A Spin TQFT Related to the Ising Categories

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Abstract

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Most interesting 3d Topological Quantum Field Theories (TQFTs) are constructed by starting with algebraic data, usually in the form of some kind of category [Tur16]. This category typically comes from an area of mathematics different from 3-manifold topology, and its topological nature can be hard to understand. This dissertation reverses the process, at least in one simple example, by constructing a Spin TQFT from pure topology and then uncovering some interesting categories.

The topology used to construct the Spin TQFT is entirely classical. If $(M, s)$ is a closed Spin 3-manifold, then an embedded surface $\Sigma \subset M$ inherits a $\text{Pin}^-$ structure, $s|_\Sigma$, from $(M, s)$. If $\Sigma$ and $\Sigma'$ represent the same class in $H_2(M; \mathbb{Z}/2)$, then $s|_\Sigma$ and $s|_{\Sigma'}$ are isomorphic. If $t$ is a $\text{Pin}^-$ structure on $\Sigma$, there is a classical invariant of the isomorphism type of $(\Sigma, t)$, denoted $\beta(\Sigma, t)$, that is an eight root of unity. One can therefore form the Spin 3-manifold invariant

$$Z_3(M, s) := \frac{1}{2^{b_0(M)}} \sum_{[\Sigma] \in H_2(M;\mathbb{Z}/2)} \beta(\Sigma, s|_\Sigma).$$

It turns out that this invariant fits into a Spin TQFT. The detailed construction of this TQFT is the subject of this dissertation.

By a theorem of Kirby and Melvin ([KM91]), $Z_3$ is very much related to the Ising categories ([DGNO10], Appendix B). In extending the TQFT for $Z_3$, one encounters a category (associated with the bounding Spin circle) which has most of the same properties as the Ising categories. One also encounters a category (associated with the interval) from which $Z_3$ can be reconstructed in the style of Turaev-Viro and Barrett-Westbury. Because $Z_3$ is a Spin TQFT, these categories are linear over super vector spaces. In fact, they are realized as the module categories of certain explicit super algebras.

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# Contents

# 1 Introduction

In [RT91] and [Tur16], Reshetikhin and Turaev construct 3-dimensional topological quantum field theories from certain tensor categories called "modular tensor categories". Roughly speaking, a modular tensor category consists of several pieces of data:

- A rigid braided fusion category $\mathcal{C}$. In particular, $\mathcal{C}$ is semisimple linear category with finitely many simples.
- A "twist" automorphism $\theta$ of the identity functor $\mathcal{C} \to \mathcal{C}$.
- A matrix $S = \{S_i^j\}_{i,j \in I}$ ($I$ is the set of isomorphism classes of simple objects) that satisfies a certain nondegeneracy condition.

To a topologist looking at this list of algebraic data, it can be a little difficult to understand what is "3-dimensional" about a modular tensor category. This paper constructs, in a topologically motivated way, a related sort of category. It comes from a simple extended 3d Spin TQFT.

Fix a primitive 8th root of unity $\zeta$. One can define a modular tensor category $\mathcal{I}_\zeta$ called an Ising category (see [DGNO10], Appendix B). It has three simple objects ($X_0$, $X_1$, and $X_2$), a monoidal product with multiplication table

$$\begin{array}{c|ccc} & X_0 & X_1 & X_2 \\ \hline X_0 & X_0 & X_1 & X_2 \\ X_1 & X_1 & X_0 & X_2 \\ X_2 & X_2 & X_2 & X_0 \oplus X_1 \end{array}, \tag{1}$$

twists $\theta_{X_0} = 1$, $\theta_{X_1} = -1$, and $\theta_{X_2} = \zeta^{1/2}$ and, with respect to the basis $(X_0, X_1, X_2)$, an $S$-matrix

$$\begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}. \tag{2}$$

The Ising category for $\zeta = e^{3\pi i/4}$ can be constructed from the quantum group $U_q(\mathfrak{sl}_2)$ when $q$ is a 4th root of unity using the usual construction of modular tensor categories from quantum groups [Tur16], [RT91], [CP94]. For some detailed discussion on (the restricted version of) this quantum group, see [GR17]. When $\zeta = e^{2\pi i/8}$, $\mathcal{I}_\zeta$ can also be constructed more topologically (but less rigorously) from the free fermion conformal field theory, see for example [Sch] Section 7. The conformal field theoretic perspective makes it clear that the category is related to Spin structures on surfaces. For example, $\dim(Z_{\mathcal{I}_\zeta}(\Sigma_g)) = \frac{1}{2}(4^g + 2^g)$ is the number of bounding Spin structures on $\Sigma_g$ and also the number of bounding (resp. nonbounding) classes in $H_1(\Sigma_g; \mathbb{Z}/2)$ on a bounding (resp. nonbounding) Spin structure on $\Sigma_g$. However, the conformal field theory perspective does not make it very clear if the category is related to Spin structures on 3-manifolds. In fact, it is, as Kirby and Melvin showed:

**Theorem 1.1** ([KM91] Theorem 7.1)**.** *If $\zeta = e^{3\pi i/4}$ and $M$ is a closed oriented 3-manifold then*

$$Z_{\mathcal{I}_\zeta}(M) = \frac{1}{2^{b_0(M)}} \sum_{s \in \mathrm{Spin}(M)} \zeta^{R(M,s)/2}$$

*where $R(M,s)$ is Rokhlin's $\mathbb{Z}/16$-valued invariant of a closed Spin 3-manifold.*

Kirby and Melvin's normalization has been changed to match the normalizations of this paper. Their theorem is proved using the quantum group description of $\mathcal{I}_\zeta$ which requires $\zeta = e^{3\pi i/4}$. It is also probably true for the other primitive 8th roots of unity.

Kirby and Melvin do not extend their theorem to give an intrinsic description of $Z_{\mathcal{I}_\zeta}(M)$ for non-closed $M$. Exactly how such an extension might work remains an interesting question. However, if you combine $Z_{\mathcal{I}_\zeta}(M)$ with $R(M,s)$ for a specific Spin structure, you can fully define an analogous Spin TQFT. This is the main result of the present paper. For concreteness, fix $\zeta = e^{2\pi i/8}$, $\zeta^{1/2} = e^{2\pi i/16}$, and write $\mu(M,s) = \zeta^{R(M,s)/2}$. Other TQFTs can be defined analogously for other primitive 8th roots of unity, but one has to be careful about the right choice of square root.

**Theorem 1.2.** *There exists a 3d Spin TQFT $\hat{Z}_3$, purely topologically constructed, with the following properties:*

*(a) If $(M,s)$ is a closed Spin 3-manifold then*

$$\hat{Z}_3(M,s) = \frac{\mu(M,s)}{2^{b_0(M)}} \sum_{t \in \mathrm{Spin}(M)} \mu(M,t)^{-1}.$$

*(b) If $(\Sigma_g, s)$ is a closed oriented Spin surface of genus $g$ then $\dim \hat{Z}_3(\Sigma, s) = \frac{1}{2}(4^g + 2^g)$ and $\hat{Z}_3(\Sigma_g, s)$ admits a basis in natural bijection with bounding (resp. nonbounding) classes in $H_1(\Sigma_g; \mathbb{Z}/2)$ if $s$ is bounding (resp. nonbounding).*

*(c) In the construction of $\hat{Z}_3$, the bounding Spin structure on the circle is naturally assigned a linear monoidal category $\mathcal{C}_0$ that has multiplication table (1). It also naturally comes with an $S$-matrix of the form (2).*

*(d) In the construction of $\hat{Z}_3$, the unique Spin structure on the interval is naturally assigned a fusion category that has properties very similar to a spherical fusion category. One can construct $\hat{Z}_3$ from this category in the style of Turaev-Viro-Barrett-Westbury (see* [Tur16] *Chapter VII for more on the TVBW construction).*

*(a) is proved in Section 17, (b) follows from Proposition 12.24, (c) is the content of Section 16, and (d) is the content of Section 14.*

$\hat{Z}_3$ is an extended TQFT in the sense that if $(\Sigma, s)$ is a Spin surface with boundary there is a suitably compatible vector space, called $Z_3(\Sigma, s)$, that is an object in the category assigned to the boundary. The details of this extension are not spelled out, but anyone with an interest in the extension of TQFTs should be able to work out these details from Section 16.

The category $\mathcal{C}_0$ is realized as the category of modules of an algebra $\mathcal{A}_0$ in $\mathcal{SV}$, the category of finite-dimensional super vector spaces. As such there is an endofunctor of $\mathcal{C}_0$ given by $X \mapsto \widetilde{X}$ where $\widetilde{X}$ is the same underlying module as $X$ but with $\mathcal{A}_0$ action twisted by the degree involution $a \mapsto (-1)^{\deg(a)}a$. The category $\mathcal{C}_0$ is not quite braided, but there is a natural isomorphism of the functors $(X, Y) \mapsto X \odot Y$ and $(X, Y) \mapsto \widetilde{Y} \odot X$ that does satisfy the braid relations. Here $\odot$ stands for the monoidal product on $\mathcal{C}_0$. Similarly, there is a natural "twist" isomorphism of the identity functor to $X \mapsto \widetilde{X}$. This isomorphism plays the role of the twist $\theta$ in the definition of a modular tensor category.

Presumably, the Spin TQFT $\hat{Z}_3$ can be understood in terms of [GK16] and [BGK17]. In particular, the category assigned to the interval is probably a spherical super-fusion category in the sense of [GK16] Appendix C and the TQFT $\hat{Z}_3$ is probably recovered by their construction applied to this category. This direction is not pursued in this paper.

Rokhlin's invariant is closely related to Brown's invariant of $\mathrm{Pin}^-$ surfaces (for reference see [KT90]). Given a closed Spin 3-manifold $M$ and $\Sigma \subset M$ a surface, the restriction $s|_\Sigma$ defines a $\mathrm{Pin}^-$ structure on $\Sigma$ (note that $\Sigma$ might be nonorientable). If $\Sigma$ and $\Sigma'$ are $\mathbb{Z}/2$-homologous in $M$, then their $\mathrm{Pin}^-$ structures are cobordant. The 2d $\mathrm{Pin}^-$ cobordism group is isomorphic to $\mathbb{Z}/8$, and generated by one of the two $\mathrm{Pin}^-$ structures on $\mathbb{R}P^2$. Identifying one of these two with $\zeta$ defines an isomorphism $\beta$ from the $\mathrm{Pin}^-$ cobordism group to $\langle \zeta \rangle \subset \mathbb{C}^\times$. (Note that if $(\Sigma, s)$ is an orientable surface then $\beta(\Sigma, s)$ is 1 if $s$ is bounding and $-1$ if not.) One can then form the following invariant of $(M, s)$:

$$\frac{1}{2^{b_0(M)}} \sum_{x \in H_2(M;\mathbb{Z}/2)} \beta(x, s|_x). \tag{3}$$

Here a class $x$ is represented by an embedded surface and $\beta$ is evaluated on that embedded surface. If $x^\vee \in H^1(M;\mathbb{Z}/2)$ is Poincare dual to $x \in H_2(M;\mathbb{Z}/2)$ then it is also true that

$$\mu(M, s + x^\vee) = \mu(M, s)\beta(x, s|_x)^{-1}$$

so (3) is equal to

$$\frac{1}{2^{b_0(M)}} \mu(M, s) \sum_{t \in \mathrm{Spin}(M)} \mu(M, t)^{-1}$$

The "integral" (3) is the starting point for defining $\hat{Z}_3$. Curiously, it bears resemblance to another expression for $\beta$ ([KT90] section 3):

$$\beta(\Sigma, s) = \frac{2^{\chi(\Sigma)/2}}{2^{b_0(\Sigma)}} \sum_{x \in H_1(\Sigma;\mathbb{Z}/2)} \lambda(x, s|_x) \tag{4}$$

where $\lambda \in \{\pm 1, \pm i\}$ is a certain quantity associated to closed Spin 1-manifolds. Here $x$ is represented by an embedded submanifold.

The "integral" (4) can be used as the starting point for defining a 2d $\mathrm{Pin}^-$ TQFT.

**Theorem 1.3.** *There exists a purely topologically constructed* $\mathrm{Pin}^-$ *TQFT* $\hat{Z}_2$ *such that if* $(\Sigma, s)$ *is a closed* $\mathrm{Pin}^-$ *surface then* $\hat{Z}_2(\Sigma, s) = \beta(\Sigma, s)$. *In particular, if* $\Sigma$ *is closed and orientable,* $\hat{Z}_2(\Sigma, s) = 1$ *if* $s$ *is bounding and* $-1$ *if* $s$ *is nonbounding. The Spin TQFT* $\hat{Z}_2$ *extends to* $\mathrm{Pin}^-$ *surfaces with corners.*

This existence of this $\mathrm{Pin}^-$ TQFT is not new (see for example [DG]). However, its construction here seems to be new. It is fairly combinatorial in flavor and amenable to direct computation.

Spin and $\mathrm{Pin}^-$ structures exist in the smooth category, so their extension to surfaces with corners requires some comment.

**Construction 1.4.** *Let* $(\Sigma, \tau)$ *be an oriented surface with triangulation* $\tau$. *Mark some of the edges of* $\tau$. *Define a 0-chain in the* $\Delta$*-chain group* $C_0(\Sigma; \frac{1}{2}\mathbb{Z}/2)$ *by*

$$\mathbf{v} = \sum_{\sigma \text{ a 2-simplex in } \tau} \frac{1}{2}[1]_\sigma + \sum_{e \text{ a marked edge}} \partial e \mod 2$$

*where* $[1]_\sigma$ *is the vertex* $[1]$ *in the 2-simplex* $\sigma$. *If* $\mathbf{v}$ *is the sum of all the vertices in* $\Sigma$, *then there is a well-defined Spin structure corresponding to this marking.*

With this construction, Spin structures can be written down via triangulations of surfaces. There is an analogous construction for triangulated 3-manifolds where faces are marked. The case for surfaces seems similar to a construction used by Cimasoni and Reshetikhin [CR07] in their study of dimer models, but their construction only works for some surface triangulations.

There is a standard way, called a state-sum construction, to construct oriented 2d TQFTs from special symmetric Frobenius algebras (see [LP07]). The construction essentially involves assigning the multiplication and comultiplication to the two different kinds of oriented 2-simplex in the triangulation. Let $A$ be the super Frobenius algebra with basis $a_0, a_1$ where $a_0$ is even and $a_1$ is odd and with multiplication and comultiplication

$$a_0 \otimes a_0 \mapsto 2^{-1/4} a_0,\ a_0 \otimes a_1 \mapsto 2^{-1/4} a_1,\ a_1 \otimes a_0 \mapsto 2^{-1/4} a_1,\ a_1 \otimes a_1 \mapsto 2^{-1/4} a_0$$

$$a_0 \mapsto 2^{-1/4}(a_0 \otimes a_0 + a_1 \otimes a_1),\ a_1 \mapsto 2^{-1/4}(a_0 \otimes a_1 + a_1 \otimes a_0).$$

This algebra is, of course, isomorphic to the Clifford algebra $\mathrm{Cl}_1$.

**Corollary 1.5.** *Let* $(\Sigma, s)$ *be a Spin surface described by a marked triangulation. Apply the state-sum construction for* $A$ *to this triangulation modified by the operator* $x \mapsto (-1)^{\deg(x)} x$ *at each marked edge. If* $\Sigma$ *is closed, the resulting number is* $\beta(\Sigma, s)$, *i.e.,* $1$ *if* $s$ *bounds and* $-1$ *is* $s$ *does not bound.*

This construction is discussed in [GK16] Section 6.1 from the state-sum perspective, but it appears naturally in the construction of $\hat{Z}_2$.

A large part of the paper is devoted to describing a general way to construct a TQFT (not a Spin TQFT) from certain invariants of manifolds with corners. Here the construction is called an "extended pre-TQFT" and it consists (essentially) of the combinatorics of the Turaev-Viro-Barrett-Westbury construction of TQFTs ([Tur16] Chapters VI and VII, [TV92], [BW96]). The TVBW construction centers around dual vector spaces

$$Z(\pm\Delta^2)$$

assigned to the two oriented 2-simplices and a vector

$$Z(\pm\Delta^3) \in Z(\pm\Delta^2) \otimes Z(\pm\Delta^2) \otimes Z(\mp\Delta^2) \otimes Z(\mp\Delta^2)$$

assigned to oriented 3-simplices. Here each tensor factor corresponds to the one of the 2-simplices in the boundary of the 3-simplex. If $(M,\tau)$ is an oriented triangulated 3-manifold then there is a number $Z(M,\tau)$ obtained by starting with

$$\bigotimes_{\pm\Delta^3 \in \tau} Z(\pm\Delta^3)$$

and contracting pairs of tensor factors corresponding to edge labelings, with such contractions weighted in a crucial way. With the correct sort of weightings, $Z(M,\tau)$ turns out not to depend on the triangulation $\tau$. With a small amount of extra work, this construction can be used to give a 3d TQFT, not just a closed 3-manifold invariant. An extended pre-TQFT, as defined here, is essentially this construction but $\pm\Delta^2$ is generalized to any oriented surface with corners, $\pm\Delta^3$ is generalized to any oriented 3-manifold with with an embedded graph in its boundary, and $(M,\tau)$ is generalized to any 3-manifold decomposed as a union of 3-manifolds with such graphs.

In Section 5, the framework of an extended pre-TQFT helps provide some topological motivation behind the apperance of modular tensor categories in TQFTs. More importantly for this paper, though, it provides the right context for constructing the Pin$^-$ TQFT $\hat{Z}_2$ and the Spin TQFT $\hat{Z}_3$. Specifically, Sections 10 and 12 construct Spin extended pre-TQFTs, called $Z_2$ and $Z_3$, respectively, from which the existence of $\hat{Z}_2$ and $\hat{Z}_3$ follow easily (Section 17). All of the interesting algebraic properties of $\hat{Z}_2$ and $\hat{Z}_3$ come from algebraic properties of $Z_2$ and $Z_3$.

The following is an outline of the paper:

- Section 2 recalls important notions about $\Delta$-complexes and fixes notation for spaces that will be used repeatedly, like the circle, cylinder, pair of pants, etc.

- Section 3 recalls the definition of a TQFT. The constructions in this paper will involve something like a TQFT, called an (extended) pre-TQFT. Proposition 3.6 proves that given a pre-TQFT one can construct a TQFT. This section describes both 2d and 3d extended pre-TQFTs.

- There is little algebra involved in the definitions of 2d and 3d extended pre-TQFTs, and this is by design. Algebraic structures arise as a consequence of these axioms, and Sections 4 and 5 describe these algebraic structures for 2d and 3d pre-TQFTs. For example, the existence of monoidal categories "assigned" to the interval and circle are consequences of the axioms of an extended 3d pre-TQFT.

- Section 6 recalls classical theorems about Spin structures and classical invariants of Spin and $\text{Pin}^-$ manifolds.

- Section 7 describes Construction 1.4 and its analogue for 3-manifolds. It may be of interest independent of the rest of (TQFT-centric) results of this paper.

- Section 8 recalls the definitions of Spin and $\text{Pin}^-$ TQFTs.

- One of the Spin invariants defined in Section 6 is the invariant $(S, s) \mapsto \lambda(S, s)$, which (in the simplest possible situation) assigns to a closed Spin 1-manifold $(S, s)$ the number 1 if $s$ bounds and the number $-1$ if it doesn't. Section 9 extends $\lambda$ to non-closed Spin 1-manifolds. Essentially a Spin 1-manifold $(S, s)$ is assigned an element in $\bigotimes_{p\in\partial S} L_p$ where $L_p$ are certain odd complex lines. Gluing two boundary points of $S$ together corresponds to contracting the two corresponding complex lines to a copy of $\mathbb{C}$. For example, a Spin interval $(S, s)$ is assigned a vector in a tensor product of two lines, and gluing the two ends of this interval to form a Spin circle $(S', s')$ corresponds to contracting those two lines. The resulting number is $\lambda(S', s')$. There are details to be worked out regarding the gluing of Spin intervals. In particular, one must pick sections of the boundary of the Spin 1-manifolds.

- Section 10 describes the extended $\text{Pin}^-$ pre-TQFT $Z_2$ that extends $\beta$ to $\text{Pin}^-$ surfaces with corners. Essentially, a $\text{Pin}^-$ surface $(\Sigma, s)$ is assigned a vector $Z_2(\Sigma, s)$ in a certain tensor product of super vector spaces, one tensor factor for each piece of the boundary of $\Sigma$. Since $\Sigma$ can have corners, these pieces might be intervals.

- Section 11 gives examples of the extension of $\beta$ to surfaces with boundary. The vector space assigned to an interval is the super Frobenius algebra $A$ mentioned earlier.

- For Spin 3-manifolds $(M, s)$, Section 12 extends the quantity

$$Z_3(M, s) = \frac{1}{2^{b_0(M)}} \sum_{x\in H_2(M;\mathbb{Z}/2)} \beta(x, s|_x)$$

to 3-manifolds with corners. The construction at first depends on a choice of curves in the boundary of the 3-manifold. After giving some examples, some work is done to remove this dependency.

- Section 13 discusses a certain mapping class group representation related to $Z_3$.
- The extension of $Z_3$ to Spin 3-manifolds with corners includes, in particular, vector spaces $Z_3(\Sigma, s)$ for spin surfaces $\Sigma$. The vector spaces for the bigon and the cylinder each give algebras. Sections 14, 15, and 16 describe these algebras and their categories of modules. The category of modules of the cylinder algebra is particularly interesting. It has many of the properties of a modular tensor category.
- Section 17 uses $Z_2$ and $Z_3$ to define the $\mathrm{Pin}^-$ and Spin TQFTs $\hat{Z}_2$ and $\hat{Z}_3$ mentioned in Theorems 1.3 and 1.2.
- Section 18 contains some open questions and speculation.

# 2 Δ-Complexes

See [Hat02], Section 2.1 for the definition of a Δ-complex. Let $\Delta^n$ denote the standard $n$-simplex, together with an ordering of its vertices. Its subsimplices can be expressed in terms of subsets of its vertices. For example, $\Delta^2$ has edges [01], [12], and [02]. There are two orientations on $\Delta^n$, denoted $+\Delta^n = \Delta^n$ and $-\Delta^n$. If $X$ is an oriented manifold with a Δ-complex structure, each top-dimensional simplex inherits a sign $\pm$ depending on whether its inclusion into $X$ is orientation reversing or not. Conversely, if each top-dimensional simplex is assigned a sign $\pm$ such that the boundaries of adjacent simplices cancel algebraically, then this determines a class in $H_n(X, \partial X; \mathbb{Z})$ and hence an orientation.

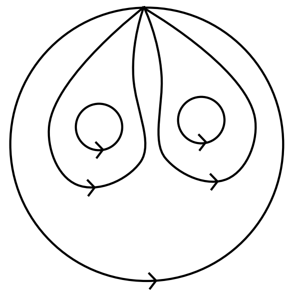
The oriented boundary of $+\Delta^n$ is

$$\partial(\pm\Delta^n) = \bigcup_{i=0}^{n} \pm(-1)^i [0 \cdots \hat{i} \cdots n].$$

For example

$$\begin{aligned}
\partial(+\Delta^2) &= +[01] \cup +[12] \cup -[02] \\
\partial(-\Delta^2) &= -[01] \cup -[12] \cup +[02] \\
\partial(+\Delta^3) &= +[123] \cup -[023] \cup +[013] \cup -[012] \\
\partial(-\Delta^3) &= -[123] \cup +[023] \cup -[013] \cup +[012]
\end{aligned}$$

Some standard oriented Δ-complexes will appear repeatedly and so it will be helpful to fix notation for these complexes. Let $\mathsf{S}$ denote the oriented circle obtained from $+\Delta^1$ by gluing its two ends together. Let $\mathsf{B}$ denote the bigon obtained from $\Delta^1 \times \Delta^1$ by collapsing $\Delta^1 \times \partial\Delta^1$ to $\partial\Delta^1$. Note that $\partial\mathsf{B} = (-\Delta^1) \cup (+\Delta^1)$ where the copy of $+\Delta^1$ corresponds to $\{1\} \times \Delta^1 \subset \mathsf{B}$. Let $\mathsf{C}$ denote the cylinder $\Delta^1 \times \mathsf{S}$. Note that $\partial\mathsf{C} = (-\mathsf{S}) \sqcup (+\mathsf{S})$ where the copy of $+\mathsf{S}$ corresponds to $\{1\} \times \mathsf{S} \subset \mathsf{C}$. Let $\mathsf{P}$ denote the pair of pants obtained by gluing two copies of $\mathsf{C}$ to the negative sides of $-\Delta^2$:



Let $\mathsf{D}$ denote the disk obtained by gluing two oppositely oriented sides of $+\Delta^2$ together. Forget the $\Delta$-complex structure on the interior of $\mathsf{D}$. The important part is that $\partial\mathsf{D} = \mathsf{S}$. Let $\mathsf{T}$ denote $\mathsf{S} \times \mathsf{S}$. Then $\mathsf{T}$ can be obtained by gluing the two ends of $\mathsf{C}$ together.

It will also help to fix orderings of the boundary pieces of several $\Delta$-complexes. Let $\partial(+\Delta^2)$ be ordered as [02], [01], [12] and give $\partial(-\Delta^2)$ the reverse ordering. Let $\partial(+\Delta^3)$ be ordered as [012], [023], [013], [123] and give $\partial(-\Delta^3)$ the reverse ordering. Let the ordering of $\partial\mathsf{P}$ be induced from the ordering on $-\Delta^2$ and let the ordering on $\partial(-\mathsf{P})$ be induced from the ordering on $+\Delta^2$. Let the orderings on $\partial\mathsf{B}$ and $\partial\mathsf{C}$ be given by the negatively oriented side first.

# 3 TQFTs and pre-TQFTs

This section recalls the definition of an $n$-dimensional TQFT and defines an similar sort of object called a pre-TQFT. Essentially, a TQFT assigns linear maps to 3-manifolds. A linear map $V \to W$ is (assuming $V$ and $W$ are finite dimensional) the same thing as an element of $V^* \otimes W$. A pre-TQFT takes the latter approach and assigns to 3-manifolds elements of tensor products of vector spaces.

Assume that $n \leq 3$. This dimension restriction relieves the burden of a detailed discussion of smooth structures and collar neighborhoods of boundary components.

**Definition 3.1.** If $\Sigma_1$ and $\Sigma_2$ are closed oriented $(n-1)$-manifolds, an $n$-dimensional cobordism from $\Sigma_1$ to $\Sigma_2$ consists of

- An oriented $n$-manifold $M$
- A partition of $\partial M$ into two disjoint sets: $\partial M = (\partial M)_- \sqcup (\partial M)_+$
- Orientation preserving diffeomorphisms

$$\phi_- : (\partial M)_- \to -\Sigma_1$$

$$\phi_+ : (\partial M)_+ \to \Sigma_2.$$

Here $-\Sigma_1$ denotes $\Sigma_1$ with the opposite orientation and $\partial M$ inherits an orientation via the "outward normal first" rule.

A cobordism $(M_{12}, (\phi_{12})_-, (\phi_{12})_+)$ from $\Sigma_1$ to $\Sigma_2$ can be combined with a cobordism $(M_{23}, (\phi_{23})_-, (\phi_{23})_+)$ from $\Sigma_2$ to $\Sigma_3$ by gluing $(\partial M_{12})_+$ to $(\partial M_{23})_-$ along $(\phi_{23})_-^{-1} \circ (\phi_{12})_+$. The result is a cobordism from $\Sigma_1$ to $\Sigma_3$.

**Definition 3.2.** Say two cobordisms $(M_1, (\phi_1)_-, (\phi_1)_+)$ and $(M_2, (\phi_2)_-, (\phi_2)_+)$ are equivalent if there is a diffeomorphism $f : M_1 \to M_2$ such that $(\phi_2)_\pm \circ f = (\phi_1)_\pm$.

**Definition 3.3.** Define the monoidal category $n\mathrm{Cob}$ as follows. Let $\mathrm{Obj}(n\mathrm{Cob})$ be the the set[1] of closed oriented $(n-1)$-manifolds. Let $\mathrm{Mor}(\Sigma_1, \Sigma_2)$ be the set of equivalence classes of cobordisms from $\Sigma_1$ to $\Sigma_2$. Composition is given by gluing together representative cobordisms. The monoidal product on $n\mathrm{Cob}$ is given by disjoint union.

Let $\mathcal{V}$ be the category of finite-dimensional vector spaces.

**Definition 3.4.** A $n$-dimensional TQFT is a monoidal functor $\hat{Z} : n\mathrm{Cob} \to \mathcal{V}$.

The one-line Definition 3.4 is very attractive, but the meat of a TQFT is in the gluing laws that make up the definition of the cobordism category. Because of this, Definition 3.4 can be a little cumbersome to generalize to more complicated gluing patterns. It will be helpful to instead consider the following set of assignments, which I will call a "pre-TQFT".

**Definition 3.5.** A pre-TQFT is an assignment $Z$ constructed as follows and with the following properties:

- To every closed $(n-1)$-manifold $\Sigma$, assign a vector space $Z(\Sigma)$ with $Z(\emptyset) := \mathbb{C}$.
- If $f : \Sigma \to \Sigma'$ is a diffeomorphism, assign a linear map $f_* : Z(\Sigma) \to Z(\Sigma')$ such that $(fg)_* = f_* g_*$.
- Assign a canonical isomorphism $Z(\Sigma \sqcup \Sigma') \cong Z(\Sigma) \otimes Z(\Sigma')$ compatible with the other properties in the obvious way.
- Assign a nondegenerate pairing $\langle , \rangle : Z(\Sigma) \otimes Z(-\Sigma) \to \mathbb{C}$ for each closed surface $\Sigma$ such that $\langle f_* v, f_* w \rangle = \langle v, w \rangle$ for a diffeomorphism $f$.
- To every oriented compact $n$-manifold $M$, assign a vector $Z(M) \in Z(\partial M)$. Use an outward normal first rule to orient $\partial M$. If $f : M \to N$ is a diffeomorphism, then $(f|_\partial)_* Z(M) = Z(N)$.

[1] Not really a set, but if one is worried about set-theoretic difficulties one can restrict to a set of representatives of each diffeomorphism type.

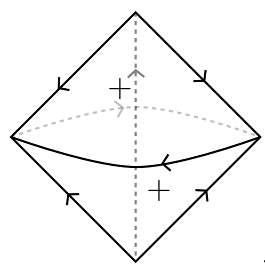
- The above assignments must satisfy the following "gluing law". Let $M$ be a compact oriented $n$-manifold, and write its boundary as union of three closed pieces $\partial M = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Let $f$ be an orientation-preserving diffeomorphism $f : \Sigma_1 \to -\Sigma_2$. Write $M_f$ for $M$ glued along $f$. Then $Z(M) \in Z(\Sigma_1) \otimes Z(\Sigma_2) \otimes Z(\Sigma_3)$. Require that $Z(M_f)$ is the result of applying $f_* \otimes \mathrm{id} \otimes \mathrm{id}$ to $Z(M)$ then contracting the first two tensor factors using the canonical pairing $Z(-\Sigma_2) \otimes Z(\Sigma_2) \to \mathbb{C}$.

- Assert that $Z(I \times \Sigma) \in Z(-\Sigma) \otimes Z(\Sigma)$ is a left and right dual to the nondegenerate pairing $Z(\Sigma) \otimes Z(-\Sigma) \to \mathbb{C}$.

**Proposition 3.6.** *A pre-TQFT determines a TQFT.*

*Proof.* Given a pre-TQFT $Z$ as above, define a functor $\hat{Z} : n\mathrm{Cob} \to \mathcal{V}$ as follows. Set $\hat{Z}(\Sigma) = Z(\Sigma)$. Let $(M, \phi_-, \phi_+)$ be a cobordism from $\Sigma_1$ to $\Sigma_2$. Then $Z(\partial M) \cong Z((\partial M)_-) \otimes Z((\partial M)_+)$ so $((\phi_-)_* \otimes (\phi_+)_*) Z(M) \in Z(-\Sigma_1) \otimes Z(\Sigma_2)$. Use the nondegenerate pairing to identify $Z(-\Sigma_1)$ with $Z(\Sigma_1)^*$, so $((\phi_-)_* \otimes (\phi_+)_*) Z(M) \in Z(\Sigma_1)^* \otimes Z(\Sigma_2) \cong \mathrm{Hom}(Z(\Sigma_1), Z(\Sigma_2))$. Set $\hat{Z}(M, \phi_-, \phi_+) = ((\phi_-)_* \otimes (\phi_+)_*) Z(M)$. By the gluing law, $\hat{Z}$ is functorial. The condition on $Z(I \times \Sigma)$ is ensures that $\hat{Z}(I \times \Sigma, \mathrm{id}, \mathrm{id})$ is the identity map. □

In the busisness of constructing manifolds, however, one often wants to glue along pieces of the boundary other than connected components. For example here is an oriented triangulation of a ball which, when glued antipodally, produces $\mathbb{R}P^3$:




For this it is necessary to "extend" the definition of a pre-TQFT.

**Definition 3.7.** A 3-manifold with generalized corners is a 3-manifold with an embedded graph on its boundary.

If $M$ is a 3-manifold with generalized corners, write $M^0$ for the vertices of $M$. 3-manifolds with corners are examples of 3-manifolds with generalized corners. 3-manifolds with generalized corners should be thought of as those 3-manifolds obtained by gluing together polyhedra along faces. The boundary of such a 3-manifold inherits a graph from the 1-skeletons of the polyhedra.

**Definition 3.8.** Let $M$ be a 3-manifold with generalized corners and let $M^1$ denote the graph on its boundary. A "piece" of the boundary of $M$ is a component of $\partial M \setminus M^1$ compactified with a boundary by pulling back the closure of its inclusion into $\partial M$. A piece is therefore a surface with corners. Let $\Sigma$ be a surface with corners and $\Sigma^0$ be its vertex set. Also use the word "piece" to describe a component of $\partial\Sigma \setminus \Sigma^0$ compactified with boundary by pulling back the closure of its inclusion into $\partial\Sigma$. A piece of $\partial\Sigma$ is therefore a 1-manifold with boundary.

**Example 3.9.** If you take the unit cube and glue two opposite faces together then you end up with a manifold with generalized corners whose boundary has four pieces, each of which is a square.

Note that if $X$ is oriented then each piece of $\partial X$ inherits an orientation by an "outward normal first" convention.

It will be convenient to describe manifolds with (generalizd) corners as pairs $(X, \mathcal{P})$ where $X$ is a manifold with boundary and $\mathcal{P}$ is a decomposition of its boundary into pieces. If $\mathcal{P}$ is such a decomposition, write $P \in \mathcal{P}$ to mean that $P$ is one of the pieces of the decomposition. The "vertices" and "edges" of $\mathcal{P}$ are the vertices and edges of $X$, considered as a manifold with (generalized) corners. For example, in Example 3.9, there are four vertices and eight edges. The vertices and edges of $\mathcal{P}$ are not (usually) the vertices and edges of $\bigsqcup_{P\in\mathcal{P}} P$.

**Definition 3.10.** Let $(Y, \mathcal{P})$ and $(Y, \mathcal{Q})$ be two different boundary decompositions for the same manifold. For example $\mathcal{P}$ and $\mathcal{Q}$ could be two different triangulations of a surface. Let $(c(Y), \mathcal{P} \sqcup \mathcal{Q})$ be the $n$-manifold with (generalized) corners obtained from $I \times Y$ by collapsing each component of $I \times \partial Y$ to a copy of $\partial Y$ and by decomposing $\{0\} \times Y$ into $\mathcal{P}$ and $\{1\} \times Y$ into $\mathcal{Q}$. $(c(Y), \mathcal{P} \sqcup \mathcal{Q})$ can be thought of as something like a canonical cobordism from $\mathcal{P}$ to $\mathcal{Q}$.

**Definition 3.11.** If $X$ is a manifold, write $\mathcal{P}_X$ for the decomposition of $X$ into a single piece.

**Definition 3.12.** A 2d extended pre-TQFT is a the following collection of data:

- A nonzero complex number $\Gamma$.
- To every oriented 1-manifold $S$ (possibly with boundary) assign a vector space $Z(S)$ with $Z(\emptyset) := \mathbb{C}$.
- If $\mathcal{P}$ is a decomposition of $S$, set $Z(S, \mathcal{P}) = \bigotimes_{P\in\mathcal{P}} Z(P)$.
- If $f : S \to S'$ is an orientation-preserving diffeomorphism, assign a linear map $f_* : Z(S) \to Z(S')$ such that $(fg)_* = f_* g_*$.
- Assign a canonical isomorphism $Z(S \sqcup S') \cong Z(S) \otimes Z(S')$ compatible with the other properties in the obvious ways.

- Assign a nondegenerate pairing $\langle,\rangle : Z(S)\otimes Z(-S) \to \mathbb{C}$ for each 1-manifold $S$ such that $\langle f_* v, f_* w\rangle = \langle v, w\rangle$ for an orientation-preserving diffeomorphism $f$.

- To every oriented compact 2-manifold with corners $(\Sigma, \mathcal{P})$ assign a vector $Z(\Sigma, \mathcal{P}) \in Z(\partial\Sigma, \mathcal{P})$. Use an outward normal first rule to orient $\partial\Sigma$. If $f : (\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ is a diffeomorphism of manifolds with corners, then $(f|_\partial)_* Z(\Sigma) = Z(\Sigma')$.

- The above assignments must satisfy the following "gluing law". Let $(\Sigma, \mathcal{P})$ be an oriented 2-manifold with corners, and write its boundary as union of three parts $\partial\Sigma = S_1 \cup S_2 \cup S_3$ where $S_i \cap S_j$ is contained in the vertices of $\mathcal{P}$. Let $f$ be an orientation-preserving diffeomorphism $f : S_1 \to -S_2$. Write $(\Sigma_f, \mathcal{P}_f)$ for $(\Sigma, \mathcal{P})$ glued along $f$. Then $Z(\Sigma, \mathcal{P}) \in Z(S_1, \mathcal{P}|_{S_1}) \otimes Z(S_2, \mathcal{P}|_{S_2}) \otimes Z(S_3, \mathcal{P}|_{S_3})$. Require that $Z(\Sigma_f, \mathcal{P}_f)$ is the result of applying $f_* \otimes \mathrm{id} \otimes \mathrm{id}$ to $Z(\Sigma, \mathcal{P})$, contracting the first two tensor factors with the canonical map $Z(-S_1, \mathcal{P}|_{S_1}) \otimes Z(S_1, \mathcal{P}|_{S_1}) \to \mathbb{C}$, then dividing by $\Gamma$ for each vertex of $\mathcal{P}$ that becomes interior in $\mathcal{P}_f$.

- For every compact oriented 1-manifold $S$, assert that

$$Z(c(S), \mathcal{P}_S \sqcup \mathcal{P}_S) \in Z(-S) \otimes Z(S)$$

  is inverse to the pairing $Z(S) \otimes Z(-S) \to \mathbb{C}$.

Note that the restriction of a 2d extended pre-TQFT to closed 1-manifolds is a pre-TQFT.

**Definition 3.13.** Let $Z$ be a 2d extended pre-TQFT and let $(S, \mathcal{P})$ and $(S, \mathcal{Q})$ be two decompositions of the same 1-manifold. Define a linear map

$$Z(S, \mathcal{P}) \to Z(S, \mathcal{Q})$$

as follows. Start with $Z(c(S), \mathcal{P} \sqcup \mathcal{Q}))$ in

$$Z(-S, \mathcal{P}) \otimes Z(S, \mathcal{Q})$$

Identify this space with

$$Z(S, \mathcal{P})^* \otimes Z(S, \mathcal{Q})$$

using the pairing in the definition of the extended pre-TQFT and then apply a map $\delta_\mathcal{P}$ (defined momentarily) to the result. Call this map $\phi_{S;\mathcal{P},\mathcal{Q}}$. The map

$$\delta_\mathcal{P} : Z(S, \mathcal{P}) \to Z(S, \mathcal{P})$$

is multiplication by $\Gamma^{-n}$ where $n$ is the number of interior vertices in $\mathcal{P}$.

**Proposition 3.14.** *Let $Z$ be a 2d extended pre-TQFT. The following are true:*

1. $\phi_{S;\mathcal{P}_2,\mathcal{P}_3} \circ \phi_{S;\mathcal{P}_1,\mathcal{P}_2} = \phi_{S;\mathcal{P}_1,\mathcal{P}_3}$.

2. *$\phi_{S;\mathcal{P}_S,\mathcal{P}}$ is an injection*

3. *$\phi_{S;\mathcal{P},\mathcal{P}_S}$ is a surjection with image canonically isomorphic to $Z(S)$.*

4. *If $(\Sigma, \mathcal{Q})$ is an oriented surface with corners then $Z(\Sigma, \mathcal{Q}) \in \mathrm{im}(\phi_{\partial\Sigma;\mathcal{Q},\mathcal{Q}})$.*

*Proof.* 1 follows from the gluing law. 2 follows the fact that $\phi_{S;\mathcal{P},\mathcal{P}_S} \circ \phi_{S;\mathcal{P}_S,\mathcal{P}}$ is the identity on $Z(S)$ by part 1. 3 follows by factoring $\phi_{S;\mathcal{P},\mathcal{P}} = \phi_{S;\mathcal{P}_S,\mathcal{P}} \circ \phi_{S;\mathcal{P},\mathcal{P}_S}$. 4 follows from the gluing law: gluing a collar onto a manifold produces a copy of the same manifold. □

There is a pairing $Z(S) \otimes Z(-S) \to \mathbb{C}$. If $\mathcal{P}$ is a nontrivial decomposition of $S$, then there is an inclusion $\phi_{S;\mathcal{P}_S,\mathcal{P}} : Z(S) \to Z(S, \mathcal{P})$. If $x \in Z(S)$ and $y \in Z(-S)$ then, in general

$$\langle x, y\rangle \neq \langle \phi_{S;\mathcal{P}_S,\mathcal{P}}(x), \phi_{-S;\mathcal{P}_S,\mathcal{P}}(y)\rangle$$

However:

**Proposition 3.15.** *If $x \in Z(S)$ and $y \in Z(-S)$, then*

$$\langle x, y\rangle = \langle \delta_{\mathcal{P}} \circ \phi_{S;\mathcal{P}_S,\mathcal{P}}(x), \phi_{-S;\mathcal{P}_S,\mathcal{P}}(y)\rangle$$

*Proof.* Let $\langle z\rangle_{12}$ denote the tensor contraction of the first and second tensor factors of $z$. Since

$$\phi_{S;\mathcal{P}_S,\mathcal{P}}(x) = \langle x, Z(c(S), \mathcal{P}_S \sqcup \mathcal{P})\rangle_{12}$$

then

$$\begin{aligned}\langle \delta_{\mathcal{P}} \circ \phi_{S;\mathcal{P}_S,\mathcal{P}}(x), \phi_{-S;\mathcal{P}_S,\mathcal{P}}(y)\rangle \\ = \langle\langle x \otimes (\delta_{\mathcal{P}} \otimes \mathrm{id}_{Z(S)})(Z(c(S), \mathcal{P}_S \sqcup \mathcal{P})))\rangle_{12} \otimes \langle y \otimes Z(c(-S), \mathcal{P}_S \sqcup \mathcal{P}))\rangle_{12}\rangle\end{aligned} \tag{5}$$

Since $(c(S), \mathcal{P}_S \sqcup \mathcal{P})$ and $(c(-S), \mathcal{P}_S \sqcup \mathcal{P})$ glue together to form $(c(S), \mathcal{P}_S \sqcup \mathcal{P}_S)$ and the appearance of $\delta$ in (5) is just right for application of the gluing law, then (5) is equal to

$$\langle x \otimes y, Z(c(S), \mathcal{P}_S \sqcup \mathcal{P}_S)\rangle = \langle x, y\rangle$$

where the equality here follows from the fact that $Z(c(S), \mathcal{P}_S, \mathcal{P}_S)$ is dual to the pairing $Z(S) \otimes Z(-S) \to \mathbb{C}$ (this is one of the pre-TQFT axioms). □

The definition of a 2d extended pre-TQFT generalizes to the 3d case, though for interesting applications one must put certain gradings on the vector spaces. The introduction of gradings might seem unnatural, and it probably is. However, it works for all relevant examples, including those in this paper.

