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Los Angeles

Global Optimization of Sequential Processes

A thesis submitted in partial satisfaction

of the requirements for the degree Master of Science

in Chemical Engineering

by

Omar Zahid Sheikh

2018

ABSTRACT OF THE THESIS

Global Optimization of Sequential Processes

by

Omar Zahid Sheikh

Master of Science in Chemical Engineering

University of California, Los Angeles, 2018

Professor Vasilios Manousiouthakis, Chair

This work presents a novel method to identify the global optimum of a general class of single parameter optimization problems, typically arising in optimizing sequential processes. Capitalizing on the problem's necessary conditions of optimality, the algorithm identifies arbitrarily tight upper and lower bounding envelopes for the graph of the global optimum as a function of the optimization problem's single parameter. A challenging case study is presented illustrating the algorithm's global optimum identification capabilities.

The M.S. thesis of Omar Sheikh is approved.

Yunfeng Lu

Eliezer Gafni

Vasilios Manousiouthakis, Committee Chair

University of California, Los Angeles

2018

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1. INTRODUCTION

The globally optimal solution of optimization problems featuring nonconvex objectives and/or constraints is notoriously difficult to identify in an efficient manner. To overcome these fundamental computational limitations, strategies are often pursued which seek to identify the global optimum for classes of nonconvex optimization problems, that are general enough to have real life applications, and lend themselves to the development of efficient solution procedures that can guarantee the identification of the global optimum.

The global optimization of sequential processes gives rise to such special optimization problem classes, having significant applications, and a special structure that can be exploited for their global solution. Bellman [1] put forward Dynamic Programming, as a tailor made methodology for the global optimization of sequential processes. Although useful for certain problem subclasses (linear quadratic optimal control), the method suffers from the curse of dimensionality. In a similar manner, geometric programming was shown (e.g. Wilde, [2]) to be applicable to certain sequential process optimization subclasses featuring posynomial inequality constraints. More recently, Conner and Manousiouthakis presented efficient solution procedures for the global optimization of compressor sequences. The resulting optimization problems featured box constraints, a single nonconvex multinomial equality constraint describable as the product of several first order polynomials in several variables, and either linear [3], or separable power law objective functions [4].

In this paper, we present a novel methodology for the identification of the global optimum for a general class of problems that arise in the optimization of sequential processes. The considered class is a generalization of that presented in [3, 4], as it involves a single nonconvex equality constraint, describable as the product of single variable functions (in different variables), and an objective function that is the sum of single variable functions (in different variables). An efficient global solution methodology is developed, based on the problem's necessary conditions of optimality. A case study is presented to illustrate the proposed solution method, and conclusions are drawn.

2. PROBLEM STATEMENT

Consider the feasible, finite dimensional, infimization problem

$$
\begin{aligned}
v(C) = \inf_{\{x_j\}_1^n \in \mathbb{R}^n} & \left[\sum_{j=1}^{n} f_j(x_j) \right] \\
s.t. \quad & \prod_{j=1}^{n} g_j(x_j) = C > 0 \\
& x_j \in \Omega_j \quad \forall j = 1, n
\end{aligned}
$$

where $\Omega_j \triangleq \left\{ x_j \in \mathbb{R} : -\infty < x_j^l \leq x_j \leq x_j^u < \infty \right\} \ \forall j = 1, n$

and $f_j : \Omega_j \to \mathbb{R} \quad \forall j = 1, n; \ g_j : \Omega_j \to \mathbb{R} \quad \forall j = 1, n$ satisfy the following properties:

Property 1: $f_j : \Omega_j \to \mathbb{R}$ and $g_j : \Omega_j \to \mathbb{R}$ are differentiable functions over $\Omega_j \quad \forall j = 1, n$, with derivatives designated as $\dot{f}_j : \Omega_j \to \mathbb{R}; \ \dot{g}_j : \Omega_j \to \mathbb{R}; \ \forall j = 1, n$

Property 2: $g_j : \Omega_j \to \mathbb{R}$ is such that $g_j(x) \neq 0 \ \forall x \in \Omega_j \quad \forall j = 1, n$

