb{Z}} B(v,m)^{2n} e^{2\pi i\gamma_1\tau Q(m)} e^{2\pi i \gamma_2 z B(m,h)} = \theta_h(Q, v, 2n, \gamma_1\tau, \gamma_1 z).$$

Replacing c and d in the right hand side of (4.3.0.20) with $-c$ and $-d$, respectively, gives

$$(-1)^r \epsilon(d) e^{2\pi i \frac{cz^2}{c\tau+d}Q(h)}(-1)^r(c\tau+d)^r \exp\left(\frac{2cQ(v)X}{c\tau+d}\right)\Theta_h(Q, v, \tau, z, X)$$

$$= \epsilon(d) e^{2\pi i \frac{cz^2}{c\tau+d}Q(h)}(c\tau+d)^r \exp\left(\frac{2cQ(v)X}{c\tau+d}\right)\Theta_h(Q, v, \tau, z, X).$$

Here we used that $\epsilon(-d) = (-1)^r \epsilon(d)$ for $d > 0$. Therefore (4.3.0.20) is unchanged due to the negative. The proof of Theorem 4.3.0.5 is now complete.

4.3.2 Transformation laws of $\mathbf{SL}_2(\mathbb{Z})$ on Ψ_h

When v is a unit vector, Theorem 4.3.0.5 tells us that $\Theta_h(Q, v, \tau, z, X)$ is a Jacobi-like form. Though this does not imply that θ_h is a Jacobi form, as alluded to above we can multiply the Jacobi-like form Θ_h with another Jacobi-like form that will provide us with Jacobi forms. This is how we establish (4.3.0.18) for the functions Ψ_h in Theorem 4.3.0.4.

Consider the holomorphic function

$$\widehat{E}_2(\tau, X) = \sum_{n \geq 0} (-1)^n \frac{E_2(\tau)^n}{n!} (2\pi i X)^n, \tag{4.3.2.1}$$

where $E_2(\tau)$ is as in (4.1.2.6). Due to the transformation of E_2 given in (4.1.2.7) we find that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$\begin{aligned} \widehat{E}_2\left(\gamma\tau, \frac{X}{(c\tau+d)^2}\right) &= \exp\left(-2\pi i \left((c\tau+d)^2 E_2(\tau) - \frac{c(c\tau+d)}{2\pi i}\right) \frac{X}{(c\tau+d)^2}\right) \\ &= \exp\left(\frac{cX}{c\tau+d}\right) \widehat{E}_2(\tau, X). \end{aligned}$$

It follows that $\widehat{E}_2(\tau, X)$ is a holomorphic Jacobi-like form of weight 0 and index 1. We find using Theorem 4.3.0.5 that when $\langle v, v\rangle = 2Q(v) = 1$,

$$\begin{aligned} &\widehat{E}_2\left(\gamma\tau, \frac{-X}{(c\tau+d)^2}\right) \Theta_h\left(Q, v, \gamma\tau, \gamma z, \frac{X}{(c\tau+d)^2}\right) \\ &= \exp\left(\frac{-cX}{c\tau+d}\right) \widehat{E}_2(\tau, -X)(c\tau+d)^r e^{2\pi i \frac{cz^2}{c\tau+d} Q(h)} \exp\left(\frac{cX}{c\tau+d}\right) \Theta_h(Q, v, \tau, z, X) \\ &= (c\tau+d)^r e^{2\pi i \frac{cz^2}{c\tau+d} Q(h)} \widehat{E}_2(\tau, -X) \Theta_h(Q, v, \tau, z, X). \end{aligned}$$

That is, $\widehat{E}_2(\tau,-X)\Theta_h(Q,v,\tau,z,X)$ is a Jacobi-like form of weight r, index $Q(h)$, and which satisfies $\ell = 0$. Following [10] we write

$$\widehat{E}_2(\tau,-X)\Theta_h(Q,v,\tau,z,X) = \sum_{k\geq 0} f_k(\tau,z)(2\pi i X)^k,$$

where

$$f_k(\tau,z) = \frac{2^k}{(2k)!}\sum_{t=0}^{k}\gamma(t,2k)E_2(\tau)\theta_h(Q,v,2k-2t,\tau,z). \tag{4.3.2.2}$$

By our discussions above, this implies (4.3.0.18).

4.3.3 Transformation laws of $\mathbb{Z}\times\mathbb{Z}$ on θ_h and Ψ_h

We now discuss transformation laws (4.3.0.17) and (4.3.0.19). Note that

$$2Q(m+\lambda h) = B(m+\lambda h, m+\lambda h) = 2Q(m) + 2\lambda B(m,h) + 2\lambda^2 Q(h).$$

In this case, we find

$$\begin{aligned}
\theta_h(Q,v,k,\tau,z+\lambda\tau+\mu) &= \sum_{m\in\mathbb{Z}^f} B(v,m)^k e^{2\pi i\tau Q(m)} e^{2\pi i(z+\lambda\tau+\mu)B(m,h)} \\
&= \sum_{m\in\mathbb{Z}^f} B(v,m)^k e^{\pi i\tau(2Q(m)+2\lambda B(m,h))} e^{2\pi i z B(m,h)} \\
&= \sum_{m\in\mathbb{Z}^f} B(v,m)^k e^{\pi i\tau(2Q(m+\lambda h)-2\lambda^2 Q(h))} e^{2\pi i z B(m,h)} \\
&= e^{-2\pi i(\lambda^2 Q(h)\tau)} \sum_{m\in\mathbb{Z}^f} B(v,m)^k e^{2\pi i\tau Q(m+\lambda h)} e^{2\pi i z B(m,h)} \\
&= e^{-2\pi i(\lambda^2 Q(h)\tau - zB(\lambda h,h))} \sum_{m+\lambda h\in\mathbb{Z}^f} B(v,m+\lambda h-\lambda h)^k e^{2\pi i\tau Q(m+\lambda h)} e^{2\pi i z B(m+\lambda h,h)} \\
&= e^{-2\pi i(\lambda^2 Q(h)\tau - zB(\lambda h,h))} \sum_{m\in\mathbb{Z}^f} B(v,m-\lambda h)^k e^{2\pi i\tau Q(m)} e^{2\pi i z B(m,h)}.
\end{aligned} \tag{4.3.3.1}$$

In the case $B(v,h)=0$ (which is our assumption), (4.3.3.1) becomes

$$\theta_h(Q,v,k,\tau,z+\lambda\tau+\mu) = e^{-2\pi i Q(h)(\lambda^2\tau-2z\lambda)}\theta_h(Q,c,k,\tau,z). \tag{4.3.3.2}$$

This establishes (4.3.0.17). Equation (4.3.0.19) follows immediately from this result. We have now completed the proof of Theorem 4.3.0.4.

4.3.4 Transformation laws for Jacobi theta functions

For the remainder of this chapter we change notation and take $2Q(x) = \langle x, x\rangle$. This will be the notation used later. Let G be the gram matrix associated with the bilinear form $\langle\cdot,\cdot\rangle$. We note that

$$\begin{aligned}\theta_{\underline{h}}(Q, v, k, \tau, \underline{z}) &= \sum_{m\in\mathbb{Z}^f} \langle v, m\rangle^k \, q^{\langle m,m\rangle/2}\zeta_1^{\langle m,h_1\rangle}\cdots\zeta_n^{\langle m,h_n\rangle} \\ &= \sum_{m\in\mathbb{Z}^f} \langle v, m\rangle^k \, q^{\langle m,m\rangle/2}e^{2\pi i\langle m,h_1z_1+\cdots h_nz_n\rangle}.\end{aligned}$$

We want to extend Theorem 4.3.0.4 to the matrix Jacobi theta series case.

Theorem 4.3.4.1 *Let the notation be as above, and recall that $f = 2r$ is the rank of the matrix representing the quadratic form Q. Suppose $\langle v, h_i\rangle = 0$ for $1 \le i \le n$ and let* $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(N)$ *and $(\underline{\lambda}, \underline{\mu}) \in \mathbb{Z}^n \times \mathbb{Z}^n$.*

1. *If $\langle v, v\rangle = 0$, then*

$$\theta_{\underline{h}}(Q, v, 2n, \gamma\tau, \gamma\underline{z}) = \epsilon(d)e^{-\pi i\frac{c}{c\tau+d}G[\underline{z}]}(c\tau + d)^{2n+r}\theta_{\underline{h}}(Q, v, 2n, \tau, \underline{z}), \tag{4.3.4.1}$$

and

$$\theta_{\underline{h}}(Q, v, 2n, \tau, \underline{z} + \underline{\lambda}\tau + \underline{\mu}) = e^{\pi i(G[\underline{\lambda}]\tau+2\underline{z}^tG\underline{\lambda})}\theta_{\underline{h}}(Q, v, 2n, \tau, \underline{z}). \tag{4.3.4.2}$$

2. *If $\langle v, v\rangle = 1$, then*

$$\Psi_{\underline{h}}(Q, v, \gamma\tau, \gamma\underline{z}) = \epsilon(d)e^{-\pi i\frac{c}{c\tau+d}G[\underline{z}]}(c\tau + d)^{2n+r}\Psi_{\underline{h}}(Q, v, \tau, \underline{z}), \tag{4.3.4.3}$$

and

$$\Psi_{\underline{h}}(Q,v,\tau,\underline{z}+\underline{\lambda}\tau+\underline{\mu}) = e^{\pi i(G[\underline{\lambda}]\tau+2\underline{z}^tG\underline{\lambda})}\Psi_{\underline{h}}(Q,v,\tau,\underline{z}). \tag{4.3.4.4}$$

Proof The most difficult part of this proof is again establishing (4.3.4.1). However, this follows just as in the proof of Theorem (4.3.0.16) except we take $2Q(x) = xAx = \langle x,x\rangle$ immediately. The proof is identical until the step between (4.3.1.11) and (4.3.1.12) where we had replaced h with zh. We now replace h with $z_1h_1 + \cdots z_nh_n$ so that (4.3.1.12) reads

$$\begin{aligned}
&e^{\frac{\pi i}{\tau}G[\underline{z}]}\frac{(2\pi i)^k\tau^rN^{-k}}{i^r\sqrt{D}}\sum_{\substack{m_1\in\mathbb{Z}^f\\ Am_1\equiv 0(N)}}(\ell^tAm_1)^ke^{2\pi i\tau m_1^tAm_1/N^2+2\pi i\langle m_1,z_1h_1+\cdots+z_nh_n\rangle/N+2\pi i m_1^tAp/N^2}\\
&=\sum_{j=0}^{[k/2]}\sum_{\substack{m\in\mathbb{Z}^f\\ n\equiv p(N)}}N^{2j-k}\left(-\frac{2\pi i}{\tau}\right)^{k-j}\gamma(j,k)(2Q(\ell))^j(\ell^tAm+z_1\ell^tAh_1+\cdots z_n\ell^tAh_n)^{k-2j}\\
&\qquad\cdot e^{-\frac{2\pi i}{\tau}(m^tAm/N^2+z_1\langle m,h_1\rangle/N+\cdots+z_n\langle m,h_n\rangle/N)}.
\end{aligned} \tag{4.3.4.5}$$

Here we used that

$$e^{\frac{\pi i}{\tau}\langle z_1h_1+\cdots+z_nh_n,z_1h_1+\cdots+z_nh_n\rangle} = e^{\frac{\pi i}{\tau}G[\underline{z}]},$$

and G is the Gram matrix associated with $h_1,\ldots,h_n$. The rest of the proof remains the same making the appropriate (and clear) changes. It should be noted that when we replace τ with $-1/\tau$ and z_j with z_j/τ for each $1\le j\le n$ after (4.3.1.26), we have

$$\frac{c}{\tau}G[\underline{z}]\mapsto\frac{c}{\tau(d\tau-c)}G[\underline{z}].$$

We now develop the analogous result of Theorem 4.3.0.5. We find

$$\Theta_{\underline{h}}\left(Q, v, \gamma\tau, \gamma\underline{z}, \frac{X}{(c\tau+d)^2}\right) = \sum_{n\geq 0} \frac{2^n \theta_{\underline{h}}(Q, v, 2n, \gamma\tau, \gamma\underline{z})}{(2n)!} \left(\frac{2\pi i X}{(c\tau+d)^2}\right)^n$$

$$= \epsilon(d) e^{\pi i \frac{c}{c\tau+d} G[\underline{z}]} (c\tau+d)^r \sum_{n\geq 0} \sum_{j=0}^{n} \frac{2^n}{(2n)!} \left(\frac{\langle v, v\rangle\, c}{2\pi i (c\tau+d)}\right)^j$$

$$\cdot\, \gamma(j, 2n)(c\tau+d)^{2n} \theta_{\underline{h}}(Q, v, 2n-2j, \tau, \underline{z}) \left(\frac{2\pi i X}{(c\tau+d)^2}\right)^n$$

$$= \epsilon(d) e^{\pi i \frac{c}{c\tau+d} G[\underline{z}]} (c\tau+d)^r$$

$$\cdot \sum_{n\geq 0} \sum_{j=0}^{n} \frac{2^{n-j}}{(2n-2j)!} \theta_{\underline{h}}(Q, v, 2n-2j, \tau, \underline{z}) (2\pi i X)^{n-j} \left(\frac{\langle v, v\rangle\, c}{c\tau+d}\right)^j \frac{X^j}{j!}$$

$$= \epsilon(d) e^{\pi i \frac{c}{c\tau+d} G[\underline{z}]} (c\tau+d)^r \exp\left(\frac{\langle v, v\rangle\, cX}{c\tau+d}\right) \Theta_{\underline{h}}(Q, v, \tau, \underline{z}, X),$$

as before. When $\langle v, v\rangle = 0$, we have (4.3.4.1).

As in Theorem 4.3.0.4, when $\langle v, v\rangle = 1$ we find

$$\widehat{E}_2\left(\gamma\tau, \frac{-X}{(c\tau+d)^2}\right) \Theta_{\underline{h}}\left(Q, v, \gamma\tau, \frac{\underline{z}}{c\tau+d}, \frac{X}{(c\tau+d)^2}\right)$$

$$= \exp\left(\frac{-cX}{c\tau+d}\right) \widehat{E}_2(\tau, -X)(c\tau+d)^r e^{\pi i \frac{c}{c\tau+d} G[\underline{z}]} \exp\left(\frac{cX}{c\tau+d}\right) \Theta_{\underline{h}}(Q, v, \tau, \underline{z}, X)$$

$$= (c\tau+d)^r e^{\pi i \frac{c}{c\tau+d} G[\underline{z}]} \widehat{E}_2(\tau, -X) \Theta_{\underline{h}}(Q, v, \tau, \underline{z}, X).$$

That is, $\widehat{E}_2(\tau, -X)\Theta_{\underline{h}}(Q, v, \tau, \underline{z}, X)$ is a Jacobi-like form of weight r, index $G/2$, and which satisfies $\ell = 0$. Just as before, we write

$$\widehat{E}_2(\tau, -X)\Theta_{\underline{h}}(Q, v, \tau, \underline{z}, X) = \sum_{k\geq 0} f_k(\tau, \underline{z})(2\pi i X)^k,$$

where

$$f_k(\tau, \underline{z}) = \frac{2^k}{(2k)!} \sum_{t=0}^{k} \gamma(t, 2k) E_2(\tau) \theta_{\underline{h}}(Q, v, 2k-2t, \tau, \underline{z}). \tag{4.3.4.6}$$

Again, this implies (4.3.4.3).

We now consider the $\underline{z}$-variable transformations. We find

$$\theta_{\underline{h}}(Q,v,k,\tau,\underline{z}+\underline{\lambda}\tau+\underline{\mu}) = \sum_{m\in\mathbb{Z}^f} \langle v,m\rangle^k\, e^{\pi i\tau\langle m,m\rangle} e^{2\pi i\langle m, h_1(z_1+\lambda_1\tau+\mu_1)+\cdots h_n(z_n+\lambda_n\tau+\mu_n)\rangle}$$

$$= \sum_{m\in\mathbb{Z}^f} \langle v,m\rangle^k\, e^{\pi i\tau(\langle m,m\rangle+2\langle m,\lambda_1h_1+\cdots+\lambda_nh_n\rangle)} e^{2\pi i(z_1\langle m,h_1\rangle+\cdots+z_n\langle m,h_n\rangle)}$$

$$= \sum_{m\in\mathbb{Z}^f} \langle v,m\rangle^k\, e^{\pi i\tau(\langle m+\lambda_1h_1+\cdots+\lambda_nh_n, m+\lambda_1h_1+\cdots+\lambda_nh_n\rangle h)-G[\underline{\lambda}])} e^{2\pi i(z_1\langle m,h_1\rangle+\cdots+z_n\langle m,h_n\rangle)}$$

$$= e^{-\pi i(G[\underline{\lambda}]\tau)} \sum_{m\in\mathbb{Z}^f} \langle v,m\rangle^k\, e^{\pi i\tau(\langle m+\lambda_1h_1+\cdots+\lambda_nh_n, m+\lambda_1h_1+\cdots+\lambda_nh_n\rangle)} e^{2\pi i\langle m, z_1h_1+\cdots z_nh_n\rangle}$$

$$= e^{-\pi i(G[\underline{\lambda}]\tau-2\underline{z}^tG\underline{\lambda})} \sum_{m+\lambda_1h_1+\cdots+\lambda_nh_n\in\mathbb{Z}^f} \langle v, m+\lambda_1h_1+\cdots+\lambda_nh_n-(\lambda_1h_1+\cdots+\lambda_nh_n)\rangle^k$$

$$\cdot\, e^{\pi i\tau\langle m+\lambda_1h_1+\cdots+\lambda_nh_n, m+\lambda_1h_1+\cdots+\lambda_nh_n\rangle} e^{2\pi iB(m+\lambda_1h_1+\cdots+\lambda_nh_n, z_1h_1+\cdots z_nh_n)}$$

$$= e^{-\pi i(G[\underline{\lambda}]\tau-2\underline{z}^tG\underline{\lambda})} \sum_{m\in\mathbb{Z}^f} \langle v, m-(\lambda_1h_1+\cdots+\lambda_nh_n)\rangle^k\, q^{\langle m,m\rangle/2}\zeta_1^{\langle m,h_1\rangle}\cdots\zeta_n^{\langle m,h_n\rangle}. \tag{4.3.4.7}$$

In the case $\langle v,h_j\rangle = 0$ for $1\le j\le n$, (4.3.4.7) becomes

$$\theta_{\underline{h}}(Q,v,k,\tau,\underline{z}+\underline{\lambda}\tau+\underline{\mu}) = e^{-\pi i(G[\underline{\lambda}]\tau-2\underline{z}^tG\underline{\lambda})}\theta_{\underline{h}}(Q,v,k,\tau,\underline{z}). \tag{4.3.4.8}$$

This establishes (4.3.4.2). Equation (4.3.4.4) follows immediately from this result. We have now completed the proof of Theorem 4.3.4.1. □

Note that in the above proof we also proved the following analogue of Theorem 4.3.0.5 for $2Q(x)=\langle x,x\rangle$ and complex variables $z_1,\ldots,z_n$.