**Definition 3.16.** A 3d extended pre-TQFT is the following collection of data and properties:

- A nonzero complex number $\Gamma$.
- A finite set $I$ together with a function $d : I \to \mathbb{C}^\times$.
- Let $\Sigma$ be a 2-manifold with corners with a boundary parametrized by copies of $\pm\Delta^1$ or $\pm\mathsf{S}$. The decomposition of the boundary into these pieces and the parametrizations are omitted from the notation. Assign a vector space $Z(\Sigma)$ with $Z(\emptyset) := \mathbb{C}$. Require that $Z(\Sigma)$ is graded by labelings of the pieces of $\partial\Sigma$ by elements of $I$. Write $Z(\Sigma)_\ell$ for the graded part corresponding to a labeling $\ell$.
- Let $\mathcal{P}$ be a decomposition of a surface $\Sigma$ into pieces so that the 1-dimensional part of $\mathcal{P}$ is a union of intervals, and each interval is identified with $\Delta^1$. Then

  $$Z(\Sigma, \mathcal{P}) := \bigotimes_{P \in \mathcal{P}} Z(P)$$

  If $\ell$ is a labeling of the edges of $\mathcal{P}$, let $Z(\Sigma, \mathcal{P})_\ell$ denote the subspace

  $$\bigotimes_{P \in \mathcal{P}} Z(P)_{\ell|_P}$$

- If $f : \Sigma \to \Sigma'$ is an orientation-preserving diffeomorphism that takes boundary parametrizations to boundary parametrizations, assign a grading-preserving linear map $f_* : Z(\Sigma) \to Z(\Sigma)$ such that $(fg)_* = f_* g_*$.
- Assign a canonical isomorphism $Z(\Sigma \sqcup \Sigma') \cong Z(\Sigma) \otimes Z(\Sigma')$ suitably compatible with the other properties.
- Assign a nondegenerate pairing $\langle , \rangle : Z(\Sigma)_\ell \otimes Z(-\Sigma)_\ell \to \mathbb{C}$ for each 2-manifold with corners $\Sigma$ such that $\langle f_* v, f_* w \rangle = \langle v, w \rangle$ for an orientation-preserving diffeomorphism $f$. Extend this to a nondegenerate pairing $Z(\Sigma) \otimes Z(-\Sigma) \to \mathbb{C}$ by mapping $Z(\Sigma)_\ell \otimes Z(-\Sigma)_{\ell'}$ to zero if $\ell \neq \ell'$.
- To every oriented compact 3-manifold $(M, \mathcal{P})$ with corners assign a vector $Z(M, \mathcal{P}) \in Z(\partial M, \mathcal{P})$. Write $Z(M)_\ell$ to denote the projection of $Z(M)$ to $\bigotimes_{P \in \mathcal{P}} Z(P)_{\ell|_P}$. Use an outward normal first rule to orient $\partial M$.
- If $f : M \to M'$ is a diffeomorphism of manifolds with corners, then $(f|_\partial)_* Z(M, \mathcal{P}) = Z(M', \mathcal{P}')$.
- The above assignments must satisfy the following "gluing law". Let $(M, \mathcal{P})$ be a compact 3-manifold with generalized corners, and write its boundary as union of three parts $\partial M = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ where $\Sigma_i \cap \Sigma_j$ is contained in the dimension 1 part of $\mathcal{P}$. Let $f$ be an orientation-preserving diffeomorphism $f : \Sigma_1 \to -\Sigma_2$ compatible with the structure of $\Sigma_1$ and $\Sigma_2$ as surfaces with corners. Write $(M_f, \mathcal{P}_f)$ for $(M, \mathcal{P})$ glued

along $f$. Then $Z(M,\mathcal{P}) \in Z(\Sigma_1, \mathcal{P}|_{\Sigma_1}) \otimes Z(\Sigma_2, \mathcal{P}|_{\Sigma_2}) \otimes Z(\Sigma_3, \mathcal{P}|_{\Sigma_3})$. For a labeling $\ell$, apply $f_* \otimes \mathrm{id} \otimes \mathrm{id}$ to $Z(M,\mathcal{P})_\ell$. Contract the first two tensor factors, multiply by $d(\ell(e))$ for each edge in $\mathcal{P}$ that becomes interior in $\mathcal{P}_f$, then divide by $\Gamma$ for each vertex of $\mathcal{P}$ that becomes interior in $\mathcal{P}_f$. Repeat this for each labeling $\ell$ and sum the results. Require that this sum be $Z(M_f, \mathcal{P}_f)$.

- For every compact oriented 2-manifold with corners $\Sigma$ assert that $Z(c(\Sigma), \mathcal{P}_\Sigma \sqcup \mathcal{P}_\Sigma) \in Z(-\Sigma) \otimes Z(\Sigma)$ is inverse to the pairing $Z(\Sigma) \otimes Z(-\Sigma) \to \mathbb{C}$.

Note that restricting an extended 3d pre-TQFT to closed surfaces defines a 3d pre-TQFT.

**Definition 3.17.** Let $Z$ be a 3d extended pre-TQFT and let $(\Sigma, \mathcal{P})$ and $(\Sigma, \mathcal{Q})$ be two different decompositions of the same surface. Define a linear map

$$Z(\Sigma, \mathcal{P}) \to Z(\Sigma, \mathcal{Q})$$

as follows. Start with $Z(c(\Sigma), \mathcal{P} \sqcup \mathcal{Q}))$ in

$$Z(-\Sigma, \mathcal{P}) \otimes Z(\Sigma, \mathcal{Q})$$

Identify this space with

$$Z(\Sigma, \mathcal{P})^* \otimes Z(\Sigma, \mathcal{Q})$$

using the pairing in the definition of the extended pre-TQFT. Precompose this with a map $\delta_\mathcal{P}$ defined momentarily. Call the result $\phi_{\Sigma;\mathcal{P},\mathcal{Q}}$. The map

$$\delta_\mathcal{P} : Z(\Sigma, \mathcal{P}) \to Z(\Sigma, \mathcal{P})$$

is defined to be multiplication by $\Gamma^{-1}$ for each vertex in $(\Sigma, \mathcal{P})$, and multiplication by

$$\prod_{\text{interior edges } e \text{ of } \mathcal{P}} d(\ell(e))$$

on $Z(\Sigma, \mathcal{P})_\ell$. Note that $\delta_\mathcal{P}$ is invertible.

**Proposition 3.18.** *The following are true:*

1. $\phi_{\Sigma;\mathcal{P}_2,\mathcal{P}_3} \circ \phi_{\Sigma;\mathcal{P}_1,\mathcal{P}_2} = \phi_{\Sigma;\mathcal{P}_1,\mathcal{P}_3}$.

2. $\phi_{\Sigma;\mathcal{P}_\Sigma,\mathcal{P}}$ *is an injection*

3. $\phi_{\Sigma;\mathcal{P},\mathcal{P}_\Sigma}$ *is a surjection with image canonically isomorphic to* $Z(\Sigma)$.

4. *Let* $(M, \mathcal{P})$ *be an oriented 3-manifold with generalized corners. Then* $Z(M, \mathcal{P}) \in \mathrm{im}(\phi_{\partial M;\mathcal{P},\mathcal{P}})$

*Proof.* Analogous to the proof of Proposition 3.14. ∎

**Proposition 3.19.** *If $x \in Z(\Sigma)$ and $y \in Z(-\Sigma)$, then*

$$\langle x, y\rangle = \langle \delta_{\mathcal{P}} \circ \phi_{\Sigma;\mathcal{P}_\Sigma,\mathcal{P}}(x), \phi_{-\Sigma;\mathcal{P}_\Sigma,\mathcal{P}}(y)\rangle$$

*Proof.* Identical to the proof of Proposition 3.15. □

**Proposition 3.20.** *If $Z$ is a 3d TQFT, $\Sigma$ an oriented surface with corners, and $f : \Sigma \to \Sigma$ is a diffeomorphism isotopic to the identity rel boundary, then $f_* = \mathrm{id}_{Z(\Sigma)}$.*

*Proof.* Suppose that $\Sigma$ is closed. Let $h : \Delta^1 \times \Sigma \to \Sigma$ be such that $h_0 = \mathrm{id}_\Sigma$ and $h_1 = f$. Let $F : \Delta^1 \times \Sigma \to \Delta^1 \times \Sigma$ be $F(t, x) = (t, h(t, x))$. Then $F|_\partial$ is $\mathrm{id}_\Sigma \sqcup f$. Also $Z(\Delta^1 \times \Sigma) = (F|_\partial)_* Z(\Delta^1 \times \Sigma)$. On the left is $\mathrm{id}_{Z(\Sigma)}$ in $Z(-\Sigma) \otimes Z(\Sigma) \cong Z(\Sigma)^* \otimes Z(\Sigma)$. On the right is $f$ in $Z(-\Sigma) \otimes Z(\Sigma) \cong Z(\Sigma)^* \otimes Z(\Sigma)$.

If $\Sigma$ has boundary, the same proof works except collapse the components of $\Delta^1 \times \partial\Sigma$ to $\partial\Sigma$. □

**Corollary 3.21.** *The action of a $\mathrm{Diff}^+(\Sigma)$ on $Z(\Sigma)$ descends to an action of the mapping class group of $\Sigma$.*

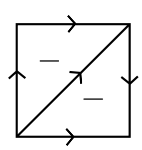
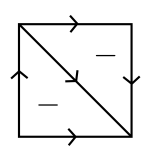
# 4 Algebraic Properties of 2d pre-TQFTs

In this section fix $Z$ a 2d extended pre-TQFT. This section discusses the natural algebraic structures that arise from the 2d extended pre-TQFT axioms. For example, $Z(+\Delta^1)$ turns out to be a Frobenius algebra with multiplication given by $Z(-\Delta^2)$. The section concludes with a discussion of the 2d finite group pre-TQFT.

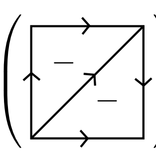
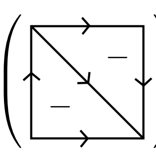
Write $A = Z(+\Delta^1)$. Use the nondegenerate pairing from the TQFT to identify $Z(-\Delta^1)$ with $A^*$. By definition $Z(\mathsf{B})$ is the identity map in $A^* \otimes A \cong \mathrm{End}(A)$.

To simplify notation, write $Z(\pm\Delta^2)$ for $Z(\pm\Delta^2, \mathcal{P})$ where $\mathcal{P}$ is the standard decomposition of the boundary of $\Delta^2$ into three 1-simplices.

$Z(-\Delta^2)$ is an element in $A^* \otimes A^* \otimes A$ and thus defines a multiplication in $A$. In order to make things consistent, use the regular ordering of edges from Section 2 to order the tensor factors of $Z(-\Delta^2)$: [12], [01], [02]. This multiplication is associative: since

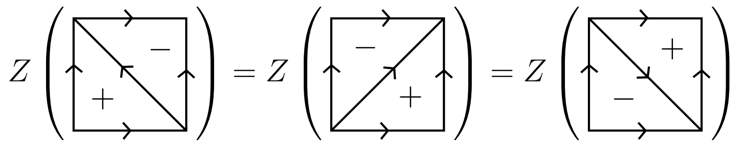
 and  (6)

glue together to give the same oriented 2-manifold with corners then the gluing law of the pre-TQFT implies

$Z$ (  ) $= Z$ (  ). (7)

Similarly $Z(+\Delta^2)$ defines a coassociative comultiplication in $A^* \otimes A \otimes A$. Again use the ordering of the edges from Section 2 to order the tensor factors.

There's a compatibility condition between the multiplication and comultiplication:



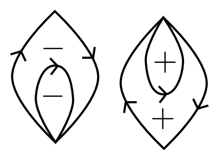

(8)

Furthermore $Z(\mathsf{D})$ is a unit for the algebra and $Z(-\mathsf{D})$ is a counit. This can be seen by gluing $\mathsf{D}$ to $-\Delta^2$ to create a copy of $\mathsf{B}$: $Z(\mathsf{B})$ is the identity map $A \to A$ and, by the gluing law, plugging $Z(\mathsf{D})$ into one of the two multiplication inputs produces $Z(\mathsf{B})$. Reversing the orientations shows that $Z(-\mathsf{D})$ is a counit.

**Proposition 4.1.** *$A$ is a Frobenius algebra with multiplication $Z(-\Delta^2)$, comultiplication $Z(+\Delta^2)$, unit $Z(\mathsf{D})$ and counit $Z(-\mathsf{D})$.*

*Proof.* This follows from (6), (7), and (8). □

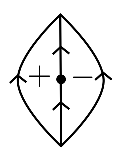
Every Frobenius algebra has a pairing given by composing the multiplication with the counit and a copairing given composing the unit with the comultiplication. These are inverses of each other. This can be seen by gluing




(9)

Each bigon in (9) has a rotational symmetry. This implies that the Frobenius pairing is symmetric. Such a Frobenius algebra is called symmetric.

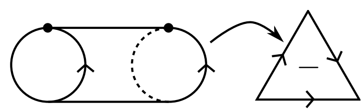
You can glue $\pm\Delta^2$ together to get a copy of $\mathsf{B}$:



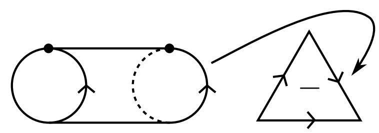

By the gluing law, the composition $A \to A \otimes A \to A$ of comultiplication then multiplication is scalar multiplication by $\Gamma$. A Frobenius algebra with such a condition is called special. Special symmetric Frobenius algebras produce 2d TQFTs via a construction called the state-sum construction [LP07]. The state-sum construction is as follows. Given a special symmetric Frobenius algebra $A'$, let $\Gamma'$ be the scalar element from the special condition. Given an oriented triangulation of a 2-manifold, assign the multiplication of $A'$ to each $-\Delta^2$, the comultiplication of $A'$ to each $+\Delta^2$, contract tensor factors corresponding to glued edges, and divide by $\Gamma'$ for each interior vertex. This precisely mirrors the gluing law for a pre-TQFT.

**Proposition 4.2.** *Let $\mathcal{P}$ be the decomposition of the circle $\mathsf{S}$ into a single copy of $+\Delta^1$. Then $\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}$, as defined in Definition 3.13, is the projection onto the center of $A$.*

*Proof.* Proposition 3.14 implies that $\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}$ is a projection. The image lies in the center because

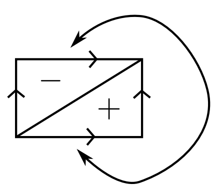


and



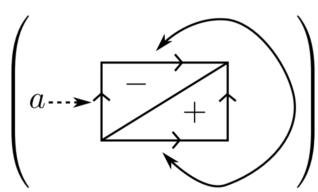
are the same manifold with boundary. If you apply $Z$ to the first one, you get the projection $\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}$ followed by right multiplication, and if you apply $Z$ to the second one you get the projection $\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}$ followed by left multiplication. Hence, for all $a, b \in A$, then $a\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}(b) = \phi_{\mathsf{S};\mathcal{P},\mathcal{P}}(b)a$.

It remains to show that if $a$ is in the center of $A$, then $\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}(a) = a$. The cylinder used in defining $\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}$ can be triangulated as



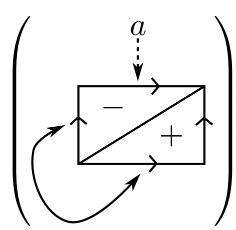
Then

$$\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}(a) = \frac{1}{\Gamma} Z$$




where the inclusion of $a$ next to an edge indicates that $a$ has been been contracted with the tensor factor correpsonding to that edge. Since $a$ is in the center then this is equal to

$$\frac{1}{\Gamma} Z$$




which, by the gluing law, is $a$ contracted with $Z(\mathsf{B})$. Since $Z(\mathsf{B})$ is the identity element in $A^* \otimes A$, then $\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}(a) = a$. ☐

**Corollary 4.3.** *Let $\mathcal{P}$ be the decomposition of $\mathsf{S}$ into a single copy of $\Delta^1$. Then $Z(\mathsf{S})$ is isomorphic to the center of $A = Z(\mathsf{S}, \mathcal{P})$. $Z(-\mathsf{S}, \mathcal{P})$ is isomorphic to the center of $A^* = Z(-\mathsf{S}, \mathcal{P})$, where the multiplication on $A^*$ is given by $Z(+\Delta^2)$.*

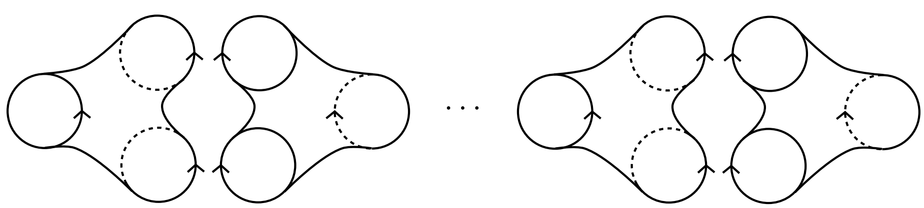
*Proof.* The second sentence follows from Propositions 4.2 and 3.14. The third follows by reversing the orientations. □

**Corollary 4.4.** *Let $\mathcal{P}$ be the decomposition of $\mathsf{S}$ into a single copy of $\Delta^1$. If you realize $Z(\mathsf{S}) \subset Z(\mathsf{S}, \mathcal{P})$ as the center of $A$ and $Z(-\mathsf{S}) \subset Z(-\mathsf{S}, \mathcal{P})$ as the center of $A^*$, then the pairing $Z(\mathsf{S}) \otimes Z(-\mathsf{S}) \to \mathbb{C}$ is the standard pairing $A \otimes A^* \to \mathbb{C}$ multiplied by $\Gamma^{-1}$.*

*Proof.* This is an application of Proposition 3.15, where $S = \mathsf{S}$ and $\mathcal{P}$ is the decomposition into a single piece that is a copy of $\Delta^1$. Then $\delta_{\mathcal{P}}$ is multiplication by $\Gamma^{-1}$. □

Let $\hat{Z}$ be the TQFT corresponding to $Z$. Then $\hat{Z}(\mathsf{S})$ can be realized as the center of a Frobenius algebra $A$. The identification $Z(-\mathsf{S}) \cong Z(\mathsf{S})^*$ picks up a factor of $\Gamma^{-1}$. With this factor, then $\hat{Z}(I \times \mathsf{S}, \mathrm{id}_{\mathsf{S}}, \mathrm{id}_{\mathsf{S}})$ becomes the map $\phi_{\mathsf{S};\mathcal{P}_{\mathsf{S}},\mathcal{P}_{\mathsf{S}}}$ and hence, the identity on $Z(\mathsf{S})$. Therefore the pair of pants with two inputs and one output is the multiplication on the center of $A$ and the pair of pants with one input and two outputs is the comultiplication on the center of $A$. So $\hat{Z}(S)$ is a commutative Frobenius algebra.

**Remark 4.5.** Given a commutative Frobenius algebra, one can find a basis $e_1, \ldots, e_N$ such that $e_i e_j = \delta_{i=j} e_i$ with comultiplication $e_i \mapsto \alpha_i e_i \otimes e_i$. Then a simple argument involving stringing several pairs of pants together



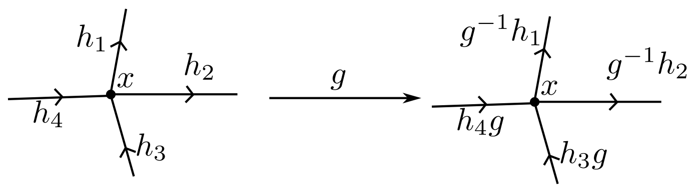
then gluing one end to the other shows that $\hat{Z}(\Sigma_g) = \sum_{i=1}^{N} \alpha_i^{g-1}$. In particular, $\hat{Z}(T^2)$ is the dimension of the center of $A$. In fact, Frobenius algebras over $\mathbb{C}$ are semisimple (the nondegenerate pairing precludes any nontrivial nilpotent ideals), and the $e_i$ can be taken to be the projections onto the simple factors of $A$.

**Remark 4.6.** If $\hat{Z}$ is a 3d TQFT, then $X \mapsto \hat{Z}(S^1 \times X)$ defines a 2d TQFT. In particular, if $\hat{Z}$ is a 3d TQFT then $\hat{Z}(S^1 \times S^1)$ is a commutative Frobenius algebra with multiplication given by $\hat{Z}(S^1 \times P)$ where $P$ is the pair of pants with two inputs identified with $S^1$ and one output identified with $S^1$.

**Example 4.7.** This is an example of a 2d extended pre-TQFT $Z_G$, where $G$ is a finite group. If $p$ is a finite subset of $X$, let $\pi_1(X,p)$ denote the fundamental groupoid of $X$ with respect to $p$. In particular, $\mathrm{Obj}(\pi_1(X,p)) = p$, and if $p$ is single point then $\pi_1(X,p)$ is the fundamental group. Let $\Sigma$ be a closed oriented surface and let $p$ be a finite subset of $\Sigma$ that intersects each component of $\Sigma$. Define

$$Z_G(\Sigma) = \frac{|\operatorname{Hom}(\pi_1(\Sigma,p),G)|}{|G|^{|p|}}.$$

The right hand side, it turns out, does not depend on $p$ as long as $p$ intersects every component of $\Sigma$. To see this, note that $G^p$ acts on $\operatorname{Hom}(\pi_1(\Sigma,p),G)$: an element in $\operatorname{Hom}(\pi_1(\Sigma,p),G)$ is given by a certain $G$-labeling of the homotopy classes of paths between points in $p$ and the factor of $G^p$ corresponding to $x \in p$ acts on this labeling by multiplying the labeling on incoming edges to $x$ by on the right by $g$ and outgoing edges on the left by $g^{-1}$:




The action is free at $x$ if $x$ is not the only point of $p$ in its connected component.

Note that $\operatorname{Hom}(\pi_1(\pm\Delta^1, \partial(\pm\Delta^1)), G)$ is canonically identified with $G$: given $g \in G$, take the unique homotopy class of path in the direction of the ordering of the vertices of $\pm\Delta^1$ and map it to $g$. If $S$ is an oriented interval, define $Z(S) = \mathbb{C}\operatorname{Hom}(\pi_1(S,\partial S),G)$. Let $\Sigma$ be an oriented surface with corners, each of whose components has nonempty boundary and such that each boundary component contains a vertex. Let $p$ denote the vertices. In particular, $\partial\Sigma$ is decomposed into a collection of pieces $\mathcal{P}$, and each piece is an oriented interval. Define

$$Z(\Sigma) := \sum_{\rho \in \operatorname{Hom}(\pi_1(\Sigma,p),G)} \rho|_{\partial\Sigma} \in \bigotimes_{P \in \mathcal{P}} Z(P).$$

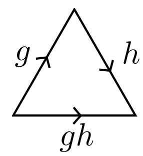
For example, if $\Sigma$ is the torus with a hole removed and a single vertex on the boundary then

$$Z(\Sigma) = \sum_{g,h \in G} ghg^{-1}h^{-1}.$$

Define a pairing $Z(S) \otimes Z(-S) \to \mathbb{C}$ so $\rho \otimes \rho' \mapsto 1$ if $\rho = \rho'$ and otherwise $\rho \otimes \rho' \mapsto 0$. Set $\Gamma = |G|$. Then the gluing law holds.

If $S$ is a circle, let $\mathcal{P}$ be the decomposition with a single vertex. Define $Z(S) = \operatorname{im} \phi_{S;\mathcal{P},\mathcal{P}}$. With this definition, $Z(\Sigma)$ can be defined on surfaces with boundary whose boundary components might not have vertices (see Proposition 3.14 part 4).

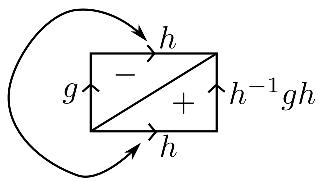
$Z(+\Delta^1)$ is an algebra with multiplication given by $Z(-\Delta^2)$. Identify $Z(+\Delta^1)$ with $\mathbb{C}G$ as above. Then $Z(-\Delta^2) \in (\mathbb{C}G)^* \otimes (\mathbb{C}G)^* \otimes \mathbb{C}G$ is the usual group algebra multiplication:




so that $Z(+\Delta^1)$, as a Frobenius algebra, is isomorphic to the group algebra of $G$. $Z(\mathsf{S})$ should be the center of $\mathbb{C}G$, which has a basis given by

$$\underline{\alpha} = \sum_{g \in \alpha} g$$

for each conjugacy class $\alpha$. By inspecting




$Z(\mathsf{S})$ is the image of the map (here $\mathcal{P}$ is the decomposition of $\mathsf{S}$ into $\Delta^1$)

$$\phi_{\mathsf{S};\mathcal{P},\mathcal{P}} : \mathbb{C}G \to \mathbb{C}G$$

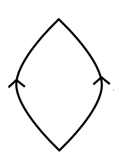
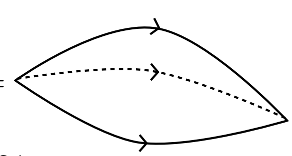
$$g \mapsto \frac{1}{|G|} \sum_{h \in G} h^{-1} g h$$

which projects $g$ onto $\underline{[g]}$ where $[g]$ is the conjugacy class of $g$. Notice that $Z(\Delta^1 \times \mathsf{S})$ is $\sum_{g,h} g \otimes \overline{h^{-1}gh}$ and that the factor of $\frac{1}{|G|}$ comes from the definition of $\phi_{\mathsf{S};\mathcal{P},\mathcal{P}}$.

# 5 Algebraic Properties of 3d pre-TQFTs

In this section let $Z$ be a fixed extended pre-TQFT. This section describes some algebraic consequences of the pre-TQFT axioms. For example, $Z(\mathsf{B})$ and $Z(\mathsf{C})$ become algebras and $Z$ applied to surfaces with (suitably decomposed) boundaries gives modules for these algebras. $Z(\Delta^2)$ provides a monoidal product on the category of right modules of $Z(\mathsf{B})$ and $Z(\Delta^3)$ provides an associator for this monoidal product. Similarly $Z(\mathsf{P})$ provides a monoidal product on the category of right modules of $Z(\mathsf{C})$.

Let $M_{\mathsf{B}}$ be the three-manifold obtained from $+\Delta^2 \times \Delta^1$ by collapsing each component of $\Delta^2 \times \partial\Delta^1$ to a point.

$\mathsf{B} =$ , $M_{\mathsf{B}} =$  (10)

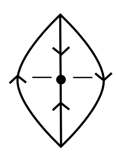
Note that $\partial M_{\mathsf{B}} = (-\mathsf{B}) \cup \mathsf{B} \cup \mathsf{B}$ and this provides a decomposition $\mathcal{P}_{M_{\mathsf{B}}}$. Each piece corresponds to an edge of $\Delta^2$. Identify $Z(-\mathsf{B})$ with $Z(\mathsf{B})^*$ via the nondegenerate pairing $Z(\mathsf{B}) \otimes Z(-\mathsf{B}) \to \mathbb{C}$. Then $Z(M_{\mathsf{B}}, \mathcal{P}_{M_{\mathsf{B}}}) \in Z(\mathsf{B})^* \otimes Z(\mathsf{B}) \otimes Z(\mathsf{B})$ defines a comultiplication on $Z(\mathsf{B})$. Here the first $Z(\mathsf{B})$ tensor factor corresponds to the face of $M_{\mathsf{B}}$ that correpsonds to edge $[01]$ in $\Delta^2$. Similarly, $Z(-M_{\mathsf{B}}, \mathcal{P}_{M_{\mathsf{B}}}) \in Z(\mathsf{B})^* \otimes Z(\mathsf{B})^* \otimes Z(\mathsf{B})$ defines a multiplication where the first $Z(\mathsf{B})^*$ is taken to correspond to the $[12]$ edge of $-\Delta^2$. An argument along the lines of Section 4 can show that this multiplication and comultiplication turns $Z(\mathsf{B})$ into a Frobenius algebra. In this case, one needs to take a little care about the fact that $Z(\mathsf{B})$ is graded by $I \times I$ and the extended pre-TQFT $Z$ involves the function $d : I \to \mathbb{C}$. This direction will not be pursued here. What is important, however, is the category of modules of $Z(\mathsf{B})$.

**Definition 5.1.** Considering $Z(\mathsf{B})$ as an algebra with multiplication $Z(-M_{\mathsf{B}}, \mathcal{P}_{M_{\mathsf{B}}})$, write $Z(\Delta^1)$ for the category of its right modules and $Z(-\Delta^1)$ be its category of left modules.

Identify $Z(-\mathsf{C})$ with $Z(\mathsf{C})^*$ using the nondenegerate pairing $Z(\mathsf{C}) \otimes Z(-\mathsf{C}) \to \mathbb{C}$. Let $M_{\mathsf{C}}$ be the oriented 3-manifold obtained from $\Delta^2 \times \Delta^1$ by identifying the two components of $\Delta^2 \times \partial\Delta^1$. Let $\mathcal{P}_{M_{\mathsf{C}}}$ be the obvious decomposition of $\partial M_{\mathsf{C}}$ into three cylinders. Then $Z(M_{\mathsf{C}}, \mathcal{P}_{M_{\mathsf{C}}})$ is in $Z(\mathsf{C})^* \otimes Z(\mathsf{C}) \otimes Z(\mathsf{C})$, where the first $Z(\mathsf{C})$ corresponds to the $[01]$ edge of $\Delta^2$. And $Z(-M_{\mathsf{C}}, \mathcal{P}_{M_{\mathsf{C}}})$ is in $Z(\mathsf{C})^* \otimes Z(\mathsf{C})^* \otimes Z(\mathsf{C})$, where the first $Z(\mathsf{C})^*$ factor corresponds to the $[12]$ edge of $-\Delta^2$. $Z(M_{\mathsf{C}}, \mathcal{P}_{M_{\mathsf{C}}})$ and $Z(-M_{\mathsf{C}}, \mathcal{P}_{M_{\mathsf{C}}})$ define a Frobenius algebra structure just as for $Z(\mathsf{B})$.

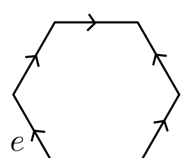
**Definition 5.2.** Considering $Z(\mathsf{C})$ as an algebra with multiplication $Z(-M_{\mathsf{C}})$, write $Z(\mathsf{S})$ for the category of right modules of $Z(\mathsf{C})$ and $Z(-\mathsf{S})$ for the category of left modules of $Z(\mathsf{C})$.

**Definition 5.3.** Consider the following oriented complex:

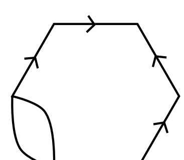
$K =$ 

Let $M'_{\mathsf{B}}$ be $K \times \Delta^1$ with the components of $K \times \partial\Delta^1$ collapsed to points. Then $\partial M'_{\mathsf{B}} = (-\mathsf{B}) \cup (-\mathsf{B})$ Let $M'_{\mathsf{C}}$ be $K \times \Delta^1$ with the components of $K \times \partial\Delta^1$ identified. Then $\partial M'_{\mathsf{C}} = (-\mathsf{C}) \cup (-\mathsf{C})$.

**Proposition 5.4.** *Let $\Sigma$ be an oriented surface with boundary parametrized by copies of $\pm\Delta^1$. Let $e$ be a copy of $+\Delta^1$ in $\partial\Sigma$ and let $M(e)$ denote the 3-manifold obtained from $\Delta^1 \times \Sigma$ by collapsing all of the boundary except $e$:*



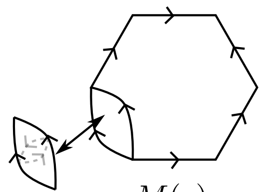

$\Sigma$



$M(e)$

*Then $Z(M(e)) \in Z(\mathsf{B})^* \otimes Z(\Sigma)^* \otimes Z(\Sigma)$ defines a right action of $Z(\mathsf{B})$ on $Z(\Sigma)$.*

*Let $e$ be a copy of $-\Delta^1$ in $\partial\Sigma$ and let $M(e)$ denote the 3-manifold obtained from $\Delta^1 \times \Sigma$ by collapsing all of the boundary except $e$ and gluing on a copy of $M'_{\mathsf{B}}$:*

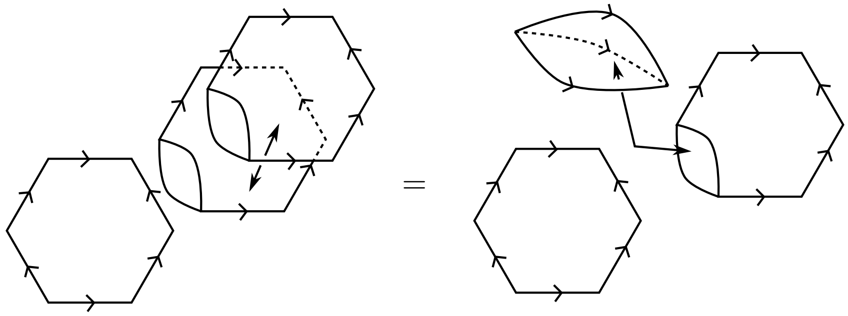


$M(e)$

*Then $Z(M(e)) \in Z(\Sigma)^* \otimes Z(\mathsf{B})^* \otimes Z(\Sigma)$ defines a left action of $Z(\mathsf{B})$ on $Z(\Sigma)$.*

**Remark 5.5.** The purpose of gluing in $M'_{\mathsf{B}}$ is to change from a right $Z(\mathsf{B})^*$ action to a left $Z(\mathsf{B})$ action.

*Proof of Proposition 5.4.* For case of $e = +\Delta^1$, the statement follows by applying $Z$ to both sides of the equality:



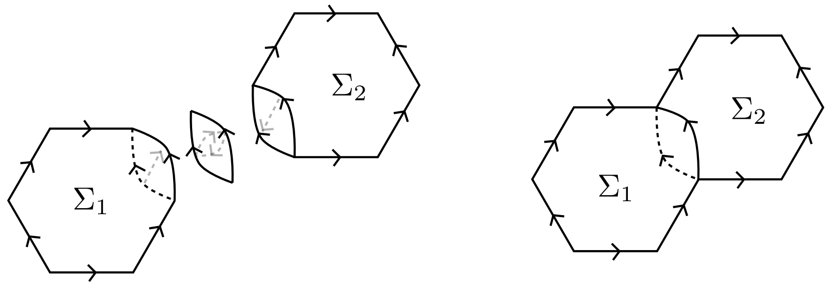
The left side represents acting on the right by $a$ then $b$. The right side represents multiplying $a$ and $b$ then acting by the result.

The case of $e = -\Delta^1$ works by a similar picture.

□

**Proposition 5.6.** *Let $\Sigma_1$ and $\Sigma_2$ be two surfaces with boundaries parametrized by copies of $\pm\Delta^1$. Let $\Sigma$ be the result of gluing $\Sigma_1$ and $\Sigma_2$ along a copy of $+\Delta^1$ in $\partial\Sigma_1$ and a copy of $-\Delta^1$ in $\partial\Sigma_2$. Let $\mathcal{P}$ be the splitting of $\Sigma$ into pieces $\Sigma_1$ and $\Sigma_2$. Then the image of $\phi_{\Sigma;\mathcal{P},\mathcal{P}}$ (which by Proposition 3.18 is isomorphic to $Z(\Sigma)$) is isomorphic to $Z(\Sigma_1) \otimes_{Z(\mathsf{B})} Z(\Sigma_2)$ where the action of $Z(\mathsf{B})$ on each factor is via the action on each of the glued edges.*

*Proof.* The left 3-manifold glues together to form the right 3-manifold:



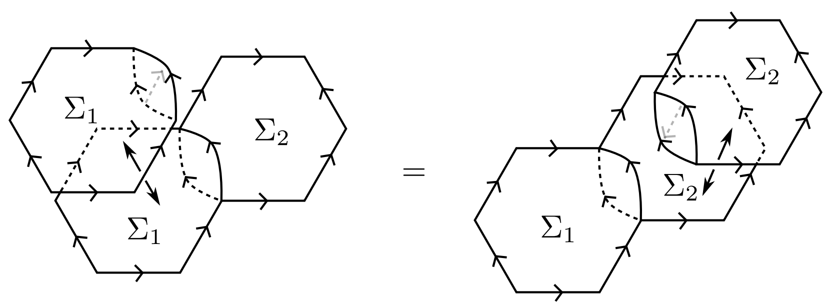

Here the piece in the middle on the left is $+M'_{\mathsf{B}}$. The gray arrows indicate the parametrization of that face by $\pm\mathsf{B}$, where the arrow points from the negative side to the positive side on $+\mathsf{B}$ and from the positive side to the negative side on $-\mathsf{B}$. From these arrows it is clear that $\Sigma_1$ has a right action by $\mathsf{B}$ and $\Sigma_2$ has a left action.

Let $M$ be the 3-manifold on the right. Then

$$Z(M) \in Z(\Sigma_2)^* \otimes Z(\Sigma_1)^* \otimes Z(\Sigma_1) \otimes Z(\Sigma_2),$$

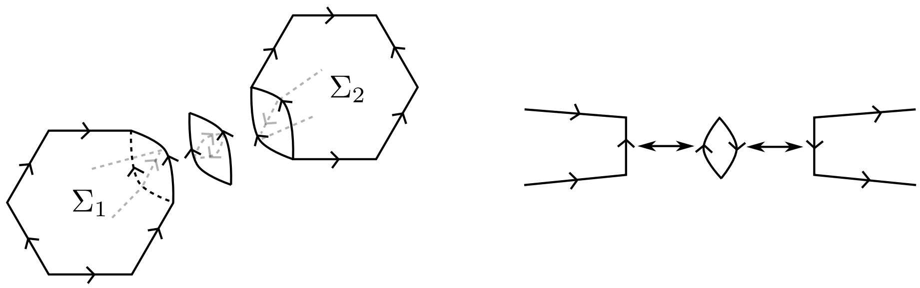
so $Z(M)$ can be viewed as a map $Z(\Sigma_1) \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2)$. In fact, $\phi_{\Sigma;\mathcal{P},\mathcal{P}}$ is this map precomposed by $\delta_{\mathcal{P}}$.

Identify $Z(\Sigma_1) \otimes_{Z(\mathsf{B})} Z(\Sigma_2)$ with the subset of $Z(\Sigma_1) \otimes Z(\Sigma_2)$ consisting of elements $\sum_k x_k \otimes y_k$ such that $\sum_k x_k a \otimes y_k = \sum_k x_k \otimes a y_k$ for all $a \in Z(\mathsf{B})$. To see that the image of $\phi_{\Sigma;\mathcal{P},\mathcal{P}}$ is contained in $Z(\Sigma_1) \otimes_{Z(\mathsf{B})} Z(\Sigma_2)$ it is enough to apply $Z$ to both sides of the equality:

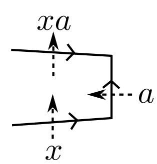



Up to the invertible map $\delta_{\mathcal{P}}$, the left represents $\phi_{\Sigma;\mathcal{P},\mathcal{P}}$ followed by the right action of $Z(\mathsf{B})$ and the right represents $\phi_{\Sigma;\mathcal{P},\mathcal{P}}$ followed by the left action. Therefore on $\operatorname{im} \phi_{\Sigma;\mathcal{P},\mathcal{P}}$, these two actions are the same.

