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Structural Models of Coefficient of Variation Matrices

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Structural Models of Coefficient of Variation Matrices

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by

Kirsten Alexandra Trickey

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ABSTRACT OF THE DISSERTATION

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by

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Doctor of Philosophy in Psychology

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Professor Peter M. Bentler, Chair

The coefficient of variation (CV) measures variability relative to the mean, and can be useful when increases in the mean correspond with systematic increases in variability. The univariate CV has been studied and applied extensively, but until recently it was not possible to study the structure of relative covariation occurring in multivariate data. However, Boik and Shirvani (2009) demonstrated that relative covariation could be modeled using estimators they developed to describe the sampling distribution of the CV matrix. This matrix, denoted Ψ, is defined as

$$\Psi = D_{\boldsymbol{\mu}}^{-1}\, \Sigma\, D_{\boldsymbol{\mu}}^{-1},$$

where D_{μ} is diagonal matrix containing variable means and Σ is the covariance matrix. The present research builds on this previous work by considering a more general class of structure models of the CV matrix.

Specifically, we investigated how structural equation models of the CV matrix could be estimated and applied. First, a statistical theory for the estimation and evaluation of structural equation models of CV matrices was developed for both normally and arbitrarily distributed variables using generalized least squares. Computational algorithms were then written to implement the theory and to allow CV models to be estimated. Using these algorithms, a series of simulation studies were conducted to determine the quality of the estimators proposed by Boik and Shirvani (2009) and the quality of the subsequent model parameters, standard errors, and test statistics, which rely on those estimators. The simulations considered a range of sample sizes, normal and log-normal data, different numbers of variables, and models with either one or two factors. It was found that Boik and Shirvani's theoretical estimators of the variance of the sampling distribution of the Ψ converged very slowly to their expected values and that they were particularly unreliable for log-normal data. That said, the estimation methods relying on these estimators, were generally able to estimate factor loadings accurately across conditions and when the sample sizes were fairly large and the number of variables was small, they also produced reasonably accurate estimates of the variance parameters, standard errors and test-statistics. However, in small samples with large numbers of variables, the variance estimates and the model fit statistics tended to be too low and the standard errors were typically overly conservative. In addition, when the data were log-normal the model fit statistics were problematic regardless of whether the estimator relied on normal theory. This general pattern of results was observed in both one-factor and two-factor models. The discussions below address some possible explanations for the estimation problems noted here and propose future work that should be done to better understand and potentially correct these problems.

Some of this work was initiated here and included in a short series of follow-up studies. Specifically, we addressed the numerical stability of the CV matrix and its sampling distribution covariance in terms of condition numbers. It was found that these matrices were both typically less stable than their counterparts in structural covariance modeling. In addition, we observed that Winsorizing the data used for the estimation produced modest improvements in the numerical stability. It remains to be seen how this might affect model estimation.

Finally, a one-factor CV model was fit to a longitudinal dataset assessing alcohol use over four years. Although each estimation method seemed to be able to reproduce the sample CV matrix with some accuracy, the model fit statistics indicated the model should be rejected. Given the non-normal distributions of the variables in the model, the appropriate interpretation of these results is ambiguous, but the interpretation and implications are discussed.

The dissertation of Kirsten Alexandra Trickey is approved.

Li Cai

Martin M. Monti

Frederic Paik Schoenberg

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University of California, Los Angeles

2015

To my wonderful Hika… I am pleased to report that there will be more mystical yellow orbs for you to retrieve in the future.

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This work would not have been possible without the support and guidance of my advisor Dr. Peter M. Bentler. Peter went above and beyond as an advisor and I would not have made it through without his patience and persistent encouragement. In addition, I would like to thank the members of my committee, as well as Dr. Albert Satorra for their suggestions and guidance along the way. The collaborative and supportive environment fostered by the psychology graduate students, faculty and staff has been critical to my success as a graduate student.

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Chapter 1. Introduction and Motivation

In empirical investigations of variability, measures of *absolute variation*, such as the standard deviation and the variance, are generally used by default without consideration of other options. While these measures are typically sufficiently informative, in some instances it may be more meaningful to consider variability relative to the mean, that is, to consider a measure of *relative variation* (Van Valen, 2005). One popular index of relative variation is the coefficient of variation (CV), which can be defined as the standard deviation divided by the mean. Although this coefficient is not the default measure of variability, its unique properties have resulted in widespread use across most scientific and academic disciplines.

However, the vast majority of these applications have only considered the CV as a univariate summary statistic. This is not ideal because, as we will discuss below, there are often many variables in studies with meaningful CVs. In recognition of this, Boik and Shirvani (2009) developed a principal components technique to analyze relative variation using a multivariate extension of the CV. Presently, we propose a more general structural equation modeling (SEM) approach to this problem that will allow researchers to construct structural models of the multivariate CV whose fit and parameters can be tested.

This proposal is structured as follows: In this introductory section, we will first provide a basic definition and description of the CV, followed by a discussion of several potential applications of structural CV models. In the second section, we describe the existing methods of performing statistical inference for the univariate and multivariate CVs. Third, we will provide a detailed definition of the CV matrix and describe its sampling distribution, which will be used in the fourth section to develop the theory of structural modeling of coefficient of variation matrices. Finally, we will propose a series of simulation studies and applications to examine the empirical value of this type of model.

Background and Basic Definition

Although it is somewhat rare to see measures of variation aside from the standard deviation and variance, there are actually many measures of inequality and diversity (Allison, 1978; Bedeian & Mossholder, 2000). In addition to the CV and the usual variance, these classes include the Gini coefficient, Theil's coefficient, the logarithmic variance, the generalized variance, and entropy among others (Allison, 1978; Budescu & Budescu, 2012). In terms of its properties, the CV has the most in common with the Gini coefficient, which is defined as the average of the absolute deviations between pairs of observations, scaled by the mean. However, these alternative inequality measures will not be elaborated upon here. (See Allison (1978) for a comparison of several of these techniques.)

Before proceeding, it may be useful to define the CV more formally.[1] As described above, the univariate CV (ψ) can be defined as the standard deviation divided by the mean:

$$\psi = \frac{\sigma}{\mu}. \tag{1.1}$$

As we will describe in later sections, there are several distinct multivariate analogs of this formula, but for our purposes we are primarily interested in the matrix form. The CV matrix (Ψ) has diagonal elements equal to the squared coefficients of variation for each variable and off-diagonal elements equal to the covariance of the two variables divided by their means. That is

[1] Throughout this paper notation will be introduced as it is used. However, for your reference most of the symbols and other notion used are listed with definitions in Appendix A.

$$\Psi_{ii} = \frac{\sigma_i^2}{\mu_i^2} \tag{1.2}$$

and

$$\Psi_{ij} = \frac{\sigma_{ij}}{\mu_i \mu_j}. \tag{1.3}$$

This particular form is useful because it includes both measures of relative variation and measures of relative covariation. Also, it should be note that the CV is clearly undefined when the mean is 0. For this and additional reasons (discussed below), it is generally required that the data be positive with means that are bounded away from 0.

Properties and Appropriate Use of the CV

One well known feature of the coefficient of variation is that it is invariant to rescaling (Allison, 1978). That is, if k is a scalar constant and $k > 0$, then the CV of kX will be the same as the CV of X. This means that the relative distances between observations are preserved by multiplication. This property is beneficial because it means that CV values from different samples can be compared even if different units of measurement were used (Sorensen, 2002).

Although we may multiply the CV by a constant without effect, we cannot add a constant without changing its value. Following Bedeian & Mossholder (2000) we refer to this property as location sensitivity. Specifically, if a positive value is added to all of the scores the CV will decrease. This may seem strange, but in some contexts it has an intuitive appeal. For instance, one might consider investigations of income inequality (Sorensen, 2002). Suppose one group of individuals has an average annual income of $20K and another group has average annual income of $100K. Does a deviation of 1K mean the same thing to both groups? Intuitively, the answer is certainly not, but the usual (absolute) measures of variation would treat the $1K difference identically regardless of the group's average annual income. On the other hand, the CV would assign more weight to a $1K change in the former group relative to the latter, and thus would be smaller for the group with the higher mean. Although these properties are useful in some contexts, they do suggest a need for caution when applying and interpreting the CV.

In particular, it should be apparent that the level of measurement is important in determining the meaningfulness of the CV. That is, generally the data should be of the ratio level, so that the zero value will represent a true absence of the quantity being measured. If the data were interval, the scale could be shifted arbitrarily. However, because of the location sensitivity property of the CV, these arbitrary shifts would change the value of the CV and thus change the meaning of the statistic. This has resulted in recurrent misapplications of the CV which have led some to suggest that the CV should not be used in general (Livers, 1942; Sorensen, 2002). However, others advise a more nuanced approach and have shown that the CV can be meaningfully interpreted even with interval-level measures if the arbitrary zero-point is fixed, and the underlying trait can be assumed to be positive with a true zero (Allison, 1978; Bedeian & Mossholder, 2000). However, whenever possible it is of course preferred to have true ratio-level variables.

Another source of controversy regarding CV applications involves the use of the CV to compare variation across groups (Fiske & Baer, 1955; Sorensen, 2002). First, it has been noted that the CV depends on the sample size and that appropriate adjustments should be made prior to comparing across groups of different sizes (Bedeian & Mossholder, 2000). Also, when comparisons across groups are made, it is implicitly assumed that the groups are from a common population (Fiske & Baer, 1955; Sorensen, 2002). However, this assumption may not be valid or testable, so one must be cautious when interpreting any differences observed (Fiske & Baer, 1955).

Potential Applications

Being a fairly simple and intuitive measure of relative variation, the CV has seen applications in a broad range of fields from chemistry to sociology (Wilson & Payton, 2002; Sorensen, 2002). In this section, we describe some of the existing applications in a small subset of these disciplines and emphasize those for which structural equation models may be particularly useful. Most of the applications described used the only the univariate statistic. However, in each case we elaborate on the potential advantages of the proposed approach to multivariate CV modeling.

Biology. To begin, we will briefly address some of the applications of the CV in biology, where much of the existing work on the multivariate CV originated (Albert & Zhang, 2010; Reyment, 1960; Van Valen, 1974, 2005). Rement (1960) first described a form of the multivariate CV as a means of assessing relative fossil sizes of different species of ostracods, a class of crustaceans. Others have applied versions of the multivariate CV to assess the quality of various electrophoresis techniques, to compare cranial sizes of male and female gophers, and to summarize high-dimensional microarray data assessing gene expression in the context of acute leukemia (Albert & Zhang, 2010).

Although the previous examples may not lend themselves well to SEM, other biologists have considered methods that could translate into structural models. In particular, another study of gene expression in leukemia considered the principal components (PC) of CV matrices (Nawaz & Ali, 2010). They used the results of the principal components analysis (PCA) to identify clustering in the data. In addition, they compared this analysis with a PCA on the correlation matrix. They concluded that the PCs obtained from the CV matrix were more informative than those obtained from the correlation matrix, but oddly they did not compare these with the covariance matrix PCs. Although an exploratory approach was taken in this application, we note that a structural modeling procedure for the CV matrix may provide an avenue for future hypothesis tests regarding shared relative variation (in terms of a factor-analytic framework rather than a cluster analysis framework).

Economics. The fields of economics and finance have applied the CV statistic extensively in contexts which might readily benefit from a multivariate structural modeling framework. In particular, in economics there are circumstances in which the mean and variance vary systematically for a large number of variables of interest (Holgersson, Karlsson, Mansoor, 2011). Common economics measures such as "risk," "marginal utility," "future cash flow" and "excess return" are examples of such variables (Aizenman, 1998; Holgersson, Karlsson, Mansoor, 2011; Rajgopal & Shevlin, 2002). For example, the risk of an investment can be thought of as the ratio of return variability to the average return. That is, more stable investments will have low variability and high average returns whereas riskier investments will tend to have high variance relative to the mean. This suggests that fitting a structural model on the basis of relative variation may be more substantively meaningful (in some contexts) than fitting a model based on the raw covariance. Others have constructed models to address this issue using alternative statistical methods, but thus far none of these would allow for the inclusion of latent variables (Holgersson, Karlsson, Mansoor, 2011). Therefore, the introduction of structural equation models of CV matrices may be preferable relative to existing methods.

Cognition. Several cognitive phenomena, such as learning (or conversely cognitive decline), intelligence and attention, have mean values that are related to their variances (Dodge, Mattek, Austin, Hayes, & Kaye, 2012; Klein, Wendling, Huetter, Ruder, & Peper, 2006; Segalowitz, Segalowitz, & Wood, 1998; Wisdom, Mignogna, & Collins, 2012). For instance, in models of second-language learning it has been observed that as skills are learned, both the mean and variance of reactions times associated with various skills tests decline (Hulstijn, van Gelderen, & Schoonen, 2009). However, as language skills become more automatic it is speculated that the variance will decrease faster than the mean leading to a decline in the CV of the reaction times (Segalowitz, et al., 1998). A latent growth curve model of the CV matrix might be able to reveal (or falsify) this hypothetical pattern. Similar CV models may be useful in testing hypotheses regarding changes in intelligence over time or in assessing cognitive decline in the elderly (Dodge et al., 2012; Wisdom et al., 2012).

In addition, the coefficient of variation has been used in studies of attention. In particular, the CV and other measures of intra-subject variability (ISV) have been used to distinguish between those with ADHD and controls (Klein et al., 2006). With the goal of developing a diagnostic aid for clinicians, these researchers computed different ISV statistics for several measures of attention and impulsivity. They then conducted separate principal components analyses for each of the ISVs. The loadings were calculated for the ADHD group and for the controls. The authors noted a few differences in the loadings between the analysis for the CV and that for the usual standard deviation. Although the authors were hesitant to endorse the CV as a standalone measure of variability, they suggested that the CV and the standard deviation be considered in tandem (Klein et al., 2006). Although this investigation does not directly translate into a structural model of attention, it does suggest that differences might be found between models based on relative versus absolute variance. This pattern suggests that structural models of the CV matrix may provide new information that complements the findings of traditional structural equation models of ADHD and related constructs.

Neuroscience. Interesting applications of the CV have also appeared in the neuroscience literature. For instance, in an fMRI study intended to ignite a "functional connectome" project, Bharat et al. (2010) used voxelwise CVs of various connectivity maps to find (putative) functional boundaries and sharp anatomical boundaries between regions (particularly, for the fALFF map). They also showed that the CV values tended to be more similar within regions. Although this might suggest that a model considering the relative covariance of regions might be useful for connectivity studies, it should be noted that the validity of this application of the CV is somewhat dubious. The CV values calculated by Bharat et al. (2010) could take on negative values; that is, the scale of measurement was interval which may mean that valid comparisons cannot be made. They addressed (or possibly ignored) this potential issue by taking the absolute value of the CVs. However, given that their CV maps were in agreement with known anatomical structures, the analyses they presented are clearly not completely meaningless, but perhaps should be interpreted with care.

Statistics. Finally, the CV has general statistical utility as well. For instance, within the context of Absolute Simplex Theory (AST), the CV matrix can be used to parameterize a model of Guttman data (Bentler, 1971; Bentler, 2012). Guttman (1944) described a type of scale composed of binary items ordered from most to least difficult, such that once a participant answered one question correctly all of the following items would also be answered correctly. If there were "errors" in this data (e.g. endorsing an item and failing to endorse a later item), these observations would be discarded. This of course is a less than ideal solution, which combined with the lack of a statistical estimation and testing framework, resulted in this method falling into disuse (Bentler, 2012).

However, Bentler (1971) showed that a set of error-free Guttman data (a.k.a. an "absolute simplex") could be completely represented in terms of its CV matrix, which has the form

$$\Psi_{ij} = \Psi_{ii} \text{ for } i \geq j. \tag{1.4}$$

Furthermore, it was shown that a set of Guttman data containing errors (a.k.a. a "quasi-simplex"), which cannot be completely recovered from the CV matrix, could be described in terms of a factor analytic model parameterized in terms of the CV. That is, the CV model $\Psi(\boldsymbol{\theta})$ could have the form

$$\Psi(\boldsymbol{\theta}) = D_{\boldsymbol{\mu}}^{-1}\Lambda\Phi\Lambda D_{\boldsymbol{\mu}}^{-1} + D_{\boldsymbol{\mu}}^{-1}\mathcal{E}D_{\boldsymbol{\mu}}^{-1}, \tag{1.5}$$

where $\boldsymbol{\theta}$ is a vector of model parameters, Λ is a matrix of factor loadings, Φ is the covariance matrix of the latent factors, and $\mathcal{E}$ is the covariance matrix describing the unique factors. With modern SEM techniques it has become possible estimate and test parameters of a model with this struture. This is useful, in part, because it brings the Guttman Scaling method into a more realistic context, but also because the CV parameters (once estimated) can be used to obtain person scores comparable to those of item response theory (Bentler, 1971; Lewis, 2010).

Chapter 2. Existing Statistical Methods

The Univariate Case

In this section we briefly summarize the existing statistical literature regarding inference on the univariate CV. McKay (1932) was the first to consider the distribution of ψ. He showed that for normally distributed data, the distribution of the CV followed non-central t-distribution, and this was later used to construct a confidence interval for the CV (Johnson and Welch, 1940). However, Koopmans, Owen, and Rosenblatt (1964) note that this confidence interval can sometimes have infinite length. Furthermore, they showed that by using some *a priori* information about the population mean, one could construct an interval that was guaranteed to have finite bounds. They also showed that for log-normal data, which has the property that the CV only depends on the variance (and not the mean), the situation simplifies and finite intervals can be guaranteed without *a priori* information.

A variety of alternative methods of estimating confidence intervals for the CV have been developed. In particular, intervals for the CV have been constructed for data drawn from various distributions, including the inverse Gaussian, Poisson, and Gamma distributions (Hsieh, 1990; Linhart, 1965; Panichitkosolkul, 2010). More recently, other scholars constructed large-sample aymptotic approximations of the intervals and considered the finite-sample bias of the resulting intervals (Albrecher, Ladoucette & Teugels, 2010; Bao, 2009; Curto & Pinto, 2009). Finally, small sample theories have been used to construct confidence intervals for the CV (Vangel, 1996; Wong & Wu, 2002). (For a comparison of several of these methods see Gulhar, Golam Kibria, Albatineh, and Ahmed (2012).)

Aside from simple confidence intervals, little has been done to model the univariate CV. However, one study investigated how a few of the previously developed approximate distributions for the CV could be used to construct tests of factorial models (Wilson & Payton, 2002). Specifically, they considered factorial models in which the data to be modeled consisted of a set of univariate CVs and they applied the generalized linear model to estimate model parameters and test model fit.

The Multivariate Case

Most work on generalizing the CV to multiple variables has focused on creating a scalar index which can summarize the relative variation of all of the variables simultaneously. The first such index, developed by Reyment (1960), can be regarded as an analog to the generalized variance (the determinant of the covariance matrix) (Reyment, 1960; Van Valen, 2005). Adopting the subscript notation of Albert and Zhang (2010), the index can be expressed as

$$CV_{RR} = \left(\frac{|\Sigma|^{\frac{1}{p}}}{\boldsymbol{\mu}^{T}\boldsymbol{\mu}}\right)^{\frac{1}{2}}, \tag{2.1}$$

where Σ is the covariance martrix, μ is the vector of means, p is the number of variables, and $|\cdot|$ is the matrix determinant. Reyment (1960) also developed a formula for the standard deviation of his CV index that can be used with large samples and briefly examined the performance of this measure relative to correlation and univariate CV methods.

More recently, others have developed similar indices to address issues with Reyment's formulation. Van Valen (1974, 2005) noted that because the previous definition depends on the determinant of the covariance matrix it may be zero or very close to zero when the matrix is ill-conditioned even if there is substantial variation in one or more of the variables. He suggested replacing the numerator with the trace of the covariance matrix:

$$CV_{VV} = \left(\frac{tr(\Sigma)}{\boldsymbol{\mu}^{T}\boldsymbol{\mu}}\right)^{\frac{1}{2}}. \tag{2.2}$$

Albert and Zhang (2010) noted that although this version is more stable than the original formulation, it neglects information about the covariance terms entirely. They compared CV_{RR} and CV_{VV} with a generalization of the (inverted) Mahalanobis distance

$$CV_{VN} = (\boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu})^{-1/2}, \tag{2.3}$$

and an index which they developed

$$CV_{AZ} = \left(\frac{\boldsymbol{\mu}^T \Sigma \boldsymbol{\mu}}{(\boldsymbol{\mu}^T \boldsymbol{\mu})^2} \right)^{\frac{1}{2}} \tag{2.4}$$

(Albert & Zhang, 2010; Budsaba & Smith, 2006). Albert and Zhang (2010) commented that although CV_{VN} was scale invariant, it required matrix inversion and as result it would break down similarly to Reyment's (1976) index when the covariance matrix was ill-conditioned or singular. On the other hand, they argued that their formulation was superior because it did not require matrix inversion. Accordingly, they proceeded to compare these methods and their corresponding confidence interval estimates in a series of applications.

Beyond the consideration of scalar indices, it seems that there has been little investigation into the distribution of the multivariate CV matrix. In a pair of brief papers, Bennett (1977, 1980) used normal theory to derive a joint distribution of the multivariate coefficient of variation. However, there was no investigation into the applications or properties of the distribution. A later work by Boik and Shirvani (2009) independently developed an alternative distribution of the CV matrix which can be applied in normal and non-normal conditions. The distribution they developed (described in detail below) was then used to construct principal component models of the CV matrix. The modeling procedure was applied to women's Olympic track data and the validity of hypothesis tests of the models were assessed via simulations.

Chapter 3. The Coefficient of Variation Matrix

Definition

Let Y be an $n \times p$ matrix containing n observations drawn from a p-variate distribution with the population mean vector $\boldsymbol{\mu}$ and the population covariance matrix Σ. Denote the transpose of each row of Y as Y_i, so that Y_i is a $p \times 1$ vector consisting of one observation. Assuming that the expected values of the means are positive and bounded away from 0, we can define the population coefficient of variation matrix as

$$\Psi = D_{\boldsymbol{\mu}}^{-1} \, \Sigma \, D_{\boldsymbol{\mu}}^{-1}, \tag{3.1}$$

where $D_{\boldsymbol{\mu}}$ is the $p \times p$ diagonal matrix of means. Then, as described in (1.2) and (1.3), the elements of the coefficient of variation matrix Ψ_{ij} have the form $\Sigma_{ij}/(\mu_i \mu_j)$.

Similarly, we may define the corresponding sample statistics. Let $\hat{\mu}$ be the p-vector of sample means and let $\hat{\Sigma}$ be the sample covariance matrix. Then we can define the sample CV matrix as

$$\widehat{\Psi} = D_{\widehat{\boldsymbol{\mu}}}^{-1} \, \hat{\Sigma} \, D_{\widehat{\boldsymbol{\mu}}}^{-1}, \tag{3.2}$$

where $D_{\widehat{\boldsymbol{\mu}}}$ is the $p \times p$ diagonal matrix of sample means. The elements of this matrix ($\widehat{\Psi}_{ij}$) will have the analogous form $\hat{\Sigma}_{ij}/(\hat{\mu}_i \hat{\mu}_j)$.

Sampling Distribution

Boik and Shirvani (2009) derived the asymptotic distribution and finite sample bias of the CV matrix using the delta method. That is, they first calculated the first and second partial derivatives of the of the CV matrix with respect to the mean and covariance. Then, these were used to expand a second-order Taylor series of $\sqrt{n}\, vec(\widehat{\Psi} - \Psi)$ around the population mean and covariance (where the function $vec(\cdot)$ simply stacks the columns of the matrix). Finally, they applied central limit theorem to the Taylor expansion to obtain the asymptotic distribution of $\sqrt{n}\, vec(\widehat{\Psi} - \Psi)$. The main results of this procedure are summarized below; however, for details refer to Boik and Shirvani (2009).

First it is helpful to describe preliminary results regarding the sampling distributions of $\widehat{\boldsymbol{\mu}}$ and $\widehat{\Sigma}$. In particular, Boik and Shirvani (2009) derived the covariance matrix of $\sqrt{N}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})$ and $\sqrt{N}(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = vec(\Sigma)$ and $\widehat{\boldsymbol{\sigma}} = vec(\widehat{\Sigma})$. They showed that this covariance has the block form

$$cov\begin{bmatrix}\sqrt{N}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})\\ \sqrt{N}(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\sigma})\end{bmatrix} = \begin{pmatrix}\Sigma & \Omega_{12}\\ \Omega_{12}^{T} & \Omega_{22}\end{pmatrix} \tag{3.3}$$

where

$$\Omega_{12} = \mathrm{E}\left[(Y_i - \boldsymbol{\mu})(Y_i - \boldsymbol{\mu})^{T\,\otimes 2}\right] \tag{3.4}$$

and

$$\Omega_{22} = \mathrm{E}\left[(Y_i - \boldsymbol{\mu})^{\otimes 2}(Y_i - \boldsymbol{\mu})^{T\,\otimes 2}\right] - \boldsymbol{\sigma}\boldsymbol{\sigma}' + \frac{2N_p(\Sigma \otimes \Sigma)}{n-1}. \tag{3.5}$$

In the above equations, the operator $\otimes$ represents the Kronecker product and $A^{\otimes k}$ is the k^{th} power of the Kronecker product of matrix A (see Appendix). In addition, N_p is a matrix which projects onto the space of symmetric matrices and can be expressed as a product of duplication matrices (Boik & Shirvani, 2009; Harville, 1997). The duplication matrix D_p is the unique $p^2 \times p^*$ matrix that maps $vech(A)$ to $vec(A)$ and $N_p = D_p\left(D_p^T D_p\right)^{-1} D_p^T$ (Boik & Shirvani, 2009; Harville, 1997; Magnus and Neudecker, 1999).

Importantly, it was shown that when $\left\|E\left(Y_i^{\otimes 4}\right)\right\|$ is finite, $\sqrt{n}\, vec\left(\widehat{\Psi} - \Psi\right)$ converges in distribution to $\mathcal{N}(\mathbf{0}_p, \Sigma_\Psi)$ as n approaches ∞, where

$$\Sigma_\Psi = 2N_p(\Psi \otimes I_p)L_p\Psi L_p^T(\Psi \otimes I_p)2N_p - {\Omega_{12}^*}^T L_p^T(\Psi \otimes I_p)2N_p$$
$$-2N_p(\Psi \otimes I_p)L_p\Omega_{12}^* + \Omega_{22}^*, \tag{3.6}$$

where

$$\Omega_{12}^* = D_{\boldsymbol{\mu}}^{-1}\Omega_{12}(D_{\boldsymbol{\mu}}^{-1} \otimes D_{\boldsymbol{\mu}}^{-1}) \tag{3.7}$$

and

$$\Omega_{22}^* = (D_{\boldsymbol{\mu}}^{-1} \otimes D_{\boldsymbol{\mu}}^{-1})\Omega_{22}(D_{\boldsymbol{\mu}}^{-1} \otimes D_{\boldsymbol{\mu}}^{-1}). \tag{3.8}$$

Note that in the above expressions $\|\cdot\|$ denotes the usual vector norm and L_p is a $p^2 \times p$ matrix of zeros and ones that inserts rows and/or columns of zeros into another matrix via multiplication (details in Appendix). Boik and Shirvani (2009) also approximated the expected finite sample bias of the CV matrix estimate:

$$E[vec(\widehat{\Psi} - \Psi)]$$
$$\approx \frac{1}{n}\{(I_p \otimes L_p^T)vec[\Psi L_p^T(\Psi \otimes I_p)2N_p - 2\Omega_{12}^*] \tag{3.9}$$
$$+ (\Psi \otimes I_p)L_p vec(\Psi)\}.$$

They note that the preceding formula holds when $\|E(Y_i^{\otimes 5})\|$ is finite and the error of the approximation is on the order of $1/n^2$.

If it can be assumed that the observations Y_i are independent and identically distributed following the multivariate normal distribution $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$, then the above results simplify somewhat. Specifically, Boik and Shirvani (2009) showed that when normality assumptions were met, the variance of the sampling distribution of the CV matrix becomes

$$\Sigma_\Psi = 2N_p(\Psi \otimes I_p)L_p\Psi L_p^T(\Psi \otimes I_p)2N_p + 2N_p(\Psi \otimes \Psi). \tag{3.10}$$

The bias estimate simplifies similarly with the Ω_{12}^* dropping out.

Boik and Shirvani (2009) also proposed estimators for the variance of the sampling distribution Σ_Ψ. To construct an estimator according to (3.6) estimators are needed for the covariance matrices Ω_{12}^* and Ω_{22}^*. The authors suggested using the root-n consistent estimators

$$\widehat{\Omega}_{12}^* = D_{\widehat{\boldsymbol{\mu}}}^{-1}\widehat{\Omega}_{12}(D_{\widehat{\boldsymbol{\mu}}}^{-1} \otimes D_{\widehat{\boldsymbol{\mu}}}^{-1}) \tag{3.11}$$

where

$$\widehat{\Omega}_{12} = \frac{n\sum_{i=1}^{n}(Y_i - \widehat{\mu_i})(Y_i - \widehat{\mu_i})^{T^{\otimes 2}}}{(n-1)(n-2)}, \tag{3.12}$$

and

$$\widehat{\Omega}_{22}^* = \left(D_{\widehat{\boldsymbol{\mu}}}^{-1} \otimes D_{\widehat{\boldsymbol{\mu}}}^{-1}\right)\widehat{\Omega}_{22}\left(D_{\widehat{\boldsymbol{\mu}}}^{-1} \otimes D_{\widehat{\boldsymbol{\mu}}}^{-1}\right). \tag{3.13}$$

where

$$\widehat{\Omega}_{22} = \frac{\sum_{i=1}^{n}(Y_i - \widehat{\mu_i})^{\otimes 2}(Y_i - \widehat{\mu_i})^{T^{\otimes 2}} - (n-2)\widehat{\boldsymbol{\sigma}}^T\widehat{\boldsymbol{\sigma}}}{n - p^* - 1} \tag{3.14}$$

Then substituting these into (3.6), they obtained the root-n consistent estimator

$$\begin{aligned}\widehat{\Sigma}_\Psi = 2N_p\left(\widehat{\Psi} \otimes I_p\right)L_p\widehat{\Psi}L_p^T\left(\widehat{\Psi} \otimes I_p\right)2N_p - {\widehat{\Omega}_{12}^*}{}^T L_p^T\left(\widehat{\Psi} \otimes I_p\right)2N_p \\ -2N_p\left(\widehat{\Psi} \otimes I_p\right)L_p\widehat{\Omega}_{12}^* + \widehat{\Omega}_{22}^*.\end{aligned} \tag{3.15}$$

Alternatively, if normality assumptions are met, (3.10) can be used to obtain the estimator

$$\widehat{\Sigma}_\Psi = 2N_p\left(\widehat{\Psi} \otimes I_p\right)L_p\widehat{\Psi}L_p^T\left(\widehat{\Psi} \otimes I_p\right)2N_p + 2N_p\left(\widehat{\Psi} \otimes \widehat{\Psi}\right) \tag{3.16}$$

which is also root-n consistent.

In addition, it should be noted that the above variances correspond to the sampling distribution of the large p^2-vector $vec(\Psi)$, whereas the computations in the following chapters will require the equivalent matrices for the reduced p^*-vector $vech(\Psi)$. More precisely, the matrices represented in (3.15) and (3.16) contain redundant elements corresponding to the redundant elements in $vec(\Psi)$ and thus they are non-invertible. These redundancies can be removed using the duplication matrix D_p and/or the elimination matrix H_p, which maps $vec(A)$ to $vech(A)$, and can be expressed in terms of the duplication matrix: $H_p = \left(D_p^T D_p\right)^{-1} D_p^T$ (Magnus & Neudecker, 1999). In order to produce generalized least squares (GLS) estimators of principal component models of the CV matrix, Boik and Shirvani (2009) needed to "invert" the matrices in (3.15) and (3.16). To do this, they addressed the redundancy problem using the transformation

$$\hat{\Sigma}_{\Psi}^{+} = D_p \left(D_p^T \hat{\Sigma}_{\Psi} D_p\right)^{-1} D_p^T \quad (3.17)$$

to obtain the Moore-Penrose inverse $\hat{\Sigma}_{\Psi}^{+}$ (which is also a large $p^2 \times p^2$ matrix). Below we will also consider a more intuitive (but ultimately equivalent) approach using the elimination matrix to obtain a smaller invertible version of $\hat{\Sigma}_{\Psi}$, which is defined as follows[2]:

$$\hat{\Sigma}_{\boldsymbol{\psi}} = H_p \hat{\Sigma}_{\Psi} H_p^T. \quad (3.18)$$

This form of the variance of the sampling distribution of the CV matrix is invertible and therefore readily compatible with the existing literature that we will use to develop more general reduced parameter models of the CV matrix.

[2] Recall, that we use $\boldsymbol{\psi}$ to denote $vech(\Psi)$, so this notation represents the estimated variance of the distribution of $vech(\Psi)$.

Chapter 4. Structural Models

A General Structural Modeling Method

Although historically SEM has dealt primarily with covariance structures, the general principle may be applied to a wide range of statistics. For instance, one might wish to model means, polychoric or tetrachoric correlations, frequencies, higher-order moments, or in our case, coefficient of variation matrices (Bentler & Dijkstra, 1985). In this section, we introduce the theory behind this more general SEM framework, and in the next sections we consider how these methods apply to the covariance matrix and the CV matrix.

First, let $\boldsymbol{s}$ be a vector with length p^* of asymptotically normal sample statistics obtained from n independent observations, such that $\boldsymbol{s}$ converges in probability to $\boldsymbol{s}_0$ as n approaches ∞. Also, let Θ be a parameter space contained in $\mathbb{R}^q$ where $q \leq p^*$, and let $\boldsymbol{\theta}$ be an arbitrary element of Θ. Then a structural model $\mathscr{s}(\boldsymbol{\theta})$ is said to hold if there exists $\boldsymbol{\theta}_0 \in \Theta$ such that $\mathscr{s}(\boldsymbol{\theta}_0) = \boldsymbol{s}_0$, where $\mathscr{s}(\boldsymbol{\theta})$ is a continuous vector function defined on Θ and $\boldsymbol{s}_0$ contains the population values of the statistics in $\boldsymbol{s}$ (Browne, 1984).

Next we consider the parameters $\boldsymbol{\theta}$ and some of their properties. One general method of estimating these parameters is to minimize a discrepancy function, such as the GLS function

$$F(\boldsymbol{s}, \mathscr{s}(\boldsymbol{\theta})|W) = \left(\boldsymbol{s} - \mathscr{s}(\boldsymbol{\theta})\right)^T W\left(\boldsymbol{s} - \mathscr{s}(\boldsymbol{\theta})\right), \tag{4.1}$$

with respect to $\boldsymbol{\theta}$, where W is a symmetric weight matrix. Although not addressed here, constraints may be placed on the values of θ, and this procedure is addressed in detail by Bentler and Dijkstra (1985). To estimate the set of parameters $\widehat{\boldsymbol{\theta}}$ at which $F\left(\boldsymbol{s}, \mathscr{s}\left(\widehat{\boldsymbol{\theta}}\right)|W\right)$ is minimized, generally some sort of iterative algorithm is employed (such as the Gauss-Newton or Newton-Raphson algorithms). Bentler and Dijkstra (1985) showed that when $\boldsymbol{s} \rightarrow \boldsymbol{s}_0$ in probability and W is appropriately chosen, then $\widehat{\boldsymbol{\theta}}$ will be a consistent estimator under the model $\mathscr{s}(\boldsymbol{\theta})$. To be specified appropriately, W can either be a fixed positive definite matrix W_0 that is constant across n or it can be allowed to vary across n if the sequence converges in probability to the positive definite matrix W_0. Next, Bentler and Dijkstra (1985) showed that if

$$\sqrt{n}\begin{bmatrix}(\boldsymbol{s}-\boldsymbol{s}_0)\\ vec(W-W_0)\end{bmatrix} \to \mathcal{N}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}V_{ss} & V_{sw}\\ V_{ws} & V_{ww}\end{bmatrix}\right) \tag{4.2}$$

in distribution and the model is correctly specified,

$$\sqrt{n}\left(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right) \to \mathcal{N}(\mathbf{0}, (\dot{\mathscr{s}}(\boldsymbol{\theta}_0)^T W \dot{\mathscr{s}}(\boldsymbol{\theta}_0))^{-1}) \tag{4.3}$$

in distribution, where $\dot{\mathscr{s}}(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}'}\mathscr{s}(\boldsymbol{\theta})$. Furthermore, for the estimation to be asymptotically efficient, it is sufficient to select a W that converges in probability to V_{ss}^{-1}. These results provide a foundation for conducting hypothesis tests of the model $\mathscr{s}(\boldsymbol{\theta})$ and the parameters $\boldsymbol{\theta}$.

To test hypotheses regarding model fit, we can construct a test statistic by evaluating the GLS function in (4.1) at $\hat{\theta}$ and setting $W = \hat{V}_{ss}^{-1}$. Then

$$T_{GLS} = nF\left(\boldsymbol{s}, \mathscr{s}(\hat{\boldsymbol{\theta}})\middle| W = \hat{V}_{ss}^{-1}\right) \tag{4.4}$$

has an approximate χ^2-distribution with $p^* - q$ degrees of freedom (Browne, 1982; 1984). In the case that W does not depend on any particular distributional assumptions, we will call this estimation method Arbitrary-distribution GLS (or AGLS) or equivalently we could say it is asymptotically distribution free (ADF). If W is derived using normal theory assumptions, we will call this method Normal-theory GLS (or NGLS).

In addition, the result in (4.3) allows us to construct tests of individual parameters θ_i using a z-test. In particular, (4.3) shows that the variance of the sampling distribution of θ is

$$\Delta = \left(\dot{\mathscr{s}}(\boldsymbol{\theta}_0)^T W \dot{\mathscr{s}}(\boldsymbol{\theta}_0)\right)^{-1}, \tag{4.5}$$

with $W = \hat{V}_{ss}^{-1}$. By selecting the appropriate diagonal element of this matrix we may calculate the standard errors of the parameters via

$$SE(\hat{\theta}_i) = \sqrt{\Delta_{ii}/n}\,. \tag{4.6}$$

Modeling Coefficient of Variation Matrices

Now we may apply this framework to models of the CV matrix. Boik and Shirvani (2009) showed that the asymptotic sampling distribution of the CV matrix meets the requirements put forth by Bentler and Dijkstra (1985) for structural modeling via GLS. Thus when we appropriately specify CV models, we use the procedures described above to obtain hypothesis tests for the model fit and for the parameter estimates.

More concretely, given some CV matrix Ψ and an appropriate model $\Psi(\boldsymbol{\theta})$, we can develop a version of the GLS function in (4.1). Let the vector sample statistics $\boldsymbol{s} = \widehat{\boldsymbol{\psi}}$, where $\widehat{\boldsymbol{\psi}} = vech(\widehat{\Psi})$ (the half-vectorization of $\widehat{\Psi}$), and let $\mathcal{s}(\boldsymbol{\theta}) = \boldsymbol{\psi}(\boldsymbol{\theta})$, where $\boldsymbol{\psi}(\boldsymbol{\theta}) = vech(\Psi(\boldsymbol{\theta}))$. Then (4.1) becomes

$$F(\widehat{\boldsymbol{\psi}}, \boldsymbol{\psi}(\boldsymbol{\theta}) | W) = \left(\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}(\boldsymbol{\theta})\right)^T W \left(\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}(\boldsymbol{\theta})\right), \tag{4.7}$$

which can be minimized to obtain parameter estimates. Also, the value of V_{ss} for the full vectorization of $\widehat{\Psi}$ is given by (3.6) for data with an arbitrary distribution and (3.10) for multivariate normal data. Moreover, there are two corresponding estimators of V_{ss}. First, to perform AGLS estimation we can use the arbitrary-distribution form of $\hat{\Sigma}_\Psi$ given in (3.15) and calculate $\hat{\Sigma}_{\boldsymbol{\psi}}$ as shown in (3.8) and let $W = \hat{\Sigma}_{\boldsymbol{\psi}}^{-1}$. Alternatively, to perform NGLS estimation, we can use the normal-theory form of $\hat{\Sigma}_\Psi$ and again calculate $\hat{\Sigma}_{\boldsymbol{\psi}}$ as shown in (3.8) and let $W = \hat{\Sigma}_{\boldsymbol{\psi}}^{-1}$. Substituting these weight matrices into (4.4) along with $\widehat{\Psi}$ and an appropriate model, we can obtain AGLS and NGLS χ^2-test statistics, respectively. Finally, to perform hypothesis tests of the individual parameters of the model, we would need to evaluate the derivative $\dot{\boldsymbol{\mathcal{s}}}(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} \psi(\boldsymbol{\theta})$ and then calculate the standard errors using (4.5) and (4.6).

Modeling Covariance Matrices

In the modeling of covariance matrices, the GLS function has been studied and applied using a variety of different choices of W and/or estimators of V_{SS}. One option is to estimate the variance of the asymptotic sampling distribution of the covariance matrix, denoted $\hat{\Sigma}_S$, without assuming any particular distribution for the data. This matrix can be computed by first calculating

$$Z_i = vech\{(Y_i - \hat{\mu})(Y_i - \hat{\mu})^T\} \tag{4.8}$$

and then

$$\hat{\Sigma}_S = cov(Z). \tag{4.9}$$

Then, the corresponding weight matrix W would be $\hat{\Sigma}_S^{-1}$ and this would yield an AGLS test statistic. It is known that this test-statistic generally does not perform well in empirical applications, and that in the case of structural covariance modeling, a simple adjustment can substantially improve the test's accuracy in finite samples (Yuan and Bentler, 1997b). Furthermore, it is known that the empirical standard errors resulting from the ADF procedure tend to be substantially lower than their true values, and that this bias can be substantially reduced with a simple correction (Yuan & Bentler, 1997a). However, it is not clear whether similar corrections might be relevant to models of the CV matrix.

In addition, it has been shown that when the data are normally distributed, the weight matrix obtained from $\hat{\Sigma}_S$ can be reduced to a simpler form, allowing the GLS function in (4.1) to simplify (Browne 1982; 1984). That is, in structural models of covariance matrices with normal data,

$$W = \frac{1}{2} D_p^T (W^* \otimes W^*) D_p, \tag{4.10}$$

which means that (4.1) can be expressed in terms of a $p \times p$ weight matrix W^* rather than the much larger $p^* \times p^*$ matrix W:

$$F(\boldsymbol{s}, \boldsymbol{\sigma}(\boldsymbol{\theta})|W^*) = \frac{1}{2} tr\big[\big(S - \Sigma(\boldsymbol{\theta})\big)W^*\big]^2 \qquad (4.11)$$

(Jennrich, 1970; Browne, 1974). Using this form, we can obtain a normal theory GLS χ^2-statistic by setting $W^* = S^{-1}$. In addition, it is possible to perform reweighted least squares (RLS) estimation by allowing W^* to be updated iteratively as $\boldsymbol{\theta}$ is estimated. To do this, we set $W^* = \Sigma^{-1}(\widehat{\boldsymbol{\theta}})$, where $\Sigma(\boldsymbol{\theta})$ is the model of the covariance matrix (as a function of $\boldsymbol{\theta}$**).** In other words, the weight matrix is the inverse of the model's predicted covariance values given the current estimate of $\boldsymbol{\theta}$. This method is equivalent to normal theory maximum likelihood (Lee & Jennrich, 1979; Bentler, 2006). Because these normal-theory estimators use smaller more easily estimated matrices they can be more computationally stable, but like the AGLS estimates, may also be biased (Yuan & Bentler, 1997a). However, if the "sandwich" covariance estimator is used, this bias can be reduced and the resulting estimates can be quite robust to violations (Yuan & Bentler, 1997a). The need for this or a similar correction may also exist in more general structural equation models. Finally, it should be noted that the substitution $W^* = I_p$ can be made here to get the least squares estimates; however, this choice of W^* will not result in a χ^2-test statistic (Bentler, 2006). Therefore, this strategy will not be addressed any further here.

Applying Covariance Methods to CV Matrix Modeling?

Although technically the use of the simplified form of the GLS function given in (4.11) would be misspecified if applied to CV matrix model estimation, the reduced complexity and increased stability of these methods relative to AGLS may nevertheless make them useful for modeling the CV matrix. Therefore, in addition to the correctly specified AGLS and NGLS methods for modeling the CV matrix, we also consider variations of the covariance NGLS and RLS procedures described in the previous section. That is, we will also estimate CV models by modifying (4.11) for use with CV matrices as follows:

$$F(\Psi, \Psi(\boldsymbol{\theta})|W^*) = \frac{1}{2} tr\left[\left(\Psi - \Psi(\boldsymbol{\theta})\right)W^*\right]^2 \qquad (4.12)$$

First, we will consider a misspecified-NGLS estimation procedure (abbreviated MGLS below) that uses (4.12) with $W^* = \widehat{\Psi}^{-1}$. Second, we will consider a misspecified-RLS (MRLS) estimation procedure that uses (4.12) with $W^* = \Psi^{-1}(\widehat{\boldsymbol{\theta}})$, where $\widehat{\boldsymbol{\theta}}$ is updated iteratively. After minimizing the GLS function, the test-statistics and standard errors will be computed with the same method described previously.

Defining Models of Coefficient of Variation Matrices

There are a variety of possible choices for the model of the CV matrix. For example, we could define $\Psi(\boldsymbol{\theta})$ using a variation of the confirmatory factor analytic model in (1.5) (Jöreskog, 1969). Although here we primarily consider confirmatory factor models, the theory could be applied to any the structural equation model of the form

$$\boldsymbol{\eta} = \beta\boldsymbol{\eta} + \gamma\boldsymbol{\xi} \qquad (4.13)$$

where $\boldsymbol{\eta}$ contains dependent variables, which may be latent or observed, β contains the set of coefficients for the regression of variables in $\boldsymbol{\eta}$ on other variables in $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ contains a set of strictly independent variables, and γ is a set of coefficients for the regression of $\boldsymbol{\eta}$ on $\boldsymbol{\xi}$ (Bentler & Weeks, 1980). Let B be a matrix containing β (and known 0 elements), Γ be a matrix containing γ (and known 0 and 1 elements), and Φ be the covariance matrix for the independent variables. Then we can express the modeled CV matrix in terms of the linear model as

$$\Psi(\boldsymbol{\theta}) = G(I - B)^{-1}\Gamma\Phi^{\mathrm{T}}(1 - B)^{T^{-1}}G \qquad (4.14)$$

where $\boldsymbol{\theta}$ contains the unknown elements of B, Γ, Φ, and $D_{\boldsymbol{\mu}}$ and G is a selection matrix which selects the observed variables. As always we will need to select models that are appropriately identified, to ensure that there is a unique solution for the parameters.

Chapter 5. General Simulation Method

A series of studies were conducted to assess the quality of various statistics related to the estimation of the CV models proposed in the previous chapter. First, a preliminary study was conducted to examine the accuracy of the formulas for $\hat{\Sigma}_{\Psi}$ proposed by Boik and Shirvani (2009) (described in Chapter 4). However, the primary goal of this study was to assess the quality of the parameter estimates, test statistics, and standard errors obtained from each of the estimation methods proposed in the previous chapter across a range of data types and models. These simulations are described in Chapters 7 through 11. In addition, follow-up studies were conducted to address concerns regarding the computational stability and accuracy of some of the statistics central to the CV model estimation process. These follow-up simulation are described in Chapters 12 and 13.

The purpose of the current chapter is to describe the general methods that were used in all or most of the simulation studies mentioned above. This includes descriptions of the models, the data generation procedures, the estimation procedures and the replication process. Brief methods are also provided in each of the subsequent chapters to describe additional study-specific methods and deviations from the general method.

Models

In each simulation, a particular structural model needed to be specified. The specified model (a) served as the *true* model that gave rise to the population CV matrix, which was needed to generate data, and (b) was used to determine what model should be fit to the data. In each case, we assumed that the model fit to the data was correctly specified. However, we did vary the type of model and the numbers of variables included in each model.

One factor model. A simple one factor model was considered containing either 5 or 20 observed variables (p). In each case, the true models used to generate the data for the simulations had all path coefficients and variances set to 1. This is illustrated for the 5-variable case in Figure 5.1(a). Furthermore, these population parameter values give rise to a $p \times p$ population CV matrix of the form

$$\Psi = \Psi(\boldsymbol{\theta}_0) = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{pmatrix}. \tag{5.1}$$

The corresponding model that was fit to the data was specified such that the factor variance, the error variances and all but one of the factor loadings were free parameters, while the remaining path coefficients were fixed to 1 to ensure that the model was identified. Figure 5.1(b) shows the model that was specified and estimated in simulation for the 5-variable case. The models in the 20-variable case were defined similarly and thus are not depicted.

Two factor model. A two factor model was also considered with either 6 or 20 observed variables. The variables were split evenly so that each factor had equal numbers of indicators. All model parameters excluding the factor covariance were set to 1.0, and the factor covariance was set to 0.3. Figure 5.2(a) depicts the true model for the 6 variable condition. These parameters result in a population CV matrix that has the following block structure, which can be expressed in terms of two $\frac{p}{2} \times \frac{p}{2}$ matrices C and D, as follows:

$$\Psi = \Psi(\boldsymbol{\theta}_0) = \begin{pmatrix} D & C \\ C & D \end{pmatrix}, \tag{5.2}$$

where

$$C = \begin{pmatrix} 0.3 & \dots & 0.3 \\ \vdots & \ddots & \vdots \\ 0.3 & \dots & 0.3 \end{pmatrix}, \tag{5.3}$$

and

$$D = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{pmatrix}. \tag{5.4}$$

In order to identify the models that were estimated in samples, one loading per factor was fixed to 1.0 and all paths from the error terms had coefficients fixed to 1.0. The remaining parameters were left as free for estimation. The parameters that were fixed and free are also depicted in Figure 5.2(b) for the 6 variable condition, where the free parameters are marked with asterisks. Again, the models in the 20-variable case were defined similarly (but with 10 variables loading on each factor) and therefore are not diagrammed.

Data Generation

To generate the data, first, the true model was used to construct the corresponding population CV matrix Ψ (as described above). A vector $\boldsymbol{\mu}$ was also specified, and this was used together with Ψ to calculate the population covariance matrix Σ. Given these population parameters, samples could be drawn from the distribution of interest. Two types of multivariate distributions were considered: normal and log-normal. In addition, a variety of sample sizes were considered ranging from 100 to 100,000.

Normal data. In order to generate normal data, $\boldsymbol{\mu}$ was fixed and Σ was calculated from Ψ and $\boldsymbol{\mu}$, using the relation

$$\Sigma = \mathrm{D}_{\boldsymbol{\mu}}\Psi(\boldsymbol{\theta_0})\mathrm{D}_{\boldsymbol{\mu}}. \tag{5.5}$$

Then each observation was drawn from the multivariate normal distribution such that $Y_i \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma = \mathrm{D}_{\mu}\Psi D_{\mu})$. In one condition, we considered a simple case in which the population mean of each observed variable was set to 1 (i.e. $\boldsymbol{\mu} = \mathbf{1}_p$).[3] As a consequence, the population values of Σ and Ψ were identical. In another condition, we considered population means that differed across the variables. Namely, the means varied from 1.0 to 3.0 in 4 evenly spaced steps (of size $1/2$) in the $p = 5$ condition and 19 evenly spaced steps (of size $2/19$) in the $p = 20$ condition. That is, $\boldsymbol{\mu} = (1.0, 1.5, 2.0, \ldots, 3.0)^T$ and $\boldsymbol{\mu} \approx (1.00, 1.11, 1.22, \ldots, 3.00)^T$ in the 5-variable and 20-variable conditions, respectively. The specific type or types of data considered in particular simulations are listed in specific method sections in the chapters below.

Log-normal data. Next, to generate log-normal data we needed to first obtain a matrix X of observations X_i drawn from a normal distribution with appropriate population parameters, i.e. $X_i \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, and then transform these observations according to the relation

$$Y_{ij} = e^{X_{ij}}. \tag{5.6}$$

The resulting observations Y_i have a multivariate log-normal distribution. However, in order to ensure that Y_i was drawn from a population with the specified value of Ψ, we first needed to use some of the special properties of log-normal data to determine what values of $\boldsymbol{\mu}$ and Σ should be used to generate the data X_i. In particular, Tarmast (2001) showed that the elements of the mean vector and covariance matrix of multivariate log-normal data can be expressed as functions of the mean and covariance of the untransformed data as follows:

$$E[Y]_i = e^{\mu_i + \frac{1}{2}\Sigma_{ii}} \tag{5.7}$$

and

[3] Note that while 1 is not a large positive number, given the samples sizes considered here, it is substantially larger than 0 in terms of the standard error of the mean.

$$Cov[Y]_{ij} = e^{\mu_i+\mu_j+\frac{1}{2}(\Sigma_{ii}+\Sigma_{jj})}\left(e^{\Sigma_{ij}} - 1\right). \tag{5.8}$$

Therefore, the corresponding CV matrix of the log-normal data would have elements

$$\Psi_{ij} = e^{\Sigma_{\mathrm{ij}}} - 1, \tag{5.9}$$

which like the univariate analog do not depend on the mean. Consequently, the elements of the covariance matrix of the untransformed data can be expressed in terms of the elements of Ψ as follows:

$$\Sigma_{ij} = \log\left(\Psi_{ij} + 1\right). \tag{5.10}$$

Therefore, we generated log-normal data from a population with the CV matrix Ψ by first calculating Σ according to (5.10) and setting $\boldsymbol{\mu} = \mathbf{1}_p$, then generating observations $X_i \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ and finally transforming X to Y via (5.6).

Estimation

Each sample was fit to the models described above using each of the four CV-model estimation methods described in Chapter 4. More specifically, the models were fit by minimizing the GLS function in (4.1) using four different choices of the weight matrix W. Two correctly specified forms of GLS estimation were performed by setting $W = \hat{\Sigma}_{\Psi}^{-1}$ (defined in Chapter 3). The first version, using $\hat{\Sigma}_{\Psi}$ as it is defined in (3.15), is intended for data with an unknown or arbitrary distribution and, as noted previously, will be referred to as the AGLS method. The other version, which uses $\hat{\Sigma}_{\Psi}$ as it is defined in (3.16), is intended for normally distributed data and, as noted previously, will be referred to as the NGLS method. In addition, two misspecified estimation methods, MGLS and MRLS, were employed. These are also described in Chapter 4 and used a weight matrix of the form given in (4.10) with $W^* = \hat{S}^{-1}$ for the MGLS procedure and $W^* = \mathcal{S}\left(\hat{\theta}\right)^{-1}$for the MRLS procedure.

The minimization was accomplished using optimizers built in to the statistical programming language R (version 3.1.0). In particular, Gauss-Newton and quasi-Newton methods, as implemented in the *nlm* and *optim* functions of the R *stats* package, were used (R Core Team, 2014). The initial values of the free parameters $\boldsymbol{\theta}$, found in various components of the structural models defined in (4.13) and (4.14), were set to the default EQS start values as described by Bentler (2006). For each sample, these start values and the appropriate GLS function (i.e. the criterion function) were provided to the optimizers, and the final parameter estimates were taken from whichever method yielded the lowest estimate of the criterion. The minimization algorithms also returned a value for the GLS function in (4.7), which was multiplied by the sample size to obtain a χ^2-test statistic with $p^* - q$ degrees of freedom as shown in (4.8).

Once the parameter estimates were computed, the standard errors were calculated. The derivative needed to compute Δ (and the standard errors), shown in (4.5), was computed numerically using the *jacobian* function found in the R *numDeriv* package (Gilbert & Varadhan, 2012). Finally, (4.5) and (4.6) were used to obtain the standard errors of the parameter estimates.

Replications

Above we have described a variety of conditions, each with a particular model, number of observed variables, data type, and sample size. In each of these conditions, we drew 500 samples of size N from the specified distribution and calculated a set of results and/or fit the appropriate model. The 500 sets of results were recorded and analyzed. The details of these procedures are described in the following chapters.

Figures

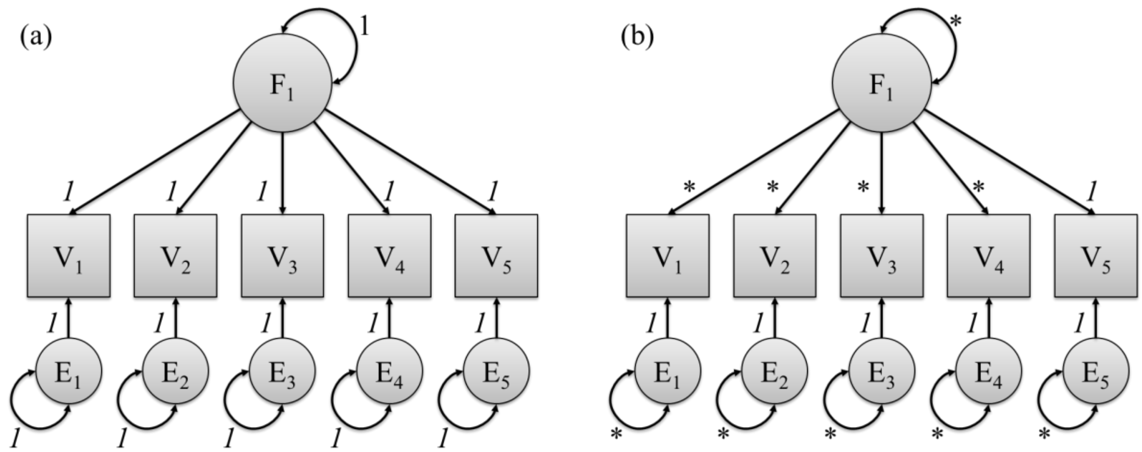

Figure 5.1. The 5-variable one-factor model is depicted, with (a) showing the true model and (b) showing the model that was estimated with the asterisks (*) indicating free parameters.