Theorem 4.3.4.2 *Suppose $B(v,h)=0$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have*

$$\Theta_{\underline{h}}\left(Q, v, \gamma\tau, \gamma\underline{z}, \frac{X}{(c\tau+d)^2}\right) = \epsilon(d) e^{\pi i \frac{c}{c\tau+d} G[\underline{z}]} (c\tau+d)^r \exp\left(\frac{c\langle v,v\rangle X}{c\tau+d}\right) \Theta_{\underline{h}}(Q, v, \tau, \underline{z}, X). \tag{4.3.4.9}$$

Part II

Trace Functions and Recursion Formulas

Chapter 5

Correlation Functions

A number of "trace" functions are investigated in the literature and discussed in this thesis. This section provides an overview of such functions and provides recursion formulas that are (a) used to reduce the study of n-point functions to the study of 1-point functions and, (b) used in computations later in this thesis.

5.1 n-point functions

Let V be a vertex operator algebra of central charge c and M a V-module. For $n \geq 0$, the *n-point correlation function*, or simply *n-point function*, for states $v_1, \ldots, v_n \in V$ is defined as the formal expression

$$\begin{aligned} F_M((v_1, w_1), \ldots, (v_n, w_n), \tau) :&= \mathrm{Tr}_M\, Y(q_1^{L(0)} v_1, q_1) \cdots Y(q_n^{L(0)} v_n, q_n) q^{L(0)-c/24} \\ &= \mathrm{Tr}_M\, Y(v_1, q_1) \cdots Y(v_n, q_n) q_1^{\mathrm{wt}\, v_1} \cdots q_n^{\mathrm{wt}\, v_n} q^{L(0)-c/24}, \end{aligned} \tag{5.1.0.1}$$

where $q_i = e^{2\pi i w_i}$ for variables $w_1, \ldots, w_n \in \mathbb{C}$. In the case $n = 1$ and for homogeneous $v \in V$, (5.1.0.1) becomes

$$\begin{aligned}
F_M((v,w),\tau) &= \mathrm{Tr}_M\, Y(e^{2\pi i w \,\mathrm{wt}\, v} v, e^{2\pi i w}) q^{L(0)-c/24} \\
&= \mathrm{Tr}_M \sum_{n\in\mathbb{Z}} (q_w^{\mathrm{wt}\, v} v)(n) q_w^{-n-1} q^{L(0)-c/2} \\
&= q_w^{\mathrm{wt}\, v} \sum_{n\in\mathbb{Z}} \mathrm{Tr}_M\, v(n) q_w^{-n-1} q^{L(0)-c/24} \\
&= q_w^{\mathrm{wt}\, v}\, \mathrm{Tr}_M\, v(\mathrm{wt}\, v - 1) q_w^{-\,\mathrm{wt}\, v} q^{L(0)-c/24} \\
&= \mathrm{Tr}_M\, o(v) q^{L(0)-c/24}.
\end{aligned}$$

That is, the 1-point function for an element $v \in V$ is independent of w. 1-point functions will be treated often as a special case and will be denoted

$$Z_M(v;\tau) = \mathrm{Tr}_M\, o(v) q^{L(0)-c/24}.$$

When $v = \mathbf{1}$, $o(\mathbf{1})$ is the identity operator on each graded space $M_{\lambda+n}$, where λ is the conformal weight of M. It follows that

$$\begin{aligned}
Z_M(\mathbf{1};\tau) = \mathrm{Tr}_M\, o(\mathbf{1}) q^{L(0)-c/24} &= \sum_{n\geq 0} \mathrm{Tr}_{M_{\lambda+n}}\, q^{L(0)-c/24} \\
&= q^{\lambda-c/24} \sum_{n\geq 0} (\dim M_{\lambda+n}) q^n.
\end{aligned}$$

This is the 0-point function which is typically called the *graded trace* or *partition function* of M. The notations F_j and Z_j will be used to denote the respective trace functions for a module M^j.

5.1.1 Zhu's Theorem

As mentioned in the introduction, a number of results detailing the modular-invariance of n-point functions have been developed. The following theorem due to Zhu [46] establishes transformation laws for functions of the form (5.1.0.1) under the action of the modular group.

Theorem 5.1.1.1 (Zhu) *Let V be a rational, C_2-cofinite vertex operator algebra, and $M^1, \ldots, M^r$ be its finitely many inequivalent irreducible admissible modules. Let $v_1, \ldots, v_n$ be homogeneous elements in V. Then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, there exists scalars $A^j_{k,\gamma}$ such that*

$$F_j\left(\left(v_1, \frac{z_1}{c\tau+d}\right), \ldots, \left(v_n, \frac{z_n}{c\tau+d}\right), \frac{a\tau+b}{c\tau+d}\right)$$
$$= (c\tau+d)^{\sum_{i=1}^{n}[\mathrm{wt}]v_i} \sum_{k=1}^{r} A^k_{j,\gamma} F_k((v_1, z_1), \ldots, (v_n, z_n), \tau).$$

Moreover, $F_j((v_1, z_1), \ldots, (v_n, z_n), \tau)$ is a meromorphic function on the domain

$$\left\{(z_1, \ldots, z_n, q) \mid z_i \neq 0, z_i \neq z_j q^k (k \in \mathbb{Z}), |q| < 1\right\}.$$

Generalizations of this result include the modularity of trace functions associated to orbifold models [7], n-point functions associated to $\mathbb{R}$-graded super VOAs [37], and intertwining operators [39].

5.1.2 The J-trace functions

This subsection introduces the functions which are the focus of this thesis. For $v_1, \ldots, v_d \in V$, a V-module M, and $h_1, \ldots, h_n \in V_1$ that satisfy Condition **H**, define

$$J_{M,\underline{h}}(v_1, \ldots, v_d; \tau, \underline{z}) := \mathrm{Tr}_M\, Y(q_1^{L(0)}, q_1) \cdots Y(q_d^{L(0)}, q_d) \zeta_1^{h_1(0)} \cdots \zeta_n^{h_n(0)} q^{L(0)-c/24}. \quad (5.1.2.1)$$

Here, $q_j = e^{2\pi i w_j}, w_j \in \mathbb{C}$ and $\zeta_j = e^{2\pi i z_j}, z_j \in \mathbb{C}$. Again, the 1-point function reduces so that

$$J_{M,\underline{h}}(v;\tau,\underline{z}) := \mathrm{Tr}_M\, o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}. \tag{5.1.2.2}$$

5.2 Recursion formulas

We first prove the following lemma.

Lemma 5.2.0.1 *Let $v \in V_k$ and $v_1,\ldots,v_m \in V$. When $\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}v = v$, we have*

$$\sum_{r=1}^{m} J_{M,\underline{h}}(v_1,\ldots,v[0]v_r,\ldots,v_m;\tau,\underline{z}) = 0. \tag{5.2.0.3}$$

Proof It is known (for example [37]) that

$$[o(v), Y(q_{w_r}^{L(0)}v_r, q_{w_r})] = Y(q_{w_r}^{L(0)}v[0]v_r, q_{w_r}).$$

It follows that

$$\begin{aligned}\mathrm{Tr}_M[o(v),&Y(q_{w_1}^{L(0)}v_1,q_{w_1})\cdots Y(q_{w_m}^{L(0)}v_m,q_{w_m})]\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\ &=\sum_{r=1}^{m} J_{M,\underline{h}}(v_1,\ldots,v[0]v_r,\ldots,v_m;\tau,\underline{z}).\end{aligned}$$

Let $h_j(0)v = \mu_j v$, $1 \le j \le n$, where $\mu_j \in \mathbb{C}$. Since $\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}$ is an automorphism, we have

$$\begin{aligned}v(k-1)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)} &= \zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}\left(\zeta_1^{-h_1(0)}\cdots\zeta_n^{-h_n(0)}v\right)(k-1)\\ &= \zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}o(v)\zeta_1^{-\mu_1}\cdots\zeta_n^{-\mu_n}.\end{aligned} \tag{5.2.0.4}$$

Therefore,

$$\mathrm{Tr}_M[o(v), Y(q_{w_1}^{L(0)}v_1, q_{w_1})\cdots Y(q_{w_m}^{L(0)}v_m, q_{w_m})]\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$$

$$= \mathrm{Tr}_M\, o(v)Y(q_{w_1}^{L(0)}v_1, q_{w_1})\cdots Y(q_{w_m}^{L(0)}v_m, q_{w_m})\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$$

$$- \mathrm{Tr}_M\, Y(q_{w_1}^{L(0)}v_1, q_{w_1})\cdots Y(q_{w_m}^{L(0)}v_m, q_{w_m})o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$$

$$= \mathrm{Tr}_M\, o(v)Y(q_{w_1}^{L(0)}v_1, q_{w_1})\cdots Y(q_{w_m}^{L(0)}v_m, q_{w_m})\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$$

$$- \mathrm{Tr}_M\, Y(q_{w_1}^{L(0)}v_1, q_{w_1})\cdots Y(q_{w_m}^{L(0)}v_m, q_{w_m}\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}o(v)e^{-2\pi i(\mu_1 z_1+\cdots\mu_n z_n)}$$

$$= \mathrm{Tr}_M\, o(v)Y(q_{w_1}^{L(0)}v_1, q_{w_1})\cdots Y(q_{w_m}^{L(0)}v_m, q_{w_m})\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$$

$$- e^{-2\pi i(\mu_1 z_1+\cdots\mu_n z_n)}\,\mathrm{Tr}_M\, o(v)Y(q_{w_1}^{L(0)}v_1, q_{w_1})\cdots Y(q_{w_m}^{L(0)}v_m, q_{w_m}\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$$

$$= (1 - e^{-2\pi i(\mu_1 z_1+\cdots\mu_n z_n)})J_{M,\underline{h}}(v, v_1, \ldots, v_n; \tau, \underline{z}), \tag{5.2.0.5}$$

where we've used the cyclic property of the trace function.

Combining (5.2.0.4) and (5.2.0.5), it follows that

$$\sum_{r=1}^{m} J_{M,\underline{h}}(v_1, \ldots, v[0]v_r, \ldots, v_m; \tau, \underline{z}) = 0,$$

so long as $e^{-2\pi i(\mu_1 z_1+\cdots\mu_n z_n)} = 1$. That is, so long as $\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}v = v$. This proves the lemma. □

5.2.1 2-point recursion formulas

The analysis necessary to develop a recursion formula expressing n-point functions of the form (5.1.2.1) as a sum of $(n-1)$-point functions with coefficients that are quasi-modular forms is already in the literature [46]. Few additional concerns must be addressed. The proofs of two lemmas which allow for 2-point functions to be written as 1-point functions will be presented first. These lemmas will be of use in later sections. The proofs of these lemmas follow along the lines of those in [35, 37]. Generalizing these constructions provides the proofs of a more general n-point recursion formula which is essentially found in [37]. Let $\underline{z} \cdot \underline{\mu}$ denote the dot product of $\underline{z}$ and $\underline{\mu}$.

Lemma 5.2.1.1 *Let $u, v \in V$ and M be a V-module. Suppose $h_i(0)u = \mu_i u$, $\mu_i \in \mathbb{C}$, for each $1 \leq i \leq n$. Then*

$$J_{M,\underline{h}}(u, v; \tau, \underline{z}) = \delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}} \operatorname{Tr}_M o(u)o(v)\zeta_1^{h_1(0)} \cdots \zeta_n^{h_n(0)} q^{L(0)-c/24} + \sum_{t\geq 0}(-1)^{t+1}\tilde{P}_{t+1}((z_2 - z_1), \tau, \underline{z} \cdot \underline{\mu}) J_{M,\underline{h}}(u[t]v; \tau, \underline{z}), \qquad (5.2.1.1)$$

where $\delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}}$ is 1 when $\underline{z} \cdot \underline{\mu} \in \mathbb{Z}$, and 0 otherwise.

Proof It follows using (3.1.0.5) that

$$[u(s), Y(q_2^{L(0)}v, q_2)] = \sum_{r\geq 0} \binom{s}{r} Y(u(r)q_2^{L(0)}v, q_2)q_2^{s-r} = q_2^{s+1-\operatorname{wt} u} Y\left(q_2^{L(0)} \sum_{r\geq 0} \binom{s}{r} u(r)v, q_2\right).$$

Applying (3.1.1.2) gives

$$[u(s), Y(q_2^{L(0)}v, q_2)] = q_2^{s+1-\operatorname{wt} u} \sum_{t\geq 0} \frac{(s + 1 - \operatorname{wt} u)^t}{t!} Y(q_2^{L(0)}u[t]v, q_2).$$

Therefore,

$$\begin{aligned}
&\mathrm{Tr}_M u(s)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad= \mathrm{Tr}_M\left(Y(q_2^{L(0)}v,q_2)u(s)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\right)\\
&\qquad\quad+\mathrm{Tr}_M[u(s),Y(q_2^{\mathrm{wt}\,v}v,q_2)]\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad= q^{s+1-\mathrm{wt}\,u}\zeta_1^{-\mu_1}\cdots\zeta_n^{-\mu_n}\,\mathrm{Tr}_M\left(Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}u(s)\right)\\
&\qquad\quad+q_2^{s+1-\mathrm{wt}\,u}\sum_{t\geq 0}\frac{(s+1-\mathrm{wt}\,u)^t}{t!}\,\mathrm{Tr}_M\,Y(q_2^{L(0)}u[t]v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad= q^{s+1-\mathrm{wt}\,u}e^{-2\pi i(\underline{z}\cdot\underline{\mu})}\,\mathrm{Tr}_M\left(u(s)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\right)\\
&\qquad\quad+q_2^{s+1-\mathrm{wt}\,u}\sum_{t\geq 0}\frac{(s+1-\mathrm{wt}\,u)^t}{t!}\,\mathrm{Tr}_M\,Y(q_2^{L(0)}u[t]v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}.
\end{aligned}$$

Solving for $\mathrm{Tr}_M\,u(s)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$ establishes

$$\begin{aligned}
&\mathrm{Tr}_M\,u(s)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad=\frac{q_2^{s+1-\mathrm{wt}\,u}}{1-q^{s+1-\mathrm{wt}\,u}q_{\underline{z}\cdot\underline{\mu}}^{-1}}\sum_{t\geq 0}\frac{(s+1-\mathrm{wt}\,u)^t}{t!}\,\mathrm{Tr}_M\,Y(q_2^{L(0)}u[t]v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24},
\end{aligned} \tag{5.2.1.2}$$

so long as $(\underline{z}\cdot\underline{\mu},s)\neq(\lambda,\mathrm{wt}\,u-1)$, $\lambda\in\mathbb{Z}$. Setting $m=s+1-\mathrm{wt}\,u$, (5.2.1.2) reads (for $(\underline{z}\cdot\underline{\mu},m)\neq(\lambda,0),\lambda\in\mathbb{Z}$)

$$\begin{aligned}
&\mathrm{Tr}_M u(m-1+\mathrm{wt}\,u)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad=\frac{q_2^{m}}{1-q^{m}q_{\underline{z}\cdot\underline{\mu}}^{-1}}\sum_{t\geq 0}\frac{m^t}{t!}\,\mathrm{Tr}_M\,Y(q_2^{L(0)}u[t]v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}.
\end{aligned} \tag{5.2.1.3}$$

Finally, (5.2.1.3) gives

$$
\begin{aligned}
&J_{M,\underline{h}}(u,v;\tau,\underline{z})\\
&\quad=\sum_{s\in\mathbb{Z}} q_1^{\mathrm{wt}\,u-s-1}\operatorname{Tr}_M u(s)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad=\sum_{m\in\mathbb{Z}} q_1^{-m}\operatorname{Tr}_M u(m+\mathrm{wt}\,u-1)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad=\delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}}\operatorname{Tr}_M u(\mathrm{wt}\,u-1)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\qquad+\sum_{m\in\mathbb{Z}}{}' q_1^{-m}\operatorname{Tr}_M u(m+\mathrm{wt}\,u-1)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad=\delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}}\operatorname{Tr}_M o(u)Y(q_2^{L(0)}v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\qquad+\sum_{m\in\mathbb{Z}}{}' q_1^{-m}\frac{q_2^m}{1-q^m q_{\underline{z}\cdot\underline{\mu}}^{-1}}\sum_{t\geq 0}\frac{m^t}{t!}\operatorname{Tr}_M Y(q_2^{L(0)}u[t]v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad=\delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}}\operatorname{Tr}_M o(u)o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\qquad+\sum_{t\geq 0}\operatorname{Tr}_M Y(q_2^{L(0)}u[t]v,q_2)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\frac{1}{t!}\sum_{m\in\mathbb{Z}}{}'\frac{m^t}{1-q^m q_{\underline{z}\cdot\underline{\mu}}^{-1}}\left(\frac{q_2}{q_1}\right)^m\\
&\quad=\delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}}\operatorname{Tr}_M o(u)o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\qquad+\sum_{t\geq 0}J_M(u[t]v;\tau,z)(-1)^{t+1}\tilde{P}_{t+1}((z_2-z_1),\tau,\underline{z}\cdot\underline{\mu}).
\end{aligned}
$$

This completes the proof of the Lemma. □

Using (5.2.1.1) the following Lemma can be obtained.