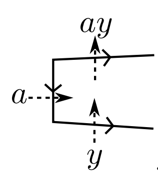
To see that that $Z(\Sigma_1) \otimes_{Z(\mathsf{B})} Z(\Sigma_2)$ is contained in $\operatorname{im} \phi_{\Sigma;\mathcal{P},\mathcal{P}}$, it is enough to show that $\delta_{\mathcal{P}}(\phi(v)) = \delta_{\mathcal{P}}(v)$ if $v \in Z(\Sigma_1) \otimes_{Z(\mathsf{B})} Z(\Sigma_2)$. It will be helpful to take the following cross section of $M$:



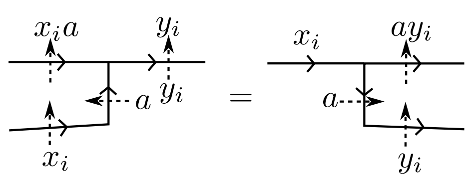

The right action of $a$ on $x \in Z(\Sigma_1)$ can be indicated



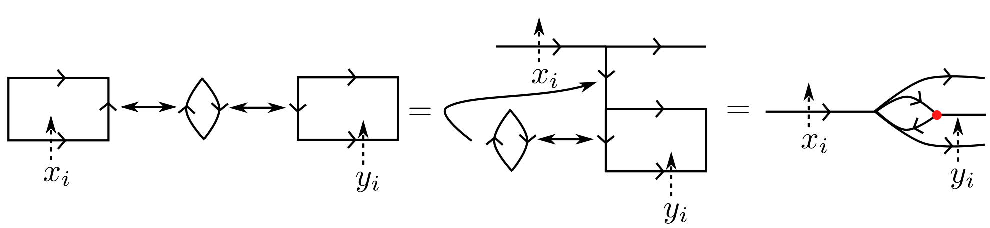

and the left action of $a$ on $y \in Z(\Sigma_2)$ can be indicated




Then the statement that $\sum_i x_i a \otimes y_i = \sum_i x_i \otimes a y_i$ can be indicated (summation sign dropped):




Suppose that $\sum_i x_i a \otimes y_i = \sum_i x_i \otimes a y_i$. Then if you plug $\sum_i x_i \otimes y_i$ into $Z(M)$ (thought of as a map $Z(\Sigma_1) \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2)$) you get




Now the right side is almost the identity map $\sum_i x_i \otimes y_i \to \sum_i x_i \otimes y_i$. It would be except for a factor involving $d$ that comes from the internal edge (labeled by the red dot). This factor is precisely the same one that comes post-composing with $\delta$. Hence, on $Z(\Sigma_1) \otimes_{Z(\mathsf{B})} Z(\Sigma_2)$, $\delta_{\mathcal{P}} \circ Z(M)$ acts as the identity. Hence $\delta_{\mathcal{P}} \circ Z(M) \circ \delta_{\mathcal{P}} = \delta_{\mathcal{P}} \circ \phi_{\Sigma;\mathcal{P},\mathcal{P}}$ acts as $\delta_{\mathcal{P}}$. Hence $\phi_{\Sigma;\mathcal{P},\mathcal{P}}$ acts as the identity. $\square$

There are analogues of Propositions 5.4 and 5.6 for $Z(\mathsf{C})$:

**Proposition 5.7.** *Let $\Sigma$ be a surface with boundary parametrized by copies of $\pm\Delta^1$.*

*If $S$ is a circle in $\partial\Sigma$ given by a copy of $+\Delta^1$ with its endpoints identified, let $M(S)$ denote the 3-manifold obtained from $\Delta^1 \times \Sigma$ by collapsing all of the boundary except $S$. Then $Z(M(S)) \in Z(\mathsf{C})^* \otimes Z(\Sigma)^* \otimes Z(\Sigma)$ defines a right action of $Z(\mathsf{C})$ on $Z(\Sigma)$.*

*If $S$ is a circle in $\partial\Sigma$ given by a copy of $-\Delta^1$ with its endpoints identified, let $M(S)$ denote the 3-manifold obtained from $\Delta^1 \times \Sigma$ by collapsing all of the boundary except $S$ and gluing on a copy of $M'_{\mathsf{C}}$. Then $Z(M(S)) \in Z(\Sigma)^* \otimes Z(\mathsf{C})^* \otimes Z(\Sigma)$ defines a left action action of $Z(\mathsf{C})$ on $Z(\Sigma)$.*

*Proof.* Analogous to the proof of Proposition 5.4. □

**Proposition 5.8.** *Let $\Sigma_1$ and $\Sigma_2$ be two surfaces with boundaries parametrized by copies of $\pm\Delta^1$. Let $S \subset \Sigma_1$ be a circle parametrized by a single copy of $+\Delta^1$. Let $S' \subset \Sigma_2$ be a circle parametrized by a single copy of $-\Delta^1$. Let $\Sigma$ be the result of gluing $\Sigma_1$ and $\Sigma_2$ together by identifying $S$ with $S'$. Let $\mathcal{P}$ be the splitting of $\Sigma$ into pieces $\Sigma_1$ and $\Sigma_2$. Then the image of $\phi_{\Sigma;\mathcal{P},\mathcal{P}}$ (which by Proposition 3.18 is isomorphic to $Z(\Sigma)$) is isomorphic to $Z(\Sigma_1) \otimes_{Z(\mathsf{C})} Z(\Sigma_2)$ where the action of $Z(\mathsf{C})$ on each factor is via the actions corresponding to $S$ and $S'$, respectively.*

*Proof.* Analogous to the proof of Proposition 5.6. □

**Example 5.9.** $Z(-\Delta^2)$ can be considered as left $Z(\mathsf{B})$-module from edge [12], a left $Z(\mathsf{B})$-module from edge [01], and a right $Z(\mathsf{B})$-module from edge [02]. Hence you can write[2]

$$Z(-\Delta^2) \in \big({}_{Z(\mathsf{B})}\,\mathrm{Mod}\big)\boxtimes\big({}_{Z(\mathsf{B})}\,\mathrm{Mod}\big)\boxtimes\big(\mathrm{Mod}_{Z(\mathsf{B})}\big) = Z(-\Delta^1)\boxtimes Z(-\Delta^1)\boxtimes Z(+\Delta^1)$$

Here "$\in$" means "is an object of". This justifies the definitions of $Z(\pm\Delta^1)$.

**Example 5.10.**

$$Z(\mathsf{P}) \in \big({}_{Z(\mathsf{C})}\,\mathrm{Mod}\big) \boxtimes \big({}_{Z(\mathsf{C}))}\,\mathrm{Mod}\big) \boxtimes \big(\mathrm{Mod}_{Z(\mathsf{C})}\big) = Z(-\mathsf{S}) \boxtimes Z(-\mathsf{S}) \boxtimes Z(\mathsf{S})$$

You could also consider $\mathsf{P}$ a module for $Z(\mathsf{B})$ corresponding to any of the three boundary components since these components are written in terms of $\pm\Delta^1$, but this won't be done.

**Definition 5.11.** Let $\mathcal{V}$ be the category of finite-dimensional complex vector spaces. Given an algebra $A$ define a pairing $\langle , \rangle : \mathrm{Mod}_A \boxtimes_A \mathrm{Mod} \to \mathcal{V}$ by $(X, Y) \mapsto X \otimes_A Y$. Applying this pairing will be called a "contraction" on the two categories.

[2] $\boxtimes$ is the Deligne tensor product of linear categories. See [EGNO15] 1.11 for a definition.

**Remark 5.12.** Propositions 5.6 and 5.8 form a sort of "gluing law" for surfaces in the extended 3d pre-TQFT $Z$. For example, $Z$ applied to $\mathsf{P} \sqcup \mathsf{P}$ is an object in a (Deligne) tensor product of six linear categories:

$$Z(\mathsf{P} \sqcup \mathsf{P}) \in (Z(\mathsf{S}))^{\boxtimes 4} \boxtimes (Z(-\mathsf{S}))^{\boxtimes 2}$$

If you glue two oppositely oriented circles of $\partial(\mathsf{P} \sqcup \mathsf{P})$ together to form a four-holed sphere $Q$, then

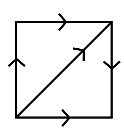
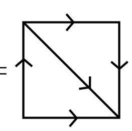
$$Z(Q) \in (Z(\mathsf{S}))^{\boxtimes 3} \boxtimes Z(-\mathsf{S})$$

is (up to isomorphism) obtained from $Z(\mathsf{P} \sqcup \mathsf{P})$ by contracting along two the tensor factors.

The pairing in Definition 5.11 defines a functor from ${}_A\operatorname{Mod} \to \operatorname{Func}(\operatorname{Mod}_A, \mathcal{V})$ given by $Y \mapsto (X \mapsto X \otimes_A Y)$. After making this identification, then $Z(-\Delta^2)$ is a functor $Z(\Delta^1) \boxtimes Z(\Delta^1) \to Z(\Delta^1)$ and $Z(\mathsf{P})$ is a functor $Z(\mathsf{S}) \boxtimes Z(\mathsf{S}) \to Z(\mathsf{S})$. It is also easy to see that, viewed as an endofunctor of $Z(\Delta^1)$, $Z(\mathsf{B})$ is the functor $X \mapsto X \otimes_{Z(\mathsf{B})} Z(\mathsf{B})$ and hence is canonically isomorphic to the identity functor. Similarly $Z(\mathsf{C})$ can be viewed as an endofunctor of $Z(\mathsf{S})$ canonically isomorphic to identity functor.

Write $Z(\pm\Delta^3)$ for $Z(\pm\Delta^3, \mathcal{P})$ where $\mathcal{P}$ is the standard decomposition of $\partial\Delta^3$ into four 2-simplices.

**Proposition 5.13.** *The boundary of the 3-simplex leads to two different decompositions of the square:*

$\mathcal{P}_1 =$  $\qquad \mathcal{P}_2 =$ 

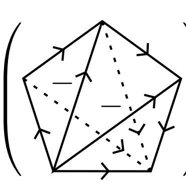
*If $\Sigma$ is the square let $\alpha := \phi_{\Sigma;\mathcal{P}_1,\mathcal{P}_2}$. Note that $\alpha$ is $Z(-\Delta^3) \circ \delta_{\mathcal{P}_1}$. Write $X \odot Y$ for $Z(-\Delta^2)$ applied to $X \boxtimes Y \in Z(\Delta^1) \boxtimes Z(\Delta^1)$. Then $\odot$ is a monoidal product on $Z(\Delta^1)$ with associator $\alpha$ and unit $Z(\mathsf{D})$.*

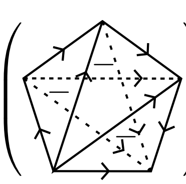
*Proof.* The associator needs to be such that the following diagram commutes (here $XY$ means $X \odot Y$:

$$\begin{array}{ccccc} ((XY)Z)W & \longrightarrow & (XY)(ZW) & \longrightarrow & X(Y(ZW)) \\ & \searrow & & \nearrow & \\ & (X(YZ)W) & \longrightarrow & X((YZ)W) & \end{array} \tag{11}$$

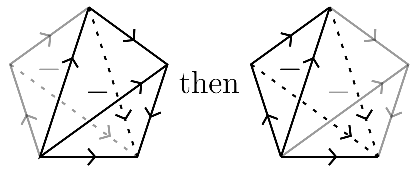
where here the arrows are all induced by the associator. This diagram commutes because of the following identity:

$$Z\left( \text{ } \right) = Z\left( \text{ } \right). \tag{12}$$

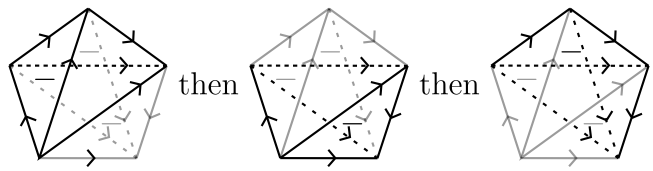




where the left side is the composition

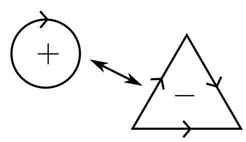



and the right side is the composition of




These are precisely the top and bottom of (11).

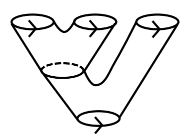
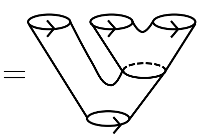
Proposition 5.6 and the following decomposition of the bigon:



imply that $Z(\mathsf{D})$ is a left unit. Moving the disk to the other side implies that $Z(\mathsf{D})$ is also a right unit. $\square$

Similarly:

**Proposition 5.14.** *Two different decompositions of the four-holed sphere can be obtained by gluing two copies of* $\mathsf{P}$ *in two different ways:*

$\mathcal{P}_1 =$  $\mathcal{P}_2 =$ 

*If* $\Sigma$ *is the four-holed sphere, let* $\alpha := \phi_{\Sigma;\mathcal{P}_1,\mathcal{P}_2}$. *Let* $X \odot Y$ *denote* $X \boxtimes Y \in Z(\mathsf{S}) \boxtimes Z(\mathsf{S})$ *applied to* $Z(\mathsf{P})$. *Then* $\odot$ *forms a monoidal product on* $Z(\mathsf{S})$ *with associator* $\alpha$ *and unit* $Z(\mathsf{D})$.

*Proof.* The proof is analogous to that of Proposition 5.13. $\square$

**Proposition 5.15.** *With the notation from Proposition 5.14,* $Z(\mathsf{S})$ *is a braided monoidal category.*

*Proof.* The braiding is a natural transformation from $\odot$ to $\odot \circ \sigma$ where $\sigma : Z(\mathsf{S}) \boxtimes Z(\mathsf{S}) \to Z(\mathsf{S}) \boxtimes Z(\mathsf{S})$ is the functor which switches the factors. Since $Z(\mathsf{P})$ realizes $\odot$, such a braiding can be realized by an appropriate linear map $Z(\mathsf{P}) \to Z(\mathsf{P})$. Fix a diffeomorphism $f : \mathsf{P} \to \mathsf{P}$ that switches the two legs of $\mathsf{P}$ corresponding to the inputs of $\odot$ while keeping the third leg fixed. Corollary 3.21 ensures that the linear map $f_*$ does not depend on the choice of such an $f$. Then $f_*$ is desired braiding. That it satisfies the braid relations is a consequence of the fact that it fits into an action of the mapping class group of $\mathsf{P}$. $\square$

**Remark 5.16.** The category $Z(\Delta^1)$ also seems to be a rigid category. Let $B'$ be the bigon obtained by reversing the parametrization on the negative side of $\mathsf{B}$, so $Z(B') \in Z(-\Delta^1) \boxtimes Z(-\Delta^1)$. Then $Z(B')$ defines a functor $Z(\Delta^1) \to Z(-\Delta^1)$, i.e., a functor $\mathrm{Mod}_{Z(\mathsf{B})} \to {}_{Z(\mathsf{B})}\mathrm{Mod}$. Any left module can be considered as a right module by taking its linear dual. Therefore $X \mapsto (Z(B')(X))^*$ is an endofunctor of $Z(\Delta^1)$. It seems like $(Z(B')(X))^*$ is a dual to $X$ as a right $Z(\mathsf{B})$-module, though the details are a little complicated to sort out. In examples applicable to this paper rigidity can be checked by hand, so a proof of rigidity of $Z(\Delta^1)$ is not included here. An analogous construction seems to show that $\mathrm{Mod}_{Z(\mathsf{C})}$ is also rigid.

The braiding in Proposition 5.15 represents an important algebraic relation in the category $Z(\mathsf{S})$ derived from the action of a mapping class. Here are two others:

**Definition 5.17.** Recall that $Z(\mathsf{C})$ can be interpreted as an endofunctor of $Z(\mathsf{S})$ canonically isomorphic to the identity. The Dehn twist on the cylinder $\mathsf{C}$ induces a map $Z(\mathsf{C})$ that gives an automorphism of this functor and hence an automorphism of the identity endofunctor. Call this map $\theta$.

**Definition 5.18.** Suppose that the center of $Z(\mathsf{C})$ is spanned by idempotents that project onto simple modules of $Z(\mathsf{C})$. Since $X \mapsto Z(\mathsf{S} \times X)$ forms a 2d pre-TQFT, then Corollary 4.3 implies that $Z(\mathsf{S} \times \mathsf{S})$ is isomorphic to the center of $Z(\mathsf{S})$ and so has a basis that corresponds to the simple projectors. Call this basis $(\pi_1, \ldots, \pi_n)$. The Frobenius algebra structure also implies a nondegenerate pairing $Z(\mathsf{S} \times \mathsf{S}) \otimes Z(\mathsf{S} \times \mathsf{S}) \to \mathbb{C}$ such that $\pi_i \otimes \pi_j \mapsto 0$ if $i \neq j$. Let $(\omega_1, \ldots, \omega_n)$ be the basis $(\pi_1, \ldots, \pi_n)$ rescaled to be orthonormal.

Let $\mathcal{S} : \mathsf{S} \times \mathsf{S} \to \mathsf{S} \times \mathsf{S}$ be the torus automorphism

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and let $\mathcal{S}_* : Z(\mathsf{S} \times \mathsf{S}) \to Z(\mathsf{S} \times \mathsf{S})$ be the induced map. of the torus. Let $(\mathcal{S}_i^j)$ denote its matrix representation with respect to the basis $(\omega_1, \ldots, \omega_n)$

**Proposition 5.19.** [3] *The rigid braided monoidal category $Z(\mathsf{S})$ equipped with twist $\theta$ and $S$-matrix $(\mathcal{S}_i^j)$ is a modular tensor category.*

*Proof.* This result will not be needed later and details are left to the interested reader. □

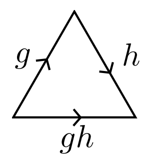
**Example 5.20.** This is an example of a 3d extended pre-TQFT $Z_G$, where $G$ is a finite group. This is analogous to the TQFT $Z_G$ defined in Example 4.7. The same notation $Z_G$ is used for the 3d version for simplicity.

[3]This is only true modulo questions about rigidity, as mentioned in Remark 5.16.

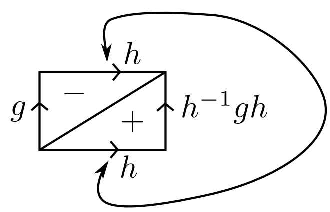
Let $M$ be a closed oriented 3-manifold and let $p$ be a finite subset of $M$ that intersects every component of $M$. As in Example 4.7, set

$$Z_G(M) := \frac{|\operatorname{Hom}(\pi_1(\Sigma, p), G)|}{|G|^{|p|}}.$$

Let $\Sigma$ be an oriented 2-manifold whose boundary is parametrized by copies of $\pm\Delta^1$ and such that every component of $\Sigma$ has nonempty boundary. Let $p$ denote the vertices, and set $Z(\Sigma) := \mathbb{C}\operatorname{Hom}(\pi_1(\Sigma, p), G)$. Given $\rho \in \operatorname{Hom}(\pi_1(\Sigma, p), G)$, each copy of $\pm\Delta^1$ determines an element of $G$ by restriction of $\rho$. For example, on $\Delta^2$ a representation $\rho$ might restrict like

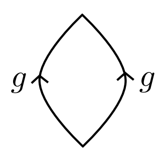



This restriction provides the grading function $\ell : \{\text{intervals in } \partial\Sigma\} \to G$ used in the definition of the pre-TQFT. For example, $\operatorname{Hom}(\pi_1(\mathsf{C}, p), G) \cong G \times G$:




where $p$ is the set of two vertices on the endpoints. From the picture, one can see that $\rho \in \operatorname{Hom}(\pi_1(\mathsf{C}, p), G)$ is determined by its restriction to one boundary and to the longitudinal curve going across the cylinder. The representation $\rho$ pictured is in grading $(g, h^{-1}gh)$.

As a simpler example, consider the bigon $\mathsf{B}$ and let $p$ be its two vertices. Then $\operatorname{Hom}(\pi_1(\mathsf{B}, p), G) \cong G$, determined by the $G$-labeling on one of its boundary components (they're both labeled by the same element):

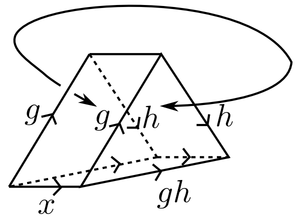



Define the pairing $Z(\Sigma) \otimes Z(-\Sigma) \to \mathbb{C}$ on the natural basis as $\rho \otimes \rho' \mapsto 1$ if $\rho = \rho'$ and $\rho \otimes \rho' \mapsto 0$ otherwise. Set $d : G \to \mathbb{C}$ to be the constant function 1 and set $\Gamma = |G|$.

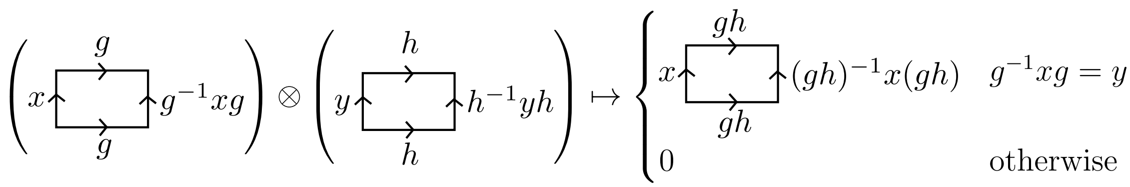
Let $M$ be a 3-manifold with boundary decomposed into a set of pieces $\mathcal{P}$. Suppose that each component of $M$ has nonempty boundary and that each piece of $\mathcal{P}$ has nonempty boundary. Let $p$ be the set of vertices. Then set

$$Z(M, \mathcal{P}) := \sum_{\rho \in \operatorname{Hom}(\pi_1(M,p),G)} \rho|_{\partial M} \in \bigotimes_{P \in \mathcal{P}} Z(P) = Z(\partial M, \mathcal{P})$$

For example, for $-\Delta^2 \times \mathsf{S}$ each term in the sum corresponds to one of the following $G$-labelings:




leading to a multiplication on the algebra $Z(\mathsf{C})$ of the form

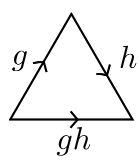



This algebra is called the Drinfeld double of $G$.

It is easy to see that the algebra structure on $Z(\mathsf{B})$ is isomorphic to the commutative algebra of functions on $G$. Its modules are therefore vector spaces graded over $G$. The multiplication on $G$ induces a monoidal product on the category of $G$-graded vector spaces:

$$\mathbb{C}_g \boxtimes \mathbb{C}_h \mapsto \mathbb{C}_{gh}.$$

Since $Z(-\Delta^2)$ has a basis consisting of the the following elements




it is not hard to see that the monoidal product on $Z(\Delta^1)$ is the usual monoidal product on $G$-graded vector spaces.

For a closed connected surface $\Sigma$, write $\Sigma = \Sigma_1 \cup \Sigma_2$ where $\Sigma_2$ is small closed disk in $\Sigma$ and $\Sigma_1$ is the closure of the complement. Let $\mathcal{P}$ be this decomposition of $\Sigma$. Then set $Z(\Sigma) = \operatorname{im}(\phi_{\Sigma;\mathcal{P},\mathcal{P}})$. It is not hard to see that $Z(\Sigma_1) \cong \mathbb{C}\operatorname{Hom}(\pi_1(\Sigma_1, p), G)$ where $p$ is the point on the boundary. The conjugation action of $G$ induces a $G$-module structure on $\mathbb{C}\operatorname{Hom}(\pi_1(\Sigma, p), G)$ and $\phi_{\Sigma,\mathcal{P},\mathcal{P}}$ projects onto the trivial submodule. Therefore $Z(\Sigma)$ is this trivial submodule. Alternatively, $Z(\Sigma) \cong \mathbb{C}\operatorname{Hom}(\pi_1(\Sigma, p), G)/G$.

One could in fact take any other decomposition $\mathcal{P}'$ of $\Sigma$, since $\phi_{\Sigma;\mathcal{P},\mathcal{P}'}$ canonically identifies $\operatorname{im}\phi_{\Sigma;\mathcal{P},\mathcal{P}}$ with $\operatorname{im}\phi_{\Sigma;\mathcal{P}',\mathcal{P}'}$.

# 6 Spin and Pin$^-$ Structures

This section recalls some of the basics of Spin and Pin$^-$ structures on manifolds of dimension less than 3. In particular, it discusses explicit constructions of

Spin structures on some important manifolds and discusses various classical Spin and $\mathrm{Pin}^-$ invariants.

Over a space $X$, let $\underline{F}$ denote the trivial $F$-bundle $X \times F$. The base $X$ should be clear from context.

**Definition 6.1.** Let $E$ be an oriented rank 3 real vector bundle with a fiber metric over a space $X$. Let $F(E)$ be the oriented frame bundle of $E$. A Spin lift of $F(E)$ is a $\mathrm{Spin}(3)$-bundle $P \to F(E)$ which restricts to a nontrivial covering map on each fiber. In particular, a Spin lift implies a factorization $X \to B\,\mathrm{Spin}(3) \to B\,\mathrm{SO}(3)$ of the classifying map of the frame bundle. Two Spin lifts are isomorphic if the classifying maps $X \to B\,\mathrm{Spin}(3)$ are homotopic through homotopies preserving the factorization.

**Definition 6.2.** Let $X$ be an oriented 0-, 1-, 2- or 3-manifold. Put a Riemannian metric on $X$. A Spin structure on $X$ is an isomorphism class of Spin lift of $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$ where $a + b = 3 - \dim(X)$. Whether to put the stabilization directions before or after the tangent directions is entirely conventional, though for this paper we will need to use both conventions at the same time. Since the space of metrics is contractible, the Spin structure is independent of the chosen metric. Let $\mathrm{Spin}(X)$ denote the set of Spin structures on $X$. The letters $s$ and $t$ will be used to denote a Spin structure, e.g., $(X, s)$ denotes a pair of manifold $X$ and a Spin structure $s$ on $X$. Somewhat abusively, $s$ will also be used to indicate a choice of Spin lift of $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$ representing the class $s$.

Let $E$ be a rank 3 real metric bundle. Let $p \subset X$ be a finite collection of points and let $\overline{\sigma}$ be a trivialization of $F(E)$ over $p$. This defines a map of pairs $(X, p) \to (B\,\mathrm{SO}(3), *)$ where $*$ is a basepoint in $B\,\mathrm{SO}(3)$. A pointed Spin lift of $(F(E), \overline{\sigma})$ is a Spin lift $P \to F(E)$ together with a trivialization $\sigma$ of $P|_p$ lifting $\overline{\sigma}$. In particular, this defines a factorization $(X, p) \to (B\,\mathrm{Spin}(3), *) \to (B\,\mathrm{SO}(3), *)$. Two pointed Spin lifts are said to be isomorphic if the corresponding maps $(X, p) \to (B\,\mathrm{Spin}(3), *)$ are homotopic as maps of pairs through homotopies preserving the factorization.

**Definition 6.3.** Let $p$ be a finite collection of points in an oriented 0-, 1-, 2-, or 3-manifold. A pointed Spin structure on $(X, p)$ is an isomorphism class of Spin lift of $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$ for some Riemannian metric on $X$. Let $\mathrm{Spin}(X, \overline{\sigma})$ denote the set of pointed Spin structures on $X$ relative to $\overline{\sigma} : p \to F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$. Sometimes $\overline{\sigma}$ will be dropped and these will be called Spin structures pointed over $p$.

The following result is classical:

**Proposition 6.4.** *Spin structures exist on all oriented 0-, 1-, 2-, and 3-manifolds. Spin structures on $(X, p)$ form a torsor for $H^1(X, p; \mathbb{Z}/2)$. Automorphisms of a fixed Spin structure are in bijection with $H^0(X, p; \mathbb{Z}/2)$.*

**Remark 6.5.** Note that $H^0(X, p; \mathbb{Z}/2)$ is isomorphic to $(\mathbb{Z}/2)^k$ where $k$ is the number of components of $X$ that do not contain a point of $p$. On these components, the automorphism acts by switching the two fibers of the Spin bundle over the frame bundle. The action of $\alpha \in H^1(X, p; \mathbb{Z}/2)$ on the set of pointed Spin structures can be seen explicitly as follows. Let $Y \subset X$ be an embedded codimension-1 submanifold Poincare dual to $\alpha$ and avoiding $p$. Cut open the Spin bundle along $Y$ and reglue with the nontrivial Spin automorphism.

**Example 6.6.** There are two Spin structures on the oriented circle $S^1$. There are $2^{2g}$ Spin structures on a closed oriented surface of genus $g$.

Perhaps the easiest way to construct a Spin structure is via a trivialization of $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$. A trivialization defines an isomorphism $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b) \to \underline{\mathrm{SO}(3)}$. One can pull back the covering $\underline{\mathrm{Spin}(3)} \to \underline{\mathrm{SO}(3)}$ to get a Spin structure on $X$. Note that such trivializations form a torsor over $\mathrm{Map}(X, \mathrm{SO}(3))$.

**Definition 6.7.** If $s$ is a Spin structure that comes from a trivialization, it has a canonical section, given by pulling back the constant identity section on $\underline{\mathrm{Spin}(3)}$. Call this section the trivial section.

**Example 6.8.** There's a canonical trivialization of $F(TS^1 \oplus \underline{\mathbb{R}}^2)$ induced by the orientation on $S^1$. If you modify this trivialization by a map in the nontrivial connected component of $\mathrm{Map}(S^1, \mathrm{SO}(3))$, then you get a different trivialization. These two trivializations lead to the two isomorphism classes of Spin structures on $S^1$.

**Example 6.9.** Similarly to Example 6.8, if $\Delta^1$ is the interval then there's a canonical trivialization of $F(\Delta^1 \oplus \underline{\mathbb{R}}^2)$ induced by the orientation on $\Delta^1$. This trivialization defines a pointed Spin structure for $(\Delta^1, \partial\Delta^1)$. To get the other pointed Spin structure, modify the trivialization by a homotopically nontrivial map $(\Delta^1, \partial\Delta^1) \to (\mathrm{SO}(3), e)$ where $e$ is the identity in $\mathrm{SO}(3)$.

Let $(X, s)$ be a Spin manifold such that $s$ is a Spin lift of $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$. Recall that, by convention, the boundary is oriented with an "outward normal first" convention. That is, if $i : \partial X \to X$ is the inclusion then $i^*TX$ is isomorphic to $\nu \oplus T\partial X$ and an orientation on $\nu$, given by the outward normal, plus the orientation on $TX$, induce an orientation on $\partial X$. If $s$ is a Spin structure lift of $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$, then $i^*s$ is a Spin lift of $F(\underline{\mathbb{R}}^a \oplus \nu \oplus T\partial X \oplus \underline{\mathbb{R}}^b)$. An outward normal identifies this with a Spin lift of $F(\underline{\mathbb{R}}^a \oplus \underline{\mathbb{R}} \oplus T\partial X \oplus \underline{\mathbb{R}}^b)$.

Let $(X, s)$ be a Spin manifold such that $s$ is a Spin lift of $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$. Let $Y \subset X$ be an embedded submanifold intersecting $\partial X$ orthogonally and let $i : Y \to X$ be the inclusion. Then $i^*TX \cong TY \oplus \nu$ where $\nu$ is the normal bundle to $Y$. Note that this convention is different from the inclusion of boundary: now the normal bundle comes after the tangent bundle. Then a trivialization of the normal bundle of $Y$ turns $s|_Y$ into a Spin structure on $Y$, where there are additional stabilization directions after the tangent directions.

It should be emphasized that there are two different conventions here: for inducing Spin structures on the boundary, the normal bundle comes first, and for inducing Spin structures on embedded submanifolds transverse to the boundary, the normal bundle comes second. This is for the following geometric reason. Suppose $X$ is 3-dimensional and $Y$ is 1-dimensional. If $Y$ intersects $\partial X$ in a point, then the normal directions of $Y$ in $X$ coincide with the tangent directions $\partial X$, and the normal direction of $X$ coincides (if the intersection is isotoped to be orthogonal) with the tangent direction of $Y$. Hence $\nu_{\partial X} \oplus T\partial X$ and $TY \oplus \nu_Y$ are consistent.

**Definition 6.10.** If $(Y, t)$ is a Spin manifold, one can ask if there exists $(X, s)$ such that $\partial X \cong Y$ and $s|_{\partial X} = t$. If such an $s$ exists, one says that $t$ is a bounding Spin structure. Otherwise, it is not.

**Example 6.11.** The Spin structure on $S^1$ arising from the canonical trivialization as in Example 6.8 does not bound. The other Spin structure on $S^1$ does bound. In fact, it comes from embedding $S^1$ into $\mathbb{R}^3$ equipped with its usual Spin structure.

If $X$ is an oriented manifold, then $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$ and $F(\underline{\mathbb{R}}^a \oplus T(-X) \oplus \underline{\mathbb{R}}^b)$ are different spaces, since these are oriented frames. In particular, if $s$ is a Spin structure of $X$, then $s$ is not a Spin structure of $-X$.

**Definition 6.12.** Let $(X, s)$ be a Spin manifold where $s$ is a Spin lift of $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$. Note that $F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$ is different from $F(\underline{\mathbb{R}}^a \oplus T(-X) \oplus \underline{\mathbb{R}}^b)$ because these are oriented frames. If $a$ and $b$ are not zero, then there are two maps

$$F(\underline{\mathbb{R}}^a \oplus T(-X) \oplus \underline{\mathbb{R}}^b) \to F(\underline{\mathbb{R}}^a \oplus TX \oplus \underline{\mathbb{R}}^b)$$

obtained by reversing either the last stabilization direction before the tangent directions, or reversing the first stabilization direction after the tangent directions. Let $(-X, -s)$ be the pullback of $s$ along one of these maps. Which map is used should be clear from context. For example, if $(X, s)$ is induced from the boundary of a manifold, then there is a stabilization direction before $X$ and that is the one that is reversed. If $(X, s)$ is induced from a (boundary-transverse) embedding into a larger manifold, then there is a stabilization direction after the tangent directions and that is the one that is used.

**Example 6.13.** Let $X = \Delta^1 \times Y$. Let $t$ be a Spin structure on $Y$ and let $s$ be the Spin structure on $X$ induced by pulling $t$ back along the projection $\Delta^1 \times Y \to Y$. Let $i : \partial X \to X$ denote the inclusion. Then $i^* s$ is isomorphic to $t \sqcup t$ if you use the canonical trivialization of $T\Delta^1$ to trivialize the stabilization direction. However $s|_{\partial X}$ is isomorphic to $(-t) \sqcup t$ because the outward normal on $\{0\} \times Y$ points opposite the orientation direction of $\Delta^1$. Therefore one can write $\partial(X, s) = (-Y, -t) \sqcup (Y, t)$.

**Proposition 6.14.** *Let $X$ be an oriented $n$-manifold and $Y \subset X$ a closed oriented immersed $(n-1)$-submanifold. Suppose $X$ has a Spin structure $s$ and $Y$ is $0$ in $H_{n-1}(X;\mathbb{Z}/2)$. Then the pullback of $s$ to $Y$ is bounding (i.e., it is isomorphic to $t|_{\partial W}$ where $(W,t)$ is some Spin $n$-manifold whose boundary is identified with $Y$).*

*Proof.* It is sufficient to restrict to the case where $X$ is connected. Surger the immersion of $Y \subset X$ along its self intersections to get an embedded submanifold $Y' \subset X$ cobordant to $Y$. Since $Y$ is obtained from $Y'$ by surgery, there is an oriented cobordism $W \subset \Delta^1 \times X$ between $\{0\} \times Y$ and $\{1\} \times Y'$. The Spin structure on $X$ induces a Spin structure on $\Delta^1 \times X$ via pullback. Pulling back the Spin structure on $\Delta^1 \times X$ along the inclusion of $W$ shows that $Y'$ is Spin cobordant to $Y$. In $H_{n-1}(X;\mathbb{Z}/2)$, $Y$ and $Y'$ represent the same class. Since $Y'$ is null-homologous, in the long exact sequence

$$0 \to H_n(X;\mathbb{Z}/2) \to H_n(X,Y';\mathbb{Z}/2) \to H_{n-1}(Y';\mathbb{Z}/2) \to H_{n-1}(X;\mathbb{Z}/2)$$

the rightmost map has the orientation class of $Y'$ in its kernel, hence $H_n(X,Y';\mathbb{Z}/2)$ is strictly bigger than $H_n(X;\mathbb{Z}/2) = \mathbb{Z}/2$. Excision shows that $H_n(X,Y';\mathbb{Z}/2) \cong H_n(X - Y', \partial(X - Y');\mathbb{Z}/2)$. Hence $Y'$ bounds a submanifold of $X$. □

It is will be essential to extend the notion of a Spin structure to nonorientable surfaces. An excellent reference is [KT90].

**Definition 6.15.** Let $\Sigma$ be a (possibly nonorientable) surface. Let $K$ be a real line bundle such that $w_1(K) = w_1(\Sigma)$. In particular, $K$ is isomorphic to the trivial bundle if $\Sigma$ is orientable. A Pin$^-$ structure on $\Sigma$ is an isomorphism class of Spin lift of $F(T\Sigma \oplus K)$ with respect to some metric on $\Sigma$ and some orientation on $T\Sigma \oplus K$. If $p$ is a collection of points in $\Sigma$, a pointed Pin$^-$ structure on $(\Sigma, p)$ is defined analogously.

**Remark 6.16.** Definition 6.15 is a nonstandard definition. The equivalence to the traditional definition is the first line of Lemma 1.7 in [KT90]. The usual notion of Pin$^-$ structure on a rank-3 metric bundle $E$ is an isomorphism class of lift of the (nonoriented) frame bundle $F_O(E)$ to a Pin$^-(3)$ bundle. Here Pin$^-(3)$ is a group covering O(3). There's another group Pin$^+(3)$, also covering O(3), which is of no importance to this paper. For details on these groups, see [KT90]. A Pin$^-$ structure on a surface $\Sigma$ is usually defined as a Pin$^-$ structure on the (possibly nonorientable) bundle $T\Sigma \oplus \underline{\mathbb{R}}$.

**Remark 6.17.** Pointed Pin$^-$ structures on $(\Sigma, p)$ also form a torsor for $H^1(\Sigma, p;\mathbb{Z}/2)$ and the automorphism group of such a pointed Spin structure is isomorphic to $H^0(\Sigma, p;\mathbb{Z}/2)$.

**Example 6.18.** Let $(M, s)$ be a Spin 3-manifold and let $\Sigma \subset M$ be an embedded surface intersecting $\partial M$ transversely. Let $\nu$ be the normal bundle to $\Sigma$. Then, by the conventions set out earlier, $s|_\Sigma$ is a Spin lift of $F(T\Sigma \oplus \nu)$. Since $TM$ is orientable, then $w_1(\nu) = w_1(T\Sigma)$, so $s|_\Sigma$ is a Pin$^-$ structure on $\Sigma$.

Let $(X, s, \sigma)$ be a Spin interval pointed relative to its boundary. This means that $X$ is oriented and that $\sigma$ is a section $\partial X \to s$. Allow for the case where $s$ is a Spin lift of $F(TX \oplus K_1 \oplus K_2)$ where $K_1$ and $K_2$ are real line bundles on $X$ and $TX \oplus K_1 \oplus K_2$ is oriented.

Define the holonomy $\mathrm{hol}(X, s, \sigma)$ as follows. First extend $\sigma$ to a section of $s$ on all of $X$. Construct a section of $F(TX \oplus K_1 \oplus K_2)$ by setting the first vector to be the orientation vector of $TX$, the second to be some unit vector in $K_1$, and the third is determined by these two and the orientation on $TX \oplus K_1 \oplus K_2$. Lift this to a section $\sigma_{\text{path}}$ of $s$. There are some choices made here: a choice of unit section of $K_1$ and a choice of lift of the section of frames to $\sigma_{\text{path}}$. Any two of these choices are related by multiplication by a constant element $r \in \mathrm{Spin}(3)$:

$$\sigma'_{\text{path}} = \sigma_{\text{path}} r$$

where $r$ covers either the identity or a rotation around the $x$-axis by $\pi$. Then

$$\sigma = \sigma_{\text{path}} \cdot \gamma$$

for some path $\gamma : X \to \mathrm{Spin}(3)$.

**Definition 6.19.** With the notation above, let $x_0, x_1$ be the start and endpoints of $X$ relative to its orientation. Set $\mathrm{hol}(X, s, \sigma) = \gamma(x_0)^{-1}\gamma(x_1)$. This is well-defined since multiplying $\sigma_{\text{path}}$ by a fixed element in $\mathrm{Spin}(3)$ does not change the product $\gamma(x_0)^{-1}\gamma(x_1)$.

**Lemma 6.20.** *Let $f : (X, s, \sigma) \to (X', s', \sigma')$ be an isomorphism of pointed Spin bundles covering a map on frames $F(TX \oplus K_1 \oplus K_2) \to F(TX' \oplus K_1' \oplus K_2')$ that is induced by an isometry $X \to X'$. (Because $f$ is an isomorphism of pointed Spin bundles, $f$ takes $\sigma$ to $\sigma'$.) Then* $\mathrm{hol}(X, s, \sigma) = \mathrm{hol}(X', s', \sigma')$.

*Proof.* First suppose that the map on frames is induced by isomorphisms $K_1 \to K_1'$ and $K_2 \to K_2'$ as well. Then one can simultaneously identify $(X, s, \sigma)$ and $(X', s', \sigma')$ simultaneously with a Spin structure on $F(T\Delta^1 \oplus \underline{\mathbb{R}}^2)$. Clearly an automorphism of this bundle preserves the holonomy. If the map on frames does not respect the direct sum decomposition $K_1 \oplus K_2 \to K_1' \oplus K_2'$, then (because $f$ sends $\sigma$ to $\sigma'$) it can be made to do so by multiplying $s$ by a map $\gamma' : X \to \mathrm{Spin}(3)$ which is the identity on the boundary. This too does not change the holonomy. □

**Lemma 6.21.** *Given $(X, s, \sigma)$ a pointed Spin interval, let $(-X, -s, -\sigma)$ be the result of reversing the orientation and reflecting $s$ and $\sigma$ across the first stabilization direction. Then* $\mathrm{hol}(-X, -s, -\sigma) = \mathrm{hol}(X, s, \sigma)^{-1}$.

*Proof.* In going from $X$ to $-X$ the tangent direction is reversed, and in going from $s$ to $-s$ the first stabilization direction is also reversed, so in going from $(X, s, \sigma)$ to $(-X, -s, -\sigma)$ then $\sigma_{\text{path}}$ is multiplied by an element of $\mathrm{Spin}(3)$ covering a constant rotation around the $z$-axis (the second stabilization direction) and this does not change the product $\gamma(x_0)^{-1}\gamma(x_1)$ in the definition of the holonomy (Definition 6.19). However the reversal of the orientation on $X$ reverses $\gamma$, so the roles of $\gamma(x_0)$ and $\gamma(x_1)$ are switched. □

**Lemma 6.22.** *Let $(X, s, \sigma)$ and $(X', s', \sigma')$ be two pointed Spin intervals with endpoints $x_0, x_1$ and $x_0', x_1'$, respectively. Let $f : s|_{x_1} \to -s|_{x_0'}$ be a Spin map covering a map on frames that is either the identity or a rotation by $\pi$ around the $x$-axis. Then $s$ and $s'$ glue together along $f$ to form a new pointed Spin interval with endpoints $x_0$ and $x_1'$. Call this new interval $(X'', s'', \sigma'')$. Then*

$$\mathrm{hol}(X'', s'', \sigma'') = \mathrm{hol}(X, s, \sigma)\,\mathrm{hol}(X', s', \sigma')$$

*Proof.* $f$ covers a map on frames which sends the orientation direction on $X$ with the orientation direction $X'$ and also identifies $K_1$ at $x_1$ with $K_1'$ at $x_0'$. Hence the two bundles $K_1 \oplus K_2$ and $K_1' \oplus K_2'$ glue together to form a new bundle $K_1'' \oplus K_2''$ on $X \cup X'$. $s''$ is a Spin lift of the frames of this bundle. The product formula follows from the definition of holonomy. □

**Example 6.23.** Use the orientation vector field on $TX$ to identify $F(T\Delta^1 \oplus \underline{\mathbb{R}}^2) \cong F(\underline{\mathbb{R}}^3)$. A section of$F(\mathbb{R}^3)$ defines a trivialization and hence a Spin structure $s$. Let $\sigma$ be the trivial section of this trivialization restricted to $\partial\Delta^1$.