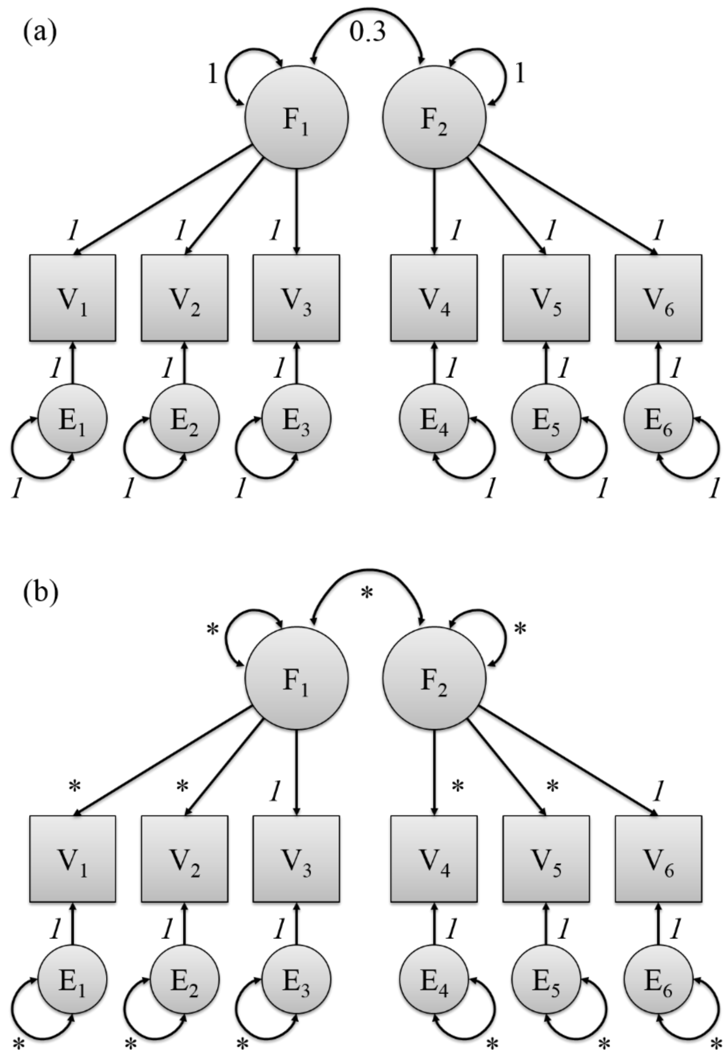

Figure 5.2. The 6-variable two-factor model is depicted, with (a) showing the true model and (b) showing the model that was estimated with the asterisks (*) indicating free parameters.

Chapter 6. Analysis of the Convergence of $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\psi}}$ to $\boldsymbol{\Sigma}_{\boldsymbol{\psi}}$

In order to examine the asymptotic behavior of our two estimators of $\Sigma_{\boldsymbol{\psi}}$, we compared the estimates produced from a large set of simulated samples to the population values. In the case of normally distributed data, with known parameters, the population value can be computed exactly using (3.10). However, when the data are non-normal this is not possible, so a very-large-sample estimate was used in place of the population value. The goals of this simulation were to confirm that the estimators did converge to their expected values, and to considered the samples sizes required to produce reasonable estimates of $\Sigma_{\boldsymbol{\psi}}$.

Method

Conditions. In this simulation, the simple one-factor model with 5 variables (with means of 1) was examined as the sample size increased. The sample size values considered were 100, 300, 500, 1000, 10,000, and 100,000 and both normal and log-normal data were analyzed.

Calculating the Population Values of $\boldsymbol{\Sigma}_{\boldsymbol{\psi}}$. When the data are normally distributed with known parameters, Σ_{Ψ} can be calculated exactly using (3.10). Because (3.10) contains redundant elements (as described in Chapter 3), this matrix was reduced using the elimination matrix, as is demonstrated in (3.18), for an estimator of Σ_{Ψ}. This yielded a $p^* \times p^*$ matrix, called $\Sigma_{\boldsymbol{\psi}}$, that is the population covariance matrix of $vech(\boldsymbol{\psi})$. Therefore, in the normal data conditions, our estimators ($\Sigma_{\boldsymbol{\psi}}$) could be compared directly with the population matrix $\Sigma_{\boldsymbol{\psi}}$.

However, for non-normal data there is no known way to calculate the population value of Σ_{Ψ}. Specifically, this is not known because (3.10) does not apply for non-normal data, and the arbitrary distribution form of Σ_{Ψ}, given in (3.6), requires knowledge of the population values of Ω_{12} and Ω_{22}, which are not known (descriptions of Ω_{12} and Ω_{22} may be found in Chapter 3).

Therefore, when the data were log-normal, instead of calculating an exact population value of Σ_{ψ} we produced a very-large-sample estimate of Σ_{ψ} using a sample size of 500,000 and used this in place of Σ_{ψ} in the analyses described below. In addition, to provide some validation for this approach, a very-large-sample estimate of Σ_{ψ} was also produced in the case of normal data and we examined the difference between this estimate and the known population value of Σ_{ψ}.[4]

Comparison of the $\hat{\Sigma}_{\psi}$ and Σ_{ψ}. To assess the difference between an estimate of $\hat{\Sigma}_{\psi}$ and the population matrix Σ_{ψ} we took the difference between the two matrices and because the matrices are both symmetric, we then selected only the unique elements of the difference matrix (by taking the half-vectorization). That is, we calculated a set of residuals (res) corresponding to the unique elements of Σ_{ψ} such that

$$res = vech(\hat{\Sigma}_{\psi} - \Sigma_{\psi}). \tag{6.1}$$

Note that since $\hat{\Sigma}_{\psi}$ and Σ_{ψ} have dimensions $p^* \times p^*$, res is a vector of length $p^{**} = \frac{p^*(p^*+1)}{2}$. Next, the elements of res were squared and the average of the elements was taken to produce a form of mean squared error (MSE) as follows:

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[4] Each of these matrices described in this section are also reported in Appendix B. The exact value of Σ_{ψ}, as calculated for the normal data condition considered here, is reported in Table B.1. The very-large-sample approximation of Σ_{ψ} that was calculated in the normal data condition is reported in Table B.2. In addition, the differences between the corresponding elements of Σ_{ψ} and its very-large-sample approximation are reported in Table B.3. Finally, the very-large-sample approximation of Σ_{ψ} that was calculated in condition using log-normal data is reported in Table B.4.

$$MSE = \frac{\sum_{i=i}^{p^{**}} res_i^2}{p^{**}} \qquad (6.2)$$

Note, the MSE in the above formula is a scalar index summarizing the average error of the elements of the matrix. This index was calculated in each sample using both the arbitrary distribution and normal theory versions of the estimator $\hat{\Sigma}_{\boldsymbol{\psi}}$. When the data were drawn from a normal distribution, then the known population value of $\Sigma_{\boldsymbol{\psi}}$ was used in (6.1). When the data were drawn from a log-normal distribution, the population value of $\Sigma_{\boldsymbol{\psi}}$ was not known, so the very-large-sample estimate was substituted into (6.1) in place of $\Sigma_{\boldsymbol{\psi}}$.[5]

Analyses. In each of the 500 samples, an arbitrary distribution theory and a normal theory value of the MSE were computed using the appropriate value of $\Sigma_{\boldsymbol{\psi}}$ (described above). This produced a set of 500 scalar MSE values for each method and condition. For each of these sets, the mean, standard deviation and five-number summaries were calculated and reported below.

Results

The summary statistics describing the MSE values in the normal data condition are presented in Table 6.1 for the arbitrary distribution form of the estimator of $\Sigma_{\boldsymbol{\psi}}$, and in Table 6.2 for the normal theory form of the estimator of $\Sigma_{\boldsymbol{\psi}}$. As shown in the tables, for the smallest samples (N=100) the average MSE value for the arbitrary distribution estimator was about 727 and that for the normal theory estimator was about 643. As the sample sizes increased, the average MSE values consistently decreased and in the largest sample size (N=100,000) these

[5] The value of Σ_{ψ} used to calculated the res and subsequently the MSE in the normal data condition is reported in Table B.1 of Appendix B and that used in the log-normal data condition is reported in Table B.4 of Appendix B.

values were about 0.15 for the arbitrary distribution estimator and 0.14 for the normal theory estimator. In this condition, with a few exceptions, the minimums, maximums, and quartiles also tended to decrease.

The summary statistics describing the MSE values in the log-normal data condition are presented in Table 6.3 for the arbitrary distribution form of the estimator of $\Sigma_{\boldsymbol{\psi}}$, and in Table 6.4 for the normal theory form of the estimator of $\Sigma_{\boldsymbol{\psi}}$. Unlike the normal data case, the average MSE values did not neatly converge towards 0. The average values seemed to be much larger than they were for normal data and the values went up and down as the sample size increased. This may be due to the skew and extreme values in the distribution. Extremely large maximum values of the MSE were particularly common when the arbitrary distribution estimator was used. However, the median values did consistently decrease for both methods. The median MSEs of the arbitrary distribution estimator, decreased from 7779 (with N=100) to 2260 (with N=100,000). However, those of the normal theory estimator only decreased from 7490 (with N=100) to 6650. Given these results, it seems that for this sort of non-normal data, the normal theory estimates are more stable, but typically less accurate than the arbitrary distribution theory estimators.

Discussion

In summary, these results show some instances where the estimators of $\Sigma_{\boldsymbol{\psi}}$ seem to behave appropriately and other instances where they break down. Specifically, the estimators seem to work fairly well when the data are normally distributed. In this case, both the arbitrary distribution and normal theory estimators converged asymptotically to the theoretical value of $\Sigma_{\boldsymbol{\psi}}$. However, the errors in these estimates may be quite large in samples with a size less than 1000 and it is difficult to say what effect this might have on the estimation process and the resulting parameters and test statistic values. In contrast to the case with normal data, the estimators had more difficultly when the data were log-normally distributed. The typical errors arising from the use of the arbitrary distribution estimator of $\Sigma_{\boldsymbol{\psi}}$ seemed to gradually decline, but the magnitudes of these errors at the observed sample sizes were in general very large, which may indicate that even if the arbitrary distribution estimator converges to $\Sigma_{\boldsymbol{\psi}}$, it may do this too slowly to be of any real use in estimation. Furthermore, extremely large values of the MSE were still prevalent at large sample sizes, indicating that for at least some samples the estimated values of $\Sigma_{\boldsymbol{\psi}}$ (particularly those given by the arbitrary distribution theory) were wildly inaccurate. Lastly, this simulation suggests that the normal theory estimator is not robust to violations of normality. Although the estimation errors produced for the log-normal data were often smaller than those of the arbitrary distribution estimator, the improvements seen with increases in sample size, seemed to stagnate quickly. This suggests that these estimators will not perform well in non-normal data, but how this will effect model estimation is not yet clear. The following chapters will begin to assess this question.

Additionally, there are some other possible explanations for the large discrepancies seen between the observed and theoretical values of $\Sigma_{\boldsymbol{\psi}}$ seen in the condition with log-normal data. The first and most obvious possibility, is that there is an error in the either the theoretical formulas or the implementation of the method. Extensive testing has not revealed a coding error, but that does not preclude the possibility. If there is a theoretical error in the formulas for $\Sigma_{\boldsymbol{\psi}}$ or $\hat{\Sigma}_{\boldsymbol{\psi}}$, that may be particularly difficult to detect. Besides human error, the discrepancies could be caused by the use of the very-large-sample approximation. The apparent convergence rate of the arbitrary distribution estimator shown in Table 6.3 is very slow but does seem to occur at least for median values. Perhaps when the data are log-normally distributed a sample larger than

500,000 is required to obtain an accurate fix on the population value. Moreover, there could be some unstated or unknown assumption or condition required by the estimators that is not met by some forms of non-normal data. For instance, there could be an implicit distributional assumption that is violated, or perhaps there is numerical instability is leading to violations of boundary conditions or division by near-zero numbers. The large and non-decreasing nature of the maximum values of the MSEs resulting from the arbitrary distribution estimator (relative to those of the normal theory estimators) lends some support for the possibility of a numerical instability issue. However, in any case, additional work is required to determine the source of the deviations between the theoretical and estimated values of $\Sigma_{\boldsymbol{\psi}}$.

This work could be extended to address the concerns raised above. For example, perhaps it is possible to find or construct a method of calculating the exact population value $\Sigma_{\boldsymbol{\psi}}$for arbitrary distributions. If future theoretical research into this was successful, it would be possible to see if the problem persists when the true value is used rather than a very-large-sample approximation. Moreover, work could be done to examine the computation stability of the estimated matrices. This could help determine whether there is a problem with division by a near-zero number and to determine when these matrices will be numerically stable enough to be inverted (as is required for CV model estimation). This topic is partially addressed in chapters 12 and 13, which follows up on some of the questions raised here and in the next few chapters.

Future work might also consider whether the individual elements of the $\hat{\Sigma}_{\boldsymbol{\psi}}$ exhibit differential performance. It could be the case that some of the elements converge more quickly than others, or that some have more potential variability. For instance, do the elements describing variances of elements of the CV matrix behave differently than those describing covariances of elements of the CV matrix? If it turned out that some of the elements were behaving poorly, this may help isolate a theoretical issue or help identify means of producing more robust estimates.

As noted above, it also remains to be seen how these estimators will perform in the task of model estimation. This question is considered in depth for both normal and non-normal data with different structures in the next few chapters.

Tables

Table 6.1. *Summary statistics for the MSEs of the arbitrary distribution version of* $\hat{\Sigma}_{\psi}$ *for normal data.*

N	Mean	SD	Min	Q1	Med	Q3	Max
100	727.38	2958.60	6.02	47.86	84.09	259.52	36829.15
300	81.97	157.52	2.36	16.19	29.69	69.13	1835.26
500	48.15	105.25	1.74	10.52	19.01	41.99	1514.97
1000	17.82	26.11	0.88	5.24	10.03	19.34	269.62
10,000	1.47	1.57	0.06	0.53	0.94	1.80	11.55
100,000	0.15	0.14	0.01	0.05	0.10	0.18	0.86

Table 6.2. *Summary statistics for the MSEs of the normal theory version of* $\hat{\Sigma}_{\psi}$ *for normal data.*

N	Mean	SD	Min	Q1	Med	Q3	Max
100	643.14	2620.74	6.35	45.36	79.02	243.69	39613.26
300	78.74	154.17	1.26	14.78	28.82	61.52	1467.97
500	45.03	103.11	0.74	9.98	17.59	39.62	1555.40
1000	17.11	25.55	0.52	5.25	9.41	17.82	255.36
10,000	1.40	1.50	0.04	0.50	0.88	1.68	10.13
100,000	0.14	0.14	0.00	0.05	0.09	0.18	0.83

Table 6.3. *Summary statistics for the $MSEs$ of the arbitrary distribution version of $\hat{\Sigma}_{\psi}$ for log-normal data.*

N	Mean	SD	Min	Q1	Med	Q3	Max
100	15339	113995	3202	6915	7779	8281	2465508
300	37038	195759	2846	6350	7244	8043	2292974
500	222368	2157830	3339	5871	6784	7836	29423057
1000	94859	1096735	2860	5271	6291	7647	23688653
10,000	881873	16697903	933	3139	4046	8604	372671488
100,000	107042	1473562	523	1476	2260	8944	31426238

Table 6.4. *Summary statistics for the $MSEs$ of the normal theory version of $\hat{\Sigma}_{\psi}$ for log-normal data.*

N	Mean	SD	Min	Q1	Med	Q3	Max
100	364848	7879649	2304	6804	7490	8037	176197165
300	11529	49026	3054	6407	7113	7624	854765
500	32510	311359	2691	6398	7035	7444	4363580
1000	6838	3235	3133	6308	6814	7196	72923
10,000	6628	520	4208	6456	6686	6847	14668
100,000	6632	112	5977	6571	6650	6703	6944

Chapter 7. Quality of Parameter, Standard Error and Test Statistic Estimates: One Factor Models of Normal Data with Variables with Equal Population Means

In this chapter and the following four chapters we examined the performance of the AGLS, NGLS, MGLS and MRLS methods of estimating CV models in terms of the parameter estimates, standard error estimates and test-statistics in finite samples. These chapters are organized by model and data type, with Chapters 7 through 9 addressing one-factor models and Chapters 10 and 11 addressing two-factor models. In addition, the current chapter, along with Chapters 8 and 10 consider models of normal data, whereas Chapters 9 and 11 consider data with a log-normal distribution. The current chapter specifically addresses simple one-factor models of normal data with all variable means equal to 1.

Method

Conditions. Normally distributed data were generated according to the structure shown in Figure 5.1. The models included either 5 or 20 variables, and the population means of the variables were either fixed to 1 or allowed to vary from 1 to 3 (see Chapter 5 for details). The sample sizes in the 5-variable case ranged from 100 to 100,000, whereas in the 20-variable case, they ranged from 300 to 100,000.

Analyses. For each condition and estimation method, 500 samples were drawn, and for each sample, χ^2-test statistics, parameter estimates and standard errors were calculated, as described in Chapters 4 and 5. In addition, each estimated χ^2-test statistic was compared to a critical value obtained from the appropriate asymptotic χ^2 distribution in order to conduct a hypothesis test to assess the model fit at the .05-level. The percentage of models rejected was then compared to the expected 5% rejection rate. Finally, the means and standard deviations of the parameter estimates and their corresponding standard errors were calculated. This allowed the average parameter estimates to be compared to their true values and the average estimated standard errors to be compared to the empirical standard errors (i.e. the standard deviations of the set of 500 parameter estimates).

Results

Convergence. In some replications, not all of the estimation methods were able to converge. In the 5-variable case with N=100, the AGLS method did not converge for two samples and the NGLS method did not converge for three samples. These cases were excluded from the analyses below. In addition, in the 20-variable case with N=300, there were two cases of non-convergence for the MGLS method.

Parameter Estimates and Standard Errors. Tables 7.1 through 7.4 show the means and standard deviations of the factor loadings, factor variance and errors variances for each of the four methods in the 5-variable condition. Each table displays the results for one sample size (100, 300, 500, and 1000, respectively). In the population, each parameter (including the factor loadings, the factor variance and the error variances) has a population value of 1 and the tabled results were contrasted with this value. In particular, all methods seemed to yield reasonable but slightly high estimates of the factor loadings and these estimates got closer to 1 as the sample size increased. For the first factor loading, this trend is depicted with boxplots in Figure 7.1. However, the factor and error variance estimates obtained from the AGLS and NGLS methods tended to be a low, particularly in the N=100 and N=300 conditions. This is illustrated in Figure 7.2 for the factor variance and Figure 7.3 for the first error variance. The figures also reveal how the variance estimates improved with increases in sample size.

The theoretical standard errors are also displayed in Tables 7.1 through 7.4. The quality of these estimates was assessed by contrasting them with the standard deviations of the parameters (i.e. the empirical standard errors). It seems that while the MGLS and MRLS methods tended to produce underestimates of the standard errors, the AGLS and NGLS methods produced slightly conservative standard error estimates. As the sample size increased, all methods yielded estimates that were progressively closer to the empirical standard errors. However, the AGLS and NGLS standard errors continued to be a bit high and the MGLS and MRLS methods continued to be a bit low.

Tables 7.5 through 7.8 show the means and standard deviations of some of the parameter estimates of each of the four methods in the 20-variable condition. Again, each table displays the results for one sample size (300, 500, 1000 and 10,000, respectively), and the tables show the results for the first 6 factor loadings, the factor variance and the first 7 error variances. In this case, the factor loading estimates were still quite accurate in all methods. The largest deviations from the population values of the factor loadings were seen in the N=300 condition for the AGLS estimates, which were somewhat higher than those of the other methods (see Table 7.5). In addition, there were differences in the variability of the estimates across methods, which is apparent in the boxplots for the first factor loading shown in Figure 7.4 (as well as in tables). While the factor loading estimates were generally good, the variance estimates for the factor and the errors were quite poor for the AGLS and NGLS methods. For example, Table 7.4 shows that in the smallest sample size, these values tended to be in the 0.2-0.4 range, well below the population value of 1.0. Interestingly, the NGLS variance estimates were lower than the AGLS, in spite of the data following a normal distribution. The variance estimates obtained from the NGLS and NRLS methods were substantially better (e.g. they were in the range of 0.8 to 1.1 in the N=300 condition). The underestimation of the factor and error variances by the AGLS and NGLS methods persisted into the largest samples sizes. With N=1000, the estimated variances were in the 0.5-0.6 range, and with N=10,000 they were in the 0.9-1.0 range. These trends can be seen in the boxplots of the factor variance and first error variance in Figures 7.5 and 7.6, respectively.

Again, the theoretical standard errors for the 20-variable conditions are displayed in Tables 7.5 through 7.8, and these were compared against the standard deviations of the parameters (i.e. the empirical standard errors). Similarly to the 5-variable condition, in the 20-variable condition the MGLS and MRLS methods tended to underestimate the standard errors, while the AGLS and NGLS methods tended to overestimate the standard errors. As the sample size increased, the all methods yielded better estimates, but the estimates from the AGLS and NGLS converged more quickly with the corresponding standard deviations of the parameters, and with N=10,000, the estimates were still noticeably better than the MGLS and MRLS estimates.

Test Statistics. In the 5-variable condition, the population/theoretical value of the χ^2-test statistic was 5. As shown in Table 7.9, in this condition the AGLS and NGLS methods underestimated the χ^2 value, but improved as the sample size increased. Specifically, with N=100, the average χ^2 value was 3.669 for AGLS and 3.820 for NGLS, and with N=1000, the average χ^2 value was 4.835 for AGLS and 4.823 for NGLS. In contrast, the MGLS and MRLS methods were consistently close to the population value for all sample sizes. Table 7.10 provides another look at the discrepancies between the observed and population χ^2 values by considering the results of hypothesis tests of the χ^2 value. Specifically, the table shows the percentage of replications that would have resulted in an incorrect rejection of the model, which should happen 5% of the time with $\alpha = 0.05$. While the MGLS method consistently provided rejection rates that were close to the correct value, with estimates between 4.0% and 6.0%, the AGLS, NGLS, and MRLS methods dramatically under-rejected at low sample sizes. The AGLS and NGLS methods did not reject any models when the sample size was 100, but they generally got closer to the appropriate value as the sample size increased. The MRLS method was a bit better, rejecting 3.6% of models in the N=100 condition and gradually improving. It is also worth noting that all methods over-rejected the model in the N=100,000 condition, with rejection rates of 5.8% for the AGLS method and 6.0% for the other methods. Figures 7.7 through 7.10 depict histograms of the χ^2-test statistics as the sample size increases. In particular, Figures 7.7 and 7.8 show the initial underestimation of the χ^2 value for the AGLS and NGLS methods, respectively. On the other hand, Figures 7.9 and 7.10 show that the MGLS and MRLS methods, respectively, generally produced χ^2 estimates that more closely matched the expected distribution.

The test statistics in the 20-variable condition followed the same trends as those observed in the 5-variable condition, but the discrepancies between the observed values and population values were more pronounced, and larger sample sizes were required to obtain approximately correct values. Table 7.11 shows the means and the standard deviations of the χ^2-test statistics for each method and sample size and Table 7.12 shows the rejection rates resulting from hypothesis tests of the model for each method and sample size. Once again, most of the methods underestimated the χ^2-test statistics. The underestimation was particularly profound for the AGLS and NGLS procedures, which yielded χ^2 values close to 50 in the N=100 condition and close to 100 in the N=1000 condition. Eventually, with N=100,000 the χ^2 values were closer to the expected value of 170. This issue was also reflected in the rejection rates, as the AGLS and NGLS methods did not reject any models in sample sizes less than 10,000. The extent of the deviation from the expected distribution is also displayed graphically in Figure 7.11 for the AGLS method and in Figure 7.12 for the NGLS method. The MGLS and MRLS procedures performed much better. The MGLS method had the best performance: the average χ^2 estimates were close to the population χ^2 value of 170, and the model rejection rate was close to the expected rejection rate of 5%. The histograms of the estimated χ^2 values obtained through MGLS also appear consistent with the expected χ^2 distribution, as shown in Figure 7.13. However, the MRLS method still somewhat underestimated the χ^2 values. With a sample size of 100, the average MRLS χ^2 value was about 161, and with a sample size of 1000, it was about 168. However, even this seemingly small deviation seen in the N=1000 condition was enough to produce a rather concerning issue with under-rejection. Specifically, only 2.60% of models were rejected. In Figure 7.14 shows histograms of the MRLS χ^2 estimates. The underestimation is particularly apparent in the N=300 and N=500 conditions.

Discussion

The results of this study seem to suggest that AGLS and NGLS may be viable methods for estimating CV models when there only a few variables and the sample size is quite large. However, they also suggest that when there are many variables, the estimation procedures will not yield accurate χ^2 values or parameter variance estimates unless the sample sizes are unrealistically large. In addition, this study showed that the MGLS and MRLS methods may be good choices for estimating CV models when the data are normal, even though they are misspecified. In particular, the MGLS method performed quite well in both the large and small samples with both 5 and 20 variables. This is interesting because it suggests that MGLS was able to provide stable parameter estimates and accurate test statistics in conditions when the correctly specified methods could not. The MRLS method also performed fairly well relative the AGLS and NGLS methods. However, it did not perform as well as the MGLS method particularly in terms of estimating the parameters and χ^2-test statistics in cases with a large number of variables.

It is also noteworthy that while all of the estimation methods seemed to be able to accurately estimate the factor loadings, the AGLS and NGLS methods tended to underestimate the variance parameters. This tendency was exacerbated when the model contained a large number of variables. This tendency has also been observed in ADF estimation of structural models of covariance matrices. In the present simulations, the models were identified by fixing path coefficients (i.e. for the error variances and for one of the factor loadings); however, perhaps the AGLS and NGLS estimation methods would do better if the models were identified by fixing variance components and estimating the path coefficients. This could be investigated by re-running these simulations using a slightly different model specification and comparing the results. If this is effective, it is also possible that a similar trick might be used in structural covariance models.

Also, with regards to the estimation of the test statistics for models containing a large number of variables, the AGLS and NGLS estimation methods produced χ^2 values that converged to their expected distributions at a comically slow rate. Possibly, theoretical work could be done to improve this convergence rate so that the χ^2 values were approximately correct within the range of realistic sample sizes. Without a solution along these lines, the AGLS and NGLS estimated model fit statistics will not be useful when there are more than a few variables to be modeled.

Furthermore, it should be noted that initially these simulations were planned to be completed in samples sizes ranging from 100 to 1000 for both the 5-variable and 20-variable conditions. However, in the 20-variable condition, it quickly became apparent that the $\hat{\Sigma}_{\boldsymbol{\psi}}$

matrices were numerically non-invertible with a sample size of 100. That is, in small samples the $\hat{\Sigma}_{\boldsymbol{\psi}}$ matrices were not positive semi-definite. Therefore, the simulations were not run in the smallest sample size condition in the 20-variable case, and the sample size range was increased to include larger values. However, in order to assess the source of this problem, additional simulations were run to consider potential causes of the numerical stability in $\hat{\Sigma}_{\boldsymbol{\psi}}$ and to examine a potential solution relying on Winsorization. Because it seems plausible that the root of the problem is that the CV matrix itself is unstable, in Chapter 12 we examined the condition numbers of the CV matrix relative to the sample covariance matrix. Then in Chapter 13 we examined the condition numbers of the $\hat{\Sigma}_{\boldsymbol{\psi}}$ relative to $\hat{\Sigma}_{\boldsymbol{S}}$ (the covariance matrix describing the sampling distribution of the sample covariance matrix).

Tables

Table 7.1. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of normal data with 5 variables with equal population means and N=100.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.015	1.015	1.011	1.017	0.893	0.816	0.802	0.811	0.821	0.823
		SD	0.241	0.242	0.239	0.244	0.369	0.303	0.299	0.295	0.307	0.318
	$SE_{\hat{\theta}}$	M	0.275	0.276	0.276	0.277	0.424	0.344	0.334	0.337	0.347	0.347
		SD	0.111	0.113	0.097	0.101	0.215	0.169	0.175	0.166	0.180	0.176
NGLS	$\hat{\theta}$	M	1.011	1.017	1.016	1.011	0.865	0.799	0.786	0.799	0.806	0.807
		SD	0.199	0.215	0.211	0.213	0.330	0.278	0.276	0.288	0.289	0.303
	$SE_{\hat{\theta}}$	M	0.281	0.281	0.284	0.283	0.424	0.349	0.338	0.343	0.350	0.350
		SD	0.082	0.087	0.086	0.080	0.202	0.169	0.174	0.171	0.181	0.176
MGLS	$\hat{\theta}$	M	1.008	1.017	1.018	1.018	1.108	1.011	0.985	0.997	1.014	1.015
		SD	0.225	0.236	0.238	0.241	0.472	0.367	0.375	0.374	0.388	0.379
	$SE_{\hat{\theta}}$	M	0.166	0.168	0.168	0.169	0.288	0.186	0.181	0.183	0.186	0.186
		SD	0.047	0.052	0.050	0.049	0.104	0.064	0.066	0.065	0.067	0.065
MRLS	$\hat{\theta}$	M	1.008	1.017	1.017	1.016	1.093	1.091	1.064	1.076	1.098	1.100
		SD	0.227	0.238	0.239	0.239	0.468	0.396	0.402	0.398	0.419	0.407
	$SE_{\hat{\theta}}$	M	0.171	0.173	0.173	0.174	0.290	0.195	0.189	0.191	0.194	0.195
		SD	0.049	0.054	0.052	0.050	0.105	0.067	0.069	0.067	0.071	0.068

Table 7.2. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of normal data with 5 variables with equal population means and N=300.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.007	1.021	1.010	1.019	0.955	0.922	0.929	0.938	0.938	0.929
		SD	0.125	0.131	0.129	0.125	0.228	0.188	0.184	0.185	0.202	0.187
	$SE_{\hat{\theta}}$	M	0.136	0.138	0.136	0.137	0.232	0.193	0.194	0.197	0.197	0.194
		SD	0.021	0.022	0.022	0.022	0.066	0.051	0.052	0.051	0.054	0.051
NGLS	$\hat{\theta}$	M	1.006	1.019	1.009	1.018	0.956	0.923	0.928	0.936	0.937	0.927
		SD	0.125	0.129	0.126	0.120	0.227	0.184	0.182	0.183	0.200	0.186
	$SE_{\hat{\theta}}$	M	0.136	0.138	0.137	0.138	0.232	0.194	0.195	0.198	0.198	0.196
		SD	0.019	0.020	0.020	0.020	0.065	0.051	0.051	0.050	0.054	0.051
MGLS	$\hat{\theta}$	M	1.006	1.020	1.010	1.019	1.035	0.998	1.003	1.012	1.014	1.002
		SD	0.129	0.134	0.131	0.126	0.258	0.198	0.198	0.198	0.215	0.201
	$SE_{\hat{\theta}}$	M	0.095	0.096	0.095	0.096	0.158	0.104	0.104	0.105	0.105	0.104
		SD	0.015	0.016	0.015	0.015	0.033	0.020	0.019	0.019	0.021	0.019
MRLS	$\hat{\theta}$	M	1.007	1.020	1.010	1.019	1.030	1.023	1.027	1.039	1.040	1.028
		SD	0.129	0.134	0.131	0.126	0.257	0.202	0.202	0.202	0.220	0.204
	$SE_{\hat{\theta}}$	M	0.096	0.097	0.096	0.097	0.158	0.105	0.105	0.107	0.106	0.106
		SD	0.015	0.016	0.016	0.015	0.033	0.020	0.020	0.019	0.021	0.020

Table 7.3. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of normal data with 5 variables with equal population means and N=500.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.005	1.004	1.005	1.007	0.973	0.955	0.961	0.951	0.955	0.957
		SD	0.092	0.098	0.097	0.094	0.166	0.146	0.157	0.151	0.143	0.146
	$SE_{\hat{\theta}}$	M	0.102	0.102	0.102	0.102	0.177	0.150	0.151	0.149	0.150	0.150
		SD	0.012	0.013	0.012	0.012	0.036	0.029	0.032	0.031	0.028	0.029
NGLS	$\hat{\theta}$	M	1.004	1.004	1.005	1.008	0.973	0.956	0.961	0.953	0.955	0.958
		SD	0.092	0.098	0.096	0.093	0.165	0.145	0.156	0.151	0.141	0.144
	$SE_{\hat{\theta}}$	M	0.102	0.102	0.102	0.102	0.178	0.150	0.151	0.150	0.150	0.151
		SD	0.011	0.012	0.011	0.011	0.035	0.029	0.032	0.031	0.028	0.029
MGLS	$\hat{\theta}$	M	1.005	1.005	1.005	1.008	1.021	1.002	1.007	0.999	1.000	1.004
		SD	0.093	0.100	0.098	0.095	0.177	0.151	0.164	0.158	0.145	0.150
	$SE_{\hat{\theta}}$	M	0.073	0.073	0.073	0.073	0.121	0.080	0.081	0.080	0.080	0.080
		SD	0.008	0.009	0.009	0.008	0.018	0.011	0.012	0.012	0.011	0.011
MRLS	$\hat{\theta}$	M	1.005	1.004	1.005	1.008	1.018	1.017	1.023	1.015	1.015	1.019
		SD	0.094	0.100	0.098	0.095	0.177	0.153	0.166	0.159	0.146	0.151
	$SE_{\hat{\theta}}$	M	0.074	0.074	0.074	0.074	0.122	0.081	0.081	0.081	0.081	0.081
		SD	0.008	0.009	0.008	0.008	0.018	0.011	0.012	0.012	0.011	0.011

Table 7.4. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of normal data with 5 variables with equal population means and N=1000.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.007	0.999	1.008	1.005	0.982	0.979	0.976	0.976	0.977	0.977
		SD	0.065	0.070	0.067	0.065	0.117	0.102	0.102	0.105	0.105	0.101
	$SE_{\hat{\theta}}$	M	0.070	0.070	0.071	0.070	0.124	0.106	0.106	0.105	0.106	0.106
		SD	0.005	0.006	0.006	0.006	0.017	0.015	0.014	0.015	0.015	0.014
NGLS	$\hat{\theta}$	M	1.006	0.999	1.008	1.004	0.983	0.980	0.977	0.976	0.978	0.977
		SD	0.064	0.070	0.066	0.064	0.117	0.101	0.102	0.104	0.105	0.101
	$SE_{\hat{\theta}}$	M	0.070	0.070	0.071	0.070	0.124	0.106	0.106	0.106	0.106	0.106
		SD	0.005	0.006	0.006	0.005	0.017	0.014	0.014	0.015	0.015	0.014
MGLS	$\hat{\theta}$	M	1.006	0.999	1.008	1.004	1.007	1.003	1.000	1.000	1.001	1.001
		SD	0.065	0.070	0.067	0.065	0.122	0.105	0.105	0.107	0.107	0.104
	$SE_{\hat{\theta}}$	M	0.052	0.052	0.052	0.052	0.085	0.057	0.057	0.056	0.057	0.056
		SD	0.004	0.004	0.004	0.004	0.009	0.006	0.006	0.006	0.006	0.006
MRLS	$\hat{\theta}$	M	1.006	0.999	1.008	1.004	1.005	1.011	1.008	1.007	1.009	1.008
		SD	0.065	0.071	0.067	0.065	0.122	0.106	0.106	0.107	0.108	0.104
	$SE_{\hat{\theta}}$	M	0.052	0.052	0.052	0.052	0.085	0.057	0.057	0.057	0.057	0.057
		SD	0.004	0.004	0.004	0.004	0.009	0.006	0.006	0.006	0.006	0.006

Table 7.5. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of normal data with 20 variables with equal population means and N=300.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.029	1.022	1.031	1.024	1.026	1.012	0.398	0.359	0.364	0.361	0.364	0.366	0.365	0.365
		SD	0.293	0.227	0.243	0.229	0.228	0.217	0.133	0.080	0.090	0.087	0.084	0.086	0.088	0.084
	$SE_{\hat{\theta}}$	M	0.308	0.280	0.293	0.280	0.286	0.282	0.182	0.147	0.148	0.147	0.148	0.148	0.148	0.150
		SD	0.777	0.164	0.393	0.138	0.234	0.213	0.047	0.035	0.037	0.035	0.036	0.036	0.037	0.039
NGLS	$\hat{\theta}$	M	1.005	1.003	1.003	1.002	1.005	1.002	0.306	0.265	0.269	0.270	0.269	0.271	0.270	0.271
		SD	0.068	0.073	0.068	0.070	0.068	0.065	0.042	0.055	0.059	0.060	0.056	0.057	0.057	0.058
	$SE_{\hat{\theta}}$	M	0.342	0.341	0.342	0.342	0.342	0.342	0.178	0.145	0.145	0.143	0.144	0.145	0.145	0.147
		SD	0.051	0.050	0.048	0.050	0.050	0.050	0.031	0.036	0.036	0.035	0.035	0.036	0.037	0.038
MGLS	$\hat{\theta}$	M	1.003	0.998	1.002	0.998	0.999	1.003	1.049	0.905	0.906	0.899	0.905	0.906	0.905	0.917
		SD	0.115	0.114	0.118	0.113	0.115	0.112	0.230	0.170	0.176	0.172	0.168	0.175	0.175	0.181
	$SE_{\hat{\theta}}$	M	0.084	0.083	0.084	0.084	0.083	0.084	0.149	0.083	0.083	0.082	0.083	0.083	0.083	0.084
		SD	0.011	0.011	0.011	0.011	0.011	0.011	0.029	0.015	0.015	0.015	0.015	0.016	0.016	0.016
MRLS	$\hat{\theta}$	M	1.003	0.998	1.002	0.998	0.999	1.003	1.039	1.084	1.082	1.078	1.086	1.086	1.085	1.094
		SD	0.115	0.114	0.118	0.113	0.115	0.113	0.229	0.199	0.203	0.198	0.200	0.207	0.207	0.207
	$SE_{\hat{\theta}}$	M	0.087	0.087	0.087	0.087	0.087	0.087	0.152	0.093	0.093	0.093	0.093	0.093	0.093	0.094
		SD	0.012	0.011	0.012	0.011	0.012	0.012	0.029	0.017	0.017	0.017	0.017	0.018	0.018	0.018

Table 7.6. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of normal data with 20 variables with equal population means and N=500.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.004	1.009	1.008	1.010	1.005	1.004	0.481	0.448	0.449	0.449	0.446	0.448	0.447	0.450
		SD	0.113	0.109	0.113	0.114	0.109	0.110	0.095	0.076	0.075	0.071	0.074	0.073	0.075	0.073
	$SE_{\hat{\theta}}$	M	0.175	0.177	0.176	0.176	0.176	0.176	0.145	0.114	0.113	0.115	0.112	0.114	0.114	0.114
		SD	0.033	0.035	0.035	0.035	0.034	0.032	0.026	0.022	0.021	0.023	0.023	0.021	0.022	0.022
NGLS	$\hat{\theta}$	M	0.999	0.997	0.999	1.000	1.000	1.001	0.416	0.385	0.384	0.384	0.384	0.383	0.385	0.386
		SD	0.041	0.041	0.043	0.039	0.043	0.043	0.040	0.056	0.056	0.057	0.056	0.055	0.058	0.055
	$SE_{\hat{\theta}}$	M	0.211	0.212	0.211	0.211	0.212	0.211	0.151	0.120	0.119	0.121	0.119	0.120	0.120	0.120
		SD	0.023	0.024	0.024	0.022	0.023	0.023	0.022	0.023	0.023	0.024	0.025	0.023	0.024	0.023
MGLS	$\hat{\theta}$	M	1.002	1.007	0.998	1.007	1.006	1.004	1.029	0.943	0.938	0.950	0.935	0.944	0.945	0.944
		SD	0.093	0.094	0.091	0.090	0.093	0.092	0.173	0.138	0.136	0.141	0.144	0.136	0.142	0.137
	$SE_{\hat{\theta}}$	M	0.065	0.065	0.065	0.065	0.065	0.065	0.114	0.065	0.065	0.066	0.065	0.065	0.065	0.065
		SD	0.007	0.007	0.007	0.007	0.007	0.007	0.017	0.009	0.009	0.010	0.010	0.009	0.010	0.009
MRLS	$\hat{\theta}$	M	1.002	1.007	0.998	1.007	1.005	1.005	1.023	1.053	1.046	1.059	1.044	1.054	1.055	1.049
		SD	0.093	0.094	0.091	0.090	0.093	0.092	0.173	0.151	0.153	0.156	0.156	0.153	0.156	0.151
	$SE_{\hat{\theta}}$	M	0.066	0.067	0.066	0.067	0.067	0.067	0.115	0.070	0.070	0.070	0.069	0.070	0.070	0.070
		SD	0.007	0.007	0.007	0.007	0.007	0.007	0.017	0.010	0.010	0.010	0.010	0.010	0.010	0.010

Table 7.7. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of normal data with 20 variables with equal population means and N=1000.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.001	0.998	1.001	1.002	0.999	1.002	0.626	0.604	0.604	0.602	0.601	0.599	0.599	0.603
		SD	0.061	0.057	0.060	0.061	0.059	0.060	0.073	0.061	0.061	0.063	0.061	0.058	0.064	0.060
	$SE_{\hat{\theta}}$	M	0.099	0.099	0.099	0.099	0.099	0.099	0.109	0.087	0.087	0.086	0.087	0.086	0.086	0.087
		SD	0.010	0.010	0.010	0.010	0.010	0.010	0.014	0.012	0.012	0.012	0.012	0.011	0.012	0.012
NGLS	$\hat{\theta}$	M	1.000	0.998	1.000	0.999	0.999	0.999	0.587	0.566	0.567	0.565	0.565	0.560	0.562	0.565
		SD	0.044	0.039	0.041	0.041	0.042	0.041	0.052	0.052	0.052	0.054	0.051	0.051	0.054	0.050
	$SE_{\hat{\theta}}$	M	0.110	0.110	0.110	0.110	0.110	0.110	0.113	0.091	0.091	0.090	0.091	0.090	0.090	0.091
		SD	0.009	0.009	0.009	0.009	0.009	0.009	0.014	0.012	0.012	0.013	0.012	0.012	0.013	0.012
MGLS	$\hat{\theta}$	M	1.000	0.997	1.001	1.000	0.999	1.000	1.014	0.975	0.974	0.970	0.973	0.966	0.967	0.973
		SD	0.068	0.062	0.064	0.066	0.066	0.065	0.124	0.098	0.096	0.101	0.099	0.097	0.103	0.097
	$SE_{\hat{\theta}}$	M	0.046	0.046	0.046	0.046	0.046	0.046	0.080	0.047	0.047	0.046	0.047	0.046	0.046	0.047
		SD	0.004	0.003	0.004	0.004	0.004	0.004	0.008	0.005	0.005	0.005	0.005	0.005	0.005	0.005
MRLS	$\hat{\theta}$	M	1.000	0.997	1.001	1.000	0.999	1.000	1.012	1.029	1.027	1.023	1.027	1.021	1.021	1.027
		SD	0.068	0.062	0.064	0.066	0.066	0.065	0.124	0.103	0.101	0.105	0.106	0.102	0.108	0.101
	$SE_{\hat{\theta}}$	M	0.047	0.046	0.047	0.046	0.046	0.047	0.080	0.048	0.048	0.048	0.048	0.048	0.048	0.048
		SD	0.004	0.003	0.004	0.004	0.004	0.004	0.008	0.005	0.005	0.005	0.005	0.005	0.005	0.005

Table 7.8. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of normal data with 20 variables with equal population means and N=10,000.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.000	1.000	1.001	1.001	1.001	1.002	0.933	0.931	0.931	0.930	0.932	0.932	0.932	0.932
		SD	0.020	0.019	0.019	0.020	0.019	0.019	0.034	0.030	0.028	0.030	0.029	0.028	0.029	0.030
	$SE_{\hat{\theta}}$	M	0.022	0.022	0.022	0.022	0.022	0.022	0.037	0.031	0.031	0.031	0.031	0.031	0.031	0.031
		SD	0.0006	0.0005	0.0005	0.0006	0.0005	0.0005	0.0016	0.0014	0.0013	0.0014	0.0014	0.0013	0.0013	0.0013
NGLS	$\hat{\theta}$	M	1.001	1.000	1.001	1.001	1.001	1.002	0.932	0.930	0.930	0.929	0.931	0.932	0.931	0.931
		SD	0.020	0.019	0.019	0.020	0.019	0.019	0.033	0.029	0.028	0.030	0.029	0.028	0.028	0.029
	$SE_{\hat{\theta}}$	M	0.022	0.022	0.022	0.022	0.022	0.022	0.037	0.031	0.031	0.031	0.031	0.031	0.031	0.031
		SD	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0016	0.0013	0.0013	0.0014	0.0013	0.0013	0.0013	0.0013
MGLS	$\hat{\theta}$	M	1.001	1.000	1.001	1.001	1.001	1.002	0.999	0.996	0.997	0.995	0.997	0.998	0.997	0.997
		SD	0.021	0.020	0.020	0.021	0.020	0.020	0.037	0.032	0.030	0.032	0.032	0.031	0.030	0.032
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.015	0.015	0.025	0.015	0.015	0.015	0.015	0.015	0.015	0.015
		SD	0.0003	0.0003	0.0003	0.0004	0.0003	0.0003	0.0008	0.0005	0.0004	0.0005	0.0005	0.0005	0.0004	0.0005
MRLS	$\hat{\theta}$	M	1.001	1.000	1.001	1.001	1.001	1.002	0.998	1.001	1.002	1.000	1.003	1.003	1.003	1.003
		SD	0.021	0.020	0.020	0.021	0.020	0.020	0.037	0.032	0.031	0.032	0.032	0.031	0.031	0.032
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.015	0.015	0.025	0.015	0.015	0.015	0.015	0.015	0.015	0.015
		SD	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0008	0.0005	0.0005	0.0005	0.0005	0.0005	0.0004	0.0005

Table 7.9. *Means and standard deviations of estimated* $\chi^2(df = 5)$ *test statistics of the one-factor model of normal data in the condition with 5 variables with equal population means.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
100	3.67	1.84	3.82	1.80	5.13	3.01	5.03	2.94
300	4.54	2.53	4.51	2.44	5.01	2.96	4.97	2.90
500	4.70	2.81	4.70	2.81	5.03	3.19	4.99	3.13
1000	4.84	2.85	4.82	2.81	4.99	2.99	5.00	3.02

Table 7.10. *Percent of replications in which the model of normal data was rejected by a* $\chi^2(df = 5)$ *test with* $\alpha = .05$ *in the condition with 5 variables with equal population means.*

N	AGLS	NGLS	MGLS	MRLS
100	0.0	0.0	5.6	3.6
300	1.4	1.4	4.0	3.6
500	2.4	2.8	4.6	4.0
1000	2.8	3.0	4.2	4.6
10,000	3.6	3.8	3.8	4.0
100,000	5.8	6.0	6.0	6.0

Table 7.11. *Means and standard deviations of estimated* $\chi^2(df = 170)$ *test statistics of the one-factor model of normal data in the condition with 20 variables with equal population means.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
300	49.66	5.61	49.86	5.05	169.19	18.29	161.26	15.86
500	75.25	7.07	68.99	5.34	170.22	18.75	163.98	16.92
1000	103.70	8.97	98.81	7.09	171.06	18.50	167.87	17.53
10,000	158.18	15.74	158.06	15.55	169.48	17.87	169.22	17.92
100,000	169.12	17.95	169.13	18.01	170.36	18.27	170.38	18.28

Table 7.12. *Percent of replications in which the one-factor model of normal data was rejected by a* $\chi^2(df = 170)$ *test with* $\alpha = .05$ *in the condition with 20 variables with equal population means.*

N	AGLS	NGLS	MGLS	MRLS
300	0.00	0.00	5.22	0.40
500	0.00	0.00	5.60	1.60
1000	0.00	0.00	6.20	2.60
10,000	0.20	0.20	3.60	4.40
100,000	4.00	4.00	5.00	5.20

Figures

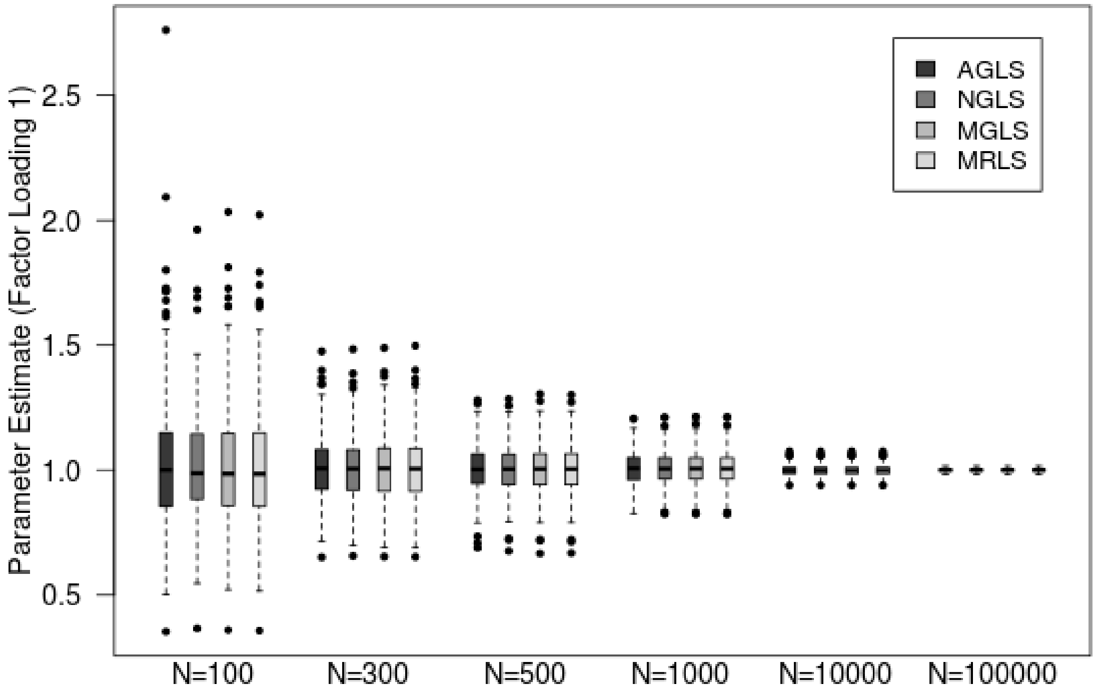

Figure 7.1. Boxplots of the values of the first factor loading estimated by the different methods across sample sizes (in the normal data condition with 5 variables with equal means).

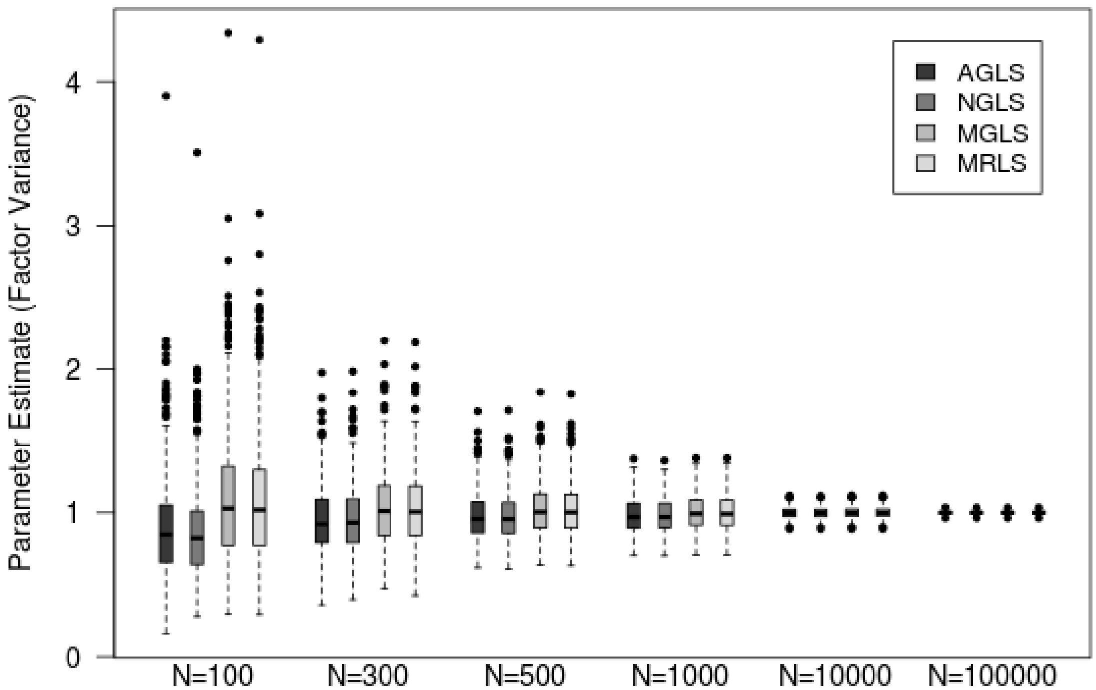

Figure 7.2. Boxplots of the values of the factor variance estimated by the different methods across sample sizes (in the normal data condition with 5 variables with equal means).

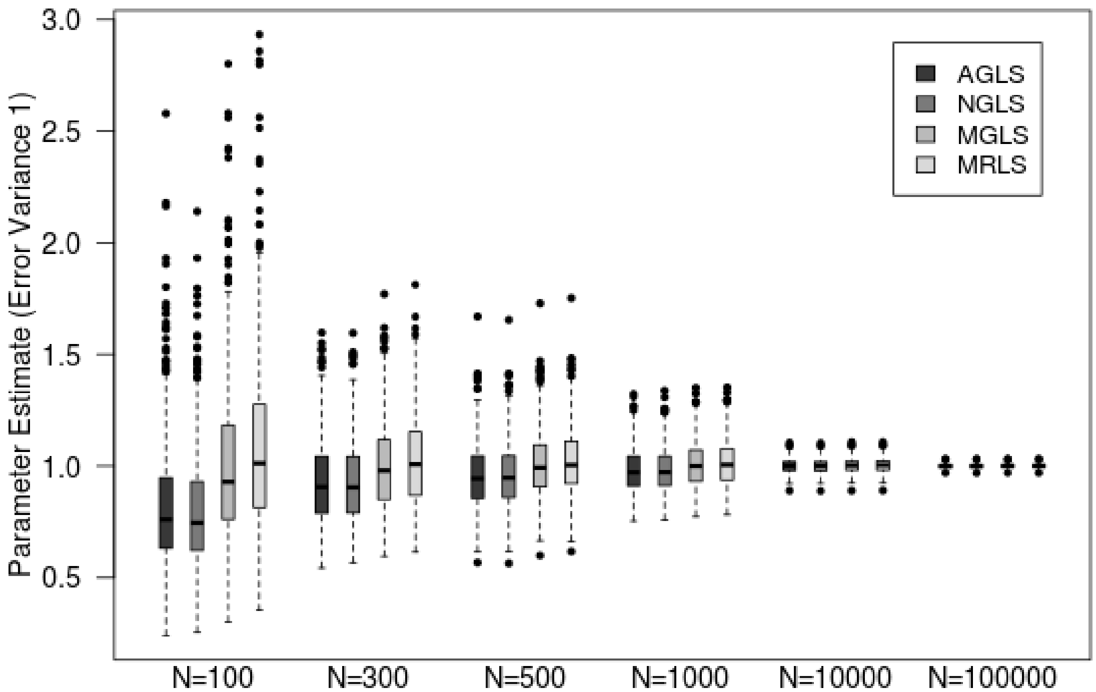

Figure 7.3. Boxplots of the values of the first error variance estimated by the different methods across sample sizes (in the normal data condition with 5 variables with equal means).

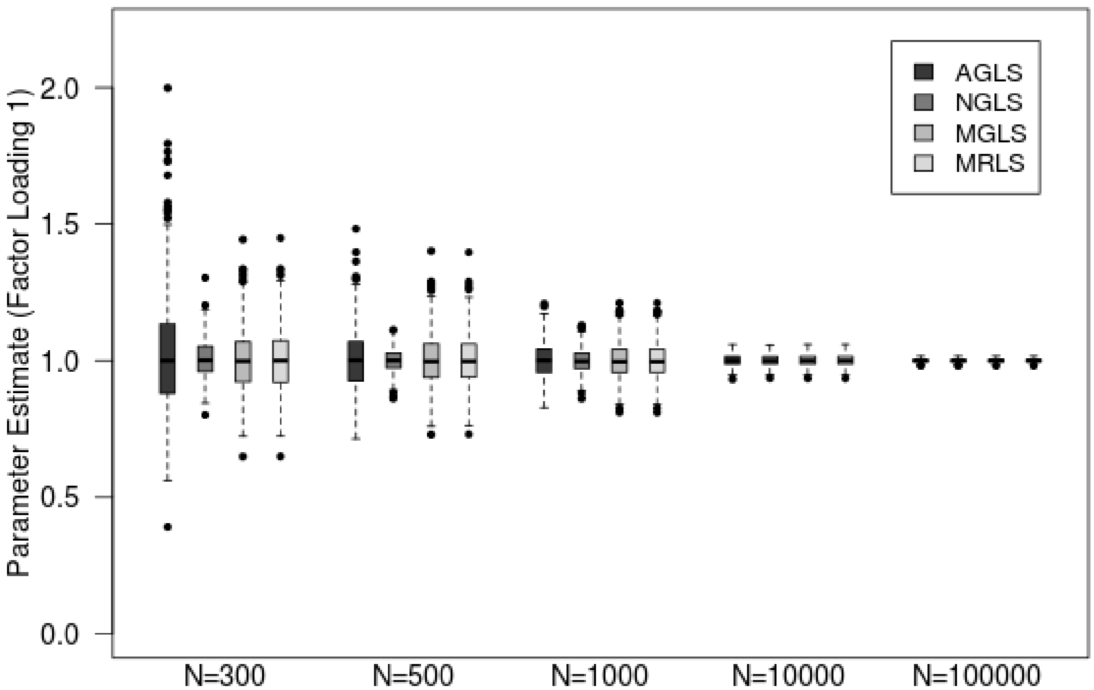

Figure 7.4. Boxplots of the values of the first factor loading estimated by the different methods across sample sizes (in the normal data condition with 20 variables with equal means).