Lemma 5.2.1.2 *Let $u,v\in V$ and M be a V-module. Suppose $h_i(0)u=\mu_i u$, $\mu_i\in\mathbb{C}$, for each $1\leq i\leq n$. Then for $m\geq 1$,*

$$
\begin{aligned}
J_{M,\underline{h}}(u[-m]v;\tau,\underline{z}) &= \delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}}\delta_{m,1}\operatorname{Tr}_M o(u)o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}\\
&\quad+\sum_{k\geq 1}(-1)^{m+1}\binom{k+m-1}{k}\tilde{E}_{k+m}(\tau,\underline{z}\cdot\underline{\mu})J_{M,\underline{h}}(u[k]v;\tau,\underline{z}). \quad (5.2.1.4)
\end{aligned}
$$

Proof Although $Y(u, z_1)Y(v, z_2)$ does not necessarily equal $Y(Y(u, z_1)v, z_2)$, it is true that (cf. (3.1.0.9))

$$\mathrm{Tr}_M Y(u, z_1 + z_2)Y(v, z_2)\zeta_1^{h_1(0)} \cdots \zeta_n^{h_n(0)} q^{L(0)-c/24}$$
$$= \mathrm{Tr}_M\, Y(Yu, z_1)v, z_2)\zeta_1^{h_1(0)} \cdots \zeta_n^{h_n(0)} q^{L(0)-c/24}.$$

Along with (3.1.0.14), this gives (see Lemma 1 in [36] for a complete proof)

$$J_{M,\underline{h}}(u, v; \tau, \underline{z}) = \mathrm{Tr}_M\, Y^M Y(q_1^{L(0)}u, q_1 - q_2)q_2^{L(0)}v, q_2)\zeta_1^{h_1(0)} \cdots \zeta_n^{h_n(0)} q^{L(0)-c/24}$$
$$= \mathrm{Tr}_M\, Y_M(q_2^{L(0)}Y(q_{z_1-z_2}^{L(0)}u, q_{z_1-z_2} - 1)v, q_2)\zeta_1^{h_1(0)} \cdots \zeta_n^{h_n(0)} q^{L(0)-c/24}$$
$$= J_{M,\underline{h}}(Y[u, z_1 - z_2]v, \tau, \underline{z}).$$

Therefore, since $(z_1 - z_2)^m = (-1)^m(z_2 - z_1)^m$ for $m \geq 0$ it follows

$$J_{M,\underline{h}}(u, v; \tau, \underline{z}) = J_{M,\underline{h}}(Y[u, z_1 - z_2]v, \tau, \underline{z})$$
$$= \sum_{m\in\mathbb{Z}} J_{M,\underline{h}}(u[m]v, \tau, \underline{z})(z_1 - z_2)^{-m-1}$$
$$= \sum_{m\in\mathbb{Z}} J_{M,\underline{h}}(u[-m]v, \tau, \underline{z})(z_1 - z_2)^{m-1}$$
$$= \sum_{m\geq 1}(-1)^{m-1} J_{M,\underline{h}}(u[-m]v, \tau, \underline{z})(z_2 - z_1)^{m-1} + \sum_{m\leq 0} J_{M,\underline{h}}(u[-m]v, \tau, \underline{z})(z_1 - z_2)^{m-1}.$$

Combining this with Lemma 5.2.1.1 and using Lemma 4.2.4.2, we find

$$\sum_{m\geq 1}(-1)^{m-1}J_{M,\underline{h}}(u[-m]v,\tau,\underline{z})(z_1-z_2)^{m-1}+\sum_{m\leq 0}J_{M,\underline{h}}(u[-m]v,\tau,\underline{z})(z_1-z_2)^{m-1}$$

$$= J_{M,\underline{h}}(u,v;\tau,\underline{z})$$

$$= \delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}}\operatorname{Tr}_M o(u)o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$$

$$+\sum_{t\geq 0}J_M(u[t]v;\tau,\underline{z})(-1)^{t+1}\tilde{P}_{t+1}((z_2-z_1),\tau,\underline{z}\cdot\underline{\mu})$$

$$= \delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}}\operatorname{Tr}_M o(u)o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$$

$$+\sum_{t\geq 0}J_M(u[t]v;\tau,\underline{z})\left(\frac{(-1)^{t+1}}{(z_2-z_1)^{t+1}}+\sum_{k\geq t+1}\binom{k-1}{t}\tilde{E}_k(\tau,\underline{z}\cdot\underline{\mu})(z_2-z_1)^{k-t-1}\right)$$

$$= \delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}}\operatorname{Tr}_M o(u)o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24}$$

$$+\sum_{t\geq 0}J_M(u[t]v;\tau,\underline{z})\left(\frac{(-1)^{t+1}}{(z_2-z_1)^{t+1}}+\sum_{k\geq 1}\binom{k+t-1}{t}\tilde{E}_{k+t}(\tau,\underline{z}\cdot\underline{\mu})(z_2-z_1)^{k-1}\right).$$

Equating coefficients of the (z_2-z_1) terms establishes (5.2.1.2) for $m\geq 1$. No information is obtained for $m\leq 0$. □

5.2.2 n-point recursion formulas

In this subsection, two lemmas are quoted which together provide a general recursion formula for n-point functions. The proofs follow similarly to the lemmas above, and a detailed proof is nearly identical to that found in Section 3.3 of [37].

Lemma 5.2.2.1 *Let $v\in V$ and suppose $h_i(0)v=\mu_i v$, $\mu_i\in\mathbb{C}$, for each $1\leq i\leq n$. Then* *for any* $v_1, \dots, v_m \in V$*,*

$$\begin{aligned} & J_{M,\underline{h}}(v, v_1, \dots, v_n; \tau, \underline{z}) \\ & = \delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}} \operatorname{Tr}_M o(v) Y^M(q_1^{L(0)} v_1, q_1) \cdots Y(q_m^{L(0)} v_m, q_m) \zeta_1^{h_1(0)} \cdots \zeta^{h_n(0)} q^{L(0)-c/24} \\ & \quad + \sum_{s=1}^{m} \sum_{k\geq 0} \tilde{P}_{k+1}(z - z_s, \tau, \underline{z}\cdot\underline{\mu}) J_{M,\underline{h}}(v_1, \dots, v[k]v_r, \dots, v_m; \tau, \underline{z}). \end{aligned} \tag{5.2.2.1}$$

□

Due to the first term on the right hand side of (5.2.2.1), however, the left hand side is not written in terms of $(n-1)$-point functions. The following lemma remedies this problem.

Lemma 5.2.2.2 *Let the conditions be the same as in the previous lemma. Then for* $p \geq 1$*,*

$$\begin{aligned} & J_{M,\underline{h}}(v[-p]v_1, \dots, v_m; \tau, \underline{z}) \\ & = \delta_{\underline{z}\cdot\underline{\mu},\mathbb{Z}} \delta_{p,1} \operatorname{Tr}_M o(v) Y^M(q_1^{L(0)} v_1, q_1) \cdots Y(q_m^{L(0)} v_m, q_m) \zeta_1^{h_1(0)} \cdots \zeta^{h_n(0)} q^{L(0)-c/24} \\ & \quad + \sum_{k\geq 0} (-1)^{k+1} \binom{k+p-1}{k} \tilde{E}_{k+p}(\tau, \underline{z}\cdot\underline{\mu}) J_{M,\underline{h}}(v[k]v_1, \dots, v_m; \tau, \underline{z}) \\ & \quad + \sum_{s=2}^{m} \sum_{k\geq 0} (-1)^{p+1} \binom{k+p-1}{k} \tilde{P}_{k+p}(z_1 - z_s, \tau, \underline{z}\cdot\underline{\mu}) J_{M,\underline{h}}(v_1, \dots, v[k]v_s, \dots, v_m; \tau, \underline{z}). \end{aligned} \tag{5.2.2.2}$$

□

A number of remarks should be made regarding these lemmas. To begin, note that Lemma 5.2.1.1 is a special case of Lemma 5.2.2.1, while Lemma 5.2.1.2 follows from Lemma 5.2.2.2. The proofs of Lemmas 5.2.1.1 and 5.2.1.2 are included here to indicate how the presence of the $\zeta_1^{h_1(0)} \cdots \zeta_n^{h_n(0)}$ affects the outcome.

From discussion in Subsection 4.2.4 we have that convergence of the n-point functions exist on the domain $|q| < |q_{z_1-z_s}| < 1$ when $\zeta_{\underline{z}\cdot\underline{\mu}} \neq 1$. In the case $\zeta_{\underline{z}\cdot\underline{\mu}} = 1$, the $\tilde{P}_{m+1}(z_1 - z_r, \tau, \underline{z} \cdot \underline{\mu})$ are simply the elliptic functions $P_{m+1}(z_1 - z_r, \tau)$.

5.3 Heisenberg partition function

Recall the notation and discussion in Subsection 3.2.5. In particular, H is a d-dimensional space, M_H is the corresponding Heisenberg VOA, and $M_H \cong S(H \otimes t^{-1}\mathbb{C}[t^{-1}])$. The central charge of $(M_H, Y, \mathbf{1}, \omega_{M_H})$ is 1. In the case H is 1-dimensional, $S(H \otimes t^{-1}\mathbb{C}[t^{-1}])$ is isomorphic to $S(\coprod_{n\geq 1} \mathbb{C}[t^{-n}])$ and is a polynomial algebra in the variables t^{-n}. The graded dimension of the Fock space M_H can then be computed as

$$\begin{aligned} Z_{M_H}(\mathbf{1}, \tau) &= q^{-1/24} \prod_{n\geq 1} (q - \text{dimension of } \mathbb{C}[t^{-n}]) = q^{-1/24} \prod_{n\geq 1} (1 + q^n + q^{2n} + \cdots) \\ &= q^{-1/24} \prod_{n\geq 1} (1 - q^n)^{-1} \\ &= \eta(\tau)^{-1}. \end{aligned}$$

Since the partition function is multiplicative over tensor products, in the case $\dim H = d$, we find

$$Z_{M_H}(\mathbf{1}, \tau) = \eta(\tau)^{-d}. \tag{5.3.0.3}$$

Consider $J_{M_H(\alpha),\underline{h}}(\mathbf{1}; \tau, \underline{z})$, where $M_H(\alpha) = M_H \otimes e^{\alpha}$ is as in (3.2.5.1). Recall that $L(0)e^{\alpha} = \frac{1}{2}\langle \alpha, \alpha \rangle e^{\alpha}$, $h_i(0)e^{\alpha} = \langle h_i, \alpha \rangle e^{\alpha}$, while $h_i(0)M_H = 0$ for all $1 \leq i \leq n$. In this context, we have

$$
\begin{aligned}
J_{M_H(\alpha),\underline{h}}(\mathbf{1};\tau,\underline{z}) &= \mathrm{Tr}_{M_H\otimes e^\alpha}\,\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-1/24}\\
&= \left(\mathrm{Tr}_{M_H}\,\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-1/24}\right)\left(\mathrm{Tr}_{e^\alpha}\,\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-1/24}\right)\\
&= \left(Z_{M_H}(\mathbf{1},\tau)\right)\zeta_1^{\langle h_1,\alpha\rangle}\cdots\zeta_n^{\langle h_n,\alpha\rangle}q^{\frac{1}{2}\langle\alpha,\alpha\rangle}\\
&= \frac{\zeta_1^{\langle h_1,\alpha\rangle}\cdots\zeta_n^{\langle h_n,\alpha\rangle}q^{\frac{1}{2}\langle\alpha,\alpha\rangle}}{\eta(\tau)^d}. \qquad (5.3.0.4)
\end{aligned}
$$

Chapter 6

A Result of Miyamoto

A portion of the proof of Theorem 1.0.0.2 relies on extending a result of Miyamoto [38] to include zero modes of elements in a VOA. The proof of this refinement differs only slightly from the original. The purpose of this chapter is to establish the proof of Theorem 1.0.0.1.

6.1 Miyamoto's function Φ

Let V be a vertex operator algebra and M be a V-module. For $u, v \in V_1$ and $w \in V$ define the function

$$\Phi_M(v; u, w, \tau) := \mathrm{Tr}_M\, o^M(v) e^{2\pi i (w^M(0) + \langle u, w\rangle/2)} q^{L^M(0) + u^M(0) + \langle u, u\rangle/2 - c/24}, \qquad (6.1.0.1)$$

where as usual, $q = e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$. The function Φ is the same as the function Z found in [38], except here, the roles of u and v have been reversed, the bilinear form has a normalization which is negative of that in [38], and the zero mode of w is included. Φ is the 1-point analogue of the function considered by Miyamoto.

After establishing Theorem 1.0.0.1, it is possible to prove the 1-point analogue of the Main Theorem in [38] with only a little more effort. As this result is not needed in this thesis, it is not included. It should be mentioned, however, that the Main Theorem in [38] is displayed twice, the first incorrectly.

We now prove Theorem 1.0.0.1.

6.2 Proof of Theorem 1.0.0.1

Recall the definition of the n-point functions, (5.1.0.1). In this section, consider n-point functions that also include a formal power series endomorphism

$$\psi \in \mathrm{End}(M^j)((q_{z_1}, \ldots, q_{z_n}))$$

which preserves the grading of M^j. We will call such endomorphisms *grade-preserving*. For homogeneous elements $v_1, \ldots, v_n \in V$ set

$$S_j(\psi; z_1, \ldots, z_n, \tau) := \mathrm{Tr}_{M^j}\, \psi Y^j(v_1, q_{z_1}) \cdots Y^j(v_n, q_{z_n}) q_{z_1}^{[\mathrm{wt}]v_1} \cdots q_{z_n}^{[\mathrm{wt}]v_n} q^{L(0)-c/24}. \quad (6.2.0.2)$$

The z_i in the left hand side of (6.2.0.2) will sometimes be replaced with (v_i, z_i) to emphasize the role of v_i.

Throughout this proof take $v_1, \ldots, v_n \in V_1$ such that $v_i(0)v_j = 0$ and $v_i(1)v_j = \langle v_r, v_j \rangle \mathbf{1} \in \mathbb{C}\mathbf{1}$ for all $1 \leq i, j \leq n$. If an element $v \in V_{[\mathrm{wt}]v}$ is also included, (6.2.0.2) reads

$$\begin{aligned} &S_j(\psi; z_1, \ldots, z_n, (v, z), \tau) \\ &\quad = \mathrm{Tr}_{M^j}\, \psi Y^j(v_1, q_{z_1}) \cdots Y^j(v_n, q_{z_n}) Y^j(v, z) q_1 \cdots q_n q_z^{[\mathrm{wt}]v} q^{L(0)-c/24}. \end{aligned} \quad (6.2.0.3)$$

Applying Theorem 5.1.1.1 to (6.2.0.3) establishes the equality

$$S_j\left(1; \frac{z_1}{c\tau+d}, \ldots, \frac{z_n}{c\tau+d}, \left(v, \frac{z}{c\tau+d}\right), \gamma\tau\right)$$
$$= (c\tau+d)^{[\mathrm{wt}]v+n} \sum_{i=1}^{r} A^i_{j,\gamma} S_i(1; z_1, \ldots, z_n, (v,z), \tau), \tag{6.2.0.4}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Before tackling the proof of Theorem 1.0.0.1, a number of lemmas must be collected whose proofs rely on keeping track of permutations of sets. The notation here is kept the same to that in [38]. Let Σ_n denote the symmetric group of a set $\Omega = \{1, \ldots, n\}$. For $\sigma \in \Sigma_n$ set

$$m(\sigma) = \{i \in \Omega \mid \sigma(i) \neq i\} \text{ and} \tag{6.2.0.5}$$

$$f(\sigma) = \{i \in \Omega \mid \sigma(i) = i\}. \tag{6.2.0.6}$$

Elements σ_1 and σ_2 in Σ_n are called *disjoint* if $m(\sigma_1) \cap m(\sigma_2) = \emptyset$. The notation $\sigma = \sigma_1 + \cdots + \sigma_n$ signifies that $\sigma_1, \ldots, \sigma_n$ are disjoint and $\sigma = \sigma_1 \cdots \sigma_n$.

Let (i, j) denote the element of Σ_n that switches i and j and fixes the remaining elements of Ω. Such an element is called a transposition. An *involution* of Σ_n is an element in Σ_n with order 2. Every transposition is an involution. Set $I(n) := \left\{\sigma \in \Sigma_n \mid \sigma^2 = 1\right\}$ and note that along with all involutions, $I(n)$ also includes the identity. Since every element $\sigma \in \Sigma_n$ can be written as a product of transpositions, σ is in $I(n)$ if and only if these transpositions are mutually disjoint. That is, if $\sigma \in \Sigma_n$ is in $I(n)$, then $\sigma = (i_{11}, i_{12}) + \cdots + (i_{t1}, i_{t2})$ (i.e., $i_{j1} < i_{j2}$ for all $1 \leq j \leq t$ and $i_{ab} \neq i_{cd}$ for $(a, b) \neq (c, d)$).

Versions of the following lemma are found in both [38] and [46].

Lemma 6.2.0.3 *Let $v_1, \ldots, v_n$ be defined as above and let $v \in V$ be such that $v_i(m)v = 0$ for $m \geq 0$ and all $1 \leq i \leq n$. Assume also that $[\psi, v_i(n)] = 0$ and $[\psi, v] = 0$ for each i. Then for a V-module M,*

$$
\begin{aligned}
&S_M(\psi; z_1, z_2, \ldots, z_n, (v, z), \tau) \\
&\quad = \sum_{\sigma \in I(n)} \prod_{i < \sigma(i)} \left(\langle v_i, v_{\sigma(i)} \rangle \frac{P_2(z_{\sigma(i)-i}, \tau)}{(2\pi i)^2} \right) S_M \left(\psi \prod_{s \in f(\sigma)} o(v_s); (v, z), \tau \right).
\end{aligned}
$$

Proof For $k \in \mathbb{Z}$, note that $\psi v_1(k) q_{z_1}^{-k}$ is again a grade-preserving endomorphism. Using (3.1.0.5) gives

$$
\begin{aligned}
&S_M(\psi v_1(k) q_{z_1}^{-k}; z_2, \ldots, z_n, (v, z), \tau) \\
&\quad = \mathrm{Tr}_M\, \psi v_1(k) q_{z_1}^{-k} Y(v_2, q_{z_2}) \cdots Y(v_n, q_{z_n}) Y(v, q_z) q_{z_1} \cdots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} \\
&\quad = \mathrm{Tr}_M\, \psi [v_1(k), Y(v_2, q_{z_2}) \cdots Y(v_n, q_{z_n}) Y(v, q_z)] q_{z_1}^{-k} q_{z_2} \ldots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} \\
&\qquad + \mathrm{Tr}_M\, \psi Y(v_2, q_{z_2}) \cdots Y(v_n, q_{z_n}) Y(v, q_z) v_1(k) q_{z_1}^{-k} q_{z_2} \ldots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} \\
&\quad = \sum_{j=2}^{n} \sum_{i \geq 0} \binom{k}{i} q_{z_j}^{k-i}\, \mathrm{Tr}_M\, \psi Y(v_2, q_{z_2}) \cdots Y(v_1(i) v_j, q_{z_j}) \cdots Y(v_n, q_{z_n}) Y(v, q_z) \\
&\qquad\quad \cdot q_{z_1}^{-k} q_{z_2} \ldots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} \\
&\qquad + \sum_{i \geq 0} \binom{k}{i} q_z^{[\mathrm{wt}]v+k-i}\, \mathrm{Tr}_M\, \psi Y(v_2, q_{z_2}) \cdots Y(v_n, q_{z_n}) Y(v_1(i) v, q_z) \\
&\qquad\quad \cdot q_{z_1}^{-k} q_{z_2} \ldots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} \qquad (6.2.0.7) \\
&\qquad + \mathrm{Tr}_M\, \psi Y(v_2, q_{z_2}) \cdots Y(v_n, q_{z_n}) Y(v, q_z) q_{z_1}^{-k} q_{z_2} \ldots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} v(k) q^k.
\end{aligned}
$$

Because $v_1(i)v = 0$ for all $i \geq 0$, the term (6.2.0.7) is zero. Recalling that $v_i(m)v_j = \delta_{m,1} \langle v_i, v_j \rangle \mathbf{1}$, we have

$$
\begin{aligned}
&S_M(\psi v_1(k) q_{z_1}^{-k}; z_2, \ldots, z_n, (v,z), \tau) \\
&\quad = \sum_{j=2}^{n} k q_{z_j}^{k-1} q_{k_1}^{-k} \operatorname{Tr}_M \psi Y(v_2, q_{z_2}) \cdots Y(\langle v_j, v_1 \rangle \mathbf{1}, q_{z_j}) \cdots Y(v_n, q_{z_n}) Y(v, q_z) \\
&\qquad\quad \cdot q_{z_2} \ldots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} \\
&\qquad + \operatorname{Tr}_M \psi v_1(k) q_{z_1}^{-k} Y(v_2, q_{z_2}) \cdots Y(v_n, q_{z_n}) Y(v, q_z) q_{z_2} \ldots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} q^k \\
&\quad = \sum_{j=2}^{n} \operatorname{Tr}_M \langle v_j, v_1 \rangle k q_{z_j - z_1}^{k} \psi Y(v_2, q_{z_2}) \cdots \widehat{Y(v_j, q_{z_j})} \cdots Y(v_n, q_{z_n}) Y(v, q_z) \\
&\qquad\quad \cdot q_{z_2} \ldots \widehat{q_{z_j}} \ldots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} \\
&\qquad + \operatorname{Tr}_M \psi v_1(k) q_{z_1}^{-k} Y(v_2, q_{z_2}) \cdots Y(v_n, q_{z_n}) Y(v, q_z) q_{z_2} \ldots q_{z_n} q_z^{[\mathrm{wt}]v} q^{L(0)-c/24} q^k \\
&\quad = \sum_{j=2}^{n} \langle v_1, v_j \rangle k q_{z_j - z_1}^{k} S_M(\psi; z_2, \ldots \widehat{z_j}, \ldots, z_n, (v,z), \tau) \\
&\qquad + S_M(\psi v_1(k) q_{z_1}^{-k}; z_2, \ldots, z_n, (v,z), \tau) q^k,
\end{aligned}
$$

where the notation $\widehat{\alpha}$ signifies the deletion of the α term.