If the trivialization is the constant trivialization on the standard basis, then $\mathrm{hol}(\Delta^1, s, \sigma)$ is the identity in Spin(3).

If the trivialization is path through frames that rotates once around one axis, then $\mathrm{hol}(\Delta^1, s, \sigma)$ is the nontrivial central element in Spin(3).

If the trivialization is a path through frames that rotates by $\pi$ around the $x$-axis, then the holonomy is one of two elements covering a rotation by $\pi$ around the $x$-axis. One of them comes from a path through counterclockwise rotations and the other comes from a path of clockwise rotations.

**Definition 6.24.** If $X$ is a circle and $K_1$ and $K_2$ are nontrivial real line bundles on $S$, let a twisted Spin structure on $s$ be a Spin lift of $F(TX \oplus K_1 \oplus K_2)$ for some orientation on $TX \oplus K_1 \oplus K_2$.

There are two isomorphism classes of twisted Spin structures on the circle, obtained by gluing up a pointed Spin interval whose holonomy in Spin(3) covers a path from the identity to the rotation around the $x$-axis by $\pi$. The two different structures come from whether this path is through counterclockwise rotations or through clockwise rotations.

**Definition 6.25.** Let $(S^1, s)$ be an oriented circle with a (twisted) Spin structure. Define

$$\lambda(S^1, s) = \begin{cases} 1 & s \text{ bounding Spin} \\ i & s \text{ counterclockwise twisted Spin} \\ -1 & s \text{ nonbounding Spin} \\ -i & s \text{ clockwise twisted Spin} \end{cases}$$

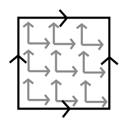
Alternatively, if $\lambda(S^1, s) = i^n$ where $n$ is the number of half twists in a trivialization representing $s$. Extend $\lambda$ to other closed 1-manifolds by $\lambda(S_1 \sqcup S_2) = \lambda(S_1)\lambda(S_2)$.

The following is classical:

**Proposition 6.26.** *Let $(\Sigma, s)$ be a $\mathrm{Pin}^-$ surface. If $S_1, S_2 \subset \Sigma$ represent the same class in $H_1(\Sigma; \mathbb{Z}/2)$, then $\lambda(S_1) = \lambda(S_2)$.*

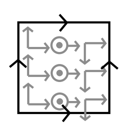
*Proof.* This follows from Proposition 6.14. □

**Example 6.27.** Consider the Spin structure induced from the canonical trivialization of $\Delta^1 \times \Delta^1$:



Here directions 1 and 2 are pictured and the stabilization direction is determined by these two. If you glue opposite sides together, you get the Spin structure from the canonical trivialization on $\mathsf{T} = \mathsf{S} \times \mathsf{S}$. It is non-bounding because its restriction to each nontrivial class in $H_1(\mathsf{T}; \mathbb{Z}/2)$ is nonbounding. If you glue any pair of sides together with the nontrivial Spin automorphism, you get a bounding Spin structure. This produces 3 bounding Spin structures on $\mathsf{T}$. In each bounding Spin structure, there is precisely one nontrivial class in $H_1(\mathsf{T}; \mathbb{Z}/2)$ that is nonbounding. This class is represented by the curve on which the gluing was done with the nontrivial Spin automorphism.

**Example 6.28.** Consider the trivialization on $\Delta^1 \times \Delta^1$ that twists halfway around clockwise in one direction:



If you glue the top and bottom together and the left and right together with a flip, you get the Klein bottle $\Sigma$ with a trivialization of $F(T\Sigma \oplus K)$ where $K$ is a line bundle that is nontrivial when restricted to homology classes $(1,0)$ and $(1,1)$. This trivialization produces a $\mathrm{Pin}^-$ structure on $\Sigma$. The homology class $(0,1)$ inherits a nonbounding Spin structure. The homology classes $(1,0)$ and $(1,1)$ inherit counterclockwise twisted Spin structures. If you glue the sides together with the nontrivial Spin automorphism you get the other three $\mathrm{Pin}^-$ structures on the Klein bottle. For example, if you glue the left and right sides together with the nontrivial Spin automorphism and the top and bottom together as usual, then $(0,1)$ is still a nonbounding class, but $(1,0)$ and $(1,1)$ have clockwise twisted Spin structures. If you glue the left right as usual and the top and bottom with the nontrivial Spin automorphism, then $(0,1)$ is bounding, $(1,0)$ has a counterclockwise twisted Spin structure, and $(1,1)$ has a clockwise twisted Spin structure. This last $\mathrm{Pin}^-$ structure on the Klein bottle bounds.

**Example 6.29.** There are two $\mathrm{Pin}^-$ structures on $\mathbb{R}P^2$, depending on whether the nontrivial homology class is represented by a curve with a clockwise or counterclockwise twisted Spin structure.

**Example 6.30.** On $(\mathsf{T}, s)$, if $s$ is bounding then there are three bounding homology classes and there is one nonbounding class. If $s$ is nonbounding, then there is one bounding homology class and there are three nonbounding classes. By taking connect sums, it follows that there are $\frac{1}{2}(4^g + 2^g)$ bounding homology classes in $(\Sigma_g, s)$ if $s$ bounds, and there are $\frac{1}{2}(4^g + 2^g)$ nonbounding homology classes in $(\Sigma_g, s)$ if $s$ doesn't bound. Therefore

$$\sum_{x \in H_1(\Sigma_g;\mathbb{Z}/2)} \lambda(x, s|_x) = \begin{cases} 2^g & \text{if } s \text{ bounds} \\ -2^g & \text{if } s \text{ doesn't bound} \end{cases}.$$

**Definition 6.31.** Let $(\Sigma, s)$ be a closed $\mathrm{Pin}^-$ surface. Define

$$\beta(\Sigma, s) = \frac{2^{\chi(\Sigma)/2}}{2^{b_0(\Sigma)}} \sum_{x \in H_1(\Sigma;\mathbb{Z}/2)} \lambda(x, s|_x).$$

In the sum, a 1-manifold representing $x$ is chosen. Exactly which is chosen doesn't matter, by Proposition 6.14.

**Example 6.32.** The two $\mathrm{Pin}^-$ structures on $\mathbb{R}P^2$ have $\beta$ invariant $\zeta^{\pm 1}$. $\beta(\Sigma_g, s)$ is 1 if $s$ is bounding and $-1$ if it is not bounding. The four $\mathrm{Pin}^-$ structures on the Klein bottle have $\beta$ invariants $1, 1, i, -i$.

**Proposition 6.33** ([KT90] Section 3)**.** *$\beta(\Sigma, s)$ is an 8th root of unity. If $(\Sigma, s)$ and $(\Sigma', s')$ are $\mathrm{Pin}^-$ cobordant then $\beta(\Sigma, s) = \beta(\Sigma', s')$. In fact, $\beta$ defines an isomorphism from the 2d $\mathrm{Pin}^-$ cobordism group to $\mathbb{Z}/8$.*

$\beta$ is therefore to $\mathrm{Pin}^-$ surfaces what $\lambda$ is to (twisted) Spin 1-manifolds. By Example 6.18, an embedded surface in a Spin 3-manifold inherits a $\mathrm{Pin}^-$ structure. It is a classical fact that $\beta$ is invariant within homology classes in a Spin 3-manifold:

**Proposition 6.34.** *Let $(M, s)$ be a closed Spin 3-manifold. Suppose $\Sigma_1, \Sigma_2 \subset M$ are embedded surfaces representing the same class in $H_2(M;\mathbb{Z}/2)$. Then $\beta(\Sigma_1, s|_{\Sigma_1}) = \beta(\Sigma_2, s|_{\Sigma_2})$.*

*Proof.* This follows from Proposition 6.14. □

In analogy with the definition of $\beta$, one can construct the following invariant of $(M, s)$:

$$\frac{1}{2^{b_0(M)}} \sum_{x \in H_2(M)} \beta(x, s|_x) \tag{13}$$

This quantity is not a root of unity but, as with the Euler characteristic factor in the definition of $\beta$, if you divide by a purely topological factor you get a

16th root of unity. As indicated in the introduction, the topological factor is $\overline{Z_{\mathcal{I}_\zeta}(M)}$ where $Z_{\mathcal{I}_\zeta}$ is the TQFT from the Ising category for $\zeta$. The 16th root of unity is a classical Spin invariant, explained below.

Because the inclusion $\mathrm{Spin}(3) \to \mathrm{Spin}(4)$ is an isomorphism on fundamental groups, a $\mathrm{Spin}(4)$ structure on $F(\mathbb{R} \oplus E)$ gives a $\mathrm{Spin}(3)$ structure on $F(E)$. Not every 4-manifold, however, has a Spin structure:

**Theorem 6.35** (Rokhlin)**.** *Let $W$ be a closed Spin 4-manifold. Then the signature of $W$ is divisible by 16.*

It is true, however, that every Spin 3-manifold bounds a Spin 4-manifold. Then

**Corollary 6.36.** *Let $(M, s)$ be a Spin 3-manifold. The mod 16 signature of a bounding Spin 4-manifold is a $\mathbb{Z}/16$-valued invariant of $(M, s)$. Call it Rokhlin's invariant, denoted $R(M, s)$.*

**Definition 6.37.** Set $\mu(M, s) = e^{2\pi i R(M,s)/16} = \zeta^{R(M,s)/2}$. This will, somewhat abusively, also be called Rokhlin's invariant.

**Theorem 6.38** ([KT90] Theorem 4.11)**.** *Let $x \in H_2(M; \mathbb{Z}/2)$ and let $x^\vee$ be its Poincare dual in $H^1(M; \mathbb{Z}/2)$. Then for $(M, s)$ a closed Spin 3-manifold:*

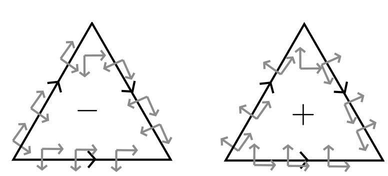
$$\mu(M, s + x^\vee) = \mu(M, s)\beta(x, s|_x)^{-1}$$

**Corollary 6.39.** *If $(M, s)$ is a closed Spin 3-manifold, then*

$$\frac{1}{2^{b_0(M)}} \sum_{x \in H_2(M)} \beta(x, s|_x) = \frac{\mu(M, s)}{2^{b_0(M)}} \sum_{t \in \mathrm{Spin}(M)} \mu(M, t)^{-1} \quad = \mu(M, s)\overline{Z_{I_\zeta}(M)}$$

# 7 Spin Structures on Triangulated Manifolds

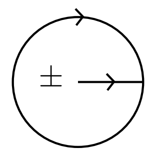
It will be helpful to construct standard trivializations of the stabilized frame bundles on $\pm\Delta^2$. These trivializations will be defined only on the complement of the vertices. These are constructed as follows. On positive boundary edges, set the first basis vector to point out and the second basis vector point from the lower vertex to the higher vertex. On negative boundary edges, have the first basis vector point in and the second basis vector point from the lower vertex to the higher vertex. Around each vertex, have the trivialization trace out the shortest path it can between the trivializations at each incident edge:



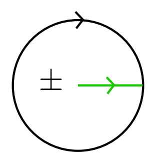

(14)

The trivializations define Spin structures on the two oriented 2-simplices away from the vertices. Call these structures $s_{\pm\Delta^2}$. There is an ambiguity here, however. One can place the stabilization direction to $\Delta^2$ before or after the tangent directions. Both of situations will occur: by convention, the stabilization direction comes before the two tangent directions if $\Delta^2$ appears on the boundary of a 3-manifold and the stabilization comes after the two tangent directions otherwise. Both situations will be denoted with the same notation $s_{\pm\Delta^2}$ even though they are, strictly speaking, different Spin structures.

Let $(\Sigma, \tau)$ be an oriented surface with triangulation $\tau$. Assigning $s_{\pm\Delta^2}$ to each copy of $\pm\Delta^2$ in $\tau$ defines a Spin structure on the complement of the vertices of $\tau$. It may or may not extend over an interior vertex. For example, if $\Sigma$ is the disk with the following triangulation:




then the Spin structure does not extend over the interior vertex. One can modify the Spin structure by, along some of the edges, gluing the Spin structures together with the nontrivial Spin automorphism. Denote these edges in green and call such an edge a "marked" edge. For example,




is $(\pm\Delta^2, s_{\pm\Delta^2})$ with the Spin structures on edges [01] and [02] glued together with the nontrivial Spin automorphism. One can check that this Spin structure does extend over the interior vertex.

**Proposition 7.1.** *Let $(\Sigma, \tau)$ be an oriented surface with triangulation $\tau$. Mark some of the edges of $\tau$ so that $\tau$ defines a Spin structure on the complement of the vertices of $\tau$. Define a 0-chain in the $\Delta$-chain group $C_0(\Sigma; \frac{1}{2}\mathbb{Z}/2\mathbb{Z})$ by*

$$\mathbf{v} = \sum_{\sigma \text{ a 2-simplex in } \tau} \frac{1}{2}[1]_\sigma + \sum_{e \text{ a marked edge}} \partial e \mod 2$$

*where $[1]_\sigma$ is the vertex [1] in the 2-simplex $\sigma$. Then the Spin structure defined by $\tau$ and the marked edges extends over an interior vertex if and only if the coefficient of that vertex in $\mathbf{v}$ is 1.*

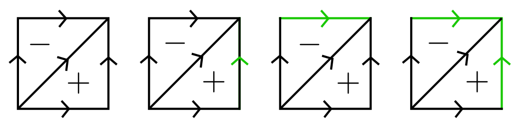
*Proof.* Assume there are no marked edges. If you rotate around a vertex, the trivialization will rotate either always counterclockwise or always clockwise, or it won't rotate at all. It does a half twist when it passes around a [1] vertex. The induced Spin structure on a small circle around the vertex will bound if and only if the trivialization twists an odd number of times.

If you take a Spin structure on a circle, cut it open and reglue it with the nontrivial Spin automorphism, it will change from bounding to nonbounding or vice versa. So the Spin structure on a small circle around a vertex will bound if and only if number of twists plus the number of marked edges incident to that vertex is 1 mod 2. □

**Proposition 7.2.** *Let $\tau$ be a triangulation of an oriented surface $\Sigma$. Any spin structure can be obtained by marking some edges of $\tau$.*

*Proof.* Let $s$ be a Spin structure on $\Sigma$ and let $\tau$ be a triangulation. Pick a trivialization. Let $\sigma$ be a 2-simplex. There is a unique Spin structure on $\sigma$, so $s|_\sigma$ is isomorphic to $s_{\pm\Delta^2}$. Pick such an isomorphism. If $\sigma$ and $\sigma'$ share an edge $e$, the isomorphisms to $s_{\pm\Delta^2}$ agree up to an automorphism of $s|_e$. Mark $e$ if that automorphism is nontrivial. □

**Example 7.3.** Here are the four Spin structures on the torus, in terms of an oriented triangulation with marked edges (identify opposite edges on each square):




One can also construct standard trivializations on $\pm\Delta^3$ defined on the complement of its edges. Do this as follows. On the interior of each negative face set the first basis vector to point in and the other two basis vectors as in $-\Delta^2$. On the interior of each positive face set the first vector to point out and the other two basis vectors as in $-\Delta^2$ (not $+\Delta^2$). Around each edge, connect the trivializations by shortest possible path. These two trivializations induce Spin structures on $\pm\Delta^3$. Call these Spin structures $s_{\pm\Delta^3}$.

**Remark 7.4.**

$$s_{+\Delta^3}|_{\partial\Delta^3} = ([123], -s_{-\Delta^2}) \cup ([023], s_{-\Delta^2}) \cup ([013], -s_{-\Delta^2}) \cup ([012], s_{-\Delta^2})$$

$$s_{-\Delta^3}|_{\partial\Delta^3} = ([123], s_{-\Delta^2}) \cup ([023], -s_{-\Delta^2}) \cup ([013], s_{-\Delta^2}) \cup ([012], -s_{-\Delta^2})$$

Here $s_{-\Delta^2}$ is the Spin structure where the stabilization direction comes first, because these copies of $\pm\Delta^2$ come from the boundary of a 3-manifold.

Let $(M, \tau)$ be an oriented 3-manifold with triangulation $\tau$. Assign $s_{\pm\Delta^3}$ to each copy of $\pm\Delta^3$ in $\tau$. These glue together to form a Spin structure on the complement of the edges of $\tau$. The Spin structure may or may not extend over each edge. One can modify the Spin structure by gluing along a face with the nontrivial Spin automorphism. Color such a face green and call it a "marked" face.

The following two Propositions are proved analogously to the 2-dimensional case:

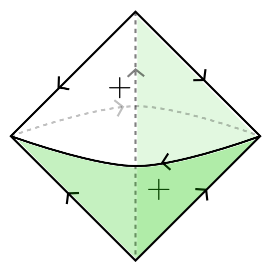
**Proposition 7.5.** *Let $(M, \tau)$ be an oriented 3-manifold with triangulation $\tau$. Mark some of the faces of $\tau$ so that $\tau$ defines a Spin structure on the complement of the edges. Define a 1-chain in the $\Delta$-chain group $C_1(\Sigma; \frac{1}{2}\mathbb{Z}/2)$ by*

$$\mathbf{e} = \sum_{\sigma \text{ a 3-simplex in } \tau} \frac{1}{2}\left([02]_\sigma + [13]_\sigma\right) + \sum_{f \text{ a marked face}} \partial f \mod 2$$

*where $[ij]_\sigma$ is the edge $[ij]$ in the 3-simplex $\sigma$. Then the Spin structure defined by $\tau$ and the marked faces extends over an interior edge if and only if the coefficient of that edge in $\mathbf{e}$ is 1.*

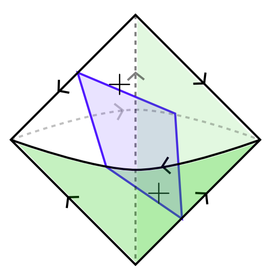
**Proposition 7.6.** *Let $\tau$ be a triangulation on an oriented manifold $M$. Any spin structure can be obtained by marking some faces of $\tau$.*

For example, here is a marked oriented triangulation of the ball which, when antipodal boundary points are glued, gives a Spin structure on $\mathbb{R}P^3$:




(15)

These sorts of presentations of Spin structures can be useful for computing the invariants $\lambda$ and $\beta$. For example, here's a copy of $\mathbb{R}P^2$ in $\mathbb{R}P^3$:




One can see that, in this $\mathbb{R}P^2$, the nontrivial curve has $\lambda$ invariant $-i$, so the $\beta$ invariant of the embedded $\mathbb{R}P^2$ is $\zeta^{-1}$.

It will also be helpful later to fix Spin structures on the standard shapes used in this paper. Let $s_{\Delta^1}$ be the Spin structure obtained from the section of $F(\underline{\mathbb{R}} \oplus T\Delta^1 \oplus \underline{\mathbb{R}})$ where the first basis vector points along the first stabilization direction, the second basis vector points along $T\Delta^1$ from lower vertex to higher vertex, and the third basis vector points along the third stabilization direction. Set $s_{-\Delta^1}$ to be $-s_{\Delta^1}$ where the $-s_{\Delta^1}$ is obtained from $s_{\Delta^1}$ by reversing the first stabilization direction. Then $s_{\partial\Delta^2} = s_{-\Delta^1} \cup s_{+\Delta^1} \cup s_{+\Delta^1}$. On the other hand, it is not true that $-s_{\Delta^2} = s_{-\Delta^2}$. This is simply a quirk of the trivializations used to construct these Spin structures.

Let $s_{\mathsf{S},\mathrm{nb}}$ be the Spin structure obtained from $s_{\Delta^1}$ by gluing the two ends together. Let $s_{\mathsf{S},\mathrm{b}}$ be the Spin structure obtained from $s_{\Delta^1}$ by gluing the two ends together with the nontrivial Spin automorphism. Then $s_{\mathsf{S},\mathrm{b}}$ is bounding and $s_{\mathsf{S},\mathrm{nb}}$ is nonbounding.

Let $s_{\mathsf{C},\mathrm{b}}$ and $s_{\mathsf{C},\mathrm{nb}}$ be the Spin structures obtained by pulling back $s_{\mathsf{S},\mathrm{b}}$ and $s_{\mathsf{S},\mathrm{nb}}$ along the projections $\mathsf{C} = \Delta^1 \times \mathsf{S} \to \mathsf{S}$. These will be called the bounding and nonbounding Spin structures on the cylinder. To be specific, these are Spin lifts of $F(T\mathsf{C} \oplus \underline{\mathbb{R}})$.

Let $s_{\mathsf{B}}$ be the Spin structure on $\mathsf{B}$ given by pulling back $s_{\Delta^1}$ via the canonical map $\mathsf{B} \to \Delta^1$ induced by the projection $\Delta^1 \times \Delta^1 \to \Delta^1$. This Spin structure is not defined on the two vertices of $\mathsf{B}$. It is a Spin lift of $F(\mathsf{B} \oplus \underline{\mathbb{R}})$ away from these two points.

There are four Spin structures on the pair of pants, call them $s_{\mathsf{P},000}$, $s_{\mathsf{P},101}$, $s_{\mathsf{P},011}$, $s_{\mathsf{P},110}$. Here $s_{\mathsf{P},101}$ is the Spin structure whose restriction to the first boundary component of $\mathsf{P}$ does not bound, whose restriction to the second does, and whose restriction to the third does not. The other three are defined similarly. Explicitly, construct these by gluing $s_{\mathsf{C},(\mathrm{n})\mathrm{b}}$ to $s_{-\Delta^2}$ where here $s_{-\Delta^2}$ is the Spin lift of $F(T(-\Delta^2) \oplus \underline{\mathbb{R}})$.

There are four Spin structures on $\mathsf{T}$. Let $s_{\mathsf{T},(0,0)}$ denote the nonbouding one obtained by identifying the two sides of $s_{\mathsf{C},\mathrm{nb}}$. Let $s_{\mathsf{T},(1,0)}$ denote the Spin structure obtained by identifying the two sides of $s_{\mathsf{C},\mathrm{b}}$. Let $s_{\mathsf{T},(0,1)}$ denote the Spin structure obtained by identifying the two sides of $s_{\mathsf{C},\mathrm{nb}}$ with the nontrivial Spin automorphism. Let $s_{\mathsf{T},(1,1)}$ denote the Spin structure obtained by identifying the two sides of $s_{\mathsf{C},\mathrm{b}}$ with the nontrivial Spin automorphism. On $s_{\mathsf{T},(a,b)}$, the $\mathbb{Z}/2$-homology class $(a, b)$ is the one whose restriction to the circle is not Spin cobordant to the other three classes.

# 8 Spin TQFTs

A Spin TQFT can be defined analogously to a TQFT, using the fact that Spin structures on an $n$-manifold restrict to Spin structures on the boundary.

**Proposition 8.1.** *Let $(X, s)$ be a Spin $n$-manifold with boundary. Suppose the boundary is divided into three disjoint parts $\partial X = Y_1 \cup Y_2 \cup Y_3$ and let $f : (Y_1, s|_{Y_1}) \to (-Y_2, -s|_{Y_2})$ be a Spin isomorphism such that $f$ covers an orientation-preserving diffeomorphism $g : Y_1 \to -Y_2$ and its induced map on frames $g_* : F(\underline{\mathbb{R}}^a \oplus TY_1 \oplus \underline{\mathbb{R}}^b) \to F(\underline{\mathbb{R}}^a \oplus T(-Y_2) \oplus \underline{\mathbb{R}}^b)$. Then $(X, s)$ glues together along $f$ to form a Spin manifold $(X_f, s_f)$.*

*Proof.* $-s|_{Y_2}$ is the same thing as the Spin structure on $Y_2$ induced by setting the inward pointing (as opposed to outward) normal direction to the stabilization direction. Then $-s|_{Y_2}$ extends over a collar neighborhood of $Y_2$ where its stabilization direction points inward. $s|_{Y_1}$ extends over a collar neighborhood of $Y_1$ where its stabilization direction points outward. Since $f$ identifies the tangent directions of these two Spin structures, the Spin structures on the two collar neighborhoods glue together to form a collar neighborhood of a Spin structure on $Y_1 = Y_2$ in $M_f$. $\square$

**Definition 8.2.** If $(\Sigma_1, t_1)$ and $(\Sigma_2, t_2)$ are closed Spin $(n-1)$-manifolds, an $n$-dimensional cobordism from $(\Sigma_1, t_1)$ to $(\Sigma_2, t_2)$ consists of

- A Spin $n$-manifold $(M, s)$
- A partition of $\partial M$ into two disjoint sets: $\partial M = (\partial M)_- \sqcup (\partial M)_+$
- Spin isomorphisms

$$\phi_- : ((\partial M)_-, s|_{(\partial M)_-}) \to (-\Sigma_1, -t_1)$$

$$\phi_+ : ((\partial M)_+, s|_{(\partial M)_+}) \to (\Sigma_2, t_2).$$

  such that $\phi_\pm$ covers maps on frames induced by diffeomorphisms and the identity map on the stabilization direction.

A cobordism $((M_{12}, s_{12}), (\phi_{12})_-, (\phi_{12})_+)$ from $(\Sigma_1, t_1)$ to $(\Sigma_2, t_2)$ can be combined with a cobordism $((M_{23}, s_{23}), (\phi_{23})_-, (\phi_{23})_+)$ from $(\Sigma_2, t_2)$ to $(\Sigma_3, t_3)$ by gluing $(M_{12}, s_{12})$ to $(M_{23}, s_{23})$ along $(\phi_{23})_-^{-1} \circ (\phi_{12})_+$. The result is a cobordism from $(\Sigma_1, t_1)$ to $(\Sigma_3, t_3)$.

**Definition 8.3.** Say two Spin cobordisms

$$((M_1, s_1), (\phi_1)_-, (\phi_1)_+) \text{ and } ((M_2, s_2), (\phi_2)_-, (\phi_2)_+)$$

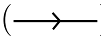
are equivalent if there is an isomorphism $f : (M_1, s_1) \to (M_2, s_2)$ such that $(\phi_2)_\pm \circ f|_\partial = (\phi_1)_\pm$.

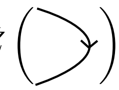
**Definition 8.4.** Define the monoidal category $n\mathrm{Cob}_{\mathrm{Spin}}$ as follows. Let the objects of $n\mathrm{Cob}_{\mathrm{Spin}}$ be the the set of closed Spin $(n-1)$-manifolds. Let $\mathrm{Mor}((\Sigma_1, t_1), (\Sigma_2, t_2))$ be the set of equivalence classes of cobordisms from $(\Sigma_1, t_1)$ to $(\Sigma_2, t_2)$. Composition is given by gluing together representative cobordisms. The monoidal product on $n\mathrm{Cob}_{\mathrm{Spin}}$ is given by disjoint union.

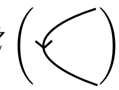
Let $\mathcal{SV}$ be the symmetric monoidal category of finite-dimensional super vector spaces. Each object in this category has an "grading involution" defined on elements of pure degree by the linear map $v \mapsto (-1)^{\deg(v)} v$. Note that the symmetric braiding on $\mathcal{SV}$ is the map $V \otimes W \to W \otimes V$ given by $v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v$.

**Definition 8.5.** A $n$-dimensional Spin TQFT is a monoidal functor $\hat{Z} : n\mathrm{Cob}_{\mathrm{Spin}} \to \mathcal{SV}$.

**Example 8.6.** Here is a simple 1d Spin TQFT $\hat{Z}$. There is a single Spin bundle $(p, s)$ on any oriented point obtained by restricting the Spin bundle over the frame bundle in $\mathbb{R}^3$ to the origin. Let $\hat{Z}(p, s) = L$ where $L$ is an odd copy of $\mathbb{C}$. Let $\hat{Z}(-p, -s)$ be $L^*$. Let $v$ be a basis of $L$ and $\xi$ a dual basis such that $\langle v, \xi \rangle = 1$. Then $\langle \xi, v \rangle = -1$. If $t$ is the standard Spin structure on $\Delta^1$ obtained by embedding $\Delta^1$ in $\mathbb{R}^3$ then there are canonical identifications at each point of $\partial\Delta^1$ with the Spin bundle on a point. In all the following assignments, parametrize the boundary points by these identifications:

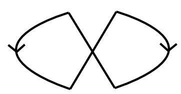
$\hat{Z}$ (  ) $= v \mapsto v$

$\hat{Z}$  $= v \otimes \xi \mapsto 1$ (16)

$\hat{Z}$  $= 1 \mapsto \xi \otimes v$ (17)

where here the order of the tensor factors is from top to bottom. If you change any boundary parametrizations by the nontrivial Spin automorphism of $(p, s)$, multiply by $-1$.

In order to compose (16) and (17), you have to switch the two tensor factors:



Since $L$ and $L^*$ are odd, this leads to a sign of $-1$. Therefore $\hat{Z}(\mathsf{S}, s_{\mathsf{S},\mathrm{nb}}) = -1$. Changing one of the boundary parametrizations of (16) or (17) by the nontrivial Spin automorphism changes the circle's Spin structure to the bounding one, so $\hat{Z}(\mathsf{S}, s_{\mathrm{b}}) = 1$.

One can define a 2d Pin$^-$ TQFT by tweaking the definition of a Spin TQFT. If $\Sigma$ has a Pin$^-$ structure, then $\partial\Sigma$ need not be oriented but $\nu \oplus T\partial\Sigma \oplus K$ is oriented, where $\nu$ is the normal bundle to $\partial\Sigma$. The outward normal makes the frames of this bundle into $F(\underline{\mathbb{R}} \oplus \partial\Sigma \oplus K)$. Therefore the objects of the Pin$^-$ cobordism category are pairs $(S, t)$ where $S$ is a 1-manifold and $t$ is a Spin lift of $F(\underline{\mathbb{R}} \oplus TS \oplus K)$ for some line bundle $K$ on $S$ and orientation on $\underline{\mathbb{R}} \oplus TS \oplus K$. If $s$ is such a Spin lift, then $-s$ is obtained from $s$ by pulling back along the map which reverses the first stabilization direction. $-s$ is then a Spin lift of the oppositely oriented frame bundle. $\hat{Z}(S, s)$ and $\hat{Z}(S, -s)$ are dual super vector spaces. It doesn't make sense to write $-S$ since $S$ is not oriented. A cobordism $(\Sigma, s, \phi_-, \phi_+)$ consists of a Pin$^-$ surface and boundary parametrizations

$$\phi_- : ((\partial\Sigma)_-, s|_{(\partial\Sigma)_-}) \to (S_1, -t_1)$$
$$\phi_+ : ((\partial\Sigma)_+, s|_{(\partial\Sigma)_+}) \to (S_2, t_2)$$

where $\phi_\pm$ cover maps of (oriented) frames that are induced by a diffeomorphism on the tangent direction, the identity map on the stabilization direction, and sends the third summand to itself. For example, you can parametrize an oriented circle with an orientation reversing map as long as you flip the $K$ direction at the same time. These objects and cobordisms (up to equivalence) form the category $2\mathrm{Cob}_{\mathrm{Pin}^-}$. Its monoidal product is given by disjoint union.

**Definition 8.7.** A 2d Pin$^-$ TQFT is a monoidal functor $2\mathrm{Cob}_{\mathrm{Pin}^-} \to \mathcal{SV}$.

# 9 Extending $\lambda$

The goal of this section is to extend $\lambda$ (Definition 6.25) to 1-manifolds with boundary. The construction will depend on two complex lines $L$ and $L^*$. A Spin interval with a section on its boundary will be assigned an element in $L \otimes L$ or $L \otimes L^*$ or $L^* \otimes L$ or $L^* \otimes L^*$, depending on the boundary section. Each of the two tensor factors will correspond to one of the boundary points. Gluing two Spin intervals by identifying two of their boundary points will result in contracting the two tensor factors corresponding to those two points.

It is necessary to recall some basic facts about the 2-dimensional representation of Spin(3). Fix once and for all an isomorphism between Spin(3) and SU(2) so that the matrices

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

maps to a rotation of $2\theta$ counterclockwise about the $x$-axis and

$$\begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{pmatrix}$$

maps to a rotation of $2\theta$ counterclockwise around the $y$-axis and

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

maps to a rotation of $2\theta$ counterclockwise about the $z$-axis.[4]

The action of SU(2) on $\mathbb{C}^2$ preserves the following antisymmetric bilinear form:

$$(a, b) \otimes (c, d) \mapsto ad - bc.$$

In particular, the subgroup $H < \mathrm{SU}(2)$ consisting of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}$$

preserves this form. $H$ also preserves the decomposition of $\mathbb{C}^2$ into a direct sum of two copies of $\mathbb{C}$: $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$. Write $L$ for the first copy of $\mathbb{C}$ and $L^*$ for the second, identifying $\mathbb{C}$ with $\mathbb{C}^*$ in the usual way. Make $L$ and $L^*$ odd complex lines, so the isomorphism $L \otimes L^* \to L^* \otimes L$ (for example) is the usual isomorphism $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ multiplied by $-1$. Let $v$ be a basis vector of $L$ and let $\xi \in L^*$ be such that $\langle v, \xi \rangle = 1$. Note that then $\langle \xi, v \rangle = -1$. There is a natural pairing

$$(L \oplus L^*) \otimes (L \oplus L^*) \to \mathbb{C}$$

$$(av \oplus b\xi) \otimes (cv \otimes d\xi) \mapsto ad\langle v, \xi \rangle + bc\langle \xi, v \rangle = ad - bc$$

[4]These might not be the matrices one might expect for counterclockwise rotations, but it will be useful to think of SU(2) acting on $\mathbb{C}^2$ on the right, since holonomies compose left to right.

Therefore the subgroup $H$ acts on $L \oplus L^*$ and preserves the natural pairing $(L \oplus L^*) \otimes (L \oplus L^*) \to \mathbb{C}$. Note that every element of $H$ either sends $L$ to $L$ or switches $L$ and $L^*$.

Identify $\mathrm{End}(L \oplus L^*)$ with $(L \oplus L^*) \otimes (L \oplus L^*)$ using the canonical pairing. Note that identifying $\mathrm{End}(L)$ with $L^* \otimes L$ means the identity transformation gets identified with $\xi \otimes v$ but that identifying $\mathrm{End}(L^*)$ with $L \otimes L^*$ means that the identity transformation gets identified with $-v \otimes \xi$. Therefore, thinking of $2 \times 2$ matrices acting on $L \oplus L^*$ on the right,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a\xi \otimes v + b\xi \otimes \xi - cv \otimes v - dv \otimes \xi \tag{18}$$

In particular,

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto a\xi \otimes v - a^{-1} v \otimes \xi$$

$$\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \mapsto a\xi \otimes \xi + a^{-1} v \otimes v$$

**Definition 9.1.** Let $(p, s, \sigma)$ be a Spin point with a section $\sigma : p \to s$ such that $\sigma(p)$ covers a frame whose first vector is $\pm u_1$ where $(u_1, u_2, u_3)$ is the standard basis of $\mathbb{R}^3$. Set

$$Z_1(p, s, \sigma) = L$$

if the first vector of the frame covered by $\sigma$ is $u_1$ and set

$$Z_1(p, s, \sigma) = L^*$$

if the first vector of the frame covered by $\sigma$ is $-u_1$. Extend $Z_1$ to a disjoint union of points by tensor product: $Z_1(p \sqcup q, s \sqcup t, \sigma_1 \sqcup \sigma_2) := Z_1(p, s, \sigma_1) \otimes Z_1(q, t, \sigma_2)$.

**Remark 9.2.** Importantly, note that the order of the tensor factors of $Z_1(p \sqcup q, s \sqcup t, \sigma_1 \sqcup \sigma_2)$ is not canonical, since disjoint union does not recognize order: $p \sqcup q = q \sqcup p$. However, any two orderings of the tensor factor are canonically related by an isomorphism in $\mathcal{SV}$.

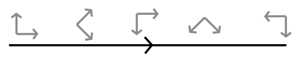
**Definition 9.3.** There is a pairing $Z_1(p, s, \sigma) \otimes Z_1(-p, -s, -\sigma) \to \mathbb{C}$ induced by the pairings $L^* \otimes L \to \mathbb{C}$ and $L \otimes L^* \to \mathbb{C}$. If $\sigma$ covers a frame on $s$ whose first vector is $u_1$, then $-\sigma$ covers a frame on $-p$ whose first vector is $-u_1$.

**Definition 9.4.** Let $(p, s, \sigma)$ and $(p', s', \sigma')$ be two Spin points with sections. Precisely, $s$ and $s'$ are Spin lifts of $F(\mathbb{R} \oplus K_1 \oplus K_2)$ and $F(\mathbb{R} \oplus K_1' \oplus K_2')$ for some real lines $K_1$ and $K_2$ and orientation on $\mathbb{R} \oplus K_1 \oplus K_2$. Let $f : (p, s, \sigma) \to (p', s', \sigma')$ be a map of Spin points with sections that covers a map on frames which is the identity on $\mathbb{R}$ and preserves the direct sum decomposition $K_1 \oplus K_2 \to K_1' \oplus K_2'$. Define $f_* : Z_1(p, s, \sigma) \to Z_1(p', s', \sigma')$ to be the identity map on $Z_1(p, s, \sigma) = Z_1(p', s', \sigma')$.

**Definition 9.5.** Let $(X, s, \sigma)$ be a Spin interval with boundary section. Write $h = \mathrm{hol}(X, s, \sigma)$. Then $h$ determines an element in $(L \oplus L^*) \otimes (L \oplus L^*)$ as in (18). Let $Z_1(X, s, \sigma)$ be the projection of this element to $Z_1(\partial X, s|_{\partial X}, \sigma|_{\partial X})$, where the first tensor factor corresponds to the startpoint of $X$ and the second tensor factor corresponds to the endpoint of $X$.

For example, if $Z_1(\partial X, s|_{\partial X}, \sigma|_{\partial X}) = L^* \otimes L^*$, then $Z_1(X, s, \sigma)$ is the projection of $h$ to the $L^* \otimes L^*$ summand of $(L \oplus L^*) \otimes (L \oplus L^*)$.

**Example 9.6.** Let $(X, s, \sigma)$ be the Spin structure determined by the following trivialization:



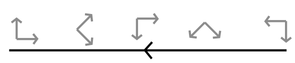
The first vector points inward at the boundary, the second vector is also pictured, and the third vector is determined by those two (following the right hand rule, it would come out of the page). Here $\sigma$ comes from the trivial section of this trivialization.

Extend $\sigma$ to the interior of $X$ along the trivial section from the trivialization. Use the obvious lift $\sigma_{\mathrm{path}}$. Then $\sigma$ rotates clockwise around the $z$-axis as you move across $X$, so the holonomy is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This matrix is $\mathrm{hol}(X, s, \sigma)$.

**Example 9.7.** Change Example 9.6 so that the interval is oriented the other way

.

The trivialization has remained the same though of course the first stabilization direction must be reversed. Then $\sigma$ rotates counterclockwise around the $z$-axis as you move across the interval. So the holonomy is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

**Example 9.8.** Let $(X, s, \sigma)$ be as in Example 9.6. Then $Z_1(\partial X, s|_{\partial X}, \sigma|_{\partial X}) = L^* \otimes L^*$. The holonomy is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which corresponds to the sum

$$-\xi \otimes \xi - v \otimes v$$

so $Z_1(X, s, \sigma) = -\xi \otimes \xi$.

**Example 9.9.** Let $(X, s, \sigma)$ be as in Example 9.7. Then $Z_1(\partial X, s|_{\partial X}, \sigma|_{\partial X}) = L^* \otimes L^*$. The holonomy is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which corresponds to the sum

$$\xi \otimes \xi + v \otimes v$$

so $Z_1(X, s, \sigma) = \xi \otimes \xi$.

**Proposition 9.10.** *Let $(X, s, \sigma)$ be a pointed Spin interval and let $(-X, -s, -\sigma)$ be as in Lemma 6.21. Then $Z_1(-X, -s, -\sigma)$ is obtained from $Z_1(X, s, \sigma)$ by applying the braiding in $\mathcal{SV}$ to its two tensor factors.*

*Proof.* This follows from the fact that $\mathrm{hol}(-X, -s, -\sigma) = \mathrm{hol}(X, s, \sigma)^{-1}$ and switching tensor factors corresponds to taking a transpose. Because $\mathrm{hol}(X, s, \sigma)$ preserves the canonical bilinear form on $\mathbb{C}^2$, it is equal to its own inverse transpose in the SU(2) representation. □

**Example 9.11.** An example of Proposition 9.10 can be seen in Examples 9.8 and 9.9

**Proposition 9.12.**

1. *Let $(X, s, \sigma)$, $(X', s', \sigma')$, $(X'', s'', \sigma'')$, and $f$ be as in Lemma 6.22. Recall that this means $f$ maps the right endpoint of $(X, s, \sigma)$ to the left endpoint of $(X', s', \sigma')$ and $(X'', s'', \sigma'')$ is the result of gluing these two Spin intervals together along $f$. Then $Z_1(X'', s'', \sigma'')$ is obtained from $Z_1(X, s, \sigma) \otimes Z_1(X', s', \sigma')$ by contracting the two tensor factors corresponding to the glued points.*

2. *Let $(X, s, \sigma)$ be a pointed Spin interval and let $f$ be a Spin map identifying the two boundary sections of $X$. Let $(X', s')$ be the (twisted) Spin circle obtained by gluing $s$ along $f$. Then $\lambda(X', s')$ is obtained from $Z_1(X, s, \sigma)$ by contracting the two tensor factors of $Z_1(X, s, \sigma)$.*

*Proof.* Part 1 follows from the Lemma 6.22 which says that

$$\mathrm{hol}(X'', s'', \sigma'') = \mathrm{hol}(X, s, \sigma)\,\mathrm{hol}(X', s', \sigma')$$

and the fact that contraction of tensor factors corresponds to composition of linear maps.