Note that in the AGLS condition with *N*=300, there was one outlier (with a value of 5.7) that is not displayed above.

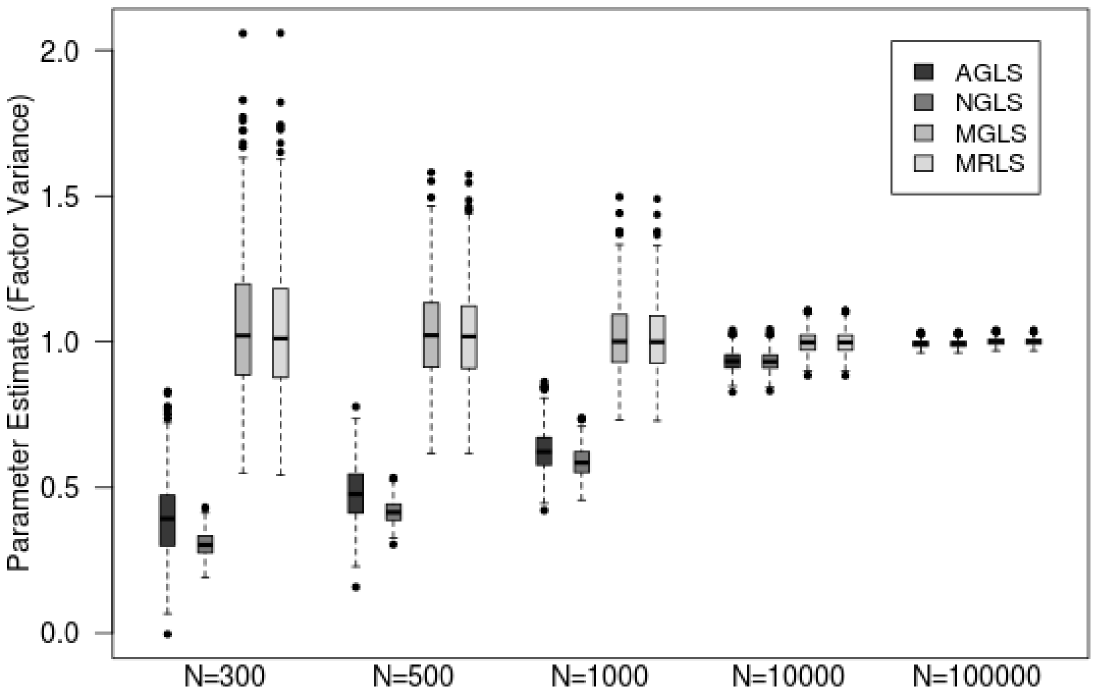

Figure 7.5. Boxplots of the values of the factor variance estimated by the different methods across sample sizes (in the normal data condition with 20 variables with equal means).

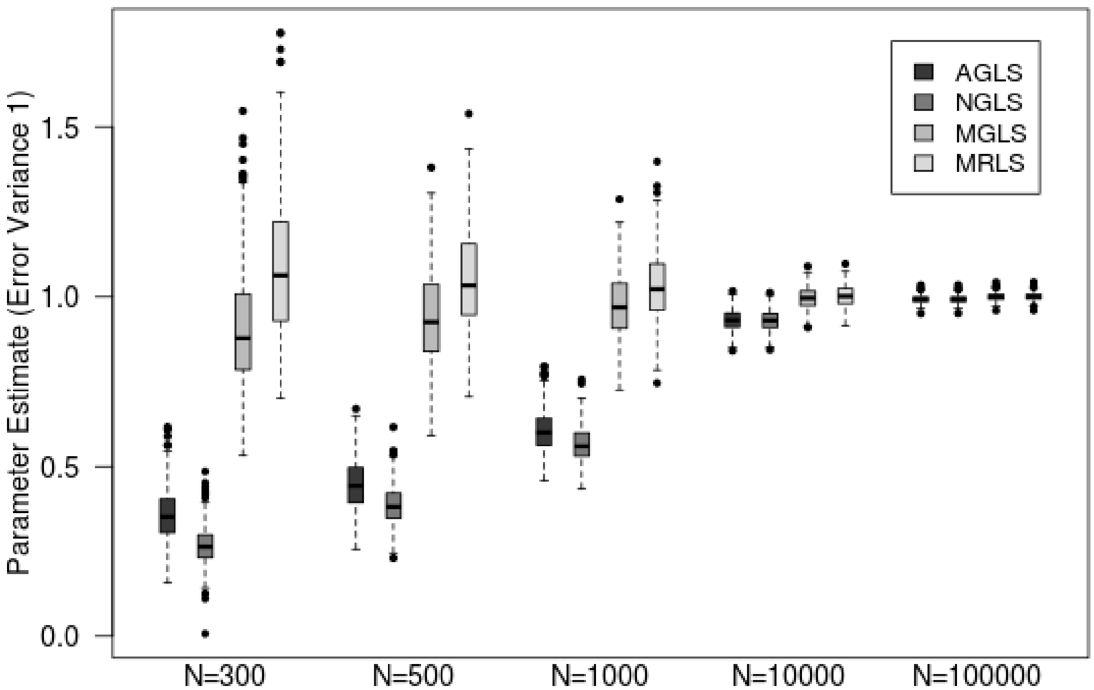

Figure 7.6. Boxplots of the values of the first error variance estimated by the different methods across sample sizes (in the normal data condition with 5 variables with equal means).

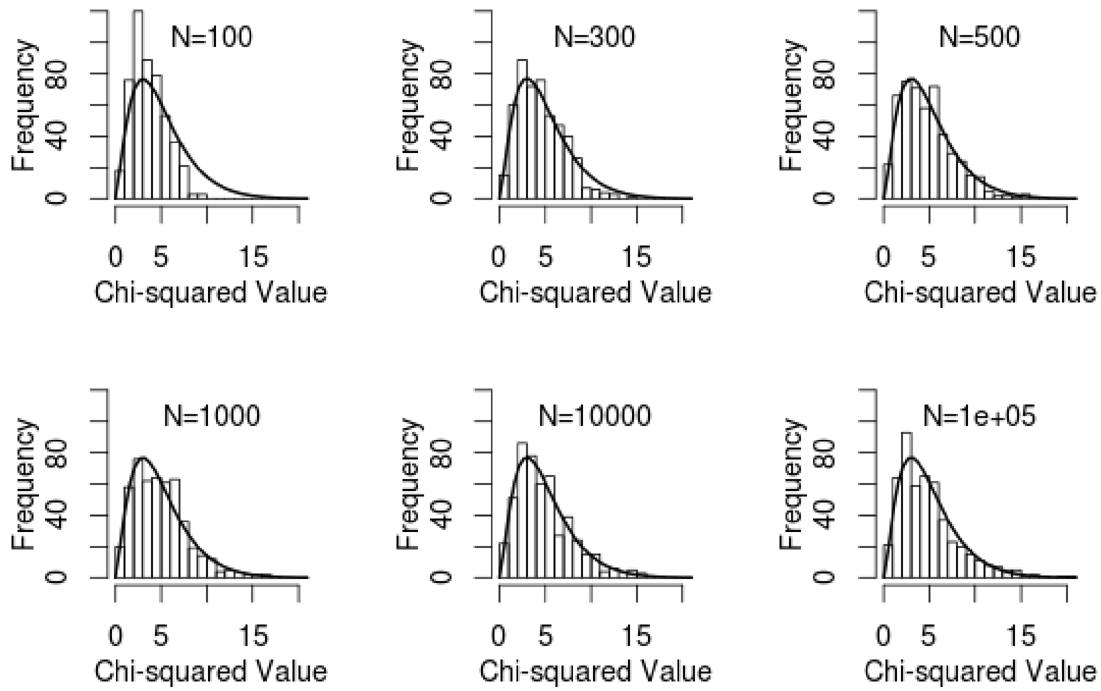

Figure 7.7. Histogram of the values of the χ^2-test statistic produced through AGLS estimation across sample sizes (in the normal data condition with 5 variables with equal means).

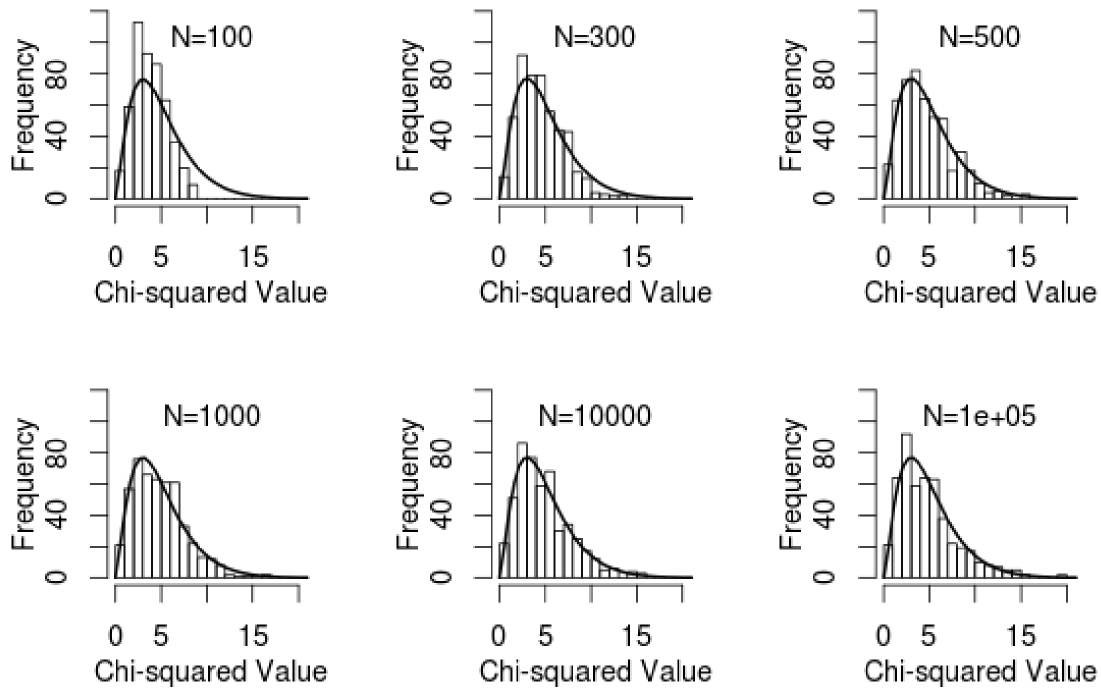

Figure 7.8. Histogram of the values of the χ^2-test statistic produced through NGLS estimation across sample sizes (in the normal data condition with 5 variables with equal means).

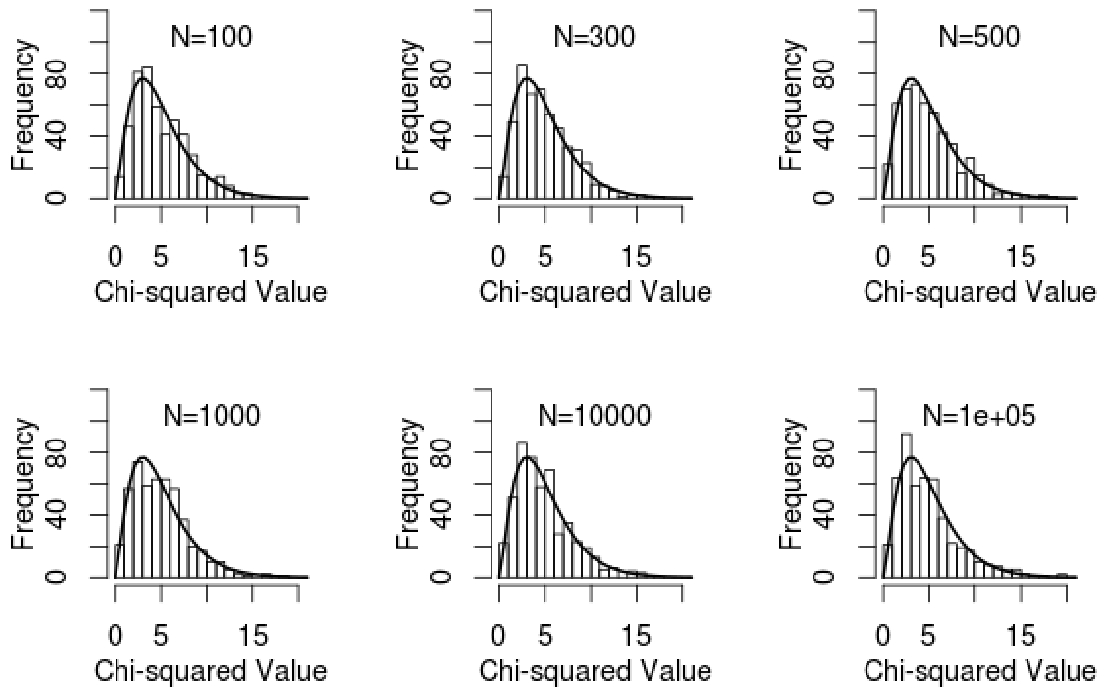

Figure 7.9. Histogram of the values of the χ^2-test statistic produced through MGLS estimation across sample sizes (in the normal data condition with 5 variables with equal means).

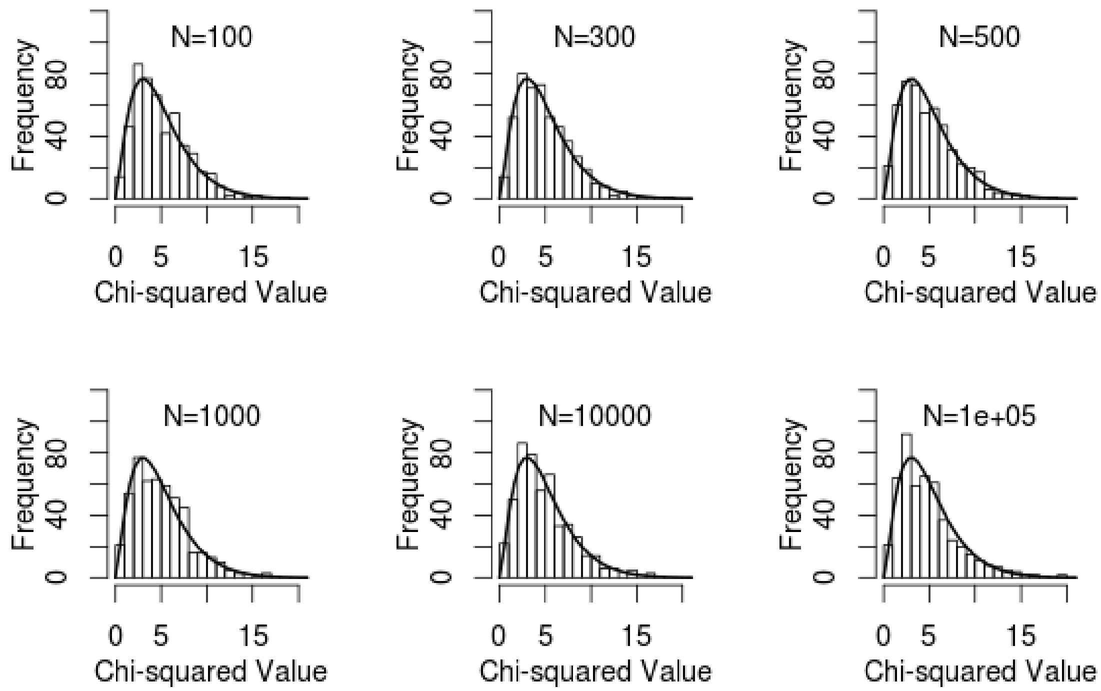

Figure 7.10. Histogram of the values of the χ^2-test statistic produced through MRLS estimation across sample sizes (in the normal data condition with 5 variables with equal means).

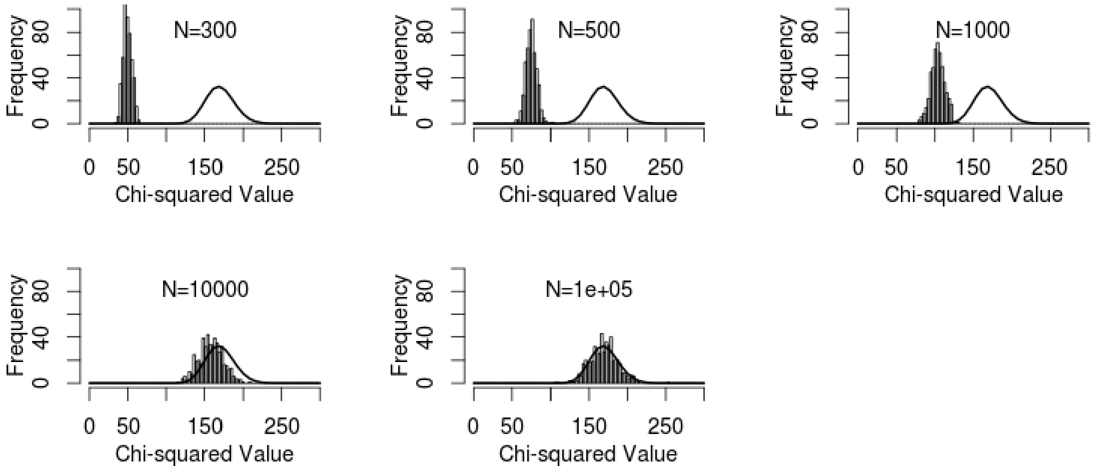

Figure 7.11. Histogram of the values of the χ^2-test statistic produced through AGLS estimation across sample sizes (in the normal data condition with 20 variables with equal means).

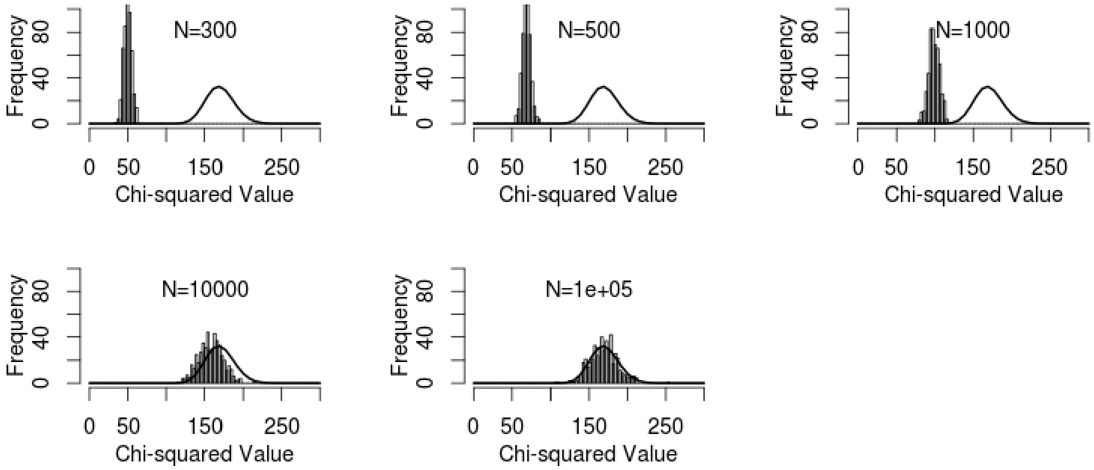

Figure 7.12. Histogram of the values of the χ^2-test statistic produced through NGLS estimation across sample sizes (in the normal data condition with 20 variables with equal means).

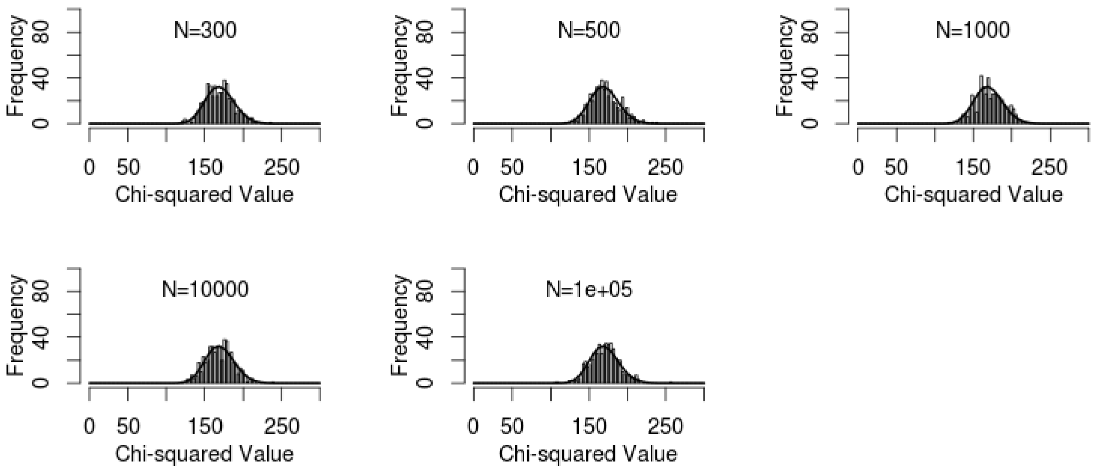

Figure 7.13. Histogram of the values of the χ^2-test statistic produced through MGLS estimation across sample sizes (in the normal data condition with 20 variables with equal means).

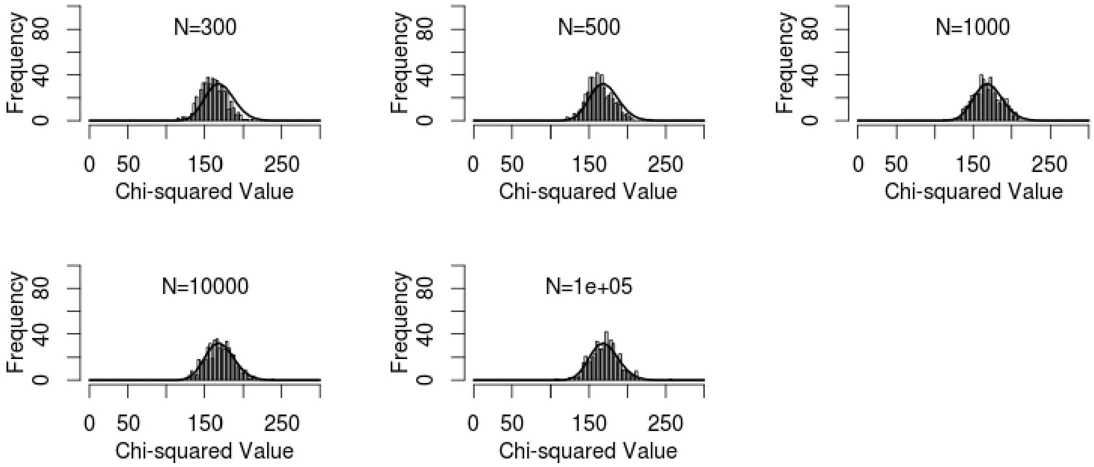

Figure 7.14. Histogram of the values of the χ^2-test statistic produced through MRLS estimation across sample sizes (in the normal data condition with 20 variables with equal means).

Chapter 8. Quality of Parameter, Standard Error and Test Statistic Estimates: One Factor Models of Normal Data with Variables with Unequal Population Means

In the previous chapter, we addressed simple one-factor models of normal data that had all variable means equal to 1 in the population. As described previously, in that case the covariance and the CV matrix are identical in the population. This makes it difficult to empirically verify whether the misspecified estimation methods (MGLS and MRLS), which are based on covariance theory, are in fact modelling the CV matrix appropriately. Therefore, this chapter repeats the analyses of the previous chapter in a population with variables that have means that deviate from 1 to provide additional verification of the accuracy of the estimation methods.

Method

Conditions. Here, normally distributed data were generated according to the structure shown in Figure 5.1. The models included either 5 or 20 variables, and the population means of the variables were allowed to vary from 1 to 3 (see Chapter 5 for details). The sample sizes in the 5-variable case ranged from 100 to 100,000, whereas in the 20-variable case, they ranged from 300 to 100,000.

Analyses. The analyses employed here were identical to those used in Chapters 7, so you may refer to the Chapter 7 methods for details.

Results

Convergence. In the condition with 5 variables and a sample size of 100, AGLS did not converge for one sample and NGLS did not converge for one sample. However, at larger sample sizes all methods were able to converge on a solution.

Parameter Estimates and Standard Errors. In general, the pattern of results observed here seemed comparable to that observed in the equal means condition in Chapter 7. The means and standard deviations of the parameter estimates and standard errors for the 5-variable condition are displayed in Tables 8.1 through 8.4 and those for the 20-variable condition are shown in Tables 8.5 through 8.8. (As in the previous chapter, the tables for the 20 variable condition show the results for the first 6 factor loadings, the factor variance and the first 7 error variances.) Again, each table contains the results of a different sample size condition. Comparing these tables to Table 7.1 through 7.8, respectively, reveals no systematic differences in the average estimates. For example, consider the results of the equal and unequal means conditions with a sample size of 100 shown in Table 7.1 and 8.1, respectively. Initially, it appears that the average estimates in the AGLS and NGLS methods deviate more from their population values in the unequal means condition than those in the equal means condition. However, this pattern does not persist in larger sample sizes or in the 20-variable condition, suggesting the deviations are spurious. Also, the boxplots of the of the parameter estimates for the first factor loading, the factor variance and the first error variance were contrasted with the those of Chapter 7. Because the distributions were close matches to the corresponding distributions in Figures 7.1 through 7.6, these are not provided.

Test Statistics. In addition, for the 5-variable condition, Table 8.9 shows the means and standard deviation of the χ^2 values and Table 8.10 shows the model rejection rates for each sample size and estimation method. Similarly, for the 20-variable, Table 8.11 shows the means and standard deviations of the χ^2 estimates and Table 8.12 shows the model rejection rates for each sample size and estimation method. The values in these tables are generally a close match for the comparable values in Tables 7.9 through 7.12. In the 5-variable condition, there appear to be some small differences between the values in the equal means condition and the unequal means condition shown here, but again, similar differences are not observed in the 20-variable condition. This suggests that the differences are probably not meaningful. In addition, the trends observed in the histograms of the χ^2 distributions for each method were the same as those shown in the corresponding equal means conditions in Figures 7.7 through 7.14. Therefore, these are not displayed below.

Discussion

The consistency of these results with the previous chapter's results is encouraging as it suggests that the all of the estimation methods are estimating the CV structure taking the mean structure into account appropriately. If the MGLS and MRLS methods, which are derived from covariance-theory methods, were still estimating a covariance structure, it may not have been apparent in the equal means condition, but it would have been apparent in the unequal means condition. However, since no substantial differences were apparent in the results it seems that this is not the case.

Tables

Table 8.1. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of normal data with 5 variables with unequal population means and N=100.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.022	1.034	1.020	1.016	0.872	0.807	0.789	0.810	0.816	0.795
		SD	0.250	0.258	0.246	0.233	0.366	0.319	0.285	0.289	0.304	0.317
	$SE_{\hat{\theta}}$	M	0.273	0.279	0.275	0.272	0.407	0.341	0.328	0.336	0.339	0.332
		SD	0.110	0.145	0.106	0.100	0.212	0.191	0.155	0.162	0.172	0.193
NGLS	$\hat{\theta}$	M	1.017	1.028	1.026	1.020	0.848	0.796	0.781	0.793	0.810	0.788
		SD	0.210	0.204	0.216	0.202	0.330	0.307	0.271	0.271	0.291	0.307
	$SE_{\hat{\theta}}$	M	0.276	0.282	0.281	0.278	0.410	0.343	0.330	0.339	0.344	0.334
		SD	0.073	0.079	0.082	0.075	0.206	0.191	0.155	0.159	0.175	0.197
MGLS	$\hat{\theta}$	M	1.024	1.035	1.029	1.021	1.075	0.994	0.971	0.989	1.004	0.975
		SD	0.241	0.236	0.241	0.229	0.476	0.396	0.353	0.351	0.376	0.378
	$SE_{\hat{\theta}}$	M	0.168	0.170	0.170	0.168	0.280	0.182	0.178	0.182	0.183	0.179
		SD	0.047	0.046	0.047	0.045	0.110	0.069	0.060	0.060	0.064	0.065
MRLS	$\hat{\theta}$	M	1.024	1.034	1.029	1.021	1.060	1.071	1.049	1.067	1.080	1.052
		SD	0.240	0.235	0.241	0.231	0.471	0.424	0.385	0.373	0.404	0.402
	$SE_{\hat{\theta}}$	M	0.173	0.175	0.175	0.173	0.283	0.190	0.186	0.190	0.191	0.187
		SD	0.047	0.048	0.049	0.048	0.111	0.072	0.063	0.063	0.067	0.067

Table 8.2. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of normal data with 5 variables with unequal population means and N=300.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.015	1.014	1.006	1.013	0.943	0.916	0.932	0.923	0.912	0.921
		SD	0.126	0.129	0.134	0.126	0.211	0.165	0.180	0.188	0.193	0.166
	$SE_{\hat{\theta}}$	M	0.137	0.137	0.136	0.136	0.226	0.189	0.193	0.191	0.187	0.191
		SD	0.023	0.023	0.024	0.022	0.056	0.044	0.049	0.051	0.051	0.047
NGLS	$\hat{\theta}$	M	1.015	1.013	1.005	1.013	0.940	0.915	0.929	0.924	0.910	0.919
		SD	0.124	0.126	0.130	0.123	0.207	0.162	0.178	0.185	0.191	0.164
	$SE_{\hat{\theta}}$	M	0.137	0.137	0.136	0.137	0.227	0.190	0.194	0.193	0.189	0.192
		SD	0.021	0.021	0.022	0.021	0.056	0.043	0.048	0.050	0.051	0.046
MGLS	$\hat{\theta}$	M	1.015	1.013	1.005	1.014	1.014	0.987	1.001	0.995	0.979	0.990
		SD	0.129	0.130	0.135	0.128	0.228	0.175	0.192	0.200	0.203	0.177
	$SE_{\hat{\theta}}$	M	0.096	0.096	0.095	0.096	0.155	0.102	0.104	0.103	0.102	0.103
		SD	0.015	0.015	0.016	0.015	0.028	0.017	0.018	0.019	0.020	0.018
MRLS	$\hat{\theta}$	M	1.016	1.013	1.005	1.014	1.009	1.012	1.025	1.020	1.003	1.015
		SD	0.129	0.130	0.135	0.128	0.228	0.181	0.197	0.203	0.206	0.182
	$SE_{\hat{\theta}}$	M	0.097	0.097	0.096	0.097	0.156	0.104	0.105	0.105	0.103	0.104
		SD	0.015	0.015	0.016	0.015	0.028	0.017	0.019	0.019	0.020	0.018

Table 8.3. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of normal data with 5 variables with unequal population means and N=500.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.005	0.998	1.002	1.004	0.962	0.952	0.950	0.938	0.945	0.947
		SD	0.100	0.099	0.095	0.101	0.168	0.144	0.143	0.133	0.143	0.140
	$SE_{\hat{\theta}}$	M	0.102	0.101	0.101	0.102	0.175	0.148	0.148	0.145	0.147	0.147
		SD	0.012	0.012	0.013	0.013	0.035	0.029	0.029	0.027	0.029	0.029
NGLS	$\hat{\theta}$	M	1.005	0.997	1.003	1.004	0.961	0.950	0.949	0.936	0.944	0.946
		SD	0.100	0.097	0.094	0.099	0.166	0.143	0.142	0.131	0.142	0.139
	$SE_{\hat{\theta}}$	M	0.102	0.101	0.102	0.102	0.175	0.149	0.149	0.146	0.148	0.148
		SD	0.012	0.012	0.012	0.012	0.035	0.029	0.029	0.026	0.029	0.028
MGLS	$\hat{\theta}$	M	1.006	0.997	1.003	1.004	1.007	0.996	0.994	0.980	0.989	0.992
		SD	0.102	0.099	0.096	0.101	0.178	0.152	0.149	0.138	0.150	0.147
	$SE_{\hat{\theta}}$	M	0.074	0.073	0.074	0.074	0.120	0.080	0.080	0.078	0.079	0.079
		SD	0.009	0.009	0.009	0.009	0.018	0.011	0.011	0.010	0.011	0.011
MRLS	$\hat{\theta}$	M	1.005	0.997	1.003	1.004	1.004	1.010	1.009	0.994	1.004	1.008
		SD	0.102	0.099	0.096	0.101	0.178	0.155	0.151	0.140	0.152	0.151
	$SE_{\hat{\theta}}$	M	0.074	0.074	0.074	0.074	0.120	0.080	0.080	0.079	0.080	0.080
		SD	0.009	0.009	0.009	0.009	0.018	0.011	0.011	0.010	0.011	0.011

Table 8.4. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of normal data with 5 variables with unequal population means and N=1000.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.001	1.004	0.999	0.999	0.985	0.975	0.974	0.976	0.969	0.963
		SD	0.066	0.069	0.069	0.068	0.123	0.108	0.108	0.108	0.102	0.103
	$SE_{\hat{\theta}}$	M	0.070	0.070	0.070	0.070	0.124	0.106	0.105	0.106	0.104	0.104
		SD	0.006	0.006	0.006	0.006	0.018	0.015	0.015	0.015	0.014	0.014
NGLS	$\hat{\theta}$	M	1.001	1.004	0.999	1.000	0.985	0.976	0.975	0.976	0.968	0.963
		SD	0.066	0.068	0.069	0.067	0.122	0.109	0.108	0.107	0.101	0.102
	$SE_{\hat{\theta}}$	M	0.070	0.070	0.070	0.070	0.124	0.106	0.106	0.106	0.105	0.104
		SD	0.006	0.006	0.006	0.005	0.017	0.015	0.015	0.015	0.014	0.014
MGLS	$\hat{\theta}$	M	1.001	1.004	0.999	1.000	1.009	1.000	0.999	1.000	0.992	0.986
		SD	0.067	0.069	0.070	0.068	0.126	0.111	0.110	0.109	0.103	0.104
	$SE_{\hat{\theta}}$	M	0.052	0.052	0.052	0.052	0.085	0.056	0.056	0.056	0.056	0.056
		SD	0.004	0.004	0.004	0.004	0.009	0.006	0.006	0.006	0.005	0.006
MRLS	$\hat{\theta}$	M	1.001	1.004	0.999	1.000	1.007	1.007	1.007	1.007	0.999	0.994
		SD	0.067	0.069	0.069	0.068	0.126	0.112	0.110	0.110	0.105	0.104
	$SE_{\hat{\theta}}$	M	0.052	0.052	0.052	0.052	0.085	0.057	0.057	0.057	0.056	0.056
		SD	0.004	0.004	0.004	0.004	0.009	0.006	0.006	0.006	0.006	0.006

Table 8.5. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of normal data with 20 variables with unequal population means and N=300.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.022	1.040	1.026	1.019	1.032	1.033	0.391	0.367	0.358	0.363	0.364	0.363	0.362	0.365
		SD	0.212	0.227	0.206	0.194	0.219	0.227	0.123	0.086	0.083	0.085	0.091	0.086	0.085	0.087
	$SE_{\hat{\theta}}$	M	0.285	0.283	0.279	0.278	0.281	0.282	0.179	0.146	0.144	0.145	0.148	0.145	0.146	0.147
		SD	0.242	0.150	0.108	0.104	0.136	0.126	0.043	0.036	0.035	0.036	0.036	0.035	0.035	0.037
NGLS	$\hat{\theta}$	M	1.004	1.003	0.999	0.998	1.002	1.000	0.307	0.271	0.266	0.269	0.268	0.271	0.268	0.268
		SD	0.070	0.068	0.068	0.066	0.067	0.067	0.042	0.059	0.056	0.057	0.062	0.061	0.056	0.060
	$SE_{\hat{\theta}}$	M	0.338	0.338	0.338	0.337	0.338	0.338	0.177	0.142	0.140	0.143	0.144	0.141	0.143	0.144
		SD	0.047	0.046	0.045	0.045	0.046	0.046	0.031	0.035	0.035	0.036	0.036	0.035	0.036	0.037
MGLS	$\hat{\theta}$	M	1.000	1.005	1.005	1.002	1.006	1.006	1.024	0.896	0.884	0.895	0.901	0.890	0.896	0.902
		SD	0.176	0.114	0.135	0.112	0.142	0.130	0.220	0.171	0.173	0.176	0.174	0.172	0.170	0.176
	$SE_{\hat{\theta}}$	M	0.092	0.086	0.088	0.088	0.089	0.088	0.146	0.082	0.081	0.082	0.082	0.081	0.082	0.082
		SD	0.180	0.053	0.086	0.087	0.105	0.092	0.027	0.015	0.015	0.016	0.015	0.015	0.015	0.016
MRLS	$\hat{\theta}$	M	1.006	1.005	1.008	1.000	1.010	1.009	1.018	1.071	1.062	1.076	1.084	1.066	1.068	1.083
		SD	0.109	0.111	0.117	0.111	0.113	0.111	0.215	0.195	0.197	0.205	0.203	0.198	0.199	0.210
	$SE_{\hat{\theta}}$	M	0.087	0.087	0.088	0.087	0.087	0.088	0.149	0.092	0.091	0.092	0.093	0.092	0.092	0.093
		SD	0.012	0.012	0.012	0.011	0.012	0.012	0.027	0.016	0.017	0.017	0.017	0.017	0.017	0.018

Table 8.6. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of normal data with 20 variables with unequal population means and N=500.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.001	1.001	1.012	1.005	1.008	1.007	0.481	0.455	0.447	0.448	0.447	0.453	0.451	0.453
		SD	0.105	0.112	0.109	0.104	0.118	0.108	0.090	0.075	0.071	0.075	0.071	0.073	0.073	0.076
	$SE_{\hat{\theta}}$	M	0.173	0.173	0.174	0.172	0.173	0.174	0.144	0.114	0.111	0.112	0.112	0.113	0.114	0.113
		SD	0.028	0.029	0.030	0.027	0.030	0.029	0.024	0.021	0.020	0.021	0.022	0.022	0.023	0.022
NGLS	$\hat{\theta}$	M	0.999	0.996	1.001	1.002	1.000	0.999	0.418	0.389	0.386	0.387	0.384	0.386	0.387	0.389
		SD	0.042	0.042	0.046	0.044	0.042	0.042	0.041	0.054	0.055	0.057	0.057	0.056	0.055	0.056
	$SE_{\hat{\theta}}$	M	0.209	0.208	0.209	0.209	0.209	0.209	0.150	0.119	0.117	0.118	0.118	0.119	0.120	0.119
		SD	0.021	0.022	0.022	0.022	0.023	0.022	0.022	0.022	0.021	0.022	0.024	0.024	0.024	0.023
MGLS	$\hat{\theta}$	M	1.001	0.999	1.005	1.007	1.005	1.002	1.017	0.940	0.929	0.931	0.931	0.937	0.941	0.939
		SD	0.088	0.086	0.094	0.091	0.092	0.087	0.162	0.133	0.128	0.133	0.140	0.141	0.141	0.138
	$SE_{\hat{\theta}}$	M	0.065	0.065	0.065	0.065	0.065	0.065	0.113	0.065	0.064	0.064	0.064	0.065	0.065	0.065
		SD	0.007	0.006	0.007	0.007	0.007	0.006	0.016	0.009	0.009	0.009	0.010	0.010	0.010	0.009
MRLS	$\hat{\theta}$	M	1.001	0.999	1.005	1.007	1.005	1.002	1.012	1.047	1.035	1.037	1.036	1.042	1.047	1.048
		SD	0.088	0.086	0.094	0.091	0.092	0.087	0.162	0.144	0.145	0.149	0.156	0.156	0.153	0.151
	$SE_{\hat{\theta}}$	M	0.066	0.066	0.067	0.067	0.067	0.067	0.114	0.070	0.069	0.069	0.069	0.069	0.070	0.070
		SD	0.007	0.006	0.007	0.007	0.007	0.006	0.016	0.009	0.010	0.010	0.010	0.010	0.010	0.010

Table 8.7. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of normal data with 20 variables with unequal population means and N=1000.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.000	1.000	0.999	0.998	0.998	0.999	0.630	0.603	0.601	0.601	0.602	0.602	0.597	0.602
		SD	0.060	0.056	0.058	0.058	0.059	0.059	0.072	0.065	0.063	0.063	0.060	0.060	0.066	0.066
	$SE_{\hat{\theta}}$	M	0.099	0.099	0.099	0.099	0.099	0.099	0.110	0.087	0.087	0.087	0.087	0.086	0.086	0.087
		SD	0.010	0.010	0.010	0.010	0.010	0.009	0.014	0.012	0.012	0.012	0.011	0.012	0.012	0.012
NGLS	$\hat{\theta}$	M	1.000	1.000	0.999	0.998	1.001	1.001	0.590	0.566	0.563	0.565	0.566	0.567	0.562	0.567
		SD	0.040	0.040	0.042	0.040	0.042	0.043	0.054	0.054	0.053	0.055	0.052	0.052	0.056	0.056
	$SE_{\hat{\theta}}$	M	0.109	0.110	0.109	0.109	0.109	0.110	0.113	0.091	0.091	0.091	0.091	0.091	0.090	0.091
		SD	0.009	0.009	0.009	0.009	0.009	0.009	0.015	0.013	0.013	0.013	0.012	0.012	0.013	0.013
MGLS	$\hat{\theta}$	M	1.002	1.001	0.999	0.997	1.002	1.002	1.018	0.975	0.969	0.972	0.972	0.972	0.968	0.974
		SD	0.062	0.062	0.065	0.064	0.066	0.065	0.128	0.103	0.100	0.101	0.093	0.096	0.103	0.102
	$SE_{\hat{\theta}}$	M	0.046	0.046	0.046	0.046	0.046	0.046	0.080	0.047	0.046	0.047	0.047	0.047	0.046	0.047
		SD	0.004	0.004	0.004	0.004	0.003	0.004	0.009	0.005	0.005	0.005	0.004	0.005	0.005	0.005
MRLS	$\hat{\theta}$	M	1.002	1.001	0.999	0.997	1.002	1.002	1.015	1.029	1.024	1.027	1.026	1.025	1.022	1.028
		SD	0.062	0.062	0.065	0.063	0.066	0.065	0.128	0.107	0.105	0.106	0.098	0.103	0.108	0.105
	$SE_{\hat{\theta}}$	M	0.046	0.046	0.046	0.046	0.046	0.046	0.080	0.048	0.048	0.048	0.048	0.048	0.048	0.048
		SD	0.004	0.004	0.004	0.004	0.004	0.004	0.009	0.005	0.005	0.005	0.005	0.005	0.005	0.005

Table 8.8. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model with 20 variables with unequal population means and N=10,000.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.000	0.999	1.000	1.000	0.998	1.001	0.935	0.931	0.929	0.931	0.929	0.929	0.927	0.931
		SD	0.020	0.020	0.020	0.020	0.021	0.020	0.036	0.030	0.029	0.029	0.030	0.031	0.028	0.030
	$SE_{\hat{\theta}}$	M	0.022	0.022	0.022	0.022	0.022	0.022	0.037	0.031	0.031	0.031	0.031	0.031	0.031	0.031
		SD	0.001	0.001	0.001	0.001	0.001	0.001	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001
NGLS	$\hat{\theta}$	M	1.000	1.000	1.000	1.000	0.999	1.001	0.934	0.930	0.928	0.930	0.928	0.928	0.926	0.930
		SD	0.020	0.020	0.020	0.020	0.021	0.020	0.036	0.029	0.029	0.028	0.029	0.030	0.028	0.030
	$SE_{\hat{\theta}}$	M	0.022	0.022	0.022	0.022	0.022	0.022	0.037	0.031	0.031	0.031	0.031	0.031	0.031	0.031
		SD	0.001	0.001	0.001	0.000	0.001	0.001	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001
MGLS	$\hat{\theta}$	M	1.000	0.999	1.000	1.000	0.999	1.001	1.000	0.996	0.994	0.996	0.994	0.994	0.992	0.996
		SD	0.020	0.021	0.021	0.021	0.022	0.020	0.039	0.032	0.031	0.031	0.032	0.032	0.031	0.032
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.015	0.015	0.025	0.015	0.015	0.015	0.015	0.015	0.015	0.015
		SD	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
MRLS	$\hat{\theta}$	M	1.000	0.999	1.000	1.000	0.999	1.001	1.000	1.002	1.000	1.001	1.000	0.999	0.998	1.001
		SD	0.020	0.021	0.021	0.021	0.022	0.020	0.039	0.032	0.031	0.031	0.032	0.032	0.031	0.032
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.015	0.015	0.025	0.015	0.015	0.015	0.015	0.015	0.015	0.015
		SD	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 8.9. *Means and standard deviations of estimated* $\chi^2(df = 5)$ *test statistics of the one-factor model of normal data in the condition with 5 variables with unequal population means.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
100	3.523	1.867	4.359	2.482	4.662	2.675	4.956	3.034
300	3.674	1.863	4.418	2.476	4.676	2.642	4.941	2.979
500	4.926	3.227	4.899	3.011	4.987	2.994	5.112	3.186
1000	4.788	2.915	4.853	2.953	4.987	3.030	5.103	3.178

Table 8.10. *Percent of replications in which the one-factor model was rejected by a* $\chi^2(df = 5)$ *test with* $\alpha = .05$ *in the condition with 5 variables with unequal population means.*

N	AGLS	NGLS	MGLS	MRLS
100	0.0	0.0	5.0	2.8
300	1.2	1.6	4.0	4.2
500	2.2	2.2	4.0	4.6
1000	4.4	4.2	6.0	5.6
10,000	4.4	4.6	4.6	4.6
100,000	5.6	5.6	5.6	5.2

Table 8.11. *Means and standard deviations of estimated* $\chi^2(df = 170)$ *test statistics of the one-factor model of normal data in the condition with 20 variables with unequal population means.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
300	49.75	5.53	49.97	5.32	170.16	19.67	161.17	16.32
500	75.22	7.16	69.00	5.53	168.38	19.10	162.85	17.21
1000	103.18	9.01	98.35	7.15	170.00	18.86	167.49	18.18
10,000	158.27	15.20	158.18	14.78	169.59	16.99	169.35	16.85
100,000	168.50	18.26	168.45	18.26	169.67	18.52	169.64	18.49

Table 8.12. *Percent of replications in which the one-factor model was rejected by a* $\chi^2(df = 170)$ *test with* $\alpha = .05$ *in the condition with 20 variables with unequal population means.*

N	AGLS	NGLS	MGLS	MRLS
300	0.0	0.0	5.8	0.4
500	0.0	0.0	4.4	1.0
1000	0.0	0.0	5.6	3.2
10,000	0.0	0.2	3.2	3.2
100,000	4.6	4.6	4.6	4.8

Chapter 9. Quality of Parameter, Test Statistic, and Standard Error Estimates: One Factor Models of Log-Normal Data

In this chapter we continued to examine the performance of the AGLS, NGLS, MGLS and MRLS methods of estimating CV models. Specifically, here we assessed the performance of the parameter estimates, standard error estimates and the test-statistics in finite samples of log-normal data with a one-factor population CV structure.

Method

Conditions. In this case, log-normal data were generated using the procedure described in Chapter 5. Once again, the structure shown in Figure 5.1 was used to form the population CV matrices including either 5 or 20 variables. The means were each set to 1 in this condition. The sample sizes in the 5-variable case ranged from 100 to 100,000, whereas in the 20-variable case, ranged from 300 to 100,000.

Analyses. The analyses employed here were identical to those used in Chapters 7 and 8. For details refer to the analyses section of Chapter 7.

Results

Convergence. In the conditions with 5 variables and log-normal data, each of the estimation methods had some samples for which they failed to converge on solutions. First, the AGLS method did not converge for 7 samples in the N=100 condition, 18 samples in the N=300 condition and 1 sample in the N=500 condition. Second, the NGLS method did not converge for 5 samples in the N=300 condition. Next, the MGLS method did not converge in 1 sample in the N=100 condition and 1 sample in the N=300 condition. Finally, the MRLS method did not converge for 5 samples in the N=100 condition, 2 samples in the N=300 condition, 2 samples in the N=1000 condition and 1 sample in the N=10,000 condition.

In the 20-variable condition, again each method had some issues with convergence. In particular, the AGLS method did not converge for 16 samples in the N=300 condition and 1 sample in the N=500 condition. The NGLS method did not converge for 3 samples in the N=300 condition and 1 sample in the N=500 condition. Next the MGLS method did not converge for 1 sample in the N=300 condition. Finally the MRLS method did not converge in 13 samples in the N=300 case, 3 samples in the N=500 case, and 1 sample in the N=1000 case.

Parameter Estimates and Standard Errors. For the 5-variable condition, the means and standard deviations of the parameter estimates and standard error estimates for each of the four estimation methods are shown in Tables 9.1 through 9.4. Each table shows one sample size (100, 300, 500, and 1000, respectively). As shown in the tables, each estimation method seemed to produce slightly high, but reasonable estimates of the factor loadings and these estimates improved as the sample size increased. Boxplots illustrate this trend for the first factor loading in Figure 9.1. However, the factor variance and error variance estimates were all too low. This is also illustrated with boxplots in Figure 9.2 for the estimates of the factor variance and in Figure 9.3 for the estimates of the first error variance. Specifically, the average error variances resulting from MRLS method were the closest to the population values, and this method also produced reasonable factor variance estimates. On the other hand, the MGLS method produced the best average factor variance estimates, but the average error variance estimates were much lower. Lower still were the average variance estimates resulting from the AGLS and NGLS methods, with the AGLS method performing somewhat better than the NGLS method.

In addition, the standard error estimates in the 5-variable condition produced by each method were also too low. The AGLS and NGLS methods produced standard errors that were better matches for the standard deviations of the parameters than those of the MGLS and MRLS methods which tended to be very low. In particular, in the N=100 condition, some of the standard deviations of the standard error estimates were exceeding large variance (e.g. see the value corresponding to the factor variance in the MGLS condition). These standard deviations became smaller as the sample size increased revealing increasing stability in the estimated standard errors. However, with the exception of the AGLS method, this increasing stability was not associated with an increase in accuracy.

For the 20-variable condition, the means and standard deviations of the parameter estimates and standard errors for each of the four estimation methods are shown in Tables 9.5 through 9.8. Once again, these tables show the results for the first 6 factor loadings, the factor variance and the first 7 error variances. Each table shows one sample size (300, 500, 1000 and 10,000, respectively). These tables reveal some troubling issues with estimation. Most notably, in the N=100 and N=300 the MRLS method seem to produce volatile and inaccurate average parameter estimates (e.g. one of the average error variance values was 257.97 with a standard deviation of 5663.62). However, the boxplots of the distributions of the estimates of the first factor loading, shown in Figure 9.4, suggest that some of these issues may be driven by extreme outliers as it appears that all of the methods had median factor loadings close to 1.0. Also, the AGLS and NGLS methods produced average error variance estimates that were very low (e.g. around 0.07 to 0.09 rather than 1.00 in the N=100 condition). The average error variances estimated by the MGLS method were somewhat better, but were still quite low (e.g. these were in the 0.51 to 0.53 range in the N=100 condition). In addition, while the average factor variance obtained through the MGLS method was fairly accurate, those obtained through the AGLS and MGLS were very low. The boxplots of the estimates of the factor variance in Figure 9.5 and of the first error variance in Figure 9.6, exemplify these issues with the variance estimates.

Test Statistics. The test statistics found in the 5-variable case were also quite poor. As shown in Table 9.9, only the AGLS method was able to produce reasonable average χ^2 values. This was also apparent when examining the model rejection rates (see Table 9.8). The other three methods each had average χ^2 values that increased with sample size and this resulted in most models being rejected by hypothesis tests. Figures 9.7 to 9.8, show the histograms of the χ^2 estimates, and from these plots it is apparent that only the AGLS method begins to approximate the theoretical χ^2 distribution as the sample size increased. On the other hand, the χ^2 statistics resulting from the NGLS, MGLS, MRLS procedures, had distributions that became flatter and more dispersed as the sample size increased. This trend is shown in Figures 9.8, 9.9 and 9.10 for sample sizes ranging from 100 to 1000. In the N=10,000 and N=100,000 conditions, the trend of flattening and spreading away from the χ^2 distribution continued, so these distributions are not displayed in the figures.

The test-statistics in the 20-variable condition behaved even more erratically. First, as shown in Table 9.11, the AGLS method produced average χ^2 values that were initially (with N=300) far below the predicted χ^2 values. These values then seemed to get closer to the expected value with N=500 only to overshoot in the N=1000 condition. The values finally seemed to begin to converge in the N=10,000 and N=100,000. The extent of this strange behavior is shown by the histograms in Figure 9.11. The other three estimation methods seemed to produce χ^2-test statistics that followed a similar pattern to those in the 5-variable case. That is the average χ^2 values became larger, more and more models were rejected (see Tables 9.11 and 9.12) and the histograms of the χ^2 values became increasingly dispersed (see Figures 9.12, 9.13 and 9.14).

Discussion

The results of the estimation of models of log-normal data were a bit ambiguous. When the number of variables in the model was small, the factor loadings were accurately estimated, but the variance parameters were generally too low, at least in small samples. These got better as the sample size increased, but the AGLS and NGLS variance parameter estimates were still low in large samples. This problem was worse in the condition with a large number of variables, and there was also some trouble estimating the factor loadings. The standard error estimates were also unreliable. The standard errors given MGLS and MRLS appeared to be very low, and they were also highly variable, making them unreliable for use in testing parameters. The AGLS and MGLS standard errors were less bad, but they were still usually too low (in contrast to what was seen in the normal data conditions). Furthermore, the test statistic estimates were inaccurate. The AGLS estimates seemed slow to converge to the expected χ^2 distribution, and those produced by the NGLS, MGLS, and MRLS methods seemed to outright diverge from the expected χ^2 distribution.

The results for the condition with a large number of variables were even messier. For the most part, increasing the number of variables simply resulted in an exaggeration of the problems described in the small variable case. However, in the case of the AGLS test statistic estimates, this also revealed a new problem. That is, the estimated χ^2-test statistics given by the AGLS method seemed to wobble up and down with the sample size before seeming to settle in near the expected distribution. From the perspective of applying this method, this fluctuation is particularly troubling, because a research would not have any way to know whether a statistic was too high or too low. It seems when we have non-normal data and models containing a large number of variables, we will not be able to rely upon the test statistics given by any of these estimation methods.

These results, particularly those relating to the distributions of the estimated test-statistics, confirm that the AGLS method is the only correctly specified method. The NGLS method makes normal theory assumptions as do the MGLS and MRLS methods, which are based on normal-theory estimation of covariance models. This explains why the NGLS, MGLS, and MRLS methods all seemed to fail in consistent ways in terms of the estimated test statistics. That is, they became misspecified when normality was violated.

Finally, more work should be done to determine why the estimation methods did not perform well here in the case of log-normal data. Perhaps this data was problematic because of the tendency to have extreme values which can influence the mean, and of course the covariance and CV matrices. In structural equation modeling of covariance matrices, the use of case-robust estimators, which are not influenced by outlying values, have been shown to be beneficial when extreme cases are present (e.g. Bentler, Satorra, Yuan, 2009). This strategy might also be applied here by obtaining case-robust estimators of the mean and covariance matrix and using these to produce a potentially more stable CV matrix. Further, it might be the case that these estimation methods perform better in other forms of non-normal data that do not contain frequent extreme values. Along those lines, maybe the skew of the log-normal data leads to problems and the estimators would perform better in symmetric (but not necessarily normal) distributions. This is yet another avenue that future research might explore.