In the case $k \neq 0$, the previous calculation may be rewritten so that

$$
\begin{aligned}
&S_M(\psi v_1(k) q_{z_1}^{-k}; z_2, \ldots, z_n, (v,z), \tau) \\
&\quad = \sum_{j=2}^{n} \langle v_j, v_1 \rangle \frac{k q_{z_j - z_1}^{k}}{1 - q^k} S_M(\psi; z_2, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n, (v,z), \tau).
\end{aligned} \tag{6.2.0.8}
$$

Since

$$
S_M(\psi; z_1, z_2, \ldots, z_n, (v,z), \tau) = \sum_{k \in \mathbb{Z}} S_M(\psi v_1(k) q_{z_1}^{-k}; z_2, \ldots, z_n, (v,z), \tau),
$$

the $k = 0$ term can be separated from the sum. Applying (6.2.0.8) we find

$$S_M(\psi; z_1, z_2, \ldots, z_n, (v, z), \tau)$$

$$= S_M(\psi v_1(0); z_2), \ldots, z_n, (v, z), \tau) + \sum_{k \neq 0} S_M(\psi v_1(k) q_{z_1}^{-k}; z_2, \ldots, z_n, (v, z), \tau)$$

$$= S_M(\psi v_1(0); z_2, \ldots, z_n, (v, z), \tau)$$

$$+ \sum_{k \neq 0} \sum_{j=2}^{n} \langle v_j, v_1 \rangle \frac{k q_{z_j - z_1}^k}{1 - q^k} S_M(\psi; z_2, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n, (v, z), \tau)$$

$$= S_M(\psi v_1(0); z_2, \ldots, z_n, (v, z), \tau)$$

$$+ \sum_{j=2}^{n} \langle v_j, v_1 \rangle \frac{P_2(q_{z_j - z_1}, \tau)}{(2\pi i)^2} S_M(\psi; z_2, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n, (v, z), \tau).$$

Next, take $\psi v_1(0)$ on the right hand side to be the grade-preserving endomorphisms and repeat the steps for $S_M(\psi v_1(0); z_2, \ldots, z_n, (v, z), \tau)$. Repeating this process for the terms with such endomorphisms establishes the desired result. □

Set

$$D(r, j) = \langle v_r, v_j \rangle \left(\frac{1}{2\pi i} \right)^2 (c\tau + d)^2 P_2(q_{z_r - z_j}, \tau), \tag{6.2.0.9}$$

$$E(r, j) = \langle v_r, v_j \rangle \left(\frac{1}{2\pi i} \right)^2 (c\tau + d)^2 P_2(q_{z_r - z_j}, \tau) - \left(\frac{1}{2\pi i} \right) c(c\tau + d), \tag{6.2.0.10}$$

$$D_\sigma = \prod_{j < \sigma(j)} D(\sigma(j), j), \text{ and} \tag{6.2.0.11}$$

$$E_\sigma = \prod_{j < \sigma(j)} E(\sigma(j), j). \tag{6.2.0.12}$$

In [38] there is a typo where the $(1/2\pi i)^2$ factor in (6.2.0.10) is omitted. The following lemma from [38] will be useful in simplifying notation.

Lemma 6.2.0.4 *If $|m(\sigma)| = 2p$, then*

$$\sum_{\sigma_1 + \cdots + \sigma_t = \sigma} (-1)^t E_{\sigma_t} \cdots E_{\sigma_2} E_{\sigma_1} = (-1)^p E_\sigma. \tag{6.2.0.13}$$

Proof Recall that $\sigma = \sigma_1 + \cdots + \sigma_t$ means $m(\sigma_1) \cap \cdots \cap m(\sigma_t) = \emptyset$ and $\sigma = \sigma_1 \cdots \sigma_t$. It follows that

$$\begin{aligned} E_{\sigma_t} \cdots E_{\sigma_1} &= \left(\prod_{i<\sigma_t(i)} E(\sigma_t(i), i) \right) \cdots \left(\prod_{i<\sigma_1(i)} E(\sigma_1(i), i) \right) \\ &= \prod_{i<\sigma_t(i)} \cdots \prod_{i<\sigma_1(i)} E(\sigma_t(i), i) \cdots E(\sigma_1(i), i) \\ &= \prod_{i<\sigma(i)} E(\sigma(i), i) = E_\sigma . \end{aligned}$$

Therefore, the proof reduces to counting how many σ arise on the left hand side of (6.2.0.13). This is performed by induction on p. For $p = 1$, $|m(\sigma)| = 2$ and σ is a single transposition. That is, $\sigma = \sigma_1$ and the result follows trivially.

Assume the result holds for $p - 1$ and consider an arbitrary

$$\sigma = (r_1, r_2) \cdots (r_{2p-1}, r_{2p})$$

such that $|m(\sigma)| = 2p$. The question becomes how many ways σ can be decomposed as $\sigma = \sigma_1 + \cdots + \sigma_t$. Without loss of generality, take $\sigma_1 \neq 1$ and set $\sigma' = \sigma_2 + \cdots + \sigma_t$ so that $|m(\sigma_1)| \geq 2$ and $\sigma = \sigma_1 + \sigma'$. Suppose that $|m(\sigma_1)| = 2j$ with $1 \leq j \leq p$.

Of the p many pairs of elements $\{(r_1, r_2), \cdots, (r_{2p-1}, r_{2p})\}$ in a set Ω that are permuted by σ, σ_1 accounts for j many of them. Therefore, there are $\binom{p}{j}$ many different σ_1 that satisfy $|m(\sigma_1)| = 2j$ and $\sigma_1 + \sigma' = \sigma$. Since for each of these σ_1, $|m(\sigma_1)| = 2j$, it follows that $|m(\sigma')| = |m(\sigma_2 + \cdots + \sigma_t)| = 2(p-j)$ so that the coefficient of the $E_{\sigma_t} \cdots E_{\sigma_2} E_{\sigma_1}$ term for each j is $(-1)^{|m(\sigma_2+\cdots+\sigma_t)|/2} = (-1)^{p-j}$. Utilizing the induction hypothesis establishes

$$\begin{aligned}
\sum_{\sigma_1+\cdots+\sigma_t=\sigma}(-1)^t E_{\sigma_t}\cdots E_{\sigma_2}E_{\sigma_1} &= \sum_{j=1}^{p}\sum_{\substack{\sigma_1,|m(\sigma_1)|=2j\\ \sigma_1+\sigma'=\sigma}}\sum_{\sigma_2+\cdots+\sigma_t=\sigma'}(-1)^t E_{\sigma_t}\cdots E_{\sigma_2}E_{\sigma_1}\\
&= -\sum_{j=1}^{p}\binom{p}{j}\sum_{\sigma_2+\cdots+\sigma_t=\sigma'}(-1)^{t-1}E_{\sigma_t}\cdots E_{\sigma_2}E_{\sigma_1}\\
&= -\sum_{j=1}^{p}\binom{p}{j}(-1)^{p-j}E_\sigma\\
&= -(-(-1)^p)E_\sigma = (-1)^p E_\sigma,
\end{aligned}$$

as desired. □

Most of the work required to establish Theorem 1.0.0.1 is contained in the proof of the next lemma. The proof presented here follows that in [38], but with more detail and the inclusion of the zero mode.

Lemma 6.2.0.5 *Let $v_1,\ldots,v_n \in V_1$ be as before. For $v \in V_{[\mathrm{wt}]v}$ and a module M^h, $1 \le h \le r$, we have*

$$S_h\left(\prod_{s=1}^{n} o(v_s); \left(v, \frac{z}{c\tau+d}\right), \gamma\tau\right) = (c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r}A^k_{h,\gamma}\sum_{\sigma\in I(n)}\prod_{j<\sigma(j)}\left(\frac{c(c\tau+d)}{2\pi i}\right) \cdot \left\langle v_j, v_{\sigma(j)}\right\rangle S_k\left(\prod_{s\in f(\sigma)}(c\tau+d)o(v_s); (v,z), \tau\right)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Proof Consider the function

$$S_h\left(1; \frac{z_1}{c\tau+d}, \ldots, \frac{z_n}{c\tau+d}, \left(v, \frac{z}{c\tau+d}\right), \gamma\tau\right). \qquad (6.2.0.14)$$

Let γz_i denote $\frac{z_i}{c\tau+d}$. On one hand, Lemma 6.2.0.3 gives

$$S_h\left(1;\gamma z_1,\ldots,\gamma z_n,(v,\gamma z),\gamma\tau\right)$$
$$=\sum_{\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left\langle v_{\sigma(s)},v_s\right\rangle\left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{\gamma z_{\sigma(s)}-\gamma z_s},\gamma\tau\right)\right)S_h\left(\prod_{j\in f(\sigma)}o(v_j);(v,\gamma z),\gamma\tau\right),$$

while on the other, applying (6.2.0.4) and Lemma 6.2.0.3 to (6.2.0.14) establishes

$$S_h\left(1;\gamma z_1,\ldots,\gamma z_n,(v,\gamma z),\gamma\tau\right)$$
$$=(c\tau+d)^{[\mathrm{wt}\,v]+n}\sum_{k=1}^{r}A_{h,\gamma}^{k}S_k(1;z_1,\ldots,z_n,(v,z),\tau)$$
$$=(c\tau+d)^{[\mathrm{wt}]v+n}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left(\left\langle v_s,v_{\sigma(s)}\right\rangle\left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{z_{\sigma(s)}-z_s},\tau\right)\right)\right)$$
$$\cdot S_k\left(\prod_{j\in f(\sigma)}o(v_j);(v,z),\tau\right).$$

Setting these two identities equal to one another, we find

$$\sum_{\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left\langle v_{\sigma(s)},v_s\right\rangle\left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{\gamma z_{\sigma(s)}-\gamma z_s},\gamma\tau\right)\right)S_h\left(\prod_{j\in f(\sigma)}o(v_j);(v,\gamma z),\gamma\tau\right)$$
$$=(c\tau+d)^{[\mathrm{wt}]v+n}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left(\left\langle v_s,v_{\sigma(s)}\right\rangle\left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{z_{\sigma(s)}-z_s},\tau\right)\right)\right)$$
$$\cdot S_k\left(\prod_{j\in f(\sigma)}o(v_j);(v,z),\tau\right). \tag{6.2.0.15}$$

Recall that $I(n)$ contains the identity element 1. However, when $\sigma=1$ in (6.2.0.15) there is no product over $s<\sigma(s)$. The term associated to $\sigma=1$, which is

$$S_h\left(\prod_{j\in f(\sigma)}o(v_j);(v,\gamma z),\gamma\tau\right)=S_h\left(\prod_{s=1}^{n}o(v_s);(v,\gamma z),\gamma\tau\right),$$

may be pulled from (6.2.0.15). Display (6.2.0.15) may now be written as

$$S_h\left(\prod_{j=1}^{n} o(v_j);(v,\gamma z),\gamma\tau\right)+\sum_{1\neq\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left\langle v_{\sigma(s)},v_s\right\rangle\left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{\gamma z_{\sigma(s)}-\gamma z_s},\gamma\tau\right)\right)$$
$$\cdot S_h\left(\prod_{j\in f(\sigma)} o(v_j);(v,\gamma z),\gamma\tau\right)$$
$$=(c\tau+d)^{[\mathrm{wt}]v+n}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left(\left\langle v_s,v_{\sigma(s)}\right\rangle\left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{z_{\sigma(s)}-z_s},\tau\right)\right)\right)$$
$$\cdot S_k\left(\prod_{j\in f(\sigma)} o(v_j);(v,z),\tau\right).$$

Solving for $S_h\left(\prod_{j=1}^{n} o(v_j);(v,\gamma z),\gamma\tau\right)$ gives

$$S_h\left(\prod_{j=1}^{n} o(v_j);(v,\gamma z),\gamma\tau\right)$$
$$=(c\tau+d)^{[\mathrm{wt}]v+n}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left(\left\langle v_s,v_{\sigma(s)}\right\rangle\left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{z_{\sigma(s)}-z_s},\tau\right)\right)\right)$$
$$\cdot S_k\left(\prod_{j\in f(\sigma)} o(v_j);(v,z),\tau\right)$$
$$-\sum_{1\neq\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left\langle v_{\sigma(s)},v_s\right\rangle\left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{\gamma z_{\sigma(s)}-\gamma z_s},\gamma\tau\right)\right)$$
$$\cdot S_h\left(\prod_{j\in f(\sigma)} o(v_j);(v,\gamma z),\gamma\tau\right).$$

Note that $|m(\sigma)|+|f(\sigma)|=n$. Since $m(\sigma)=\{i\in\Omega\mid\sigma(i)\neq i\}$, the set $\{i<\sigma(i)\}$ contains half the elements that the set $m(\sigma)$ contains. It follows that $2\left|\{s<\sigma(s)\}\right|+|f(\sigma)|=n$, and thus

$$(c\tau+d)^n=(c\tau+d)^{2|\{s<\sigma(s)\}|}(c\tau+d)^{|f(\sigma)|}.$$

Utilizing this identity we find

$$
\begin{aligned}
&S_h\left(\prod_{j=1}^{n} o(v_j);(v,\gamma z),\gamma\tau\right)\\
&=(c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\sigma\in I(n)}\\
&\quad\cdot\left((c\tau+d)^{2|\{s<\sigma(s)\}|}\prod_{s<\sigma(s)}\left(\left\langle v_s,v_{\sigma(s)}\right\rangle\left(\frac{1}{2\pi i}\right)^{2}P_2\left(q_{z_{\sigma(s)}-z_s},\tau\right)\right)\right)\\
&\qquad\qquad\qquad\qquad\cdot S_k\left((c\tau+d)^{|f(\sigma)|}\prod_{j\in f(\sigma)}o(v_j);(v,z),\tau\right)\\
&\quad-\sum_{1\neq\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left\langle v_{\sigma(s)},v_s\right\rangle\left(\frac{1}{2\pi i}\right)^{2}P_2\left(q_{\gamma z_{\sigma(s)}-\gamma z_s},\gamma\tau\right)\right)S_h\left(\prod_{j\in f(\sigma)}o(v_j);(v,\gamma z),\gamma\tau\right)\\
&=(c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left(\left\langle v_s,v_{\sigma(s)}\right\rangle(c\tau+d)^{2}\left(\frac{1}{2\pi i}\right)^{2}P_2\left(q_{z_{\sigma(s)}-z_s},\tau\right)\right)\right)\\
&\qquad\qquad\qquad\qquad\cdot S_k\left(\prod_{j\in f(\sigma)}(c\tau+d)o(v_j);(v,z),\tau\right)\\
&\quad-\sum_{1\neq\sigma\in I(n)}\left(\prod_{s<\sigma(s)}\left\langle v_{\sigma(s)},v_s\right\rangle\left(\frac{1}{2\pi i}\right)^{2}P_2\left(q_{\gamma z_{\sigma(s)}-\gamma z_s},\gamma\tau\right)\right)S_h\left(\prod_{j\in f(\sigma)}o(v_j);(v,\gamma z),\gamma\tau\right).
\end{aligned}
\tag{6.2.0.16}
$$

We now apply the transformation law for P_2,

$$P_2\left(\gamma z,\gamma\tau\right)=(c\tau+d)^{2}P_2(z,\tau)-2\pi ic(c\tau+d),$$

to the term (6.2.0.16) on the right hand side of the equality. This gives

$$\begin{aligned}
&S_h\left(\prod_{j=1}^{n} o(v_j); (v, \gamma z), \gamma\tau\right)\\
&= (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{\sigma\in I(n)} \left(\prod_{s<\sigma(s)} \left(\langle v_s, v_{\sigma(s)}\rangle (c\tau+d)^2 \left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{z_{\sigma(s)}-z_s}, \tau\right)\right)\right)\\
&\qquad\qquad \cdot S_k\left(\prod_{j\in f(\sigma)} (c\tau+d) o(v_j); (v, z), \tau\right)\\
&\quad - \sum_{1\neq\sigma\in I(n)} \left(\prod_{s<\sigma(s)} \langle v_{\sigma(s)}, v_s\rangle \left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{\gamma z_{\sigma(s)}-\gamma z_s}, \gamma\tau\right)\right) S_h\left(\prod_{j\in f(\sigma)} o(v_j); (v, \gamma z), \gamma\tau\right)\\
&= (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{\sigma\in I(n)} \left(\prod_{s<\sigma(s)} \left(\langle v_s, v_{\sigma(s)}\rangle (c\tau+d)^2 \left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{z_{\sigma(s)}-z_s}, \tau\right)\right)\right)\\
&\qquad\qquad \cdot S_k\left(\prod_{j\in f(\sigma)} (c\tau+d) o(v_j); (v, z), \tau\right)\\
&\quad - \sum_{1\neq\sigma\in I(n)} \left(\prod_{s<\sigma(s)} \langle v_{\sigma(s)}, v_s\rangle \left[(c\tau+d)^2 \left(\frac{1}{2\pi i}\right)^2 P_2\left(q_{z_{\sigma(s)}-z_s}, \tau\right) - \left(\frac{1}{2\pi i}\right) c(c\tau+d)\right]\right)\\
&\qquad\qquad \cdot S_h\left(\prod_{j\in f(\sigma)} o(v_j); (v, \gamma z), \gamma\tau\right). \qquad (6.2.0.17)
\end{aligned}$$

Using the notation in (6.2.0.9)—(6.2.0.12), equation (6.2.0.17) may be rewritten as

$$\begin{aligned}
&S_h\left(\prod_{j=1}^{n} o(v_j); (v, \gamma z), \gamma\tau\right)\\
&\quad= (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{\sigma\in I(n)} \left(\prod_{s<\sigma(s)} D(\sigma(s), s)\right) S_k\left(\prod_{j\in f(\sigma)} (c\tau+d) o(v_j); (v, z), \tau\right)\\
&\qquad - \sum_{1\neq\sigma\in I(n)} \left(\prod_{s<\sigma(s)} E(\sigma(s), s)\right) S_h\left(\prod_{j\in f(\sigma)} o(v_j); (v, \gamma z), \gamma\tau\right)\\
&\quad= (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{\sigma\in I(n)} D_\sigma S_k\left(\prod_{j\in f(\sigma)} (c\tau+d) o(v_j); (v, z), \tau\right)\\
&\qquad - \sum_{1\neq\sigma\in I(n)} E_\sigma S_h\left(\prod_{j\in f(\sigma)} o(v_j); (v, \gamma z), \gamma\tau\right). \qquad (6.2.0.18)
\end{aligned}$$

The idea is to repeat this process on the last term, $S_h\left(\prod_{j\in f(\sigma)} o(v_j);(v,\gamma z),\gamma\tau\right)$. In this case the product $\prod_{j\in f(\sigma)}$ is encountered instead of $\prod_{j=1}^{n}$. Therefore, the sum must be over another element $\sigma' \in I(n)$ such that $m(\sigma)\cap m(\sigma') = \emptyset$. This can be seen as follows. Suppose σ is an element of $I(n)$ on the set $\{1,\dots,n\}$ that fixes the first p many elements. That is, $\sigma(i) = i$ for $1 \le i \le p$, where $p < n$. Then $f(\sigma)$ equals $\{1,\dots,p\}$, and the next $\sigma' \in I(n)$ to arise in the computation should affect only this set, i.e., σ' needs to satisfy $\sigma'(j) = j$ for all $j \in \{p+1,\dots,n\}$. This is equivalent to saying that $m(\sigma) = \{p+1,\dots,n\}$ and $m(\sigma') \subseteq \{1,\dots,p\}$. Therefore, the requirement $m(\sigma)\cap m(\sigma') = \emptyset$ to assure this condition holds is established.