Part 2 follows by reducing to the case where $s$ and $\sigma$ are obtained from a trivialization that twists around the tangent direction of $X$ some half integral number of times and whose first vector points along the orientation direction of $X$. One can reduce to this case by applying Lemma 6.20 and possibly Lemma

6.21 to reverse the orientation of $X$. In this situation, the holonomy is of the form

$$\begin{pmatrix} i^n & 0 \\ 0 & (-i)^n \end{pmatrix}$$

and $\lambda(X', s') = -(i^n)$. If the first vector of the trivialization points along the tangent direction, then $Z_1(X, s, \sigma) = i^n \xi \otimes v$, whose contraction is $-i^n$. $\square$

Because of the above Proposition, make the following definition:

**Definition 9.13.** If $(X, s)$ is a Spin or twisted Spin circle, set $Z_1(X, s) = \lambda(X, s)$.

Extend $Z_1$ to disconnected Spin 1-manifolds with boundary sections by the asserting that

$$Z_1(X \sqcup X', s \sqcup s', \sigma \sqcup \sigma') = Z_1(X, s, \sigma) \otimes Z_1(X', s', \sigma')$$

Proposition 9.12 says that $Z_1$ obeys a very simple gluing law: if you glue two Spin 1-manifolds together at two points, contract the tensor factors corresponding to those points.

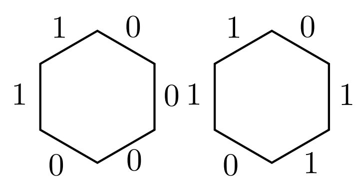
# 10 Extending $\beta$

The goal of this section is to extend $\beta$ (Definition 6.31) to $\text{Pin}^-$ surfaces with corners. The way to do this is straightforward: the definition of $\beta$ for closed surfaces uses the definition of $\lambda$ for closed 1-manifolds, so replace $\lambda$ in the definition of $\beta$ by $Z_1$. Before doing so it is worthwhile to recall some facts about homology and cohomology with $\mathbb{Z}/2$ coefficients.

Let $\Sigma$ be a surface and $\mathcal{P}$ a decomposition of $\partial\Sigma$ so that every component of $\partial\Sigma$ contains a vertex of $\mathcal{P}$. For example, $\Sigma$ might be $\Delta^2$ and $\mathcal{P}$ might be its three boundary intervals. Let $V$ be the vertex set of $\mathcal{P}$. Recall that the vertices of $\mathcal{P}$ are the vertices in $\Sigma$, not the union of the vertices of the pieces of $\mathcal{P}$. For example if $\mathcal{P}$ is the usual boundary decomposition of $\Delta^2$ then $\mathcal{P}$ has three vertices. As in Example 4.7,

$$H^1(\Sigma, V; \mathbb{Z}/2) \cong \text{Hom}(\pi_1(\Sigma, V), \mathbb{Z}/2)$$

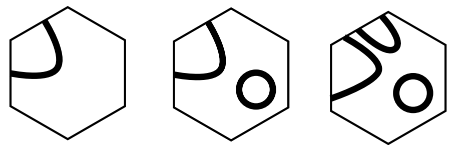
and each class in $H^1(\Sigma, V; \mathbb{Z}/2)$ determines a labeling of each component of $\partial\Sigma \setminus V$ given by whether or not that arc or circle is assigned 0 or 1 in $\mathbb{Z}/2$. For example, if $\Sigma$ is a hexagon and $V$ its six vertices, then here are two classes in $H^1(\Sigma, V; \mathbb{Z}/2)$:



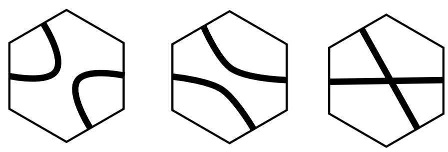

(19)

There's a Lefschetz duality isomorphism $H^1(\Sigma, V; \mathbb{Z}/2) \cong H_1(\Sigma, \partial\Sigma \setminus V; \mathbb{Z}/2)$ and the classes in the latter space are represented by (nonunique) curves

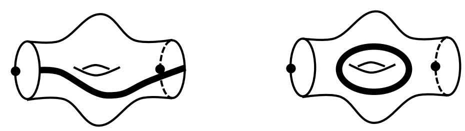
Poincare dual to the classes in the first. For example the first example (19) corresponds to curves



or the second example in (19) corresponds to curves

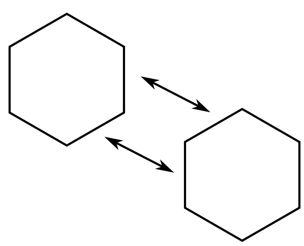


As another example, let $\Sigma$ be a two-holed torus and let $V$ be two vertices, one on each component. Here are some curves representing classes in $H_1(\Sigma, \partial\Sigma \setminus V; \mathbb{Z}/2)$:

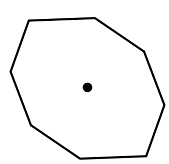


Note that one can always ensure that each class is represented by an embedded curve that meets each component of $\partial\Sigma \setminus V$ in at most a single point.

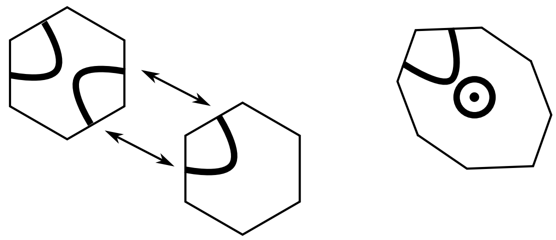
Consider a map $f$ from part of the boundary of $(\Sigma, \mathcal{P})$ to itself. For example, $\Sigma$ might be a union of two hexagons, $\mathcal{P}$ might be the twelve edges, and $f$ might identify some edges as follows



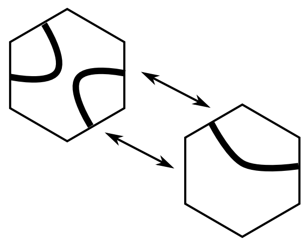
Let $(\Sigma_f, \mathcal{P}_f)$ be $(\Sigma, \mathcal{P})$ glued together along $f$, e.g.



Note that in this example, one of the exterior vertices of $\Sigma$ becomes interior in $\mathcal{P}_f$. Let $V_f$ denote the vertices of $\mathcal{P}_f$. Some classes in $H_1(\Sigma, \Sigma \setminus V; \mathbb{Z}/2)$ glue together to form classes in $H_1(\Sigma_f, \Sigma_f \setminus V_f; \mathbb{Z}/2)$, e.g.,



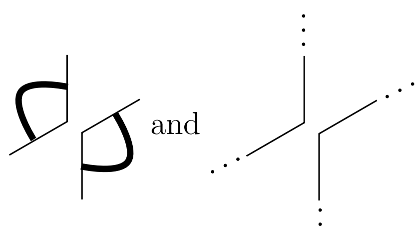
and some classes in $H_1(\Sigma, \Sigma \setminus V; \mathbb{Z}/2)$ do not glue together to form classes in $H_1(\Sigma_f, \Sigma_f \setminus V_f; \mathbb{Z}/2)$, e.g.,



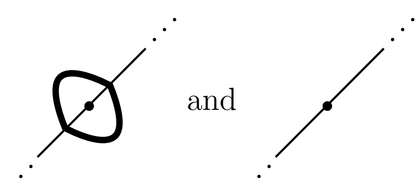
Therefore there's a partially defined map

$$H_1(\Sigma, \partial\Sigma \setminus V; \mathbb{Z}/2) \to H_1(\Sigma_f, \partial\Sigma_f \setminus V_f; \mathbb{Z}/2). \tag{20}$$

**Remark 10.1.** If $V_f$ is nonempty, then the map (20) is $2^n : 1$ where $n$ is the number of interior points of $\Sigma_f$ corresponding to vertices of $\Sigma$. This is essentially because




represent different classes in $H_1(\Sigma, \partial\Sigma \setminus V; \mathbb{Z}/2)$ but




represent the same class in $H_1(\Sigma_f, \partial\Sigma_f \setminus V_f; \mathbb{Z}/2)$.

**Definition 10.2.** Let $S$ be an open interval and $K$ a real line bundle on $S$. Suppose that $\underline{\mathbb{R}} \oplus TS \oplus K$ is oriented and let $s$ be a Spin lift of $F(\underline{\mathbb{R}} \oplus TS \oplus K)$. Given $p \in S$ let $\sigma : p \to s$ be a section of $s$ over $p$ such that $\sigma$ covers a frame whose first vector is $\pm u_1$ where $u_1$ is the standard basis of $\mathbb{R}$. Define

$$Z_2(S, s, \sigma) = \mathbb{C} \oplus Z_1(p, s|_p, \sigma)$$

This is $\mathbb{C} \oplus L$ if the first vector of $\sigma$ is $u_1$ and it is $\mathbb{C} \oplus L^*$ if the first vector of $\sigma$ is $-u_1$. Here $\mathbb{C}$ is the usual (even) copy of $\mathbb{C}$ and $L$ and $L^*$ are as in the definition of $Z_1$.

Extend this definition to disjoint unions by tensor product:

$$Z_2(S \sqcup S', s \cup s', \sigma \cup \sigma') = Z_2(S, s, \sigma) \otimes Z_2(S', s', \sigma')$$

Note that the order on the right is arbitrary. It is canonically isomorphic to

$$Z_2(S', s', \sigma') \otimes Z_2(S, s, \sigma)$$

and this canonical isomorphism is implicit in the notation $Z_2(S \sqcup S', s \cup s', \sigma \cup \sigma')$ since disjoint unions are not ordered.

**Definition 10.3.** Let $S$ be 1-manifold and let $\mathcal{P}$ be a decomposition of $S$ into intervals. Let $K$ be a real line bundle on $S$, let $\underline{\mathbb{R}} \oplus TS \oplus K$ be oriented and let $s$ be a Spin lift of $F(\underline{\mathbb{R}} \oplus TS \oplus K)$. Let $\sigma$ be a section of $s$ over points $p \subset S$ where there is precisely one point of $p$ in each piece of $\mathcal{P}$. Define

$$Z_2(S, s, \mathcal{P}, \sigma) = \bigotimes_{P \in \mathcal{P}} Z_2(P, s|_P, \sigma|_P)$$

The order of the tensor factors on the right is not specified, but different orderings are canonically isomorphic.

Note that if the first vector of a frame covered by $\sigma$ is in $\pm u_1$, then the first vector of the frame covered by $-\sigma$ is $\mp u_1$. Therefore there's a pairing $Z_2(S, s, \sigma) \otimes Z_2(S, -s, -\sigma) \to \mathbb{C}$ defined by the sum of the natural pairing on $L$ and $L^*$ factors, and the canonical pairing $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$. The pairing between the even and odd parts is defined to be zero.

**Definition 10.4.** Let $(\Sigma, \mathcal{P}, s)$ be a connected $\mathrm{Pin}^-$ surface with boundary decomposition $\mathcal{P}$ such that $\partial\Sigma$ is nonempty and each component of $\partial\Sigma$ contains a vertex of $\mathcal{P}$. That $\Sigma$ is a $\mathrm{Pin}^-$ surface means that there is a line bundle $K$ on $\Sigma$ and $s$ is a Spin lift of $F(T\Sigma \oplus K)$. For $S$ a boundary interval, define $s|_S$ to be the Spin lift of $F(\underline{\mathbb{R}} \oplus TS \oplus K)$ defined by using the outward normal to trivialize $\underline{\mathbb{R}}$. Then $-s|_S$ denotes the Spin lift corresponding to the inward normal.

Let $p$ be a collection of points, precisely one in each piece of $\mathcal{P}$, and let $\sigma : p \to s$ be a section. Let $\sigma|_S$ be the induced section on $s|_S$ and let $-\sigma|_S$ be the induced section on $-\sigma|_S$.

For each class $x \in H_1(\Sigma, \partial\Sigma \setminus \Sigma^0; \mathbb{Z}/2)$, fix an embedded oriented curve $\gamma_x$ with $\partial\gamma_x \subset p$. Let $n_e$ be the number of edges of $\mathcal{P}$. Let $\mathcal{P}^0$ be the set of vertices of $\mathcal{P}$. Define

$$Z_2(\Sigma, s, \mathcal{P}, \sigma) = \frac{2^{n_e/4} 2^{\chi(\Sigma)/2}}{2^{|\mathcal{P}^0|/2}} \sum_{x \in H_1(\Sigma, \partial\Sigma \setminus \mathcal{P}^0; \mathbb{Z}/2)} Z_1(\gamma_x, s|_{\gamma_x}, \sigma|_{\partial\gamma_x}) \qquad (21)$$

and note that

$$Z_2(\Sigma, s, \mathcal{P}, \sigma) \in Z_2(\partial\Sigma, s, \mathcal{P}, \sigma). \tag{22}$$

Proposition 10.5 below is crucial to the well-definedness of $Z_2$. If $(\Sigma, s)$ is a closed $\mathrm{Pin}^-$ surface, set

$$Z_2(\Sigma, s) = Z_2(\Sigma, s, \emptyset, \emptyset) = \frac{2^{\chi(\Sigma)/2}}{2^{b_0(\Sigma)}} \sum_{x \in H_1(\Sigma;\mathbb{Z}/2)} Z_1(x, s|_x) = \beta(\Sigma, s)$$

Extend the definition of $Z_2$ to disconnected surfaces by disjoint union:

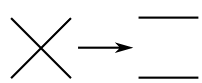
$$Z_2(\Sigma \sqcup \Sigma', s \sqcup s', \mathcal{P} \sqcup \mathcal{P}', \sigma \sqcup \sigma') = Z_2(\Sigma, s, \mathcal{P}, \sigma) \otimes Z_2(\Sigma', s', \mathcal{P}', \sigma')$$

For some simple examples of $Z_2$ applied to Spin surfaces, see Propositions 11.2 and 11.3 in the next section.

**Proposition 10.5.** *$Z_2(\Sigma, s, \mathcal{P}, \sigma)$ does not depend on the choice of curves $\gamma_x$.*

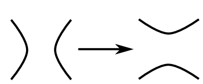
*Proof.* Proposition 9.10 implies that $Z_1(\gamma_x, s|_{\gamma_x}, \sigma)$ is invariant under simultaneously reversing the orientation on $\gamma_x$ and switching the tensor factors of $Z_1(\partial\gamma_x, s|_{\gamma_x}, \sigma)$. Therefore for a fixed ordering of the tensor factors of (22) $Z_1(\Sigma, s, \sigma)$ does not depend on the orientations of the curves $\gamma_x$.

If two different immersed unoriented curves represent the same class in $H_1(\Sigma, \partial\Sigma \setminus \mathcal{P}^0; \mathbb{Z}/2)$, then they differ by isotopy, addition/removal of bounding circles, and the move



(23)

Invariance under isotopy follows from the invariance of $Z_1$ under Spin isomorphism, invariance under addition of bounding circles follows from the fact that $Z_1(\mathsf{S}, s_{\mathsf{S},\mathrm{b}}) = 1$ or, more sophisticatedly, Proposition 6.14. If $\gamma_x$ and $\gamma_x'$ differ by the move (23), then $Z_1(\gamma_x, s|_{\gamma_x}, \sigma) = -Z_1(\gamma_x', s|_{\gamma_x'}, \sigma)$. This can be seen by isotoping the Spin structure to come from a constant trivialization in a neighborhood of the crossing, and then $Z_1$ applied to the crossing is the map that switches the tensor factors of $L \otimes L$ or $L \otimes L^*$ or $L^* \otimes L$ or $L^* \otimes L^*$, depending on how the curves are oriented. By applying this twice, there is invariance under the move



Any two embedded curves representing the same class are related via this move, isotopy, and addition/removal of bounding circles. □

Let $S$ and $S'$ be intervals and let $f : (S, s, \sigma) \to (S', s', \sigma')$ be a map of Spin lifts that covers a map on frames $F(\underline{\mathbb{R}} \oplus TS \oplus K) \to F(\underline{\mathbb{R}} \oplus TS \oplus K)$ that is the identity on the $\underline{\mathbb{R}}$ summand and induced by an isometry $S \to S'$ on the $TS$ summand. Suppose that $f(\sigma) = \sigma'$. Then $Z_2(S, s, \sigma) = Z_2(S', s', \sigma')$. Set $f_*$ to be the identity map. This extends to a disjoint union of intervals in the obvious way.

$Z_2$ obeys the following "gluing law":

**Proposition 10.6.** *Let $(\Sigma, s, \mathcal{P}, \sigma)$ be a* $\mathrm{Pin}^-$ *surface with $\mathcal{P}$ a decomposition of $\partial\Sigma$ into intervals. Here $\sigma : p \to s$ is a section and $p$ a union of points, one on each piece of $\mathcal{P}$. Let $X, Y$ be subsets of $\partial\Sigma$ that are unions of pieces of $\mathcal{P}$ and such that $X$ and $Y$ have disjoint interiors. Let $f : s|_X \to -s|_Y$ be such that $f(\sigma|_X) = -\sigma|_Y$. Let $(\Sigma_f, s_f)$ denote $s$ glued along $f$ and let $\sigma_f$ be the restriction of $\sigma$ to $\partial\Sigma_f$. Let $\mathcal{P}_f$ be the induced decomposition of $\partial\Sigma_f$ into intervals. Suppose that $s_f$ extends over the interior points in $\Sigma_f$ coming from the vertices of $\mathcal{P}$.*

*Then $Z_2(\Sigma_f, s_f, \mathcal{P}_f, \sigma_f)$ is obtained from $Z_2(\Sigma, s, \mathcal{P}, \sigma)$ by the following process. Apply $f_*$ to the tensor factors of $Z_2(\partial\Sigma, s|_{\partial\Sigma}, \mathcal{P}, \sigma|_{\partial\Sigma})$ corresponding to $X$, contract the tensor factors that correspond under the map $f$, then divide by $2^{m/2}$ where $m$ is the number of vertices of $\mathcal{P}$ that become interior in $\Sigma_f$.*

**Remark 10.7.** If you change the section on a boundary interval $S$ by the nontrivial central element in Spin(3), then $Z_2(\Sigma, s, \mathcal{P}, \sigma)$ changes by $x \mapsto (-1)^{\deg(x)}x$ applied to the tensor factor corresponding to $S$. This is because one way of achieving such a change of section involves gluing on a Spin bigon whose holonomy across is the nontrivial central element in Spin(3). The nontrivial central element acts on $L$ and $L^*$ by multiplication by $-1$. This remark enters into gluing computations in this paper when edges are marked in green.

*Proof of Proposition 10.6.* Suppose first that $\Sigma_f$ is connected and that $\mathcal{P}_f^0 \neq \emptyset$. The classes $x \in H_1(\Sigma, \partial\Sigma \setminus \mathcal{P}^0; \mathbb{Z}/2)$ that descend to classes in $H_1(\Sigma_f, \partial\Sigma_f \setminus \mathcal{P}_f^0; \mathbb{Z}/2)$ are those that are represented by curves whose intersections with $X$ and $Y$ correspond under $f$. The terms in (21) that do not correspond to these curves vanish in the pairing of $Z_2(Y, -s|_Y, \mathcal{P}, -\sigma|_Y) \otimes Z_2(Y, s|_Y, \mathcal{P}, \sigma|_Y) \to \mathbb{C}$. Therefore if you apply $f_*$, contract, then divide by $2^{m/2}$ you end up with

$$\frac{2^{n_e(\Sigma)/4} 2^{\chi(\Sigma)/2} 2^m}{2^{m/2} 2^{|\mathcal{P}^0|/2}} \sum_{x \in H_1(\Sigma_f, \partial\Sigma_f \setminus \mathcal{P}_f^0; \mathbb{Z}/2)} Z_1(\gamma_x, s|_{\gamma_x}, \sigma|_{\partial\gamma_x})$$

Here the factor of $2^m$ comes from Remark 10.1. To apply Remark 10.1, the assumption that $\mathcal{P}_f^0 \neq \emptyset$ is needed.

It is easy to see that

$$\chi(\Sigma_f) = \chi(\Sigma) - \chi(X)$$

$$|\mathcal{P}_f^0| = |\mathcal{P}^0| - 2m - 2\chi(X)$$

$$\chi(X) = \frac{n_e(\Sigma) - n_e(\Sigma_f)}{2} - m$$

and from these it follows that

$$\frac{2^{n_e(\Sigma)/4} 2^{\chi(\Sigma)/2} 2^m}{2^{m/2}} = \frac{2^{n_e(\Sigma_f)} 2^{\chi(\Sigma_f)/2}}{2^{|\mathcal{P}_f^0|/2}}$$

Next suppose that $\Sigma_f$ is connected and closed. Then the map $H_1(\Sigma, \partial\Sigma \setminus \mathcal{P}^0; \mathbb{Z}/2) \to H_1(\Sigma_f; \mathbb{Z}/2)$ is, where defined, a $2^{m-1} : 1$ map. Then if you apply

$f_* \otimes \mathrm{id}$ to $Z_2(\Sigma, s, \mathcal{P}, \sigma)$, contract the factors that correspond under $f$, and divide by $2^{m/2}$, you end up with

$$\frac{2^{n_e(\Sigma)/4} 2^{\chi(\Sigma)/2} 2^{m-1}}{2^{m/2} 2^{|\mathcal{P}^0|/2}} \sum_{x \in H_1(\Sigma_f;\mathbb{Z}/2)} Z_1(\gamma_x, s|_{\gamma_x}).$$

In this case, $|\mathcal{P}^0| = n_e(\Sigma) = 2m$ and $\chi(\Sigma) = \chi(\Sigma_f)$, so

$$\frac{2^{n_e(\Sigma)/4} 2^{\chi(\Sigma)/2} 2^{m/2}}{2^{|\mathcal{P}^0|/2} \cdot 2} = \frac{2^{\chi(\Sigma)/2}}{2^{b_0(\Sigma)}}$$

□

Let $(S, t)$ be a Spin interval. Then $t$ induces a Spin structure $s$ on $c(S)$. Recall that $c(S)$ is obtained from $\Delta^1 \times S$ and collapsing $(\partial \Delta^1) \times S$ to points. Let $\mathcal{P}$ be the decomposition of $\partial c(S)$ into the two intervals. Note that $\mathcal{P}^0$ is two points and that $s$ is not defined at these two points. Let $p, p'$ be two points in $S$ and let $\sigma : p \to s$ and $\sigma' : p' \to s$ be sections at these points. Then $\sigma \sqcup \sigma'$ defines a section on $p \sqcup p' \subset c(S)$, where $p$ is on $\{0\} \times S$ and $p'$ is on $\{1\} \times S$. Then

$$\begin{aligned} Z_2(c(S), s, \mathcal{P}, \sigma \sqcup \sigma') \in & Z_2(S, -t, -\sigma) \otimes Z_2(S, t, \sigma') \\ & \cong Z_2(S, t, \sigma)^* \otimes Z_2(S, t, \sigma') \\ & \cong \mathrm{Hom}(Z_2(S, t, \sigma), Z_2(S, t, \sigma')) \end{aligned}$$

defines a canonical isomorphism from $Z_2(S, t, \sigma)$ to $Z_2(S, t, \sigma')$, call it $\psi_{S,t;\sigma,\sigma'}$. Define $Z_2(S, t)$ to be the vector spaces $\{Z_2(S, t, \sigma)\}$ identified via these maps where here $\sigma$ runs over all sections $\sigma : p \to t$ for points $p \in S$.

**Proposition 10.8.** *Let $(\Sigma, s)$ be a $\mathrm{Pin}^-$ surface and $\sigma : p \to s$ a section where $p \subset \partial \Sigma$ is a collection of points, one on each boundary piece of $\Sigma$. Suppose $\sigma'$ differs from $\sigma$ on one boundary piece, call it $S$. Then if you apply $\psi_{S,s|_S;\sigma,\sigma'}$ to the tensor factor of $Z_2(\Sigma, s, \sigma)$ corresponding to $S$, you get $Z_2(\Sigma, s, \sigma')$.*

*Proof.* This is a result of the gluing law for $Z_2$, where you glue $c(S)$ onto $\Sigma$ at $S$. □

**Corollary 10.9.** *Let $(\Sigma, s, \mathcal{P}, \sigma)$ be a $\mathrm{Pin}^-$ surface with a section $\sigma$ at points on its boundary pieces. Then by definition*

$$Z_2(\Sigma, s, \mathcal{P}, \sigma) \in \bigotimes_{S \in \mathcal{P}} Z_2(S, s|_S, \sigma|_S)$$

*but as $\sigma$ ranges over all such boundary sections, then these give a well-defined element of*

$$\bigotimes_{S \in \mathcal{P}} Z_2(S, s|_S).$$

**Definition 10.10.** Set $Z_2(S, s, \mathcal{P}) = \bigotimes_{S\in\mathcal{P}} Z_2(S, s|_S)$. By the last corollary, $Z_2(\Sigma, s, \mathcal{P}, \sigma)$ gives a well-defined element in $Z_2(\partial\Sigma, s|_{\partial\Sigma}, \mathcal{P})$. Call this element $Z_2(\Sigma, s, \mathcal{P})$.

Let $(S, t)$ be a Spin circle. Let $s$ be the corresponding Spin structure on $c(S) = \Delta^1 \times S$. Given a point $x \in S$, let $\mathcal{P}$ be the decomposition of $\{0, 1\} \times S$ into the two intervals $\{0, 1\} \times (S \setminus x)$.

$$\begin{aligned}\frac{1}{\sqrt{2}} Z_2(c(S), s, \mathcal{P}) \in & Z_2(-S \setminus x, -s|_{S\setminus x}) \otimes Z_2(S \setminus x, s|_{S\setminus x}) \\ & \cong Z_2(S \setminus x, t)^* \otimes Z_2(S \setminus x, t) \\ & \cong \mathrm{Hom}(Z_2(S \setminus x, t), Z_2(S \setminus x, t)) \end{aligned} \tag{24}$$

defines a projection. This leads to the following definition:

**Definition 10.11.** Let $Z_2(S, t)$ be the image of $Z_2(S \setminus x, t)$ under the above projection. In particular, $Z_2(S, t)$ is defined to be a subspace of $Z_2(S \setminus x, t)$ for some $x \in S$. One can check that different choices of $x$ lead to canonically isomorphic spaces. If $\mathcal{P}$ is a decomposition of a Spin 1-manifold that involves circles and intervals, set

$$Z_2(S, t, \mathcal{P}) = \bigotimes_{P\in\mathcal{P}} Z_2(P, t)$$

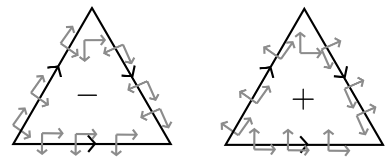
Let $(\Sigma, s, \mathcal{P})$ be a Pin surface with boundary decomposed by $\mathcal{P}$. $Z_2(\Sigma, s, \mathcal{P})$ has been defined in the case where all the pieces $\mathcal{P}$ of $\partial\Sigma$ are intervals. Suppose $\mathcal{P}$ contains circles. In order to define $Z_2(\Sigma, s, \mathcal{P})$, remove points from each circle in $\mathcal{P}$ to turn them into intervals and let $\mathcal{P}'$ be this decomposition. Suppose for simplicity that $\mathcal{P}$ consists of a single circle $S$ so that $\mathcal{P}'$ consists of a single interval $S \setminus x$. Set $Z_2(\Sigma, s, \mathcal{P}) := Z_2(\Sigma, s, \mathcal{P}')$. Since gluing on a copy of $(c(S), s)$ to $\partial\Sigma$ results in the same $\mathrm{Pin}^-$ manifold, it follows that $Z_2(\Sigma, s, \mathcal{P}')$ is in the image of the projection (24) and thus $Z_2(\Sigma, s, \mathcal{P}') \in Z_2(\partial\Sigma, s|_{\partial\Sigma}, \mathcal{P})$. The case where $\Sigma$ has multiple circle boundary components generalizes easily.

Summing up, $Z_2(\Sigma, s, \mathcal{P}) \in Z_2(\partial\Sigma, s, \mathcal{P})$ is defined for any $\mathrm{Pin}^-$ surface with boundary decomposition. Its precise computation relies on making all boundary pieces into intervals, and picking a section $\sigma : p \to s$ where $p$ is a collection of points, one on each piece of the boundary.

# 11 Some Computations of $Z_2$

This section concerns itself with the computation of $Z_2$ on various explicit $\mathrm{Pin}^-$ surfaces. Typically the boundary decomposition of such surfaces is clear from context (for example, $\partial\Delta^2$ should be split into three intervals). Furthermore, all the $\mathrm{Pin}^-$ structures come from trivializations, so the trivial section can be used as a section for points on each boundary piece. Therefore $\mathcal{P}$ and/or $\sigma$ will often be omitted from the notation.

Consider the following standard trivializations of $\pm\Delta^2$ from (14)



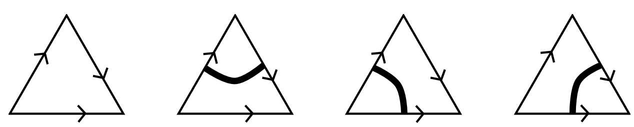

Let $(\pm\Delta^2, s_{\pm\Delta^2}, \sigma_{\pm\Delta^2})$ be the associated Spin surfaces with section. Here $s_{\pm\Delta^2}$ is the Spin structure where the stabilization comes after the tangent space. Recall that $s_{\pm\Delta^2}$ is constructed by pulling back the canonical Spin(3) lift of $\underline{\mathrm{SO}(3)}$ via the trivialization, and the section $\sigma$ is obtained by pulling back the trivial section of $\underline{\mathrm{Spin}(3)}$.

**Definition 11.1.** Set $A = \mathbb{C} \oplus L$ and $A^* = \mathbb{C}^* \oplus L^*$, with the obvious pairing between $A$ and $A^*$. Let $a_0, a_1$ be a basis of $A$ where $a_0$ is even and $a_1$ is odd. Let $a^0, a^1$ be a dual basis such that $\langle a_0, a^0\rangle = \langle a_1, a^1\rangle = 1$. Then $\langle a^1, a_1\rangle = -1$.

**Proposition 11.2.** *Recall the ordering of the boundary components of* $+\Delta^2$ *by* [02], [01], [12] *and use this order for the ordering of the tensor factors of* $Z_2(+\Delta^2, s_{+\Delta^2}, \sigma_{+\Delta^2})$. *Then*

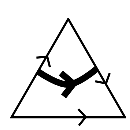
$$Z_2(+\Delta^2, s_{+\Delta^2}, \sigma_{+\Delta^2}) = 2^{-1/4}(a^0 \otimes a_0 \otimes a_0 + a^0 \otimes a_1 \otimes a_1 + a^1 \otimes a_1 \otimes a_0 + a^1 \otimes a_0 \otimes a_1) \tag{25}$$

*Proof.* Let $V$ be the vertex set of of $\Delta^2$. Then $2^{n_e/4+\chi/2-|V|/2} = 2^{-1/4}$. There are four elements in $H_1(\Delta^2, \partial\Delta^2 \setminus V; \mathbb{Z}/2)$:



(26)

Each of these corresponds to one term in (25). For example, if $\gamma$ is the empty curve, then $Z_1(\gamma, s_{+\Delta^2}|_\gamma) = 1$, which corresponds to $1 \in \mathbb{C}^* \otimes \mathbb{C} \otimes \mathbb{C} \subset A^* \otimes A \otimes A$. Suppose $\gamma$ is the second of the four curves, and orient it as the boundary intervals are ordered:



Then $Z_1(\gamma, s_{+\Delta^2}|_\gamma) \in L \otimes L$, since both boundary sections cover frames whose first vector points out of $\gamma$. The holonomy of $\gamma$ can be computed, as in Examples 9.8 and 9.9, to be[5]

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

[5] Because the standard 2-simplex is oriented so that $([01], [02])$ provides an orientation at [0], for $\Delta^2$ is drawn here in such a way that the basis vector of the stabilization direction points into the page.

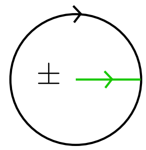
so that $Z_1(\gamma, (s_{+\Delta^2})|_\gamma, (\sigma_{+\Delta^2})|_\gamma) = v \otimes v$. In terms of the definition of $A$, this is $a_1 \otimes a_1$. Therefore the second term is $a^0 \otimes a_1 \otimes a_1$. The other terms can be computed similarly. $\square$

Analogously:

**Proposition 11.3.** *Recall the ordering of the boundary pieces of $-\Delta^2$ as* [12], [01], [02] *and order the tensor factors of $Z_2(-\Delta^2, s_{-\Delta^2})$ in the same way. Then*

$$Z_2(-\Delta^2, s_{-\Delta^2}) = 2^{-1/4}(a^0 \otimes a^0 \otimes a_0 + a^1 \otimes a^1 \otimes a_0 + a^1 \otimes a^0 \otimes a_1 + a^0 \otimes a^1 \otimes a_1) \tag{27}$$

Therefore $Z(\pm\Delta^2)$ form an algebra and coalgebra structure on $A$. In fact, this algebra is associative and coassociative and satisfies the Frobenius relations for all the same reasons as in Section 4. It is therefore a super Frobenius algebra. It is worthwhile checking the unit of $A$. The appropriate disk is obtained by gluing two of the edges of $(+\Delta^2, s_{+\Delta^2})$ together:

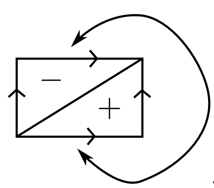
$(\mathsf{D}, s_{\mathsf{D}}) =$ 


Recall that a green edge means the Spin bundle has been glued with the nontrivial Spin automorphism along that edge. This ensures that the Spin structure extends over the interior vertex. Since the homology of the disk is trivial, it follows immediately from the definition of $Z_2$ that $Z_2(\mathsf{D}, s_{\mathsf{D}}) = 2^{1/4}a_0$. By the gluing law (Proposition 10.6) and the remark following it (Remark 10.7), this result can also be obtained by applying $(-1)^{\deg}$ to the first tensor factor of (25), contracting the first two tensor factors of the result, then multiplying by $2^{-1/2}$ for the interior vertex. This is

$$2^{-3/4}(\langle a^0, a_0\rangle a_0 + \langle a^0, a_1\rangle a_1 - \langle a^1, a_1\rangle a_0 - \langle a^1, a_0\rangle a_1) = 2^{1/4}a_0$$

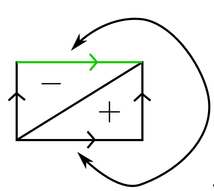
(note that $\langle a^1, a_1\rangle = -1$).

Consider the cylinder with nonbounding Spin structure on its nontrivial circle:




.

Then $Z_2$ of this cylinder is obtained from $Z_2(-\Delta^2, s_{-\Delta^2}) \otimes Z_2(+\Delta^2, s_{+\Delta^2})$ by contracting tensor factors 1 and 5 and tensor factors 3 and 4. The result is $2^{1/2}a^1 \otimes a_1$. Therefore $Z(\mathsf{S}, s_{\mathsf{S},\mathrm{nb}}) \subset A$ is the span of $a_1$. Next consider the cylinder with bounding Spin structure on its nontrivial circle:




.

$Z_2$ of this cylinder is obtained in the same way, except applying $(-1)^{\deg}$ to the first tensor factor before contracting. The result is $2^{1/2}a^0 \otimes a_0$. Therefore $Z(\mathsf{S}, s_{\mathsf{S},\mathrm{b}}) \subset A$ is the span of $a_0$.

It is satisfying to note that $Z(\mathsf{S}, s_{\mathsf{S},\mathrm{b}})$ is the center of the super Frobenius algebra $A$, just as in the case of a 2d pre-TQFT. $Z(\mathsf{S}, s_{\mathsf{S},\mathrm{nb}})$, however, is not the center of $A$.

One needs pairings

$$Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{b}}) \otimes Z_2(-\mathsf{S}, -s_{\mathsf{S},\mathrm{b}}) \to \mathbb{C}$$

and

$$Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{nb}}) \otimes Z_2(-\mathsf{S}, -s_{\mathsf{S},\mathrm{nb}}) \to \mathbb{C}.$$

Following Proposition 3.15, in order to define these pairings include $Z_2(\pm\mathsf{S}, s_{(\mathrm{n})\mathrm{b}})$ into $Z_2(\pm\Delta^1, s_{\Delta^1})$ and modify the pairing $Z_2(\Delta^1, s_{\Delta^1}) \otimes Z_2(-\Delta^1, -s_{\Delta^1}) \to \mathbb{C}$ by a factor of $\Gamma^{-1} = 2^{-1/2}$. More precisely, use the pairing $a_0 \otimes a^0 \mapsto 2^{-1/2}$ for the bounding circle and $a_1 \otimes a^1 \mapsto 2^{-1/2}$ for the nonbounding circle:

**Definition 11.4.** Define the pairing $Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{b}}) \otimes Z_2(-\mathsf{S}, -s_{\mathsf{S},\mathrm{b}}) \to \mathbb{C}$ by

$$\langle a_0, a^0 \rangle := 2^{-1/2}$$

and define the pairing $Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{nb}}) \otimes Z_2(-\mathsf{S}, -s_{\mathsf{S},\mathrm{nb}}) \to \mathbb{C}$ by

$$\langle a_1, a^1 \rangle := 2^{-1/2}$$

It will be helpful to set new notation for $Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{b}})$. Write $\widetilde{a}_0$ for the element $2^{1/4}a_0$ in $Z_2(\mathsf{S}, s_\mathrm{b})$ and write $\widetilde{a}^0$ for its dual under the pairing above, so $\widetilde{a}^0$ corresponds to $2^{1/4}a^0$ in $A^*$. Similarly, let $\widetilde{a}_1 \in Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{nb}}))$ correspond to $2^{1/4}a_1$ in $A$, and let $\widetilde{a}^1$ be its dual in $Z_2(-\mathsf{S}, -s_{\mathsf{S},\mathrm{nb}})$. So $\widetilde{a}^1$ corresponds to $2^{1/4}a^1$.

**Example 11.5.** Since $Z_2(\mathsf{C}, s_{\mathsf{C},\mathrm{b}}) = 2^{1/2}a^0 \otimes a_0$, then $Z_2(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$ is $\widetilde{a}^0 \otimes \widetilde{a}_0$ in $Z_2(\partial\mathsf{C}, s_{\mathsf{C},\mathrm{b}}|_{\partial\mathsf{C}})$, where $\partial\mathsf{C}$ is given the standard decomposition into two circles. Similarly, $Z_2(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}}) = \widetilde{a}^1 \otimes \widetilde{a}_1$.

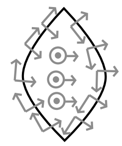
**Example 11.6.** Recall that $(\mathsf{P}, s_{\mathsf{P},000})$ is the Spin pair of pants obtained from gluing two copies of $(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$ to $(-\Delta^2, s_{-\Delta^2})$. Further glue on a disk to the output of the pair of pants to get a Spin cylinder $(C', s_{C'})$ where both boundary components are parametrized by copies of $Z_2(-\mathsf{S}, -s_{\mathsf{S},\mathrm{b}})$. Then $Z(C', s_{C'})$ is a self-pairing $Z(\mathsf{S}, s_{\mathsf{S},\mathrm{b}}) \otimes Z(\mathsf{S}, s_{\mathsf{S},\mathrm{b}}) \to \mathbb{C}$. It is the contraction of the four terms

$$(2^{1/2}a^0 \otimes a_0) \otimes (2^{1/2}a^0 \otimes a_0) \otimes (2^{-1/4}(a^0 \otimes a^0 \otimes a_0)) \otimes (2^{1/4}a^0)$$

along tensor factors 2 and 5, 4 and 6, and 7 and 8, divided by $2^{1/2}$ for the single interior vertex. The result is $2^{1/2}a^0 \otimes a^0 = \widetilde{a}^0 \otimes \widetilde{a}^0$.

A similar construction for a pair of pants with two nonbounding circles gives the pairing $\widetilde{a}^1 \otimes \widetilde{a}^1$.

**Example 11.7.** Consider next the bigon with the following trivialization:



This is the trivialization which on horizontal paths performs a counterclockwise half twist. In the trivialization the first basis vector is the one that points into the surface and the second is also pictured. The third is determined by these two. If $(\Sigma, s)$ is this Spin bigon and $\sigma$ is the trivial section from the trivialization restricted the midpoints of the two edges, then

$$Z_2(\Sigma, s, \sigma) = a^0 \otimes a_0 + ia^1 \otimes a_1$$

because the holonomy of a curve moving through the middle from left to right is

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The two sides of $\Sigma$ can be glued together antipodally, forming $\mathbb{R}P^2$. The Spin structure $s$ extends over the interior vertex. Then, writing $s'$ to denote the Spin structure on $\mathbb{R}P^2$ obtained by gluing up $s$,

$$Z_2(\mathbb{R}P^2, s') = \frac{1}{2^{1/2}}(\langle a^0, a_0\rangle + i\langle a^1, a_1\rangle) = e^{-2\pi i/8} = \zeta^{-1}$$

The other $\mathrm{Pin}^-$ structure on $\mathbb{R}P^2$ can be obtained by applying the nontrivial Spin automorphism before gluing the edges of $\Sigma$.

**Example 11.8.** Let $(\Sigma, s)$ be the Mobius band with one of its two $\mathrm{Pin}^-$ structures. $Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{b}})$ is the span of $a_0$ in $A$, so $Z_2(\Sigma, s) = za_0$ for some complex number $z$. Gluing a disk to the Mobius band produces $\mathbb{R}P^2$, and since $Z_2$ applied to the disk is $2^{1/4}a^0$, then $z = 2^{1/4}\zeta^{\pm 1}a_0$, depending on which $\mathrm{Pin}^-$ structure is selected. Therefore $Z_2(\Sigma, s) = 2^{1/4}\zeta^{\pm 1}a_0$ or $\zeta^{\pm 1}\widetilde{a}_0$

As a reality check, gluing two of these together produces either 1, 1, $i$ or $-i$, depending on which values of $\zeta^{\pm 1}$ are chosen. These are precisely the $\beta$ values for the four $\mathrm{Pin}^-$ structures on the Klein bottle.