Tables

Table 9.1. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of log-normal data with 5 variables and N=100.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.048	1.091	1.096	1.095	0.777	0.567	0.597	0.603	0.578	0.584
		SD	0.526	0.644	0.761	0.617	0.859	0.499	0.332	0.379	0.352	0.343
	$SE_{\hat{\theta}}$	M	0.264	0.303	0.316	0.271	0.352	0.299	0.199	0.214	0.198	0.204
		SD	0.604	0.848	1.850	0.579	1.868	1.857	0.138	0.268	0.144	0.142
NGLS	$\hat{\theta}$	M	1.017	1.059	1.049	1.061	0.566	0.405	0.422	0.411	0.409	0.411
		SD	0.610	0.402	0.395	0.419	0.262	0.277	0.352	0.346	0.307	0.318
	$SE_{\hat{\theta}}$	M	0.659	0.345	0.339	0.346	0.291	0.222	0.315	0.231	0.222	0.227
		SD	7.237	0.244	0.241	0.299	0.145	0.209	2.035	0.188	0.213	0.237
MGLS	$\hat{\theta}$	M	1.073	1.088	1.078	1.084	0.961	0.686	0.673	0.669	0.660	0.674
		SD	0.857	0.663	0.771	0.706	1.127	0.944	0.384	0.402	0.472	0.494
	$SE_{\hat{\theta}}$	M	0.339	0.292	0.328	0.298	2.487	2.388	0.131	0.133	0.130	0.132
		SD	3.485	2.502	3.325	2.580	50.402	50.412	0.064	0.069	0.074	0.078
MRLS	$\hat{\theta}$	M	1.047	1.063	1.068	1.073	0.949	0.877	0.906	0.934	0.908	0.903
		SD	1.138	1.074	0.948	0.825	0.700	0.648	0.751	0.755	1.296	0.830
	$SE_{\hat{\theta}}$	M	0.400	0.387	0.355	0.326	0.233	0.151	0.157	0.161	0.158	0.156
		SD	4.665	4.316	3.622	2.986	0.128	0.091	0.114	0.112	0.185	0.118

Table 9.2. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of log-normal data with 5 variables and N=300.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.023	1.027	1.035	1.031	0.774	0.729	0.727	0.752	0.727	0.724
		SD	0.267	0.277	0.269	0.254	0.286	0.256	0.277	0.309	0.251	0.259
	$SE_{\hat{\theta}}$	M	0.151	0.155	0.153	0.151	0.184	0.171	0.172	0.186	0.170	0.174
		SD	0.081	0.079	0.085	0.066	0.109	0.099	0.129	0.123	0.110	0.100
NGLS	$\hat{\theta}$	M	1.007	1.020	1.023	1.011	0.704	0.587	0.601	0.609	0.589	0.590
		SD	0.222	0.230	0.214	0.201	0.236	0.278	0.278	0.348	0.267	0.245
	$SE_{\hat{\theta}}$	M	0.165	0.168	0.166	0.165	0.195	0.155	0.156	0.164	0.152	0.151
		SD	0.071	0.080	0.068	0.063	0.074	0.096	0.101	0.125	0.094	0.078
MGLS	$\hat{\theta}$	M	1.025	1.054	1.042	1.032	0.999	0.808	0.823	0.844	0.801	0.800
		SD	0.292	0.327	0.300	0.272	0.484	0.367	0.384	0.454	0.356	0.317
	$SE_{\hat{\theta}}$	M	0.094	0.096	0.094	0.094	0.143	0.087	0.088	0.091	0.086	0.087
		SD	0.032	0.036	0.033	0.030	0.055	0.035	0.037	0.043	0.034	0.031
MRLS	$\hat{\theta}$	M	0.999	1.037	1.008	0.960	0.963	0.982	0.950	0.969	0.913	0.935
		SD	0.759	0.586	0.926	1.633	0.450	1.137	0.517	0.544	0.432	0.503
	$SE_{\hat{\theta}}$	M	0.249	0.201	0.290	0.458	0.144	0.099	0.096	0.098	0.094	0.095
		SD	3.336	2.228	4.253	8.029	0.053	0.093	0.045	0.048	0.038	0.043

Table 9.3. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of log-normal data with 5 variables and N=500.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.037	1.028	1.031	1.029	0.794	0.783	0.789	0.779	0.796	0.792
		SD	0.238	0.189	0.200	0.186	0.259	0.267	0.248	0.210	0.265	0.260
	$SE_{\hat{\theta}}$	M	0.141	0.135	0.133	0.131	0.164	0.168	0.171	0.164	0.169	0.169
		SD	0.173	0.054	0.051	0.045	0.079	0.102	0.154	0.093	0.115	0.110
NGLS	$\hat{\theta}$	M	1.023	1.018	1.017	1.025	0.755	0.690	0.679	0.683	0.684	0.685
		SD	0.178	0.179	0.176	0.170	0.207	0.270	0.251	0.243	0.254	0.270
	$SE_{\hat{\theta}}$	M	0.122	0.122	0.122	0.122	0.156	0.127	0.128	0.124	0.126	0.131
		SD	0.038	0.035	0.037	0.036	0.049	0.059	0.061	0.050	0.058	0.071
MGLS	$\hat{\theta}$	M	1.040	1.032	1.022	1.044	0.981	0.872	0.866	0.853	0.867	0.881
		SD	0.250	0.323	0.220	0.278	0.390	0.317	0.317	0.280	0.314	0.348
	$SE_{\hat{\theta}}$	M	0.074	0.075	0.073	0.075	0.112	0.071	0.072	0.070	0.071	0.073
		SD	0.024	0.034	0.020	0.029	0.035	0.023	0.023	0.019	0.022	0.026
MRLS	$\hat{\theta}$	M	1.039	1.030	1.024	1.041	0.962	0.970	0.960	0.952	0.957	0.984
		SD	0.251	0.323	0.219	0.276	0.353	0.427	0.376	0.372	0.426	0.433
	$SE_{\hat{\theta}}$	M	0.077	0.077	0.076	0.078	0.114	0.076	0.076	0.075	0.075	0.078
		SD	0.025	0.037	0.021	0.031	0.035	0.029	0.026	0.024	0.028	0.030

Table 9.4. *Means and standard deviations of parameter estimates and standard errors of a 1-factor model of log-normal data with 5 variables and N=1000.*

			Factor Loadings				Factor Variance	Error Variances				
AGLS	$\hat{\theta}$	M	1.010	1.006	1.017	1.011	0.856	0.847	0.845	0.842	0.845	0.845
		SD	0.133	0.126	0.133	0.134	0.205	0.212	0.221	0.197	0.192	0.228
	$SE_{\hat{\theta}}$	M	0.105	0.104	0.105	0.106	0.145	0.150	0.149	0.148	0.148	0.151
		SD	0.032	0.030	0.031	0.033	0.064	0.094	0.092	0.075	0.083	0.094
NGLS	$\hat{\theta}$	M	1.005	1.012	1.016	1.011	0.843	0.791	0.775	0.784	0.780	0.766
		SD	0.139	0.133	0.139	0.147	0.191	0.220	0.251	0.215	0.238	0.323
	$SE_{\hat{\theta}}$	M	0.080	0.079	0.079	0.080	0.117	0.096	0.097	0.095	0.094	0.100
		SD	0.018	0.016	0.017	0.019	0.032	0.038	0.046	0.031	0.037	0.065
MGLS	$\hat{\theta}$	M	1.015	1.014	1.017	1.027	1.003	0.924	0.923	0.915	0.906	0.937
		SD	0.180	0.163	0.159	0.187	0.313	0.275	0.309	0.241	0.268	0.357
	$SE_{\hat{\theta}}$	M	0.052	0.051	0.051	0.052	0.082	0.053	0.053	0.052	0.052	0.054
		SD	0.011	0.010	0.010	0.011	0.020	0.014	0.017	0.012	0.013	0.026
MRLS	$\hat{\theta}$	M	1.016	1.016	1.019	1.023	0.985	0.983	0.988	0.971	0.957	0.991
		SD	0.170	0.156	0.155	0.178	0.281	0.310	0.396	0.276	0.285	0.333
	$SE_{\hat{\theta}}$	M	0.053	0.053	0.053	0.053	0.082	0.055	0.055	0.054	0.054	0.055
		SD	0.011	0.010	0.010	0.011	0.020	0.015	0.020	0.013	0.014	0.016

Table 9.5. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of log-normal data with 20 variables and N=300.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.009	1.013	1.008	1.015	1.012	1.007	0.200	0.071	0.076	0.073	0.075	0.073	0.071	0.070
		SD	0.139	0.135	0.134	0.133	0.139	0.136	0.040	0.040	0.045	0.043	0.043	0.042	0.043	0.044
	$SE_{\hat{\theta}}$	M	0.128	0.127	0.128	0.128	0.128	0.128	0.048	0.043	0.045	0.044	0.043	0.044	0.044	0.044
		SD	0.028	0.028	0.029	0.027	0.028	0.029	0.007	0.014	0.015	0.014	0.013	0.015	0.012	0.012
NGLS	$\hat{\theta}$	M	1.000	1.012	1.009	1.017	1.001	0.995	0.174	0.082	0.079	0.085	0.085	0.088	0.077	0.078
		SD	0.223	0.169	0.187	0.215	0.215	0.174	0.069	0.063	0.074	0.070	0.063	0.066	0.065	0.061
	$SE_{\hat{\theta}}$	M	0.449	0.433	0.445	0.443	0.445	0.433	0.123	0.081	0.086	0.085	0.081	0.084	0.083	0.083
		SD	0.476	0.243	0.524	0.353	0.524	0.286	0.038	0.033	0.048	0.049	0.036	0.041	0.034	0.032
MGLS	$\hat{\theta}$	M	1.027	1.016	1.013	1.015	1.040	1.120	1.029	0.513	0.525	0.533	0.517	0.533	0.516	0.515
		SD	0.300	0.414	0.270	0.377	0.294	1.672	0.657	0.172	0.193	0.239	0.179	0.207	0.165	0.249
	$SE_{\hat{\theta}}$	M	0.094	0.102	0.087	0.101	0.089	0.151	0.137	0.056	0.057	0.058	0.056	0.057	0.056	0.059
		SD	0.158	0.298	0.058	0.291	0.062	1.267	0.062	0.018	0.021	0.024	0.018	0.021	0.018	0.062
MRLS	$\hat{\theta}$	M	0.359	0.822	0.848	0.803	0.077	0.738	14.457	1.319	1.450	1.404	1.279	1.363	257.974	1.372
		SD	21.245	8.290	8.906	10.197	20.710	11.752	275.822	1.304	1.622	1.722	1.221	1.907	5663.623	1.500
	$SE_{\hat{\theta}}$	M	0.133	0.114	0.114	0.117	0.122	0.120	1.256	0.112	0.123	0.119	0.109	0.116	22.427	0.117
		SD	0.537	0.215	0.239	0.271	0.320	0.305	22.524	0.107	0.133	0.141	0.101	0.157	492.424	0.123

Table 9.6. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of log-normal data with 20 variables and N=500.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.018	1.023	1.025	1.002	1.027	1.013	0.197	0.174	0.175	0.179	0.182	0.173	0.177	0.178
		SD	0.223	0.226	0.218	0.209	0.235	0.229	0.058	0.078	0.066	0.065	0.064	0.064	0.061	0.063
	$SE_{\hat{\theta}}$	M	0.131	0.132	0.131	0.129	0.133	0.132	0.042	0.048	0.047	0.048	0.046	0.048	0.047	0.047
		SD	0.087	0.071	0.073	0.075	0.092	0.083	0.008	0.017	0.012	0.013	0.013	0.013	0.014	0.012
NGLS	$\hat{\theta}$	M	1.009	1.010	1.007	1.000	1.008	1.001	0.195	0.110	0.120	0.118	0.120	0.112	0.117	0.114
		SD	0.145	0.147	0.145	0.149	0.152	0.149	0.054	0.085	0.066	0.069	0.063	0.070	0.067	0.069
	$SE_{\hat{\theta}}$	M	0.319	0.320	0.316	0.317	0.319	0.318	0.105	0.080	0.075	0.078	0.073	0.075	0.075	0.076
		SD	0.095	0.093	0.095	0.096	0.094	0.094	0.023	0.039	0.025	0.034	0.025	0.035	0.041	0.029
MGLS	$\hat{\theta}$	M	1.002	1.003	0.998	1.020	1.012	1.011	1.057	0.646	0.629	0.639	0.617	0.624	0.624	0.626
		SD	0.214	0.212	0.207	0.234	0.220	0.209	0.543	0.210	0.160	0.196	0.169	0.207	0.228	0.175
	$SE_{\hat{\theta}}$	M	0.063	0.064	0.063	0.064	0.064	0.064	0.112	0.051	0.049	0.051	0.048	0.049	0.049	0.050
		SD	0.016	0.016	0.016	0.017	0.017	0.016	0.045	0.018	0.013	0.016	0.013	0.016	0.017	0.015
MRLS	$\hat{\theta}$	M	0.566	0.710	0.489	0.100	0.588	0.335	1.068	1.613	1.184	1.231	1.155	1.320	1.175	1.246
		SD	9.661	6.486	11.222	20.261	9.354	14.971	1.159	8.158	0.737	0.824	0.824	1.546	0.796	1.164
	$SE_{\hat{\theta}}$	M	0.090	0.085	0.092	0.111	0.089	0.101	0.133	0.106	0.078	0.081	0.077	0.087	0.078	0.082
		SD	0.397	0.265	0.462	0.838	0.384	0.618	0.283	0.518	0.047	0.053	0.053	0.099	0.051	0.074

Table 9.7. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of log-normal data with 20 variables and N=1000.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.009	1.006	1.013	1.012	1.010	1.007	0.347	0.425	0.420	0.421	0.438	0.424	0.425	0.427
		SD	0.120	0.120	0.124	0.116	0.128	0.131	0.064	0.081	0.083	0.088	0.178	0.077	0.082	0.084
	$SE_{\hat{\theta}}$	M	0.075	0.075	0.077	0.076	0.076	0.076	0.042	0.050	0.049	0.049	0.052	0.048	0.050	0.050
		SD	0.012	0.013	0.014	0.013	0.014	0.014	0.006	0.014	0.013	0.013	0.058	0.013	0.013	0.014
NGLS	$\hat{\theta}$	M	1.011	1.011	1.022	1.012	1.010	1.019	0.234	0.182	0.177	0.179	0.174	0.180	0.179	0.183
		SD	0.140	0.135	0.232	0.129	0.137	0.128	0.047	0.079	0.076	0.073	0.141	0.075	0.076	0.074
	$SE_{\hat{\theta}}$	M	0.205	0.205	0.208	0.204	0.205	0.204	0.080	0.065	0.063	0.064	0.068	0.063	0.064	0.063
		SD	0.056	0.054	0.067	0.055	0.054	0.051	0.011	0.023	0.021	0.022	0.096	0.024	0.021	0.019
MGLS	$\hat{\theta}$	M	1.019	1.013	1.029	1.008	1.017	1.007	1.004	0.766	0.749	0.755	0.781	0.747	0.760	0.751
		SD	0.167	0.162	0.473	0.157	0.177	0.160	0.312	0.189	0.175	0.181	0.559	0.195	0.183	0.166
	$SE_{\hat{\theta}}$	M	0.046	0.046	0.047	0.045	0.046	0.045	0.077	0.040	0.039	0.040	0.042	0.039	0.040	0.039
		SD	0.009	0.009	0.024	0.009	0.010	0.008	0.021	0.011	0.010	0.010	0.047	0.010	0.010	0.009
MRLS	$\hat{\theta}$	M	0.961	0.975	2.022	0.957	0.956	0.947	1.052	1.139	1.137	1.133	4.632	1.103	1.333	1.133
		SD	1.195	0.804	22.688	1.111	1.301	1.314	1.598	0.708	0.559	0.493	77.944	0.499	4.809	0.992
	$SE_{\hat{\theta}}$	M	0.050	0.050	0.058	0.050	0.051	0.050	0.084	0.053	0.053	0.053	0.211	0.052	0.062	0.053
		SD	0.012	0.010	0.189	0.011	0.019	0.012	0.078	0.032	0.025	0.022	3.524	0.023	0.215	0.045

Table 9.8. *Means and standard deviations of a subset of parameter estimates and standard errors of a 1-factor model of log-normal data with 20 variables and N=10,000.*

			Factor Loadings (1-6)						Factor Variance	Error Variances (1-7)						
AGLS	$\hat{\theta}$	M	1.001	1.004	1.003	1.003	1.001	1.000	0.748	0.823	0.817	0.825	0.824	0.823	0.821	0.818
		SD	0.034	0.034	0.037	0.033	0.037	0.034	0.044	0.059	0.060	0.057	0.062	0.061	0.058	0.057
	$SE_{\hat{\theta}}$	M	0.027	0.027	0.027	0.027	0.027	0.027	0.032	0.044	0.044	0.043	0.045	0.043	0.044	0.043
		SD	0.002	0.002	0.002	0.002	0.002	0.002	0.003	0.012	0.012	0.011	0.012	0.010	0.011	0.011
NGLS	$\hat{\theta}$	M	1.002	1.002	1.000	1.001	1.001	1.002	0.571	0.548	0.550	0.549	0.550	0.548	0.550	0.545
		SD	0.042	0.042	0.045	0.044	0.045	0.042	0.057	0.082	0.082	0.077	0.087	0.080	0.081	0.076
	$SE_{\hat{\theta}}$	M	0.035	0.035	0.035	0.035	0.035	0.035	0.034	0.028	0.028	0.028	0.028	0.028	0.028	0.028
		SD	0.004	0.004	0.004	0.004	0.004	0.004	0.003	0.005	0.006	0.004	0.005	0.004	0.005	0.004
MGLS	$\hat{\theta}$	M	1.004	1.004	1.010	1.004	1.007	1.005	1.004	0.953	0.960	0.956	0.967	0.950	0.960	0.948
		SD	0.063	0.062	0.070	0.075	0.067	0.065	0.227	0.118	0.149	0.106	0.120	0.103	0.112	0.102
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.015	0.015	0.025	0.014	0.015	0.015	0.015	0.014	0.015	0.014
		SD	0.001	0.001	0.001	0.001	0.001	0.001	0.005	0.002	0.002	0.002	0.002	0.002	0.002	0.002
MRLS	$\hat{\theta}$	M	1.004	1.004	1.009	1.003	1.006	1.005	0.999	1.038	1.024	1.018	1.047	1.023	1.026	1.013
		SD	0.061	0.059	0.065	0.066	0.064	0.062	0.179	0.544	0.196	0.136	0.245	0.293	0.169	0.155
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.015	0.015	0.025	0.015	0.015	0.015	0.016	0.015	0.015	0.015
		SD	0.001	0.001	0.001	0.001	0.001	0.001	0.007	0.008	0.003	0.002	0.004	0.004	0.002	0.002

Table 9.9. *Means and standard deviations of estimated* $\chi^2(df = 5)$ *test statistics of the one-factor model of log-normal data in the condition with 5 variables.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
100	6.74	3.59	7.63	2.64	15.42	9.47	15.91	10.48
300	6.77	3.85	14.90	6.17	23.82	15.99	25.60	19.30
500	6.35	4.67	20.01	9.12	29.06	20.30	31.54	25.38
1000	5.75	3.35	27.30	15.15	36.43	30.46	38.67	34.86
10,000	5.32	3.11	58.87	49.71	62.47	57.79	64.17	62.83
100,000	5.13	3.01	80.24	82.48	80.94	84.85	82.27	91.05

Table 9.10. *Percent of replications in which the one-factor model of log-normal data was rejected by a* $\chi^2(df = 5)$ *test with* $\alpha = .05$ *in the condition with 5 variables.*

N	AGLS	NGLS	MGLS	MRLS
100	11.36	11.31	61.12	61.21
300	14.11	70.80	78.56	79.32
500	8.62	81.80	84.80	85.60
1000	9.00	86.20	87.40	88.15
10,000	6.20	94.80	95.00	94.99
100,000	4.60	97.00	97.00	97.00

Table 9.11. *Means and standard deviations of estimated* $\chi^2(df = 170)$ *test statistics of the one-factor model of log-normal data in the condition with 20 variables.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
300	83.15	7.49	72.69	8.05	565.96	84.92	677.33	152.85
500	185.04	16.09	108.42	10.27	704.36	115.47	854.97	221.64
1000	303.08	29.32	188.93	12.50	927.98	163.42	1140.73	317.98
10,000	214.24	21.95	986.19	103.88	1832.83	527.53	2103.80	1048.88
100,000	178.02	16.91	2231.84	556.03	2492.53	752.42	2591.77	983.21

Table 9.12. *Percent of replications in which the one-factor model of log-normal data was rejected by a* $\chi^2(df = 170)$ *test with* $\alpha = .05$ *in the condition with 20 variables.*

N	AGLS	NGLS	MGLS	MRLS
300	0.00	0.00	100.00	100.00
500	13.23	0.00	100.00	100.00
1000	100.00	14.80	100.00	100.00
10,000	68.40	100.00	100.00	100.00
100,000	8.60	100.00	100.00	100.00

Figures

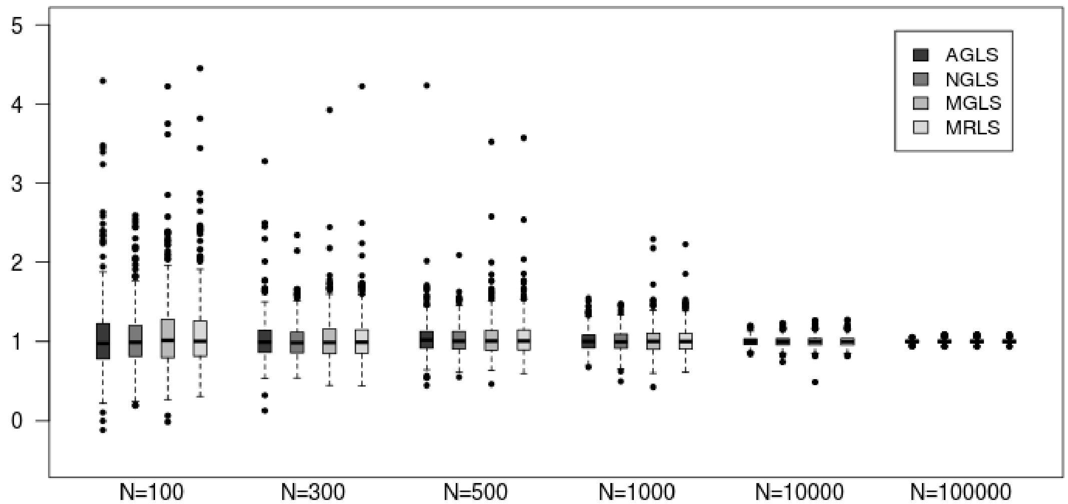

Figure 9.1. Boxplots of the values of the first factor loading estimated by the different methods across sample sizes (in the log-normal data condition with 5 variables). Note that the following extreme outliers are not depicted: In the *N*=100 case, each method had one large negative outlier between -4 (for AGLS) and -25 (for MRLS) and NGLS also had one large positive outlier (with a value of 11). In the *N*=300, the MRLS method produced one large negative outlier (about -14).

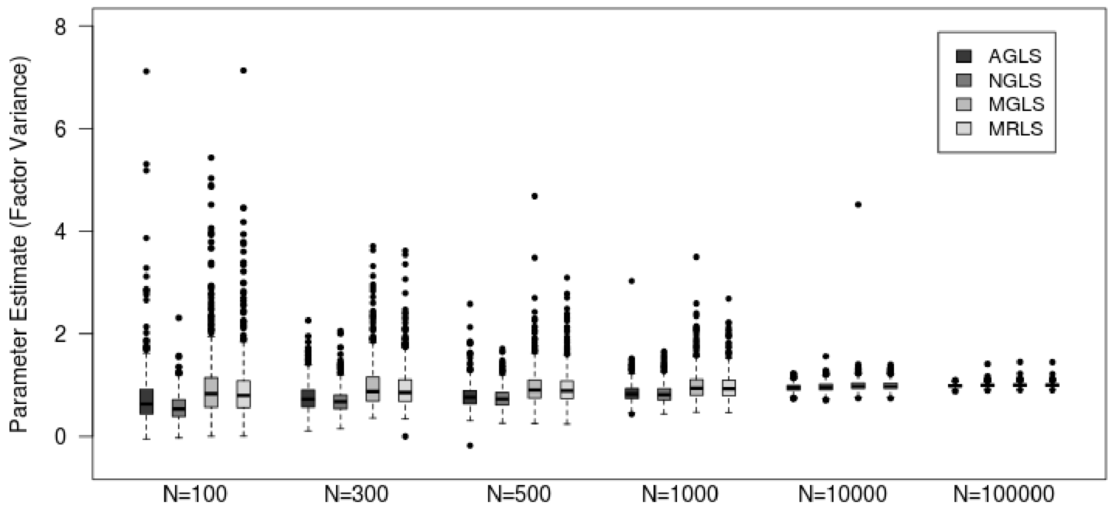

Figure 9.2. Boxplots of the values of the factor variance estimated by the different methods across sample sizes (in the log-normal data condition with 5 variables). Note that the following extreme outliers are not depicted: In the *N*=100 case the AGLS method produced one outlier (with a value of about 14) and the MGLS method produced one large negative outlier (about -18).

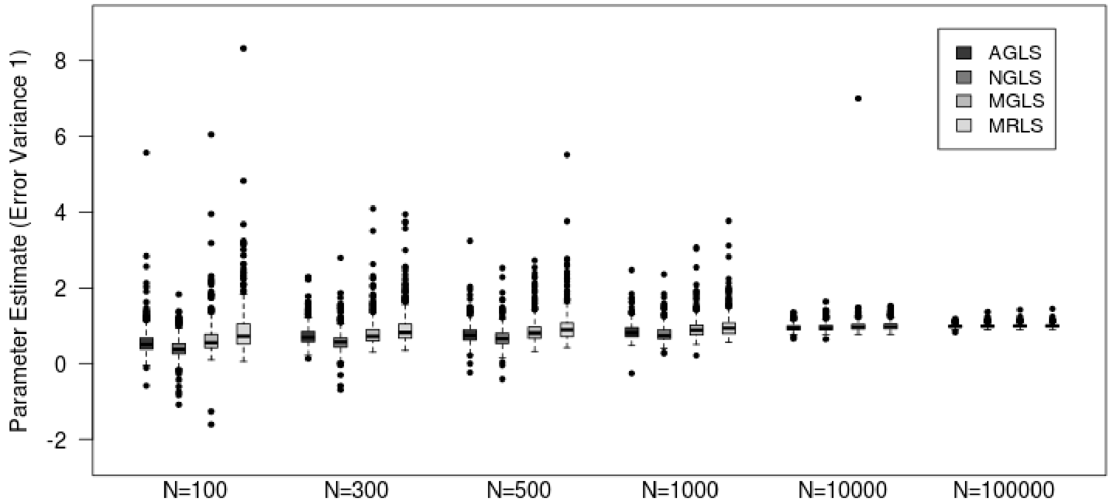

Figure 9.3. Boxplots of the values of the first error variance estimated by the different methods across sample sizes (in the log-normal data condition with 5 variables). Note that the following extreme outliers are not depicted: In the *N*=100 case the AGLS method produced one outlier (with a value of about -6) and the MGLS method produced one large positive outlier (about 20) and in the *N*=300 condition MRLS produced an outlier around 25.

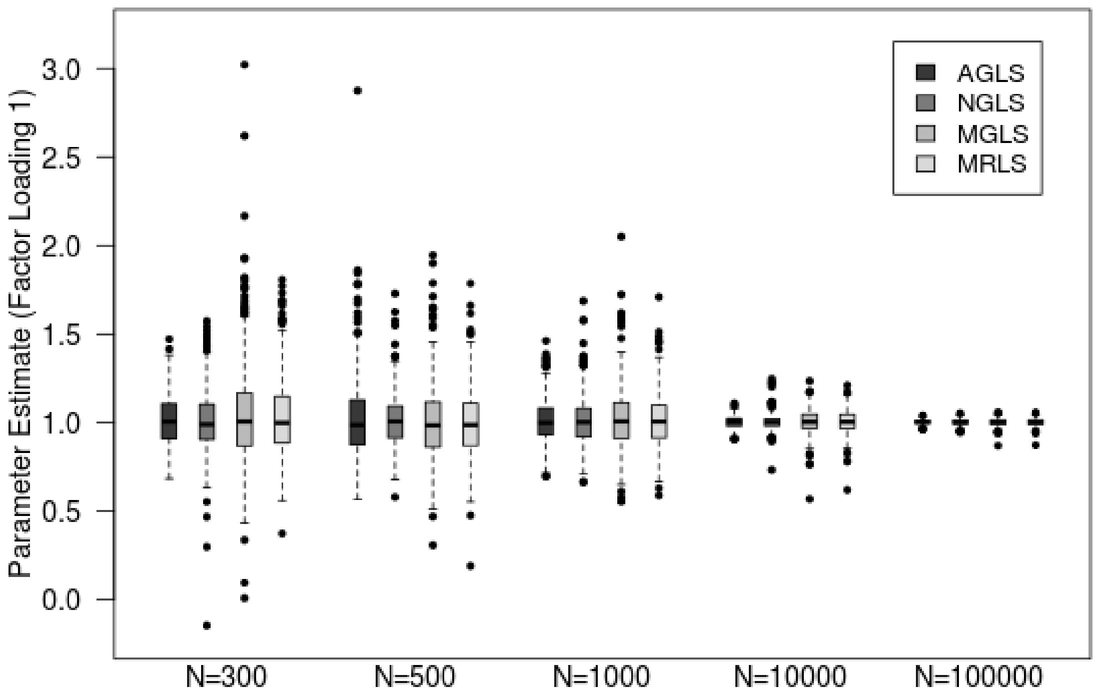

Figure 9.4. Boxplots of the values of the first factor loading estimated by the different methods across sample sizes (in the log-normal data condition with 20 variables). The following extreme outliers are not depicted: The MRLS procedure produced several extreme outliers (-450 and 150 with *N*=300, -200 with *N*=500, and -10 with *N*=1000).

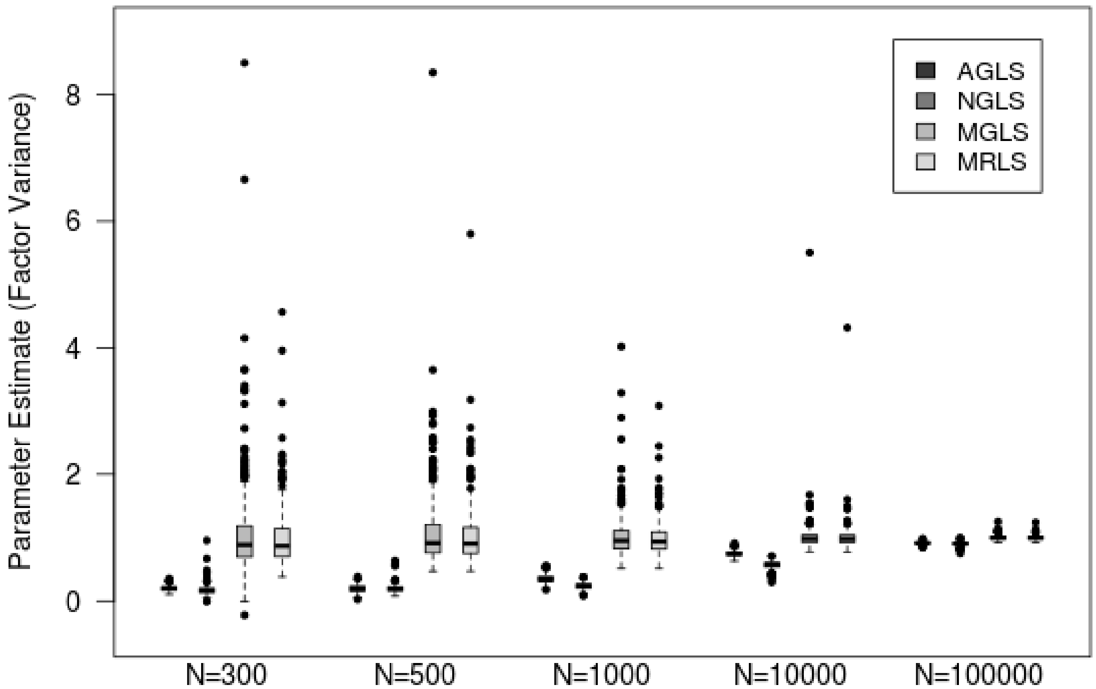

Figure 9.5. Boxplots of the values of the factor variance estimated by the different methods across sample sizes (in the normal data condition with 20 variables with equal means). The following extreme outliers are not depicted: The MRLS procedure produced several extreme outliers in the N=300 condition (at approximately 6000, 500, 45, and 25), in the N=500 condition (at approximately 25) and in the N=1000 condition (at about 35).

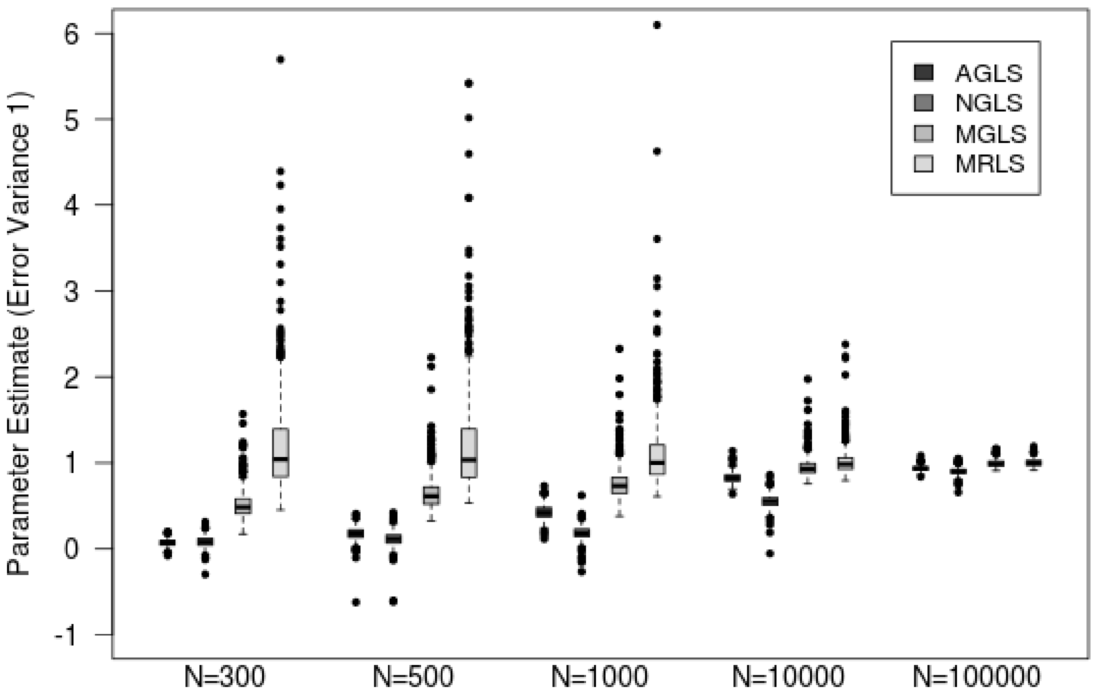

Figure 9.6. Boxplots of the values of the first error variance estimated by the different methods across sample sizes (in the log-normal data condition with 20 variables). One extreme outlier is not depicted: The MRLS procedure produced an error estimate of 175 in the *N*=500 condition.

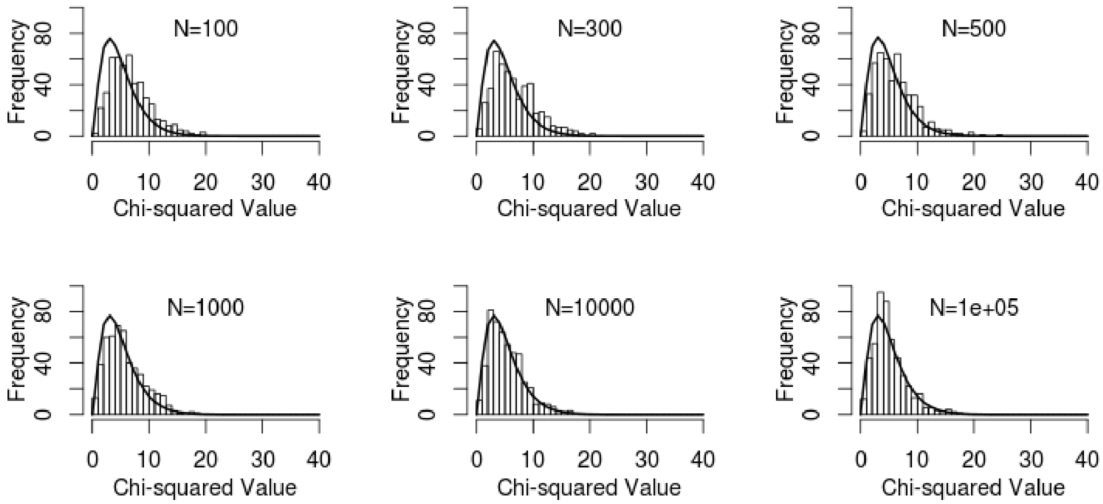

Figure 9.7. Histograms of the values of the χ^2-test statistics produced through AGLS estimation across sample sizes (in the log-normal data condition with 5 variables). Two outliers are not depicted: In the *N*=500 condition, there were outliers at 71 and 591.

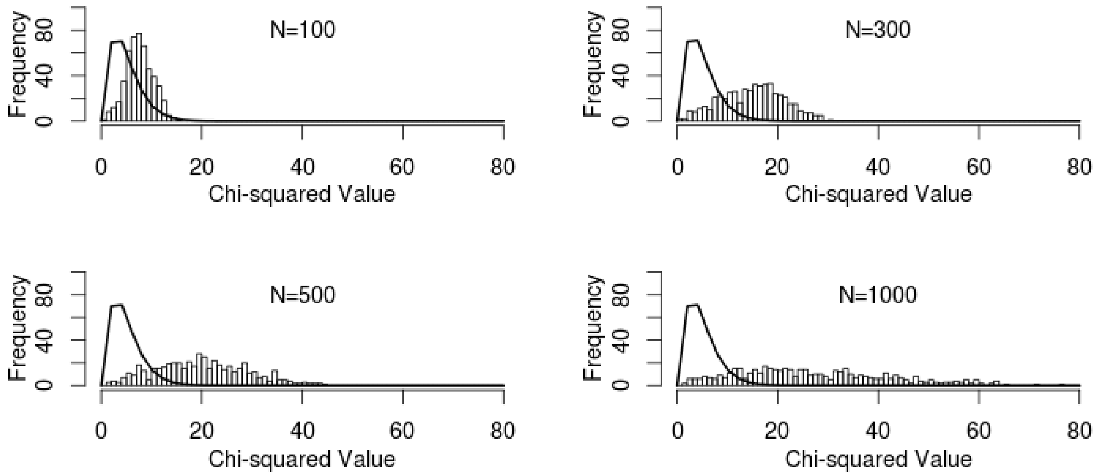

Figure 9.8. Histograms of the values of the χ^2-test statistics produced through NGLS estimation across sample sizes (in the log-normal data condition with 5 variables).

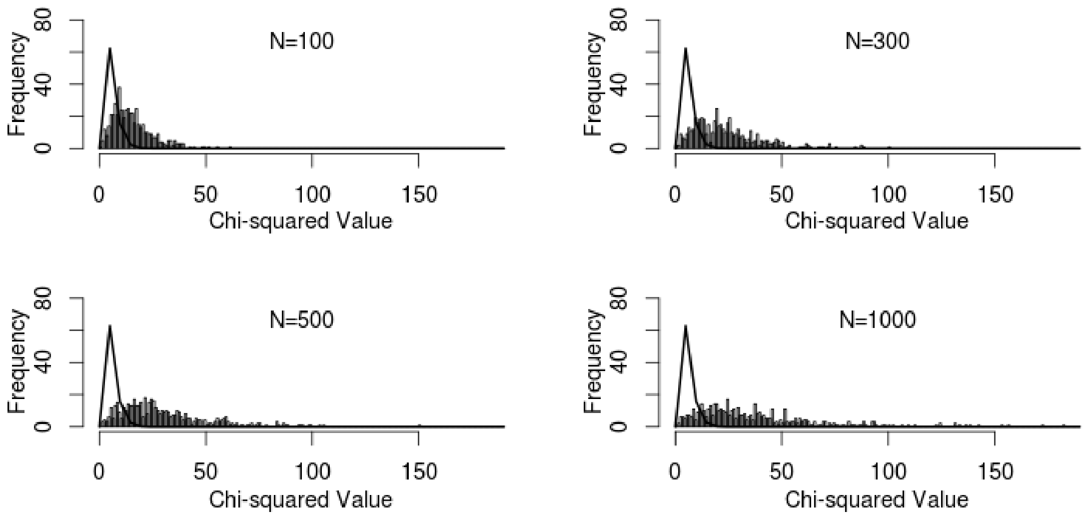

Figure 9.9. Histograms of the values of the χ^2-test statistics produced through MGLS estimation across sample sizes (in the log-normal data condition with 5 variables). In the N=1000 condition, there were two outliers at 288 and 483 which are not displayed.

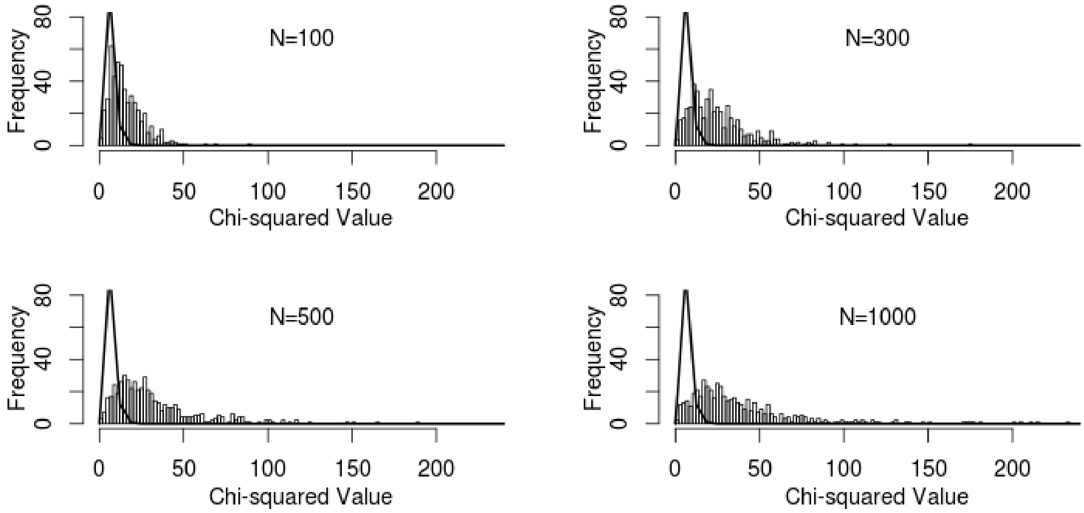

Figure 9.10. Histograms of the values of the χ^2-test statistics produced through MRLS estimation across sample sizes (in the log-normal data condition with 5 variables).

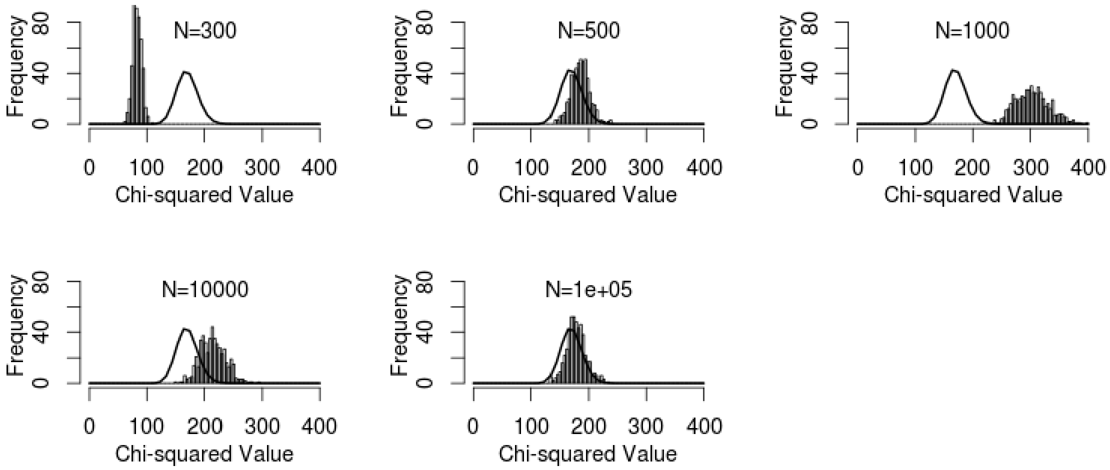

Figure 9.11. Histograms of the values of the χ^2-test statistics produced through AGLS estimation across sample sizes (in the log-normal data condition with 20 variables).

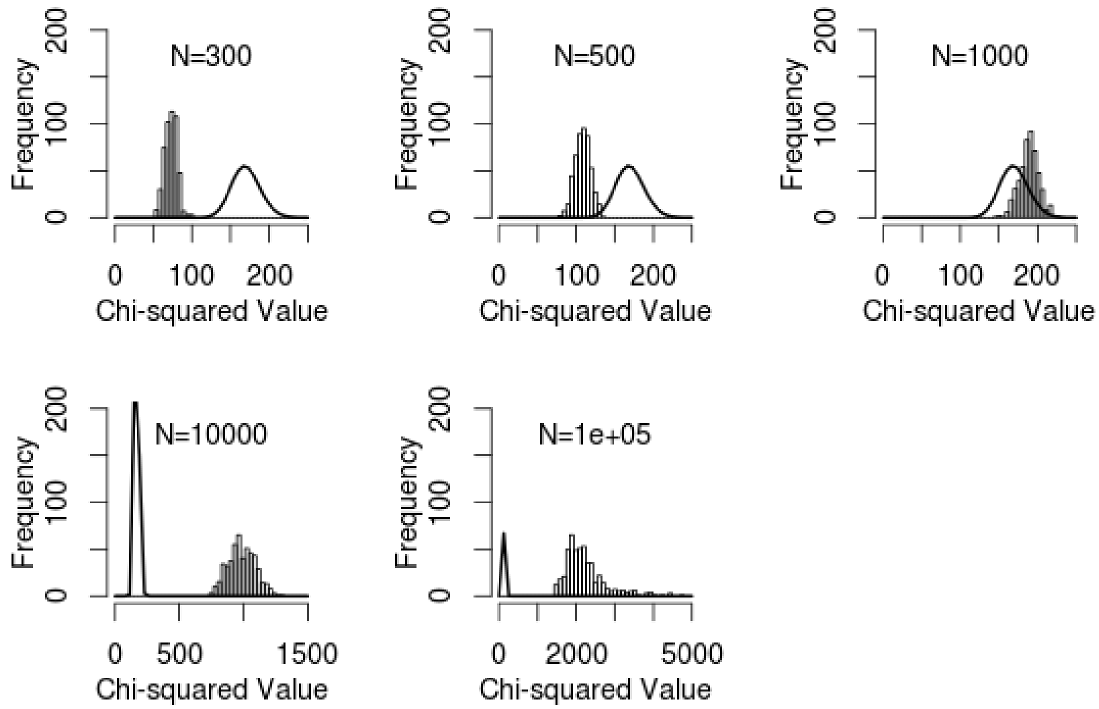

Figure 9.12. Histograms of the values of the χ^2-test statistics produced through NGLS estimation across sample sizes (in the log-normal data condition with 20 variables). In the N=100,000 condition there was an extreme value at 6064 that is not shown.

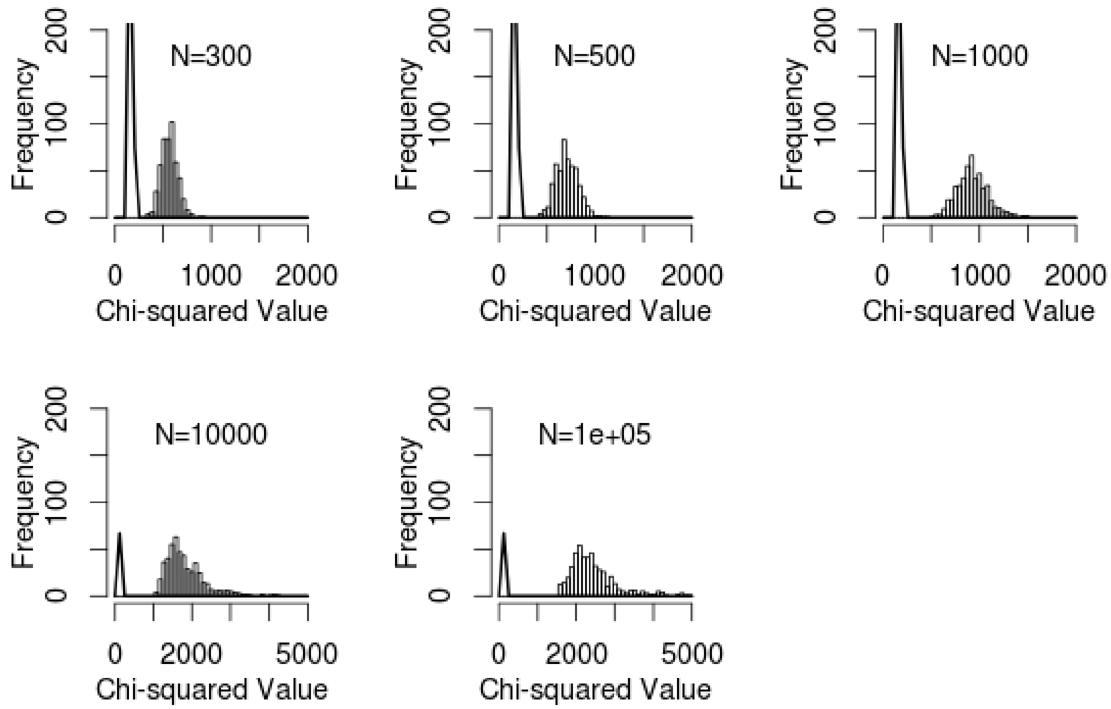

Figure 9.13. Histograms of the values of the χ^2-test statistics produced through MGLS estimation across sample sizes (in the log-normal data condition with 20 variables). Note, in the N=100,000 condition, the right tail of the distribution was quite long and could not be depicted without distorting the histogram of the more typical values. As result, seven values ranging from 5058 to 8887 are excluded.

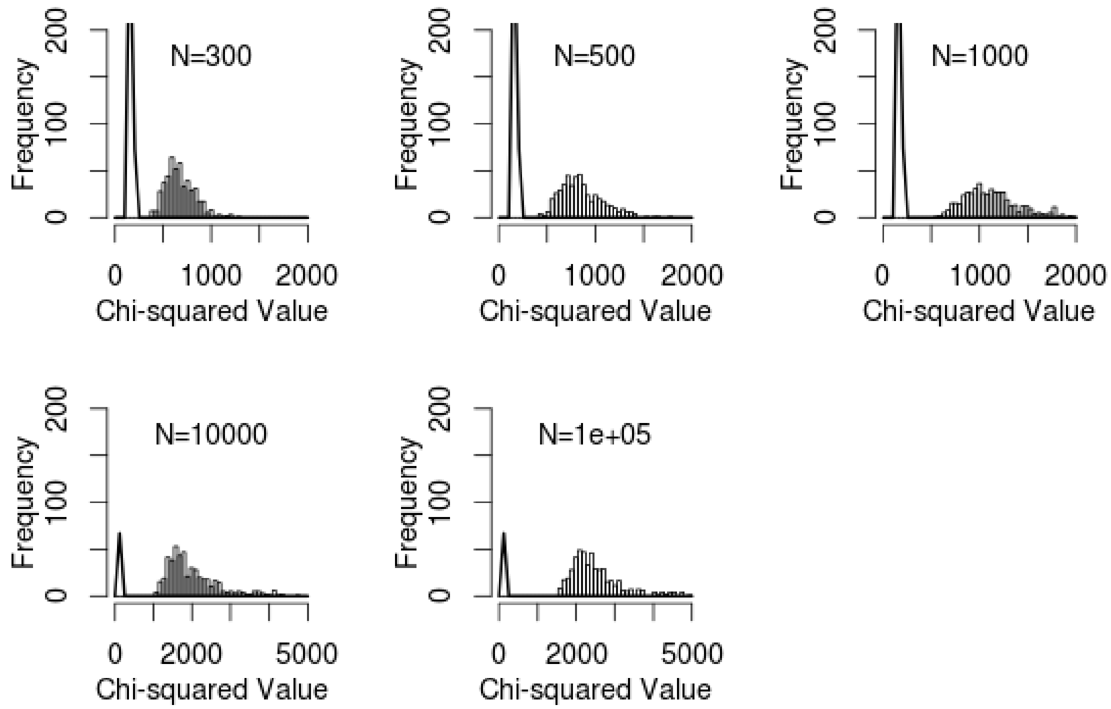

Figure 9.14. Histograms of the values of the χ^2-test statistics produced through MRLS estimation across sample sizes (in the log-normal data condition with 20 variables). Here, in several of the sample size conditions, the right tail of the distribution was very long and could not be depicted without distorting the histograms of the more typical values. As result, some values were not displayed. Specifically, in the N=1000 condition there were 11 points ranging from 2000 to 2535, in the N=10,000 condition there were 9 points ranging from 5109 to 13501, and in the N=100,000 condition there were 14 points ranging from 5024 to 14356.

Chapter 10. Quality of Parameter, Test Statistic, and Standard Error Estimates: Two Factor Models of Normal Data

In this chapter we continue to examine the performance of the AGLS, NGLS, MGLS and MRLS methods of estimating CV models by considering a model with two factors. Specifically, here we consider the performance of each method when the data are normally distributed. In the following chapter we will consider the same model, but data following a log-normal distribution.

Method

Conditions. Here, normally distributed data were generated according to the structure shown in Figure 5.2. The models included either 6 variables (3 per factor) or 20 variables (10 per factor). The sample sizes in the 6-variable case ranged from 100 to 100,000, whereas in the 20-variable case, ranged from 300 to 100,000.

Analyses. Refer to Chapter 7 for details regarding the analyses. The analyses used here were identical.

Results

Convergence. In the smallest sample size condition (i.e. the condition with 6 variables and a sample size of 100), some of the estimation methods were unable to converge to solutions for some of the replications. Specifically, the NGLS method did not converge for 3 samples and the MGLS and RLS methods each did not converge for 1 sample.

Parameter Estimates and Standard Errors. Tables 10.1 through 10.4 show means and standard deviations of the factor loadings, factor variances/covariances and errors variances for each of the four methods in the 6-variable condition. Each table displays the results for one sample size (100, 300, 500, and 1000, respectively). In the population, the factor loadings, the factor variance and the error variances each have values of 1 and the factor covariance has a value of 0.3. The tabled results were contrasted with these values.

The pattern of results observed here closely matched the previously reported results involving other small models of normal data. In particular, Tables 10.1 through 10.4 show that once again the average factor loadings seemed to be quite accurate for all estimation methods, even at small sample sizes. This is also apparent in Figure 10.1, which shows that the distributions of values of the first factor loading for the first factor was centered at 1.0 for each method and that the estimates became more precise as the sample size increased. In addition, the average variance and covariance estimates still tended to be too low for the AGLS and NGLS methods, while MGLS and MRLS produced more accurate estimates of the variance/covariance components, as was seen in previous models of normal data. For instance, in the condition with a sample size of 100, the AGLS method estimated each factor variance to be about 0.88 and the factor covariance to about 0.25. Similarly, the NGLS method estimated the factor variances to 0.83 and 0.85 and the factor covariance to be 0.24. On the other hand, the MGLS and MRLS methods yielded average variance estimates that were better, but a bit high and fairly accurate estimates of the average factor covariance. The MGLS procedure had average factor variance estimates of about 1.06, and an average factor covariance of 0.31 (in the condition with N=100). The MRLS procedure had higher factor variance estimates on average, about 1.09 and 1.10, and very accurate factor covariance estimates (0.30 on average). All methods produced more accurate parameter estimates as the sample size increased. However, the variance estimates produced by the AGLS and NGLS methods converged slowly and were still a bit low in large sample sizes (e.g. see Tables 10.3 and 10.4). In addition, Figures 10.2 and 10.3 also display the distributions of the first factor variance estimates and the factor covariance estimates, respectively, across methods and sample sizes. This figure shows the same trend regarding accuracy, but also reveals the MGLS and MRLS procedures tended to have a bit more variability in their estimates at small sample sizes. Similar, results were obtained for the error variance estimates. That is, the AGLS and NGLS methods, tended to underestimate the variances, while MGLS and MRLS methods produced more accurate estimates that had slightly higher variability in small samples. These trends are apparent in Tables 10.1 through 10.4 and in Figure 10.4, which shows the distributions of the estimates for the first error variance.

The theoretical standard errors are also displayed in Tables 10.1 through 10.4. As in previous chapters, these estimates were compared with the standard deviations of the parameters (i.e. the empirical standard errors). In small sample sizes, the AGLS and NGLS methods tended to give slightly high estimates of the standard errors relative to the variances of the parameter estimates, but these greatly improved as the sample sizes increased. On the other hand, the MGLS and MRLS procedures reliably produced estimated standard errors that were much lower than the variances of the parameters. The size of this discrepancy did decline as the sample sizes increased, but was still present in large samples. For instance, with N=1000, the standard errors of the error variances averaged about 0.07 for the MGLS and MRLS estimates, but the parameter variances were actually in the range of 0.11-0.12. The estimated standard errors for the factor loadings were a bit less biased, but the bias in the estimates of the factor variances was comparable to that in the error variances. Interestingly, the standard errors of the factor covariances were fairly accurate for all estimation methods.

Tables 10.5 through 10.8 show means and standard deviations of a subset of the parameter estimates obtained from each of the four methods in the 20-variable condition. Each table displays the results for one sample size (300, 500, 1000 and 10,000, respectively) and the parameters displayed include first 2 factor loadings for each factor, the factor variances and covariance, and the error variances for the first 3 variables loading on each factor. The estimates of the factor loadings were again fairly accurate across methods, but there were some differences in the precision of the estimates of the factor loadings. This is illustrated by Figure 10.5, which shows the distribution of the estimates of the first factor loading (on factor 1). In particular, it shows that while each method yielded factor loading estimates that were centered around 1.0, the AGLS estimates had very high variability while the NGLS method had the smallest variability, particularly at smaller sample sizes.

Next, the estimates of the variance parameters seemed to differ across methods (also shown in Tables 10.5 through 10.8). Specifically, the AGLS and NGLS methods dramatically underestimated the variance parameters. With a sample size of 300, the AGLS estimates of the factor variances had averages between 0.48 and 0.49 and average error variance estimates around 0.45 and the NGLS estimates were even worse (around 0.37 for the factor variance estimates and 0.35 for the error variance estimates). The MGLS and MRLS estimates of the variance components tended to be much more accurate, but the MGLS estimates of the error variances had averages that were still too low. These were between 0.88 and 0.90 in the N=300 case. Regarding the covariance between the two factors, the MGLS and MRLS procedures were fairly accurate, whereas the AGLS and NGLS methods produced average estimates that were much too low. Specifically, in the N=300 case, the AGLS factor covariance estimates averaged about 0.14 and the NGLS method averaged about 0.10. All estimates seemed to improve as the sample size increased, but the downward bias in the variance estimates for the AGLS and NGLS was still apparent at large sample sizes. For instance, with a sample size of 10,000 these methods produced factor and error variance estimates that were around 0.95 on average and factor covariance estimates that were between 0.28 and 0.29, which are still a bit lower than the population value of 0.30. In addition, there were noteworthy differences in the variability of estimates of the variance/covariance parameters across the methods. Specifically, the NGLS method, in addition to producing very low estimates, seemed to very reliably produce those low estimates. This is illustrated by Figures 10.6 through 10.8, which display the distributions of the estimated factor variance (for the first factor), the factor covariance and the first error variance (of the first observed variable), respectively. The figures show, that the variability of the estimates given by the NGLS method is consistently less than the variability of the other methods.