Repeating the steps above on $S_h\left(\prod_{j\in f(\sigma)} o(v_j);(v,\gamma z),\gamma\tau\right)$ in (6.2.0.18) and utilizing the appropriate notation for the σ, we have

$$S_h\left(\prod_{j=1}^{n} o(v_j);(v,\gamma z),\gamma\tau\right) \tag{6.2.0.19}$$

$$= (c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r} A_{h,\gamma}^{k}\sum_{\sigma\in I(n)} D_\sigma S_k\left(\prod_{j\in f(\sigma)}(c\tau+d)o(v_j);(v,z),\tau\right) \tag{6.2.0.20}$$

$$- (c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r} A_{h,\gamma}^{k}\sum_{1\neq\sigma\in I(n)} E_\sigma \sum_{\substack{\sigma'\in I(n)\\ m(\sigma')\cap m(\sigma)=\emptyset}}$$

$$\cdot D_{\sigma'} S_k\left(\prod_{s\in f(\sigma')\cap f(\sigma)}(c\tau+d)o(v_s);(v,z),\tau\right) \tag{6.2.0.21}$$

$$+ \sum_{1\neq\sigma\in I(n)} E_\sigma \sum_{\substack{1\neq\sigma'\in I(n)\\ m(\sigma')\cap m(\sigma)=\emptyset}} E_{\sigma'} S_h\left(\prod_{j\in f(\sigma)\cap f(\sigma')} o(v_j);(v,\gamma z),\gamma\tau\right). \tag{6.2.0.22}$$

We continue in this manner, reiterating this process on the last S_h term each time. Three "types" of terms arise during this process. One is exactly the term (6.2.0.20). Another of these terms occurs as various sums similar to those of (6.2.0.21) are collected. They are of the form

$$(c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{1\neq\sigma_1\in I(n)}\cdots\sum_{\substack{1\neq\sigma_{\ell-1}\in I(n)\\ m(\sigma_1)\cap\cdots\cap m(\sigma_{\ell-1})=\emptyset}}\cdot\sum_{\substack{\sigma_\ell\in I(n)\\ m(\sigma_1)\cap\cdots\cap m(\sigma_\ell)=\emptyset}}$$
$$\cdot(-1)^{\ell-1}E_{\sigma_1}\cdots E_{\sigma_{\ell-1}}D_{\sigma_\ell}S_k\left(\prod_{s\in f(\sigma_1)\cap\cdots\cap f(\sigma_\ell)}(c\tau+d)o(v_s);(v,z),\tau\right),\tag{6.2.0.23}$$

for various ℓ. The number of possible combinations of σ_i that satisfy the necessary conditions is dependent on the length of the set $\{1,\dots,n\}$.

The final term that occurs contains only terms E_{σ_i} and not D_{σ_i}, though it takes a different form depending on whether n is even or odd. If n is even, the intersection of all $f(\sigma_i)$ for $1\le i\le\ell$ will be empty after the last iteration since each σ_i permutes an even number of elements of the set $\{1,\dots,n\}$. On the other hand, if n is odd, the last iteration will leave one element in this intersection.

After iterating until the product in the last S_h term has either zero or one elements, (6.2.0.19) becomes

$$S_h\left(\prod_{j=1}^{n} o(v_j);(v,\gamma z),\gamma\tau\right)$$

$$= (c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r} A_{h,\gamma}^{k}\sum_{\sigma\in I(n)} D_\sigma S_k\left(\prod_{j\in f(\sigma)}(c\tau+d)o(v_j);(v,z),\tau\right) \tag{6.2.0.24}$$

$$+ (c\tau+d)^{[\mathrm{wt}]v}\sum_{\ell}\sum_{k=1}^{r} A_{h,\gamma}^{k}\sum_{1\neq\sigma_1\in I(n)}\cdots\sum_{\substack{1\neq\sigma_{\ell-1}\in I(n)\\ m(\sigma_1)\cap\cdots\cap m(\sigma_{\ell-1})=\emptyset}}$$

$$\cdot\sum_{\substack{1\neq\sigma_\ell\in I(n)\\ m(\sigma_1)\cap\cdots\cap m(\sigma_\ell)=\emptyset}}(-1)^{\ell-1}E_{\sigma_1}\cdots E_{\sigma_{\ell-1}}D_{\sigma_\ell}$$

$$\cdot S_k\left(\prod_{s\in f(\sigma_1)\cap\cdots\cap f(\sigma_\ell)}(c\tau+d)o(v_s);(v,z),\tau\right) \tag{6.2.0.25}$$

$$(n\ \textbf{even})\quad +\sum_{1\neq\sigma_1\in I(n)}\cdots\sum_{\substack{1\neq\sigma_{\ell-1}\in I(n)\\ m(\sigma_1)\cap\cdots\cap m(\sigma_{\ell-1})=\emptyset}}$$

$$\cdot\sum_{\substack{1\neq\sigma_\ell\in I(n)\\ m(\sigma_1)\cap\cdots\cap m(\sigma_\ell)=\emptyset}}(-1)^{\ell}E_{\sigma_1}\cdots E_{\sigma_{\ell-1}}E_{\sigma_\ell}S_k\left(1;(v,\gamma z),\gamma\tau\right) \tag{6.2.0.26}$$

$$(n\ \textbf{odd})\quad +\sum_{j=1}^{n}\sum_{1\neq\sigma_1\in I(n)}\cdots\sum_{\substack{1\neq\sigma_{\ell-1}\in I(n)\\ m(\sigma_1)\cap\cdots\cap m(\sigma_{\ell-1})=\emptyset}}$$

$$\cdot\sum_{\substack{1\neq\sigma_\ell\in I(n)\\ m(\sigma_1)\cap\cdots\cap m(\sigma_\ell)=\emptyset}}(-1)^{\ell}E_{\sigma_1}\cdots E_{\sigma_{\ell-1}}E_{\sigma_\ell}$$

$$\cdot S_k\left(\prod_{s\in f(\sigma_1)\cap\cdots\cap f(\sigma_\ell)=\{j\}}(c\tau+d)o(v_s);(v,\gamma z),\gamma\tau\right), \tag{6.2.0.27}$$

where the $\sum_\ell$ in (6.2.0.25) denotes the summing over various lengths of ℓ.

We now begin to simplify this equation. First we focus on (6.2.0.26) and (6.2.0.27). Note that in these cases (referred to as the "even" and "odd" cases), the summing is over all combinations of $\sigma_1, \ldots, \sigma_\ell$ such that $m(\sigma_1) \cap \cdots \cap m(\sigma_\ell) = \emptyset$, along with requirements that for all i, $\sigma_i \neq 1$. This is equivalent to summing over all $\sigma \neq 1$ such that $\sigma = \sigma_1 + \cdots + \sigma_\ell$ for various ℓ and $\sigma_i \neq 1$ for each i, since $m(\sigma_1) \cap \cdots \cap m(\sigma_\ell) = \emptyset$. Thus (6.2.0.26) and (6.2.0.27) are equal to

$$
\begin{aligned}
(n \textbf{ even}) \quad + \sum_{1 \neq \sigma \in I(n)} \; & \sum_{\substack{\sigma_1, \ldots, \sigma_\ell \in I(n) \\ \sigma_1 + \cdots + \sigma_\ell = \sigma, f(\sigma) = \emptyset}} \\
& \cdot (-1)^\ell E_{\sigma_1} \cdots E_{\sigma_{\ell-1}} E_{\sigma_\ell} S_k \left(1; (v, \gamma z), \gamma\tau\right) \\
(n \textbf{ odd}) \quad + \sum_{j=1}^{n} \sum_{1 \neq \sigma \in I(n)} \; & \sum_{\substack{\sigma_1, \ldots, \sigma_\ell \in I(n) \\ \sigma_1 + \cdots + \sigma_\ell = \sigma, f(\sigma) = \{j\}}} \\
& \cdot (-1)^\ell E_{\sigma_1} \cdots E_{\sigma_{\ell-1}} E_{\sigma_\ell} S_k \left((c\tau + d) o(v_j); (v, \gamma z), \gamma\tau\right).
\end{aligned}
$$

Applying Lemma 6.2.0.13 to the even and odd cases, and applying Zhu's Theorem to the S_k functions, we find these now equal

$$
(n \textbf{ even}) \quad + (c\tau + d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{\substack{\sigma \in I(n) \\ f(\sigma) = \emptyset}} (-1)^{n/2} E_\sigma S_h \left(1; (v, z), \tau\right) \tag{6.2.0.28}
$$

$$
(n \textbf{ odd}) \quad + (c\tau + d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{j=1}^{n} \sum_{\substack{\sigma \in I(n) \\ f(\sigma) = \{j\}}} (-1)^{(n-1)/2} E_\sigma S_h \left((c\tau + d) o(v_j); (v, z), \tau\right). \tag{6.2.0.29}
$$

Note that the $1 \neq \sigma$ notation in the last two sums may be dropped since σ cannot be 1 with the condition $f(\sigma) = \{j\}$ or $f(\sigma) = \emptyset$.

Next, consider the terms (6.2.0.24) and (6.2.0.25). In this case, since $m(\sigma_1) \cap \cdots \cap m(\sigma_{\ell-1}) = \emptyset$, the sum ranges over all σ_s such that $\sigma_s = \sigma_1 + \cdots + \sigma_{\ell-1}$. The terms (6.2.0.24) and (6.2.0.25) now take the form

$$(c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\sigma\in I(n)}D_\sigma S_k\left(\prod_{j\in f(\sigma)}(c\tau+d)o(v_j);(v,z),\tau\right)$$
$$+(c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\sigma_\ell,1\neq\sigma_t\in I(n)}\sum_{\substack{\sigma_1,\ldots,\sigma_{\ell-1}\in I(n),|f(\sigma_t)|\geq 2\\ \sigma_t=\sigma_1+\cdots+\sigma_{\ell-1}}}$$
$$\cdot(-1)^{\ell-1}E_{\sigma_1}\cdots E_{\sigma_{\ell-1}}D_{\sigma_\ell}S_k\left(\prod_{s\in f(\sigma_t)\cap f(\sigma_\ell)}(c\tau+d)o(v_s);(v,z),\tau\right). \quad (6.2.0.30)$$

The condition $|f(\sigma_t)|\geq 2$ is included in (6.2.0.30) to avoid dealing with the even and odd cases in these terms. Moreover, the fact $f(\sigma_1)\cap\cdots\cap f(\sigma_{\ell-1})=f(\sigma_t)$ is also used. Applying Lemma 6.2.0.13, these terms become

$$(c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\sigma\in I(n)}D_\sigma S_k\left(\prod_{j\in f(\sigma)}(c\tau+d)o(v_j);(v,z),\tau\right) \quad (6.2.0.31)$$
$$+(c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\substack{\sigma_\ell,1\neq\sigma_t\in I(n)\\ m(\sigma_t)\cap m(\sigma_\ell)=\emptyset}}\sum_{\substack{\sigma_t,\sigma_\ell\in I(n)\\ |f(\sigma_t)|\geq 2}}$$
$$\cdot(-1)^{|m(\sigma_t)|/2}E_{\sigma_t}D_{\sigma_\ell}S_k\left(\prod_{s\in f(\sigma_t)\cap f(\sigma_\ell)}(c\tau+d)o(v_s);(v,z),\tau\right). \quad (6.2.0.32)$$

Note that (6.2.0.31) is the case of (6.2.0.32) when $\sigma_t\neq 1$. Therefore, combining (6.2.0.31) and (6.2.0.32) creates the single term

$$(c\tau+d)^{[\mathrm{wt}]v}\sum_{k=1}^{r}A_{h,\gamma}^{k}\sum_{\substack{\sigma_\ell,\sigma_t\in I(n)\\ m(\sigma_t)\cap m(\sigma_\ell)=\emptyset}}\sum_{\substack{\sigma_t,\sigma_\ell\in I(n)\\ |f(\sigma_t)|\geq 2}}(-1)^{|m(\sigma_t)|/2}E_{\sigma_t}D_{\sigma_\ell}$$
$$\cdot S_k\left(\prod_{s\in f(\sigma_t)\cap f(\sigma_\ell)}(c\tau+d)o(v_s);(v,z),\tau\right).$$

Since $m(\sigma_t)\cap m(\sigma_\ell)=\emptyset$, the notation $\sigma=\sigma_t+\sigma_\ell$ can be used. Recalling that $f(\sigma)=$

$f(\sigma_t) \cap f(\sigma_\ell)$, the previous display becomes

$$(c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{\sigma\in I(n)} \sum_{\substack{\sigma_t,\sigma_\ell \in I(n), |f(\sigma_t)|\geq 2 \\ \sigma_t+\sigma_\ell=\sigma}} (-1)^{|m(\sigma_t)|/2} E_{\sigma_t} D_{\sigma_\ell} \cdot S_k \left(\prod_{s\in f(\sigma)} (c\tau+d) o(v_s); (v,z), \tau \right). \qquad (6.2.0.33)$$

Finally, replace (6.2.0.26) and (6.2.0.27) with (6.2.0.28) and (6.2.0.29), respectively, and (6.2.0.24) and (6.2.0.25) with the single term (6.2.0.33). This establishes

$$S_h \left(\prod_{j=1}^{n} o(v_j); (v, \gamma z), \gamma\tau \right)$$

$$= (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{\sigma\in I(n)} \sum_{\substack{\sigma_t,\sigma_\ell \in I(n), |f(\sigma_t)|\geq 2 \\ \sigma_t+\sigma_\ell=\sigma}} \cdot (-1)^{|m(\sigma_t)|/2} E_{\sigma_t} D_{\sigma_\ell} S_k \left(\prod_{s\in f(\sigma)} (c\tau+d) o(v_s); (v,z), \tau \right) \qquad (6.2.0.34)$$

$$(n \textbf{ even }) \quad + (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{\sigma\in I(n), f(\sigma)=\emptyset} (-1)^{n/2} E_\sigma S_h (1; (v,z), \tau) \qquad (6.2.0.35)$$

$$(n \textbf{ odd }) \quad + (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{j=1}^{n} \sum_{\sigma\in I(n), f(\sigma)=\{j\}} \cdot (-1)^{(n-1)/2} E_\sigma S_h ((c\tau+d) o(v_j); (v,z), \tau). \qquad (6.2.0.36)$$

It remains to discuss how this is equivalent to the final desired result.

Consider the case when n is even. The case when n is odd follows similarly. It suffices to examine

$$\sum_{\sigma\in I(n)} \sum_{\substack{\sigma_t,\sigma_\ell \in I(n), |f(\sigma_t)|\geq 2 \\ \sigma_t+\sigma_\ell=\sigma}} (-1)^{|m(\sigma_t)|/2} E_{\sigma_t} D_{\sigma_\ell} S_k \left(\prod_{s\in f(\sigma)} (c\tau+d) o(v_s); (v,z), \tau \right) + \sum_{\sigma\in I(n), f(\sigma)=\emptyset} (-1)^{n/2} E_\sigma S_h (1; (v,z), \tau).$$

Fix a $\sigma \in I(n)$ such that $|m(\sigma)| = m \leq n$. The larger m is, the more possibilities of σ_t and σ_ℓ arise. Regardless, careful observation shows that when the various identities of E_{σ_t} and D_{σ_ℓ} are expanded, all of the terms that include expressions of P_2 cancel. What remains is the term

$$\prod_{i<\sigma(i)} \left[\langle v_{\sigma(i)}, v_i \rangle \left(\frac{c\tau + d}{2\pi i} \right)^2 \right] S_k \left(\prod_{s \in f(\sigma)} (c\tau + d) o(v_s); (v, z), \tau \right).$$

Plugging this into (6.2.0.34)-(6.2.0.36) for each $\sigma \in I(n)$ obtains the desired result. □

The final piece required for the proof of Theorem 1.0.0.1 is contained in the next theorem. The proof is the same as in [38].