**Example 11.9.** Let $(\Sigma, s)$ be the bounding cylinder connect-summed with one of the two $\mathrm{Pin}^-$ structures of $\mathbb{R}P^2$ or, equivalently, the pair of pants with the Mobius band glued onto one of its ends. Then the last example shows that $Z_2(\Sigma, s) = 2^{1/2}\zeta^{\pm 1}a^0 \otimes a_0$, depending on which $\mathrm{Pin}^-$ structure is chosen. Alternatively, this is $\zeta^{\pm 1}\widetilde{a}^0 \otimes \widetilde{a}_0$.

# 12 A 3d Spin pre-TQFT

**Definition 12.1.** Let $(\Sigma, s)$ be a connected Spin surface with nonempty triangulated boundary. Write $\Sigma^0$ for the vertices of $\Sigma$. To be precise about $s$, it is a Spin lift of $F(\underline{\mathbb{R}} \oplus T\Sigma)$. Let $\mathcal{B}$ be a collection of curves representing the elements of $H_1(\Sigma, \partial\Sigma \setminus \Sigma^0; \mathbb{Z}/2)$. Assert that each of these curves intersects a boundary interval of $\Sigma$ either at its midpoint or not at all. Let $S \in \mathcal{B}$. Then $s|_S$ is, by definition, the pullback of $s$, via the inclusion $S \hookrightarrow \Sigma$, to a Spin lift of $F(\underline{\mathbb{R}} \oplus TS \oplus \nu)$ where $\nu$ is the normal bundle of $S$ in $\Sigma$. Define

$$Z_3(\Sigma, s, \mathcal{B}) := \bigoplus_{S \in \mathcal{B}} Z_2(S, s|_S)$$

This vector space has a grading by $H_0(\partial\Sigma \setminus \Sigma^0; \mathbb{Z}/2)$ given by grading the summand corresponding to $S$ by the class of $\partial S \in H_0(\partial\Sigma \setminus \Sigma^0; \mathbb{Z}/2)$. In this manner $Z_3(\Sigma, s, \mathcal{B})$ is graded by $\mathbb{Z}/2$-labelings of the edges in $\partial\Sigma$. Let $Z_3(\Sigma, s, \mathcal{B})_\ell$ denote summand corresponding to a labeling $\ell$.

**Definition 12.2.** Define a bilinear form

$$Z_3(\Sigma, s, \mathcal{B}) \otimes Z_3(-\Sigma, -s, \mathcal{B}) \to \mathbb{C}$$

by using the pairing $Z_2(S, s|_S) \otimes Z_2(S, -s|_S) \to \mathbb{C}$ on each summand.

**Definition 12.3.** If $f : (\Sigma, s) \to (\Sigma', s')$, define $f_* : Z_3(\Sigma, s, \mathcal{B}) \to Z_3(\Sigma', s', f(\mathcal{B}))$ by acting as $f_* : Z_2(S, s|_S) \to Z_2(f(S), s'|_{f(S)})$ for each $S \in \mathcal{B}$.

One can easily check that $(f \circ g)_* = f_* g_*$.

**Definition 12.4.** Let $(\Sigma, s)$ be a (possibly disconnected) Spin surface with triangulated boundary and let $\mathcal{P}$ be a decomposition of $\Sigma$ into pieces, each of which has triangulated boundary. Note that $\Sigma^0 \subset \mathcal{P}^0$. Let $\mathcal{B}$ be a collection of curves representing each class in $H_1(P, \partial P \setminus \mathcal{P}^0; \mathbb{Z}/2)$ for each $P \in \mathcal{P}$. Also, somewhat abusively, add to $\mathcal{B}$ the properly embedded curves in $\Sigma$ that can be formed from the curves in $\mathcal{B}$. Then these form representatives of each class in $H_1(\Sigma \setminus \mathcal{P}^0, \partial\Sigma \setminus \Sigma^0; \mathbb{Z}/2)$. Let $\mathcal{B}|_P$ be the curves of $\mathcal{B}$ that are in $P$. Set

$$Z_3(\Sigma, s, \mathcal{P}, \mathcal{B})_\ell = \bigotimes_{P \in \mathcal{P}} Z_2(P, s, \mathcal{B}|_P)_{\ell|_P}$$

(recall here that $\ell$ is a $\mathbb{Z}/2$-labeling of the edges of $\mathcal{P}$) and

$$Z_3(\Sigma, s, \mathcal{P}, \mathcal{B}) = \bigoplus_{\ell} Z_3(\Sigma, s, \mathcal{P}, \mathcal{B})_\ell$$

So that $Z_3(\Sigma, s, \mathcal{P}, \mathcal{B})$ is naturally graded by $\mathbb{Z}/2$-labelings of the edges of $\mathcal{P}$.

**Definition 12.5.** Let $(M, s)$ be a Spin 3-manifold such that each component of $M$ has nonempty boundary and let $\mathcal{P}$ be a decomposition of $\partial M$ into surfaces with nonempty triangulated boundary. Let $\mathcal{B}$ be a choice of homology representatives for $(\partial M, \mathcal{P})$ as above. For each class $x \in H_2(M, \partial M \setminus \mathcal{P}^0; \mathbb{Z}/2)$, fix a properly embedded surface $\Sigma_x$ representing $x$ so that $\partial\Sigma_x \in \mathcal{B}$. Let $s|_{\Sigma_x}$

be the pullback of $s$ to $\Sigma_x$ to a Spin bundle on $F(T\Sigma_x \oplus \nu)$ where $\nu$ is the normal bundle to $\Sigma_x$ in $M$. Define

$$Z_3(M, s, \mathcal{B}) = \sum_{x \in H_1(M, \partial M \setminus \mathcal{P}^0; \mathbb{Z}/2)} Z_2(\Sigma_x, s|_{\Sigma_x})$$

then

$$Z_3(M, s, \mathcal{B}) \in Z_3(\partial M, s, \mathcal{P}, \mathcal{B})$$

(That $Z_3$ is well-defined depends on Proposition 12.6 below.) Let $Z_3(M, s, \mathcal{B})_\ell$ denote the projection of $Z_3(M, s, \mathcal{B})$ to $Z_3(M, s, \mathcal{B})_\ell$. Next suppose that $(M, s)$ is a closed Spin manifold. Set

$$Z_3(M, s) := Z_3(M, s, \emptyset) := \frac{1}{2^{b_0(M)}} \sum_{x \in H_2(M; \mathbb{Z}/2)} Z_2(\Sigma, s|_\Sigma) = \frac{1}{2^{b_0(M)}} \sum_{x \in H_2(M; \mathbb{Z}/2)} \beta(\Sigma, s|_\Sigma)$$

Extend $Z_3(M, s, \mathcal{P}, \mathcal{B})$ to disconnected manifolds by tensor product:

$$Z_3(M \sqcup M', s \sqcup s', \mathcal{P} \sqcup \mathcal{P}', \mathcal{B} \sqcup \mathcal{B}') = Z_3(M, s, \mathcal{P}, \mathcal{B}) \otimes Z_3(M', s', \mathcal{P}', \mathcal{B}')$$

**Proposition 12.6.** *Given $(M, s)$ as in Definition 12.5, and $x \in H_2(M, \partial M \setminus \mathcal{P}^0; \mathbb{Z}/2)$, then if $\Sigma_x$ and $\Sigma'_x$ both represent the class $x$ and agree on $\partial M$, then $Z_2(\Sigma_x, s|_{\Sigma_x}) = Z_2(\Sigma'_x, s|_{\Sigma'_x})$.*

*Proof.* The proof is entirely analogous to that of Proposition 10.5, except instead of isotopies of curves, it's isotopies of surfaces, instead of addition/removal of bounding circles, it's addition/removal of bounding surfaces, and instead of surgeries of curves along intersection points, it's surgeries of surfaces along intersection circles. Proposition 6.14 is used to show that $s$ restricts to a bounding Spin structure on null-homologous closed surfaces. □

**Example 12.7.** For $\pm\Delta^2$, let $\mathcal{B}(\Delta^2)$ be the four curves as in (26). Let $s_{\pm\Delta^2}$ be the Spin structure where the stabilization direction comes before the tangent direction. If you were to pick, on each of the three nonempty curves of $-\Delta^2$, a point $p$ and a section $\sigma : p \to s|_{-\Delta^2}$ that covers a frame whose first vector is $u_1$, then

$$Z_3(-\Delta^2, s|_{-\Delta^2}, \mathcal{B}(\Delta^2)) \cong \mathbb{C} \oplus A \oplus A \oplus A$$

one summand for each curve. On each of the three nonempty curves of $+\Delta^2$, pick a point $p$ and a section $\sigma : p \to s|_{+\Delta^2}$ which covers a frame whose first vector is $-u_1$. Then

$$Z_3(+\Delta^2, s|_{+\Delta^2}, \mathcal{B}(\Delta^2)) \cong \mathbb{C}^* \oplus A^* \oplus A^* \oplus A^*$$

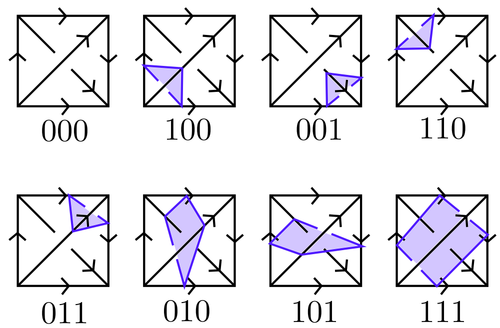
(where here as usual there is a canonical isomorphism $\mathbb{C} \cong \mathbb{C}^*$).

**Example 12.8.** Let $s_{\pm\Delta^3}$ be the Spin structures on $\pm\Delta^3$ from Section 7. Then

$$s_{+\Delta^3}|_{[023]} = s_{+\Delta^3}|_{[012]} = s_{-\Delta^2}$$

$$s_{+\Delta^3}|_{[123]} = s_{+\Delta^3}|_{[013]} = -s_{-\Delta^2}$$

The trivial section on $s_{\pm\Delta^3}$ covers a frame whose first vector is $u_1$ on the restrictions to [023] and [012] and $-u_1$ on the restrictions to [123] and [013]. Using this section to trivialize everything, letting $\mathcal{B}$ come from the curves in Example 12.7, and letting $\mathcal{P}_{\Delta^3}$ be the four pieces of $\partial\Delta^3$, then $Z_3(+\Delta^3, s_{+\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})$ is a sum of 8 terms, each corresponding to a surface slicing through $+\Delta^3$:




(28)

In order to make $Z_3(+\Delta^3, s_{+\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})$ precise, order its pieces as [123], [013], [023], [012]. Then $Z_3(+\Delta^3, s_{+\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})$ is an element of

$$Z_3(+\Delta^2, -s|_{-\Delta^2}, \mathcal{B}|_{[123]}) \otimes Z_3(+\Delta^2, -s_{-\Delta^2}, \mathcal{B}_{[013]})$$

$$\otimes Z_3(-\Delta^2, s|_{-\Delta^2}, \mathcal{B}|_{[023]}) \otimes Z_3(-\Delta^2, s_{-\Delta^2}, \mathcal{B}_{[012]})$$

The term in $Z_3(+\Delta^3, s_{+\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})$ corresponding to the first surface in (28) (the empty surface) is $1 \otimes 1 \otimes 1 \otimes 1$. The term in $Z_3(+\Delta^3, s_{+\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})$ corresponding to the second surface on the first row of (28) is[6]

$$2^{-1/4}(a^0 \otimes a_0 \otimes a_0 + a^1 \otimes a_0 \otimes a_1 + a^0 \otimes a_1 \otimes a_1 + a^1 \otimes a_1 \otimes a_0)$$

The term in $Z_3(+\Delta^3, s_{+\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})$ corresponding to the first term on the second row of (28)

$$2^{-1/4}(a^0 \otimes a_0 \otimes a_0 + a^1 \otimes a_0 \otimes a_1 - ia^1 \otimes a_1 \otimes a_0 + \zeta a^0 \otimes a_1 \otimes a_1)$$

The other terms five terms be computed similarly. In the following table, an entry $ijk$ in the left column refers to a labeling $[01] \mapsto i$, $[12] \mapsto j$, $[23] \mapsto k$ and this determines a labeling of all six edges.

[6] There really should be four tensor factors here. In order to simplify notation, the tensor factors which are 1 are dropped from the notation.

| Edge Labeling $\ell$ | $Z_3(+\Delta^3, s_{+\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})_\ell$ |
|---|---|
| 000 | 1 |
| 100 | $2^{-1/4}(a^0a_0a_0 + a^0a_1a_1 + a^1a_0a_1 + a^1a_1a_0)$ |
| 010 | $2^{-1/4}(a^0a^0a_0 + a^0a_1a_1 + a^1a_0a_1 + a^1a^1a_0)$ |
| 110 | $2^{-1/4}(a^0a^0a_0 + a^0a^1a_1 + \zeta^{-1}a^1a^0a_1 + \zeta a^1a^1a_0)$ |
| 011 | $2^{-1/4}(a^0a_0a_0 + a^1a_1a_0 + \zeta^{-1}a^1a_0a_1 + \zeta a^0a_1a_1)$ |
| 010 | $2^{-1/2}(a^0a^0a_0a_0 + a^0a^0a_1a_1 + a^0a^1a_1a_0 + a^0a^1a_0a_1$ $+a^1a^0a_1a_0 + a^1a^0a_0a_1 + a^1a^1a_0a_0 + a^1a^1a_1a_1)$ |
| 101 | $2^{-1/2}(a^0a^0a_0a_0 + a^0a^1a_1a_0 - ia^1a^0a_0a_1 + \zeta a^0a^0a_1a_1$ $+\zeta a^1a^1a_0a_0 + \zeta^{-1}a^1a^0a_1a_0 + \zeta^{-1}a^0a^1a_0a_1 + ia^1a^1a_1a_1)$ |
| 111 | $2^{-1/2}(a^0a^0a_0a_0 + \zeta a^1a^1a_0a_0 + \zeta a^0a^0a_1a_1 - ia^1a^0a_0a_1$ $+a^0a^1a_1a_0 + \zeta a^1a^0a_1a_0 + \zeta a^0a^1a_0a_1 - ia^1a^1a_1a_1)$ |

(29)

Reversing the orientation, $Z_3(-\Delta^3, s|_{-\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})$ can be computed similarly. It is convenient to set the order of the boundary faces as the reverse of the ordering for $+\Delta^3$: [012], [023], [013], [123]. Then $Z_3(-\Delta^3, s|_{-\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})$ is an element of

$$Z_3(\Delta^2, -s|_{-\Delta^2}, \mathcal{B}_{[012]}) \otimes Z_3(\Delta^2, -s_{-\Delta^2}, \mathcal{B}_{[023]})$$

$$\otimes Z_3(-\Delta^2, s|_{-\Delta^2}, \mathcal{B}_{[013]}) \otimes Z_3(-\Delta^2, s_{-\Delta^2}, \mathcal{B}_{[123]})$$

that is a sum of eight terms, one corresponding to each surface in (28). The formulas for $Z_3(-\Delta^3, s_{-\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})$ are the the complex conjugates of the corresponding formulas for $+\Delta^3$:

| Edge Labeling $\ell$ | $Z_3(-\Delta^3, s_{-\Delta^3}, \mathcal{P}_{\Delta^3}, \mathcal{B})_\ell$ |
|---|---|
| 000 | 1 |
| 100 | $2^{-1/4}(a^0a_0a_0 + a^0a_1a_1 + a^1a_0a_1 + a^1a_1a_0)$ |
| 010 | $2^{-1/4}(a^0a^0a_0 + a^0a_1a_1 + a^1a_0a_1 + a^1a^1a_0)$ |
| 110 | $2^{-1/4}(a^0a^0a_0 + a^0a^1a_1 + \zeta a^1a^0a_1 + \zeta^{-1}a^1a^1a_0)$ |
| 011 | $2^{-1/4}(a^0a_0a_0 + a^1a_1a_0 + \zeta a^1a_0a_1 + \zeta^{-1}a^0a_1a_1)$ |
| 010 | $2^{-1/2}(a^0a^0a_0a_0 + a^0a^0a_1a_1 + a^0a^1a_1a_0 + a^0a^1a_0a_1$ $+a^1a^0a_1a_0 + a^1a^0a_0a_1 + a^1a^1a_0a_0 + a^1a^1a_1a_1)$ |
| 101 | $2^{-1/2}(a^0a^0a_0a_0 + a^0a^1a_1a_0 + ia^1a^0a_0a_1 + \zeta^{-1}a^0a^0a_1a_1$ $+\zeta^{-1}a^1a^1a_0a_0 + \zeta a^1a^0a_1a_0 + \zeta a^0a^1a_0a_1 - ia^1a^1a_1a_1)$ |
| 111 | $2^{-1/2}(a^0a^0a_0a_0 + \zeta^{-1}a^1a^1a_0a_0 + \zeta^{-1}a^0a^0a_1a_1 + ia^1a^0a_0a_1$ $+a^0a^1a_1a_0 + \zeta^{-1}a^1a^0a_1a_0 + \zeta^{-1}a^0a^1a_0a_1 + ia^1a^1a_1a_1)$ |

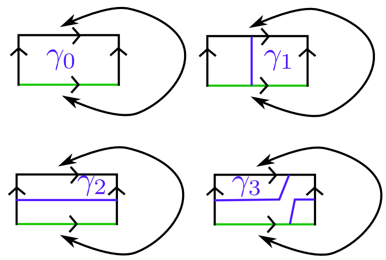
(30)

From now on, $\mathcal{P}$ and $\mathcal{B}$ will often be dropped from the notation for $Z_3$. In examples given, $\mathcal{P}$ should be clear from context. It will later be shown that $Z_3(M, s, \mathcal{P}, \mathcal{B})$ and $Z_3(M, s, \mathcal{P}, \mathcal{B}')$ are canonically equivalent for two different choices $\mathcal{B}$ and $\mathcal{B}'$.

**Example 12.9.** Since $\partial\mathsf{C}$ is composed of two copies of $+\Delta^1$, let $\mathsf{C}^0$ be two points, one on each boundary component. Recall that

$$Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}}) \cong \bigoplus_{[\gamma]\in H_1(\mathsf{C},\partial\mathsf{C}\setminus\mathsf{C}^0;\mathbb{Z}/2)} Z_2(\gamma, s_{\mathsf{C},\mathrm{b}}|_\gamma)$$

where one picks a representative curve $\gamma$ in each $\mathbb{Z}/2$-homology class. There are four such classes, and fix the following representative curves:




On each of the arcs, pick basepoints as follows

By restricting the trivial section of $s_{C,\mathrm{b}}$ to these points, one gets isomorphisms

$$Z_3(\gamma_2, s_{C,\mathrm{b}}|_{\gamma_2}) \cong A$$

$$Z_3(\gamma_3, s_{C,\mathrm{b}}|_{\gamma_3}) \cong A$$

$Z_3(\gamma_1, s_{C,\mathrm{b}}|_{\gamma_1})$ has a basis given by $\widetilde{a}_0$, and thus can be identified with $\mathbb{C}$. Then $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}}) \cong \mathbb{C} \oplus \mathbb{C} \oplus A \oplus A$.

There is an analogous decomposition for the nonbounding cylinder:

$$Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}}) \cong \bigoplus_{[\gamma]\in H_1(\mathsf{C},\partial\mathsf{C}\setminus\mathsf{C}^0;\mathbb{Z}/2)} Z_2(\gamma, s_{\mathsf{C},\mathrm{nb}}|_\gamma)$$

and the same set of curves and basepoints identifies

$$Z_3(\gamma_2, s_{C,\mathrm{nb}}|_{\gamma_2}) \cong A$$

$$Z_3(\gamma_3, s_{C,\mathrm{nb}}|_{\gamma_3}) \cong A$$

However, $Z_3(\gamma_1, s_{\mathsf{C},\mathrm{nb}}|_{\gamma_1})$ has a basis given by $\widetilde{a}_1$, and thus can be identified with an odd copy of $\mathbb{C}$ (say, $L$). Then $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}}) \cong \mathbb{C} \oplus L \oplus A \oplus A$.

$Z_3$ obeys the following "gluing law":

**Proposition 12.10.** *Let $(M, s)$ a Spin 3-manifold, let $\mathcal{P}$ be a decomposition of $\partial M$ into pieces each of which has nonempty triangulated boundary, and let $\Sigma_1$ and $\Sigma_2$ be two pieces of $\partial M$. Suppose $f : (\Sigma_1, s|_{\Sigma_1}) \to (-\Sigma_2, -s|_{\Sigma_2})$ is an isomorphism of Spin bundles and let $(M_f, s_f)$ denote the Spin 3-manifold $M$*

*glued along $f$. Let $\mathcal{P}_f$ denote the induced decomposition of $\partial M_f$. Given $\mathcal{B}$ a set of homology representatives of $H_1(\partial M \setminus \mathcal{P}^0; \mathbb{Z}/2)$, let $\mathcal{B}_f$ be the induced set of representatives of $H_1(\partial M_f \setminus \mathcal{P}_f^0; \mathbb{Z}/2)$. Then $Z_3(M_f, s_f, \mathcal{P}_f, \mathcal{B}_f)$ is obtained from $Z_3(M, s, \mathcal{P}, \mathcal{B})$ by the following process. Fix a $\mathbb{Z}/2$-labeling $\ell$ of the edges of $\mathcal{P}$. Apply $f_*$ to the tensor factor of $Z_3(M, s, \mathcal{B})_\ell$ corresponding to $\Sigma_1$ then contract the tensor factors corresponding to $\Sigma_1$ and $\Sigma_2$. Multiply by $2^{-1/2}$ for each newly interior edge labeled by 1. Divide by 2 for each newly interior vertex. Repeat this for each $\mathbb{Z}/2$-labeling $\ell$ of the edges of $\mathcal{P}$ and add them up. The result is $Z_3(M_f, s_f, \mathcal{B}_f)$.*

*Proof.* The proof is analogous to the gluing law for $Z_2$ (Proposition 10.6) except instead of pairing tensor factors of $Z_1(\gamma_x, s|_{\gamma_x})$ one pairs tensor factors of $Z_2(\Sigma_x, s|_{\Sigma_x})$. The only real difference is dividing by $2^{1/2}$ for each boundary edge labeled by 1. For a boundary edge to be labeled by 1 means that it intersects a surface $\Sigma_x$ representing a homology class, so in particular that surface has a vertex there. If that vertex becomes interior then by the gluing law for $Z_2$ one must divide by $2^{1/2}$.

The factor of 2 at each vertex comes from the fact that, if $M_f$ is connected and $\partial M_f$ is nonempty, then the partially defined map

$$H_2(M, \partial M \setminus \mathcal{P}^0; \mathbb{Z}/2) \to H_2(M_f, \partial M_f \setminus \mathcal{P}_f^0; \mathbb{Z}/2)$$

is $2^n : 1$ where $n$ is the number of newly interior vertices of $M_f$. Therefore one must divide by 2 to avoid double counting. In the case where $M_f$ is closed, this map is $2^{n-1} : 1$, leading to an extra factor of 2. These extra factors, one for each closed component, appear in the denominator of the definition of $Z_3(M, s)$ for closed $M$. □

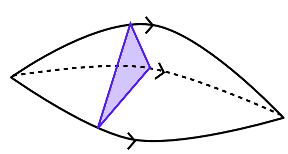
**Example 12.11.** Given the Spin bigon $(\mathsf{B}, s_{\mathsf{B}})$, let $\mathcal{B}(\mathsf{B})$ be the empty curve and the curve $\Delta^1 \times 1/2$. From now on $\mathcal{B}(\mathsf{B})$ will be omitted from the notation. Then $Z_3(B, s_B) \cong \mathbb{C} \oplus A$ where the $\mathbb{C}$ summand corresponds to the empty curve and the $A$ summand corresponds to the nontrivial arc.

Let $M_{\mathsf{B}}$ be as in (10). Then $-M_{\mathsf{B}}$ inherits a Spin structure from $-\Delta^2 \times \Delta^1$, call it $s_{-M_{\mathsf{B}}}$. Then

$$\begin{aligned} Z_3(-M_{\mathsf{B}}, s_{-M_{\mathsf{B}}}) \in & Z_3(-\mathsf{B}, -s_{\mathsf{B}}) \otimes Z_3(-\mathsf{B}, -s_{\mathsf{B}},) \otimes Z_3(\mathsf{B}, s_{\mathsf{B}}) \\ & \cong Z_3(\mathsf{B}, s_{\mathsf{B}})^* \otimes Z_3(\mathsf{B}, s_{\mathsf{B}})^* \otimes Z_3(\mathsf{B}, s_{\mathsf{B}}) \end{aligned}$$

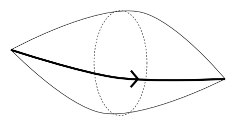
and this defines an algebra multiplication on $Z_3(\mathsf{B}, s_{\mathsf{B}})$.

More concretely, $Z_3(-M_{\mathsf{B}}, s_{-M_{\mathsf{B}}})$ is a sum of two terms, one corresponding to the empty surface and the other to the surface

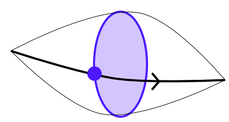


This Spin surface is identified with $(-\Delta^2, s_{-\Delta^2})$ and so $Z_2$ applied to it defines the usual algebra multiplication $A \otimes A \to A$. Associativity of this algebra follows from the associativity of $\mathbb{C}$ and $A$ but it also could be seen from the gluing law, by gluing two copies of $-M_{\mathsf{B}}$ two ways to get the same result, analogously to (6). $Z_3(M_{\mathsf{B}}, s_{M_{\mathsf{B}}})$ analogously defines a comultiplication on $Z_3(\mathsf{B})$, and these two satisfy all the relations of a Frobenius algebra.

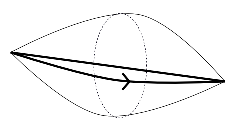
The computation of the unit for this algebra is a useful example of the application of the gluing law. Glue two edges of $M_{\mathsf{B}}$ together to get a ball with one edge on its boundary and two on its interior



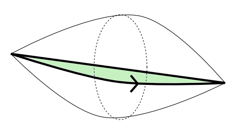
When you apply $Z_3$ to this object you also get two terms, one corresponding to the empty surface and one applied to the disk



The Spin structure does not extend over the edge in the center of this 3-manifold:



But it does if you glue the Spin structures on the two glued edges of $M_{\mathsf{B}}$ with the nontrivial Spin automorphism, denoted as usual by green

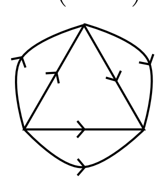


(31)

Then the process of going from the nontrivial term in $Z_3(M_{\mathsf{B}}, s_{M_{\mathsf{B}}})$ to the nontrivial term in $Z_3$ applied to (31) involves applying $(-1)^{\deg}$ to the first tensor factor of the nontrivial term in $Z_3(M_{\mathsf{B}}, s_{M_{\mathsf{B}}}, \mathcal{B})$, contracting the first two tensor factors of the result (this is contracting the tensor factors corresponding to the two pieces being glued), then dividing by $2^{1/2}$ for the newly interior edge (one needs to do this because the gluing law for $Z_2$ involves such a factor for interior vertices and the interior edge of the 3-manifold intersects the surface in an interior vertex).

**Example 12.12.** Recall that $c(-\Delta^2)$ is $\Delta^1 \times (-\Delta^2)$ with $\Delta^1 \times \partial(-\Delta^2)$ collapsed down to a copy of $\partial(-\Delta^2)$.

Divide the $\{0\} \times (-\Delta^2)$ part of $c(-\Delta^2)$ into three pieces:



The $\{1\} \times (-\Delta^2)$ part of $c(-\Delta^2)$ will be left as one piece, the copy of $-\Delta^2$. Let $\mathcal{P}$ be this decomposition of $\partial(c(-\Delta^2))$.

The Spin structure $s_{-\Delta^2}$ induces a Spin structure on $\Delta^1 \times (-\Delta^2)$ and hence on $c(-\Delta^2)$, call it $s_{c(-\Delta^2)}$. Then $Z_3(c(-\Delta^2), s|_{c(-\Delta^2)}, \mathcal{P})$ is in

$$Z_3(\Delta^2, -s_{-\Delta^2}, \mathcal{B}(\Delta^2)) \otimes Z_3(-\mathsf{B}, -s_{\mathsf{B}}) \otimes Z_3(-\mathsf{B}, -s_{\mathsf{B}})$$

$$\otimes Z_3(-\Delta^2, s_{-\Delta^2}, \mathcal{B}(\Delta^2)) \otimes Z_3(-\mathsf{B}, -s_{\mathsf{B}})$$

which is isomorphic to

$$Z_3(-\mathsf{B}, -s_{\mathsf{B}})^* \otimes A^* \otimes A^* \otimes Z_3(\mathsf{B}, s_{\mathsf{B}})^* \otimes A^*$$

Since stacking two copies of $c(-\Delta^2)$ is the same manifold with the same boundary parts as gluing on copies of $M_{\mathsf{B}}$ (from Example 12.11) to $c(-\Delta^2)$ onto each bigon piece of $\mathcal{P}$, then the same argument as in Proposition 5.4 shows that $Z_3(c(-\Delta^2), s|_{c(-\Delta^2)}, \mathcal{P})$ turns $Z_3(-\Delta^2, s_{-\Delta^2})$ into an $Z_3(\mathsf{B}, s_{\mathsf{B}}) \otimes Z_3(\mathsf{B}, s_{\mathsf{B}})$ – $Z_3(\mathsf{B}, s_{\mathsf{B}})$ bimodule.

**Example 12.13.** The last example generalizes to other surfaces. Suppose a a Spin surface $(\Sigma, s)$ has boundary parametrized by intervals $\pm\Delta^1$. Then for each copy of $+\Delta^1$ in $\partial\Sigma$ there's a right action of $Z_3(\mathsf{B}, s_{\mathsf{B}})$ on $Z_3(\Sigma, s)$ and for each copy of $-\Delta^1$ in $\partial\Sigma$ there's a left action of $Z_3(\mathsf{B}, s_{\mathsf{B}})$.

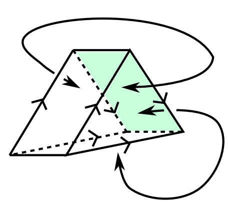
**Example 12.14.** The Spin structure $s_{-\Delta^2}$ (where the stabilization direction comes last) includes naturally into a Spin structure on $(-\Delta^2) \times \Delta^1$ and hence to a Spin structure on $-\Delta^2 \times \mathsf{S}$. Call this Spin structure $s_{-\Delta^2 \times \mathsf{S},\mathrm{nb}}$. $\partial(-\Delta^2 \times \mathsf{S})$ consists of three cylinders. Then $Z_3(-\Delta^2 \times \mathsf{S}, s|_{-\Delta^2 \times \mathsf{S},\mathrm{nb}})$ defines an element in

$$Z_3(-\mathsf{C}, -s_{\mathsf{C},\mathrm{nb}}) \otimes Z_3(-\mathsf{C}, -s_{\mathsf{C},\mathrm{nb}}) \otimes Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}}) \tag{32}$$

where here, by convention, the ordering of the tensor factors is according to the order of the edges of $-\Delta^2$ given by [12], [01], [02]. The vector space (32) is isomorphic, via the pairing from Definition 12.2, to

$$Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}})^* \otimes Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}})^* \otimes Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}})$$

and hence $Z_3(-\Delta^2 \times \mathsf{S}, s_{-\Delta^2 \times \mathsf{S},\mathrm{nb}})$ defines an algebra structure on $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}})$. Call this algebra $\mathcal{A}_1$. By the gluing law, this is an associative unital algebra. The unit, of course, is given by applying $Z_3$ to

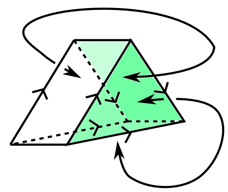


(33)

here the shape pictured is $+\Delta^2 \times \mathsf{S}$ with Spin structure $s_{+\Delta^2 \times \mathsf{S}}$. The two cylinders that are glued together have their Spin bundles identified with the nontrivial Spin automorphism, as indicated by the green.

By Example 12.9, $\dim(\mathcal{A}_1) = 6$.

**Example 12.15.** Analogously to Example 12.14, if you glue together the Spin structure on $-\Delta^2 \times \Delta^1$ with the nontrivial Spin automorphism to get a Spin structure on $-\Delta^2 \times \mathsf{S}$, call this Spin structure $s_{-\Delta^2\times\mathsf{S},\mathrm{b}}$. Then $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$ defines an algebra structure on $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$. Call this algebra $\mathcal{A}_0$. The unit is obtained by applying $Z_3$ to

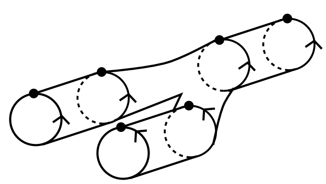


where now two face pairings involve the nontrivial Spin automorphism. By Example 12.9, $\dim(\mathcal{A}_0) = 6$.

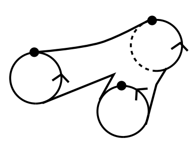
**Example 12.16.** Recall that $\mathsf{P}$ is constructed by gluing two cylinders to a copy of $-\Delta^2$. Let $\mathcal{B}(\mathsf{P})$ denote the sixteen curves on $\mathsf{P}$ whose restrictions to $\mathsf{C}$, $\mathsf{C}$, and $\Delta^2$ are in $\mathcal{B}(\mathsf{C})$, $\mathcal{B}(\mathsf{C})$, and $\mathcal{B}(\Delta^2)$, respectively. These curves can be seen in Example 13.5. $\mathcal{B}(\mathsf{P})$ will be omitted from the notation.

Recall that there are four Spin structures $s_{\mathsf{P},000}$, $s_{\mathsf{P},110}$, $s_{\mathsf{P},101}$, $s_{\mathsf{P},011}$ on $\mathsf{P}$. Since $\mathcal{B}(\mathsf{P})$ consists of four closed curves (including the empty curve) and twelve arcs, it follows that $\dim Z_3(\mathsf{P}, s_{\mathsf{P},ijk}) = (4 \cdot 1) + (12 \cdot 2) = 28$.

**Example 12.17.** Recall that $c(\mathsf{P})$ is $\Delta^1 \times \mathsf{P}$ with $\Delta^1 \times \partial\mathsf{P}$ collapsed down to a copy of $\partial\mathsf{P}$. Divide the $\{0\} \times \mathsf{P}$ part of $c(\mathsf{P})$ into three pieces: a pair of pants plus collar neighborhoods $\mathsf{C}$ of each boundary component:



The $\{1\} \times \mathsf{P}$ part of $\mathsf{P}$ will be left as one piece:



The Spin structures $s_{\mathsf{P},ijk}$ all induce Spin structures on $\Delta^1 \times \mathsf{P}$ and hence on $c(\mathsf{P})$. Call these $s_{K_\mathsf{P},ijk}$. Then $Z_3(\Delta^1 \times \mathsf{P}, s|_{\Delta^1\times\mathsf{P},ijk})$ is in

$$Z_3(-\mathsf{P}, -s_{\mathsf{P},ijk}) \otimes Z_3(-\mathsf{C}, -s_{\mathsf{C},(\mathrm{n})\mathrm{b}}) \otimes Z_3(-\mathsf{C}, -s_{\mathsf{C},(\mathrm{n})\mathrm{b}})$$

$$\otimes Z_3(\mathsf{P}, s_{\mathsf{P},ijk}) \otimes Z_3(-\mathsf{C}, -s_{\mathsf{C},(\mathrm{n})\mathrm{b}})$$

which is isomorphic to

$$Z_3(\mathsf{P}, s_{\mathsf{P},ijk})^* \otimes \mathcal{A}_i^* \otimes \mathcal{A}_j^* \otimes Z_3(\mathsf{P}, s_{\mathsf{P},ijk})) \otimes \mathcal{A}_k^*$$

Since stacking two copies of $c(\mathsf{P})$ is the same manifold with the same boundary parts as gluing on copies of $-\Delta^2 \times \mathsf{S}$ to $\Delta^1 \times \mathsf{P}$ onto each component of $\Delta^1 \times \partial\mathsf{P}$, then the same argument as in Proposition 5.4 shows that $Z_3(\Delta^1 \times \mathsf{P}, s|_{\Delta^1 \times \mathsf{P}, ijk})$ turns $Z_3(\mathsf{P}, s_{\mathsf{P},ijk})$ into an $\mathcal{A}_i \otimes \mathcal{A}_j - \mathcal{A}_k$ bimodule.

**Example 12.18.** In fact, Example 12.17 generalizes to arbitrary Spin surfaces with circle boundary components parametrized by $\pm\Delta^1$. For each bounding circle boundary component parametrized by $+\Delta^1$, there is a right action of $\mathcal{A}_0$. For each bounding circle boundary component parametrized by $-\Delta^1$, there is a left action of $\mathcal{A}_0$. For each nonbounding circle boundary component parametrized by $+\Delta^1$, there is a right action of $\mathcal{A}_1$. For each nonbounding circle boundary component parametrized by $-\Delta^1$, there is a left action of $\mathcal{A}_1$.

Thus far $Z_3$ has only been defined on manifolds such that all their components have either empty boundary or at least two pieces in each of their boundary components. Also, the definition of $Z_3$ uses a fixed collection of homology representatives $\mathcal{B}$. It is possible to define $Z_3$ in a way that does not depend on these choices, using the construction in Definition 3.17. Let $(\Sigma, s)$ be a Spin surface. Let $\mathcal{P}$ and $\mathcal{P}'$ be two decompositions of $\Sigma$ into pieces. Let $\mathcal{B}$ be a collection of homology representatives for $H_1(\Sigma, \partial\Sigma \setminus \mathcal{P}^0; \mathbb{Z}/2)$ and let $\mathcal{B}'$ be such a collection for $\mathcal{P}'$. Let $I = \mathbb{Z}/2$ and $d : I \to \mathbb{C}$ be the function $d(0) = 1$, $d(1) = 1/\sqrt{2}$. Set $\Gamma = 2$. Then Definition 3.17 applies to define maps

$$\phi_{\Sigma,s;(\mathcal{P},\mathcal{B}),(\mathcal{P}',\mathcal{B}')} : Z_3(\Sigma, s, \mathcal{P}, \mathcal{B}) \to Z_3(\Sigma, s, \mathcal{P}', \mathcal{B}')$$

Part 1 of Proposition 3.18 applies in this case, replacing $\Sigma$ by $(\Sigma, s)$ and $\mathcal{P}_i$ by $(\mathcal{P}_i, \mathcal{B}_i)$.

**Definition 12.19.** Let $(\Sigma, s)$ be a Spin surface. Let $Z_3(\Sigma, s)$ be the limit of system of vector spaces and linear maps given by $\{\phi_{\Sigma,s;(\mathcal{P},\mathcal{B}),(\mathcal{P}',\mathcal{B}')}\}$ as $(\mathcal{P}, \mathcal{B})$ and $(\mathcal{P}', \mathcal{B}')$ range over all pairs as above. If $\mathcal{P}$ is a decomposition of $\Sigma$, set

$$Z_3(\Sigma, s, \mathcal{P}) = \bigotimes_{P \in \mathcal{P}} Z_3(P, s)$$

In particular, this definition applies to closed Spin surfaces. Note how $Z_3(\Sigma, s)$ does not depend on a choice of representative homology curves.

By Part 1 of Proposition 3.18, the following is true:

**Proposition 12.20.** *$Z_3(\Sigma, s)$ is isomorphic to the image of the projection $\phi_{\Sigma,s;(\mathcal{P},\mathcal{B}),(\mathcal{P},\mathcal{B})}$ for any pair $(\mathcal{P}, \mathcal{B})$ for $\Sigma$. The map $\phi_{\Sigma,s;(\mathcal{P},\mathcal{B}),(\mathcal{P}',\mathcal{B}')}$ takes $\operatorname{im} \phi_{\Sigma,s;(\mathcal{P},\mathcal{B}),(\mathcal{P},\mathcal{B})}$ isomorphically to $\operatorname{im} \phi_{\Sigma,s;(\mathcal{P}',\mathcal{B}'),(\mathcal{P}',\mathcal{B}')}$*

By the definition of $Z_3(\Sigma, s)$ and the gluing law, the following is true:

**Proposition 12.21.** *Let $(M, s)$ be a Spin manifold with boundary decomposition $\mathcal{P}$. Let $\mathcal{B}$ be a collection of curve representatives for $H_1(\partial M \setminus \mathcal{P}^0; \mathbb{Z}/2)$. Then $Z_3(M, s, \mathcal{P}, \mathcal{B})$, which by definition is in*

$$Z_3(\partial M, s|_{\partial M}, \mathcal{P}, \mathcal{B}|_P) = \bigotimes_{P \in \mathcal{P}} Z_3(P, s|_P, \mathcal{B}|_P),$$

*is in the image of*

$$\bigotimes_{P\in\mathcal{P}} \phi_{P,s|_P;(\mathcal{P}_P,\mathcal{B}|_P),(\mathcal{P}_P,\mathcal{B}|_P)}$$

*In particular, $Z_3(M,s,\mathcal{P},\mathcal{B})$ gives a well-defined element in $Z_3(M,s,\mathcal{P})$ that does not depend on the choice of $\mathcal{B}$.*

*Moreover, if $\mathcal{P}'$ is a different decomposition of $\partial M$, then for some choices $\mathcal{B}$ and $\mathcal{B}'$, the maps $\phi_{\partial M,s|_{\partial M};(\mathcal{P},\mathcal{B}),(\mathcal{P}',\mathcal{B}')}$ map $Z_3(M,s,\mathcal{P})$ to $Z_3(M,s,\mathcal{P}')$. In particular, this map factors through $Z_3(M,s,\mathcal{P}_{\partial M})$.*

On account of Proposition 12.21, make the following definition

**Definition 12.22.** Given $(M,s)$ a 3-manifold and $\mathcal{P}$ a decomposition of $\partial M$. Declare $Z_3(M,s,\mathcal{P})$ to be $Z_3(M,s,\mathcal{P},\mathcal{B})$ for some collection $\mathcal{B}$ of representative curves. If you regard $Z_3(M,s,\mathcal{P},\mathcal{B})$ as lying in $Z_3(\partial M,s|_{\partial M},\mathcal{P})$, then this definition does not depend on the choice of $\mathcal{B}$. If $\mathcal{P}'$ is a different decomposition of the boundary, there is a canonical identification of the vectors $Z_3(M,s,\mathcal{P}) \in Z_3(\partial M,s|_{\partial M},\mathcal{P})$ and $Z_3(M,s,\mathcal{P}') \in Z_3(\partial M,s|_{\partial M},\mathcal{P}')$.