The standard errors for the 20-variable conditions are also displayed in Tables 10.5 through 10.8 and these were compared against the standard deviations of the parameters (i.e. the empirical standard errors). In this case, the AGLS method produced the standard error estimates that most closely reflected the standard deviations of the parameters estimates. However, they were once again, too conservative. The NGLS estimates of the standard errors were also fairly accurate at large sample sizes, but the tended to be much too high in smaller samples. That is, the reduced variability of the NGLS estimates of the variance/covariance parameters that was reported above, was not reflected in the standard error estimates that this method produced. Therefore, there was a large discrepancy between the estimated standard errors and the standard deviations of the parameter estimates produced by this method (refer to Table 10.5 for examples). The MGLS and MRLS procedures, on the other hand, tended to give standard error estimates that were much too low. They did get better as the sample sizes increased, but the bias was still substantial with a sample size of 10,000. (See Table 10.8 for specific values.)

Test Statistics. In the 6-variable condition the population/theoretical value of the χ^2-test statistic was 8. The means and standard deviations of the χ^2 values for the 6-variable condition are shown in Table 10.9. As observed in previous conditions, the AGLS and NGLS methods required very large sample sizes before they began producing average χ^2 values that were close to the expected value. Specifically, they were much too low in smaller samples. This trend is also shown in the histograms of the estimated χ^2 values shown in Figures 10.9 (for the AGLS method) and 10.10 (for NGLS method). A problematic result of this downward bias, is that the proportion of hypothesis tests that would result in a rejection of the null hypothesis was much too low (or even 0) in small samples. These proportions were calculated and are reported in Table 10.10. The MGLS and MRLS procedures tended to produce much more accurate χ^2 values as shown in Table 10.9 and in Figures 10.11 (for MGLS) and 10.12 (for MRLS). However, they still under-rejected the null hypothesis when the sample size was 100 as shown in Table 10.10. In larger samples, the MGLS and MRLS methods tended to reject at a rate much closer to the expected 5%, but there were still some noteworthy fluctuations in this rate (see Table 10.10).

In the 20-variable condition the population value of the χ^2-test statistic was 169 and the means and standard deviations of the estimated χ^2 values across methods and sample sizes are shown in Table 10.11. In this condition, the AGLS and NGLS methods did not (on average) produce any reasonable values of the χ^2-test statistic at a realistic sample size. These two methods began to produce estimates that were approximately correct with sample sizes of 10,000 and 100,000, but sample sizes this large generally do not occur. The slow convergence of these χ^2 values is clearly apparent in Figure 10.13 for the AGLS method and 10.14 for the NGLS method. Furthermore, as shown in Table 10.12, this resulted in these two methods being unable to reject the null hypothesis with samples sizes between 300 and 1000 and the methods only showed a reasonable rejection rate at the highest sample size (N=100,000). The MGLS and MRLS methods tended to provide less inaccurate estimates of the χ^2 values. This is effect is apparent in the averages shown in Table 10.11 and in the histograms of the estimates shown in Figure 10.15 for the MGLS method and in Figure 10.16 for the MRLS method. However, these values were still somewhat low and this produced low rejection rates in the N=300 and N=500 conditions, particularly for the MRLS procedure (see Table 10.12). It is also noteworthy that when the sample sizes were very large (i.e. in the N=100,000 condition) all of the methods had rejection rates that were a bit too high (ranging from 5.4 to 6.0).

Discussion

Overall, the results of the simulations presented here were comparable to those in previous chapters that considered normal data with similar numbers of variables. That is, the results for the 2-factor model with 6 variables presented here were a close match for those observed in Chapters 7 and 8 for the single factor model with 5 variables. This is informative, because in spite of the model being slightly more complex (and including one additional observed variable) the estimation performance seemed roughly equivalent. Similarly, the results for the 2-factor model with 20 variables described here were a close match for those observed in Chapters 7 and 8 for the single factor model with 20 variables. This suggests that while adding another factor does not seem to greatly impact the quality of the estimation, the problems observed previously are still apparent in the more complex model.

The primary difference here was that that an addition factor variance and a factor covariance needed to be estimated. Interestingly, all methods seemed to be somewhat better at estimating the factor covariance and its standard error than other variance parameters. However, the estimates of the factor covariance were still subject to the same sort of biases as the other variance parameters. Specifically, the AGLS and NGLS methods tended to underestimate these in small samples, a trend that has been consistently observed across the studies so far.

Although the estimation methods perform comparably here in another form of factor model, does not mean the results will be similar in other types of structural equation models. There may be other models where the estimation methods perform better, and some where they completely break down. Future work should continue to explore different structures of CV models and reconsider the estimation performance in a variety of new structures. This may also lead to the discovery of new applications of the CV matrix models if one of these structures is relevant to a particular kind of data.

Tables

Table 10.1. *Means and standard deviations of parameter estimates and standard errors for a 2-factor model of normal data with 6 variables with N=100.*

			Factor Loadings				Factor Covariances			Error Variances					
			V_1	V_2	V_4	V_5	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_4	E_5	E_6
AGLS	$\hat{\theta}$	M	1.045	1.039	1.024	1.026	0.877	0.877	0.247	0.814	0.778	0.786	0.789	0.789	0.790
		SD	0.335	0.301	0.280	0.281	0.382	0.392	0.149	0.314	0.311	0.318	0.324	0.309	0.318
	$SE_{\hat{\theta}}$	M	0.334	0.332	0.322	0.327	0.443	0.444	0.167	0.382	0.377	0.375	0.371	0.363	0.374
		SD	0.187	0.148	0.137	0.145	0.205	0.226	0.058	0.174	0.289	0.179	0.176	0.168	0.183
NGLS	$\hat{\theta}$	M	1.040	1.039	1.014	1.021	0.834	0.849	0.239	0.793	0.752	0.770	0.774	0.766	0.776
		SD	0.281	0.275	0.248	0.255	0.332	0.348	0.132	0.286	0.303	0.288	0.302	0.282	0.302
	$SE_{\hat{\theta}}$	M	0.331	0.332	0.320	0.324	0.437	0.442	0.163	0.384	0.375	0.373	0.374	0.364	0.374
		SD	0.114	0.128	0.107	0.113	0.198	0.212	0.055	0.174	0.273	0.175	0.179	0.164	0.186
MGLS	$\hat{\theta}$	M	1.045	1.042	1.024	1.023	1.059	1.065	0.312	0.996	0.961	0.968	0.977	0.953	0.969
		SD	0.300	0.292	0.267	0.271	0.460	0.469	0.168	0.384	0.479	0.390	0.390	0.371	0.394
	$SE_{\hat{\theta}}$	M	0.214	0.215	0.208	0.208	0.310	0.312	0.149	0.233	0.231	0.229	0.228	0.224	0.229
		SD	0.082	0.093	0.072	0.073	0.112	0.118	0.046	0.082	0.105	0.081	0.084	0.077	0.086
MRLS	$\hat{\theta}$	M	1.045	1.040	1.029	1.023	1.094	1.097	0.302	1.084	1.052	1.066	1.070	1.037	1.052
		SD	0.291	0.293	0.267	0.273	0.472	0.461	0.161	0.419	0.524	0.429	0.421	0.386	0.414
	$SE_{\hat{\theta}}$	M	0.217	0.217	0.211	0.210	0.322	0.323	0.151	0.247	0.244	0.243	0.243	0.236	0.242
		SD	0.084	0.091	0.074	0.078	0.116	0.115	0.045	0.086	0.108	0.086	0.089	0.082	0.086

Table 10.2. *Means and standard deviations of parameter estimates and standard errors for a 2-factor model of normal data with 6 variables with N=300.*

			Factor Loadings				Factor Covariances			Error Variances					
			V_1	V_2	V_4	V_5	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_4	E_5	E_6
AGLS	$\hat{\theta}$	M	1.007	1.002	1.009	1.023	0.951	0.925	0.286	0.929	0.911	0.925	0.899	0.909	0.913
		SD	0.145	0.150	0.131	0.146	0.230	0.214	0.085	0.211	0.192	0.206	0.187	0.192	0.194
	$SE_{\hat{\theta}}$	M	0.155	0.153	0.155	0.156	0.244	0.237	0.094	0.210	0.206	0.210	0.203	0.207	0.205
		SD	0.028	0.029	0.026	0.029	0.066	0.062	0.018	0.055	0.051	0.055	0.048	0.052	0.050
NGLS	$\hat{\theta}$	M	1.006	1.002	1.008	1.023	0.944	0.922	0.284	0.922	0.907	0.918	0.897	0.906	0.909
		SD	0.140	0.146	0.128	0.141	0.222	0.210	0.083	0.206	0.187	0.201	0.182	0.187	0.189
	$SE_{\hat{\theta}}$	M	0.155	0.154	0.154	0.157	0.245	0.238	0.093	0.211	0.207	0.211	0.204	0.208	0.206
		SD	0.025	0.026	0.024	0.027	0.065	0.060	0.018	0.055	0.051	0.055	0.048	0.052	0.051
MGLS	$\hat{\theta}$	M	1.006	1.002	1.009	1.024	1.029	1.003	0.310	1.002	0.986	0.998	0.975	0.985	0.987
		SD	0.144	0.148	0.132	0.144	0.254	0.237	0.094	0.224	0.205	0.220	0.199	0.209	0.208
	$SE_{\hat{\theta}}$	M	0.116	0.115	0.115	0.117	0.176	0.171	0.085	0.129	0.128	0.130	0.126	0.129	0.127
		SD	0.021	0.022	0.019	0.021	0.036	0.033	0.014	0.025	0.024	0.026	0.022	0.025	0.023
MRLS	$\hat{\theta}$	M	1.007	1.001	1.008	1.023	1.046	1.016	0.307	1.032	1.016	1.028	1.008	1.016	1.017
		SD	0.144	0.146	0.131	0.143	0.261	0.242	0.092	0.226	0.212	0.225	0.208	0.213	0.212
	$SE_{\hat{\theta}}$	M	0.116	0.115	0.116	0.118	0.178	0.174	0.085	0.132	0.130	0.132	0.128	0.131	0.129
		SD	0.021	0.021	0.019	0.021	0.037	0.034	0.014	0.025	0.024	0.026	0.022	0.026	0.024

Table 10.3. *Means and standard deviations of parameter estimates and standard errors for a 2-factor model of normal data with 6 variables with N=500.*

			Factor Loadings				Factor Covariances			Error Variances					
			V_1	V_2	V_4	V_5	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_4	E_5	E_6
AGLS	$\hat{\theta}$	M	1.006	1.003	1.004	1.007	0.965	0.959	0.290	0.955	0.939	0.945	0.947	0.948	0.942
		SD	0.106	0.112	0.114	0.111	0.176	0.164	0.069	0.158	0.155	0.164	0.159	0.161	0.154
	$SE_{\hat{\theta}}$	M	0.115	0.114	0.114	0.115	0.186	0.184	0.072	0.161	0.159	0.160	0.159	0.160	0.159
		SD	0.015	0.016	0.016	0.016	0.038	0.034	0.011	0.031	0.030	0.032	0.031	0.032	0.030
NGLS	$\hat{\theta}$	M	1.006	1.003	1.004	1.006	0.962	0.958	0.289	0.953	0.938	0.944	0.944	0.946	0.941
		SD	0.105	0.110	0.112	0.110	0.172	0.164	0.068	0.157	0.153	0.161	0.159	0.159	0.153
	$SE_{\hat{\theta}}$	M	0.115	0.115	0.115	0.115	0.186	0.185	0.072	0.162	0.159	0.161	0.160	0.161	0.160
		SD	0.014	0.015	0.015	0.015	0.037	0.034	0.010	0.031	0.030	0.032	0.031	0.032	0.030
MGLS	$\hat{\theta}$	M	1.006	1.003	1.004	1.007	1.012	1.007	0.304	1.000	0.986	0.992	0.992	0.994	0.988
		SD	0.107	0.112	0.113	0.112	0.185	0.175	0.073	0.165	0.162	0.172	0.166	0.169	0.161
	$SE_{\hat{\theta}}$	M	0.089	0.089	0.089	0.089	0.134	0.133	0.065	0.099	0.098	0.099	0.098	0.099	0.098
		SD	0.012	0.012	0.012	0.012	0.020	0.019	0.008	0.014	0.014	0.014	0.015	0.015	0.014
MRLS	$\hat{\theta}$	M	1.006	1.003	1.004	1.007	1.021	1.014	0.302	1.018	1.004	1.010	1.010	1.012	1.006
		SD	0.106	0.112	0.113	0.112	0.187	0.177	0.072	0.166	0.164	0.174	0.169	0.173	0.163
	$SE_{\hat{\theta}}$	M	0.089	0.089	0.089	0.089	0.135	0.134	0.065	0.100	0.099	0.100	0.099	0.100	0.100
		SD	0.012	0.012	0.012	0.012	0.020	0.019	0.008	0.014	0.014	0.014	0.015	0.015	0.014

Table 10.4. *Means and standard deviations of parameter estimates and standard errors for a 2-factor model of normal data with 6 variables with N=1000.*

			Factor Loadings				Factor Covariances			Error Variances					
			V_1	V_2	V_4	V_5	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_4	E_5	E_6
AGLS	$\hat{\theta}$	M	1.004	1.004	1.002	1.004	0.981	0.981	0.293	0.977	0.971	0.969	0.969	0.971	0.968
		SD	0.074	0.078	0.079	0.085	0.125	0.136	0.045	0.118	0.118	0.111	0.109	0.110	0.109
	$SE_{\hat{\theta}}$	M	0.079	0.079	0.079	0.079	0.130	0.130	0.050	0.114	0.113	0.113	0.113	0.113	0.113
		SD	0.007	0.007	0.008	0.008	0.019	0.019	0.005	0.016	0.016	0.016	0.015	0.015	0.015
NGLS	$\hat{\theta}$	M	1.003	1.003	1.002	1.004	0.981	0.980	0.293	0.977	0.970	0.970	0.968	0.970	0.968
		SD	0.073	0.078	0.078	0.084	0.124	0.135	0.045	0.118	0.118	0.112	0.108	0.110	0.108
	$SE_{\hat{\theta}}$	M	0.079	0.079	0.079	0.079	0.131	0.130	0.050	0.114	0.113	0.113	0.113	0.113	0.113
		SD	0.007	0.007	0.007	0.008	0.018	0.019	0.005	0.016	0.016	0.016	0.015	0.015	0.015
MGLS	$\hat{\theta}$	M	1.003	1.004	1.002	1.004	1.006	1.006	0.301	1.003	0.995	0.995	0.994	0.995	0.993
		SD	0.073	0.078	0.079	0.085	0.129	0.140	0.047	0.122	0.121	0.115	0.112	0.113	0.113
	$SE_{\hat{\theta}}$	M	0.062	0.062	0.062	0.063	0.094	0.094	0.046	0.070	0.070	0.070	0.069	0.070	0.069
		SD	0.006	0.006	0.006	0.007	0.010	0.010	0.004	0.008	0.007	0.007	0.007	0.007	0.007
MRLS	$\hat{\theta}$	M	1.003	1.004	1.002	1.004	1.010	1.009	0.300	1.012	1.005	1.004	1.003	1.004	1.003
		SD	0.073	0.078	0.079	0.085	0.129	0.142	0.047	0.122	0.122	0.117	0.114	0.113	0.114
	$SE_{\hat{\theta}}$	M	0.063	0.063	0.063	0.063	0.095	0.095	0.046	0.070	0.070	0.070	0.070	0.070	0.070
		SD	0.006	0.006	0.006	0.007	0.010	0.011	0.004	0.008	0.007	0.007	0.007	0.007	0.007

Table 10.5. *Means and standard deviations of a subset of parameter estimates and standard errors for a 2-factor model of normal data with 20 variables with N=300.*

			Subset of Factor Loadings				Factor Covariances			Subset of Error Variances					
			V_1	V_2	V_{11}	V_{12}	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_{11}	E_{12}	E_{13}
AGLS	$\hat{\theta}$	M	1.021	1.018	1.031	1.028	0.486	0.481	0.139	0.452	0.451	0.448	0.450	0.450	0.450
		SD	0.220	0.219	0.230	0.200	0.162	0.156	0.075	0.112	0.111	0.114	0.108	0.110	0.104
	$SE_{\hat{\theta}}$	M	0.247	0.247	0.253	0.246	0.195	0.196	0.078	0.159	0.159	0.157	0.158	0.158	0.157
		SD	0.092	0.103	0.137	0.086	0.052	0.050	0.020	0.038	0.041	0.040	0.037	0.039	0.038
NGLS	$\hat{\theta}$	M	0.997	1.005	1.005	1.004	0.377	0.376	0.103	0.352	0.352	0.349	0.346	0.350	0.353
		SD	0.090	0.090	0.094	0.092	0.061	0.060	0.026	0.077	0.077	0.077	0.070	0.076	0.065
	$SE_{\hat{\theta}}$	M	0.293	0.294	0.297	0.295	0.190	0.191	0.071	0.157	0.156	0.154	0.155	0.155	0.154
		SD	0.051	0.052	0.055	0.053	0.038	0.039	0.010	0.039	0.041	0.041	0.037	0.039	0.038
MGLS	$\hat{\theta}$	M	1.008	1.011	1.007	1.003	0.970	0.989	0.302	0.899	0.896	0.887	0.889	0.894	0.892
		SD	0.125	0.131	0.131	0.123	0.220	0.239	0.081	0.173	0.185	0.182	0.165	0.176	0.166
	$SE_{\hat{\theta}}$	M	0.090	0.089	0.089	0.088	0.145	0.147	0.071	0.087	0.086	0.086	0.086	0.086	0.086
		SD	0.013	0.014	0.014	0.014	0.028	0.030	0.011	0.016	0.017	0.017	0.015	0.016	0.016
MRLS	$\hat{\theta}$	M	1.008	1.009	1.007	1.001	1.042	1.059	0.302	1.079	1.068	1.063	1.062	1.067	1.065
		SD	0.123	0.126	0.125	0.118	0.228	0.251	0.079	0.201	0.218	0.210	0.193	0.206	0.202
	$SE_{\hat{\theta}}$	M	0.089	0.089	0.089	0.088	0.154	0.156	0.073	0.098	0.097	0.096	0.096	0.097	0.096
		SD	0.013	0.013	0.013	0.013	0.030	0.031	0.012	0.018	0.019	0.019	0.017	0.018	0.018

Table 10.6. *Means and standard deviations of a subset of parameter estimates and standard errors for a 2-factor model of normal data with 20 variables with N=500.*

			Subset of Factor Loadings				Factor Covariances			Subset of Error Variances					
			V_1	V_2	V_{11}	V_{12}	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_{11}	E_{12}	E_{13}
AGLS	$\hat{\theta}$	M	1.006	1.004	1.013	1.014	0.572	0.566	0.172	0.555	0.548	0.541	0.548	0.548	0.550
		SD	0.115	0.114	0.118	0.108	0.109	0.108	0.050	0.093	0.090	0.086	0.095	0.096	0.089
	$SE_{\hat{\theta}}$	M	0.153	0.153	0.155	0.155	0.152	0.152	0.060	0.122	0.122	0.120	0.123	0.123	0.122
		SD	0.028	0.029	0.028	0.027	0.029	0.028	0.010	0.024	0.024	0.023	0.024	0.026	0.024
NGLS	$\hat{\theta}$	M	1.001	1.000	1.006	1.009	0.501	0.499	0.147	0.486	0.486	0.478	0.484	0.484	0.484
		SD	0.068	0.067	0.068	0.064	0.062	0.066	0.029	0.070	0.070	0.068	0.072	0.069	0.070
	$SE_{\hat{\theta}}$	M	0.179	0.179	0.181	0.181	0.157	0.157	0.059	0.129	0.129	0.127	0.130	0.130	0.128
		SD	0.022	0.023	0.023	0.023	0.028	0.028	0.008	0.026	0.026	0.024	0.025	0.027	0.026
MGLS	$\hat{\theta}$	M	0.999	1.000	1.007	1.008	0.986	0.988	0.303	0.943	0.943	0.930	0.948	0.947	0.941
		SD	0.096	0.099	0.096	0.092	0.178	0.178	0.069	0.144	0.145	0.135	0.141	0.151	0.143
	$SE_{\hat{\theta}}$	M	0.068	0.068	0.069	0.069	0.113	0.113	0.055	0.069	0.069	0.068	0.069	0.069	0.068
		SD	0.008	0.008	0.008	0.008	0.018	0.017	0.007	0.010	0.010	0.009	0.010	0.011	0.010
MRLS	$\hat{\theta}$	M	1.000	1.001	1.006	1.007	1.028	1.032	0.302	1.050	1.050	1.038	1.057	1.055	1.047
		SD	0.093	0.098	0.094	0.091	0.184	0.183	0.069	0.159	0.157	0.147	0.156	0.165	0.156
	$SE_{\hat{\theta}}$	M	0.068	0.068	0.068	0.068	0.118	0.118	0.056	0.074	0.074	0.073	0.074	0.074	0.073
		SD	0.008	0.008	0.008	0.008	0.018	0.018	0.007	0.011	0.011	0.010	0.011	0.011	0.011

Table 10.7. *Means and standard deviations of a subset of parameter estimates and standard errors for a 2-factor model of normal data with 20 variables with N=1000.*

			Subset of Factor Loadings				Factor Covariances			Subset of Error Variances					
			V_1	V_2	V_{11}	V_{12}	$var(F_1)$	$var(F_2)$	$cov(F_1,F_2)$	E_1	E_2	E_3	E_{11}	E_{12}	E_{13}
AGLS	$\hat{\theta}$	M	1.006	1.004	1.001	1.000	0.702	0.706	0.214	0.688	0.688	0.689	0.689	0.688	0.687
		SD	0.066	0.067	0.064	0.064	0.084	0.086	0.035	0.071	0.069	0.074	0.074	0.075	0.077
	$SE_{\hat{\theta}}$	M	0.089	0.089	0.090	0.090	0.111	0.113	0.043	0.090	0.091	0.090	0.091	0.091	0.091
		SD	0.009	0.009	0.009	0.009	0.015	0.015	0.005	0.012	0.013	0.013	0.013	0.012	0.013
NGLS	$\hat{\theta}$	M	1.002	1.002	0.999	0.999	0.668	0.673	0.202	0.655	0.657	0.654	0.658	0.657	0.655
		SD	0.053	0.055	0.051	0.052	0.068	0.068	0.028	0.062	0.063	0.065	0.066	0.066	0.065
	$SE_{\hat{\theta}}$	M	0.097	0.097	0.097	0.097	0.115	0.116	0.043	0.095	0.095	0.095	0.096	0.096	0.095
		SD	0.008	0.008	0.008	0.008	0.015	0.015	0.004	0.013	0.013	0.014	0.013	0.013	0.013
MGLS	$\hat{\theta}$	M	1.002	1.003	0.999	1.001	0.988	0.997	0.303	0.965	0.968	0.964	0.972	0.973	0.969
		SD	0.065	0.067	0.066	0.065	0.121	0.123	0.046	0.097	0.101	0.103	0.103	0.102	0.101
	$SE_{\hat{\theta}}$	M	0.048	0.048	0.048	0.048	0.080	0.081	0.039	0.049	0.049	0.049	0.049	0.049	0.049
		SD	0.004	0.004	0.004	0.004	0.008	0.009	0.004	0.005	0.005	0.005	0.005	0.005	0.005
MRLS	$\hat{\theta}$	M	1.003	1.003	0.999	1.001	1.008	1.020	0.302	1.017	1.021	1.016	1.027	1.028	1.022
		SD	0.064	0.067	0.066	0.064	0.121	0.125	0.046	0.103	0.104	0.108	0.109	0.107	0.105
	$SE_{\hat{\theta}}$	M	0.048	0.048	0.048	0.048	0.082	0.082	0.039	0.051	0.051	0.050	0.051	0.051	0.051
		SD	0.003	0.004	0.004	0.004	0.008	0.009	0.004	0.005	0.005	0.005	0.005	0.005	0.005

Table 10.8. *Means and standard deviations of a subset of parameter estimates and standard errors for a 2-factor model of normal data with 20 variables with N=10,000.*

			Subset of Factor Loadings				Factor Covariances			Subset of Error Variances					
			V_1	V_2	V_{11}	V_{12}	$var(F_1)$	$var(F_2)$	$cov(F_1,F_2)$	E_1	E_2	E_3	E_{11}	E_{12}	E_{13}
AGLS	$\hat{\theta}$	M	0.999	1.000	1.001	1.000	0.956	0.955	0.287	0.951	0.953	0.952	0.953	0.952	0.954
		SD	0.020	0.020	0.019	0.020	0.038	0.035	0.013	0.032	0.031	0.031	0.032	0.031	0.032
	$SE_{\hat{\theta}}$	M	0.021	0.021	0.021	0.021	0.038	0.038	0.014	0.032	0.032	0.032	0.032	0.032	0.032
		SD	0.001	0.001	0.001	0.001	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001
NGLS	$\hat{\theta}$	M	0.999	1.000	1.001	1.000	0.956	0.954	0.286	0.951	0.952	0.951	0.952	0.951	0.953
		SD	0.020	0.020	0.019	0.020	0.037	0.035	0.013	0.032	0.031	0.030	0.032	0.031	0.032
	$SE_{\hat{\theta}}$	M	0.022	0.022	0.022	0.022	0.038	0.038	0.014	0.032	0.032	0.032	0.032	0.032	0.032
		SD	0.0005	0.0005	0.0005	0.0005	0.0017	0.0016	0.0005	0.0015	0.0014	0.0014	0.0014	0.0014	0.0014
MGLS	$\hat{\theta}$	M	1.000	1.000	1.001	1.000	1.001	1.000	0.300	0.996	0.998	0.996	0.997	0.996	0.998
		SD	0.020	0.020	0.019	0.020	0.040	0.038	0.014	0.034	0.033	0.032	0.034	0.033	0.033
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.025	0.025	0.012	0.016	0.016	0.016	0.016	0.016	0.016
		SD	0.0004	0.0004	0.0003	0.0004	0.0009	0.0008	0.0004	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005
MRLS	$\hat{\theta}$	M	1.000	1.000	1.001	1.000	1.003	1.002	0.300	1.001	1.003	1.002	1.003	1.002	1.003
		SD	0.020	0.021	0.019	0.020	0.040	0.038	0.014	0.034	0.033	0.033	0.034	0.033	0.033
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.026	0.025	0.012	0.016	0.016	0.016	0.016	0.016	0.016
		SD	0.0004	0.0004	0.0003	0.0004	0.0009	0.0008	0.0004	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005

Table 10.9. *Means and standard deviations of estimated* $\chi^2(df = 8)$ *test statistics of the 2-factor model of normal data with 6 variables.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
100	5.36	2.29	5.63	2.13	7.44	3.49	7.47	3.55
300	6.99	3.34	7.08	3.26	7.92	4.06	7.91	4.14
500	7.14	3.26	7.23	3.27	7.72	3.71	7.71	3.74
1000	7.68	3.67	7.70	3.65	7.97	3.90	7.96	3.89
10,000	7.96	3.67	7.97	3.68	7.99	3.71	8.00	3.71
100,000	8.16	3.96	8.16	3.96	8.16	3.96	8.16	3.96

Table 10.10. *Percent of replications in which the 2-factor model of normal data with 6 variables was rejected by a* $\chi^2(df = 8)$ *test with* $\alpha = .05$.

N	AGLS	NGLS	MGLS	MRLS
100	0.00	0.00	2.00	2.40
300	1.40	1.00	5.20	5.40
500	1.80	1.20	4.00	4.00
1000	4.20	4.00	4.80	4.60
10,000	3.60	4.00	4.20	4.40
100,000	5.00	4.80	4.80	4.80

Table 10.11. *Means and standard deviations of estimated* $\chi^2(df = 169)$ *test statistics of the 2-factor model of normal data with 20 variables.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
300	62.09	6.87	64.73	5.44	166.11	18.71	158.75	16.01
500	91.77	8.78	86.11	6.48	169.22	17.59	163.68	16.25
1000	118.45	10.80	114.35	8.90	170.15	18.83	167.17	17.61
10,000	161.38	16.44	161.45	16.09	169.27	17.69	169.04	17.56
100,000	168.05	18.23	168.02	18.27	168.83	18.45	168.81	18.46

Table 10.12. *Percent of replications in which the 2-factor model of normal data with 20 variables was rejected by a* $\chi^2(df = 169)$ *test with* $\alpha = .05$.

N	AGLS	NGLS	MGLS	MRLS
300	0.0	0.0	3.2	0.8
500	0.0	0.0	4.4	1.4
1000	0.0	0.0	6.2	2.8
10,000	0.8	1.0	4.2	4.4
100,000	5.4	5.6	6.0	6.0

Figures

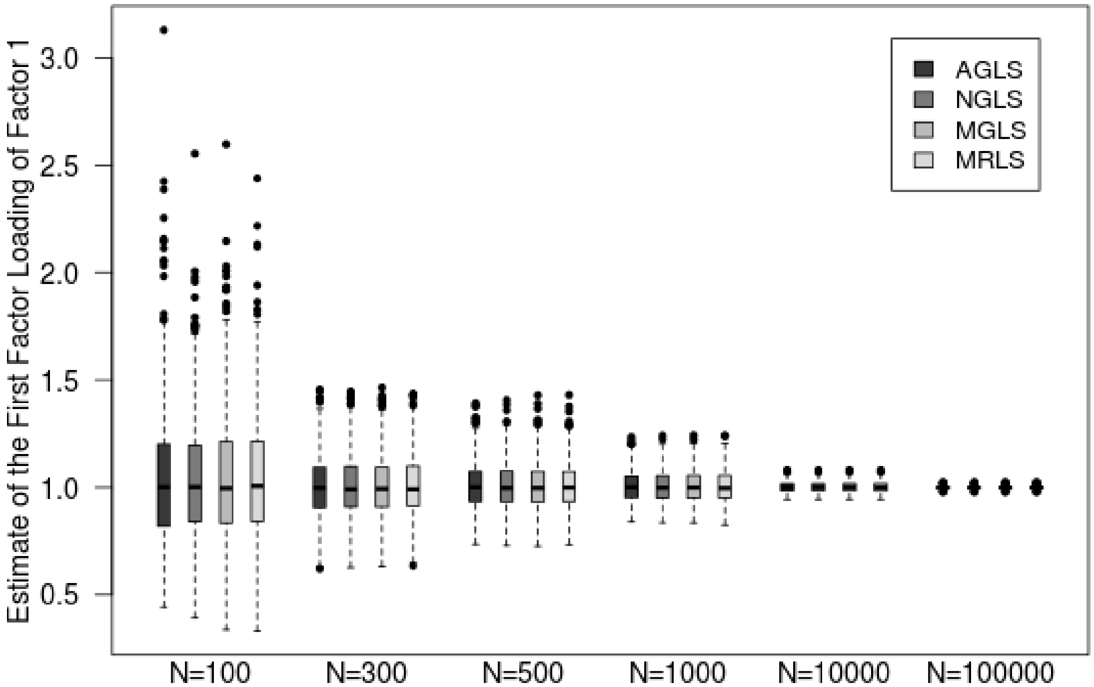

Figure 10.1. Boxplots of the values of the first factor loading of F_1 (in the 2-factor model) estimated by the different methods across sample sizes (in the normal data condition with 6 variables).

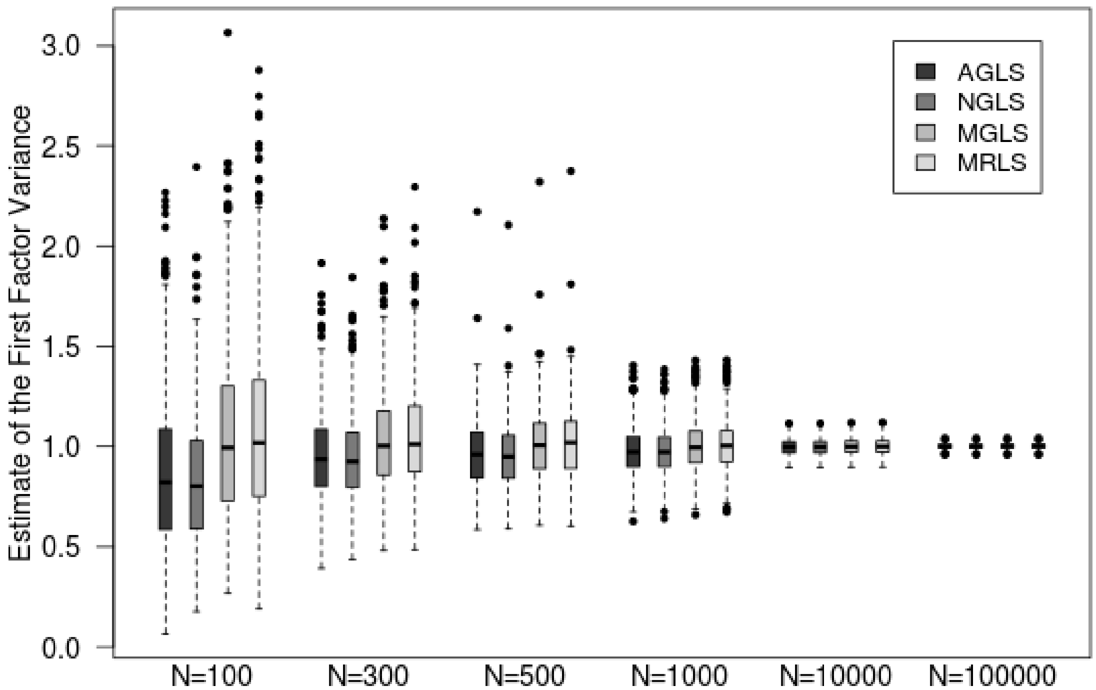

Figure 10.2. Boxplots of the values of the factor variance of F_1 (in the 2-factor model) estimated by the different methods across sample sizes (in the normal data condition with 6 variables).

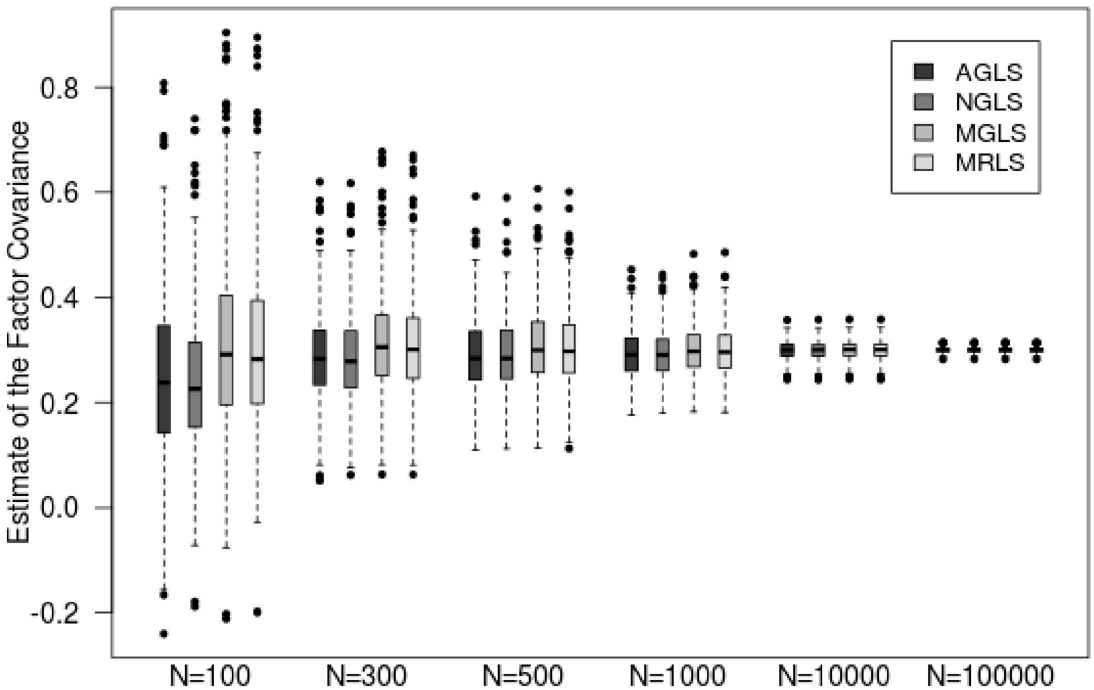

Figure 10.3. Boxplots of the values of the factor covariance between F_1 and F_2 (in the 2-factor model) estimated by the different methods across sample sizes (in the normal data condition with 6 variables).

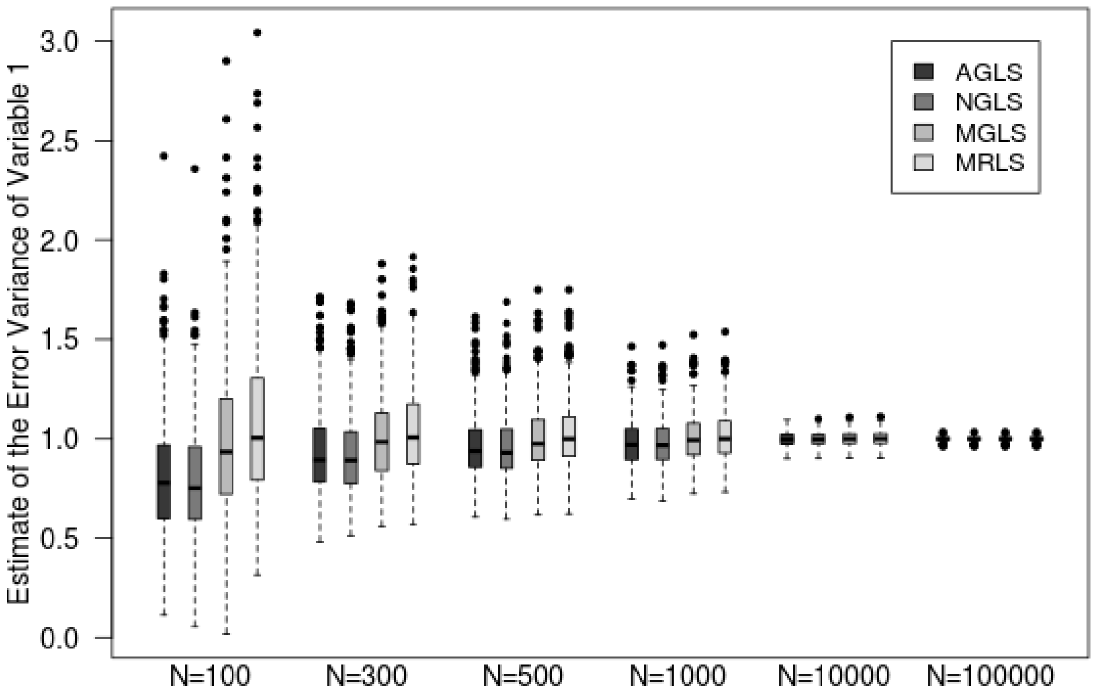

Figure 10.4. Boxplots of the values of the first error variance (of V_1 in the 2-factor model) estimated by the different methods across sample sizes (in the normal data condition with 6 variables).

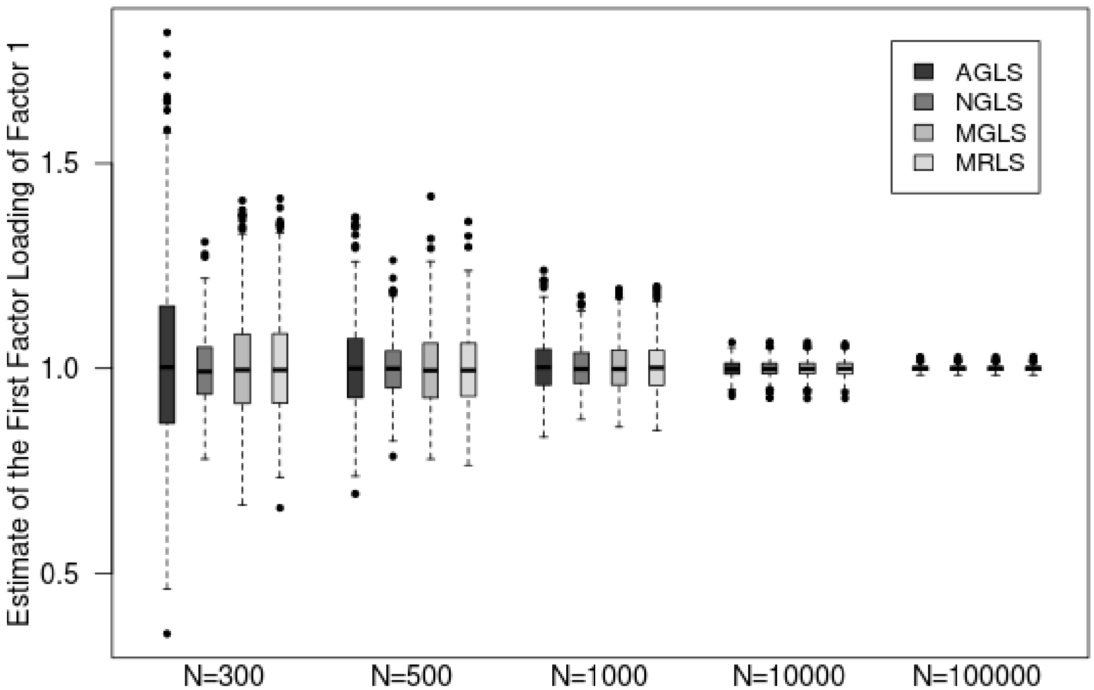

Figure 10.5. Boxplots of the values of the first factor loading of F_1 (in the 2-factor model) estimated by the different methods across sample sizes (in the normal data condition with 20 variables).

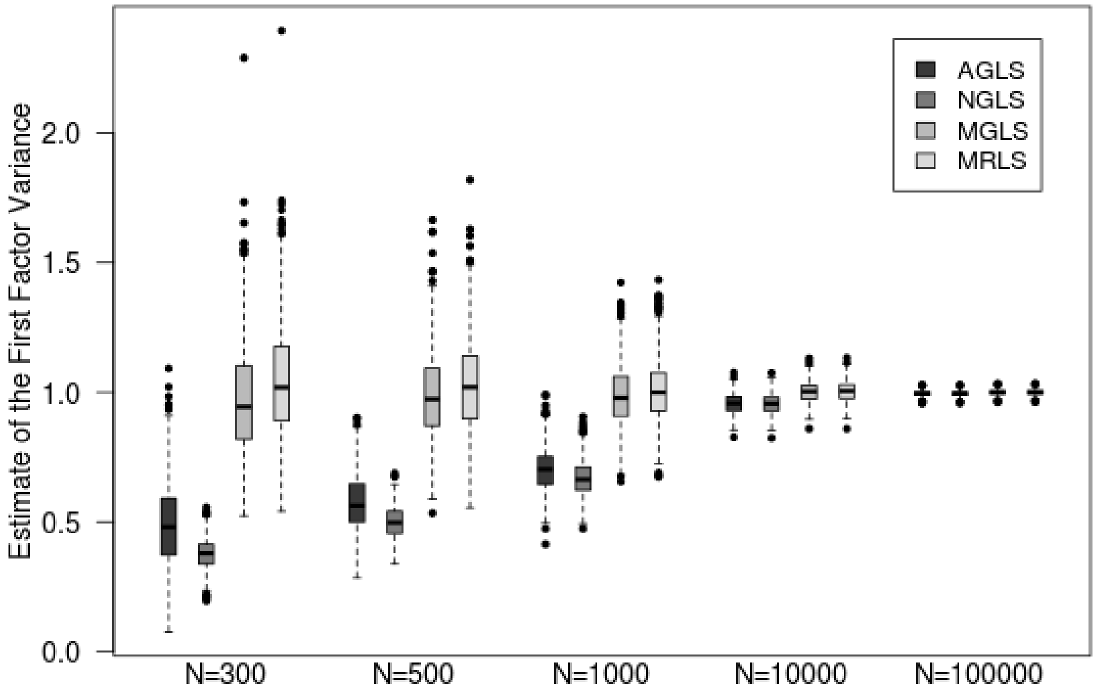

Figure 10.6. Boxplots of the values of the factor variance of F_1 (in the 2-factor model) estimated by the different methods across sample sizes (in the normal data condition with 20 variables).

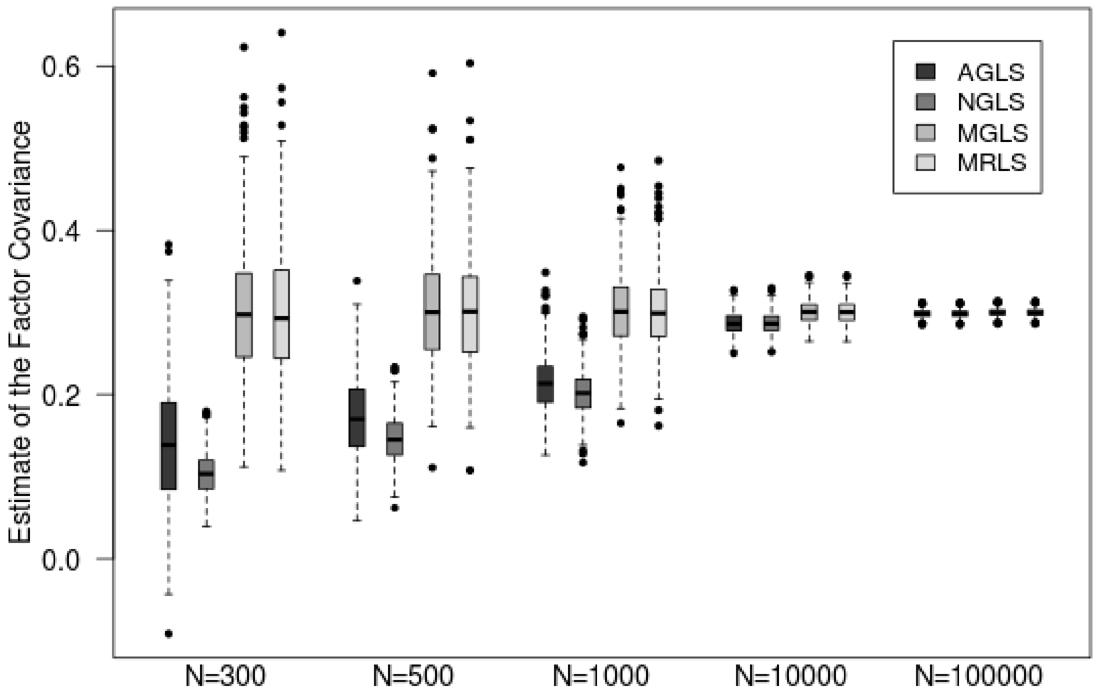

Figure 10.7. Boxplots of the values of the factor covariance between F_1 and F_2 (in the 2-factor model) estimated by the different methods across sample sizes (in the normal data condition with 20 variables).

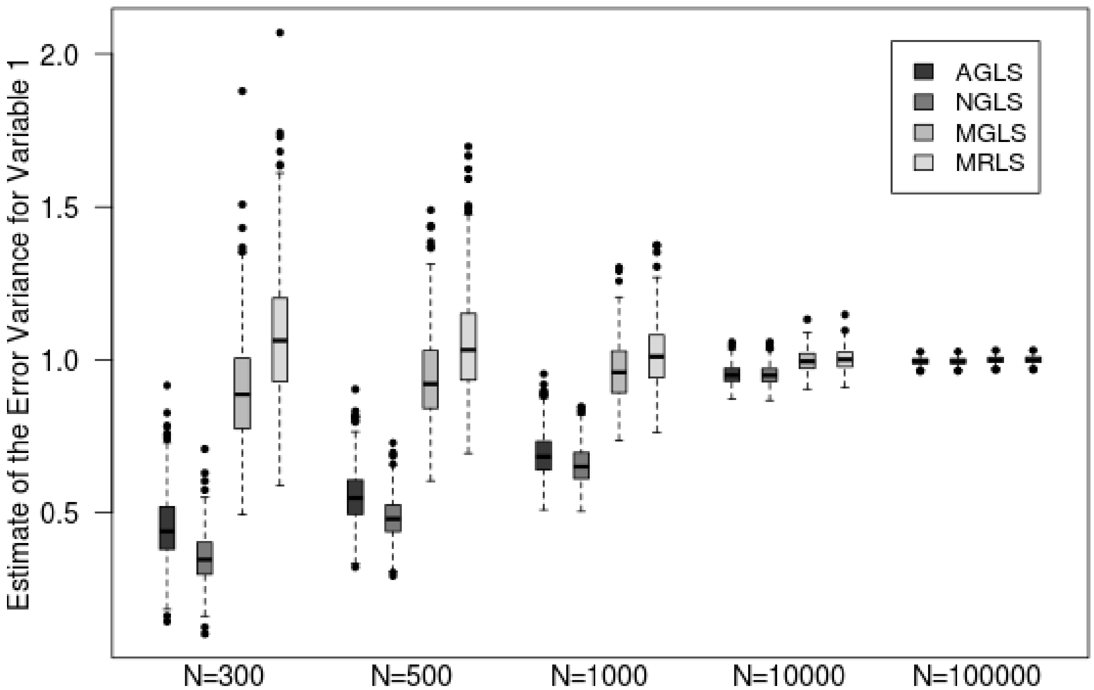

Figure 10.8. Boxplots of the values of the first error variance (of V_1 in the 2-factor model) estimated by the different methods across sample sizes (in the normal data condition with 20 variables).

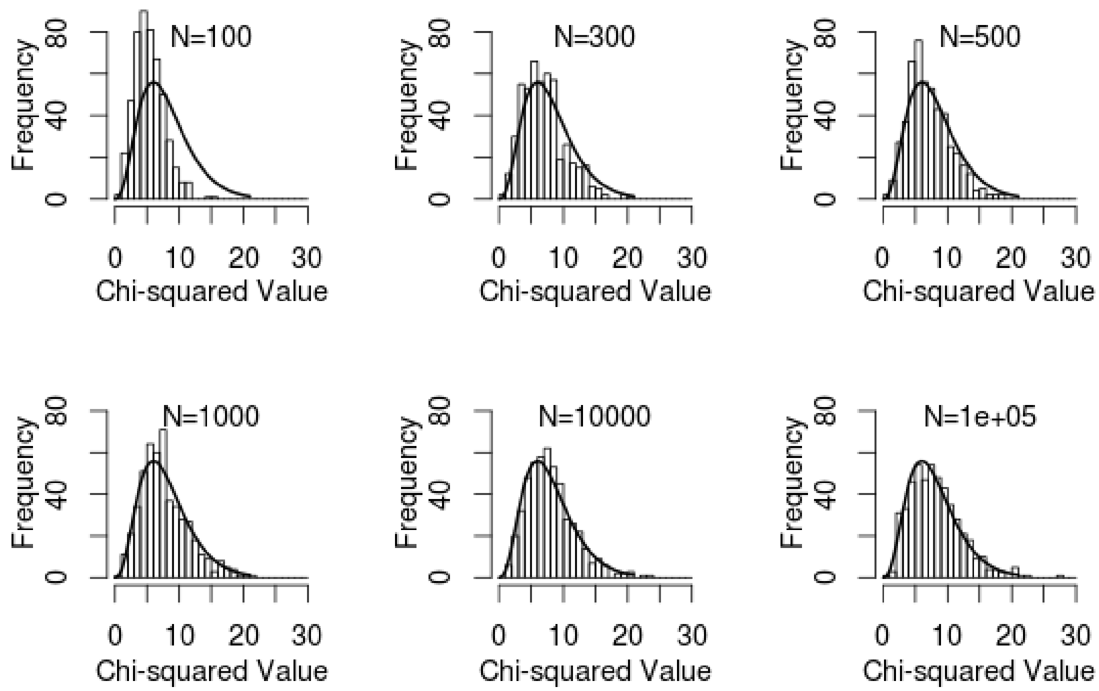

Figure 10.9. Histogram of the values of the χ^2-test statistic produced through AGLS estimation of the 2-factor model across sample sizes (in the normal data condition with 6 variables).

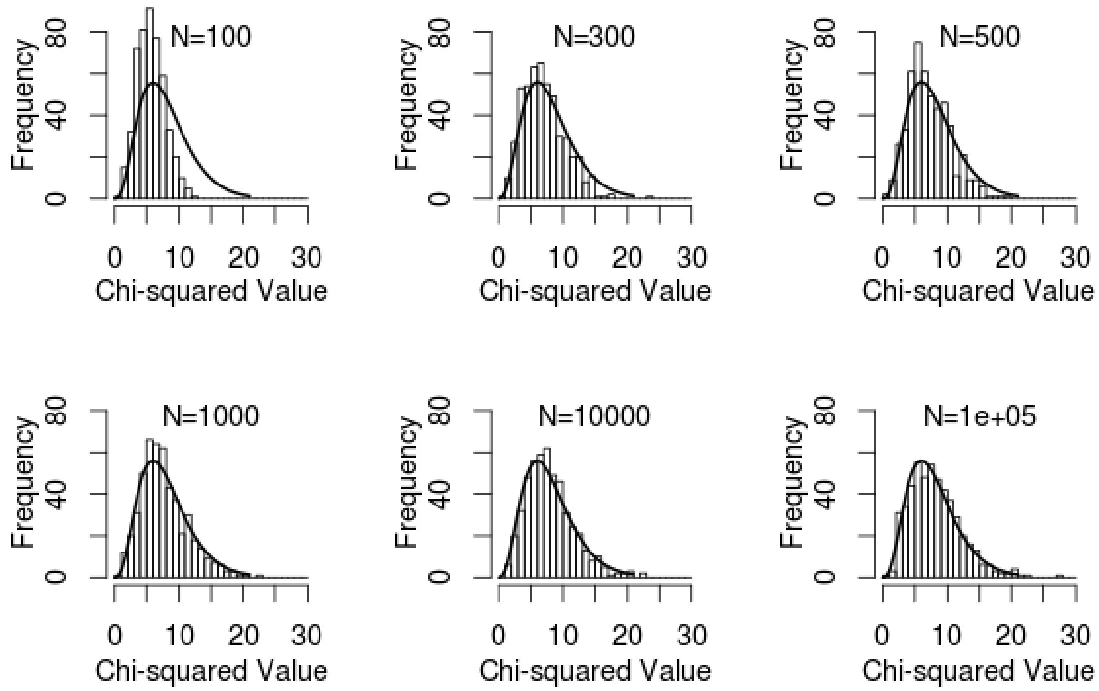

Figure 10.10. Histogram of the values of the χ^2-test statistic produced through NGLS estimation of the 2-factor model across sample sizes (in the normal data condition with 6 variables).

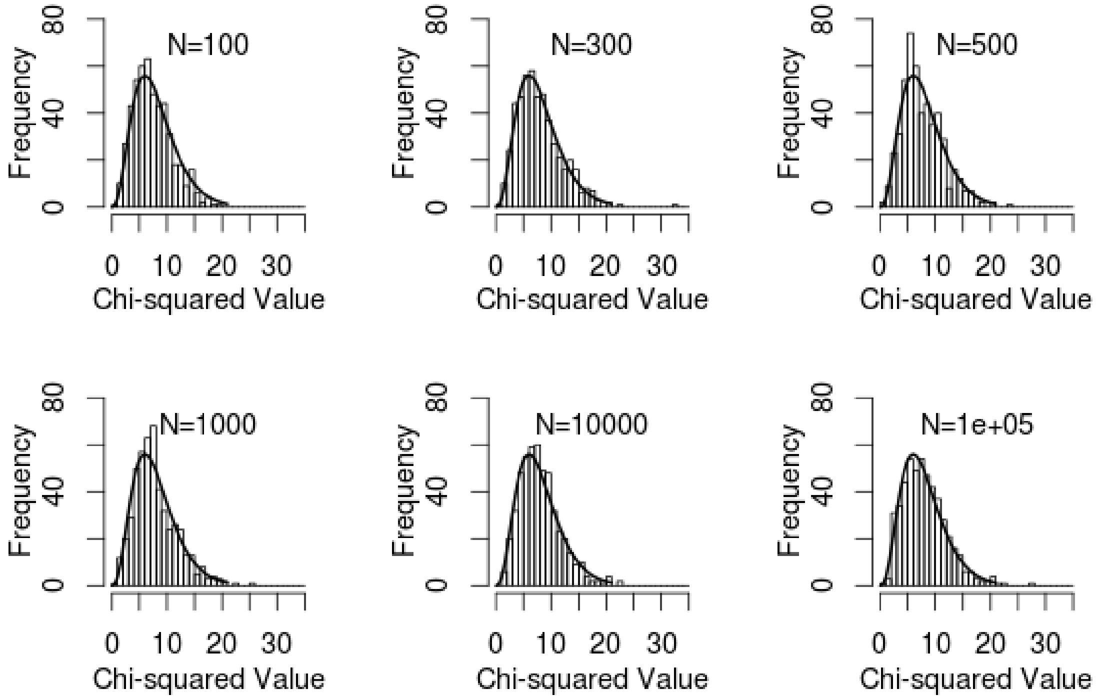

Figure 10.11. Histogram of the values of the χ^2-test statistic produced through MGLS estimation of the 2-factor model across sample sizes (in the normal data condition with 6 variables).