Theorem 6.2.0.6 *Let* $w \in V_1$ *satisfy* $w(n)w = \delta_{1,n} \langle w, w \rangle \mathbf{1}$ *and* $v \in V_{[\mathrm{wt}]v}$ *be such that* $w(m)v = 0$ *for all* $m \geq 0$. *Then*

$$S_h \left(e^{o(w)}; \left(v, \frac{z}{c\tau + d} \right), \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A^k_{h,\gamma} S_k \left(e^{\left[\frac{\langle w, w \rangle}{2} \frac{1}{2\pi i} c(c\tau+d) + (c\tau+d) o(w) \right]}, (v, z), \tau \right).$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Proof In Lemma 6.2.0.5 take $v_1 = \cdots = v_n = w$. Break n into the sum $n = 2p + j$. On a set of $2p + j$ many elements, there are $\binom{2p+j}{j} \frac{(2p+j)!}{p! 2^p}$ many involutions σ, such that

$|f(\sigma)| = r$. In this case, $|m(\sigma)| = 2p$ and

$$
\begin{aligned}
&S_h\left(e^{o(w)}; \frac{a\tau+b}{c\tau+d}, \left(\frac{z}{c\tau+d}\right)\right) = S_h\left(\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n} o(w); \gamma\tau, \gamma z\right) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!} S_h\left(\prod_{i=1}^{n} o(w); \gamma\tau, \gamma z\right) \\
&= \sum_{k=1}^{\infty} A_{h,\gamma}^{k} \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\sigma\in I(n)}\left(\langle w, w\rangle \frac{c(c\tau+d)}{2\pi i}\right)^{|m(\sigma)|/2} S_k\left([(c\tau+d)o(w)]^{|f(\sigma)|}; \tau, z\right) \\
&= \sum_{k=1}^{\infty} A_{h,\gamma}^{k} \sum_{p,j\in\mathbb{N}} \frac{1}{(2p+j)!}\sum_{\sigma\in I(2p+j)}\left(\langle w, w\rangle \frac{c(c\tau+d)}{2\pi i}\right)^{p} S_k\left([(c\tau+d)o(w)]^{j}; \tau, z\right) \\
&= \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{p,j\in\mathbb{N}} \frac{1}{(2p+j)!}\binom{2p+j}{j}\frac{(2p)!}{p!2^p}\left(\langle w, w\rangle \frac{c(c\tau+d)}{2\pi i}\right)^{p} \\
&\qquad \cdot S_k\left([(c\tau+d)o(w)]^{j}; \tau, z\right). \qquad (6.2.0.37)
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{1}{(2p+j)!}\binom{2p+j}{j}\frac{(2p)!}{p!2^p} &= \frac{1}{(2p+j)!}\frac{(2p+j)!}{(2p)!j!}\frac{(2p)!}{p!2^p} = \frac{1}{p!j!}\frac{1}{2^p} = \frac{1}{(p+j)!}\frac{(p+j)!}{p!j!}\frac{1}{2^p} \\
&= \frac{1}{(p+j)!}\binom{p+j}{j}\frac{1}{2^p},
\end{aligned}
$$

(6.2.0.37) becomes

$$
\begin{aligned}
&S_h\left(e^{o(w)}; \frac{a\tau+b}{c\tau+d}, \left(\frac{z}{c\tau+d}\right)\right) \\
&= \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{p,j\in\mathbb{N}} \frac{1}{(p+j)!}\binom{p+j}{j}\frac{1}{2^p}\left(\langle w, w\rangle \frac{c(c\tau+d)}{2\pi i}\right)^{p} S_k\left([(c\tau+d)o(w)]^{j}; \tau, z\right) \\
&= \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{p,j\in\mathbb{N}} \frac{1}{(p+j)!}\binom{p+j}{j}\left(\frac{\langle w, w\rangle}{2} \frac{c(c\tau+d)}{2\pi i}\right)^{p} S_k\left([(c\tau+d)o(w)]^{j}; \tau, z\right).
\end{aligned}
$$

(6.2.0.38)

On the other hand,

$$
\begin{aligned}
&\sum_{k=1}^{r} A_{h,\gamma}^{k} S_k \left(\exp\left[\frac{\langle w,w\rangle}{2} \frac{1}{2\pi i} c(c\tau+d) + (c\tau+d)o(w) \right]; \tau, z \right) \\
&= \sum_{k=1}^{r} A_{h,\gamma}^{k} S_k \left(\sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\langle w,w\rangle}{2} \frac{c(c\tau+d)}{2\pi i} + (c\tau+d)o(w) \right)^{m}; \tau, z \right) \\
&= \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} S_k \left(\left(\frac{\langle w,w\rangle}{2} \frac{c(c\tau+d)}{2\pi i} \right)^{m-j} ((c\tau+d)o(w))^{j}; \tau, z \right) \\
&= \sum_{k=1}^{r} A_{h,\gamma}^{k} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{1}{m!} \binom{m}{j} \left(\frac{\langle w,w\rangle}{2} \frac{c(c\tau+d)}{2\pi i} \right)^{m-j} S_k \left(((c\tau+d)o(w))^{j}; \tau, z \right). \quad (6.2.0.39)
\end{aligned}
$$

Replacing m with $p+j$ and noting

$$
\sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{1}{m!} \binom{m}{j} = \sum_{p,j\in\mathbb{N}} \frac{1}{(p+j)!} \binom{p+j}{j},
$$

(6.2.0.38) and (6.2.0.39) can be combined, which completes the proof of the theorem. □

Finally, Theorem 6.2.0.5 is used to find

$$
\begin{aligned}
&\Phi_M(v; 0, w, \gamma\tau) = \mathrm{Tr}_M\, o(v) e^{2\pi i w(0)} e^{2\pi i (L(0)-c/24)\gamma\tau} \\
&= S_M \left(e^{2\pi i w(0)}; (v,z), \gamma\tau \right) \\
&= (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{M,\gamma}^{k} S_k \left(e^{2\pi i (\frac{\langle w,w\rangle}{2} c(c\tau+d) + (c\tau+d)w(0)}; (v,z), \tau \right) \\
&= (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{M,\gamma}^{k}\, \mathrm{Tr}_{M^k}\, o(v) e^{2\pi i (\frac{\langle w,w\rangle}{2} c(c\tau+d) + (c\tau+d)w(0)} q^{L(0)-c/24} \\
&= (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{M,\gamma}^{k}\, \mathrm{Tr}_{M^k}\, o(v) e^{2\pi i (dw(0) + \langle cw, dw\rangle/2)} q^{L(0)+cw(0)+\langle cw,cw\rangle/2 - c/24} \\
&= (c\tau+d)^{[\mathrm{wt}]v} \sum_{k=1}^{r} A_{M,\gamma}^{k} \Phi_k(v; cw, dw, \tau).
\end{aligned}
$$

This completes the proof of Theorem 1.0.0.1.

Part III

Main Results

Chapter 7

Transformation Laws

7.1 Theorem 1.0.0.2

Throughout this section, take V to be a strongly regular vertex operator algebra and let $M^1, \ldots, M^r$ be its inequivalent irreducible admissible modules. Fix $h_1, \ldots, h_n \in V_1$ that satisfy Condition **H** for each module M^j and recall the notation $\underline{h} = (h_1, \ldots, h_n)$. Let G be the Gram matrix $G = (\langle h_i, h_j \rangle)$ associated with the bilinear form $\langle \cdot, \cdot \rangle$ and elements $h_1, \ldots, h_n$.

7.1.1 Proof of Theorem 1.0.0.2

We first prove the transformation law (1.0.0.7) in Theorem (1.0.0.2). Recall the functions Φ (6.1.0.1) and Theorem 1.0.0.1. Note that

$$J_{j,\underline{h}}(v; \tau, \underline{z}) = \Phi_j(v; \underline{z} \cdot \underline{h}, 0, \tau),$$

where $\underline{z} \cdot \underline{h}$ is the dot-product, $\underline{z} \cdot \underline{h} = z_1 h_1 + \cdots + z_n h_n$. By Theorem 1.0.0.1,

$$\begin{aligned} J_{j,\underline{h}}\left(v; \gamma\tau, \frac{\underline{z}}{c\tau + d}\right) &= \Phi_j\left(v; \frac{\underline{z} \cdot \underline{h}}{c\tau + d}, 0, \gamma\tau\right) \\ &= (c\tau + d)^k \sum_{\ell=1}^{r} A^r_{\ell,j} \Phi_\ell(v; c\underline{z} \cdot \underline{h}, d\underline{z} \cdot \underline{h}, \tau). \qquad (7.1.1.1) \end{aligned}$$

Manipulating the terms on the right hand side, we find

$$\begin{aligned} &\Phi_\ell(v; c\underline{z} \cdot \underline{h}, d\underline{z} \cdot \underline{h}, \tau) \\ &\quad = \mathrm{Tr}_{M^\ell}\, o(v) \exp\left[2\pi i \left(d\frac{\underline{z} \cdot \underline{h^j(0)}}{c\tau + d} + cd \sum_{s=1}^{n} \sum_{t=1}^{n} \frac{z_s \langle h_s, h_t \rangle z_t}{2(c\tau + d)^2}\right)\right] \\ &\qquad \cdot \exp\left[2\pi i\tau \left(L(0) + c\frac{\underline{z} \cdot \underline{h^j(0)}}{c\tau + d} + cd \sum_{s=1}^{n} \sum_{t=1}^{n} \frac{z_s \langle h_s, h_t \rangle z_t}{2(c\tau + d)^2} - c/24\right)\right] \\ &\quad = \mathrm{Tr}_{M^\ell}\, o(v) \exp\left(2\pi i \underline{z} \cdot \underline{h^j(0)}\right) \exp\left(2\pi i \sum_{s,t=1}^{n} \frac{c z_s \langle h_s, h_t \rangle z_t}{2(c\tau + d)}\right) q^{L(0) - c/24} \\ &\quad = \exp\left(\pi i \frac{cG[\underline{z}]}{c\tau + d}\right) \mathrm{Tr}_{M^\ell}\, o(v) \zeta_1^{h_1^j(0)} \cdots \zeta_n^{h_n^j(0)} q^{L(0) - c/24}. \qquad (7.1.1.2) \end{aligned}$$

Combining (7.1.1.1) and (7.1.1.2) establishes transformation (1.0.0.7).

While it does not appear that Condition **H** or the assumption $h_i(m)v = 0$ for all $1 \leq i \leq n$ and $m \geq 0$ are necessary for the proof of (1.0.0.7), they are used in the proof of Theorem 1.0.0.1 and are indeed necessary.

Next we prove transformation law (1.0.0.8). Recall Theorem 3.2.4.1 and its notation. In particular, we have the isomorphism (3.2.4.6) of modules,

$$(M^{j'}, Y^{j'}_{\Delta_h(z)}) \cong (M^j, Y^{M^j}).$$

Utilizing the operator $\Delta_{h_i}(z)$ on ω for any h_i, $1 \leq i \leq n$, gives

$$\begin{aligned}\Delta_{h_i}(z)\omega &= \left(z^{h_i(0)} \exp \left\{ -\sum_{k\geq 1} \frac{h_i(k)}{k}(-z)^{-k} \right\} \right) \omega \\ &= z^{h_i(0)} \left(\omega - h(1)\omega(-z)^{-1} + \frac{h(1)^2\omega}{2}(-z)^{-2} \right) \\ &= \omega + h_i z^{-1} + \langle h_i, h_i \rangle z^{-2}. \end{aligned} \tag{7.1.1.3}$$

Therefore, the modes of the conformal vector $\omega_{\Delta_{h_i}}$ on the module $(M^{j'}, Y^{j'}_{\Delta_{h_i}(z)})$ are given by

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \omega_{\Delta_{h_i}}(n) z^{-n-1} &= Y^{j'}_{\Delta_{h_i}(z)}(\omega, z) = Y(\Delta_{h_i}(z)\omega, z) \\ &= \sum_{n\in\mathbb{Z}} \left(\omega(n) z^{-n-1} + h_i(n) z^{-n-2} + \frac{\langle h_i, h_i \rangle}{2} z^{-n-3} \right). \end{aligned}$$

Taking $\mathrm{Res}_z\, z$ of both sides, we find $\omega_{\Delta_{h_i}}(1) = \omega(1) + h_i(0) + \langle h_i, h_i \rangle /2$, or

$$L_{\Delta_{h_i}}(0) = L(0) + h_i(0) + \frac{\langle h_i, h_i \rangle}{2}. \tag{7.1.1.4}$$

In the same way, we find

$$\Delta_{h_i}(z) h_j = h_j + \langle h_i, h_j \rangle z^{-1}$$

for any $1 \leq j \leq n$, and in particular,

$$(h_j)_{\Delta_{h_i}}(0) = h_j(0) + \langle h_i, h_j \rangle. \tag{7.1.1.5}$$

By linearity, (7.1.1.4) and (7.1.1.5) are used to find

$$L_{\Delta_{-\underline{\lambda}\cdot\underline{h}}}(0) = L(0) - \underline{\lambda} \cdot \underline{h} + \frac{1}{2} \sum_{s=1}^{n} \sum_{t=1}^{n} z_s \langle h_s, h_t \rangle z_t, \text{ and} \tag{7.1.1.6}$$

$$(h_j)_{\Delta_{-\underline{\lambda}\cdot\underline{h}}}(0) = h_j(0) - \sum_{s=1}^{n} \lambda_s \langle h_s, h_j \rangle. \tag{7.1.1.7}$$

Finally, with these same calculations applied to $v \in V_{[k]}$ such that $h_i(m)v = 0$ for $1 \leq i \leq n$ and $m \geq 0$, we find that the zero mode of v on $(M^{j'}, Y^{j'}_{\Delta_{-\underline{\lambda}\cdot\underline{h}}(z)})$ is $o_{\Delta_{-\underline{\lambda}\cdot\underline{h}}}(v) = v(k-1) = o(v)$.

Using (7.1.1.6), (7.1.1.7), and the isomorphism $(M^{j'}, Y^{j'}_{\Delta_{-\underline{\lambda}\cdot\underline{h}}(z)}) \cong (M^j, Y^{M^j})$, it follows that

$$\begin{aligned}
&J_{j,\underline{h}}(v;\tau,\underline{z}+\underline{\lambda}\tau+\underline{\mu}) \\
&\quad = \mathrm{Tr}_{M^j}\, o(v) \exp\left[2\pi i \sum_{s=1}^{n} (z_s + \lambda_s\tau + \mu_s)\, h_s(0)\right] \exp(2\pi i\tau(L(0)-c/24)) \\
&\quad = \mathrm{Tr}_{M^j}\, o(v) \exp\left[2\pi i \sum_{s=1}^{n} (z_s + \lambda_s\tau)\, h_s(0)\right] \exp(2\pi i\tau(L(0)-c/24)) \\
&\quad = \mathrm{Tr}_{M^{j'}}\, o(v) \exp\left[2\pi i \sum_{s=1}^{n} (z_s + \lambda_s\tau) \left(h_s(0) - \sum_{t=1}^{n} \lambda_t \langle h_t, h_s\rangle\right)\right] \\
&\qquad\quad \cdot \exp\left[2\pi i\tau\left(L(0) - \underline{\lambda}\cdot\underline{h(0)} + \frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n} \lambda_s \langle h_s, h_t\rangle \lambda_t - c/24\right)\right] \\
&\quad = \mathrm{Tr}_{M^{j'}}\, o(v) \exp\left(2\pi i\underline{z}\cdot\underline{h(0)}\right) \exp\left(2\pi i\tau\underline{\lambda}\cdot\underline{h(0)}\right) \exp\left(-2\pi i \sum_{s=1}^{n}\sum_{t=1}^{n} z_s \langle h_s, h_t\rangle \lambda_t\right) \\
&\qquad\quad \cdot \exp\left(-2\pi i\tau \sum_{s=1}^{n}\sum_{t=1}^{n} \lambda_s \langle h_s, h_t\rangle \lambda_t\right) \exp\left(-2\pi i\tau\underline{\lambda}\cdot\underline{h(0)}\right) \\
&\qquad\quad \cdot \exp\left(\pi i\tau \sum_{s=1}^{n}\sum_{t=1}^{n} \lambda_s \langle h_s, h_t\rangle \lambda_t\right) q^{L(0)-c/24} \\
&\quad = \exp\left(-\pi i\left(G[\underline{\lambda}]\tau + 2\underline{z}^t G\underline{\lambda}\right)\right) J_{j',\underline{h}}(v;\tau,z),
\end{aligned}$$

where the second equality uses the fact $\exp\left(2\pi i\underline{\mu}\cdot\underline{h(0)}\right) = 1$ since $\underline{\mu}\cdot\underline{h(0)}$ on M^j is an integer. This establishes (1.0.0.8).

It remains to prove the convergence for $J_{j,\underline{h}}(v;\tau,\underline{z})$ for any $v \in V$ and establish (1.0.0.6). The technique used to prove convergence is based on work in [8] and [18]. Note that because

$$J_{j,\underline{h}}(v;\tau,z) = q^{\lambda_j - c/24} \sum_{d\geq 0} \mathrm{Tr}_{M^j_{\lambda_j+d}}\, o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)} q^d,$$

the trace is being taken over finite-dimensional vector spaces. The Jordan decomposition can be considered so that $o(v) = o(v)_{ss} + o(v)_n$, where $o(v)_{ss}$ and $o(v)_n$ are the semisimple and nilpotent parts, respectively. Moreover, since the set of operators $\{L(0), h_i(0) \mid 1 \leq i \leq n\}$ are commuting semisimple operators on each M_n, we may choose a simultaneously diagonalizable basis for each M_n. In this case, the operator $o(v)_n \zeta_1^{h_1(0)} \cdots \zeta_n^{h_n(0)} q^{L(0)-c/24}$ is again nilpotent and so it has trace zero. The trace of $o(v)$ is then reduced to the trace of its semisimple part, so that

$$J_{j,\underline{h}}(v;\tau,\underline{z}) = q^{\lambda_j - c/24} \sum_{d \geq 0} \mathrm{Tr}_{M_d^j}\, o(v)_{ss} \zeta_1^{h_1(0)} \cdots \zeta_n^{h_n(0)} q^{L(0)}, \tag{7.1.1.8}$$

where λ_j is the conformal weight of M^j.

Consider the case $n = 1$. That is, take $\underline{h}$ to be a single element $h \in V_1$ that satisfies Condition **H**. Since $o(v)_{ss}$, $h(0)$, and $L(0)$ are mutually commuting semisimple operators on each finite dimensional $M^j_{\lambda_j+d}$, coset representatives $x_1, \ldots, x_m$ of $V/C_2(V)$ may be chosen so that $h(0)x_i = \alpha_i x_i$ and $o(v)_{ss} x_i = \beta_i x_i$ for $\alpha_i, \beta_i \in \mathbb{C}$, $1 \leq i \leq m$. A result of Gaberdiel and Neitzke [18] states that for a C_2-cofinite vertex operator algebra V and irreducible module M^j, the set

$$\left\langle x_{i_1}(s_1) \cdots x_{i_k}(s_k) M^j_{\lambda_j} \mid i_1 > \cdots > i_k \geq 1 \text{ and } s_1, \ldots, s_k \in \mathbb{Z} \right\rangle \tag{7.1.1.9}$$

generates M^j.

Consider $w \in M^j_{\lambda_j}$ and suppose $h(0)w = \alpha w$ and $o(v)_{ss} w = \beta w$ for $\alpha, \beta \in \mathbb{C}$. Set

$$W := \langle x_{i_1}(s_1) \cdots x_{i_k}(s_k) w \mid i_1 > \cdots > i_k \geq 1 \text{ and } s_1, \ldots, s_k \in \mathbb{Z} \rangle .$$

Since $M^j_{\lambda_j}$ is finite dimensional, along with the previous discussion reducing the trace to the semisimple operator $o(v)_{ss}$, it suffices to prove convergence for the function

$$\mathrm{Tr}_W\, o(v)_{ss}\zeta^{h(0)}q^{L(0)-c/24}. \tag{7.1.1.10}$$

Focusing on the case $m = 1$ for a moment, so that W is

$$W = \langle x_1(s_1)\cdots x_1(s_k)w \mid s_1, \ldots, s_k \in \mathbb{Z}\rangle\,,$$

we have the inequality (see the proof of Proposition 10 in [18] for the case $o(v)_{ss}\zeta^{h(0)} = 1$)

$$\mathrm{Tr}_W\, o(v)_{ss}\zeta^{h(0)}q^{L(0)-c/24} \le q^{\lambda_j - c/24}\beta\zeta^{\alpha}\prod_{p\ge 1}\left(1-\beta_1\zeta^{\alpha_1}q^p\right)^{-1}.$$

Incorporating $x_1, \ldots, x_m$ when $m \ge 1$ and keeping track of the additional eigenvalues, we generalize the above result (see Proposition 1.8 in [8] for the case $o(v)_{ss} = 1$) so that

$$\begin{aligned}\mathrm{Tr}_W\, o(v)_{ss}\zeta^{h(0)}q^{L(0)-c/24} &\le q^{\lambda_j - c/24}\beta\zeta^{\alpha}\prod_{k=1}^{m}\prod_{p\ge 1}\left(1-\beta_k\zeta^{\alpha_k}q^p\right)^{-1}\\ &= q^{\lambda_j - c/24}\beta\zeta^{\alpha}\prod_{k=1}^{m}\prod_{p\ge 1}\left(1-|\beta_k\zeta^{\alpha_k}q^p|\right)^{-1}.\end{aligned} \tag{7.1.1.11}$$

Therefore, to prove the convergence of $J_{j,h}(v;\tau,z)$ on $\mathbb{H}\times\mathbb{C}$, we need only prove that $\prod_{p\ge 1}\left(1-|\beta_k\zeta^{\alpha_k}q^p|\right)^{-1}$ converges on the same domain for each $1 \le k \le m$.