Property 3: $\dot{g}_j : \Omega_j \to \sim$ is such that $\dot{g}_j(x) \neq 0 \ \forall x \in \Omega_j \quad \forall j = 1, n$

Property 4: The sets

$$S_k \triangleq \left\{ x \in \left[x_k^l, x_k^u \right] : \dot{h}_k(x) = 0 \vee \dot{f}_k(x) = 0 \vee \dot{g}_k(x) = 0 \vee x = x_k^l \vee x = x_k^u \right\} \ \forall k = 1, n$$

have finite cardinality, where $h_j : \Omega_j \to \sim;\ h_j : x \to h_j(x) \triangleq \dot{f}_j(x) \dfrac{g_j(x)}{\dot{g}_j(x)} \quad \forall j = 1, n$

The feasibility of the considered optimization problem, and the finite upper and lower bounds on all problem variables, imply that the feasible region is nonempty, and bounded. Further, the problem's feasible region is a closed set, as it is formed through inequality constraints that are defined by differentiable functions and allow for equality. Since the objective function is continuously differentiable throughout the feasible region, Weierstrass' theorem [5, pp. 39-40] then ascertains the existence of the optimization problem's minimum. It is also easy to establish that the above optimization problem's feasible points are all regular [6, p. 342].

Defining the problem's Lagrangian as

$$L(x, \lambda, \mu, \nu) \triangleq \left[\sum_{i=1}^{n} f_i(x_i) + \lambda \cdot \left(\prod_{i=1}^{n} g_i(x_i) - C \right) + \sum_{i=1}^{n} \mu_i \cdot \left(x_i^l - x_i \right) + \sum_{i=1}^{n} \nu_i \cdot \left(x_i - x_i^u \right) \right]$$

then yields the following necessary optimality conditions for the above optimization problem:

$$\left\{\begin{array}{l}
\dfrac{\partial L(x,\lambda,\mu,\nu)}{\partial x_k} \triangleq \dot{f}_k(x_k) + \lambda \cdot \dot{g}_k(x_k) \cdot \prod\limits_{\substack{i=1 \\ i \neq k}}^{n} g_i(x_i) - \mu_k + \nu_k = 0 \;\; \forall k = 1,n \\
\prod\limits_{i=1}^{n} g_i(x_i) - C = 0 \\
\mu_i\left(x_i^l - x_i\right) = 0 \;\; \forall i = 1,n \\
\nu_i\left(x_i - x_i^u\right) = 0 \;\; \forall i = 1,n \\
\mu_i \geq 0 \;\; \forall i = 1,n \\
\nu_i \geq 0 \;\; \forall i = 1,n \\
x_i^l \leq x_i \leq x_i^u \;\; \forall i = 1,n \\
\end{array}\right\}$$

Then, given property 2, the above necessary conditions of optimality become:

$$\left\{\begin{array}{l}
\dot{f}_k(x_k) g_k(x_k) + \lambda C \dot{g}_k(x_k) + (\nu_k - \mu_k) g_k(x_k) = 0 \qquad \forall k = 1,n \\
\prod\limits_{i=1}^{n} g_i(x_i) - C = 0 \\
\mu_i\left(x_i^l - x_i\right) = 0 \;\; \forall i = 1,n \\
\nu_i\left(x_i - x_i^u\right) = 0 \;\; \forall i = 1,n \\
\mu_i \geq 0 \;\; \forall i = 1,n \\
\nu_i \geq 0 \;\; \forall i = 1,n \\
x_i^l \leq x_i \leq x_i^u \;\; \forall i = 1,n \\
\end{array}\right\}$$

Let the following sets be defined:

$$S_{I,k}^{x} \triangleq \left\{k : x_k^l < x < x_k^u\right\},\; S_{L,k}^{x} \triangleq \left\{k : x_k^l = x\right\},\; S_{U,k}^{x} \triangleq \left\{k : x_k^u = x\right\} \;\; \forall k = 1,n \;.$$