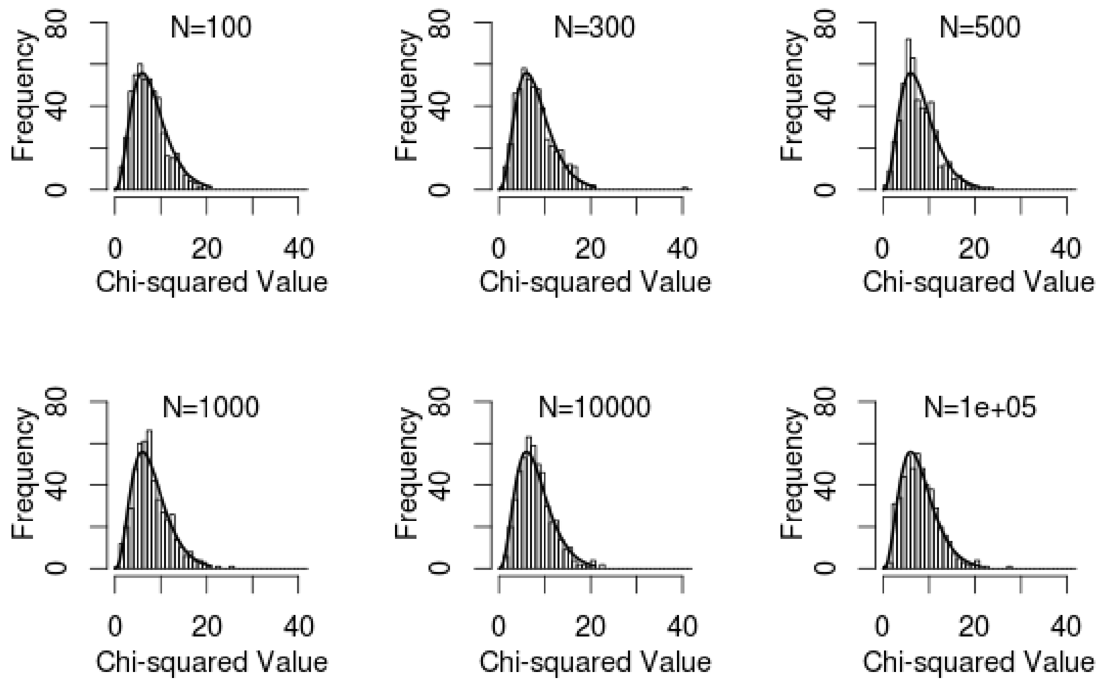

Figure 10.12. Histogram of the values of the χ^2-test statistic produced through MRLS estimation of the 2-factor model across sample sizes (in the normal data condition with 6 variables).

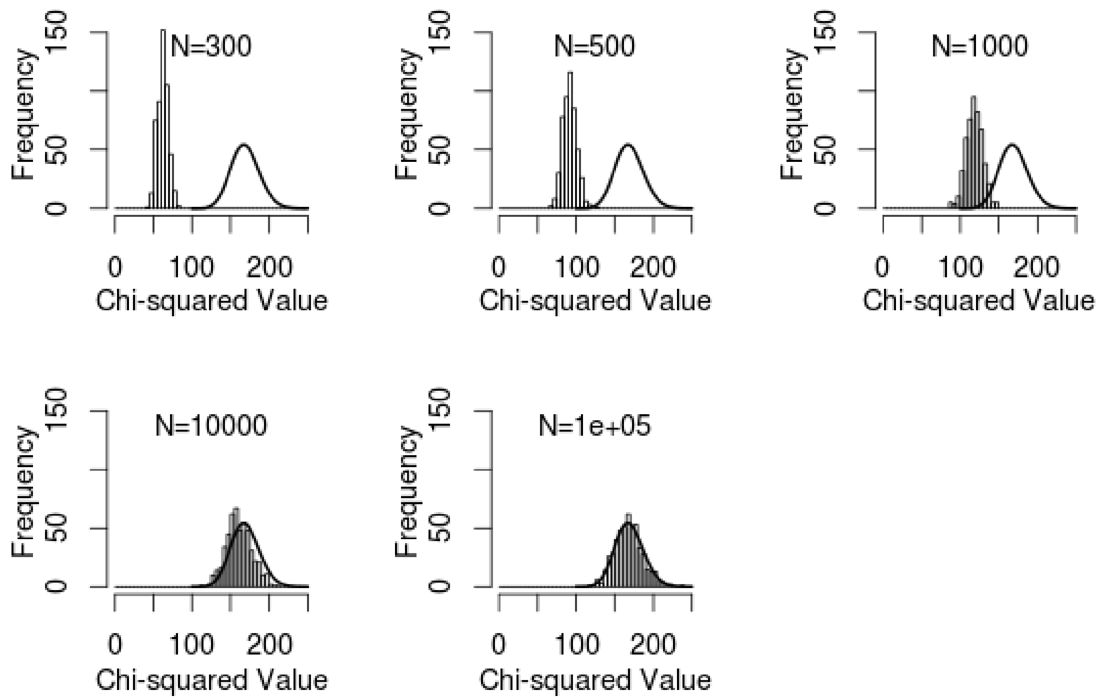

Figure 10.13. Histogram of the values of the χ^2-test statistic produced through AGLS estimation of the 2-factor model across sample sizes (in the normal data condition with 20 variables).

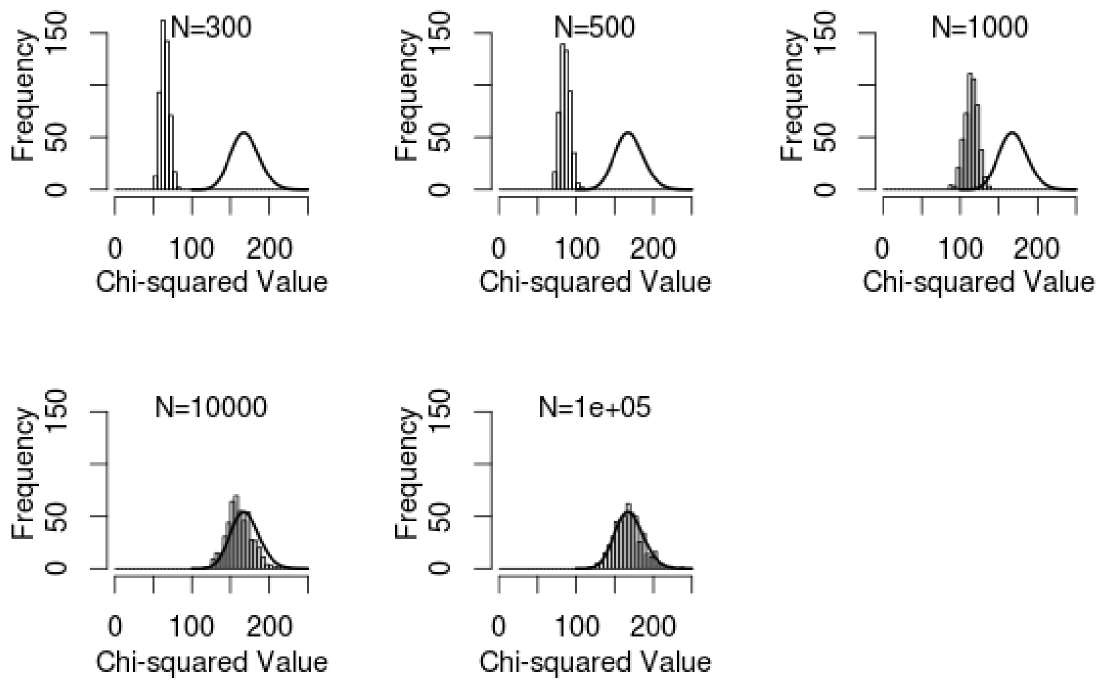

Figure 10.14. Histogram of the values of the χ^2-test statistic produced through NGLS estimation of the 2-factor model across sample sizes (in the normal data condition with 20 variables).

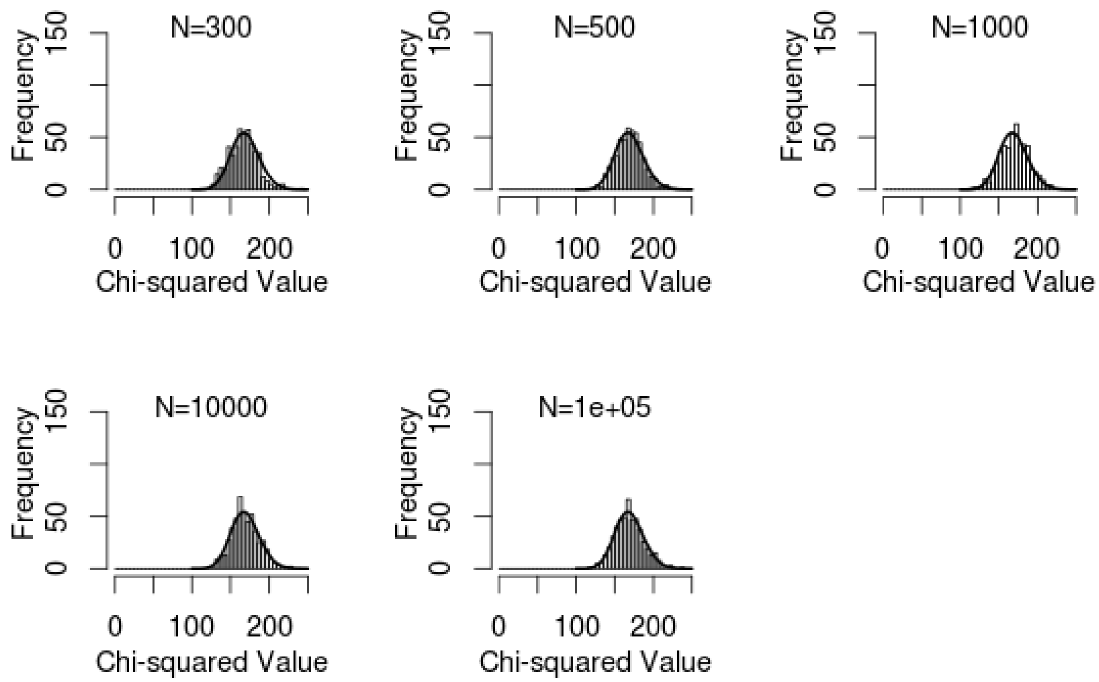

Figure 10.15. Histogram of the values of the χ^2-test statistic produced through MGLS estimation of the 2-factor model across sample sizes (in the normal data condition with 20 variables).

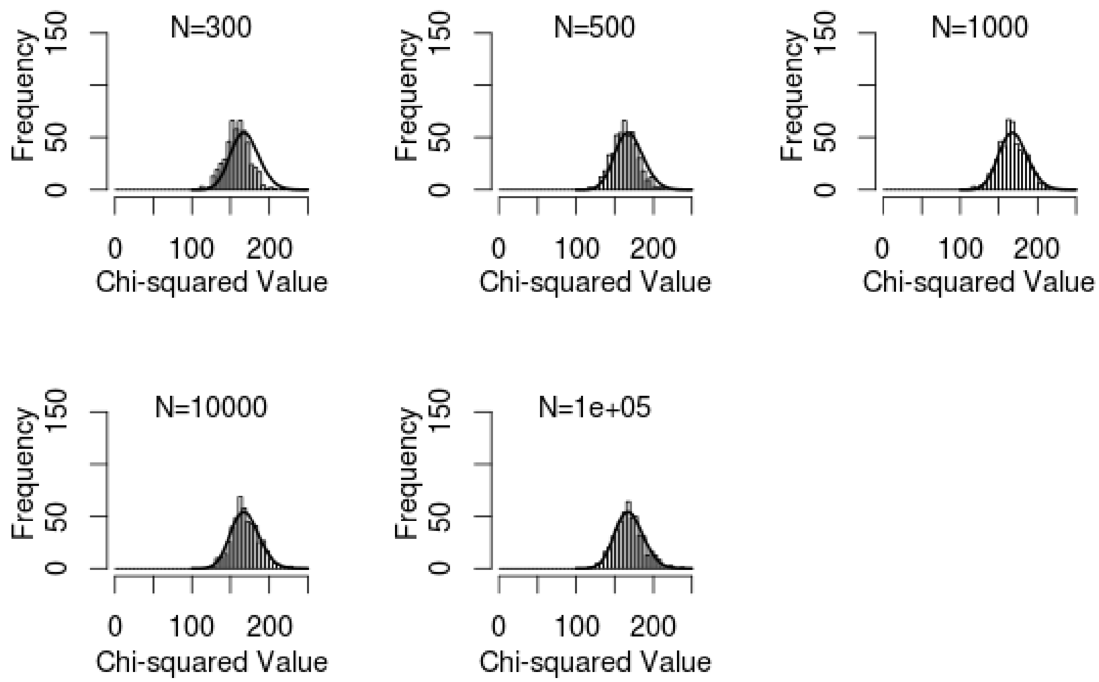

Figure 10.16. Histogram of the values of the χ^2-test statistic produced through MRLS estimation of the 2-factor model across sample sizes (in the normal data condition with 20 variables).

Chapter 11. Quality of Parameter, Test Statistic, and Standard Error Estimates: Two Factor Models of Log-Normal Data

In this chapter we continued to examine the performance of the AGLS, NGLS, MGLS and MRLS methods of estimating CV models containing two factors. Specifically, here we consider the performance of each method when the data followed a log-normal distribution.

Method

Conditions. In this chapter, we considered the same 2-factor structures as the last chapter with the structure shown in Figure 5.2. The models included either 6 variables (3 per factor) or 20 variables (10 per factor). The sample sizes in the 6-variable case ranged from 100 to 100,000, whereas in the 20-variable case, ranged from 300 to 100,000.

Analyses. Refer to Chapter 7 for details regarding the analyses. The analyses used here were identical to those in the preceding chapters.

Results

Convergence. In the present simulations, each method had some samples for which it could not converge, particularly when the sample sizes were small. However, the MRLS procedure had some difficulty finding appropriate solutions, even with relatively large sample sizes. In the 6-variable condition with a sample of 100, the AGLS procedure did not converge for 10 samples, the NGLS procedure did not converge in 8 samples, the MGLS procedure did not converge in 1 sample, and the MRLS procedure did not converge in 15 samples. The MRLS procedure was also unable to converge for 6 samples with N=300, 5 samples with N=500, and 2 samples with N=1000.

In the conditions with 20 variables, the smallest sample size was larger and as result, there were fewer issues with convergence. However, the AGLS method did not converge in 4 samples with sample of size 300 and the MRLS procedure did not converge for 12 samples of size 300, 5 samples of size 500 and 4 samples of size 1000.

Parameter Estimates and Standard Errors. As in the previous chapter, the population factor loadings, factor variances and error variances each have values of 1 and the population factor covariance has a value of 0.3 and these values were contrasted with the estimates obtained using each of the four estimation methods. The means and standard deviations of these estimates are reported in Tables 11.1 through 11.4 for the 6-variable condition. Each table displays the results for one sample size (100, 300, 500, and 1000, respectively). Unlike the results observed in the previous chapter, which examined normal data, in this case the average factor loadings tended to be too high in small samples and this appeared to be the case for each estimation method. However, these estimates did improve relatively quickly with increases in sample size. For the first factor loading, the distributions of estimates are displayed in Figure 11.1 across methods and sample sizes. In the figure it is apparent that although the median loading values were accurate, there was a large positive skew particularly in small samples.

In contrast to the factor loadings, the trends seen in the quality of the estimates of the factor variances and covariance (in the 6-variable condition) seemed very similar to those reported in the previous chapter. That is, the AGLS and NGLS techniques still yielded estimates that were much too low and the MGLS and MRLS methods tended to give factor variance and covariance estimates that were fairly accurate on average, but the estimates tended to have high variability and they were more likely to yield extreme outliers than the other two methods. For example, consider the distributions of the first factor variance and of the factor covariance given in Figures 11.2 and 11.3, respectively.

In addition, the error variance estimates produced here in the 6-variable condition tended to be worse than in the previous chapter, with all estimation methods producing values that were too low. For instance, in the condition with a sample size of 100, the average error variance estimates given by the NGLS procedure were roughly between 0.49 and 0.54 and those given by the AGLS procedure were roughly between 0.53 and 0.58. The MGLS procedure provided slightly better estimates, which were approximately between 0.69 and 0.74, and the MRLS procedure provided the best estimates, which were ranged approximately from 0.76 to 1.02. However, even in this best case, the estimates were much too low, and although there was improvement as the sample size increased, the estimates were still somewhat too low with N=1000. This tendency of the estimation methods to underestimate the error variances can be seen in Table 11.4, but is also displayed Figure 11.4, which shows the distributions of estimates of one particular error variance across methods and sample sizes.

In contrast to the previous chapter, in which the AGLS and NGLS standard errors tended to be upwardly biased and the MGLS and MRLS standard errors tended to be downwardly biased, the standard error estimates in the 6-variable log-normal data condition were all too low on average. While the AGLS method standard errors did tend to improve as the sample size increased, the other methods standard errors did not seem to improve and in some cases, appeared worse in larger samples. Furthermore, it seems that at least some of the samples/estimates produced very extreme values of the standard errors. For instance, consider the standard deviations of the standard error estimates shown in Table 11.1 for the N=100 condition. The standard deviation of the AGLS standard error estimates was 12.59 for one of the factor loadings, and the standard deviation of the MGLS standard error estimates was 31.29 for one of the error variances. This instability seemed to be resolved in the N=300 condition, but nevertheless, suggests that the standard errors produced for log-normal data are not consistently reliable.

The means and standard deviations of a subset of the parameter estimates for the 20-variable condition are displayed in Tables 11.5 through 11.8 and once again each table displays the results for all methods in one sample size (100, 300, 500, and 1000, respectively). For the most part, the average factor loadings were accurate for each method and sample size. However, the average factor loading estimates given by the MRLS procedure were unreliable and sometimes off by a substantial margin (e.g. see Tables 11.6 and 11.7). This is problem is probably driven by the occasional extreme values produced by MRLS. Figure 11.5 depicts the typical distributions of the estimates of the first factor loading (of factor 1), and contains comments about the extreme values (which are not depicted). It is noteworthy that most of these extreme values occurring in the distributions shown in Figures 11.5 through 11.8 were the result of MRLS estimation.

Next, on average the factor variance and covariance estimates (in the 20-variable condition) that were produced through MRLS and MGLS estimation were fairly accurate across sample sizes. However, the factor variance and covariance estimates given my AGLS and NGLS were extremely low, on average. Specifically, in the N=300 condition the average AGLS estimates were about 0.23 for the factor variances and about 0.14 for the covariance, while the the average NGLS estimates were about 0.21 for the factor variances and about 0.05 for the covariance. Although there was some improvement as the sample size increased, the estimates were persistently too low even in the very large sample sizes. In the N=10,000 condition the average AGLS estimates were about 0.82 for the factor variances and about 0.23 for the covariance, while the average NGLS estimates were about 0.77 for the factor variances and about 0.23 for the covariance. This underestimation is readily apparent in Figures 11.6 and 11.7 which display the distributions of the first factor variance estimates and the factor covariance estimates, respectively. In addition, these figures call attention to the large difference between the variability of the estimates given by the AGLS and NGLS methods relative to that of the estimates given by the MGLS and MRLS methods. That is, while MGLS and MRLS estimation yielded more accurate estimates of the factor variances and covariance on average, they tended to produce highly varied and sometimes extreme estimates. On the other hand, the AGLS and NGLS methods were very consistent in giving their very low estimates of the factor variance and covariance.

In addition, the error variance estimates showed a similar pattern of results to the factor variance estimates. More specifically, MRLS estimation produced the most accurate estimates on average but the tended to be a slightly high and this can probably be attributed to the large positive skew in the distributions, which is particularly apparent in Figure 10.8, which shows the distributions of the estimates of one of the error variances. MGLS estimation was yield average estimates that were somewhat too low, and the AGLS and NGLS methods reliably underestimated the error variances. For example, considering the subset of values reported in Table 11.5 for the condition with a sample size of 300, MRLS produced average estimates between 1.07 and 1.13 and MGLS produced average estimates between 0.60 and 0.63. In addition, AGLS produced average error variance estimates ranging from about 0.12 to 0.13, whereas NGLS produced average error variance estimates ranging from about 0.14 to 0.15. As the sample size increased, the estimates of all methods generally approached the true population value (see Figure 11.8 for an example).

The means and standard deviations of the standard errors for the 20-variable condition are also shown in Tables 11.5 through 11.8. The AGLS method seemed to provide standard error estimates that most closely matched the standard deviations of the parameters, but these were downwardly biased as was seen in the previous condition. By contrast, the NGLS standard errors were generally too high in small samples, but became too low in very large samples. The MRLS and MGLS methods once again produced standard error estimates that were consistently too low, and did not seem to improve as the sample size increased. In addition, there was once again trouble with excess variability in the estimated standard errors for a few of the parameters at the smallest sample size (N=300). Specifically, some factor loadings had standard errors with large standard deviations when estimated by NGLS (e.g. 6.93 and 5.34), by MGLS (e.g. 3.23 and 3.04), and by MRLS (e.g. 1.11 and 1.45) as shown in Table 11.5. In addition, the NGLS estimates of the standard error of one of the factor covariance had an extremely high standard deviation (specifically, 16.43).

Test Statistics. In the 6-variable condition the expected value of the χ^2-test statistic was 8, and Table 11.9 displays the means and standard deviations of the estimated χ^2 values by method and sample size. As can be seen in the table, only the AGLS estimation method produced χ^2 values that were consistently close to the expected value. The tendency of the AGLS χ^2 values to approximately follow the expected χ^2 distribution can also be seen in Figure 11.9, which contains histograms of the AGLS estimated χ^2 values with the expected χ^2 distribution overlaid (at different sample sizes). In addition, Table 11.10 displays the percentage of samples that would have produced χ^2 values that would result in rejection of the null hypothesis (at the 0.05 level) by each method across sample sizes. Again, only the AGLS method seemed to converge to the expected 5%. The NGLS, MGLS, and MRLS methods all tended to produce χ^2

values with averages that increased with the sample size, as shown in Table 11.9, which resulted in higher rejection rates as the sample size increased. As the sample size increased, the observed distributions of the χ^2-test statistics also became more dispersed and distant from the expected χ^2 distribution, as is shown in Figures 11.10 through 11.13.

In the 20-variable condition the expected value of the χ^2-statistic was 169, but none of the estimation methods in this condition produced χ^2 values that cleanly converged to their expected distributions. This is reflected in the means and standard deviations of the χ^2 values shown in Table 11.11. Specifically, the table shows that the mean estimates resulting from the normal theory methods (i.e. NGLS, MGLS, and MRLS) seemed to increase without bound as the sample size increased. The AGLS estimates, on the other hand, did not increase indefinitely with sample size, but instead increased well past the expected value to 276.66 and then began to approach the expected value again in very large sample sizes. The percentages of χ^2-test statistics that would have resulted in rejecting the null hypothesis are shown in 11.12, and they also demonstrate this pattern. Specifically, the rejection rates for the AGLS method was initially 0% before increasing with the sample size to 99.4% and then declining towards again towards the expected rate. The NGLS method's rejection rates are initially 0, but increase until all of the samples result in rejection and the MGLS and MRLS methods appeared to always result in a rejection of the null hypothesis. This odd behavior of the test statistics produced by each method, is consistent with what was observed in the 20-variable model of log-normal data in the 1-factor case (see Chapter 9) and is further illustrated by Figures 11.13 through 11.16. Figure 11.13 shows displays the boomerang-like behavior of the distributions of the AGLS estimates of the χ^2 values, while Figures 11.14 through 11.16 show that the distributions produced by the NGLS, MGLS, and MRLS methods (respectively) become more dispersed and have higher centers as the sample sizes increase. The expected χ^2 distribution is overlaid in each graph, so the aforementioned deviations can be easily detected.

Discussion

For the most part, the addition of a second factor did not seem to affect the estimation and in any particular way. The results observed here were generally analogous to those reported in Chapter 9. As noted previously, AGLS is the only correctly specified method in non-normal data, and the estimates of the test statistics once again reflected this. Although the results reported here for the log-normal data showed that the estimators had less than desirable performance, the fact the performance was consistent suggests that factor models may be estimated reliably using these techniques. If this finding can be extended to comparably sized non-factor models, this may allow these methods to be applied more broadly. Future research should address additional forms of models, to determine for what models this pattern of results will change.

Tables

Table 11.1. *Means and standard deviations of parameter estimates and standard errors for a 2-factor model of log-normal data with 6 variables with N=100.*

			Factor Loadings				Factor Covariances			Error Variances					
			V_1	V_2	V_4	V_5	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_4	E_5	E_6
AGLS	$\hat{\theta}$	M	1.076	1.086	1.144	1.149	0.777	0.770	0.149	0.573	0.574	0.533	0.575	0.581	0.564
		SD	0.510	0.540	1.251	0.972	0.575	0.559	0.192	0.440	0.401	0.337	0.371	0.544	0.394
	$SE_{\hat{\theta}}$	M	0.297	0.299	0.844	0.723	0.295	0.320	0.084	0.260	0.253	0.254	0.260	0.318	0.277
		SD	0.369	0.392	12.592	7.570	0.230	0.588	0.055	0.183	0.237	0.167	0.203	1.058	0.567
NGLS	$\hat{\theta}$	M	1.077	1.062	1.091	1.088	0.671	0.687	0.171	0.542	0.506	0.488	0.491	0.506	0.501
		SD	0.512	0.442	0.438	0.510	0.429	0.479	0.121	0.388	0.435	0.418	0.394	0.484	0.466
	$SE_{\hat{\theta}}$	M	0.387	0.375	0.381	0.418	0.360	0.447	0.127	0.283	0.278	0.289	0.292	0.327	0.364
		SD	0.504	0.413	0.293	0.805	0.276	1.594	0.054	0.206	0.233	0.276	0.275	0.463	1.591
MGLS	$\hat{\theta}$	M	1.059	1.060	1.123	1.107	0.943	0.958	0.270	0.720	0.693	0.686	0.694	0.739	0.685
		SD	0.464	0.453	0.529	0.511	1.060	0.704	0.241	0.496	0.519	0.929	0.520	0.610	0.536
	$SE_{\hat{\theta}}$	M	0.218	0.216	0.238	0.229	1.668	0.254	0.122	0.178	0.179	1.595	0.190	0.190	0.178
		SD	0.146	0.128	0.259	0.185	31.284	0.139	0.053	0.099	0.107	31.289	0.126	0.130	0.108
MRLS	$\hat{\theta}$	M	1.061	1.061	1.255	0.969	1.064	1.051	0.242	0.851	0.826	0.757	1.017	0.845	0.772
		SD	0.437	0.431	3.576	2.541	1.455	0.771	0.443	0.674	0.723	0.509	4.298	1.210	0.544
	$SE_{\hat{\theta}}$	M	0.218	0.213	0.245	0.234	0.277	0.274	0.128	0.197	0.195	0.196	0.244	0.220	0.195
		SD	0.146	0.115	0.536	0.374	0.231	0.138	0.088	0.116	0.120	0.117	0.923	0.362	0.097

Table 11.2. *Means and standard deviations of parameter estimates and standard errors for a 2-factor model of log-normal data with 6 variables with N=300.*

			Factor Loadings				Factor Covariances			Error Variances					
			V_1	V_2	V_4	V_5	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_4	E_5	E_6
AGLS	$\hat{\theta}$	M	1.027	1.025	1.033	1.033	0.864	0.836	0.193	0.743	0.760	0.733	0.769	0.741	0.746
		SD	0.330	0.309	0.286	0.296	0.412	0.367	0.109	0.360	0.417	0.322	0.334	0.359	0.342
	$SE_{\hat{\theta}}$	M	0.185	0.183	0.180	0.185	0.238	0.225	0.065	0.218	0.215	0.212	0.218	0.223	0.223
		SD	0.149	0.148	0.087	0.114	0.164	0.125	0.027	0.135	0.132	0.125	0.127	0.164	0.128
NGLS	$\hat{\theta}$	M	1.035	1.028	1.036	1.025	0.849	0.824	0.239	0.739	0.722	0.707	0.723	0.728	0.732
		SD	0.279	0.285	0.285	0.292	0.386	0.334	0.097	0.387	0.382	0.407	0.367	0.339	0.397
	$SE_{\hat{\theta}}$	M	0.168	0.167	0.172	0.172	0.229	0.228	0.085	0.192	0.189	0.183	0.184	0.186	0.198
		SD	0.066	0.066	0.064	0.078	0.121	0.115	0.025	0.131	0.147	0.174	0.106	0.119	0.192
MGLS	$\hat{\theta}$	M	1.038	1.031	1.041	1.021	1.011	1.004	0.295	0.874	0.865	0.831	0.846	0.865	0.889
		SD	0.293	0.292	0.316	0.301	0.503	0.571	0.146	0.488	0.517	0.505	0.413	0.455	0.566
	$SE_{\hat{\theta}}$	M	0.116	0.115	0.119	0.117	0.161	0.165	0.079	0.117	0.116	0.115	0.117	0.115	0.121
		SD	0.043	0.043	0.044	0.043	0.063	0.083	0.024	0.049	0.058	0.051	0.045	0.047	0.065
MRLS	$\hat{\theta}$	M	1.039	1.030	1.042	1.019	1.032	1.019	0.286	0.934	0.917	0.873	0.897	0.927	0.920
		SD	0.290	0.292	0.305	0.294	0.501	0.475	0.133	0.523	0.495	0.386	0.440	0.510	0.423
	$SE_{\hat{\theta}}$	M	0.117	0.116	0.118	0.116	0.164	0.166	0.079	0.121	0.119	0.117	0.122	0.120	0.122
		SD	0.043	0.044	0.042	0.042	0.056	0.060	0.022	0.050	0.049	0.038	0.047	0.049	0.044

Table 11.3. *Means and standard deviations of parameter estimates and standard errors for a 2-factor model of log-normal data with 6 variables with N=500.*

			Factor Loadings				Factor Covariances			Error Variances					
			V_1	V_2	V_4	V_5	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_4	E_5	E_6
AGLS	$\hat{\theta}$	M	1.023	1.018	1.017	1.019	0.857	0.876	0.218	0.798	0.816	0.794	0.797	0.837	0.819
		SD	0.214	0.230	0.251	0.217	0.333	0.371	0.085	0.278	0.308	0.299	0.285	0.410	0.330
	$SE_{\hat{\theta}}$	M	0.153	0.154	0.154	0.152	0.203	0.207	0.059	0.201	0.201	0.200	0.205	0.210	0.212
		SD	0.058	0.071	0.106	0.059	0.131	0.145	0.023	0.121	0.118	0.114	0.144	0.152	0.151
NGLS	$\hat{\theta}$	M	1.033	1.027	1.030	1.037	0.874	0.888	0.260	0.813	0.815	0.798	0.799	0.810	0.823
		SD	0.239	0.225	0.269	0.267	0.314	0.347	0.090	0.377	0.358	0.330	0.376	0.481	0.394
	$SE_{\hat{\theta}}$	M	0.124	0.125	0.125	0.128	0.177	0.181	0.068	0.153	0.150	0.151	0.154	0.162	0.154
		SD	0.039	0.045	0.048	0.057	0.081	0.088	0.020	0.098	0.079	0.106	0.128	0.139	0.107
MGLS	$\hat{\theta}$	M	1.041	1.024	1.039	1.048	1.005	1.013	0.299	0.907	0.907	0.903	0.904	0.943	0.913
		SD	0.260	0.235	0.296	0.295	0.571	0.541	0.136	0.431	0.406	0.448	0.534	0.589	0.479
	$SE_{\hat{\theta}}$	M	0.091	0.090	0.091	0.092	0.128	0.128	0.062	0.094	0.092	0.092	0.094	0.098	0.094
		SD	0.027	0.025	0.034	0.035	0.054	0.050	0.018	0.035	0.028	0.035	0.041	0.064	0.041
MRLS	$\hat{\theta}$	M	1.044	1.026	1.042	1.048	1.019	1.019	0.291	0.952	0.943	0.921	0.927	0.968	0.939
		SD	0.259	0.235	0.298	0.298	0.612	0.461	0.124	0.449	0.424	0.372	0.443	0.493	0.409
	$SE_{\hat{\theta}}$	M	0.091	0.090	0.092	0.092	0.128	0.129	0.062	0.096	0.094	0.093	0.096	0.098	0.095
		SD	0.027	0.025	0.035	0.035	0.048	0.043	0.017	0.035	0.029	0.027	0.036	0.038	0.032

Table 11.4. *Means and standard deviations of parameter estimates and standard errors for a 2-factor model of log-normal data with 6 variables with N=1000.*

			Factor Loadings				Factor Covariances			Error Variances					
			V_1	V_2	V_4	V_5	$var(F_1)$	$var(F_2)$	$cov(F_1,F_2)$	E_1	E_2	E_3	E_4	E_5	E_6
AGLS	$\hat{\theta}$	M	1.014	1.014	1.011	1.008	0.893	0.890	0.232	0.854	0.850	0.847	0.871	0.850	0.854
		SD	0.161	0.153	0.151	0.146	0.244	0.204	0.062	0.262	0.238	0.232	0.243	0.220	0.261
	$SE_{\hat{\theta}}$	M	0.122	0.120	0.122	0.118	0.168	0.164	0.047	0.173	0.180	0.165	0.186	0.169	0.175
		SD	0.038	0.037	0.046	0.037	0.086	0.069	0.014	0.105	0.107	0.088	0.124	0.082	0.108
NGLS	$\hat{\theta}$	M	1.021	1.021	1.010	1.005	0.929	0.946	0.270	0.880	0.882	0.863	0.912	0.863	0.876
		SD	0.177	0.178	0.184	0.174	0.332	0.271	0.071	0.311	0.349	0.331	0.382	0.276	0.359
	$SE_{\hat{\theta}}$	M	0.082	0.082	0.082	0.081	0.128	0.129	0.048	0.110	0.109	0.107	0.114	0.106	0.109
		SD	0.018	0.019	0.019	0.018	0.065	0.045	0.010	0.055	0.068	0.058	0.081	0.044	0.057
MGLS	$\hat{\theta}$	M	1.021	1.020	1.012	1.006	1.000	1.010	0.293	0.943	0.938	0.919	0.974	0.917	0.931
		SD	0.183	0.184	0.191	0.179	0.426	0.314	0.103	0.368	0.383	0.369	0.427	0.298	0.389
	$SE_{\hat{\theta}}$	M	0.063	0.063	0.062	0.062	0.091	0.092	0.044	0.067	0.067	0.066	0.068	0.066	0.067
		SD	0.014	0.014	0.014	0.013	0.031	0.024	0.009	0.020	0.021	0.025	0.025	0.018	0.020
MRLS	$\hat{\theta}$	M	1.022	1.020	1.015	1.007	1.007	1.020	0.289	0.967	0.956	0.934	0.995	0.940	0.943
		SD	0.184	0.182	0.193	0.181	0.418	0.312	0.100	0.392	0.391	0.338	0.436	0.307	0.366
	$SE_{\hat{\theta}}$	M	0.063	0.063	0.062	0.062	0.091	0.092	0.044	0.068	0.067	0.067	0.069	0.067	0.067
		SD	0.014	0.014	0.014	0.013	0.029	0.022	0.009	0.020	0.021	0.022	0.025	0.018	0.018

Table 11.5. *Means and standard deviations of a subset of parameter estimates and standard errors for a 2-factor model of log-normal data with 20 variables with N=300.*

			Subset of Factor Loadings				Factor Covariances			Subset of Error Variances					
			V_1	V_2	V_{11}	V_{12}	$var(F_1)$	$var(F_2)$	$cov(F_1,F_2)$	E_1	E_2	E_3	E_{11}	E_{12}	E_{13}
AGLS	$\hat{\theta}$	M	1.011	1.004	0.997	1.003	0.228	0.235	0.144	0.127	0.115	0.122	0.116	0.122	0.127
		SD	0.172	0.162	0.157	0.160	0.054	0.058	0.033	0.072	0.077	0.066	0.059	0.062	0.069
	$SE_{\hat{\theta}}$	M	0.146	0.145	0.142	0.143	0.057	0.058	0.033	0.063	0.063	0.063	0.060	0.060	0.063
		SD	0.036	0.036	0.036	0.035	0.010	0.011	0.005	0.023	0.033	0.021	0.019	0.019	0.021
NGLS	$\hat{\theta}$	M	1.039	1.032	1.066	1.045	0.211	0.214	0.054	0.154	0.143	0.158	0.152	0.148	0.154
		SD	0.261	0.243	0.550	0.431	0.072	0.076	0.024	0.092	0.103	0.098	0.091	0.088	0.107
	$SE_{\hat{\theta}}$	M	0.399	0.397	0.709	0.633	0.131	0.133	0.049	0.104	0.110	0.106	0.100	0.101	0.104
		SD	0.180	0.169	6.933	5.335	0.037	0.037	0.012	0.060	0.146	0.065	0.045	0.060	0.057
MGLS	$\hat{\theta}$	M	0.992	1.010	1.014	1.019	0.907	0.891	0.239	0.618	0.632	0.627	0.596	0.600	0.617
		SD	0.760	0.725	0.274	0.278	0.506	1.037	0.103	0.267	0.506	0.273	0.222	0.257	0.259
	$SE_{\hat{\theta}}$	M	0.237	0.229	0.091	0.092	0.131	0.868	0.064	0.066	0.068	0.067	0.064	0.065	0.066
		SD	3.234	3.040	0.048	0.043	0.056	16.418	0.019	0.027	0.048	0.027	0.022	0.027	0.025
MRLS	$\hat{\theta}$	M	1.001	1.000	1.010	1.003	1.039	1.106	0.286	1.111	1.129	1.104	1.068	1.110	1.127
		SD	0.722	0.890	0.241	0.230	0.547	1.017	0.170	0.678	0.602	0.552	0.563	0.672	0.775
	$SE_{\hat{\theta}}$	M	0.146	0.163	0.093	0.093	0.156	0.161	0.071	0.100	0.101	0.099	0.096	0.099	0.101
		SD	1.111	1.454	0.027	0.028	0.072	0.099	0.025	0.058	0.050	0.047	0.047	0.057	0.065

Table 11.6. *Means and standard deviations of a subset of parameter estimates and standard errors for a 2-factor model of log-normal data with 20 variables with N=500.*

			Subset of Factor Loadings				Factor Covariances			Subset of Error Variances					
			V_1	V_2	V_{11}	V_{12}	$var(F_1)$	$var(F_2)$	$cov(F_1,F_2)$	E_1	E_2	E_3	E_{11}	E_{12}	E_{13}
AGLS	$\hat{\theta}$	M	1.036	1.013	1.000	1.000	0.283	0.295	0.091	0.272	0.263	0.265	0.266	0.259	0.266
		SD	0.201	0.186	0.180	0.192	0.080	0.080	0.029	0.088	0.114	0.092	0.093	0.101	0.088
	$SE_{\hat{\theta}}$	M	0.133	0.130	0.127	0.127	0.058	0.060	0.021	0.064	0.068	0.066	0.068	0.068	0.066
		SD	0.040	0.037	0.034	0.038	0.012	0.012	0.003	0.021	0.025	0.021	0.024	0.033	0.021
NGLS	$\hat{\theta}$	M	1.020	1.021	1.012	1.015	0.264	0.268	0.075	0.205	0.206	0.207	0.207	0.199	0.213
		SD	0.196	0.199	0.190	0.184	0.073	0.074	0.022	0.097	0.122	0.094	0.126	0.114	0.102
	$SE_{\hat{\theta}}$	M	0.267	0.271	0.269	0.267	0.114	0.116	0.044	0.088	0.095	0.092	0.097	0.094	0.092
		SD	0.105	0.099	0.094	0.090	0.024	0.026	0.008	0.035	0.048	0.039	0.048	0.050	0.041
MGLS	$\hat{\theta}$	M	1.027	1.031	1.031	1.041	0.925	0.960	0.267	0.685	0.723	0.712	0.736	0.718	0.718
		SD	0.255	0.250	0.244	0.267	0.372	0.532	0.092	0.207	0.246	0.239	0.253	0.274	0.252
	$SE_{\hat{\theta}}$	M	0.069	0.070	0.071	0.071	0.104	0.107	0.052	0.055	0.058	0.057	0.059	0.058	0.057
		SD	0.023	0.022	0.022	0.023	0.037	0.047	0.013	0.017	0.021	0.018	0.020	0.025	0.018
MRLS	$\hat{\theta}$	M	1.149	1.024	1.031	1.043	1.028	1.090	0.292	1.206	1.154	1.075	1.140	1.153	1.099
		SD	2.161	1.324	0.212	0.243	0.446	1.015	0.351	3.994	0.799	0.498	0.659	0.681	0.514
	$SE_{\hat{\theta}}$	M	0.081	0.078	0.073	0.074	0.118	0.122	0.056	0.083	0.080	0.075	0.079	0.080	0.077
		SD	0.207	0.104	0.020	0.024	0.046	0.074	0.030	0.256	0.052	0.033	0.043	0.045	0.034

Table 11.7. *Means and standard deviations of a subset of parameter estimates and standard errors for a 2-factor model of log-normal data with 20 variables with N=1000.*

			Subset of Factor Loadings				Factor Covariances			Subset of Error Variances					
			V_1	V_2	V_{11}	V_{12}	$var(F_1)$	$var(F_2)$	$cov(F_1,F_2)$	E_1	E_2	E_3	E_{11}	E_{12}	E_{13}
AGLS	$\hat{\theta}$	M	1.007	1.011	1.007	1.003	0.484	0.484	0.107	0.519	0.518	0.513	0.524	0.516	0.513
		SD	0.116	0.116	0.118	0.113	0.086	0.085	0.027	0.096	0.097	0.103	0.103	0.105	0.108
	$SE_{\hat{\theta}}$	M	0.074	0.075	0.074	0.073	0.058	0.058	0.017	0.063	0.065	0.063	0.064	0.064	0.065
		SD	0.014	0.014	0.013	0.013	0.010	0.010	0.002	0.018	0.020	0.020	0.020	0.021	0.021
NGLS	$\hat{\theta}$	M	1.011	1.009	1.018	1.019	0.355	0.354	0.105	0.304	0.311	0.299	0.312	0.314	0.306
		SD	0.130	0.140	0.149	0.137	0.070	0.073	0.020	0.102	0.103	0.106	0.105	0.105	0.112
	$SE_{\hat{\theta}}$	M	0.156	0.158	0.158	0.158	0.093	0.093	0.035	0.073	0.075	0.074	0.074	0.074	0.075
		SD	0.047	0.047	0.045	0.041	0.016	0.017	0.005	0.022	0.027	0.028	0.025	0.026	0.034
MGLS	$\hat{\theta}$	M	1.017	1.006	1.006	1.003	0.978	0.971	0.292	0.797	0.819	0.803	0.808	0.811	0.814
		SD	0.206	0.167	0.160	0.161	0.314	0.294	0.075	0.188	0.216	0.217	0.202	0.212	0.241
	$SE_{\hat{\theta}}$	M	0.048	0.048	0.048	0.048	0.078	0.078	0.038	0.043	0.044	0.043	0.043	0.043	0.044
		SD	0.014	0.009	0.010	0.009	0.020	0.020	0.007	0.009	0.011	0.013	0.011	0.012	0.013
MRLS	$\hat{\theta}$	M	0.883	0.891	1.004	1.005	1.031	1.085	0.284	1.052	1.076	1.386	1.042	1.061	1.054
		SD	2.017	1.812	0.152	0.163	0.320	1.400	0.354	0.612	0.533	7.828	0.338	0.494	0.417
	$SE_{\hat{\theta}}$	M	0.095	0.092	0.049	0.049	0.083	0.086	0.040	0.052	0.053	0.067	0.051	0.052	0.052
		SD	0.956	0.895	0.010	0.009	0.023	0.065	0.021	0.028	0.025	0.355	0.016	0.023	0.019

Table 11.8. *Means and standard deviations of a subset of parameter estimates and standard errors for a 2-factor model of log-normal data with 20 variables with N=10,000.*

			Subset of Factor Loadings				Factor Covariances			Subset of Error Variances					
			V_1	V_2	V_{11}	V_{12}	$var(F_1)$	$var(F_2)$	$cov(F_1, F_2)$	E_1	E_2	E_3	E_{11}	E_{12}	E_{13}
AGLS	$\hat{\theta}$	M	1.002	1.001	1.001	1.002	0.824	0.821	0.226	0.857	0.856	0.851	0.849	0.856	0.850
		SD	0.039	0.038	0.038	0.039	0.050	0.055	0.015	0.069	0.071	0.062	0.062	0.069	0.065
	$SE_{\hat{\theta}}$	M	0.031	0.031	0.031	0.031	0.042	0.042	0.012	0.053	0.053	0.052	0.052	0.052	0.052
		SD	0.003	0.003	0.003	0.003	0.005	0.005	0.001	0.017	0.017	0.014	0.016	0.016	0.016
NGLS	$\hat{\theta}$	M	1.002	1.003	1.000	0.999	0.766	0.767	0.231	0.749	0.754	0.746	0.749	0.747	0.745
		SD	0.043	0.046	0.046	0.044	0.064	0.067	0.018	0.088	0.090	0.078	0.096	0.093	0.091
	$SE_{\hat{\theta}}$	M	0.027	0.027	0.027	0.027	0.037	0.037	0.013	0.030	0.030	0.030	0.031	0.030	0.031
		SD	0.003	0.003	0.003	0.003	0.004	0.004	0.001	0.004	0.004	0.005	0.005	0.005	0.010
MGLS	$\hat{\theta}$	M	1.003	1.002	1.002	0.999	0.995	0.999	0.300	0.971	0.975	0.970	0.975	0.970	0.975
		SD	0.059	0.059	0.064	0.064	0.104	0.104	0.026	0.109	0.110	0.115	0.128	0.128	0.205
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.025	0.025	0.012	0.015	0.015	0.015	0.015	0.015	0.015
		SD	0.001	0.001	0.001	0.001	0.003	0.002	0.001	0.002	0.002	0.002	0.002	0.002	0.003
MRLS	$\hat{\theta}$	M	1.003	1.002	1.002	1.000	1.000	1.003	0.299	1.008	1.011	1.011	1.015	1.005	1.015
		SD	0.058	0.058	0.063	0.062	0.103	0.101	0.026	0.123	0.121	0.159	0.159	0.157	0.262
	$SE_{\hat{\theta}}$	M	0.015	0.015	0.015	0.015	0.025	0.026	0.012	0.016	0.016	0.016	0.016	0.016	0.016
		SD	0.001	0.001	0.001	0.001	0.003	0.002	0.001	0.002	0.002	0.002	0.002	0.002	0.004

Table 11.9. *Means and standard deviations of estimated* $\chi^2(df = 8)$ *test statistics of the 2-factor model of log-normal data with 6 variables.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
100	9.35	3.91	7.86	2.96	12.50	7.53	12.58	8.41
300	9.81	4.86	12.66	5.91	16.50	11.03	16.32	10.87
500	9.27	4.33	14.53	8.01	17.89	14.57	17.77	14.74
1000	8.90	3.87	17.21	10.70	19.43	15.66	19.50	16.06
10,000	7.93	3.78	24.35	16.78	24.68	17.35	29.58	108.80
100,000	8.13	4.04	27.37	16.93	27.40	16.99	27.41	17.02

Table 11.10. *Percent of replications in which the 2-factor model of log-normal data with 6 variables was rejected by a* $\chi^2(df = 8)$ *test with* $\alpha = .05$.

N	AGLS	NGLS	MGLS	MRLS
100	7.14	0.61	27.86	26.39
300	11.80	28.00	41.20	40.69
500	9.00	38.40	45.40	44.04
1000	5.80	47.00	48.80	48.80
10,000	4.20	67.00	67.80	67.60
100,000	4.60	78.00	78.00	78.00

Table 11.11. *Means and standard deviations of estimated* $\chi^2(df = 169)$ *test statistics of the 2-factor model of log-normal data with 20 variables.*

	AGLS		NGLS		MGLS		MRLS	
N	M	SD	M	SD	M	SD	M	SD
300	94.44	8.76	89.86	7.77	422.95	62.28	460.50	90.63
500	193.08	16.57	130.77	9.32	517.19	90.41	573.72	132.13
1000	276.66	30.82	218.16	12.84	649.53	124.46	729.17	196.85
10,000	199.30	20.32	794.60	133.57	1072.19	294.33	1142.90	401.69
100,000	175.92	19.74	1283.00	358.71	1340.42	422.49	1365.39	484.39

Table 11.12. *Percent of replications in which the 2-factor model of log-normal data with 20 variables was rejected by a* $\chi^2(df = 169)$ *test with* $\alpha = .05$.

N	AGLS	NGLS	MGLS	MRLS
300	0.0	0.0	100.0	100.0
500	30.4	0.0	100.0	100.0
1000	99.4	92.6	100.0	100.0
10,000	47.8	100.0	100.0	100.0
100,000	11.0	100.0	100.0	100.0

Figures

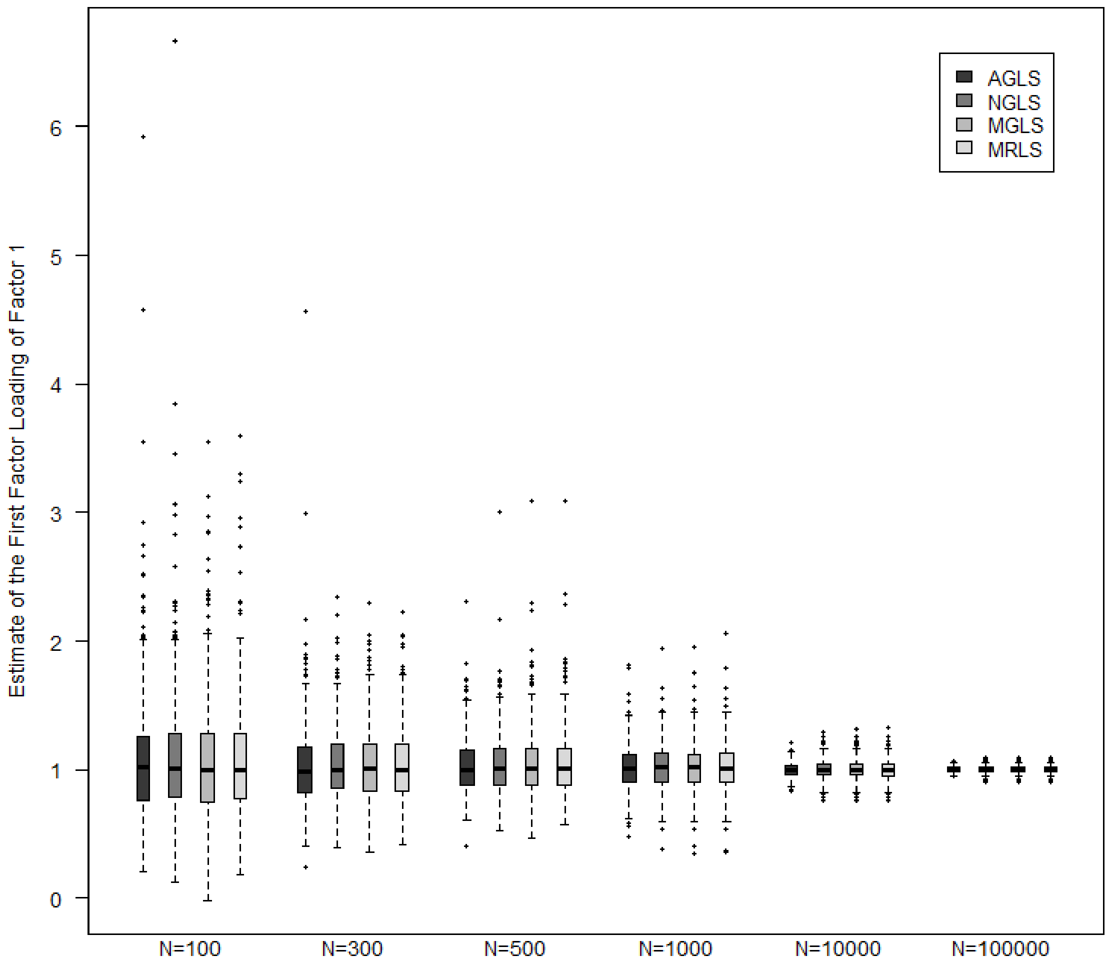

Figure 11.1. Boxplots of the values of the first factor loading of F_1 (in the 2-factor model) estimated by the different methods across sample sizes (in the log-normal data condition with 6 variables).

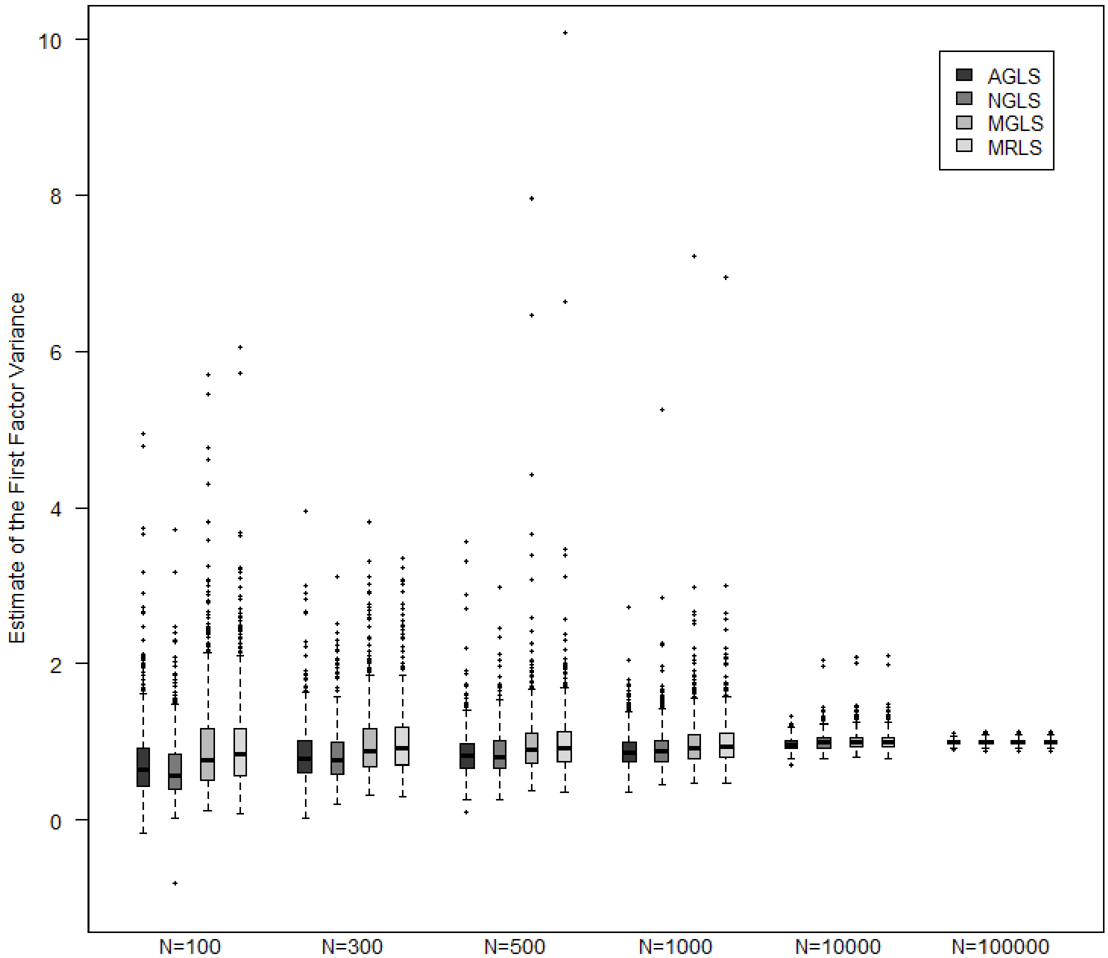

Figure 11.2. Boxplots of the values of the factor variance of F_1 (in the 2-factor model) estimated by the different methods across sample sizes (in the log-normal data condition with 6 variables). Two extreme values are not displayed above. Specifically, MGLS produced one extreme negative outlier (near -17) and MRLS produced a positive extreme outlier (near 29).