If $(\Sigma,s)$ is a closed Spin surface, one needs a pairing

$$Z(\Sigma,s)\otimes Z(-\Sigma,-s)\to\mathbb{C}$$

so, taking Proposition 3.19 as inspiration, define it as

**Definition 12.23.** Let $(\Sigma,s)$ be a closed Spin surface. Fix a decomposition $\mathcal{P}$ of $\Sigma$ and choice of representative curves $\mathcal{B}$. Then there's a canonical inclusion $i: Z(\Sigma,s)\hookrightarrow Z(\Sigma,s,\mathcal{P},\mathcal{B})$. Define the pairing

$$Z(\Sigma,s)\otimes Z(-\Sigma,s)\to\mathbb{C}$$

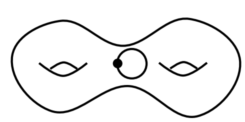
to be

$$x\otimes y\mapsto \langle\delta_{\mathcal{P}}(i(x)),i(y)\rangle$$

This is well-defined by the properties of the maps $\phi_{\Sigma,s,\mathcal{P},\mathcal{B}}$. It is perhaps, simplest, however, to take $\mathcal{P}$ to come from splitting each component of $\Sigma$ into a small disk and its complement.

**Proposition 12.24.** *Let $(\Sigma_g,s)$ be a closed connected surface of genus $g$. If $s$ is a bounding Spin structure, then $Z_3(\Sigma_g,s)$ has a natural basis given by bounding homology classes in $H_1(\Sigma_g;\mathbb{Z}/2)$. It is in even degree. If $s$ is a nonbounding Spin structure, then $Z_3(\Sigma_g,s)$ has a natural basis given by non-bounding homology classes in $H_1(\Sigma_g;\mathbb{Z}/2)$. It is in odd degree.*

*Proof.* Decompose $\Sigma_g$ into two pieces $P_1$ and $P_2$ where $P_2$ is a disk with a single vertex on its boundary and $P_1$ is the closure of $\Sigma\setminus P_1$, also with a single vertex on its boundary:

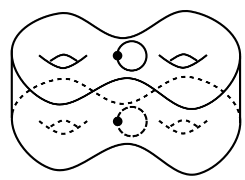


Write $\mathcal{P}$ for this decomposition and fix a collection $\mathcal{B}$ of embedded curves representing the $2^{2g}$ homology classes in $H_1(P_1, \partial P_1 \setminus P_1^0; \mathbb{Z}/2) \cong H_1(\Sigma_g; \mathbb{Z}/2)$. Then $Z_3(\Sigma_g, s) \cong \operatorname{im} \phi_{\Sigma_g,s;(\mathcal{P},\mathcal{B}),(\mathcal{P},\mathcal{B})}$.

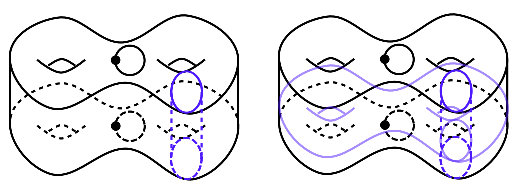
Suppose first that $s$ is bounding.

Since $Z_3(P_2, s|_{P_2}, \mathcal{B}|_{P_2})$ is just a copy of $\mathbb{C}$, it is safe to ignore this factor. $Z_3(P_1, s, \mathcal{B})$ has a basis in natural correspondence with $H_1(\Sigma_g; \mathbb{Z}/2)$ and it will turn out that the kernel of $\phi_{\Sigma_g,s;(\mathcal{P},\mathcal{B}),(\mathcal{P},\mathcal{B})}$ consists of those basis elements corresponding to nonbounding homology classes.

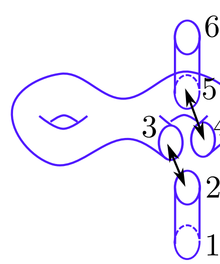
This is seen as follows. $\phi_{\Sigma_g,s;(\mathcal{P},\mathcal{B}),(\mathcal{P},\mathcal{B})}$ is $\frac{1}{2}Z_3(\Delta^1 \times \Sigma_g, s', \mathcal{P} \sqcup \mathcal{P}, \mathcal{B} \sqcup \mathcal{B})$ where $s'$ is induced from $s$ on $\Sigma_g$.



Given $\gamma \sqcup \gamma \subset \Sigma_g \sqcup \Sigma_g$ where $\gamma$ is in $\mathcal{B}$, there are two elements in $H_2(\Delta^1 \times \Sigma_g; \partial(\Delta^1 \times \Sigma_g) \setminus (\mathcal{P}^0 \sqcup \mathcal{P}^0); \mathbb{Z}/2)$ that restrict to $\gamma \sqcup \gamma$. One of them is the cylinder $\Delta^1 \times \gamma$. The other is this cylinder surgered with $\{1/2\} \times \Sigma_g$:



Recall that if $\gamma$ is bounding then $Z_2(\gamma, s|_\gamma)$ is spanned by $\widetilde{a}_0$ and $Z_2(\Delta^1 \times \gamma, s|_{\Delta^1 \times \gamma})$ is $\widetilde{a}^0 \otimes \widetilde{a}_0$. If $\gamma$ is nonbounding then $Z_2(\gamma, s|_\gamma)$ is spanned by $\widetilde{a}_1$ and $Z_2(\Delta^1 \times \gamma, s|_{\Delta^1 \times \gamma})$ is $\widetilde{a}^1 \otimes \widetilde{a}_1$. Further recall that $Z_2(\Sigma_g, s)$ is 1 if $s$ is bounding and $-1$ otherwise. In particular, if you cut $\Sigma_g$ along $\gamma$ to get a surface $\Sigma'$, then $Z_2(\Sigma', s)$ is either $\widetilde{a}^0 \otimes \widetilde{a}_0$ or $-\widetilde{a}^1 \otimes \widetilde{a}_1$, depending on whether or not $\gamma$ bounds. The picture near the surgery is




(34)

Suppose $\gamma$ is nonbounding. If you apply $Z_2$ to this picture you get

$$(\widetilde{a}^1 \otimes \widetilde{a}_1) \otimes (\widetilde{a}^1 \otimes \widetilde{a}_1) \otimes (-\widetilde{a}^1 \otimes \widetilde{a}_1)$$

where six tensor factors correspond to the numbering of the edges in (34). Therefore contract tensor factors 2 and 3 as well as 4 and 5 and you get

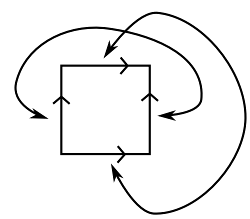
$-\widetilde{a}^1 \otimes \widetilde{a}_1$. This cancels with the $\widetilde{a}^1 \otimes \widetilde{a}_1$ coming from $\Delta^1 \times \gamma$ and so $Z_2(\gamma, s|_\gamma)$ is a subset of the kernel of $\phi_{\Sigma,s;(\mathcal{P},\mathcal{B}),(\mathcal{P},\mathcal{B})}$. If $\gamma$ is bounding, then a similar calculation shows that $Z_2(\gamma, s|_\gamma)$ is contained in the image.

If $s$ is nonbounding, there is an analogous proof. □

**Corollary 12.25.** *Let $(\Sigma_g, s)$ be a closed connected Spin surface of genus $g$. The pairing $Z_3(\Sigma_g, s) \otimes Z_3(-\Sigma_g, -s) \to \mathbb{C}$ from Definition 12.23 is a left and right inverse to $Z_3(\Delta^1 \times \Sigma_g, s) \in Z_3(-\Sigma_g, -s) \otimes Z_3(\Sigma_g, s)$.*

*Proof.* In Definition 12.23, use a splitting of $\Sigma_g$ into $D$ and $\Sigma_g \setminus D$ where $D$ is a small disk in $\Sigma_g$ with a vertex on its boundary. By Proposition 12.24, $Z_3(\Sigma_g, s)$ has a basis given by elements $\widetilde{a}_0$ or $\widetilde{a}_1$ for some classes in $H_1(\Sigma_g; \mathbb{Z}/2)$. Let $v_1, \ldots, v_n$ be this basis, and let $v^1, \ldots, v^n$ be the dual basis in $Z_3(-\Sigma_g \setminus D, -s)$ with respect to the pairing in Definition 12.2. Then the proof of Proposition 12.24 makes clear that $Z_3(\Delta^1 \times \Sigma_g, s) = 2(v^1 \otimes v_1 + \cdots + v^n \otimes v_n)$. Definition 12.23 indicates that the pairing is $v_i \otimes v^i \mapsto \frac{1}{2}$ where the factor of $\frac{1}{2}$ comes from $\delta_{\mathcal{P}}$ and the fact that $\mathcal{P}$ has a single vertex. □

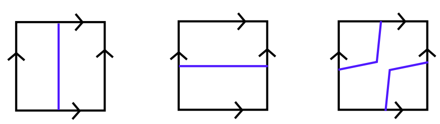
**Example 12.26.** Consider the nonbounding Spin torus $(\mathsf{T}, s_{\mathsf{T},(0,0)})$ constructed by identifying opposite sides of the square



and let $\mathcal{P}$ be the decomposition of $\mathsf{T}$ into the single piece $P$ where $P$ is the square. Then there is an inclusion $Z_3(\mathsf{T}, s_{\mathsf{T},(0,0)}) \to Z_3(P, s_{\mathsf{T},(0,0)}|_P)$ whose image is the degree 1 part of

$$Z_2(\gamma_1, s_{\mathsf{T},(0,0)}|_{\gamma_1})) \oplus Z_2(\gamma_2, s_{\mathsf{T},(0,0)}|_{\gamma_2}) \oplus Z_2(\gamma_3, s_{\mathsf{T},(0,0)}|_{\gamma_3})$$

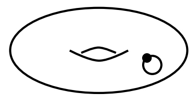
where $\gamma_1$, $\gamma_2$ and $\gamma_3$ are the three arcs



If you pick points on each of these curves (say, for example, the midpoints) for a Spin section, then $Z(\mathsf{T}, s_{\mathsf{T},(0,0)})$ can be identified with the odd part of $A \oplus A \oplus A$, each the three $A$ summands corresponding to $\gamma_1$, $\gamma_2$, and $\gamma_3$.

Therefore, $Z_3(-\mathsf{T}, -s_{\mathsf{T},(0,0)})$ includes in $Z(-\Delta^1 \times \Delta^1, -s_{\Delta^1 \times \Delta^1})$ as the odd part of $A^* \oplus A^* \oplus A^*$. The pairing $\langle (a_1, 0, 0), (a^1, 0, 0) \rangle$ (for example) is $\frac{1}{2} \cdot 2^{-1/2}$. That this pairing is not 1 comes from the fact that $\delta_{\mathcal{P}}$ is in the pairing (Definition 12.23). The factor of $\frac{1}{2}$ is from the vertex. The factor of $2^{-1/2}$ is due to the fact that the summand corresponding to $\gamma_1$ is graded by an edge labeling. There are two edges on the torus, and the $\gamma_1$ summand is graded by the labeling of 1 on the edge $\gamma_1$ intersects and 0 on the other edge. The pairing $\langle(a_1,0,0),(0,a^1,0)\rangle$ is 0, since these two vectors do not correspond to the same curve on the torus.

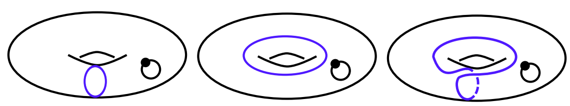
**Example 12.27.** Let $(\mathsf{T}, s_{\mathsf{T},(0,0)})$ be as in Example 12.26, except let $\mathcal{P}$ be a decomposition of $\Sigma$ into two pieces $P_1$ and $P_2$, where $P_2$ is a small disk with a vertex and $P_1$ is the complement:



Then $Z_3(P_2, s_{\mathsf{T},(0,0)}|_{P_2})$ is just a copy of $\mathbb{C}$, so its tensor factor can be ignored. $Z_3(\Sigma, s_{\mathsf{T},(0,0)})$ includes into the 4-dimensional space $Z_3(P_1, s|_{\mathsf{T},(0,0)}|_{P_1})$ as the subspace

$$Z_2(\gamma_1, s_{\mathsf{T},(0,0)}|_{\gamma_1})) \oplus Z_2(\gamma_2, s_{\mathsf{T},(0,0)}|_{\gamma_2}) \oplus Z_2(\gamma_3, s_{\mathsf{T},(0,0)}|_{\gamma_3})$$

where the $\gamma_i$ are the three circles (not arcs, as in Example 12.26)



This subspace is precisely the odd part of $Z_3(P_1, s_{\mathsf{T},(0,0)}|_{P_1})$.

Recall that $Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{nb}})$ has a basis vector $\widetilde{a}_1$. Then $Z_3(\mathsf{T}, s_{\mathsf{T},(0,0)})$ has a basis given by $(\widetilde{a}_1,0,0)$, $(0,\widetilde{a}_1,0)$, $(0,0,\widetilde{a}_1)$. $Z_3(-\mathsf{T}, -s_{\mathsf{T},(0,0)})$ has a basis given by $(\widetilde{a}^1,0,0)$, $(0,\widetilde{a}^1,0)$, $(0,0,\widetilde{a}^1)$. Recall that $\langle\widetilde{a}_1,\widetilde{a}^1\rangle = 0$ but here $\langle(\widetilde{a}_1,0,0),(\widetilde{a}^1,0,0)\rangle = \frac{1}{2}$, because of the presence of the vertex and the presence of $\delta_{\mathcal{P}}$ in Definition 12.23. $\delta_{\mathcal{P}}$ does not involve any factors of $2^{-1/2}$ from the edges because $Z_3(P_1, s|_{P_1})$ is graded by the 0-labeling of the one edge of $P_1$, since none of the curves $\gamma_i$ intersect that edge.

Given $(a,b) \in H_1(\mathsf{T};\mathbb{Z}/2)$, one can fix three curves $\gamma_1,\gamma_2,\gamma_3$ representing the three homology classes in $H_1(\mathsf{T};\mathbb{Z}/2)\setminus(a,b)$. Let $w_{\gamma_j}$ be the basis vector $\widetilde{a}_i$ correpsonding to $Z_2(\gamma_j, s_{\mathsf{T},(a,b)}|_{\gamma_j}) \cong Z_2(\mathsf{S}, s_{\mathsf{S},(\mathrm{n})\mathrm{b}})$. Then if $\{u_{\gamma_j}\}$ are the three basis vectors of $Z_3(-\mathsf{T}, -s_{\mathsf{T},(a,b)})$ corresponding $\widetilde{a}^i$ in $Z_2(-\gamma_j, -s_{\mathsf{T},(a,b)}|_{\gamma_j}) \cong Z_2(\mathsf{S}, s_{\mathsf{S},(\mathrm{n})\mathrm{b}})$, then

$$\langle w_{\gamma_i}, u_{\gamma_j}\rangle = \begin{cases} \frac{1}{2} & \gamma_i = \gamma_j \\ 0 & \text{otherwise} \end{cases}$$

Here the pairing $\langle,\rangle$ is $Z_3(\mathsf{T}, s_{\mathsf{T},(a,b)}) \otimes Z_3(-\mathsf{T}, -s_{\mathsf{T},(a,b)}) \to \mathbb{C}$. There is also a pairing

$$Z_3(\mathsf{T}, s_{\mathsf{T},(a,b)}) \otimes Z_3(\mathsf{T}, s_{\mathsf{T},(a,b)}) \to \mathbb{C}$$

given by $Z_3$ applied to a copy of $\Delta^1 \times \mathsf{T}$ where the $\{1\} \times \mathsf{T}$ boundary component is parametrized by an orientation reversing map. For example, one can obtain such a Spin manifold in a natural way by crossing the construction of Example 11.6 by an appropriate Spin circle. Under this pairing:

$$\langle w_{\gamma_i}, w_{\gamma_j} \rangle = \begin{cases} \frac{1}{2} & \gamma_i = \gamma_j \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

# 13 Mapping Class Group Actions

Let $f : \Sigma \to \Sigma$ be a self-diffeomorphism which is the identity on $\partial\Sigma$. Then if $s$ is a Spin structure on $\Sigma$, $(f^{-1})^*s$ is also a Spin structure on $\Sigma$, and $f$ is covered by a Spin isomorphism $s \to (f^{-1})^*s$. For example, $A \in \mathrm{SL}_2\mathbb{Z}$ acts on the torus and takes $(\mathsf{T}, s_{\mathsf{T},(a,b)})$ to $(\mathsf{T}, s_{\mathsf{T},A\cdot(a,b)})$. For example

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

takes $(\mathsf{T}, s_{\mathsf{T},(1,0)})$ to $(\mathsf{T}, s_{\mathsf{T},(0,1)})$ and acts as an automorphism on $(\mathsf{T}, s_{\mathsf{T},(1,1)})$.

**Definition 13.1.** Let $f : \Sigma \to \Sigma$ be a self-diffeomorphism which is the identity on $\partial\Sigma$. Let $s$ be a Spin structure on $\Sigma$ and let $\mathcal{B}$ be a set of curve representatives for $H_1(\Sigma, \partial\Sigma\backslash\Sigma^0; \mathbb{Z}/2)$. Then $f$ induces an isomorphism $(\Sigma, s) \to (\Sigma, (f^{-1})^*s)$, also called $f$. Recall the map $f_*$ from Definition 12.3:

$$f_* : Z_3(\Sigma, s, \mathcal{B}) \to Z_3(\Sigma, (f^{-1})^*s, f(\mathcal{B}))$$

defined by by $f_* : Z_2(\gamma, s|_\gamma) \to Z_2(f(\gamma), (f^{-1})^*s|_{f(\gamma)})$ on the summand of $Z_3(\Sigma, s, \mathcal{B})$ corresponding to $\gamma \in \mathcal{B}$.

Define a map

$$\rho(f) : Z_3(\Sigma, s) \to Z_3(\Sigma, (f^{-1})^*s)$$

by picking a choice $\mathcal{B}$ of homology representatives, then setting

$$\rho(f) = \phi_{\Sigma,(f^{-1})^*s;(\mathcal{P}_\Sigma,f(\mathcal{B})),(\mathcal{P}_\Sigma,\mathcal{B})} \circ f_*.$$

This doesn't depend on the choice of $\mathcal{B}$. The map $\rho(f)$ is best depicted by an example (see Example 13.3 below).

**Remark 13.2.** Definition 13.1 is obtained by having $f$ act on the vector space

$$\bigoplus_{s\in\mathrm{Spin}(\Sigma)} Z_3(\Sigma, s)$$

in the most natural way. Namely, $f_*$ maps $Z_3(\Sigma, s, \mathcal{B})$ to $Z_3(\Sigma, s', \mathcal{B}')$ for some $s'$ and $\mathcal{B}'$. There is a canonical identification of $Z_3(\Sigma, s', \mathcal{B}') \to Z_3(\Sigma, s', \mathcal{B})$, so post-compose $f_*$ by this canonical identification. Because $f_*g_* = (f \circ g)_*$, one can check that $\rho$ is a representation of the mapping class group on

$$\bigoplus_{s\in\mathrm{Spin}(\Sigma)} Z_3(\Sigma, s)$$

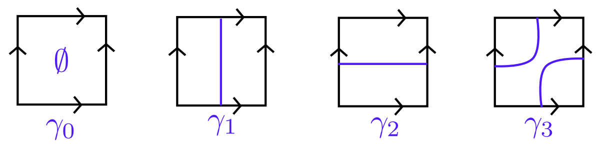
**Example 13.3.** This example will consider the action of the matrices

$$\mathcal{T}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \mathcal{T}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ \mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on the 9-dimensional vector space

$$V = Z_3(\mathsf{T}, s_{\mathsf{T},(1,1)}) \oplus Z_3(\mathsf{T}, s_{\mathsf{T},(1,0)}) \oplus Z_3(\mathsf{T}, s_{\mathsf{T},(0,1)}).$$

Fix the following curves in $\mathsf{T}$:

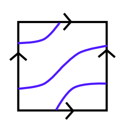



so that $w_{\gamma_0}$, $w_{\gamma_1}$, $w_{\gamma_2}$ forms a basis for $Z_3(\mathsf{T}, s_{\mathsf{T},(1,1)})$, $w_{\gamma_0}$, $w_{\gamma_1}$, $w_{\gamma_3}$ forms a basis for $Z_3(\mathsf{T}, s_{\mathsf{T},(1,0)})$, and $w_{\gamma_0}$, $w_{\gamma_2}$, $w_{\gamma_3}$ forms a basis for $Z_3(\mathsf{T}, s_{\mathsf{T},(0,1)})$. Together these form a basis for $V$. Let $\rho$ denote the action of $\mathrm{SL}_2\mathbb{Z}$ on $V$.

$\mathcal{S}$ switches the Spin structures $s_{T,(1,0)}$ and $s_{T,(0,1)}$ and switches $\gamma_1$ and $\gamma_2$, so

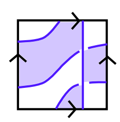
$$\rho(\mathcal{S}) = \begin{pmatrix} 1 & & & & & & & & \\ & & 1 & & & & & & \\ & 1 & & & & & & & \\ & & & & & & 1 & & \\ & & & & & & & & 1 \\ & & & & & & & 1 & \\ & & & 1 & & & & & \\ & & & & & 1 & & & \\ & & & & 1 & & & & \end{pmatrix}$$

$\mathcal{T}_1$ switches the Spin structures $s_{\mathsf{T},(0,1)}$ and $s_{\mathsf{T},(1,1)}$. $\mathcal{T}_1(\gamma_0) = \gamma_0$, $\mathcal{T}_1(\gamma_2) = \gamma_2$, $\mathcal{T}_1(\gamma_1) = \gamma_3$, but $\mathcal{T}_1(\gamma_3)$ is not in the set $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$:



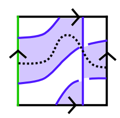
$\mathcal{T}_1(\gamma_3)$

This curve is in the homology class of $\gamma_1$, so fix a cobordism from this curve to $\gamma_1$ inside $\Delta^1 \times \mathsf{T}$:



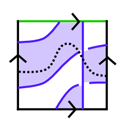
This cobordism is a cylinder connect-summed with $\mathbb{R}P^2$. Since in the Spin structures of interest here, the boundaries of the cylinder restrict to bounding Spin structures, $Z_2$ applied to this cylinder will be $\zeta^{\pm 1}\widetilde{a}^0 \otimes \widetilde{a}_0$ (see Example

11.9). Exactly what the exponent of $\zeta$ is depends on the exact Spin structure. Applying $\rho(\mathcal{T}_1)$ to the $Z_3(\mathsf{T}, s_{\mathsf{T},(1,1)})$ summand maps to the $Z_3(\mathsf{T}, s_{\mathsf{T},(0,1)})$ summand, so fix a Spin structure where $(0,1)$ is a nonbounding class by taking the Spin structure from the standard trivialization of $\Delta^1 \times T$ and regluing the edge parallel to $(0,1)$ with the nontrivial Spin automorphism, drawn here in green:



The dashed curve $\eta$ is the curve in the cobordism with a nontrivial normal bundle. The value of $\lambda(\eta, s|_\eta)$ can be computed directly by seeing how the normal bundle of $\eta$ twists around as you move around the curve. It twists halfway counterclockwise so the trivialization moves halfway clockwise. Since the curve meets the green seam, its holonomy is multiplied by $-1$. Therefore $\lambda(\eta, s|_\eta) = i$ and $Z_2$ applied to this cobordism is multiplication by $\frac{1+i}{\sqrt{2}}$, or $\zeta$.

Applying $\rho(\mathcal{T}_1)$ to the $Z_3(\mathsf{T}, s_{\mathsf{T},(1,0)})$ summand maps to the $Z_3(\mathsf{T}, s_{\mathsf{T},(1,0)})$ summand, so fix a Spin structure where the nonbounding homology class is $(1,0)$ by instead identifying both edges with the nontrivial Spin automorphism:



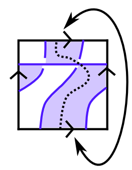
In this case $\lambda(\eta) = \zeta^{-1}$. Therefore

$$\rho(\mathcal{T}_1) = \begin{pmatrix} & & & & & & 1 & & \\ & & & & & & & & \zeta \\ & & & & & & & 1 & \\ & & & 1 & & & & & \\ & & & & & \zeta^{-1} & & & \\ & & & & 1 & & & & \\ 1 & & & & & & & & \\ & & 1 & & & & & & \\ & 1 & & & & & & & \end{pmatrix}$$

Conjugating by $\rho(\mathcal{S})$ shows that

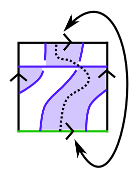
$$\rho(\mathcal{T}_2^{-1}) = \begin{pmatrix} & & & 1 & & & & & \\ & & & & & 1 & & & \\ & & & & \zeta & & & & \\ 1 & & & & & & & & \\ & & 1 & & & & & & \\ & 1 & & & & & & & \\ & & & & & & 1 & & \\ & & & & & & & & 1 \\ & & & & & & & \zeta^{-1} & \end{pmatrix}$$

**Example 13.4.** This is a calculation that will be useful later, and is similar to one of the calculations in the last example. Here is a cobordism from the curve on the cylinder that wraps twice around to the curve that runs across the cylinder:



The dashed curve is the curve $\eta$ with nonorientable normal bundle. In $(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}})$, $\lambda(\eta, s_{\mathsf{C},\mathrm{nb}}|_\eta) = i$, so the cobordism is a bigon connect-summed $\mathbb{R}P^2$ (call it $(\Sigma, t)$) with $Z_2(\Sigma, t)$ being multiplication by $\zeta$.

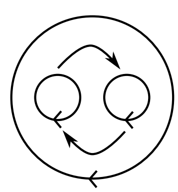
In the bounding case, $\eta$ crosses the edge colored by a nontrivial Spin automorphism:



so in this case $Z_2(\Sigma, t)$ is multiplication by $\zeta^{-1}$.

**Example 13.5.** Let $(\mathsf{P}, s_{\mathsf{P},000})$ be the pair of pants with a Spin structure that restricts to the bounding Spin structure on each boundary circle. The degree 0 part of $Z_3(\mathsf{P}, s_{\mathsf{P},000})$ is 16-dimensional, with a basis in bijection with the sixteen curves in Figure 1. $Z_3(\mathsf{P}, s_{\mathsf{P},000})$ has a degree 1 part of dimension 12, which lies in the summands corresponding to the last 12 curves, but this part of $Z_3(\mathsf{P}, s_{\mathsf{P},000})$ will be ignored for the calculations here. Let $w_{\gamma_i}$ denote the (degree 0) basis vector corresponding to $\gamma_i$.

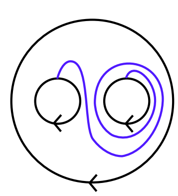
Let $f$ be the half twist, in the direction of the orientation on the outside circle, the two inside circles:



Then

$$f(\gamma_0) = \gamma_0,\ f(\gamma_1) = \gamma_2,\ f(\gamma_2) = \gamma_1,\ f(\gamma_3) = \gamma_3,\ f(\gamma_4) = \gamma_6$$

but $f(\gamma_5)$ is not in isotopic to a curve in the set $\{\gamma_0, \ldots, \gamma_{15}\}$:



$f(\gamma_5)$

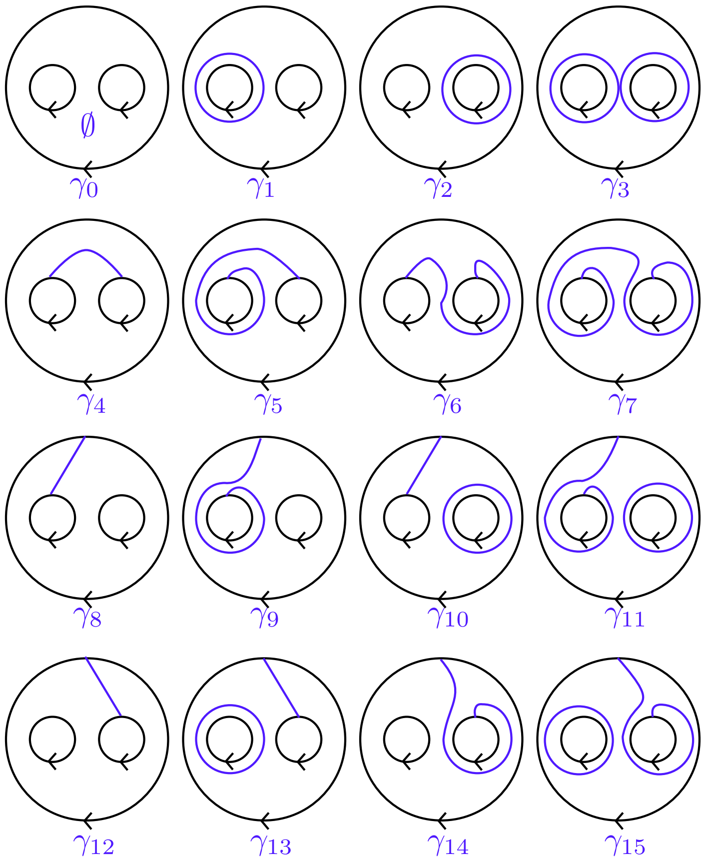



Figure 1: Sixteen curves that correspond to basis elements of the even part of $Z_3(\mathsf{P}, s_{\mathsf{P},000})$.

This curve twists twice going out of the right circle in the direction of the arrow on that circle, then connects to the left circle. Therefore this curve is in the same homology class as $\gamma_4$. Applying the bounding case of Example 13.4 near the right circle, one sees that $f_* w_{\gamma_5} = \zeta^{-1} w_{\gamma_4}$. Here's a full calculation of $f_*$:

| $x$ | $f_*(x)$ | $x$ | $f_*(x)$ | $x$ | $f_*(x)$ | $x$ | $f_*(x)$ |
|---|---|---|---|---|---|---|---|
| $w_{\gamma_0}$ | $w_{\gamma_0}$ | $w_{\gamma_4}$ | $w_{\gamma_6}$ | $w_{\gamma_8}$ | $w_{\gamma_{12}}$ | $w_{\gamma_{12}}$ | $w_{\gamma_{10}}$ |
| $w_{\gamma_1}$ | $w_{\gamma_2}$ | $w_{\gamma_5}$ | $\zeta^{-1} w_{\gamma_4}$ | $w_{\gamma_9}$ | $w_{\gamma_{15}}$ | $w_{\gamma_{13}}$ | $w_{\gamma_8}$ |
| $w_{\gamma_2}$ | $w_{\gamma_1}$ | $w_{\gamma_6}$ | $w_{\gamma_7}$ | $w_{\gamma_{10}}$ | $w_{\gamma_{13}}$ | $w_{\gamma_{14}}$ | $w_{\gamma_{11}}$ |
| $w_{\gamma_3}$ | $w_{\gamma_3}$ | $w_{\gamma_7}$ | $\zeta^{-1} w_{\gamma_5}$ | $w_{\gamma_{11}}$ | $w_{\gamma_{14}}$ | $w_{\gamma_{15}}$ | $w_{\gamma_9}$ |

# 14 The Category Assigned to the Interval

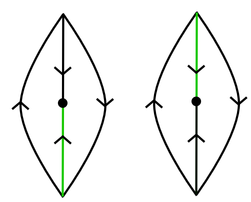
Recall that if a Spin surface $(\Sigma, s)$ has an interval $S$ on its boundary, then there is an action of the algebra $Z_3(\mathsf{B}, s_{\mathsf{B}})$ on $Z_3(\Sigma, s)$. Proposition 5.6 has an analogue in the case of $Z_3$:

**Proposition 14.1.** *Let $(\Sigma_1, s_1)$ and $(\Sigma_2, s_2)$ be two surfaces and let $S_1 \subset \partial\Sigma_1$ be a boundary interval parametrized by $+\Delta^1$ such that $s|_{S_1} = s_{+\Delta^1}$. Let $S_2 \subset$*

*$\Sigma_2$ be a boundary interval parametrized by $-\Delta^1$ such that $s_{S_2} = -s_{+\Delta^1}$. Let $(\Sigma, s)$ be the result of gluing $(\Sigma_1, s_1)$ to $(\Sigma_2, s_2)$ in the canonical way. Let $\mathcal{P}$ be the decomposition of $\Sigma$ into $\Sigma_1$ and $\Sigma_2$. Then the image of $\phi_{\Sigma;\mathcal{P},\mathcal{P}}$[7], which is isomorphic to $Z_3(\Sigma, s)$, is isomorphic to $Z_3(\Sigma_1, s_1) \otimes_{Z(\mathsf{B},s_\mathsf{B})} Z(\Sigma_2, s_2)$ where the action of $Z(\mathsf{B}, s_\mathsf{B})$ on each factor is via the actions corresponding to edges $S_1$ and $S_2$.*

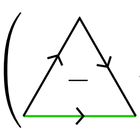
*If instead the two Spin surfaces are glued with the nontrivial Spin automorphism, then the projection is to $Z_3(\Sigma_1, s_1) \otimes_{Z(\mathsf{B},s_\mathsf{B})} Z_3(\Sigma_2, s_2)$ where the action of $Z(\mathsf{B}, s_\mathsf{B})$ on one of the two tensor factors is twisted by the automorphism of $\mathsf{B}$ given by multiplication by $(-1)^{\deg}$.*

*Proof.* The proof is analogous to the proof of Proposition 5.6, with one small detail: there are two possible Spin analogues of the 3-manifold $M'_\mathsf{B}$ in Proposition 5.6. They have cross sections

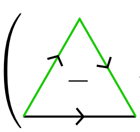


These two choices correspond to whether $(\Sigma, s)$ is obtained from gluing $(\Sigma_1, s_1)$ and $(\Sigma_2, s_2)$ with a nontrivial Spin automorphism. Details are left to the interested reader. □

Draw an edge of a Spin surface in green if that edge is identified with $(\pm\Delta^1, \pm s_{\Delta^1})$ via the nontrivial Spin automorphism. Then, for example, the right action of $Z_3(\mathsf{B}, s_\mathsf{B})$ on

$Z_3\Big($  $, s_{-\Delta^2}\Big)$ (36)

is twisted by $(-1)^{\deg}$ but the left action of $Z_3(\mathsf{B}, s_\mathsf{B}) \otimes Z_3(\mathsf{B}, s_\mathsf{B})$ is not. Applying the nontrivial Spin automorphism to the whole surface reverses this:

$Z_3\Big($  $, s_{-\Delta^2}\Big)$ (37)

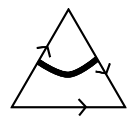
has its left action by $Z_3(\mathsf{B}, s_\mathsf{B}) \otimes Z_3(\mathsf{B}, s_\mathsf{B})$ twisted by $(-1)^{\deg}$ and its right action is the same as usual. The map $(-1)^{\deg}$ acts as an isomorphism of the modules (36) and (37). That this is an isomorphism is a straightforward exercise in the action of algebras in $\mathcal{SV}$: if $a$ is an algebra element acting on $m$ in a module, then $((-1)^{\deg(a)}a) \cdot ((-1)^{\deg(m)}m) = (-1)^{\deg(am)}am$.

[7]here a choice of $\mathcal{B}$, which is unimportant, is suppressed from the notation

**Definition 14.2.** Set $Z_3(\Delta^1, s_{\Delta^1}) = \mathrm{Mod}_{Z(\mathsf{B},s_\mathsf{B})}$ and $Z_3(-\Delta^1, -s_{\Delta^1}) = {}_{Z(\mathsf{B},s_\mathsf{B})}\mathrm{Mod}$. These are subcategories of $\mathcal{SV}$.

**Corollary 14.3.** *$Z_3(-\Delta^2, s_{-\Delta^2})$ defines a monoidal product $\odot$ on $Z_3(\Delta^1, s_{\Delta^1})$. The associator and unit are related to $Z_3(\Delta^3, s_{\Delta^3})$ and $Z_3(\mathsf{D}, s_\mathsf{D})$ just as in Proposition 5.13.*

Note that $Z_3(\mathsf{B}, s_\mathsf{B}) \cong \mathbb{C} \oplus A$ as an algebra. Let $Y_0$ be a $\mathbb{C}$-module and let $Y_1$ be an $A$-module. Then $Y_0 \odot Y_0 = Y_0 \otimes Y_0$, $Y_0 \odot Y_1 = Y_0 \otimes Y_1 \otimes_A A$, $Y_1 \odot Y_0 = (Y_1 \otimes_A A) \otimes Y_0$, and $Y_1 \odot Y_1 = (Y_1 \otimes Y_1) \otimes_{A \otimes A} A'$ where $A'$ is the left $A \otimes A$-module given by the summand of $Z_3(-\Delta_2, s_{-\Delta^2})$ corresponding to this curve:



**Remark 14.4.** The algebra $A$ is a division algebra in $\mathcal{SV}$ so $Z_3(\mathsf{B}, s_\mathsf{B})$ decomposes as a sum of two $(1 \times 1)$ matrix algebras over division algebras.

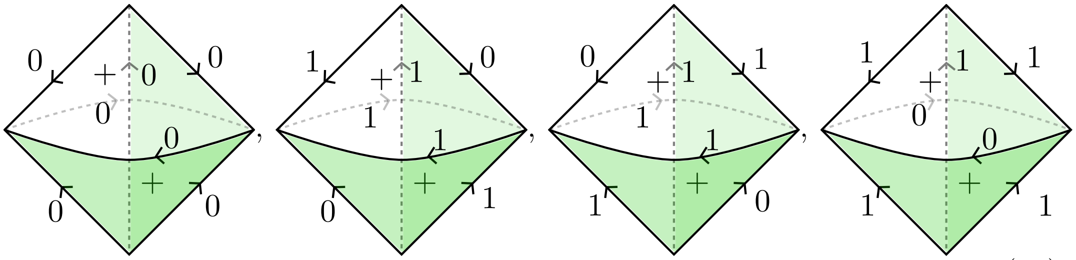
Let $(M, s)$ be a closed Spin 3-manifold. $(M, s)$ can be constructed from a marked triangulation $\tau$ of $M$. Recall that a marking of a 3d triangulation is a marking of faces indicating that the Spin structures on the adjacent tetrahedra are to be glued with the nontrivial Spin automorphism. Let

$$(M', s') = \bigsqcup_{\pm\Delta^3 \in \tau} \pm(\Delta^3, s_{\pm\Delta^3})$$

be the disjoint union of the Spin tetrahedra of $\tau$. Recall that $Z_3(\pm\Delta^3, s_{\pm\Delta^3})$ is a graded vector, the grading given by certain edge labelings of $\pm\Delta^3$ by $\mathbb{Z}/2$ or, equivalently, the grading is by the set of eight surfaces (28).

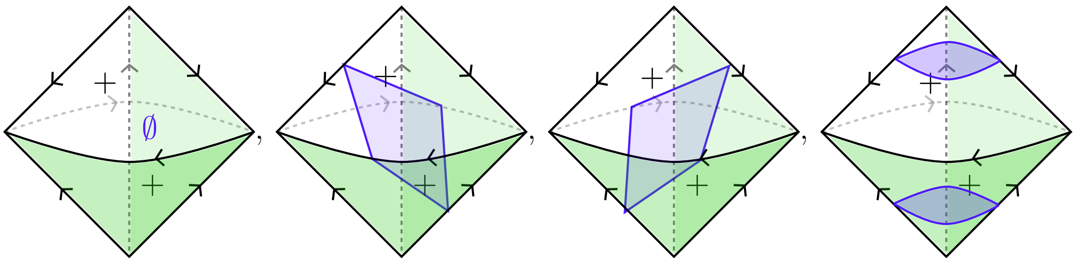
The gluing law for $M$ implies that $Z_3(M, s)$ can be constructed from $Z_3(M', s')$ by contracting tensor factors of $Z_3(M', s')$ corresponding to faces to be glued (composed with $(-1)^{\deg}$ if the corresponding face of $\tau$ is marked), weighted by a factor of $1/2$ for every vertex and a factor of $1/\sqrt{2}$ for every edge labeled by 1 in the grading by edge labelings. Apart from the factor $(-1)^{\deg}$ needed in the Spin case, this is essentially the Turaev-Viro-Barrett-Westbury construction for the category $Z_3(+\Delta^1, s_{+\Delta^1})$. This category is likely spherical in some sense (regarding rigidity see Remark 5.16) though these details are not pursued here.

**Example 14.5.** Let $(M, s)$ be the copy of $\mathbb{R}P^3$ in (15) and let $\tau$ be the marked triangulation pictured there. The triangulation $\tau$ consists of two copies of $+\Delta^3$ glued together along $[012] \leftrightarrow [013]$, $[123] \leftrightarrow [023]$. There are four edge labelings of this $\mathbb{R}P^3$:



(38)

which correspond to the four surfaces



Formulas for $Z_3(\pm\Delta^3, s_{\pm\Delta^3})$ are given in the tables (29) and (30). The ordering of the tensor factors for the positive tetrahedron is, by convention, [123], [013], [023], [012].

The first edge labeling contributes 1. For the second edge labeling in (38), the two tetrahedra give the following expression in an eightfold tensor product

$$\begin{aligned} &2^{-1/4}(a^0a^0a_0a_0 + a^0a^1a_1a_0 - ia^1a^0a_0a_1 + \zeta a^0a^0a_1a_1 \\ &\qquad + \zeta a^1a^1a_0a_0 + \zeta^{-1}a^1a^0a_1a_0 + \zeta^{-1}a^0a^1a_0a_1 + ia^1a^1a_1a_1) \\ &\otimes 2^{-1/4}(a^0a^0a_0a_0 + \zeta a^1a^1a_0a_0 + \zeta a^0a^0a_1a_1 - ia^1a^0a_0a_1 \\ &\qquad + a^0a^1a_1a_0 + \zeta a^1a^0a_1a_0 + \zeta a^0a^1a_0a_1 - ia^1a^1a_1a_1) \end{aligned}$$

apply $(-1)^{\deg}$ to tensor factors 3 and 4, then contract tensor factors 1 and 7, 2 and 8, 3 and 5, and 4 and 6. All but 8 of the 64 tensor factors vanish in the contraction, and the eight remaining terms each correspond to $\lambda$ applied to a single curve in $\mathbb{R}P^3$. The result is $2^{3/2}\zeta^{-1}$. Dividing by $2^{1/2}$ for each edge labeled by 1 results in $\zeta^{-1}$. As expected, this is $\beta$ evaluated on the copy of $\mathbb{R}P^2$ corresponding to this edge labeling.