Within the feasible region, it holds $x_j \in \Omega_j \triangleq \left\{x_j \in {}^{\sim} : -\infty < x_j^l \leq x_j \leq x_j^u < \infty\right\} \;\; \forall j = 1,n$. This implies that $S_{I,j}^{x_j} \cup S_{L,j}^{x_j} \cup S_{U,j}^{x_j} = \{j\} \;\; \forall\, x_j \in \Omega_j; \;\; \forall j = 1,n$, and

$$S_{I,j}^{x_j} \cap S_{L,j}^{x_j} = \varnothing; \;\; S_{L,j}^{x_j} \cap S_{U,j}^{x_j} = \varnothing; \;\; S_{I,j}^{x_j} \cap S_{U,j}^{x_j} = \varnothing \;\; \forall\, x_j \in \Omega_j; \;\; \forall j = 1,n \,.$$

Using these set definitions, and property 2, the necessary optimality conditions become:

$$
\left\{
\begin{array}{l}
\dot{f}_k\left(x_k\right)+\lambda C\dfrac{\dot{g}_k\left(x_k\right)}{g_k\left(x_k\right)}=0 \quad \forall k\in S_{I,k}^{x_k}=\{k\}\neq\varnothing,\quad k=1,n \\
\mu_k=\dot{f}_k\left(x_k^l\right)+\lambda C\dfrac{\dot{g}_k\left(x_k^l\right)}{g_k\left(x_k^l\right)}\geq 0 \quad \forall k\in S_{L,k}^{x_k}=\{k\}\neq\varnothing,\quad k=1,n \\
\nu_k=-\dot{f}_k\left(x_k^u\right)-\lambda C\dfrac{\dot{g}_k\left(x_k^u\right)}{g_k\left(x_k^u\right)}\geq 0 \quad \forall k\in S_{U,k}^{x_k}=\{k\}\neq\varnothing,\quad k=1,n \\
\prod\limits_{i\in S_{L,i}^{x_i^l}} g_i\left(x_i^l\right)\prod\limits_{k\in S_{I,k}^{x_k}} g_k\left(x_k\right)\prod\limits_{j\in S_{U,j}^{x_j^u}} g_j\left(x_j^u\right)-C=0 \\
x_k^l<x_k<x_k^u \quad \forall k\in S_{I,k}^{x_k},\quad k=1,n \\
x_k=x_k^l \quad \forall k\in S_{L,k}^{x_k},\quad k=1,n \\
x_k=x_k^u \quad \forall k\in S_{U,k}^{x_k},\quad k=1,n
\end{array}
\right\}
\overset{\text{Property 3}}{\Leftrightarrow}
$$

$$
\left\{
\begin{array}{ll}
\dot{f}_o\left(x_o\right)\dfrac{g_o\left(x_o\right)}{\dot{g}_o\left(x_o\right)}=-\lambda C \quad \forall o\in S_{I,o}^{x_o}=\{o\}\neq\varnothing,\quad o=1,n & \\
\mu_k=\dot{f}_k\left(x_k^l\right)-\dot{f}_o\left(x_o\right)\dfrac{g_o\left(x_o\right)}{\dot{g}_o\left(x_o\right)}\dfrac{\dot{g}_k\left(x_k^l\right)}{g_k\left(x_k^l\right)}\geq 0 & \left|\begin{array}{l}\forall o\in S_{I,o}^{x_o}=\{o\}\neq\varnothing,\quad o=1,n\\ \forall k\in S_{L,k}^{x_k^l}=\{k\}\neq\varnothing,\quad k=1,n\end{array}\right. \\
\nu_k=-\dot{f}_k\left(x_k^u\right)+\dot{f}_o\left(x_o\right)\dfrac{g_o\left(x_o\right)}{\dot{g}_o\left(x_o\right)}\dfrac{\dot{g}_k\left(x_k^u\right)}{g_k\left(x_k^u\right)}\geq 0 & \left|\begin{array}{l}\forall o\in S_{I,o}^{x_o}=\{o\}\neq\varnothing,\quad o=1,n\\ \forall k\in S_{U,k}^{x_k^u}=\{k\}\neq\varnothing,\quad k=1,n\end{array}\right. \\
\prod\limits_{i\in S_{L,i}^{x_i^l}} g_i\left(x_i^l\right)\prod\limits_{k\in S_{I,k}^{x_k}} g_k\left(x_k\right)\prod\limits_{j\in S_{U,j}^{x_j^u}} g_j\left(x_j^u\right)-C=0 & \\
x_k^l<x_k<x_k^u \quad \forall k\in S_{I,k}^{x_k},\quad k=1,n & \\
x_k=x_k^l \quad \forall k\in S_{L,k}^{x_k},\quad k=1,n & \\
x_k=x_k^u \quad \forall k\in S_{U,k}^{x_k},\quad k=1,n &
\end{array}
\right\}
$$