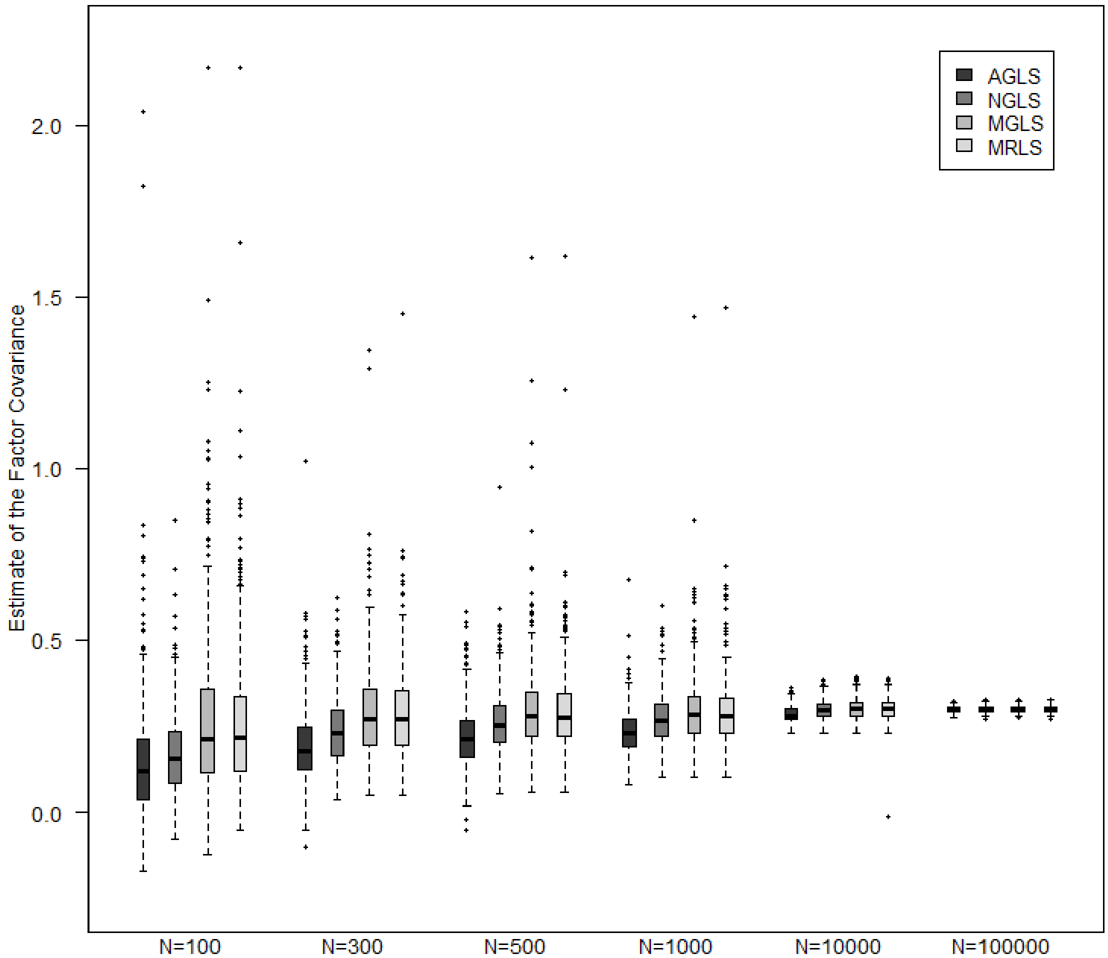

Figure 11.3. Boxplots of the values of the factor covariance between F_1 and F_2 (in the 2-factor model) estimated by the different methods across sample sizes (in the log-normal data condition with 6 variables).

With a sample size of 100, the MRLS procedure yielded one extreme value (around -8) which is not included in the figure above.

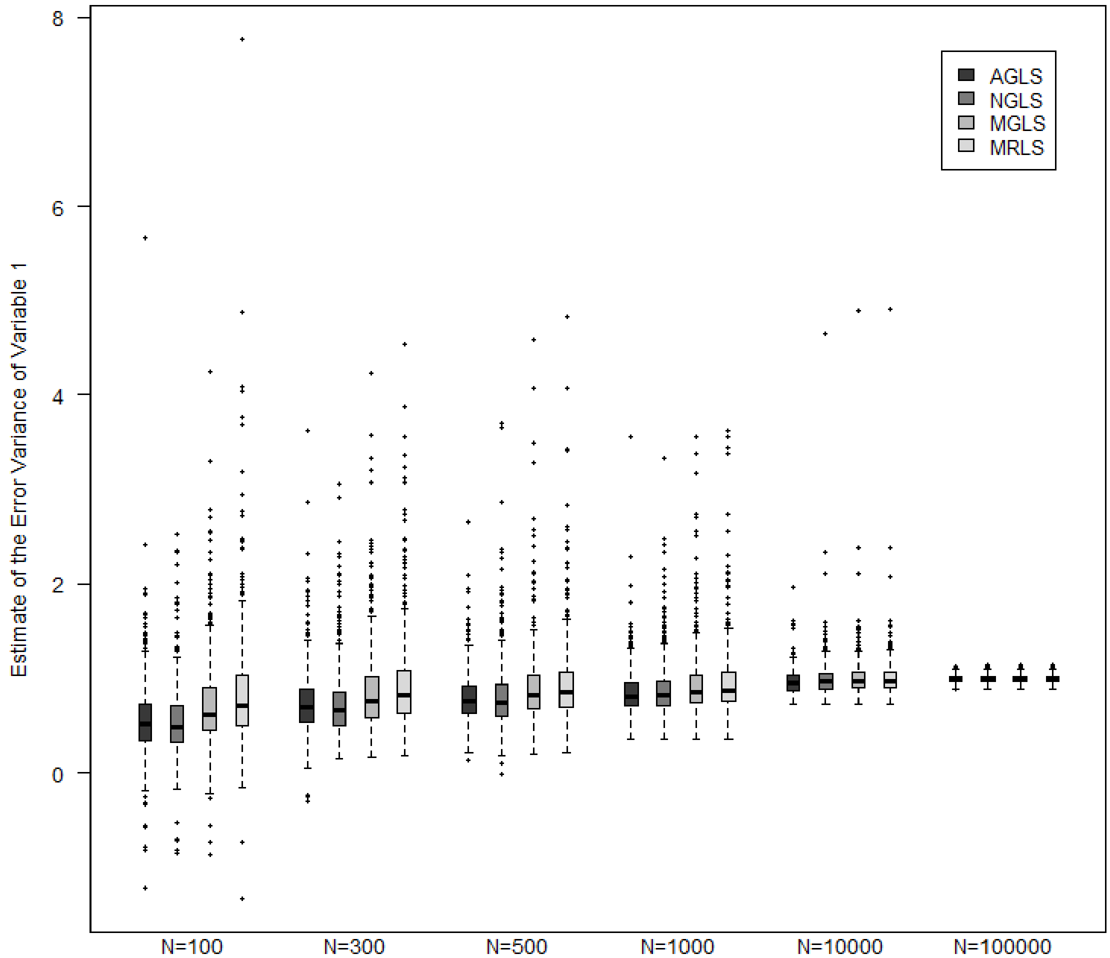

Figure 11.4. Boxplots of the values of the first error variance (of V_1 in the 2-factor model) estimated by the different methods across sample sizes (in the log-normal data condition with 6 variables).

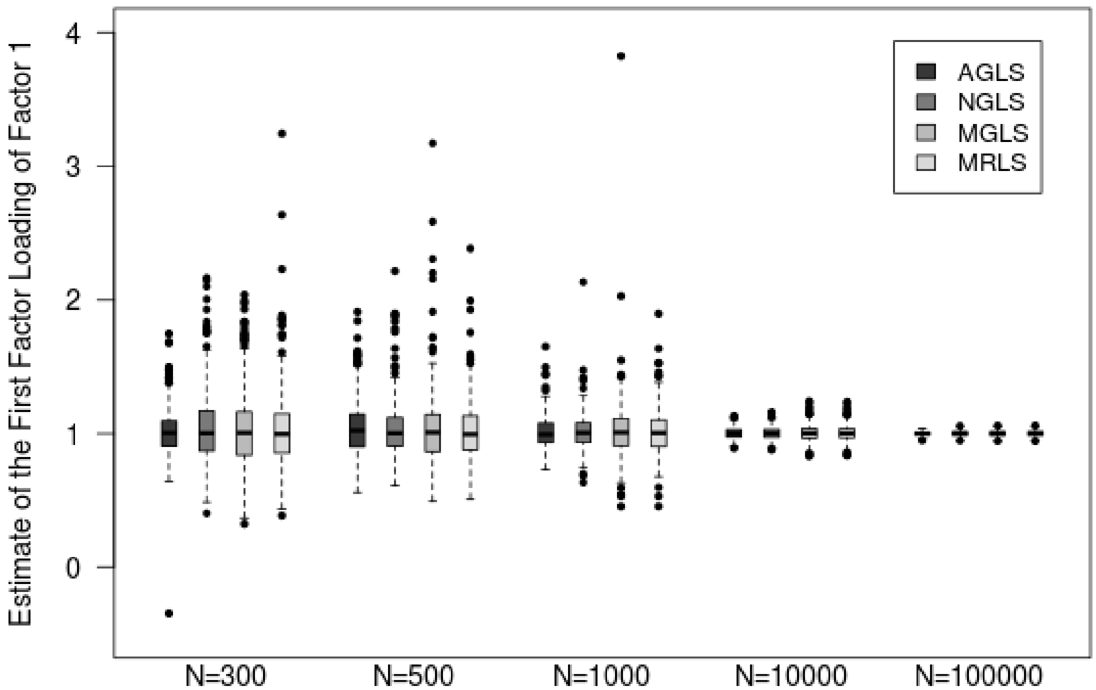

Figure 11.5. Boxplots of the values of the first factor loading of $\boldsymbol{F_1}$ (in the 2-factor model) estimated by the different methods across sample sizes (in the log-normal data condition with 20 variables).

There were several extreme outliers that are excluded from the plot above. Specifically, in the N=300 condition, the MGLS and MRLS procedures each produced outliers (around -15 and -13, respectively). In addition, the MRLS procedure had two extreme positive values (near 30 and 40) in the N=500 condition and two extreme negative values (around -25 and -33) in the N=1000 condition.

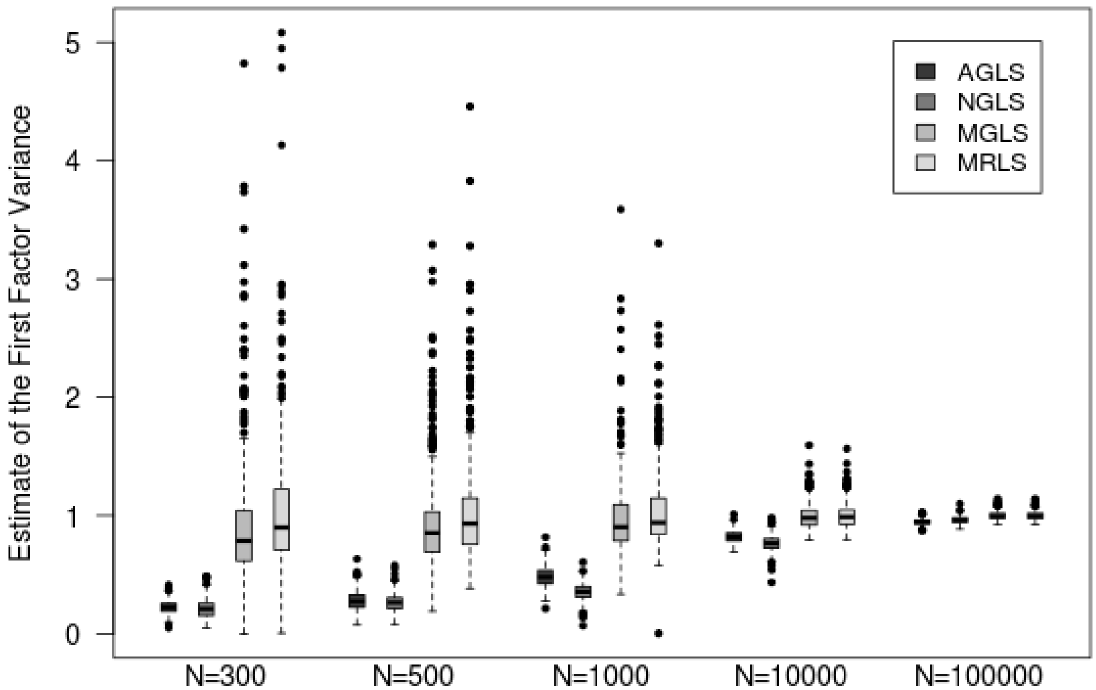

Figure 11.6. Boxplots of the values of the factor variance of F_1 (in the 2-factor model) estimated by the different methods across sample sizes (in the log-normal data condition with 20 variables).

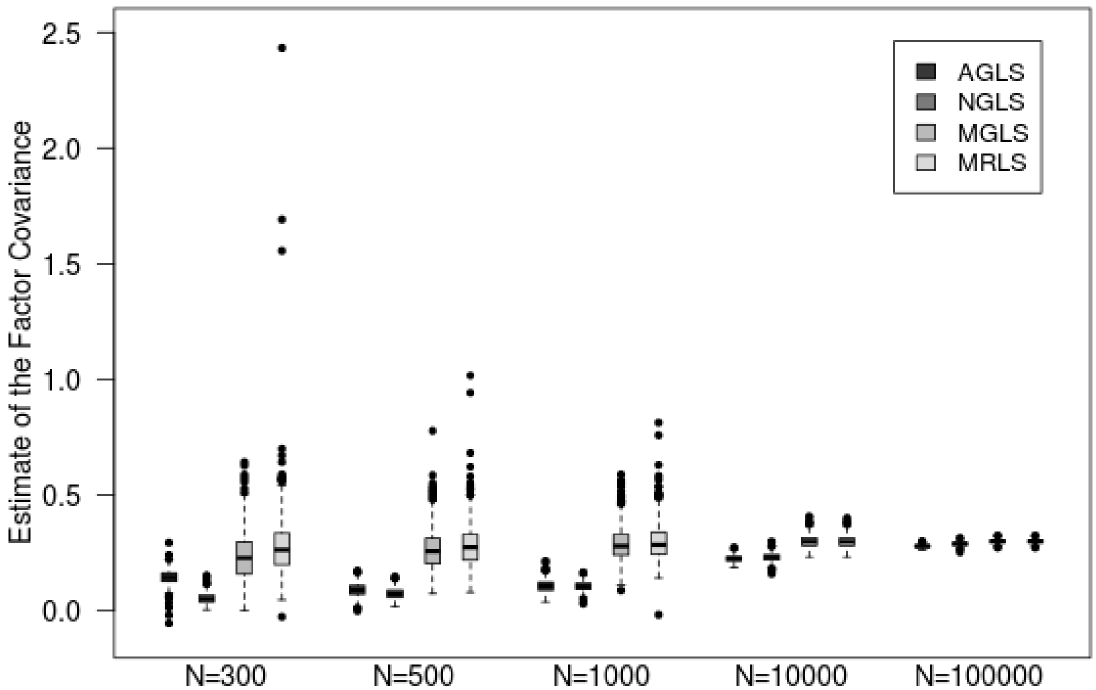

Figure 11.7. Boxplots of the values of the factor covariance between F_1 and F_2 (in the 2-factor model) estimated by the different methods across sample sizes (in the log-normal data condition with 20 variables).

The MRLS procedure produced some extreme outliers that are not depicted above. Specifically, there were two extreme values (about -3 and 7) in the N=500 condition and one extreme value (around -7) in the N=1000 condition.

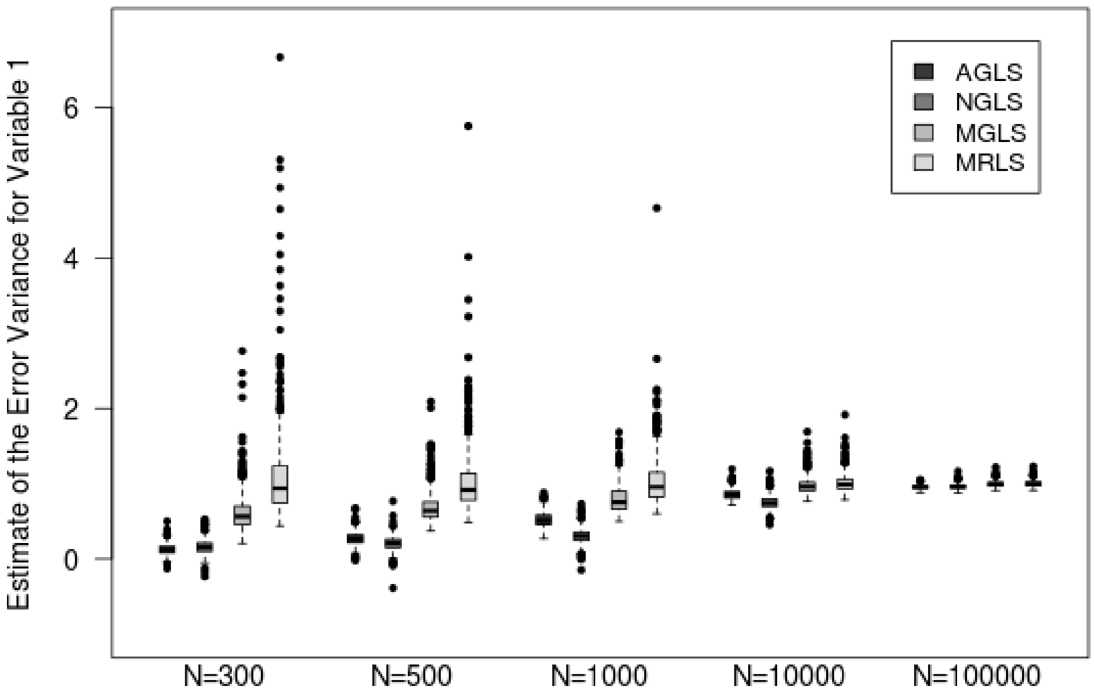

Figure 11.8. Boxplots of the values of the first error variance (of $\boldsymbol{V_1}$ in the 2-factor model) estimated by the different methods across sample sizes (in the log-normal data condition with 20 variables).

The plot above does not depict two extreme values produced by the MRLS procedure. In particular, one point (around 70) in the *N*=500 condition and one point (around 15) in the *N*=1000 condition are not shown.

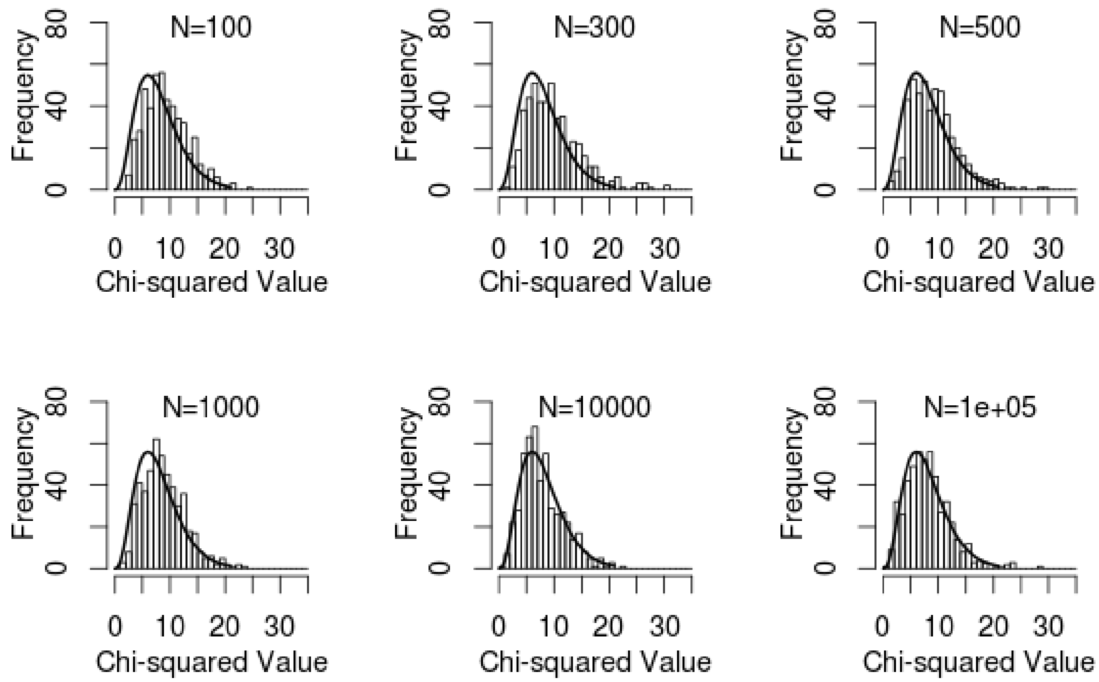

Figure 11.9. Histogram of the values of the χ^2-test statistic produced through AGLS estimation of the 2-factor model across sample sizes (in the log-normal data condition with 6 variables).

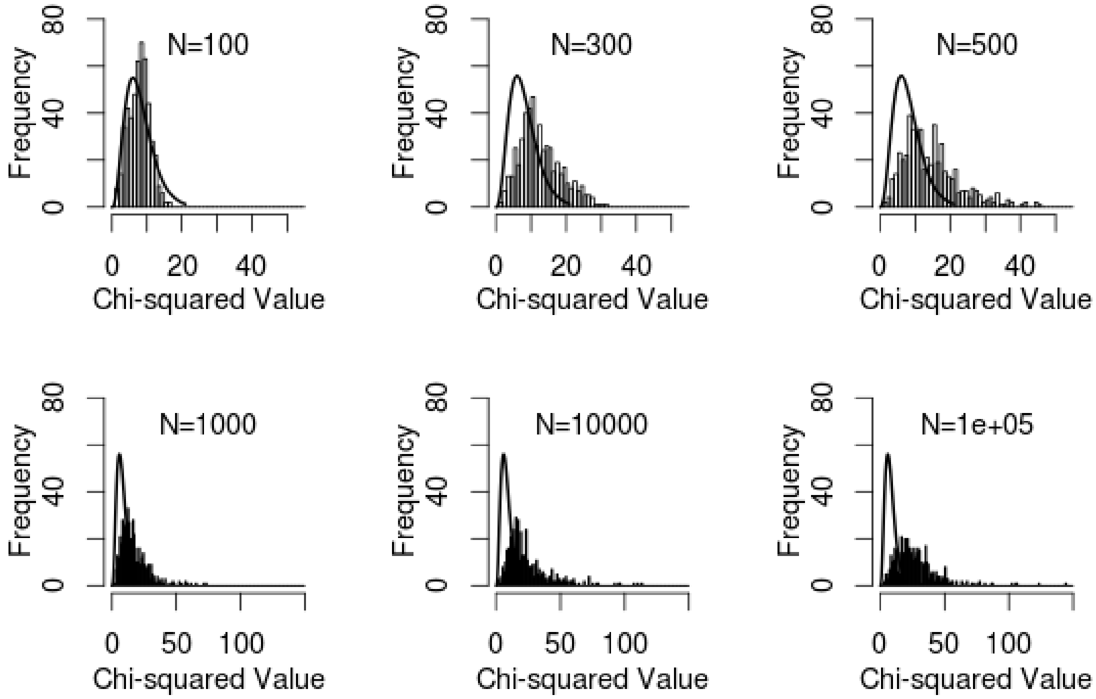

Figure 11.10. Histogram of the values of the χ^2-test statistic produced through NGLS estimation of the 2-factor model across sample sizes (in the log-normal data condition with 6 variables).

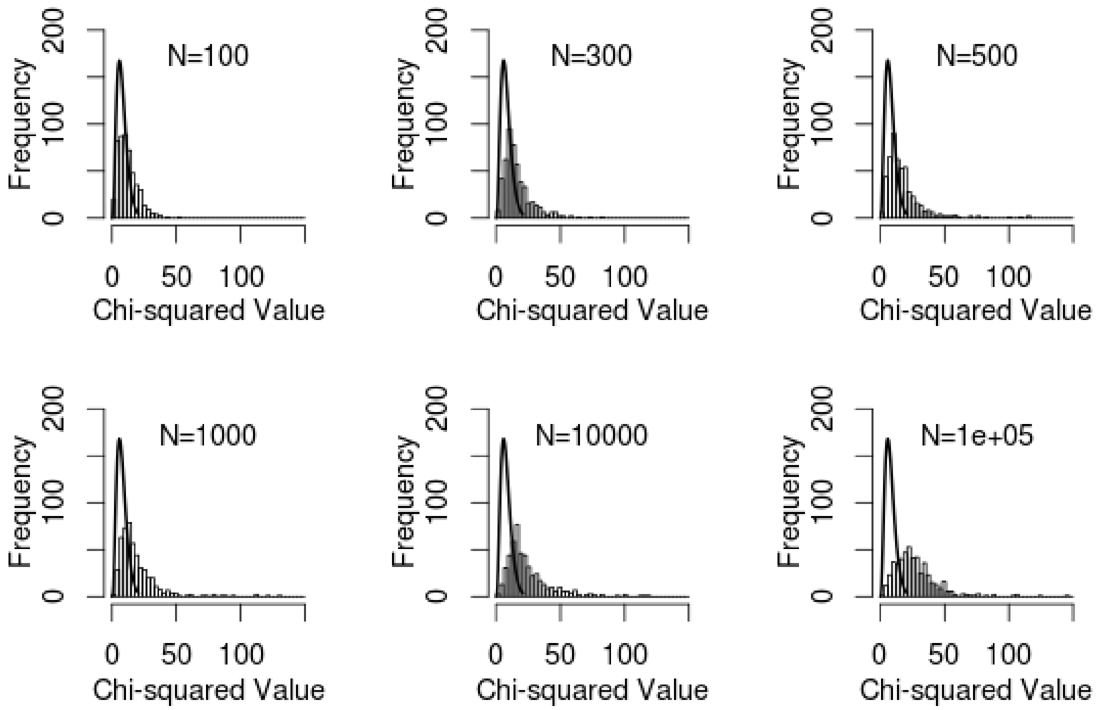

Figure 11.11. Histogram of the values of the χ^2-test statistic produced through MGLS estimation of the 2-factor model across sample sizes (in the log-normal data condition with 6 variables).

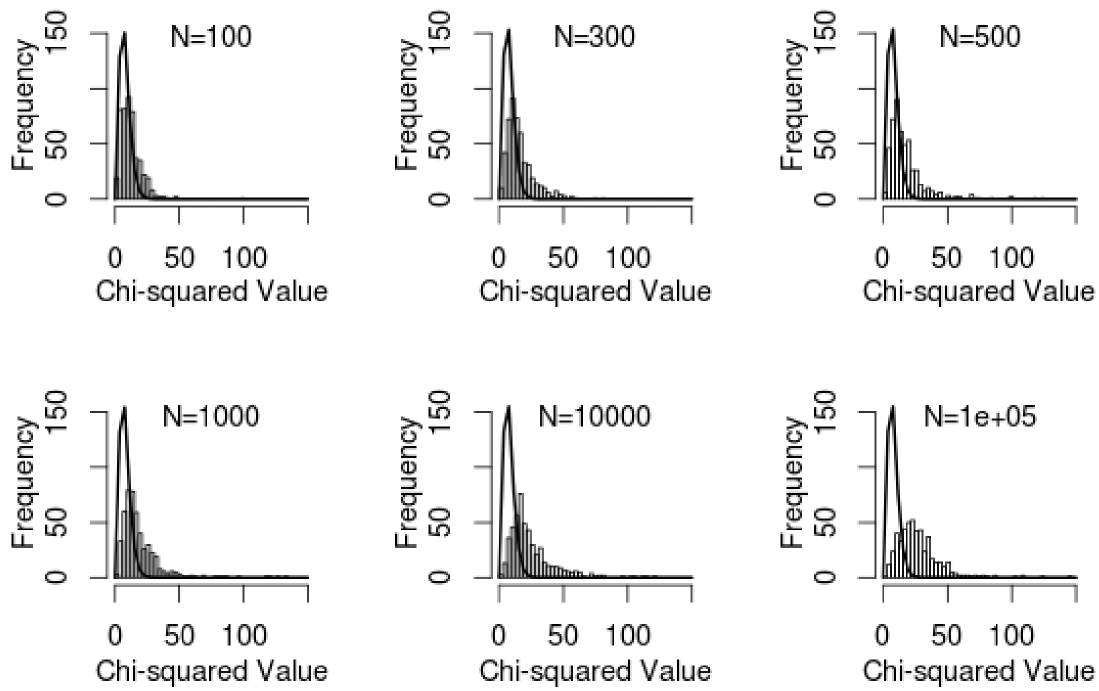

Figure 11.12. Histogram of the values of the χ^2-test statistic produced through MRLS estimation of the 2-factor model across sample sizes (in the log-normal data condition with 6 variables).

One outlier (with a value of 2426) is excluded in the N=10,000 condition above.

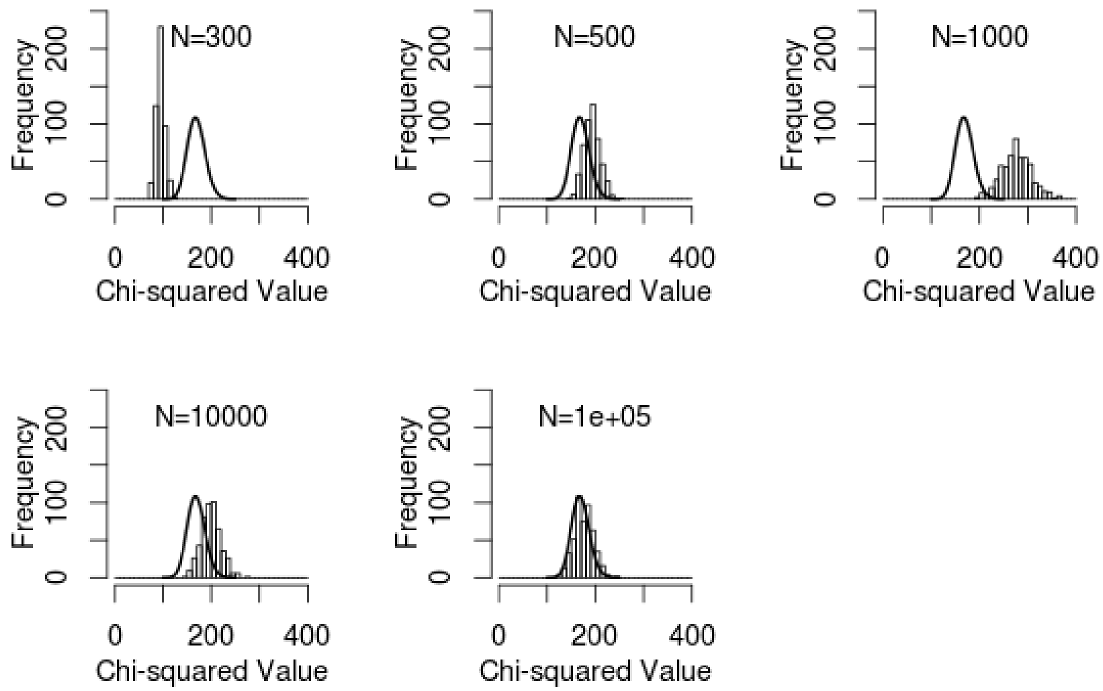

Figure 11.13. Histogram of the values of the χ^2-test statistic produced through AGLS estimation of the 2-factor model across sample sizes (in the log-normal data condition with 20 variables).

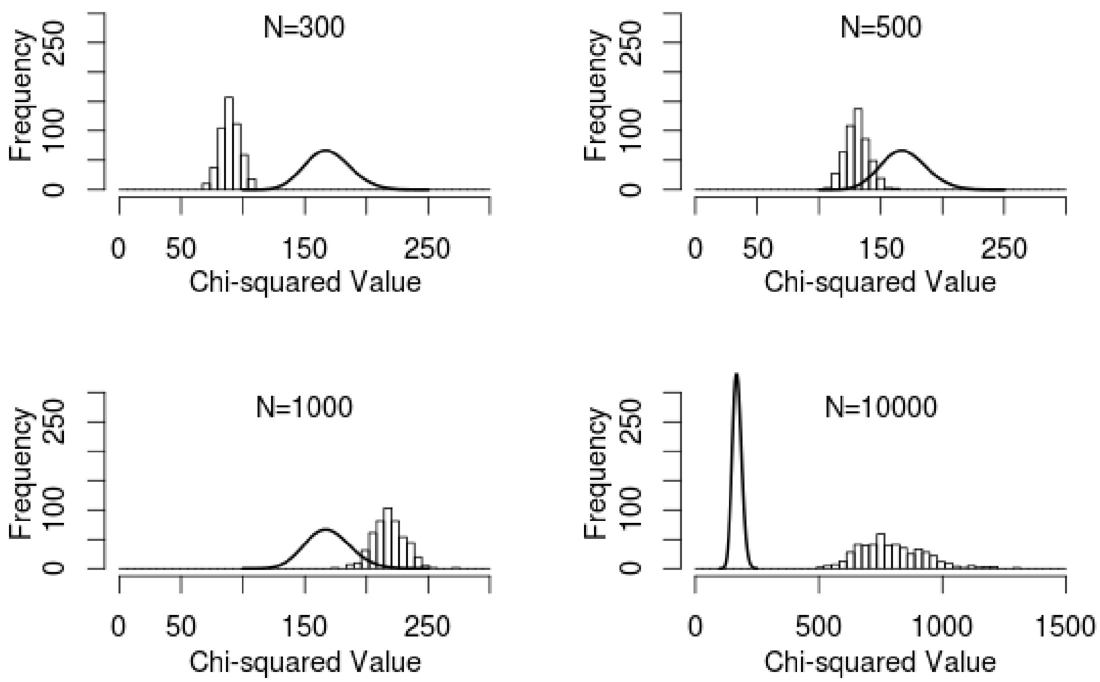

Figure 11.14. Histogram of the values of the χ^2-test statistic produced through NGLS estimation of the 2-factor model across sample sizes (in the log-normal data condition with 20 variables).

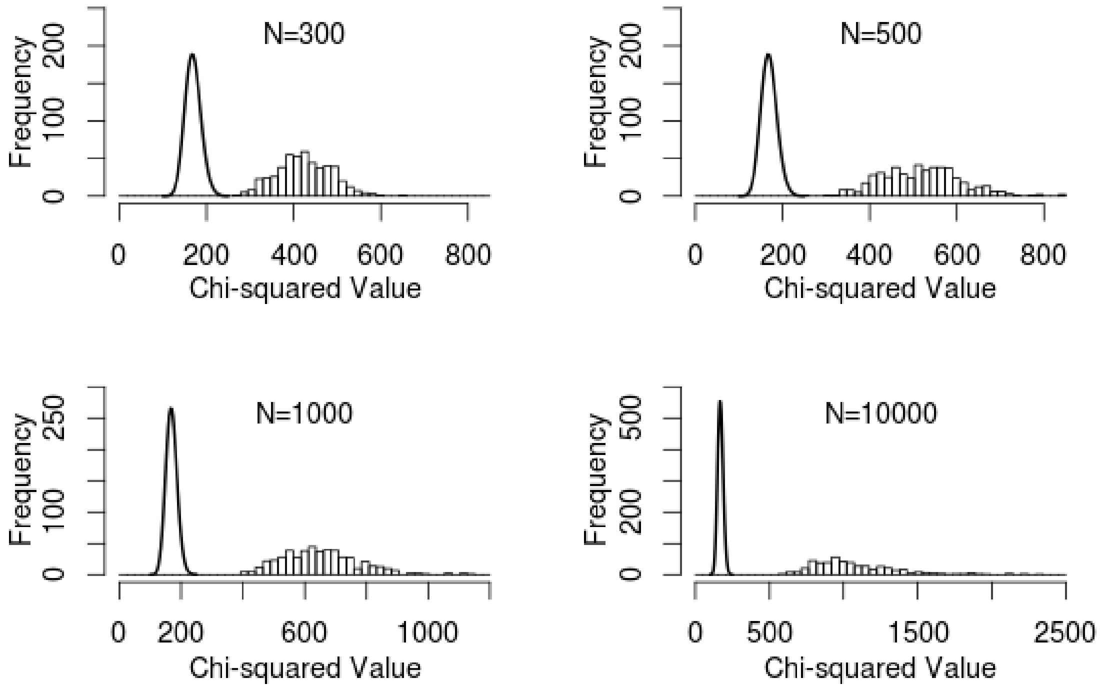

Figure 11.15. Histogram of the values of the χ^2-test statistic produced through MGLS estimation of the 2-factor model across sample sizes (in the log-normal data condition with 20 variables). The histogram for the $\boldsymbol{N}$=10,000 condition excludes one extreme value (at 2776).

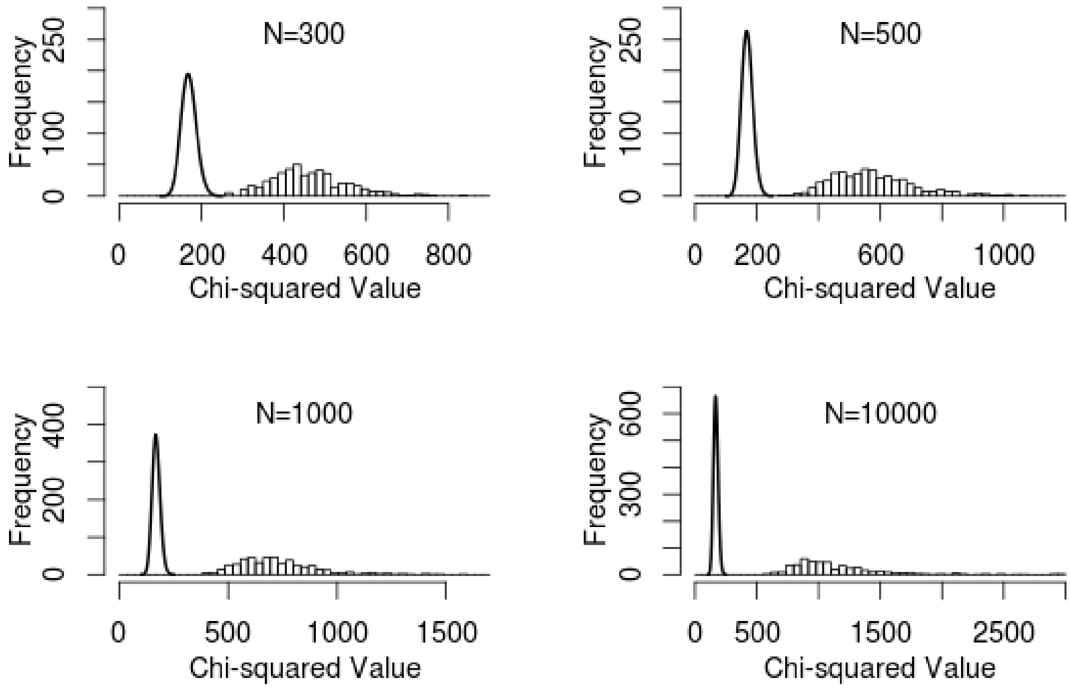

Figure 11.16. Histogram of the values of the χ^2-test statistic produced through MRLS estimation of the 2-factor model across sample sizes (in the log-normal data condition with 20 variables). Once again, some extreme values are not depicted. Namely, there were two extreme values in the N=1000 condition (around 1759 and 1814) and one in the N=10,000 condition (around 4377).

Chapter 12. Numerical Stability of $\widehat{\Psi}$ Relative to S and Potential Benefits of Winsorization

This chapter and the following chapter address the computational stability of the AGLS and NGLS estimation methods for CV models in terms of the condition numbers of some of the matrices used in the process. In the Chapter 7 discussion, we noted that we had initially intended to consider the samples size of 100 in both the 5-variable and 20-variable conditions, but that the $\hat{\Sigma}_{\boldsymbol{\psi}}$ matrices were numerically non-invertible. We speculated that because the CV matrix contains the means in the denominator, it may be more sensitive to fluctuations in sample observations than the covariance matrix. Consequently, it is possible that poor estimates of the mean, such as those that might occur in smaller samples, render the CV matrix ill-conditioned. If this occurred $\hat{\Sigma}_{\boldsymbol{\psi}}$ would also be ill-conditioned because the $\hat{\Sigma}_{\boldsymbol{\psi}}$ relies heavily on $\widehat{\Psi}$. To examine this question further, we considered the condition numbers of the both $\widehat{\Psi}$ and $\hat{\Sigma}_{\boldsymbol{\psi}}$ relative to their covariance analogs (S and $\hat{\Sigma}_{\boldsymbol{S}}$). The condition number is the ratio of the largest eigenvalue to the smallest eigenvalue of a matrix and larger values of this ratio are associated with more computational difficulties including computational singularities preventing inversion. In this chapter the condition numbers of sample CV matrices are contrasted with those of the sample covariance matrices and the next chapter will examine the condition numbers of several estimators of the V_{SS} matrix used in estimation.

Moreover, it is possible that by producing more stable estimates of the means and covariances, we could reduce this numerical instability in $\widehat{\Psi}$ and $\hat{\Sigma}_{\boldsymbol{\psi}}$. Therefore, we also examined the stability of these matrices when they were calculated from Winsorized data. It was hoped that by trimming the extreme values, the $\widehat{\Psi}$ estimates would be more stable and that the resulting $\hat{\Sigma}_{\boldsymbol{\psi}}$ would have lower condition numbers. If this were successful, it might allow for the estimation of larger models in small sample sizes.

Method

Conditions. First, we considered several one-factor models of data drawn from a normal distribution. These models contained either 5 or 20 variables and the population means were either fixed at 1 or allowed to vary from 1 to 3 as described in Chapter 5. Second, we considered two one-factor models of log-normal data: one with 5 variables and one with 20 variables. The sample size values considered were 100, 300, 500, and 1000.

Analyses. For each model and data type combination, 500 samples were drawn from the specified population and for each sample the covariance matrix and CV matrix were calculated. In addition, for each sample a Winsorized version of that sample was computed. The Winsorization was done within each variable, by trimming the lowest and highest 5% of scores (on that variable) and replacing these values with the 5th and 95th quantiles, respectively. Then, both the covariance and CV matrix were recomputed using the Winsorized version of the data.

Finally, we calculated the condition numbers of each matrix described above. That is, the eigenvalues were calculated and the condition number was computed by taking the ratio of the largest to the smallest eigenvalues. The means, standard deviations and general distributions of the condition numbers were analyzed and the results of each method were contrasted.

Results

One-factor models of normal data with equal population means. The population CV matrices for the 5 and 20 variable models, which have the form shown in (5.1), have condition numbers of 6 and 21, respectively. Because the population means are all equal to 1, the population covariance matrix has identical condition numbers. The empirical condition numbers are summarized in Tables 12.1 and 12.2 and Figures 12.1 and 12.2. As shown in Tables 12.1 and 12.2, the condition numbers of the CV matrix tended to be higher than those of the covariance matrix. In addition, the Winsorization procedure produced modest improvements in the condition number estimates. It is also apparent that the as the sample sizes increased, the condition numbers approached their true population values.

One-factor models of normal data with unequal population means. The same models and numbers of variables were considered in this condition, so the population CV matrix still has the form shown in (5.1). However, the population means of the variables were allowed to vary from 1 to 3, and therefore, the population covariance matrix differed from the CV matrix. This difference of course produced differing population condition numbers. In the 5-variable case, the CV matrix still had a condition number of 6, whereas the covariance matrix had a condition number of about 25.38. Similarly, the population condition number of the CV matrix in the 20-variable case was still 21, but that of the covariance matrix was approximately 90.58.

The empirical condition numbers are summarized in Tables 12.3 and 12.4 and Figures 12.3 and 12.4. The changes in the population condition numbers were reflected in the empirical values, which were substantially higher for covariance matrices than for CV matrices. Aside from this difference, the findings were comparable to results of the previous study of variables with equal means. That is, condition numbers tended to decrease towards their population values as the sample size increased, and Winsorization produced modest improvements.

One-factor models of log-normal data. The population CV matrices for the 5 and 20 variable models of log-normal data were identical to those for the one-factor model of normal data (shown in 5.1) and therefore have the same condition numbers: 6 and 21, respectively. The empirical condition numbers are summarized in Tables 12.5 and 12.6 and Figures 12.5 and 12.6. From these summaries, it is apparent that for log-normal data, the empirical condition numbers tended to be much larger than the theoretical values. The difference was most pronounced in small samples, but the difference was still substantial at a sample size of 1000. However, in the case of log-normal data, the benefit of Winsorization for the estimated condition number was more pronounced. As shown, in Figures 12.5 and 12.6 the distributions of the condition numbers obtained from the Winsorized data tended to have fewer extreme outliers and produced generally better estimates of the population condition numbers.

Discussion

When the data were normally distributed and the means did not vary, it was apparent that the covariance matrix was numerically more stable than the CV matrix and that these estimates improved as the sample size increase. However, the results of the studies of the one-factor models of normal data unequal means show that the condition numbers of the CV matrix and covariance matrix can be quite different. Furthermore, this difference is reflected in the sample values of the condition numbers. This suggests that when analyzing data with means that vary systematically with the variance such that there is a simple one-factor structure for the CV matrix, the CV matrix may be computationally more stable than the covariance matrix. This may have implications for the numerical stability of structural models of data and may inform our modeling decisions. In addition, the studies of log-normal data showed the condition numbers for the CV matrix were only very slightly lower than those of the covariance matrix, suggesting that the computational stability of the CV matrix will be comparable to that of the covariance matrix when modeling log-normal data.

Finally, it is noteworthy that while Winsorization only produced modest improvements in the case of normal data, it was more effective when the data followed a log-normal distribution. Although it did not substantially affect the average condition numbers, Winsorization did dramatically reduce the variability of the estimates resulting in fewer extreme outliers and more reliable values. This suggests that Winsorization may potentially be useful in enhancing the numerical stability of estimation, particularly when dealing with log-normal data. The following chapter will investigate whether these benefits carry over to improve the various estimators of $\Sigma_{\boldsymbol{\psi}}$ and discuss the implications for model estimation.

Tables

Table 12.1. *Condition number means and standard deviations of S and $\widehat{\Psi}$ for a one-factor model of normal data with 5 observed variables (equal means).*

	$N = 100$		$N = 300$		$N = 500$		$N = 1000$	
Matrix	M	SD	M	SD	M	SD	M	SD
S	8.45	1.58	7.14	0.77	6.86	0.54	6.58	0.36
$\widehat{\Psi}$	9.30	1.90	7.51	0.87	7.14	0.61	6.75	0.39
S_{Win}	8.25	1.58	6.93	0.76	6.67	0.53	6.38	0.36
$\widehat{\Psi}_{\mathrm{Win}}$	9.11	1.92	7.30	0.85	6.93	0.60	6.55	0.39

Table 12.2. *Condition number means and standard deviations of S and $\widehat{\Psi}$ for a one-factor model of normal data with 20 observed variables with equal means.*

	$N = 100$		$N = 300$		$N = 500$		$N = 1000$	
Matrix	M	SD	M	SD	M	SD	M	SD
S	61.86	10.73	35.74	3.54	31.02	2.34	27.42	1.41
$\widehat{\Psi}$	67.20	11.89	37.43	3.76	32.07	2.53	28.02	1.47
S_{Win}	61.17	11.19	34.82	3.53	30.08	2.32	26.51	1.39
$\widehat{\Psi}_{\mathrm{Win}}$	66.74	12.21	36.49	3.73	31.15	2.51	27.12	1.42

Table 12.3. *Condition number means and standard deviations of S and $\widehat{\Psi}$ for a one-factor model of normal data with 5 observed variables with means varying from 1 to 3.*

	$N = 100$		$N = 300$		$N = 500$		$N = 1000$	
Matrix	M	SD	M	SD	M	SD	M	SD
S	27.74	5.77	26.24	2.87	25.87	2.40	25.54	1.55
$\widehat{\Psi}$	9.30	1.85	7.56	0.86	7.09	0.59	6.71	0.36
S_{Win}	27.09	6.04	25.51	2.92	25.13	2.44	24.81	1.55
$\widehat{\Psi}_{\mathrm{Win}}$	9.11	1.85	7.36	0.86	6.89	0.58	6.51	0.37

Table 12.4. *Condition number means and standard deviations of S and $\widehat{\Psi}$ for a one-factor model of normal data with 20 observed variables with means varying from 1 to 3.*

	$N = 100$		$N = 300$		$N = 500$		$N = 1000$	
Matrix	M	SD	M	SD	M	SD	M	SD
S	140.83	28.48	103.24	11.52	97.48	8.73	94.00	5.58
$\widehat{\Psi}$	67.00	13.19	37.58	3.94	32.23	2.52	28.19	1.49
S_{Win}	138.68	28.62	100.02	11.93	94.41	8.69	90.71	5.66
$\widehat{\Psi}_{\mathrm{Win}}$	66.33	13.44	36.61	3.95	31.30	2.49	27.33	1.49

Table 12.5. *Condition number means and standard deviations of S and $\widehat{\Psi}$ for a one-factor model of log-normal data with 5 observed variables (equal means).*

	$N = 100$		$N = 300$		$N = 500$		$N = 1000$	
Matrix	M	SD	M	SD	M	SD	M	SD
S	19.97	14.32	12.31	4.98	10.58	3.24	8.96	1.87
$\widehat{\Psi}$	16.83	10.72	11.34	4.37	9.97	2.83	8.63	1.65
S_{Win}	13.76	3.80	10.44	1.54	9.64	1.10	8.87	0.68
$\widehat{\Psi}_{\mathrm{Win}}$	12.36	3.13	9.88	1.35	9.25	0.95	8.62	0.62

Table 12.6. *Condition number means and standard deviations of S and $\widehat{\Psi}$ for a one-factor model of log-normal data with 20 observed variables (equal means).*

	$N = 100$		$N = 300$		$N = 500$		$N = 1000$	
Matrix	M	SD	M	SD	M	SD	M	SD
S	251.84	151.28	89.56	71.55	64.02	18.02	47.09	8.99
$\widehat{\Psi}$	215.85	116.82	81.75	48.76	60.38	16.33	45.18	8.26
S_{Win}	131.52	37.04	58.98	8.38	48.39	4.67	40.51	2.65
$\widehat{\Psi}_{\mathrm{Win}}$	119.48	33.31	55.93	7.62	46.49	4.33	39.36	2.51

Figures

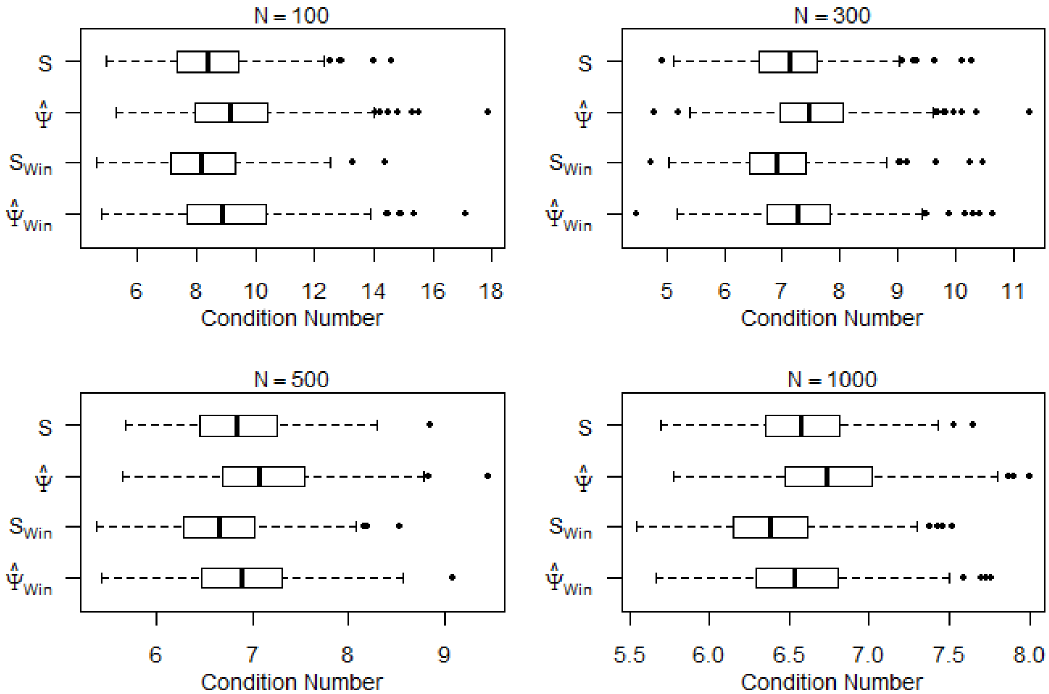

Figure 12.1. Condition numbers of S and $\hat{\Psi}$ for a one-factor model of normal data with 5 observed variables with equal means.

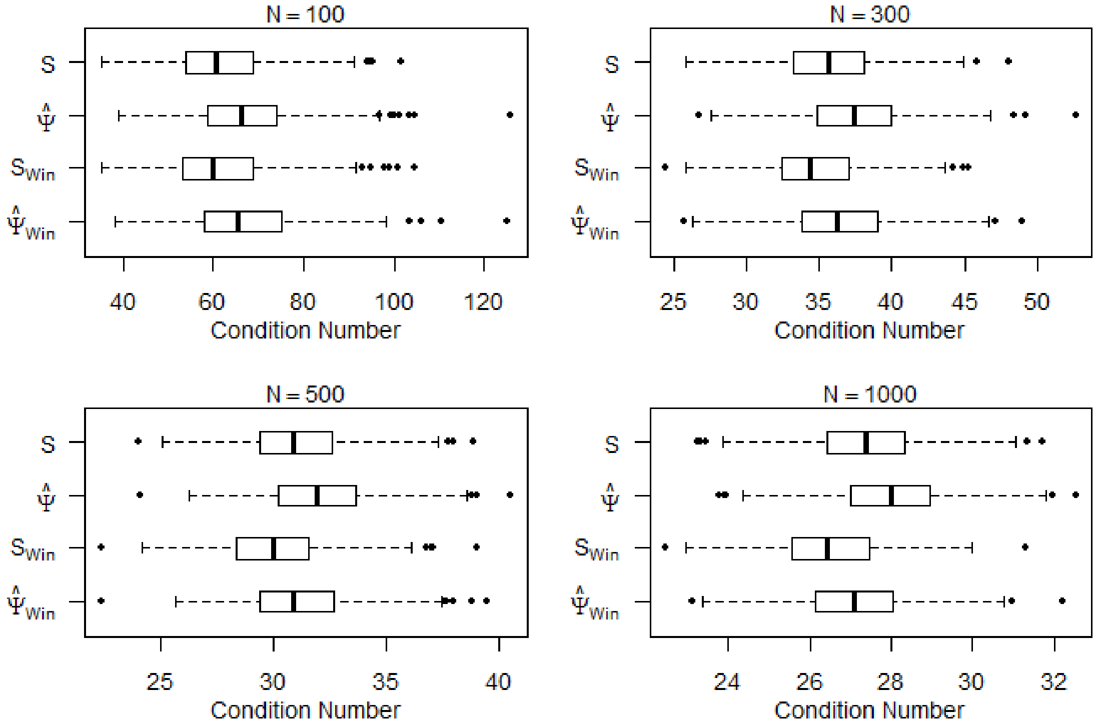

Figure 12.2. Condition numbers of S and $\widehat{\Psi}$ for a one-factor model of normal data with 20 observed variables with equal means.

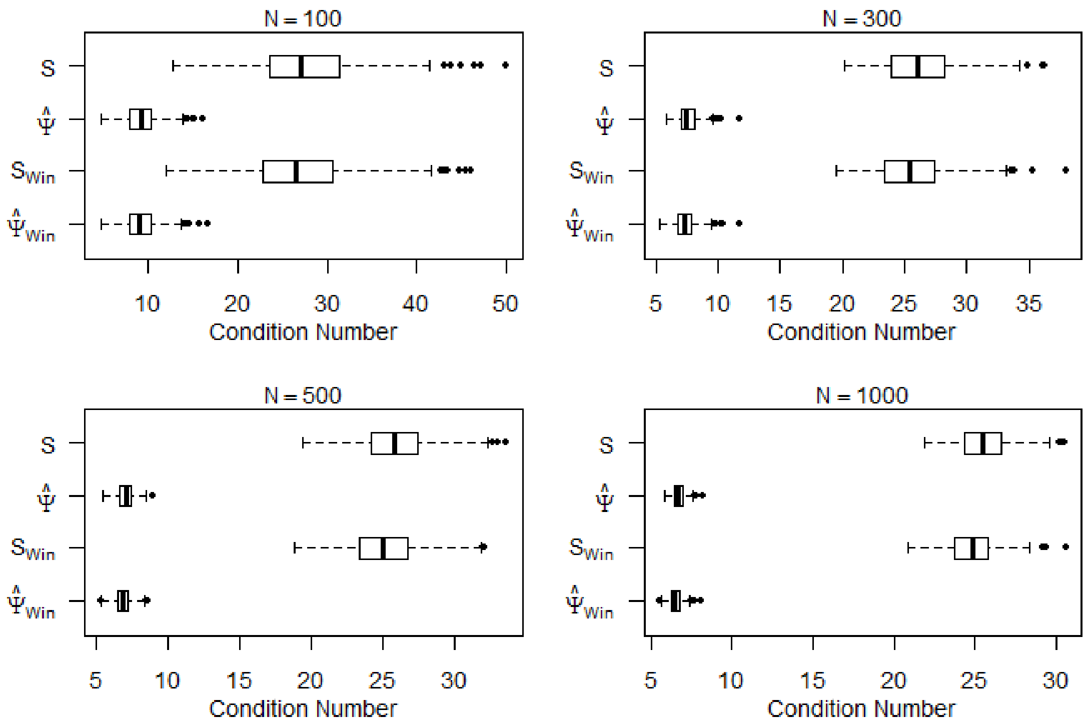

Figure 12.3. Condition numbers of S and $\hat{\Psi}$ for a one-factor model of normal data with 5 observed variables with means varying from 1 to 3.