The convergence of this product is equivalent to the logarithm of the product, which equals

$$-\sum_{p=1}^{\infty}\log(1+|\beta_k\zeta^{\alpha_k}q^p|). \tag{7.1.1.12}$$

Set $\tau = x+iy$, $z\alpha_k = a+ib$, and $\beta_k = r+is$, with $x, y, a, b, r, s \in \mathbb{R}$ and $y > 0$. Then

$$|q^p\zeta^{\alpha_k}\beta_k| = |q^p|\,|\zeta^{\alpha_k}|\,|\beta_k| = e^{-2\pi yp}e^{-2\pi b}(\sqrt{r^2+s^2}).$$

Since $|q^p\zeta^{\alpha_k}\beta_k| \to 0$ as $p \to \infty$, it follows that

$$\lim_{p\to\infty} \frac{\log(1+|q^p\zeta^{\alpha_k}\beta_k|)}{|q^p\zeta^{\alpha_k}\beta_k|} = 1.$$

By the limit comparison test, (7.1.1.12) converges if, and only if, $\sum_{p=1}^{\infty} |q^p\zeta^{\alpha_k}\beta_k|$ converges. However, this is true since

$$\sum_{p=1}^{\infty} |q^p\zeta^{\alpha_k}\beta_k| = |\beta_k\zeta^{\alpha_k}| \sum_{p=1}^{\infty} |q|^p$$

converges as $|q| < 1$. The convergence of the function $J_{j,h}(v;\tau,z)$ has now been proved.

To prove the convergence for the function $J_{j,\underline{h}}(v;\tau,\underline{z})$, we fix all but one of the complex variables $z_1,\ldots,z_n$ and apply the previous argument. Since the convergence can be established in this matter for each individual complex variable, Hartogs' Theorem in complex analysis (see for example, Theorem 2.2.8 in [21]) gives that $J_{j,\underline{h}}(v;\tau,\underline{z})$ converges on $\mathbb{H}\times\mathbb{C}^n$.

Finally since

$$\begin{aligned} J_{j,\underline{h}}(v;\tau,\underline{z}) &= \mathrm{Tr}_{M^j}\, o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24} \\ &= q^{\lambda_j-c/24}\sum_{d\geq 0}\mathrm{Tr}_{M^j_{\lambda_j+s}}\, o(v)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^d, \end{aligned}$$

and each $h_i(0)$ has integral eigenvalues on M^j, an expansion of the form (1.0.0.6) is obtained. This completes the proof of Theorem 1.0.0.2. □

7.2 Theorems 1.0.0.3 and 1.0.0.4

This section addresses the transformation laws of the functions $J_{j,\underline{h}}(v;\tau,\underline{z})$ when $h_i(m)v \neq 0$ for some $1 \leq i \leq n$ or $m \geq 0$.

7.2.1 A decomposition of a strongly regular VOA

Let V be a strongly regular vertex operator algebra and M be an irreducible admissible V-module. A result obtained by Dong and Mason [11, 34] states that the Lie algebra V_1 is *reductive* and that its action on M is *completely reducible.* The action of $u \in V_1$ on M is given by $u(0)$.

We fix a Cartan subalgebra H in V_1 of rank d. Consider the Heisenberg subVOA $M_H = (\langle H \rangle, Y, \mathbf{1}, \omega_H)$ generated by H. If $u_1, \ldots, u_d$ is a basis for H, then M_H has central charge $d = \dim H$, Fock space

$$\langle u_{i_1}(m_1) \cdots u_{i_k}(m_k)\mathbf{1} \mid i_j \in \{1, \ldots, d\}, m_j \in \mathbb{Z} \rangle,$$

and conformal vector

$$\omega_{M_H} := \frac{1}{2} \sum_{i=1}^{d} u_i(-1)u_i.$$

(Recall Subsection 3.2.5.) Let Ω_M be a subset of M defined by

$$\Omega_M := \{w \in M \mid u(n)w = 0, \text{ for } u \in H \text{ and } n \geq 1\},$$

and for $\beta \in H$ set

$$M(\beta) := \{w \in M \mid u(0)w = \langle \beta, u \rangle w, \text{ where } u \in H\}.$$

Consider the set

$$P := \{\beta \in H \mid M(\beta) \neq 0\},$$

which is a subgroup of H. Then M has a decomposition

$$M \cong M_H \otimes \Omega_M = \bigoplus_{\beta \in P} M_H \otimes \Omega_M(\beta) \tag{7.2.1.1}$$

where $\Omega_M(\beta) := \Omega_M \cap M(\beta)$ (see [34]).

We define the *commutant* of M_H in M by

$$C_M(M_H) = \ker_M L_H^M(-1).$$

Then $C_V(M_H) = \Omega_V(0) := \Omega(0)$. It is known that $\Omega(0)$ is a simple vertex operator algebra and $\Omega_M(\beta)$ are irreducible $\Omega(0)$-modules. It is also known that M_H is a simple vertex operator algebra with irreducible modules $M_H(\beta)$, where we've made the identification

$$M_H(\beta) \cong M_H \otimes e^\beta,$$

with $e^\beta \in \Omega(\beta)$. Therefore, the tensor product $M_H(\beta) \otimes \Omega_M(\beta)$ is an irreducible $M_H \otimes \Omega(0)$-module. Note also that $M(\beta) = M_H(\beta) \otimes \Omega_M(\beta)$. Set

$$L_0 = \{u \in H \mid u(0) \text{ has eigenvalues in } \mathbb{Z}\},$$

and recall the isomorphism (3.2.4.6) which holds for all $u \in L_0$. Set

$$\Lambda := \left\{u \in L_0 \mid (M^j, Y^j_{\Delta_u(z)}) \cong (M^j, Y^j)\right\}. \tag{7.2.1.2}$$

This implies

$$\Omega_M(\beta) \cong \Omega_M(\beta + u), \tag{7.2.1.3}$$

where $u \in \Lambda$ and $\beta \in P$. In the case $\beta = 0$, this gives $\Omega_M(u) \cong \Omega_M(0)$ for all $u \in \Lambda$. Therefore, $\Omega_M(u) \neq 0$ and $\Lambda \subseteq P$. In [34], it is shown that Λ is a positive-definite integral lattice of rank d and $|P\colon \Lambda|$ is finite. We set $\delta := |P\colon \Lambda|$.

The decomposition (7.2.1.1) may now be written

$$\begin{aligned} M &= \bigoplus_{i=1}^{\delta} \bigoplus_{\beta \in \Lambda} M_H(\beta + \gamma_i) \otimes \Omega_M(\gamma_i) \\ &= \bigoplus_{i=1}^{\delta} \bigoplus_{\beta \in \Lambda + \gamma_i} M_H(\beta) \otimes \Omega_M(\beta) \end{aligned} \tag{7.2.1.4}$$

where $\{\gamma_i \mid 1 \leq i \leq \delta\}$ are coset representatives of P/Λ.

7.2.2 Proof of Theorem 1.0.0.3

Let $\{u_i \mid 1 \leq i \leq d\}$ be a basis for H. By the above decomposition of V, any element in V may be written as sums of elements of the form $v = u_1[m_1] \cdots u_k[m_k] \otimes e^{\alpha} \otimes w$, $w \in \Omega(\alpha)$, for various $\alpha \in \Lambda + \gamma_j$, $1 \leq j \leq d$, $m_1, \ldots, m_k \in \mathbb{Z}$. Note that $v(n) M_H(\beta) \otimes \Omega_M(\beta) \subseteq M_H(\alpha + \beta) \otimes \Omega_M(\alpha + \beta)$. Therefore, the only v such that $J_{j,\underline{h}}(v; \tau, \underline{z}) \neq 0$, are those that are a sum containing parts which lie in $M_H(0) \otimes \Omega(0)$. It therefore suffices to consider elements of the form

$$v = u_1[-m_1] \cdots u_k[-m_k] \otimes e^0 \otimes w = u_1[-m_1] \cdots u_k[-m_k] \otimes w,$$

where $w \in \Omega(0)$, $m_1, \ldots, m_k \in \mathbb{N}$.

Since $w \in \Omega(0)$, it satisfies $h_i(0)w = \langle h_i, w \rangle w = 0$ for all $1 \leq i \leq n$, and $h_i(m)w = 0$ for all $m \geq 0$. Therefore, $h_i(m)w = 0$ for all $m \geq 0$ and $J_{j,\underline{h}}(w; \tau, \underline{z})$ satisfies the assumptions of Theorem 1.0.0.2.

The following lemma and proof mimic those found in [13].

Lemma 7.2.2.1 *Let $a \in M_H$. Consider an element $a^{\ell}[-1]w \in V$, $\ell \geq 0$, $w \in \Omega(0)$, and let $\alpha \in \Lambda + \gamma_j$ for some $1 \leq j \leq \delta$. Then there are scalars $c_{\ell,\ell-2i}$ with $0 \leq i \leq \ell/2$ and $c_{\ell,\ell} = 1$ such that*

$$\begin{aligned} &J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(a[-1]^{\ell} w; \tau, \underline{z}) \\ &\quad = \left(\sum_{0 \leq i \leq \ell/2} c_{\ell,\ell-2i} \langle a, \alpha \rangle^{\ell-2i} \left(\langle a, a \rangle E_2(\tau) \right)^i \right) J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(w; \tau, \underline{z}). \end{aligned}$$

Proof The proof is by induction on ℓ. The case $\ell = 0$ is clear. Suppose the result holds for all k, $0 \le k \le \ell$. The recursion formula in Lemma 5.2.1.2 gives

$$\begin{aligned}
&J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(a[-1]^\ell w;\tau,\underline{z}) \\
&\qquad = \mathrm{Tr}_{M_H(\alpha)\otimes\Omega_M(\alpha)}\, o(a)o(a[-1]^{\ell-1}w)\zeta_1^{h_1(0)}\cdots\zeta_n^{h_n(0)}q^{L(0)-c/24} \\
&\qquad\qquad + (\ell-1)\langle a,a\rangle E_2(\tau) J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(a[-1]^{\ell-2}w;\tau,\underline{z}) \\
&\qquad = \langle a,\alpha\rangle J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(a[-1]^{\ell-1}w;\tau,\underline{z}) \\
&\qquad\qquad + (\ell-1)\langle a,a\rangle E_2(\tau) J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(a[-1]^{\ell-2}w;\tau,\underline{z}), \qquad (7.2.2.1)
\end{aligned}$$

where the $E_2(\tau)$ occur because $h_i(0)a = 0$ for all i, so that $\tilde{E}_2(\tau,0) = E_2(\tau)$. Applying the induction hypothesis on

$$J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(a[-1]^{\ell-1}w;\tau,\underline{z}) \quad \text{and} \quad J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(a[-1]^{\ell-2}w;\tau,\underline{z}),$$

we find

$$\begin{aligned}
&J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(a[-1]^\ell w;\tau,\underline{z}) \\
&= \langle a,\alpha\rangle \left(\sum_{0\le i\le(\ell-1)/2} c_{\ell-1,\ell-1-2i}\langle a,\alpha\rangle^{\ell-1-2i}\left(\langle a,a\rangle E_2(\tau)\right)^i\right) J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z}) \\
&\quad + (\ell-1)\langle a,a\rangle E_2(\tau)\left(\sum_{0\le i\le(\ell-2)/2} c_{\ell-2,\ell-2-2i}\langle a,\alpha\rangle^{\ell-2-2i}\left(\langle a,a\rangle E_2(\tau)\right)^i\right) \\
&\qquad\qquad \cdot J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z}) \\
&= \left(\sum_{0\le i\le \ell/2} c_{\ell,\ell-2i}\langle a,\alpha\rangle^{\ell-2i}\left(\langle a,a\rangle E_2(\tau)\right)^i\right) J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z}),
\end{aligned}$$

as desired. The last equality follows because

$$\langle a,\alpha\rangle\left(\sum_{0\le i\le(\ell-1)/2} c_{\ell-1,\ell-1-2i}\,\langle a,\alpha\rangle^{\ell-1-2i}\,(\langle a,a\rangle\,E_2(\tau))^i\right)$$
$$+(\ell-1)\,\langle a,a\rangle\,E_2(\tau)\left(\sum_{0\le i\le(\ell-2)/2} c_{\ell-2,\ell-2-2i}\,\langle a,\alpha\rangle^{\ell-2-2i}\,(\langle a,a\rangle\,E_2(\tau))^i\right)$$
$$=\sum_{0\le i\le r/2}(c_{\ell-1,\ell-1-2i}+(\ell-1)c_{\ell-2,\ell-2i})\,\langle a,\alpha\rangle^{\ell-2i}\,(\langle a,a\rangle\,E_2(\tau))^i,$$

so that $c_{\ell,\ell-2i}:=c_{\ell-1,\ell-1-2i}+(\ell-1)c_{\ell-2,\ell-2i}$. $\square$

Note that

$$J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z})=J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(1\otimes w;\tau,\underline{z})$$
$$=J_{M_H(\alpha),\underline{h}}(\mathbf{1};\tau,\underline{z})\cdot J_{\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z}).$$

Equation (5.3.0.4) gives

$$J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z})=\frac{\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle}}{\eta(\tau)^d}J_{\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z}).\tag{7.2.2.2}$$

Setting

$$f_{a,\alpha,\ell(\tau)}:=\sum_{o\le i\le\ell/2} c_{\ell,\ell-2i}\,\langle a,\alpha\rangle^{\ell-2i}\,(\langle a,a\rangle\,E_2(\tau))^i,$$

and combining (7.2.2.1) and (7.2.2.2) establishes

$$J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(a[-1]^\ell w;\tau,\underline{z})$$
$$=f_{a,\alpha,\ell}(\tau)\frac{\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2}}{\eta(\tau)^d}J_{\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z}).\tag{7.2.2.3}$$

We take $u_1,\ldots,u_d$ to be an orthogonal basis for H and let $\ell_1,\ldots\ell_d$ be non-negative integers. Then for $v=u_1[-1]^{\ell_1}\cdots u_d[-1]^{\ell_d}w$, Lemma 7.2.2.1 implies

$$J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(v;\tau,\underline{z})=f_{u_1,\alpha,\ell_1}(\tau)\cdots f_{u_d,\alpha,\ell_d}(\tau)\frac{\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2}}{\eta(\tau)^d}J_{\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z}).$$

Recalling the module decomposition (7.2.1.4) for M^j, it follows that

$$\begin{aligned} &J_{j,\underline{h}}(v;\tau,\underline{z}) \\ &= \sum_{j=1}^{\delta} \sum_{\alpha\in\Lambda+\gamma_j} J_{M_H(\alpha)\otimes\Omega_M(\alpha),\underline{h}}(v;\tau,\underline{z}) \\ &= \sum_{j=1}^{\delta} \sum_{\alpha\in\Lambda+\gamma_j} f_{u_1,\alpha,\ell_1}(\tau)\cdots f_{u_d,\alpha,\ell_d}(\tau)\frac{\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2}}{\eta(\tau)^d} J_{\Omega_M(\alpha),\underline{h}}(w;\tau,\underline{z}). \end{aligned} \tag{7.2.2.4}$$

Each $\alpha\in\Lambda+\gamma_j$ may be written as $\alpha=u+\gamma_j$ for some $u\in\Lambda$. The isomorphism (7.2.1.3) then shows

$$\Omega_M(\alpha)=\Omega_M(u+\gamma_j)\cong\Omega_M(\gamma_j).$$

Therefore, (7.2.2.4) becomes

$$\begin{aligned} &J_{j,\underline{h}}(v;\tau,\underline{z}) \\ &\quad= \sum_{j=1}^{\delta} J_{\Omega_M(\gamma_j),\underline{h}}(w;\tau,\underline{z}) \sum_{\alpha\in\Lambda+\gamma_j} f_{u_1,\alpha,\ell_1}(\tau)\cdots f_{u_d,\alpha,\ell_d}(\tau)\frac{\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2}}{\eta(\tau)^d}. \end{aligned} \tag{7.2.2.5}$$

Recalling that

$$f_{a,\alpha,\ell}(\tau)=\sum_{o\le i\le \ell/2} c_{\ell,\ell-2i}\left\langle a,\alpha\right\rangle^{\ell-2i}\left(\left\langle a,a\right\rangle E_2(\tau)\right)^i,$$

we pull out the $c_{\ell,\ell-2i}(\langle a,a\rangle E_2(\tau))^i$ terms from (7.2.2.5), which are not dependent on α. Equation (7.2.2.5) is now of the form

$$\begin{aligned} &\sum_{j}^{\delta}\frac{J_{\Omega_M(\gamma_j),\underline{h}}(w;\tau,\underline{z})}{\eta(\tau)^d}\sum_{i_1,\dots,i_d=0}^{\ell/2} c_{\ell_1,\ell_1-2i_1}\cdots c_{\ell_d,\ell_d-2i_d}(\langle u_1,u_1\rangle E_2(\tau))^{i_1}\cdots(\langle u_d,u_d\rangle E_2(\tau))^{i_d} \\ &\qquad\cdot\sum_{\alpha\in\Lambda+\gamma_j}\langle u_1,\alpha\rangle^{\ell_1-2i_1}\cdots\langle u_d,\alpha\rangle^{\ell_d-2i_d}\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2}. \end{aligned} \tag{7.2.2.6}$$

However, the functions

$$\sum_{\alpha\in\Lambda+\gamma_j}\langle u_1,\alpha\rangle^{\ell_1-2i_1}\cdots\langle u_d,\alpha\rangle^{\ell_d-2i_d}\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2} \tag{7.2.2.7}$$

are linear combinations of functions of the form

$$\sum_{\alpha\in\Lambda+\gamma_j} \langle b,\alpha\rangle^{\ell_1+\cdots+\ell_d-2(i_1+\cdots+i_d)}\,\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2}, \tag{7.2.2.8}$$

for various $b\in H$. These are the functions $\theta_{\underline{h}}(Q,b,\tau,\underline{z})$ considered in Section 4.3. This proves Theorem 1.0.0.3. □

7.2.3 Proof of Theorem 1.0.0.4

Suppose $h_1,\ldots,h_n$ is an orthogonal basis for H. Then we may duplicate the previous proof with the set $\{u_i\}$ replaced with $\{h_i\}$. Display (7.2.2.7) now reads

$$\sum_{\alpha\in\Lambda+\gamma_j} \langle h_1,\alpha\rangle^{\ell_1-2i_1}\cdots\langle h_n,\alpha\rangle^{\ell_n-2i_n}\,\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2}. \tag{7.2.3.1}$$

We have

$$\begin{aligned}\sum_{\alpha\in\Lambda+\gamma_j} &\langle h_1,\alpha\rangle^{\ell_1-2i_1}\cdots\langle h_n,\alpha\rangle^{\ell_n-2i_n}\,\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2}\\ &=\left(\frac{1}{2\pi i}\frac{d}{dz_1}\right)^{\ell_1-2i_1}\cdots\left(\frac{1}{2\pi i}\frac{d}{dz_n}\right)^{\ell_n-2i_n}\sum_{\alpha\in\Lambda+\gamma_j}\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2}.\end{aligned}$$

We know the function

$$\sum_{\alpha\in\Lambda+\gamma_j}\zeta_1^{\langle\alpha,h_1\rangle}\cdots\zeta_n^{\langle\alpha,h_n\rangle}q^{\langle\alpha,\alpha\rangle/2},$$

is a Jacobi form by considering the $n=0$ case in Part 1 of Theorem 4.3.4.1 (or also by a direct proof). Since the space of quasi-Jacobi forms $\mathcal{Q}^{\mathcal{J}}$ is invariant under partial derivatives (cf. Lemma 4.2.3.1 above) and the quasi-modular form $E_2(\tau)$, it follows that $\sum_r f_r(\tau)\theta_{j,\underline{h}}(Q,b,\tau,\underline{z})$ is a sum of quasi-Jacobi form, and the theorem is proved. □

7.3 Consequences

In this section we examine some consequences from the results developed in the previous two sections. We find the following corollaries to Theorem 1.0.0.2.