Using the aforementioned definitions of the functions

$h_j : \Omega_j \to \sim;\ h_j : x \to h_j(x) \triangleq \dot{f}_j(x)\dfrac{g_j(x)}{\dot{g}_j(x)}$ $\forall j = 1, n$, allows the necessary optimality

conditions to be written as:

$$
\left\{
\begin{array}{l}
h_o(x_o) = -\lambda C \quad \forall o \in S_{I,o}^{x_o} = \{o\} \neq \varnothing, \quad o = 1, n \\
\mu_k = \dot{f}_k(x_k^l) - h_o(x_o)\dfrac{\dot{g}_k(x_k^l)}{g_k(x_k^l)} \geq 0 \quad \left| \begin{array}{l} \forall o \in S_{I,o}^{x_o} = \{o\} \neq \varnothing, \quad o = 1, n \\ \forall k \in S_{L,k}^{x_k^l} = \{k\} \neq \varnothing, \quad k = 1, n \end{array} \right. \\
\nu_k = -\dot{f}_k(x_k^u) + h_o(x_o)\dfrac{\dot{g}_k(x_k^u)}{g_k(x_k^u)} \geq 0 \quad \left| \begin{array}{l} \forall o \in S_{I,o}^{x_o} = \{o\} \neq \varnothing, \quad o = 1, n \\ \forall k \in S_{U,k}^{x_k^u} = \{k\} \neq \varnothing, \quad k = 1, n \end{array} \right. \\
\prod_{i \in S_{L,i}^{x_i^l}} g_i(x_i^l) \prod_{k \in S_{I,k}^{x_k}} g_k(x_k) \prod_{j \in S_{U,j}^{x_j^u}} g_j(x_j^u) - C = 0 \\
x_k^l < x_k < x_k^u \quad \forall k \in S_{I,k}^{x_k}, \quad k = 1, n \\
x_k = x_k^l \quad \forall k \in S_{L,k}^{x_k}, \quad k = 1, n \\
x_k = x_k^u \quad \forall k \in S_{U,k}^{x_k}, \quad k = 1, n
\end{array}
\right\} \Leftrightarrow
$$