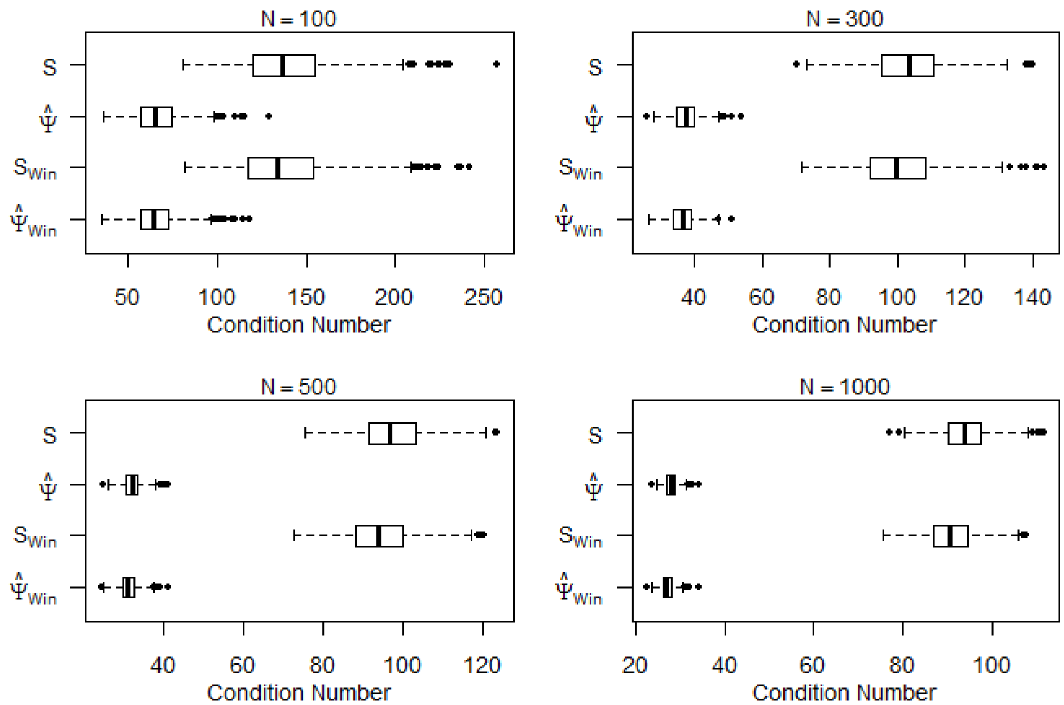

Figure 12.4. Condition numbers of S and $\hat{\Psi}$ for a one-factor model of normal data with 5 observed variables with means varying from 1 to 3.

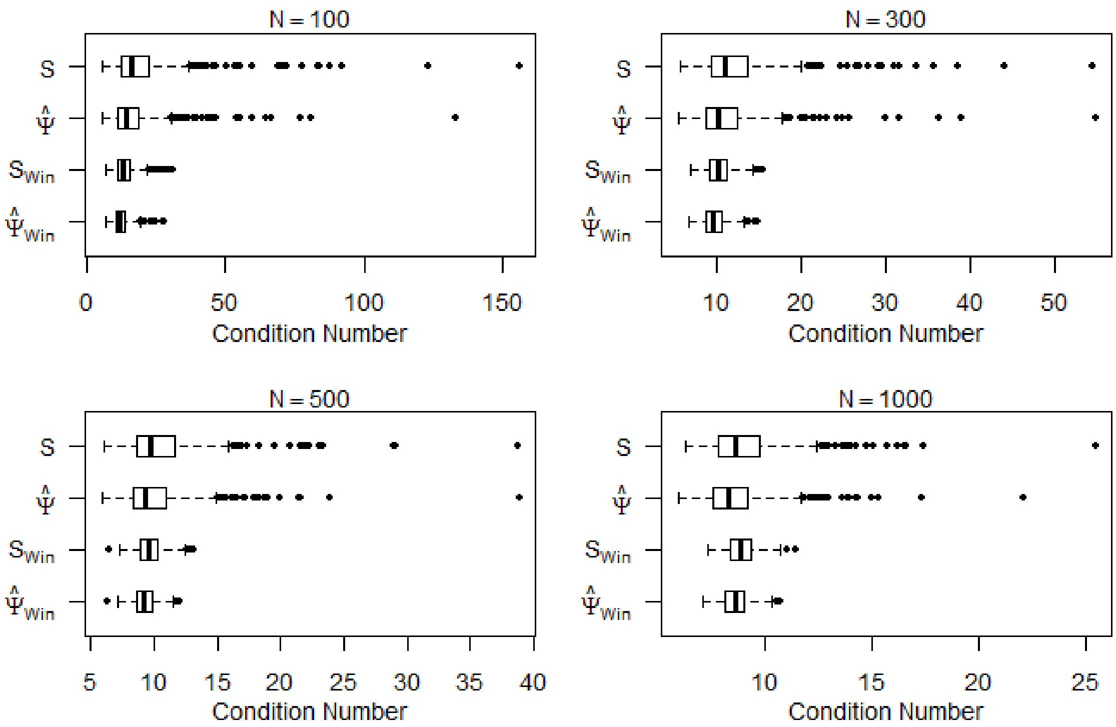

Figure 12.5. Condition numbers of S and $\widehat{\Psi}$ for a one-factor model of log-normal data with 5 observed variables (and equal means).

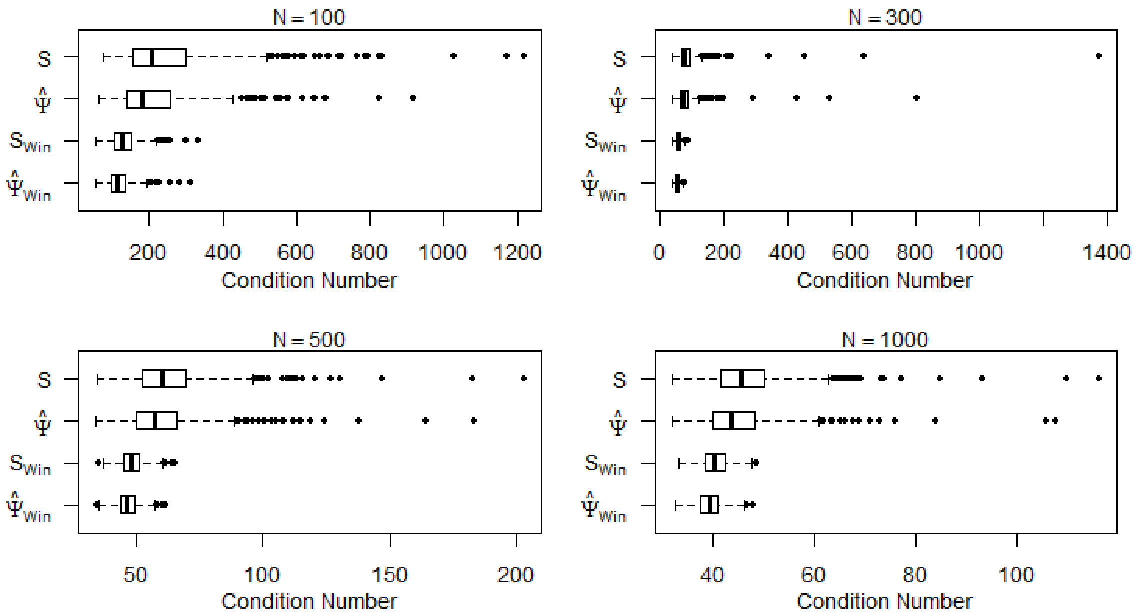

Figure 12.6. Condition numbers of S and $\widehat{\Psi}$ for a one-factor model of log-normal data with 20 observed variables (and equal means).

Chapter 13. Numerical Stability of $\hat{\Sigma}_{\psi}$ Relative to $\hat{\Sigma}_{S}$ and Potential Benefits of Winsorization

This chapter continues investigating factors contributing to the computational stability of the AGLS and NGLS estimation methods for CV models. Here we examined the condition numbers of several estimators of the V_{SS} matrix used in estimation. In particular, we considered the both the arbitrary-distribution and normal-theory versions of $\hat{\Sigma}_{\psi}$ and contrasted their condition numbers with those of $\hat{\Sigma}_{S}$. Because $\hat{\Sigma}_{\psi}$ depends on Ψ and therefore depends more heavily on the sample mean than the $\hat{\Sigma}_{S}$ and S, we expect that $\hat{\Sigma}_{\psi}$ will have higher condition numbers on average than $\hat{\Sigma}_{S}$. Again, we also expect that the condition numbers of both $\hat{\Sigma}_{\psi}$ and $\hat{\Sigma}_{S}$ to be improved by Winsorization.

In addition, we conducted an investigation of the potential differences in numerical stability between the method of inverting Σ_{Ψ} employed by Boik and Shirvani (2009) and the method employed here. These two approaches are described in (3.17) and (3.18), and they are mathematically equivalent in terms of the sum of squares they produce using different forms of the GLS function. As described previously, the first method relies on inverting the matrix $D_p^T \hat{\Sigma}_{\Psi} D_p$ and then transforming the inverse into a large $p^2 \times p^2$ matrix, whereas the second method inverts the matrix $H_p \hat{\Sigma}_{\Psi} H_p^T$ and then uses the reduced $p^* \times p^*$ matrix in estimation. Therefore, we will examine the condition numbers of these two matrices (in both the arbitrary distribution and normal theory contexts) to see if one may be more numerically stable.

Method

Conditions. This simulation examined one-factor models of the form shown in Figure 5.1 and the data were drawn from normal and log-normal distributions, respectively. The models contained either 5 or 20 variables and the population means of the variables were fixed at 1. The sample size values considered were 100, 300, 500, 1000, and 10,000.

Analyses. For each of the 500 replications, we calculated three variations of the $\mathrm{V_{SS}}$ matrix, whose inverse serves as the weight matrix during GLS estimation. The first variation we examined was the variance of the sampling distribution of the covariance matrix $\hat{\Sigma}_\mathrm{S}$. The second and third variations of $\mathrm{V_{SS}}$ examined were the two versions of the variance of the sampling distribution of the CV matrix $\hat{\Sigma}_{\boldsymbol{\psi}}$: one that does not assume any particular distribution and one that assumes normality. Each of these may be used in a GLS estimation of structural models of CV matrices and they are computed according to (3.15), (3.16), and (3.18).

As described in the previous chapter, a Winsorized version of that sample was computed by trimming the lowest and highest 5% of scores within each variable and replacing these with the 5th and 95th quantiles, respectively. Then, each of the V_{SS} matrices were recomputed using the Winsorized data.

Finally, we again calculated the condition number of each matrix by finding the eigenvalues and eigenvectors obtaining the ratio of the largest to the smallest eigenvalue. The means, standard deviations and general distributions of the condition numbers were examined at each sample size.

Results

One-factor models of normal data with equal population means. Tables 13.1 and 13.2 show the means and standard deviations of the condition numbers of the various V_{SS} matrices estimated from normally distributed data from a 5-variable one-factor covariance matrix with equal means. While Table 13.1 shows the values computed from the raw data, Table 13.2 shows the values computed from the Winsorized data. The condition numbers of $\hat{\Sigma}_S$ were lower than those of any of the CV method matrices. The highest average values of the condition numbers were observed in the arbitrary distribution method relying on $H_p\hat{\Sigma}_\Psi H_p^T$ whereas the condition numbers were lower when using normal theory and/or $D_p^T\hat{\Sigma}_\Psi D_p$. Furthermore, substantial reductions in the condition number values were obtained through Winsorization as can be seen be contrasting values in Table 13.1 to those in Table 13.2.

The means and standard deviations of the condition numbers of the various V_{SS} matrices resulting from a model containing 20-variables are shown in Table 13.3 (for the non-Winsorized data) and 13.4 (for the Winsorized data). The pattern of results found here was similar to that of the 5-variable case. However, in the 20-variable case the condition numbers tended to be very high in general and when the sample size was low (i.e. when $N = 100$) the numerical computation of the matrices was so unstable that none of the matrices, except those relying on normal theory, were positive semi-definite. For completeness, the average condition numbers are still reported in these cases but they are marked with an asterisk and should be interpreted with caution. Importantly, it should also be noted that although the average condition numbers did get smaller as the sample sizes increased, the values for the $\hat{\Sigma}_\psi$ were still very large with 10,000 observations (i.e. larger than 1500 in the for raw data and larger than 1250 for Winsorized data).

One-factor models of log-normal data. Tables 13.5 and 13.6 show the means and standard deviations of the condition numbers of the various V_{SS} matrices associated with a 5-variable one-factor covariance matrix calculated from log-normal data. The results from the non-Winsorized data are in Table 13.5, while those for the Winsorized data are reported in Table 13.6. Once more, it seems that the normal theory estimates were more stable than the arbitrary distribution theory estimates and the $D_p^T\hat{\Sigma}_\Psi D_p$ values were somewhat more stable than the $H_p\hat{\Sigma}_\Psi H_p^T$ values. Also, as was found in the results of Chapter 12, Winsorization was very effective at reducing the condition numbers of matrices derived from log-normal data. The trend in the results for the 20-variable condition with log-normal data was comparable, and the results are displayed in in Table 13.7 (for the non-Winsorized data) and Table 13.8 (for the Winsorized data).

Discussion

These results revealed a clear pattern of the stability of the $\hat{\Sigma}_\Psi$ matrices. Specifically, the normal theory estimators were more stable than the arbitrary distribution theory estimators, the D_p transformation was more stable than the H_p transformation, and Winsorization was modestly helpful with normal data and very helpful with log-normal data. However, these results do not all lead directly to estimation recommendations.

For instance, while it makes sense that the normal theory estimates would be more stable than the arbitrary distribution theory estimates given how they are computed in (3.15) and (3.16), this does not mean that the normal theory estimators are better or more appropriate. As we saw in Chapter 6, when the normal theory version of $\hat{\Sigma}_\Psi$ is calculated from log-normal data, the result tends to be a poor match for the population value. Also, we saw in Chapter 9, that model estimation using the normal theory version of $\hat{\Sigma}_\Psi$ with log-normal data leads to virtually meaningless model fit statistics. Therefore, also it is more stable and it would only be preferable in the case of normally distributed.

Next, it is interesting that the $D_p^T \hat{\Sigma}_\Psi D_p$ values were somewhat more stable than the $H_p \hat{\Sigma}_\Psi H_p^T$ values. This also makes sense, because the computation of H_p requires an addition inverse to be computed; however, we cannot necessarily conclude that using $D_p^T \hat{\Sigma}_\Psi D_p$ will lead to superior solutions. It is possible that there are other problems associated with the downstream transformations required in this method, or with estimating more elements of matrices that may have detrimental effects that are not addressed here. Additional research should compare the results of these two approaches to see if one method is superior. If it were the case that one of these methods was more reliable or accurate, this might have implications for the estimation of standard structural covariance models in addition to the coefficient of variation models considered here.

Finally, it was also apparent that Winsorizing the data enhanced the numerical stability of the estimators, especially when the data were log-normally distributed. This may lead to better and more reliable estimation and future work should investigate how Winsorization might affect the parameter estimates, standard errors and test statistics resulting from the estimation of a CV model. Other forms of Winsorization could also be considered. For instance, here we Winsorized the data within each variable, but this could also be done within cases. Alternatively, we could Winsorize the elements of the CV matrix or $\hat{\Sigma}_{\Psi}$ or employ an alternative method to produce more robust (asymptotic) covariance matrices.

Tables

Table 13.1. Condition number means and standard deviations of $\hat{\Sigma}_S$ and different estimators of $\hat{\Sigma}_{\psi}$ for a one-factor model of normal data with 5 observed variables with equal means (non-Winsorized).

	Covariance Method		Arbitrary Distribution				Normal Theory			
	$\hat{\Sigma}_S$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$	
N	M	SD	M	SD	M	SD	M	SD	M	SD
100	134.36	58.85	624.74	346.97	418.59	252.76	446.93	252.39	289.50	175.01
300	72.02	18.72	312.30	87.48	197.54	61.64	264.72	71.13	161.98	47.90
500	62.16	12.12	269.01	58.52	167.52	39.81	238.79	48.71	143.03	32.80
1000	54.93	7.28	235.35	34.30	142.17	22.52	215.87	29.31	127.09	19.51
10,000	46.06	2.13	195.88	9.37	112.71	5.95	191.69	8.28	108.74	5.34

Table 13.2. Condition number means and standard deviations of $\hat{\Sigma}_S$ and different estimators of $\hat{\Sigma}_{\psi}$ for a one-factor model of Winsorized normal data with 5 observed variables with equal means.

	Covariance Method		Arbitrary Distribution				Normal Theory			
	$\hat{\Sigma}_S$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$	
N	M	SD	M	SD	M	SD	M	SD	M	SD
100	80.88	30.81	450.14	261.29	299.53	176.84	373.81	217.13	245.43	149.03
300	43.37	9.59	222.57	64.84	140.89	43.45	218.13	59.25	136.21	40.17
500	37.68	5.98	191.54	42.20	119.51	28.05	195.83	39.35	120.16	27.56
1000	33.13	3.48	167.15	24.58	101.34	15.91	176.63	24.20	106.32	16.62
10,000	27.61	0.99	139.22	6.44	80.63	4.14	155.99	6.65	90.44	4.45

Table 13.3. Condition number means and standard deviations of $\hat{\Sigma}_S$ and different estimators of $\hat{\Sigma}_{\boldsymbol{\psi}}$ for a one-factor model of normal data with 20 observed variables with equal means (non-Winsorized). Note that the asterisks indicate matrices that were numerically non-positive semi-definite.

	Covariance Method		Arbitrary Distribution				Normal Theory			
	$\hat{\Sigma}_S$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$	
N	M	SD	M	SD	M	SD	M	SD	M	SD
100	-7.58E+15*	5.51E+15*	-13.81*	10.84*	-12.59*	10.82*	16584.60	8608.67	15289.25	8244.72
300	20993.72	5132.83	27968.62	7085.12	25631.28	6598.32	4894.66	1313.04	4366.49	1222.65
500	4250.06	747.57	9805.00	1896.33	8698.78	1720.01	3559.02	664.71	3129.15	619.39
1000	1721.65	211.24	5023.19	688.11	4381.75	619.82	2750.49	367.08	2378.77	341.76
10,000	648.00	24.92	2177.83	94.92	1836.45	83.23	1848.37	75.97	1552.90	67.83

Table 13.4. Condition number means and standard deviations of $\hat{\Sigma}_S$ and different estimators of $\hat{\Sigma}_{\boldsymbol{\psi}}$ for a one-factor model of Winsorized normal data with 20 observed variables with equal means. Note that the asterisks indicate matrices that were numerically non-positive semi-definite.

	Covariance Method		Arbitrary Distribution				Normal Theory			
	$\hat{\Sigma}_S$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$	
N	M	SD	M	SD	M	SD	M	SD	M	SD
100	-7.69E+15*	5.42E+15*	-15.12*	10.45*	-14.88*	11.74*	14096.01	7400.14	13107.45	7102.89
300	14027.04	2957.47	19780.84	4803.25	18070.39	4410.68	4059.04	1102.66	3668.41	1045.50
500	2791.86	394.84	7006.71	1334.49	6195.44	1194.36	2932.10	551.03	2610.58	522.02
1000	1112.50	104.58	3597.60	479.35	3129.92	423.49	2259.03	301.48	1979.05	284.41
10,000	408.50	12.00	1535.20	63.42	1295.07	55.44	1499.96	60.43	1279.43	56.62

Table 13.5. Condition number means and standard deviations of $\hat{\Sigma}_S$ and different estimators of $\hat{\Sigma}_{\boldsymbol{\psi}}$ for a one-factor model of log-normal data with 5 observed variables with equal means (non-Winsorized).

	Covariance Method		Arbitrary Distribution				Normal Theory			
	$\hat{\Sigma}_S$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$	
N	M	SD	M	SD	M	SD	M	SD	M	SD
100	70483.89	384249.76	8421.03	28318.14	6923.71	33587.71	4380.28	26779.34	3070.89	18021.49
300	22380.79	241804.08	3884.46	12644.99	2557.12	8513.12	2089.93	24776.36	982.09	8350.50
500	4393.97	10630.54	1966.28	4473.07	1281.37	3113.49	509.51	566.60	324.47	352.21
1000	3020.75	9480.50	1630.66	4469.89	989.38	2474.36	400.91	427.01	245.56	207.65
10,000	647.98	940.03	495.75	818.83	249.31	396.89	223.20	31.07	131.82	19.71

Table 13.6. Condition number means and standard deviations of $\hat{\Sigma}_S$ and different estimators of $\hat{\Sigma}_{\boldsymbol{\psi}}$ for a one-factor model of Winsorized log-normal data with 5 observed variables with equal means.

	Covariance Method		Arbitrary Distribution				Normal Theory			
	$\hat{\Sigma}_S$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$		$H_p\hat{\Sigma}_\Psi H_p^T$		$D_p^T\hat{\Sigma}_\Psi D_p$	
N	M	SD	M	SD	M	SD	M	SD	M	SD
100	594.38	545.46	196.55	112.92	221.77	123.52	397.42	225.94	334.06	217.72
300	163.84	63.77	70.90	19.15	98.93	22.36	249.69	70.72	196.57	61.68
500	125.47	34.70	56.03	11.04	84.80	14.84	226.78	51.56	176.05	44.98
1000	98.52	18.65	45.46	6.24	73.79	8.09	205.16	30.61	156.51	26.29
10,000	71.66	3.75	33.97	1.37	62.11	2.02	179.76	7.79	135.20	6.44

Table 13.7. Condition number means and standard deviations of $\hat{\Sigma}_S$ and different estimators of $\hat{\Sigma}_{\psi}$ for a one-factor model of log-normal data with 20 observed variables (non-Winsorized). Note that the asterisks indicate matrices that were numerically non-positive semi-definite.

	Covariance Method		Arbitrary Distribution				Normal Theory			
	$\hat{\Sigma}_S$		$H_p\hat{\Sigma}_{\Psi}H_p^T$		$D_p^T\hat{\Sigma}_{\Psi}D_p$		$H_p\hat{\Sigma}_{\Psi}H_p^T$		$D_p^T\hat{\Sigma}_{\Psi}D_p$	
N	M	SD	M	SD	M	SD	M	SD	M	SD
100	-1.43E+16*	1.29E+16*	-5.94E-03*	2.50E-03*	-5.68E-03*	2.92E-03	3.35E+05	8.41E+05	3.04E+05	7.48E+05
300	1.12E+08	3.79E+08	3.85E+07	1.34E+08	3.14E+07	1.15E+08	4.47E+04	1.59E+05	3.95E+04	1.51E+05
500	5.10E+06	1.08E+07	2.38E+06	4.59E+06	1.91E+06	3.78E+06	1.66E+04	2.01E+04	1.44E+04	1.55E+04
1000	8.67E+05	2.95E+06	4.64E+05	1.44E+06	3.69E+05	1.20E+06	9.13E+03	1.24E+04	7.98E+03	1.06E+04
10,000	2.75E+04	5.69E+04	2.12E+04	4.90E+04	1.65E+04	4.06E+04	2.74E+03	4.83E+02	2.36E+03	4.19E+02

Table 13.8. Condition number means and standard deviations of $\hat{\Sigma}_S$ and different estimators of $\hat{\Sigma}_{\psi}$ for a one-factor model of Winsorized log-normal data with 20 observed variables. Note that the asterisks indicate matrices that were numerically non-positive semi-definite.

	Covariance Method		Arbitrary Distribution				Normal Theory			
	$\hat{\Sigma}_S$		$H_p\hat{\Sigma}_{\Psi}H_p^T$		$D_p^T\hat{\Sigma}_{\Psi}D_p$		$H_p\hat{\Sigma}_{\Psi}H_p^T$		$D_p^T\hat{\Sigma}_{\Psi}D_p$	
N	M	SD	M	SD	M	SD	M	SD	M	SD
100	-1.48E+16*	1.24E+16*	-1.25E-02*	8.19E-03*	-1.16E-02*	7.69E-03*	3.16E+04	2.36E+04	3.19E+04	2.49E+04
300	7.07E+05	3.28E+05	3.19E+05	1.39E+05	3.01E+05	1.31E+05	6.84E+03	2.07E+03	6.72E+03	2.12E+03
500	4.25E+04	1.40E+04	2.36E+04	6.85E+03	2.27E+04	6.60E+03	4.72E+03	1.06E+03	4.61E+03	1.07E+03
1000	7.19E+03	1.43E+03	4.03E+03	6.38E+02	4.06E+03	6.31E+02	3.38E+03	5.21E+02	3.32E+03	5.24E+02
10,000	1.22E+03	6.17E+01	6.41E+02	2.22E+01	1.12E+03	3.46E+01	2.09E+03	9.67E+01	2.17E+03	1.10E+02

Chapter 14. Application

In this chapter, we considered how a CV model might be applied in a substantive research context. Although the preceding results indicated that the estimation methods may have trouble when certain distributional and model size conditions are not met, the effects of this in the context of real data should be investigated. Therefore, the present chapter considers a one-factor model of some longitudinal data regarding alcohol use.

Method

Data. Duncan, et al. (1997) and Duncan, Duncan, and Hops (1998) reported on a sample of 435 families containing at least one adolescent to examine development of alcohol use over time. The alcohol use of the individuals within the families (1204 total cases) was measured at 4 time points over 4 years (T_1, T_2, T_3, and T_4), as an index with values ranging from 0 to 4. A value of 0 indicted that the individual was a "lifetime abstainer," whereas a value of 4 indicated that the individual consumed 30 or more drinks per month. For additional details refer to Duncan, et al. (1998). For the purposes of the present research, one member of each family was selected to create a subset 435 independent observations to be used in the analysis below.

Model. Here a single factor model of the 4 alcohol use measures was constructed. To identify the model, the last factor loading and the paths from the error terms to the observed alcohol use variables were fixed to 1. The remaining parameters were left free to be estimated. This model is depicted in Figure 14.1.

Analyses. First, the means, the covariance matrix and the CV matrix were calculated. Next, the one factor model described above was fit to the data using each of the four estimation methods (AGLS, NGLS, MGLS, and MRLS) to produce four sets of parameter and standard error estimates as well as the four corresponding χ^2-test statistics. In addition, the parameter estimates produced by each method were used to produce a model predicted value of the CV matrix. These predicted CV matrices were contrasted with the sample CV matrix and the standardized residuals were considered. Note that we define the standardized residual to be the difference between the model-predicted elements of the CV matrix and the corresponding elements of the sample CV matrix relative to the standard deviation of the non-redundant elements of the sample CV matrix.

Results

Distribution and Summary of Data. The index of alcohol use recorded at the four time points were not normally distributed. There distributions are shown in Figure 14.2. In addition, there were slight increases in the average alcohol use over time. Specifically, the means of the alcohol use index at the 4 time points were 1.979, 2.026, 2.099, and 2.234. These average increases in alcohol use were accompanied by corresponding increases in variance, as can be seen in the sample covariance matrix

$$S = \begin{pmatrix} 1.190 & 0.858 & 0.855 & 0.797 \\ 0.858 & 1.220 & 1.019 & 0.990 \\ 0.855 & 1.019 & 1.319 & 1.141 \\ 0.797 & 0.990 & 1.141 & 1.578 \end{pmatrix}.$$

However, the coefficients of variation seemed to be more stable over time as can be seen in the sample CV matrix

$$\hat{\Psi} = \begin{pmatrix} 0.304 & 0.214 & 0.206 & 0.180 \\ 0.214 & 0.297 & 0.240 & 0.219 \\ 0.206 & 0.240 & 0.299 & 0.243 \\ 0.180 & 0.219 & 0.243 & 0.316 \end{pmatrix}.$$

In particular, the diagonal elements of S seem to increase, whereas those of $\hat{\Psi}$ do not.

Parameter Estimates. The parameter estimates and standard errors produced by each of the four estimation methods are displayed in Table 14.1. The four methods were fairly consistent in their estimation of the parameters. The first factor loading (corresponding to alcohol use at the first time point) tended to be a bit lower than the others, with estimates approximately between 0.87 and 0.89. The other two estimated factor loadings had values between 1.01 and 1.02 and 1.08 and 1.09, respectively. (The fourth factor loading was fixed at 1.) The factor variance estimates were also reasonably consistent across methods. Each method produced an estimate around 0.21 to 0.22. There was somewhat more variation in the estimates of the error variances. In particular, the estimates of the AGLS and MRLS error variances (about 0.13 and 0.14, respectively) for alcohol use at the first time point were slightly larger than those of the NGLS and MGLS (which were about 0.12). In addition, the MRLS value (0.075) of the second error variance seemed slightly larger than the others (which were closer to 0.06). The standard error estimates also tended to be fairly similar across methods. However, there were some small but potentially noteworthy differences. In particular, the NGLS method produced the highest standard error estimates for each of the factor loadings and for the factor variance and the AGLS method produced the highest standard errors for the error variances.

Test Statistics. Each estimation method produced a χ^2 value that resulted in the rejection of the model. Specifically, $\chi^2(2) = 15.32$ for AGLS, $\chi^2(2) = 19.43$ for NGLS, $\chi^2(2) = 20.28$ for MGLS, $\chi^2(2) = 24.91$ for MRLS, which are all well above the 5.99 cutoff for significance at the 0.05 level. This indicates that the one-factor CV model does not sufficiently explain the data.

Reproduced CV matrices. Each set of parameter estimates was used to produce the model-predicted value of CV matrix. The predicted values, denoted $\Psi(\hat{\theta})$, are reported along side the sample values ($\widehat{\Psi}$) in Table 14.2. Based on visual inspection of the observed sample values and those produced by the models, it appears that the models were able to reproduce the sample CV matrix. In addition, the correlations between the observed and predicted values were higher than 0.98 for each method. However, the standardized residuals suggest the prediction errors were large relative to the standard deviations of the non-redundant elements of $\widehat{\Psi}$. The distributions of these residuals are presented in Figure 14.3 for each method. It seems that MRLS estimation tended to produce the smallest errors, whereas NGLS produced the largest errors.

Discussion

While this analysis does answer some questions about the applicability of CV models to real data, it leaves many others open. In particular, we did see consistency across the estimation methods in the parameter estimates and (for the most part) the standard errors. Although in this case there is no way to determine how the estimates compared with the population, the consistency of the estimates and the ability of the model to reproduce the CV matrix suggests that these modeling techniques can provide reasonable parameter estimates in real data. On the other hand, there seemed to be large differences between the χ^2 values produced by the different estimation methods. The simulation studies suggested that these values may behave strangely when the data are non-normal. This was especially true for the three methods that make distributional assumptions (NGLS, MGLS and MRLS), but as we saw in Chapter 9, even the AGLS estimates could be unreliable if the sample size is not very large. This could explain why the models were rejected even with a seemingly high correlation between the predicted and observed CV matrix, or it could simply be the case that the residuals were large relative to the typical size of the matrix elements.

Additionally, we have so far been unable to address one of our primary questions of interest regarding the application of CV models to real data. That is, are there instances in which a structural CV model will provide simpler, more parsimonious description of the data than a traditional structural covariance model? While we have demonstrated that a CV model can be meaningful applied to real data, we have not succeeded in finding a case where this form of model will clearly be superior to a covariance model. Although the CV model used here is indeed less complex than the covariance model that would be required to model the changes in means over time, the CV model was not supported statistically. Now that a bit more is known about the behavior of the CV models and their estimation, it may be easier to understand when a CV model might be useful, and future work should certainly consider additional applications.

Tables

Table 14.1. *Parameter estimates and standard errors for the 1-factor model of alcohol use.*

		Factor Loadings			Factor Variance	Error Variances			
AGLS	$\hat{\theta}$	0.876	1.019	1.089	0.219	0.128	0.060	0.038	0.090
	$SE_{\hat{\theta}}$	0.052	0.043	0.041	0.021	0.014	0.010	0.008	0.010
NGLS	$\hat{\theta}$	0.888	1.023	1.081	0.211	0.116	0.061	0.040	0.088
	$SE_{\hat{\theta}}$	0.059	0.053	0.048	0.024	0.012	0.007	0.006	0.010
MGLS	$\hat{\theta}$	0.889	1.024	1.081	0.220	0.121	0.063	0.042	0.092
	$SE_{\hat{\theta}}$	0.052	0.046	0.043	0.021	0.010	0.006	0.006	0.008
MRLS	$\hat{\theta}$	0.879	1.014	1.086	0.219	0.141	0.075	0.041	0.099
	$SE_{\hat{\theta}}$	0.050	0.046	0.045	0.021	0.011	0.007	0.006	0.008

Table 14.2. *Values of the non-redundant elements of $\hat{\Psi}$ and the corresponding model predictions of those elements (Note, that the element ordering corresponds to the order of $vech(\Psi)$. That is, the columns of the lower triangular portion of $\hat{\Psi}$ were stacked to produce the columns below).*

$\hat{\Psi}$	$\Psi(\hat{\theta}_{AGLS})$	$\Psi(\hat{\theta}_{NGLS})$	$\Psi(\hat{\theta}_{MGLS})$	$\Psi(\hat{\theta}_{MRLS})$
0.304	0.295	0.282	0.294	0.311
0.214	0.195	0.191	0.200	0.195
0.206	0.209	0.202	0.211	0.209
0.180	0.192	0.187	0.195	0.193
0.297	0.287	0.281	0.294	0.300
0.240	0.243	0.233	0.243	0.241
0.219	0.223	0.215	0.225	0.222
0.299	0.298	0.286	0.298	0.300
0.243	0.238	0.228	0.237	0.238
0.316	0.309	0.299	0.311	0.318

Figures

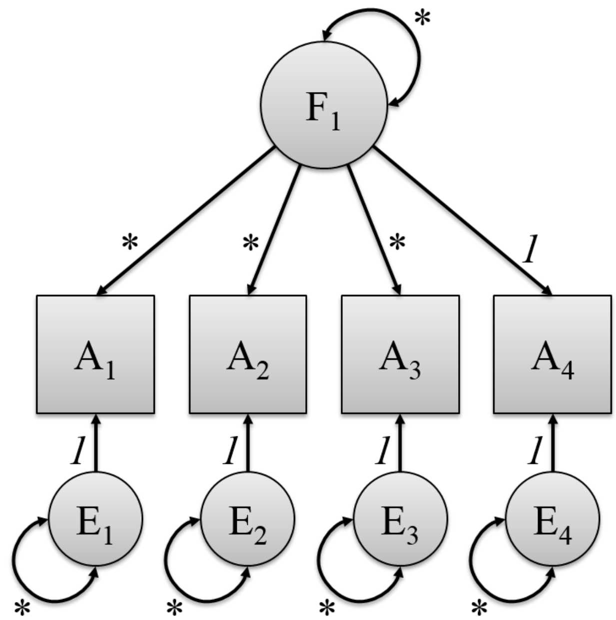

Figure 14.1. A one-factor model of the index of alcohol use at the four time points, with the values of fixed included and free parameters marked with the asterisks (*).

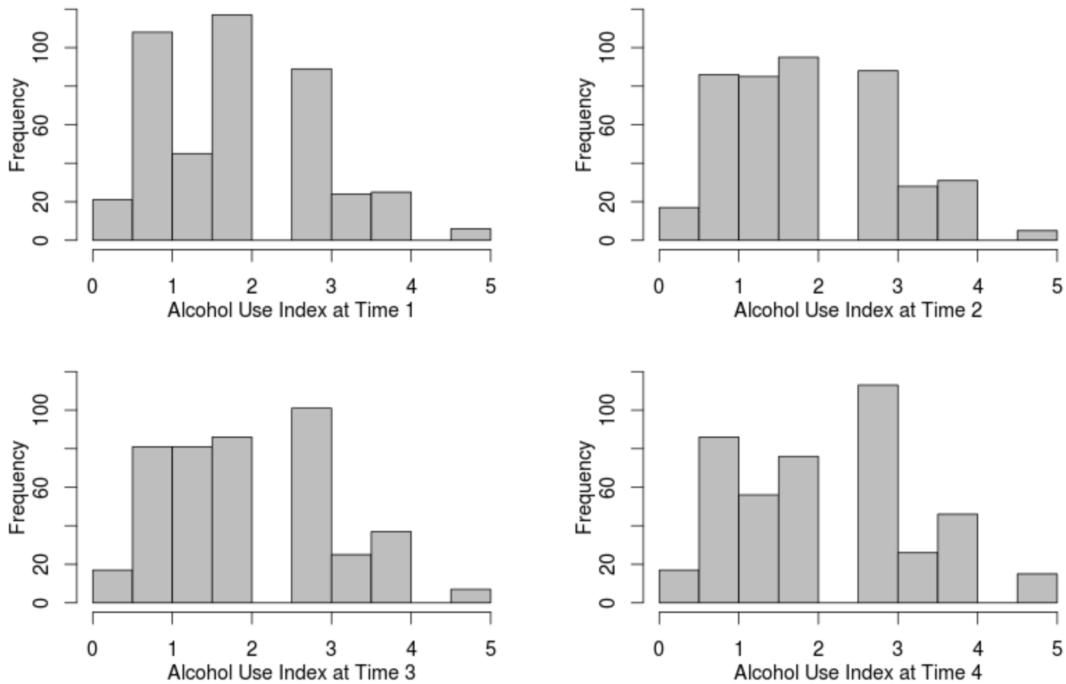

Figure 14.2. Histograms of the index of alcohol use at the four time points.

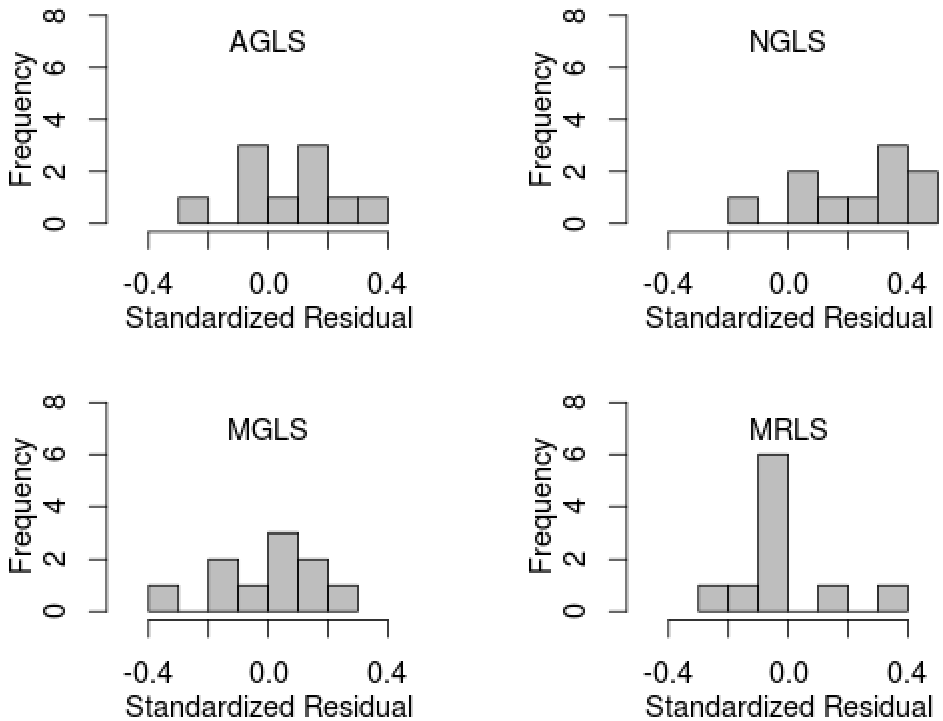

Figure 14.3. Histograms of the standardized residuals of the reproduced CV matrix.

Chapter 15. Conclusions

The simulations described here, have identified some circumstances when a CV model can be fit appropriately, and some other circumstances when a CV model will not be informative. In particular, the estimation methods preformed quite well in small models of normally distributed data. In addition, the MRLS and MGLS methods performed reasonably well in large models of normally distributed data when the sample size was large enough. In log-normal data, the results indicated that the estimation methods were generally effective at estimating parameters, but not fit statistics, for small models with large sample sizes. However, when these conditions were not met the estimation methods were unreliable for this sort of data. The results confirmed that only the AGLS method is correctly specified for non-normal data. In addition, it did not seem to matter whether one factor or two were included in the model. The type of data and the number of variables were, however, very important.

The results of the application of the data are not as informative as we had hoped. Some of the results discussed suggest that it is plausible that some data exists for which a CV model, like those described here, would provide a simpler explanation of the data than a covariance model. However, it is not yet clear what that data might be and whether the estimation methods will be able to provide accurate and informative parameters and tests explaining that data. This work has provided a foundation, for future explorations of this topic. It has identified some conditions that should be met for reliable and accurate estimation, which may also allow appropriate data to be selected and modeled. Furthermore, we have proposed numerous directions of investigation that may result in better estimation of CV models. If these topics are pursued they may allow more diverse models and data types to be fit accurately, increasing the odds that a structure or class of structures will be identified that can be better understood and more conveniently modeled in terms of the coefficient of variation matrix.

Appendix A. Notation

Distributions

$\mathcal{N}(\boldsymbol{\mu}, \Sigma)$: The multivariate normal distribution with mean μ and covariance Σ.

$log\mathcal{N}(\boldsymbol{\mu}, \Sigma)$: The multivariate log-normal distribution produced by transforming multivariate normal data with mean μ and covariance Σ.

$\chi^2(df)$: The chi-squared distribution with df degrees of freedom.

Matrix Notation: Special Matrices and Functions

A^T: The transpose of matrix A.

A^{-1}: The matrix inverse of A.

$|A|$: The determinant of A.

$tr(A)$: The trace of A.

$\|\boldsymbol{a}\|$: The ℓ_2-norm of vector a computed via $\sqrt{\sum(a_i^2)}$.

$A \otimes B$: The Kronecker product of matrix A with matrix B.

$A^{\otimes k}$: The k^{th} Kronecker product of A with itself (e.g. $A^{\otimes 3} = A \otimes A \otimes A$).

I_k: The $k \times k$ identity matrix.

$\mathbf{0}_k$: The vector of zeros of length k.

$\mathbf{1}_k$: The vector of ones of length k.

A_{ij}: The element of matrix A in the i^{th} row and j^{th} column.

A_i: Usually the i^{th} column of the matrix A (the i^{th} row in the case of Y_i)

a_i: The i^{th} element of the vector $\boldsymbol{a}$.

$D_{\boldsymbol{a}}$: The square diagonal matrix with the elements of vector a placed on the diagonal.

$vec(A)$: The vectorization of matrix A obtained by stacking the columns of A.

$vech(A)$: The half-vectorization of a symmetric matrix A obtained by stacking the elements of the columns in the lower-triangular portion of A.

$\boldsymbol{e}_i^p$: The i^{th} column of I_p.

$L_{p,qr}$: Matrix of zeros and ones defined by $\sum_{i=1}^{p}\left(\left(\boldsymbol{e}_i^p\right)^{\otimes q}\left(\boldsymbol{e}_i^p\right)^{T^{\otimes r}}\right)$.

L_p: A special case of $L_{p,qr}$ with $q = 2$ and $r = 1$. That is L_p is a $p^2 \times p$ matrix of zeros and ones defined by $\sum_{i=1}^{k}\left(\boldsymbol{e}_i^p \otimes \boldsymbol{e}_i^p\right)\left(\boldsymbol{e}_i^p\right)^T$.

D_p: The unique $p^2 \times p^*$ "duplication" matrix which converts a half-vectorization of A to a vectorization of A, i.e. $vec(A) = D_p vech(A)$ (Magnus & Neudecker, 1999).

H_p: The unique $p^* \times p^2$ "elimination" matrix which converts a vectorization of A to a half-vectorization of A, i.e. $vech(A) = H_p vec(A)$ (Magnus & Neudecker, 1999). The elimination matrix may be expressed in terms of the duplication matrix: $H_p = \left(D_p^T D_p\right)^{-1} D_p^T$ (Magnus & Neudecker, 1999).

Count Variables

n: Number of observations.

p: Number of variables.

p^*: Number of elements in $\boldsymbol{s}$. (For covariance or CV models, $p^* = p(p+1)/2$).

p^{**}: The number of unique elements in $\Sigma_{\boldsymbol{\psi}}$, which is calculated $p^{**} = p^*(p^*+1)/2$.

q: Number of model parameters.

Univariate Data, Statistics and Parameters

ψ: Population CV, i.e. $\psi = \sigma/\mu$

μ: Population mean.

σ^2: Population variance.

σ: Population standard deviation.

Multivariate Data, Statistics and Parameters

Y: An n (observations) $\times$ p(variables) matrix of observations.

Y_i: A $p \times 1$ vector consisting of one observation (row) of Y.

CV_{xx}: A scalar index summarizing the multivariate CV matrix (four types are presented).

Ψ: Population CV matrix.

$\widehat{\Psi}$: Estimated CV matrix.

$\widehat{\Psi}_{Win}$: CV matrix estimated with winsorized data (trim=5%).

$\boldsymbol{\psi}$: Vectorized population CV matrix ($vech(\Psi)$).

$\widehat{\boldsymbol{\psi}}$: Vectorized estimated CV matrix ($vech(\widehat{\Psi})$).

$\boldsymbol{\mu}$: Vector of population means.

$\widehat{\boldsymbol{\mu}}$: Vector of estimated means.

Σ: Population covariance matrix.

$\widehat{\Sigma}$: Estimated covariance matrix.

$\boldsymbol{\sigma}$: Vectorized covariance matrix ($vech(\Sigma)$).

$\widehat{\boldsymbol{\sigma}}$: Vectorized covariance matrix ($vech(\widehat{\Sigma})$).

$\Sigma_{\boldsymbol{\sigma}}$: Covariance matrix for the asymptotic sampling distribution of $vech(\mathrm{S})$.

$\widehat{\Sigma}_{\boldsymbol{\sigma}}$: Estimated covariance matrix asymptotic sampling distribution of $vech(S)$.

Σ_{Ψ}: Covariance matrix for the asymptotic sampling distribution of $vec\left(\widehat{\Psi}\right)$.

$\widehat{\Sigma}_{\Psi}$: Estimated covariance matrix asymptotic sampling distribution of $vec\left(\widehat{\Psi}\right)$.

$\Sigma_{\boldsymbol{\psi}}$: Covariance matrix for the asymptotic sampling distribution of $vech\left(\widehat{\Psi}\right)$.

$\widehat{\Sigma}_{\boldsymbol{\psi}}$: Estimated covariance matrix asymptotic sampling distribution of $vech\left(\widehat{\Psi}\right)$.

$\hat{\Sigma}_\Psi^+$: The Moore-Penrose inverse of $\hat{\Sigma}_\Psi$ (i.e. $\hat{\Sigma}_\Psi^+ = D_p\left(D_p^T \hat{\Sigma}_\Psi D_p\right)^{-1} D_p^T$).

Ω_{xy}: Sub-block of the covariance matrix defined in (3.3) through (3.5).

Ω_{xy}^*: Transformation of the covariance matrices Ω_{xy} defined in (3.6) through (3.8).

General Structural Models

$\boldsymbol{s}$: A vector of sample statistics. When applicable, $\boldsymbol{s} = vech(S)$.

$\boldsymbol{s}_0$: The population value of the population value of the statistics in $\boldsymbol{s}$.

S: A matrix of sample statistics such that $\boldsymbol{s} = vech(S)$.

$\boldsymbol{\theta}$: Vector of length q of model parameters (element of the parameter space Θ).

$\hat{\boldsymbol{\theta}}$: Estimated vector (of length q) of model parameters obtained by minimizing a discrepancy function.

$\mathcal{s}(\boldsymbol{\theta})$: A structural model with parameters $\boldsymbol{\theta}$. When applicable, $\mathcal{s}(\boldsymbol{\theta}) = vech\big(\mathcal{S}(\boldsymbol{\theta})\big)$.

$\mathcal{S}(\boldsymbol{\theta})$: A matrix such that $\mathcal{s}(\boldsymbol{\theta}) = vech\big(\mathcal{S}(\boldsymbol{\theta})\big)$.

$\dot{\mathcal{s}}(\boldsymbol{\theta})$: Derivative of $\mathcal{s}(\boldsymbol{\theta})$ relative to $\boldsymbol{\theta}$, i.e. $\dot{\mathcal{s}}(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}^T} \mathcal{s}(\boldsymbol{\theta})$.

$\boldsymbol{\theta}_0$: The set of population parameters such that $\mathcal{s}(\boldsymbol{\theta}_0) = \boldsymbol{s}_0$.

V_{SS}: The asymptotic variance of the sampling distribution of $\sqrt{n}(\boldsymbol{s} - \boldsymbol{s}_0)$ as defined in (4.2).

W: A weight matrix for the generalized least squares function which takes on different values depending on the type of estimation being conducted.

W^*: A $p \times p$ weight matrix which can be used to form an alternative expression for the weight matrix W when normality assumptions are met. See (4.10).

W_0: Either a constant-valued positive definite matrix selected to be the value of W or a positive definite matrix to which W converges in probability.

$F(\boldsymbol{s}, \mathscr{s}(\boldsymbol{\theta})|W)$: A generalized least squares discrepancy function with parameters $\boldsymbol{s}$ and $\mathscr{s}(\boldsymbol{\theta})$ and weight matrix W.

T_{XX}: Test statistic for estimation method XX (an asterisk indicates a corrected value).

Δ: Covariance of the sampling distribution of $\widehat{\boldsymbol{\theta}}$.

Models

$\Sigma(\boldsymbol{\theta})$: Model covariance matrix (function of $\boldsymbol{\theta}$).

$\Psi(\boldsymbol{\theta})$: Model CV matrix (function of $\boldsymbol{\theta}$).

$\boldsymbol{\psi}(\boldsymbol{\theta})$: Half-vectorization of the model CV matrix, i.e. $vech\big(\Psi(\theta)\big)$.

$\dot{\boldsymbol{\psi}}(\boldsymbol{\theta})$: Derivative of $\boldsymbol{\psi}(\boldsymbol{\theta})$ relative to $\boldsymbol{\theta}$, i.e. $\dot{\boldsymbol{\psi}}(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} \boldsymbol{\psi}(\boldsymbol{\theta})$.

The Bentler-Weeks Model

$\boldsymbol{\eta}$: Vector of dependent variables.

ξ: Vector of independent variables.

β: Matrix of regression coefficients relating elements of $\boldsymbol{\eta}$ to each other.

γ: Matrix of regression coefficients relating $\boldsymbol{\xi}$ to $\boldsymbol{\eta}$.

B: Matrix containing β and known 0 elements.

Γ: A matrix containing γ and known 0 and 1 elements.

Φ: The covariance matrix for the independent variables.

G: A selection matrix which selects the observed variables

Appendix B. Population Value and Large Sample Approximation of $\boldsymbol{\Sigma}_{\psi}$

Table B.1. *The population matrix Σ_{Ψ} for five normally distributed variables with parameters described in Chapter 6.*

40	16	16	16	16	18	10	10	10	18	10	10	18	10	18
16	11	8	8	8	16	8	8	8	10	6	6	10	6	10
16	8	11	8	8	10	8	6	6	16	8	8	10	6	10
16	8	8	11	8	10	6	8	6	10	8	6	16	8	10
16	8	8	8	11	10	6	6	8	10	6	8	10	8	16
18	16	10	10	10	40	16	16	16	18	10	10	18	10	18
10	8	8	6	6	16	11	8	8	16	8	8	10	6	10
10	8	6	8	6	16	8	11	8	10	8	6	16	8	10
10	8	6	6	8	16	8	8	11	10	6	8	10	8	16
18	10	16	10	10	18	16	10	10	40	16	16	18	10	18
10	6	8	8	6	10	8	8	6	16	11	8	16	8	10
10	6	8	6	8	10	8	6	8	16	8	11	10	8	16
18	10	10	16	10	18	10	16	10	18	16	10	40	16	18
10	6	6	8	8	10	6	8	8	10	8	8	16	11	16
18	10	10	10	16	18	10	10	16	18	10	16	18	16	40

Table B.2. *A large-sample estimate (N=500,000) of the population matrix* Σ_Ψ *for five normally distributed variables with parameters described in Chapter 6.*

39.56	15.87	15.82	15.93	15.83	17.85	9.91	9.96	9.89	17.83	9.92	9.87	17.90	9.92	17.80
15.87	10.92	7.93	7.97	7.92	15.88	7.94	7.97	7.91	9.92	5.96	5.92	9.96	5.95	9.88
15.82	7.93	10.89	7.94	7.90	9.91	7.93	5.96	5.92	15.86	7.92	7.89	9.91	5.93	9.87
15.93	7.97	7.94	10.96	7.95	9.95	5.96	7.98	5.95	9.92	7.93	5.93	15.91	7.94	9.91
15.83	7.92	7.90	7.95	10.89	9.87	5.93	5.95	7.90	9.87	5.93	7.89	9.91	7.92	15.81
17.85	15.88	9.91	9.95	9.87	39.74	15.89	15.94	15.84	17.86	9.91	9.86	17.93	9.91	17.78
9.91	7.94	7.93	5.96	5.93	15.89	10.92	7.95	7.91	15.88	7.92	7.89	9.92	5.93	9.86
9.96	7.97	5.96	7.98	5.95	15.94	7.95	10.97	7.94	9.91	7.94	5.93	15.94	7.94	9.89
9.89	7.91	5.92	5.95	7.90	15.84	7.91	7.94	10.87	9.86	5.92	7.88	9.91	7.91	15.79
17.83	9.92	15.86	9.92	9.87	17.86	15.88	9.91	9.86	39.69	15.82	15.79	17.80	9.87	17.75
9.92	5.96	7.92	7.93	5.93	9.91	7.92	7.94	5.92	15.82	10.88	7.89	15.84	7.91	9.86
9.87	5.92	7.89	5.93	7.89	9.86	7.89	5.93	7.88	15.79	7.89	10.84	9.87	7.88	15.77
17.90	9.96	9.91	15.91	9.91	17.93	9.92	15.94	9.91	17.80	15.84	9.87	39.72	15.85	17.79
9.92	5.95	5.93	7.94	7.92	9.91	5.93	7.94	7.91	9.87	7.91	7.88	15.85	10.89	15.80
17.80	9.88	9.87	9.91	15.81	17.78	9.86	9.89	15.79	17.75	9.86	15.77	17.79	15.80	39.53

Table B.3. *The raw differences between the elements of the population matrix* Σ_{Ψ} *and those of a large-sample estimate (N=500,000) for five normally distributed variables with parameters described in Chapter 6.*

0.44	0.13	0.18	0.07	0.17	0.15	0.09	0.04	0.11	0.17	0.08	0.13	0.10	0.08	0.20
0.13	0.08	0.07	0.03	0.08	0.12	0.06	0.03	0.09	0.08	0.04	0.08	0.04	0.05	0.12
0.18	0.07	0.11	0.06	0.10	0.09	0.07	0.04	0.08	0.14	0.08	0.11	0.09	0.07	0.13
0.07	0.03	0.06	0.04	0.05	0.05	0.04	0.02	0.05	0.08	0.07	0.07	0.09	0.06	0.09
0.17	0.08	0.10	0.05	0.11	0.13	0.07	0.05	0.10	0.13	0.07	0.11	0.09	0.08	0.19
0.15	0.12	0.09	0.05	0.13	0.26	0.11	0.06	0.16	0.14	0.09	0.14	0.07	0.09	0.22
0.09	0.06	0.07	0.04	0.07	0.11	0.08	0.05	0.09	0.12	0.08	0.11	0.08	0.07	0.14
0.04	0.03	0.04	0.02	0.05	0.06	0.05	0.03	0.06	0.09	0.06	0.07	0.06	0.06	0.11
0.11	0.09	0.08	0.05	0.10	0.16	0.09	0.06	0.13	0.14	0.08	0.12	0.09	0.09	0.21
0.17	0.08	0.14	0.08	0.13	0.14	0.12	0.09	0.14	0.31	0.18	0.21	0.20	0.13	0.25
0.08	0.04	0.08	0.07	0.07	0.09	0.08	0.06	0.08	0.18	0.12	0.11	0.16	0.09	0.14
0.13	0.08	0.11	0.07	0.11	0.14	0.11	0.07	0.12	0.21	0.11	0.16	0.13	0.12	0.23
0.10	0.04	0.09	0.09	0.09	0.07	0.08	0.06	0.09	0.20	0.16	0.13	0.28	0.15	0.21
0.08	0.05	0.07	0.06	0.08	0.09	0.07	0.06	0.09	0.13	0.09	0.12	0.15	0.11	0.20
0.20	0.12	0.13	0.09	0.19	0.22	0.14	0.11	0.21	0.25	0.14	0.23	0.21	0.20	0.47

Table B.4. *A large-sample estimate (N=500,000) for five variables log-normal distributions with parameters described in Chapter 6.*

480.65	113.94	122.44	112.76	138.08	52.90	43.12	39.29	49.56	53.50	40.01	45.62	51.42	38.72	85.54
113.94	74.84	49.40	45.30	56.20	99.44	44.38	41.36	45.89	35.77	26.70	28.45	33.22	25.10	47.94
122.44	49.40	75.22	46.03	51.91	38.07	42.07	26.67	28.43	92.22	41.22	43.08	34.82	23.58	41.48
112.76	45.30	46.03	72.27	44.56	35.48	26.72	39.24	25.08	35.23	40.96	23.58	83.26	33.81	34.96
138.08	56.20	51.91	44.56	108.67	39.36	28.47	25.08	54.36	36.84	23.59	47.64	27.90	40.65	116.13
52.90	99.44	38.07	35.48	39.36	385.03	104.61	95.08	104.39	47.94	35.73	36.40	46.03	34.66	57.46
43.12	44.38	42.07	26.72	28.47	104.61	68.49	41.62	42.47	85.66	41.05	39.49	34.49	23.48	36.13
39.29	41.36	26.67	39.24	25.08	95.08	41.62	66.18	40.52	35.07	40.45	23.44	83.44	36.71	34.82
49.56	45.89	28.43	25.08	54.36	104.39	42.47	40.52	78.94	33.36	23.45	42.04	30.68	40.64	97.95
53.50	35.77	92.22	35.23	36.84	47.94	85.66	35.07	33.36	377.38	95.78	95.61	51.50	31.56	47.68
40.01	26.70	41.22	40.96	23.59	35.73	41.05	40.45	23.45	95.78	71.67	37.22	85.54	33.37	29.48
45.62	28.45	43.08	23.58	47.64	36.40	39.49	23.44	42.04	95.61	37.22	68.05	27.60	35.02	85.46
51.42	33.22	34.82	83.26	27.90	46.03	34.49	83.44	30.68	51.50	85.54	27.60	272.27	70.99	35.75
38.72	25.10	23.58	33.81	40.65	34.66	23.48	36.71	40.64	31.56	33.37	35.02	70.99	54.90	77.91
85.54	47.94	41.48	34.96	116.13	57.46	36.13	34.82	97.95	47.68	29.48	85.46	35.75	77.91	278.01

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