Corollary 7.3.0.1 *Let V be a strongly regular VOA and let $M^1, \ldots, M^r$ its inequivalent irreducible modules. Suppose $h_1, \ldots, h_n$ are chosen in V_1 that satisfy Condition* ***H*** *and are such that G is a real, symmetric, positive definite, half-integral $n \times n$ matrix. Then for $v \in V_{[k]}$ such that $h_i(m)v = 0$ for $1 \le i \le n$ and $m \ge 0$, we have that the column vector of functions*

$$\mathbf{J}(\tau, \underline{z}) := (J_{1,\underline{h}}(v; \tau, \underline{z}), \ldots, J_{r,\underline{h}}(v; \tau, \underline{z}))^t$$

is a vector-valued weak Jacobi form of weight k and index $G/2$.

Remark: For any strongly regular vertex operator algebra V such that $V_1 \neq 0$ we may choose $h_1, \ldots, h_n \in V_1$ that satisfy Condition **H**. In particular, any subset of a Cartan subalgebra of V_1 will do once appropriately normalized. Moreover, it is always possible to find $h_1, \ldots, h_n$ that simultaneously satisfy Condition **H** and are such that G satisfies the assumptions in Corollary 7.3.0.1 (see [27, 34]).

In the special case that $r = 1$ in the corollary above, V is a holomorphic vertex operator algebra and we have the following result.

Corollary 7.3.0.2 *Let V be a holomorphic strongly regular vertex operator algebra with $h_1, \ldots, h_n$ as in the previous corollary. For $v \in V_{[k]}$ such that $h_i(m)v = 0$ for $1 \le i \le n$ and $m \ge 0$, we have the following functional equations for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and*

$[\underline{\lambda}, \underline{\mu}] \in \mathbb{Z}^n \times \mathbb{Z}^n$:

1.

$$J_{\underline{h}}\left(v; \frac{a\tau+b}{c\tau+d}, \frac{\underline{z}}{c\tau+d}\right) = \exp\left(\pi i \frac{cG[\underline{z}]}{c\tau+d}\right)\chi(\gamma)J_{\underline{h}}(v;\tau,\underline{z}) \tag{7.3.0.2}$$

(where $\chi\colon \Gamma \to \mathbb{C}^*$ is a character), and

2.

$$J_{\underline{h}}\left(v;\tau,\underline{z}+\underline{\lambda}\tau\underline{\mu}\right) = \exp\left(\pi i(G[\underline{\lambda}]\tau + 2\underline{\lambda}^t G\underline{z})\right) J_{\underline{h}}(v;\tau,\underline{z}).$$

Remark: The previous corollary says that $J_{\underline{h}}(v;\tau,\underline{z})$ is a weak Jacobi form on Γ of weight k, index $G/2$, and character χ. Multiplying $J_{\underline{h}}(v;\tau,\underline{z})$ by $\eta(\tau)^{c/24}$ eliminates the character χ and results in a holomorphic weak Jacobi form of weight $k+c/2$ and index $G/2$. The function $\eta(\tau)^{c/24}J_{\underline{h}}(v;\tau,\underline{z})$ has the Fourier-Jacobi expansion

$$\eta(\tau)^{c/24}J_{\underline{h}}(v;\tau,\underline{z}) = \sum_{n\geq 0}\sum_{t_1,\ldots,t_n\in\mathbb{Z}} c(n,t_1,\ldots,t_n)q^n\zeta_1^{t_1}\cdots\zeta_n^{t_n}.$$

The case of Corollary 7.3.0.1 in which $v = \mathbf{1}$ and $n = 1$ is considered in [27]. In this case, $h_i(m)\mathbf{1} = 0$ for all $1 \leq i \leq n$ and $m \geq 0$, while $G = \langle h, h\rangle$. Using properties of Jacobi forms, the following result is found in [27] that illustrates that (true) Jacobi forms (not necessarily weak) can occur. (See Supplement to Theorem 2 in [27].)

Theorem 7.3.0.3 *Consider the case $n = 1$ in Corollary 7.3.0.2. If $\langle h, h\rangle \leq 8$, then $\eta(\tau)^{c/24}J_h(\mathbf{1},\tau,z)$ is a Jacobi form of weight 0 and index $\langle h,h\rangle/2$ on the full Jacobi group.*

Proof Recall the notation in Chapter 4. In particular, $\mathcal{M}_k$ is the space of modular forms of weight k and $\mathcal{J}^1_{k,m}$ is the space of weak Jacobi forms of weight k and index m (with one complex variable). We define the functions

$$\tilde{\phi}_{-2,1} = \frac{\phi_{10,1}}{\Delta} = (\zeta - 2 + \zeta^{-1}) + \cdots$$

and

$$\tilde{\phi}_{0,1} = \frac{\phi_{12,1}}{\Delta} = (\zeta + 10 + \zeta^{-1}) + \cdots ,$$

where $\phi_{10,1}, \phi_{12,1}$, and Δ are defined in [14] (the details not being necessary here). The functions $\tilde{\phi}_{-2,1}$ and $\tilde{\phi}_{0,1}$ are weak Jacobi forms of index 1 and weights -2 and 0, respectively [14]. Let $\widehat{\mathcal{J}}^1_{k,m}$ denote the space of (true) Jacobi forms of weight k and index m (and only one complex variable) Note that $\widehat{\mathcal{J}}^1_{k,m} \subseteq \mathcal{J}^1_{k,m}$.

We now quote two results from [14]. First, for even k there is a linear isomorphism ([14], page 108)

$$\begin{aligned} P\colon \mathcal{M}_k \oplus \mathcal{M}_{k+2} \oplus \cdots \oplus \mathcal{M}_{k+2m} &\to \mathcal{J}^1_{k,m} \\ (f_0, f_1, \ldots, f_m) &\mapsto \sum_{i=0}^{m} f_i \tilde{\phi}^i_{-2,1} \tilde{\phi}^{m-i}_{0,1}. \end{aligned} \tag{7.3.0.3}$$

Second, for $k \geq 3$ we have (see Theorem 9.2, its corollary, and Section 10 in [14]),

$$\dim \widehat{\mathcal{J}}^1_{k,m} = \dim \mathcal{J}^1_{k,m} - \sum_{\nu=0}^{m} \left\lceil \frac{\nu^2}{4m} \right\rceil. \tag{7.3.0.4}$$

Here $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

We claim that $\phi = \sum_{n,r\in\mathbb{Z}} c(n,r) q^n \zeta^r$ is a Jacobi form if, and only if, $c(0,r) = 0$ for $r \neq 0$. Suppose ϕ is a Jacobi form. Then $c(n,r) = 0$ unless $n \geq 0$ and $r^2 \leq 4mn$. If $n = 0$, then $r^2 \leq 0$, so that $r = 0$. In other words, $c(0,r) = 0$ for $r \neq 0$. It remains to show that $c(0,r) = 0$ for $r \neq 0$ implies ϕ is a Jacobi form.

Set $E = \left\{ \phi = \sum_{n,r\in\mathbb{Z}} c(n,r) q^n \zeta^r \mid c(0,r) = 0 \text{ for } r \neq 0 \right\}$. First let us suppose that

E has codimension m in $\mathcal{J}^1_{k,m}$. If $1 \leq m \leq 4$, then

$$\sum_{\nu=0}^{m} \left\lceil \frac{\nu^2}{4m} \right\rceil = m,$$

so that (7.3.0.4) shows $\mathcal{J}^1_{k,m}$ has codimension m. By our argument in the previous paragraph, we have $\widehat{\mathcal{J}}^1_{k,m} \subseteq E \subseteq \mathcal{J}^1_{k,m}$. It follows that $E = \widehat{\mathcal{J}}^1_{k,m}$, which completes the proof of the claim. We must now prove that E has codimension m.

Recall the Eisenstein series (4.1.2.6). For convenience, we now normalize these Eisenstein series so that the constant term is 1 (as in [14]). Let E_i be the Eisenstein series $E_i = 1 + \cdots$ in $\mathcal{M}_{k+2i}$, and let $P(E_i)$, $1 \leq i \leq m$ be the weak Jacobi form defined by

$$P(E_i) = E_i \tilde{\phi}^i_{-2,1} \tilde{\phi}^{m-i}_{0,1} = (\zeta - 2 + \zeta^{-1})^i (\zeta + 10 + \zeta^{-1})^{m-i} + O(q),$$

where $O(q)$ is terms involving powers of q. Let $x = \zeta + \zeta^{-1}$. Then

$$P(E_i) = (x-2)^i (x+10)^{m-i} + O(q).$$

We claim that the polynomials $(x-2)^i(x+10)^{m-i}$, $0 \leq i \leq m$, form a basis for the polynomials of degree at most m in $\mathbb{Q}[x]$. We will prove this by induction on m. The case $m = 0$ is clear, and we assume the claim holds for all i, $0 \leq i \leq m-1$. Considering the case $0 \leq i \leq m$, we see

$$(x-2)^i(x+10)^{m-i} = x^m + Q(x),$$

where $Q(x)$ is a polynomial in $\mathbb{Q}[x]$ of degree at most $m-1$. By our induction hypothesis, $Q(x)$ is a linear combination of polynomials of the form $(x-2)^j(x+10)^{m-i}$, $0 \leq j \leq m-1$. Therefore,

$$x^m = (x-2)^i(x+10)^{m-i} - Q(x)$$

is a linear combination of elements of the form $(x-2)^i(x+10)^{m-i}$, $0 \le i \le m$, and the claim is proved.

It follows that there is some linear combination of the functions $P(E_i)$ that equals each element $x^i + O(q)$ for all $0 \le i \le m$. Moreover, these functions span all weak Jacobi forms that do not satisfy $c(0,r) = 0$ for $r \neq 0$. Since there are $m+1$ many of these elements, we must have that E has codimension m, as desired.

We have now proved that for $k \ge 4$ and $1 \le m \le 4$, $\phi = \sum_{n,r\in\mathbb{Z}} c(n,r)q^n\zeta^r$ is a Jacobi form if, and only if, $c(0,r) = 0$ for $r \neq 0$. We know that $\eta(\tau)^{c/24}J(\tau,z)$ has weight $c/2$ and index $\langle h,h\rangle/2$. Since V is holomorphic, $c/2$ is an integer divisible by 4 [47]. Therefore, the weight is indeed greater than or equal to 4. Since the index of $\eta(\tau)^{c/24}J(\tau,z)$ is $\langle h,h\rangle/2$ and $\langle h,h\rangle$ is assumed to be less than or equal to 8, $\eta(\tau)^{c/24}J(\tau,z)$ is in fact a Jacobi form, as desired. □

7.4 Examples

7.4.1 Holomorphic vertex operator algebras

Throughout this subsection let V be a holomorphic strongly regular vertex operator algebra and let $h \in V_1$ satisfy condition **H**.

It is clear that $\mathbf{1}$ satisfies the condition $h(m)\mathbf{1} = 0$ for all $m \ge 0$. Therefore (so long as $V_1 \neq 0$) h may be chosen so that $J_h(\mathbf{1};\tau,z)$ is a weak Jacobi form of weight 0 and index $\langle h,h\rangle/2$ (this is the result of Corollary 7.3.0.2). We now consider nontrivial elements whose trace functions give rise to weak Jacobi forms and weak quasi-Jacobi forms.

First, let $v \in V_{[k]}$ be such that $h(m)v = 0$ for all $m \ge 0$. Consider the element

$\left(L[-2] - \frac{1}{2\langle h,h\rangle}h[-1]^2\right) v \in V_{[k+2]}$. We find for $m \geq 0$ that

$$\begin{aligned} &h[m]\left(L[-2] - \frac{1}{2\langle h,h\rangle}h[-1]^2\right) v \\ &\quad = L[-2]h[m]v - [L[-2], h[m]]v - \frac{1}{2\langle h,h\rangle}h[-1]h[m]h[-1]v - \delta_{m,1}\frac{1}{2}h[-1]v \\ &\quad = mh[m-2]v - \delta_{m,1}h[-1]v. \end{aligned}$$

It follows that $h[m]\left(L[-2] - \frac{1}{2\langle h,h\rangle}h[-1]^2\right) v = 0$ for $m \geq 0$. By (3.1.1.2) we have

$$h(m)\left(L[-2] - \frac{1}{2\langle h,h\rangle}h[-1]^2\right) v = 0$$

for all $m \geq 0$. Theorem 1.0.0.7 now says that

$$J_j\left(\left(L[-2] - \frac{1}{2\langle h,h\rangle}h[-1]^2\right) v; \tau, z\right)$$

is a weak Jacobi form of weight $k+2$ and index $\langle h,h\rangle/2$. Take $v = \mathbf{1}$. Then $h(m)\mathbf{1} = 0$ for $m \geq 0$, and it follows that $J\left(\left(L[-2] - \frac{1}{2\langle h,h\rangle}h[-1]^2\right)\mathbf{1}; \tau, z\right)$ is a weak Jacobi form of weight 2 and index $\langle h,h\rangle/2$. Reiterating this process, we find that

$$J\left(\left(L[-2] - \frac{1}{2\langle h,h\rangle}h[-1]^2\right)^{\ell}\mathbf{1}; \tau, z\right)$$

is a weak Jacobi form of weight 2ℓ and index $\langle v,v\rangle/2$.

Note that if $v \in V_{[k]}$ satisfies $h(m)v = 0$ for $m \geq 0$ and $L[n]v = 0$ for $n \geq 1$, then

$$\begin{aligned} &J\left(\left(L[-2] - \frac{1}{2\langle h,h\rangle}h[-1]^2\right) v; \tau, z\right) \\ &\qquad = \left(\frac{1}{2\pi i}\frac{d}{d\tau} - \frac{1}{2\langle h,h\rangle}\left(\frac{1}{2\pi i}\frac{d}{dz}\right)^2 + \frac{2k-1}{2}E_2(\tau)\right) J(v;\tau,z) \\ &\qquad = \mathcal{L}_k J(v;\tau,z) \end{aligned}$$

(see [17] for another use of this operator). In particular this holds for $v = \mathbf{1}$. Here, $\mathcal{L}_k$ is the differential operator defined in (4.2.2.1) and is known by Lemma 4.2.2.1 to map weak

Jacobi forms of weight k and index m to weak Jacobi forms of weight $k+2$ and index m (see also [41]).

Consider the element $L[-2]\mathbf{1} \in V_{[2]}$. We have $h[1]L[-2]\mathbf{1} = h \neq 0$ so that we can not apply Theorem 1.0.0.2. Recall the modular derivative ∂_k (4.1.2.10). It is known (i.e., [7, 46]) that

$$J(L[-2]\mathbf{1}, \tau, z) = \partial_k J(\mathbf{1}; \tau, z) = \left(\frac{1}{2\pi i}\frac{d}{d\tau} + kE_2(\tau)\right) J(\mathbf{1}, \tau, z).$$

We know that $J(\mathbf{1}, \tau, z)$ is a weak Jacobi form of weight 0 and index $\langle h, h\rangle /2$. Recall the space of quasi-Jacobi forms discussed in Subsection 4.2.3. Since

$$\mathcal{Q}^{\mathcal{J}^1} = \mathbb{C}[E_2(\tau), E_n(\tau, z), n \geq 0],$$

we have $E_2(\tau)J(\mathbf{1}; \tau, z) \in \mathcal{Q}^{\mathcal{J}^1}$. Since $\frac{d}{dz}$ and $\frac{d}{d\tau}$ preserve $\mathcal{Q}^{\mathcal{J}^1}$ (cf. Lemma 4.2.3.1), we have $J(L[-2]\mathbf{1}, \tau, z) \in \mathcal{Q}^{\mathcal{J}^1}$.

7.4.2 Lattice vertex operator algebras

Let L be a positive-definite lattice and set $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. Let $\mathbb{C}[L]$ denote the group algebra with basis $\{e^\alpha \mid \alpha \in L\}$. It is known that $V_L = M_H \otimes \mathbb{C}[L]$ is a vertex operator algebra, with vertex operators $Y(v, z)$ for $v \in M_H$ defined as in Subsection 3.2.5. The vertex operators for the e^α are more complicated and will not be discussed here. See [15] for more details.

For each $a \in \mathfrak{h}$, $a(n)$ acts on M_H as in Subsection 3.2.5 when $n \neq 0$. When $n = 0$, $a(0)$ acts on $\mathbb{C}[L]$ by $a(0) = \langle a, \alpha\rangle e^\alpha$.

Let L° be the dual lattice of L defined by $L^\circ = \{\alpha \in \mathfrak{h} \mid \langle \alpha, L\rangle \subset \mathbb{Z}\}$ and let

$L^\circ = \bigcup_{i \in L^\circ/L}(L + \lambda_i)$ be its coset decomposition. It is known [15] that each space $V_{L+\lambda_i} = M_H \otimes \mathbb{C}[L + \lambda_i]$ is an irreducible V_L-module, and that (see [4]) $\{V_{L+\lambda_i} \mid i \in L^\circ/L\}$ is the set of all inequivalent irreducible modules of V_L. V_L is a strongly regular vertex operator algebra.

Each module $V_{L+\lambda_i}$ has the decomposition

$$V_{L+\lambda_i} = \bigoplus_{\alpha \in L+\lambda_i} M_H \otimes e^{\alpha}.$$

This decomposition is equivalent to (7.2.1.4), except in this case we have that $\Omega_{L+\lambda_i}(\alpha)$ is trivial. The functions

$$J_{\Omega_{V_{L+\lambda_i}}(\alpha),\underline{h}}(v;\tau,\underline{z})$$

in Theorem 1.0.0.4 are therefore trivial as well. It follows that for $v \in V_L$ decomposed as in Theorem 1.0.0.4, we have

$$J_{i,\underline{h}}(v;\tau,\underline{z}) = \frac{1}{\eta(\tau)^d} \sum_r f_r(\tau)\theta_{i,\underline{h}}(Q, a_r, k_r, \tau, \underline{z}). \tag{7.4.2.1}$$

Therefore,

$$\eta(\tau)^d J_{i,\underline{h}}(v;\tau,\underline{z})$$

is a sum of quasi-Jacobi forms on $\Gamma_0(N)$. Since every element in V_L is sums of elements of the form v decomposed as in Theorem 1.0.0.4, we have that $\eta(\tau)^d J_{i,\underline{h}}(u;\tau,\underline{z})$ is a sum of quasi-Jacobi form on $\Gamma_0(N)$ for any $u \in V_L$.

Appendix A

Appendix

Among other conditions, a strongly regular vertex operator algebra is C_2-cofinite, rational, and of CFT type. Below is a table listing which of these conditions are necessary for various results in this thesis.

Result	CFT type	C_2-cofinite	Rational
Translation law 1.0.0.7	No	Yes	Yes
Translation law 1.0.0.8	Yes	No	Yes
The convergence and expansion 1.0.0.6	No	Yes	No
Theorem 1.0.0.3	Yes	Yes	Yes
Theorem 1.0.0.1	No	Yes	Yes
Recursion Formulas (i.e. , 5.2.2.1)	No	No	No

Table A.1: Table of assumptions.

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