Similarly, for the the other two edge labelings, one forms an eightfold tensor product, applies the $(-1)^{\deg}$ appropriately, contracts the same tensor factors, and divides by $2^{3/2}$. The other two edge labelings produce factors of $\zeta^{-1}$ and 1, corresponding to $\beta$ evaluated on the other $\mathbb{R}P^2$ and the sphere.

Because there are two vertices, then

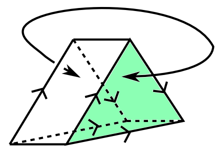
$$Z_3(M,s) = \frac{1}{2^2}(2 + \zeta^{-1} + \zeta^{-1}) = \frac{1}{2}(1 + \zeta^{-1})$$

which is consistent with the definition of $Z_3$ on closed manifolds.

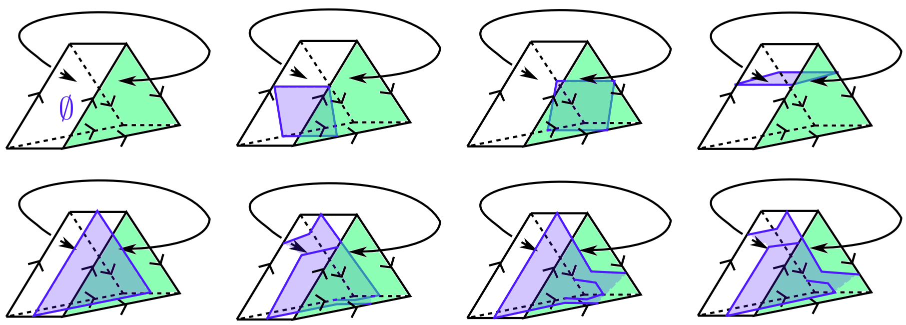
# 15 The Algebras Assigned to the Cylinders

Recall from Example 12.9 the curves $\gamma_0$, $\gamma_1$, $\gamma_2$, and $\gamma_3$. That example showed that $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}}) \cong \mathbb{C} \oplus \mathbb{C} \oplus A \oplus A$. Let $x_0$ be the element 1 in $Z_2(\gamma_0, s|_{C,\mathrm{b}}|_{\gamma_0}) = \mathbb{C}$. Let $x_1$ be the element $\widetilde{a}_0 \in Z_2(\gamma_1, s|_{C,\mathrm{b}}|_{\gamma_1}) \cong Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{b}})$. Let $x_2$ and $x_3$ correspond to $a_0$ and $a_1$, respectively, in $Z_3(\gamma_2, s_{\mathsf{C},\mathrm{b}}|_{\gamma_2})$ and let $x_4$ and $x_5$ correspond to $a_0$ and $a_1$, respectively, in $Z_3(\gamma_3, s_{C,\mathrm{b}}|_{\gamma_3})$. Then $\{x_0, \ldots, x_5\}$ forms a basis for $\mathcal{A}_0$.

Recall that applying $Z_3$ to

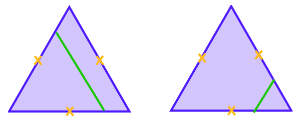


gives the multiplication in $\mathcal{A}_0$. The multiplication is therefore calculated by applying $Z_2$ to the following eight surfaces:




(The last three surfaces have a boundary that winds twice around boundary torus in one direction, and once around the other direction.) The first surface shows that $x_0 x_0 = x_0$. The second two surfaces are cylinders, where one boundary is $(-\mathsf{S}, -s_{\mathsf{S},\mathrm{b}})$ and the other is $(\mathsf{S}, s_{\mathrm{b}})$, and $Z_2$ applied to this cylinder is $\widetilde{a}^0 \otimes \widetilde{a}_0$. Therefore $x_1 x_0 = x_1$ and $x_0 x_1 = x_1$. The fourth surface is a cylinder with both boundary components equivalent to $Z_2(-\mathsf{S}, -s_{\mathsf{S},\mathrm{b}})$, so by Example 11.6, this implies that $x_1 x_1 = x_0$.

The next row of surfaces provide the multiplications for $x_2$, $x_3$, $x_4$, and $x_5$. The first of these shows that $x_2$ and $x_3$ multiply as in $A$: $x_2 x_2 = x_2$, $x_2 x_3 = x_3$, $x_3 x_2 = x_3$, $x_3 x_3 = x_2$. The next two can be unfolded to



(39)

Note that on the left the curves between the marked points on each boundary may pass through the green arcs, but on the right they do not. This ensures a certain kind of noncommutativity: $x_4 x_2 = x_4$, $x_4 x_3 = -x_5$, $x_5 x_2 = x_5$, $x_5 x_3 = -x_4$ but $x_2 x_4 = x_4$, $x_3 x_4 = x_5$, $x_2 x_5 = x_5$, $x_3 x_5 = x_4$.

The last surface is a triangle connect-summed with a Mobius band, similar to Example 13.4. Therefore $x_4 x_4 = \zeta^{-1} x_2$. These are enough calculations to produce the full multiplication table for $\mathcal{A}_0$:

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|
| $x_0$ | $x_0$ | $x_1$ | 0 | 0 | 0 | 0 |
| $x_1$ | $x_1$ | $x_0$ | 0 | 0 | 0 | 0 |
| $x_2$ | 0 | 0 | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
| $x_3$ | 0 | 0 | $x_3$ | $x_2$ | $x_5$ | $x_4$ |
| $x_4$ | 0 | 0 | $x_4$ | $-x_5$ | $\zeta^{-1}x_2$ | $-\zeta^{-1}x_3$ |
| $x_5$ | 0 | 0 | $x_5$ | $-x_4$ | $\zeta^{-1}x_3$ | $-\zeta^{-1}x_2$ |

It is not hard to see that $\pi_0 = \frac{1}{2}(x_0 + x_1)$, $\pi_1 = \frac{1}{2}(x_0 - x_1)$, and $\pi_2 = x_2$ form a complete set of simple projections of $\mathcal{A}_0$ and form a basis for the center of $\mathcal{A}_0$. The algebra $\pi_2 \mathcal{A}_0$ is isomorphic to a $2 \times 2$ matrix algebra over $\mathbb{C}$:

$$x_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_3 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$x_4 \mapsto \zeta^{-1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x_5 \mapsto \zeta^{-1/2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The algebra $\mathcal{A}_1$ has a similar basis, say $y_0, \ldots, y_5$. Here $y_i$ corresponds to the same curve as $x_i$. The only difference is that $y_1$ is in degree 1, because the nontrivial circle on the nonbounding cylinder has nonbounding Spin structure, and $Z_2(\mathsf{S}, s_{\mathsf{S},\mathrm{nb}}))$ has a basis given by $\widetilde{a}_1$. The same kind of analysis produces a multiplication table for $\mathcal{A}_1$, only now $\zeta$ appears instead of $\zeta^{-1}$ (see Example 13.4) and the analogue of (39) doesn't have any green lines in it, removing the factors of $-1$ from the multiplication table:

| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|---|
| $y_0$ | $y_0$ | $y_1$ | 0 | 0 | 0 | 0 |
| $y_1$ | $y_1$ | $y_0$ | 0 | 0 | 0 | 0 |
| $y_2$ | 0 | 0 | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
| $y_3$ | 0 | 0 | $y_3$ | $y_2$ | $y_5$ | $y_4$ |
| $y_4$ | 0 | 0 | $y_4$ | $y_5$ | $\zeta y_2$ | $\zeta y_3$ |
| $y_5$ | 0 | 0 | $y_5$ | $y_4$ | $\zeta y_3$ | $\zeta y_2$ |

A set of simple projectors is given by $y_0$, $\frac{1}{2}(y_2 \pm \zeta^{-1/2} y_4)$. Note that $y_1$ is not central because it has odd degree. $\mathcal{A}_1$ is isomorphic to a sum of three copies of $A$.

Let $\mathcal{P}_1$ be the decomposition of the torus $(\mathsf{T}, s_{\mathsf{T},(1,1)})$ obtained by gluing the two ends of the cylinder $(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$ together with the nontrivial Spin automorphism. Therefore $\mathcal{P}_1$ has one piece, the cylinder. Let $\mathcal{P}_\mathsf{T}$ be the decomposition of into itself as a single piece. Let $w_{\gamma_0}$, $w_{\gamma_1}$, $w_{\gamma_2}$, and $w_{\gamma_3}$ be as in Example 13.3. Then $\phi_{\mathsf{T},s_{\mathsf{T},(1,1)};\mathcal{P}_1,\mathcal{P}_\mathsf{T}}$ has image spanned by $w_{\gamma_0}$, $w_{\gamma_1}$, and $w_{\gamma_2}$. These are mapped to by the span of $x_0$, $x_1$, and $x_2$. Explicitly

$$x_0 \mapsto w_{\gamma_0}, \ x_1 \mapsto w_{\gamma_1}, \ x_2 \mapsto \frac{1}{\sqrt{2}} w_{\gamma_2}$$

Therefore $Z_3(\mathsf{T}, s_{\mathsf{T},(1,1)})$ is identified with the center of $\mathcal{A}_0$. Let $\omega_i$ be the projector $\pi_i$ scaled so that it has norm 1 with respect to the pairing (35). Then

$$\omega_0 = w_{\gamma_0} + w_{\gamma_1}$$
$$\omega_1 = w_{\gamma_0} - w_{\gamma_1}$$
$$\omega_2 = \sqrt{2} w_{\gamma_2}$$

$\rho(\mathcal{S})$ sends $Z_3(T_{(1,1)}, s_{T,(1,1)})$ to itself, and acts on the $\{\omega_i\}$ basis as

$$\omega_0 \mapsto w_{\gamma_0} + w_{\gamma_2} = \frac{1}{2}(\omega_0 + \omega_1) + \frac{1}{\sqrt{2}}\omega_2$$
$$\omega_1 \mapsto w_{\gamma_0} - w_{\gamma_2} = \frac{1}{2}(\omega_0 + \omega_1) - \frac{1}{\sqrt{2}}\omega_2$$
$$\omega_2 \mapsto \sqrt{2} w_{\gamma_1} = \sqrt{2}(\omega_0 - \omega_1)$$

so, with respect to this basis:

$$\rho(\mathcal{S}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{40}$$

**Remark 15.1.** One can show using Remark 4.5 that if $\Sigma_g$ is a closed surface of genus $g$ and $s$ is a bounding Spin structure on it, then $\dim(Z_3(\Sigma_g, s)) = 2^{2g-2} + (\sqrt{2})^{2g-2} + 2^{2g-2}$ where the three numbers 2, $\sqrt{2}$, and 2 show up as the inverses of the norms of the simple projectors of $\mathcal{A}_0$.

# 16 The Categories Assigned to the Circles

The following Proposition and its proof are entirely analogous to Proposition 14.1:

**Proposition 16.1.** *Let $(\Sigma_1, s_1)$ and $(\Sigma_2, s_2)$ be two Spin surfaces. Let $S_1$ be a boundary circle parametrized by $+\Delta^1$ for which $(s_1)|_{S_1}$ is bounding. Let $S_2$ be a boundary circle of $S_2$ parametrized by $-\Delta^1$ for which $(s_2)|_{S_2}$ is bounding. Let $(\Sigma, s)$ be the result of gluing $(\Sigma_1, s_1)$ to $(\Sigma_2, s_2)$ in the canonical way. Let $\mathcal{P}$ be the decomposition of $\Sigma$ into $\Sigma_1$ and $\Sigma_2$. Then the image of $\phi_{\Sigma;\mathcal{P},\mathcal{P}}$[8], which is isomorphic to $Z_3(\Sigma, s)$, is isomorphic to $Z_3(\Sigma_1, s_1) \otimes_{Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})} Z_3(\Sigma_2, s_2)$ where the action of $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$ on each factor is via the actions corresponding to edges $S_1$ and $S_2$.*

*If instead the two Spin surfaces are glued with the nontrivial Spin automorphism, then the projection is to $Z_3(\Sigma_1, s_1) \otimes_{Z(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})} Z_3(\Sigma_2, s_2)$ where the action of $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$ on one of the two tensor factors is twisted by the automorphism of $\mathsf{B}$ given by $(-1)^{\deg}$.*

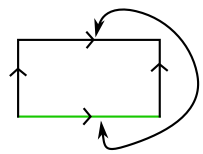
*If $S_1$ and $S_2$ are nonbounding, the same statements hold, with $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$ replaced by $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}})$.*

[8] here a choice of $\mathcal{B}$, which is unimportant, is suppressed from the notation

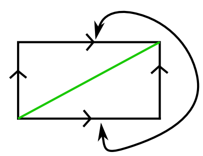
Therefore gluing together two Spin surfaces along boundary circles corresponds to tensoring over the algebra actions on those boundary circles. If you glue with the nontrivial Spin automorphism, then you twist one of the tensor factors by the degree algebra automorphism.

As the case with intervals, draw a boundary circle in green if its Spin structure is identified with $(\pm \mathsf{S}, \pm s_{(\mathrm{n})\mathrm{b}})$ via the nontrivial Spin automorphism.

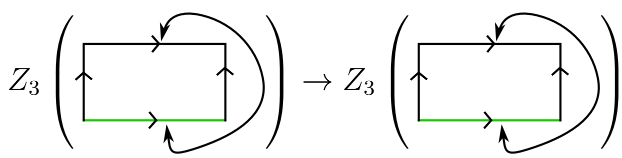
**Example 16.2.** The bounding cylinder is obtained by gluing two sides of a square with the nontrivial Spin automorphism:



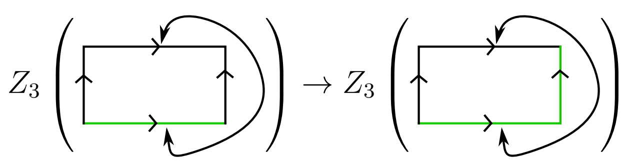
If you apply a Dehn twist $f$ to the cylinder, the seam along which the Spin structure is nontrivially identified shifts



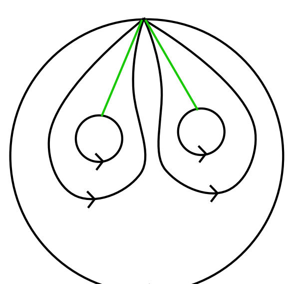
and an isomorphism back to the original Spin structure must involve applying the nontrivial Spin automorphism to precisely one of the boundary components. Therefore $\rho(f)$ is not a map of $\mathcal{A}_0 - \mathcal{A}_0$-bimodules on



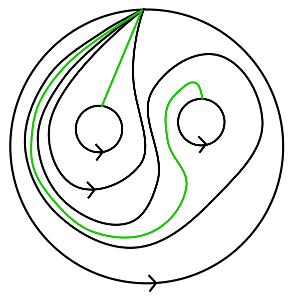

but it is a map of $\mathcal{A}_0$-$\mathcal{A}_0$-bimodules on



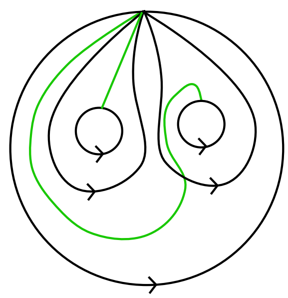

**Example 16.3.** This example is similar in flavor to the last example, though the exact Spin structures are a bit more involved. The pair of pants with three bounding boundary circles can be constructed as follows:



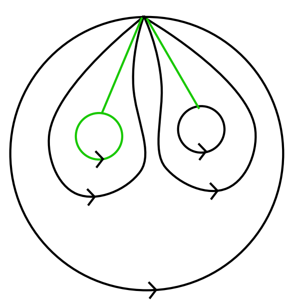
If you apply a half twist to the inside you get the following, which is not exactly the same Spin structure:



With respect to the original decomposition, this is



and any isomorphism back to the original Spin structure has to apply the nontrivial Spin automorphism either to the left circle or the other two circles:



**Definition 16.4.** Define

$$Z_3(\mathsf{S}, s_{\mathsf{S},\mathrm{b}}) := \mathrm{Mod}_{\mathcal{A}_0} = \mathrm{Mod}_{Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})}$$

$$Z_3(-\mathsf{S}, s_{\mathsf{S},\mathrm{b}}) := {}_{\mathcal{A}_0}\mathrm{Mod} = {}_{Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})}\mathrm{Mod}$$

$$Z_3(\mathsf{S}, s_{\mathsf{S},\mathrm{nb}}) := \mathrm{Mod}_{\mathcal{A}_1} = \mathrm{Mod}_{Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}})}$$

$$Z_3(-\mathsf{S}, s_{\mathsf{S},\mathrm{nb}}) := {}_{\mathcal{A}_1}\mathrm{Mod} = {}_{Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{nb}})}\mathrm{Mod}$$

For short, write $\mathcal{C}_0 = \mathrm{Mod}_{\mathcal{A}_0}$ and $\mathcal{C}_1 = \mathrm{Mod}_{\mathcal{A}_1}$.

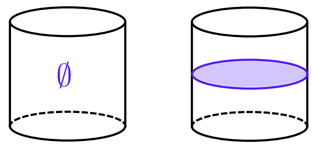
Let $X_0$, $X_1$, and $X_2$ denote the three simple modules of $\mathcal{A}_0$ corresponding to $\pi_0$, $\pi_1$, and $\pi_2$.

**Proposition 16.5.** *$Z_3(\mathsf{D}, s_{\mathsf{D}})$ is isomorphic, as a $\mathcal{A}_0$-module, to $X_0$.*

*Proof.* It is enough to calculate

$$Z_3(\Delta^1 \times \mathsf{D}, s_{\Delta^1\times\mathsf{D}}) \in Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})^* \otimes Z_3(\mathsf{D}, s_{\mathsf{D}})^* \otimes Z_3(\mathsf{D}, s_{\mathsf{D}}).$$

It is a sum of two terms, corresponding to the two surfaces




The empty surface contributes $1 \in Z_2(\gamma_0, s_{\mathsf{C},\mathrm{b}}|_{\gamma_0})^* \subset Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})^*$ and the nontrivial surface contributes $\widetilde{a}^0 \in Z_2(\gamma_1, s_{\mathsf{C},\mathrm{b}}|_{\gamma_1})^* \subset Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})^*$. The sum $1+\widetilde{a}^0$ pairs to 1 with $\pi_0$ and to zero with the other two projectors. □

**Corollary 16.6.** *Let $(\Sigma_g, s)$ be a closed Spin surface of genus $g$. Let $D$ be a disk in $(\Sigma_g, s)$. The $X_0$ part of $Z_3(\Sigma_g \setminus D, s)$ has a basis consisting of bounding (resp. nonbounding) classes in $H_1(\Sigma_g; \mathbb{Z}/2)$ if $s$ is bounding (resp. nonbounding). The $X_1$ part of $Z_3(\Sigma_g \setminus D, s)$ has a basis consisting of nonbounding (resp. bounding) classes in $H_1(\Sigma_g; \mathbb{Z}/2)$ if $s$ is bounding (resp. nonbounding).*

*Proof.* Since $Z_3(\Sigma_g, s)$ was constructed from $Z_3((\Sigma_g \setminus D) \cup D, s)$ (see the proof of Proposition 12.24), then the statement about the $X_0$ part follows from Proposition 12.24 and the earlier results of this section.

Let $\mathcal{P}$ be the decomposition of $\Sigma_g$ into $\Sigma_g \cup D$ and $D$. Applying the projector $\pi_1$ to $Z_3(\Sigma_g \setminus D, s)$ is exactly the same as applying $\phi_{\Sigma_g,s;(\mathcal{P},\mathcal{B}),(\mathcal{P},\mathcal{B})}$ as in the proof of Proposition 12.24, except $Z_2(\{1/2\} \times \Sigma_g, s)$ is weighted by a factor of $-1$ from the negative term in the projector $\pi_1$. Therefore the proof of Proposition 12.24 applies, except the roles of bounding and nonbounding homology classes are reversed. □

Note that there is no $X_2$ part in $Z_3(\Sigma_g \setminus D, s)$ because no nontrivial curves in a surface intersect the boundary in just one point.

**Corollary 16.7.** *$Z_3(\Sigma_g \setminus D, s)$ splits as a sum of a $\frac{1}{2}(4^g + 2^g)$-dimensional part and a $\frac{1}{2}(4^g - 2^g)$-dimensional part. If $s$ is bounding, these parts are the even and odd parts (respectively) and if $s$ is nonbounding these are the odd and even parts (respectively).*

Since $\mathcal{A}_0$ is a sum of matrix algebras over $\mathbb{C}$ it is, as an $\mathcal{A}_0$-$\mathcal{A}_0$-bimodule, isomorphic to $X_0^* \otimes X_0 \oplus (X_1^* \otimes X_1) \oplus (X_2^* \otimes X_2)$. The situation for $\mathcal{A}_1 \cong A \oplus A \oplus A$ is a little different because $A$, viewed as an $A$-$A$-bimodule, doesn't split as a tensor product of left and right modules.

As expected, $\mathcal{C}_0$ is a monoidal category:

**Proposition 16.8.** *The pair of pants with three bounding boundary circles, $(\mathsf{P}, s_{\mathsf{P},000})$, gives a monoidal product on $Z_3(\mathsf{S}, s_{\mathsf{S},\mathrm{b}})$. The unit is $X_0 = Z_3(\mathsf{D}, s_{\mathsf{D}})$ and the associator is given as in Proposition 5.14.*

*Proof.* The proof is similar to Propositions 5.14 and 5.13. □

**Remark 16.9.** $\mathcal{C}_1$ does not get a monoidal structure from a pair of pants since there is no Spin pair of pants with three nonbounding boundary circles. Rather $Z_3(\mathsf{P}, s_{\mathsf{P},011})$ and $Z_3(\mathsf{P}, s_{\mathsf{P},101})$ provide functors $\mathcal{C}_0 \times \mathcal{C}_1 \to \mathcal{C}_1$ and $\mathcal{C}_1 \times \mathcal{C}_0 \to \mathcal{C}_1$ that make $\mathcal{C}_1$ a left- and right-module category for $\mathcal{C}_0$. $Z_3(\mathsf{P}, s_{\mathsf{P},110})$ provides a functor $\mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_1$.

**Proposition 16.10.** *The monoidal structure on $\mathcal{C}_0$ has the following multiplication table:*

$$\begin{array}{c|ccc} & X_0 & X_1 & X_2 \\ \hline X_0 & X_0 & X_1 & X_2 \\ X_1 & X_1 & X_0 & X_2 \\ X_2 & X_2 & X_2 & X_0 \oplus X_1 \end{array}, \tag{41}$$

*This is the same multiplication table as for the Ising category.*

*Proof.* It is enough to show that the vector space of the pair of pants is, as a $\mathcal{A}_0 \otimes \mathcal{A}_0$-$\mathcal{A}_0$-bimodule, isomorphic to

$$\begin{aligned} &(X_0^* \otimes X_0^* \otimes X_0) \oplus (X_1^* \otimes X_1^* \otimes X_0) \oplus (X_1^* \otimes X_0^* \otimes X_1) \oplus (X_0^* \otimes X_1^* \otimes X_1) \\ &\oplus (X_2^* \otimes (X_0 \oplus X_1)^* \otimes X_2) \oplus ((X_0 \oplus X_1)^* \otimes X_2^* \otimes X_2) \\ &\oplus (X_2^* \otimes X_2^* \otimes (X_0 \oplus X_1)) \end{aligned}$$

To prove this, it is enough to apply projectors $\pi_i \otimes \pi_j \otimes \pi_k$ for all combinations of $i$, $j$, and $k$ to $Z_3(\mathsf{P}, s_{\mathsf{P},000})$. It is enough to do this on the even part of $Z_3(\mathsf{P}, s_{\mathsf{P},000})$ since there are no purely odd objects in $\mathcal{C}_0$.

Let $\gamma_0, \dots, \gamma_{15}$ and $w_{\gamma_0}, \dots w_{\gamma_{15}}$ be as in Example 13.5. The vectors $\{w_{\gamma_i}\}$ form a basis for the even part of $Z_3(\mathsf{P}, s_{\mathsf{P},000})$. Noting that $\gamma_3$ is homologous (via a pair of pants) to a single circle around the outside cuff, it is not hard to see that the span of $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ is isomorphic to

$$(X_0^* \otimes X_0^* \otimes X_0) \oplus (X_1^* \otimes X_1^* \otimes X_0) \oplus (X_1^* \otimes X_0^* \otimes X_1) \oplus (X_0^* \otimes X_1^* \otimes X_1)$$

Next we will show that the span of $\{\gamma_4, \dots, \gamma_7\}$ is isomorphic to the even part of $(X_2^* \otimes X_2^* \otimes (X_0 \oplus X_1))$. Let $\Pi = \pi_2 \otimes \pi_2 \otimes (\pi_0 + \pi_1)$. Because the dimensions match, it will be enough to show that $\Pi$ acts as the identity on the span of $\{\gamma_4, \dots, \gamma_7\}$. Recall that the action of $\mathcal{A}_0 \otimes \mathcal{A}_0 \otimes \mathcal{A}_0$ on $Z_3(\mathsf{P}, s_{\mathsf{P},000})$ is given by $Z_3(\Delta^1 \times \mathsf{P}, s_{\mathsf{P},000})$ and this is in turn a sum of terms each corresponding to a surface in $\Delta^1 \times \mathsf{P}$. The action of $\Pi$ on each of $\{\gamma_4, \dots, \gamma_7\}$ is via the surface $\Delta^1 \times \gamma_i$ in $\Delta^1 \times P$. It is not hard to see that part of the action from this surface is the identity when restricted to the degree 0 part of $Z_3(\mathsf{C}, s_{\mathsf{C},\mathsf{b}})^{\otimes 3}$.

There are similar arguments, more or less obtained by permuting the legs of $\mathsf{P}$, that show corresponding statements for $\{\gamma_8, \dots, \gamma_{11}\}$ and $\{\gamma_{12}, \dots, \gamma_{15}\}$. □

Explicitly, here are nonzero vectors in each summand:

$$X_0^* \otimes X_0^* \otimes X_0 : w_{\gamma_0} + w_{\gamma_1} + w_{\gamma_2} + w_{\gamma_3}$$
$$X_0^* \otimes X_1^* \otimes X_1 : w_{\gamma_0} + w_{\gamma_1} - w_{\gamma_2} - w_{\gamma_3}$$
$$X_1^* \otimes X_0^* \otimes X_1 : w_{\gamma_0} - w_{\gamma_1} + w_{\gamma_2} - w_{\gamma_3}$$
$$X_1^* \otimes X_1^* \otimes X_0 : w_{\gamma_0} - w_{\gamma_1} - w_{\gamma_2} + w_{\gamma_3}$$
$$X_2^* \otimes X_2^* \otimes X_0 : w_{\gamma_4} + \zeta w_{\gamma_7}$$
$$X_2^* \otimes X_2^* \otimes X_1 : w_{\gamma_4} - \zeta w_{\gamma_7}$$
$$X_2^* \otimes X_0^* \otimes X_2 : w_{\gamma_8} + w_{\gamma_{10}}$$
$$X_2^* \otimes X_1^* \otimes X_2 : w_{\gamma_8} - w_{\gamma_{10}}$$
$$X_0^* \otimes X_2^* \otimes X_2 : w_{\gamma_{12}} + w_{\gamma_{13}}$$
$$X_1^* \otimes X_2^* \otimes X_2 : w_{\gamma_{12}} - w_{\gamma_{13}}$$

Given $X \in \mathrm{Mod}_{\mathcal{A}_0}$, let $\widetilde{X}$ denote the same super vector space $X$ but with action twisted by the automorphism $a \mapsto (-1)^{\deg(a)}a$. The half twist $f : \mathsf{P} \to \mathsf{P}$ induces a map $\rho(f) : Z_3(\mathsf{P}, s_{P,000}) \to Z_3(\mathsf{P}, s_{\mathsf{P},000})$ which is almost a braiding for the monoidal product. By Example 16.3, $\rho(f)$ defines an isomorphism of the functors

$$X, Y \mapsto \widetilde{Y} \odot X$$

and

$$X, Y \mapsto X \odot Y$$

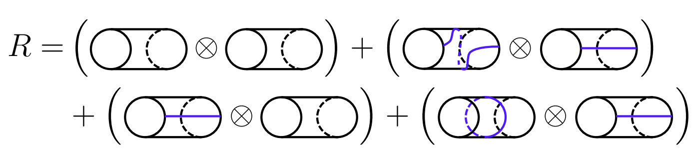
Let $\widetilde{c}_{X,Y} : X \odot Y \to \widetilde{Y} \odot X$ denote the corresponding family of maps. Then the maps $\widetilde{c}_{X,Y}$ satisfy the braid relation because $f$ satisfies the braid relation.

**Example 16.11.** The vectors $w_{\gamma_4}, w_{\gamma_5}, w_{\gamma_6}, w_{\gamma_7}$ span the space $X_2^* \otimes X_2^* \otimes (X_0 \oplus X_1)$. The eigenvalues for the action of $f_*$ on the span of these vectors are

$$\pm\zeta^{-1/2}, \ \pm i\zeta^{-1/2}$$

Note that the corresponding braiding in the Ising category (for the correct choice of $\zeta$) has eigenvalues $\zeta^{-1/2}, i\zeta^{-1/2}$.

**Remark 16.12.** If $X$ and $Y$ are simple, then a morphism $h : X \odot Y \to X \odot \widetilde{Y}$ is determined by multiplication by an element of $\mathcal{A}_0 \otimes \mathcal{A}_0$. In particular, this element can deduced just by looking at the restriction of $h$ to the degree 0 part of $X \odot Y$. In particular, the braiding is given by multiplication by



then applying the usual symmetric braiding $\sigma$ of $\mathcal{SV}$. $R$ here is deduced by looking at Example 13.5. Algebraically, it is

$$R = x_0 \otimes x_0 + x_4 \otimes x_2 + x_2 \otimes x_0 + x_1 \otimes x_2$$

Recall that $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}}) = \mathcal{A}_0 \cong (X_0^* \otimes X_0) \oplus (X_1^* \otimes X_1) \oplus (X_2^* \otimes X_2)$. Let $g : \mathsf{C} \to \mathsf{C}$ be a right-handed Dehn twist of the cylinder. Recall from example 16.2 that this twist changes the Spin parametrization of one of the boundary components by the nontrivial Spin automorphism. Then $g_* : Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}}) \to Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$ is (as a morphism of $\mathcal{A}_0$-$\mathcal{A}_0$-bimodules) a map

$$X_0^* \otimes X_0 \to X_0^* \otimes \widetilde{X}_0,\ X_1^* \otimes X_1 \to X_1^* \otimes \widetilde{X}_1,\ X_2^* \otimes X_2 \to X_2^* \otimes \widetilde{X}_2$$

If $Z_3(\mathsf{C}, s_{\mathsf{C},\mathrm{b}})$ is regarded as canonically isomorphic to the identity functor $\mathcal{C}_0 \to \mathcal{C}_0$, then $g_*$ provides an isomorphism from the identity functor to the functor $X \mapsto \widetilde{X}$.

**Example 16.13.** Let $g : \mathsf{C} \to \mathsf{C}$ be the right-handed Dehn twist. Then $g$ can be isotoped so that it acts as the identity on the curves underlying $x_0$ and $x_1$ in $\mathcal{A}_0$. In particular, $\rho(g)$ acts as the identity on $X_0^* \otimes X_0$ and $X_1^* \otimes X_1$. Because $X_0$ and $X_1$ are even vector spaces, $\widetilde{X}_0 = X_0$ and $\widetilde{X}_1 = X_1$, so $\rho(g)$ is the identity endomorphism of the identity endofunctor. This differs from the Ising category $\mathcal{I}_\zeta$ where the twist isomorphism that should correspond to a right-handed Dehn twist in the corresponding TQFT acts as multiplication by $-1$ on the nontrivial invertible simple object.

Since $\rho(g) : X_2^* \otimes X_2 \to X_2^* \otimes \widetilde{X}_2$ is a map of $\mathcal{A}_0$-$\mathcal{A}_0$-bimodules and $X_2$ is simple, $\rho(g)$ is given by multiplication on each factor by an element of $\mathcal{A}_0$. Note that multiplication by the identity is an $\mathcal{A}_0$-module map $X_2 \to X_2$ and multiplication by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{42}$$

is an $\mathcal{A}_0$-module map $X_2 \to \widetilde{X}_2$. Therefore $\rho(g)$ must be given by right multiplication by a multiple of (42). Since $g$ takes the curve underlying $x_2$ to the curve underlying $x_4$, and since $x_2$ and $x_4$ are in even degree, then $\rho(g)(x_2) = x_4$. Therefore $\rho(g)$ is multiplication by $x_4$. Note that $x_4$ acts as

$$\zeta^{-1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

# 17 Constructing TQFTs from $Z_2$ and $Z_3$

The assignments $Z_2$ and $Z_3$ are essentially “extended Spin pre-TQFTs”. Spin pre-TQFTs were not defined earlier, but there is an obvious definition building off of the definition of a pre-TQFT. One can therefore apply the idea of Propostion 3.6 in order to turn $Z_2$ into a $\mathrm{Pin}^-$ TQFT $\hat{Z}_2$ and to turn $Z_3$ into a Spin TQFT $\hat{Z}_3$.

**Proposition 17.1.** *There exists a* $\mathrm{Pin}^-$ *TQFT* $\hat{Z}_2$ *such that* $\hat{Z}_2(\Sigma, s) = \beta(\Sigma, s)$ *for closed surfaces* $\Sigma$.

*Proof.* Let $S$ be a closed 1-manifold, let $K$ be a line bundle on $S$ and suppose that $\underline{\mathbb{R}} \oplus S \oplus K$ is oriented. Let $t$ be a Spin lift of $F(\underline{\mathbb{R}} \oplus TS \oplus K)$. Set

$$\hat{Z}_2(S,t) = Z_2(S,t)$$

Let $(\Sigma, s, \phi_-, \phi_+)$ be a cobordism from $(S_1, t_1)$ to $(S_2, t_2)$. Here $t_i$ is a Spin lift of $F(\underline{\mathbb{R}} \oplus T_i \oplus K_i)$ for some line bundles $K_i$ on $S_i$, $s$ is a Spin lift of $F(T\Sigma \oplus K)$ for some line bundle $K$ such that $w_1(K) = w_1(\Sigma)$ and the maps

$$\phi_- : ((\partial\Sigma)_-, s|_{(\partial\Sigma)_+}) \to (S_1, -t_1)$$

$$\phi_+ : ((\partial\Sigma)_+, s|_{(\partial\Sigma)_+}) \to (S_2, t_2)$$

are maps of Spin bundles covering maps on frames induced by diffeomorphisms and that are the identity on the stabilization direction. Then

$$Z_2(\Sigma, s) \in Z_2((\partial\Sigma)_-, s|_{(\partial\Sigma)_-}) \otimes Z_2((\partial\Sigma)_+, s|_{(\partial\Sigma)_+})$$

so $((\phi_-)_* \otimes (\phi_+)_*)(Z_3(\Sigma, s))$ is in

$$Z_2(-S_1, -t_1) \otimes Z_2(S_2, t_2) \cong Z_2(S_1, t_1)^* \otimes Z_2(S_2, t_2)$$

where here $Z_2(-S_1, -t_1)$ is identified with $Z_2(S_1, t_1)^*$ via the pairing from Definition 11.4. By Example 11.5, this pairing is inverse to $Z_2(\Delta^1 \times S_1, t_1) \in Z_2(-S_1, -t_1) \otimes Z_3(S_1, t_1)$. Therefore $((\phi_-)_* \otimes (\phi_+)_*)(Z_3(\Sigma, s))$ gives an element in $\mathrm{Hom}(Z_2(S_1, t_1), Z_2(S_2, t_2))$. Set $\hat{Z}_2(\Sigma, s)$ to be equal to this element. By the gluing law for $Z_2$, Proposition 10.6, $\hat{Z}_2$ is compatible with composition of cobordisms. Because the pairing used to identify $Z_2(-S, -t)$ with $Z_2(S, t)^*$ is an inverse to $Z_2(\Delta^1 \times S, t)$, then $\hat{Z}_2$ sends identity functors to identity morphisms. ◻

**Proposition 17.2.** *There exists a* Spin *TQFT* $\hat{Z}_3$ *such that* $\hat{Z}_3(M, s) = \frac{1}{2^{b_0(M)}} \sum_{x \in H_2(M;\mathbb{Z}/2)} \beta(x, s|_x)$ *for closed 3-manifolds* $M$*.*

*Proof.* Set $\hat{Z}_3(\Sigma, s) = Z_3(\Sigma, s)$. If $(M, s, \phi_-, \phi_+)$ is a Spin cobordism, then set

$$\hat{Z}_3(M, s, \phi_-, \phi_+) = ((\phi_-)_* \otimes (\phi_+)_*)(Z_3(M, s))$$

realized as an element of $\mathrm{Hom}(Z_3((\partial M)_-, s|_{(\partial M)_-}), Z_3((\partial M)_+, s|_{(\partial M)_+}))$.

The proof is entirely analogous to Proposition 17.1. The pairing used between Spin surfaces is Definition 12.23 and the fact that it is inverse to $Z_3(\Delta^1 \times \Sigma, s)$ is Corollary 12.25. The gluing law for $Z_3$, which is required to prove the composition properties of $\hat{Z}_3$, is Proposition 12.10. ◻

# 18 Further Directions

It would be good to construct the Ising TQFT $Z_{\mathcal{I}_\zeta}$ topologically, instead of the related Spin TQFT $Z_3$ constructed here. There is a significant problem, however. The Ising TQFT is a TQFT with anomaly: for two closed oriented surfaces $\Sigma$ and $\Sigma'$ there is no canonical isomorphism $Z_{\mathcal{I}_\zeta}(\Sigma \sqcup \Sigma') \cong Z_{\mathcal{I}_\zeta}(\Sigma) \otimes Z_{\mathcal{I}_\zeta}(\Sigma')$. Rather, a canonical isomorphism exists only after picking Lagrangian subspaces in $H^1(\Sigma;\mathbb{Z})$ and $H^1(\Sigma';\mathbb{Z})$. Rokhlin's invariant enjoys a similar sort property. If $W_{12}$ and $W_{23}$ are oriented 4-manifolds with boundaries $M_1 \cup (-M_2)$ and $M_2 \cup (-M_3)$, and $W_{13} = W_{12} \cup_{M_2} W_{23}$, then the signatures of these three manifolds are nonadditive [Wal69]:

$$\sigma(W_{13}) = \sigma(W_{12}) + \sigma(W_{23}) + m(L_1, L_2, L_3)$$

where $m(L_1, L_2, L_3)$ is a term that depends on the Lagrangians

$$L_i = \mathrm{im}(H^1(M_i;\mathbb{R}) \to H^1(\partial M_i;\mathbb{R})).$$

This same term, the "Maslov index", appears in the mapping class group action in the Ising TQFT. The reason no such Lagrangian considerations were needed in the construction of $Z_3$ is that these dependencies cancel in the product $\mu(M,s)\overline{Z_{\mathcal{I}_\zeta}(M)}$.

Kirby and Melvin [KM91] show that all the TQFTs corresponding to SU(2) at level $k$ for $k \equiv 2 \mod 4$ also satisfy, for closed oriented $M$,

$$Z_{\mathrm{SU}(2)_k}(M) = \bigoplus_{s \in \mathrm{Spin}(M)} I_k(M,s)$$

for some family of Spin invariants $I_k(M,s)$. $I_2(M,s)$ is (up to multiplication by $2^{b_0(M)}$) Rokhlin's invariant, but the others are not as obviously related to classical topology. However, the product $I_k(M,s)\overline{Z_{\mathrm{SU}(2)_k}(M)}$ should admit a construction which, like the one in this paper, can be extended down to points. There should be a (super) spherical category from which it can be constructed in the Turaev-Viro-Barrett-Westbury style. What is this category? It is perhaps relevant to note that the dimension of the vector space $Z_{\mathrm{SU}(2)_k}(\Sigma_g)$ is a positive integral linear combination of $\frac{1}{2}(4^g + 2^g)$ and $\frac{1}{2}(4^g - 2^g)$ ([BM96] Section 19 or [AM99] Section 4). For example,

$$\dim(Z_{\mathrm{SU}(2)_k}(\mathsf{T})) = \frac{k+2}{4}\left(\frac{1}{2}(4^1 + 2^1)\right) + \frac{k-2}{4}\left(\frac{1}{2}(4^1 - 2^1)\right)$$

If $D$ is a small disk in $\Sigma_g$, then Proposition 16.5 says that $Z_3(\Sigma_g \setminus D, s)$ splits as a sum of two spaces, of dimensions $\frac{1}{2}(4^g + 2^g)$ and $\frac{1}{2}(4^g - 2^g)$. We know these are the dimensions because a basis of $Z_3(\Sigma_g \setminus D, s)$ is given in terms of curves in the surface $\Sigma_g$. These curves, in turn, fit into the picture as the boundary curves of surfaces embedded in a 3-manifold. Is there be an interpretation of $I_k(M,s)\overline{Z_{\mathrm{SU}(2)_k}(M)}$ in terms of a sum over surfaces in $M$?

A topological construction of the Ising TQFT will probably come from a construction of a 4d invertible TQFT, i.e., one for which 3-manifolds are assigned complex lines. Here is the start of a construction of such a 4d TQFT which may or may not be related to the Ising TQFT. For each closed 3-manifold, let $\mathcal{L}(M)$ be the trivial line bundle over $\mathrm{Spin}(M)$. Lift the action of $H^1(M;\mathbb{Z}/2)$ on $\mathrm{Spin}(M)$ to $\mathcal{L}(M)$ by

$$x \cdot (s,v) = (s+x, \beta(x^\vee, s|_{x^\vee})^{-1}v)$$

Let $L(M)$ be the sections of $\mathcal{L}(M)$ invariant under this action. For example, $s \mapsto \mu(M,s)$ is an invariant section. Since $\beta$ is extended, via $Z_2$, to manifolds with boundary (and even with corners), one might be able to use $Z_2$ to extend this TQFT down to surfaces and even circles.

It would also be good to answer the following question: given a Spin 3-manifold $(M,s)$ presented as a marked triangulation of a 3-manifold $M$ as in Section 7, is there a good way to compute $\mu(M,s)$ from the marked triangulation?

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