$$
\left\{
\begin{array}{l}
h_o\left(x_o\right) = -\lambda C \quad \forall o \in S_{I,o}^{x_o} = \{o\} \neq \varnothing, \quad o = 1,n \\
\mu_k \dfrac{g_k\left(x_k^l\right)}{\dot{g}_k\left(x_k^l\right)} = h_k\left(x_k^l\right) - h_o\left(x_o\right) \left\{
\begin{array}{l}
\geq 0 \; if \; \dfrac{g_k\left(x_k^l\right)}{\dot{g}_k\left(x_k^l\right)} > 0 \\
\leq 0 \; if \; \dfrac{g_k\left(x_k^l\right)}{\dot{g}_k\left(x_k^l\right)} < 0
\end{array}
\right\} \left| \begin{array}{l} \forall o \in S_{I,o}^{x_o} = \{o\} \neq \varnothing, \quad o = 1,n \\ \forall k \in S_{L,k}^{x_k^l} = \{k\} \neq \varnothing, \quad k = 1,n \end{array} \right. \\
\nu_k \dfrac{g_k\left(x_k^u\right)}{\dot{g}_k\left(x_k^u\right)} = -h_k\left(x_k^u\right) + h_o\left(x_o\right) \left\{
\begin{array}{l}
\geq 0 \; if \; \dfrac{g_k\left(x_k^u\right)}{\dot{g}_k\left(x_k^u\right)} > 0 \\
\leq 0 \; if \; \dfrac{g_k\left(x_k^u\right)}{\dot{g}_k\left(x_k^u\right)} < 0
\end{array}
\right\} \left| \begin{array}{l} \forall o \in S_{I,o}^{x_o} = \{o\} \neq \varnothing, \quad o = 1,n \\ \forall k \in S_{U,k}^{x_k^u} = \{k\} \neq \varnothing, \quad k = 1,n \end{array} \right. \\
\prod_{i \in S_{L,i}^{x_i^l}} g_i\left(x_i^l\right) \prod_{k \in S_{I,k}^{x_k}} g_k\left(x_k\right) \prod_{j \in S_{U,j}^{x_j^u}} g_j\left(x_j^u\right) - C = 0 \\
x_k^l < x_k < x_k^u \quad \forall k \in S_{I,k}^{x_k}, \quad k = 1,n \\
x_k = x_k^l \quad \forall k \in S_{L,k}^{x_k}, \quad k = 1,n \\
x_k = x_k^u \quad \forall k \in S_{U,k}^{x_k}, \quad k = 1,n
\end{array}
\right\}
$$

3. SOLUTION ALGORITHM

Based on the above necessary conditions for optimality, the following algorithm is proposed for the globally optimal solution of the considered optimization problem:

The following function values are computed

$$
g_k\left(x_k^l\right), g_k\left(x_k^u\right), \dot{g}_k\left(x_k^l\right), \dot{g}_k\left(x_k^u\right), h_k\left(x_k^l\right), h_k\left(x_k^u\right), \frac{g_k\left(x_k^l\right)}{\dot{g}_k\left(x_k^l\right)}, \frac{g_k\left(x_k^u\right)}{\dot{g}_k\left(x_k^u\right)} \quad \forall k = 1,n
$$

The following sets are identified:

$$G_L^+ \triangleq \left\{ k = 1, n : \frac{g_k\left(x_k^l\right)}{\dot{g}_k\left(x_k^l\right)} > 0 \right\},\ G_L^- \triangleq \left\{ k = 1, n : \frac{g_k\left(x_k^l\right)}{\dot{g}_k\left(x_k^l\right)} < 0 \right\}$$

$$G_U^+ \triangleq \left\{ k = 1, n : \frac{g_k\left(x_k^u\right)}{\dot{g}_k\left(x_k^u\right)} > 0 \right\},\ G_U^- \triangleq \left\{ k = 1, n : \frac{g_k\left(x_k^u\right)}{\dot{g}_k\left(x_k^u\right)} < 0 \right\}$$

The following sets, with finite cardinality $N_k < \infty \ \ \forall k = 1, n$ (as per Property 4),

$$S_k \triangleq \left\{ x \in \left[x_k^l, x_k^u\right] : \left\{ \begin{array}{l} \dot{h}_k(x) = 0 \vee \dot{f}_k(x) = 0 \vee \dot{g}_k(x) = 0 \vee \\ x = x_k^l \vee x = x_k^u \end{array} \right\} \right\} \triangleq \left\{x_{k,p}\right\}_{p=1}^{N_k} \quad \forall k = 1, n$$ are identified,

where $x_{k,p} < x_{k,p+1}$, by using a modified Brent method [7]).

The images of the elements of each S_k set under each corresponding $h_k(\cdot)$ function are incorporated into the set $\bigcup_{k=1,n} H_k$ of points defining intervals on the h axis, where

$$H_k \triangleq \left\{ y \in \sim : y = h_k(x),\ \ x \in S_k \right\}.$$

The interval defined as $sp\left[\bigcup_{k=1,n} H_k\right]$ is uniformly discretized across the h-axis. The resulting sub-intervals, with edges either the elements of $\bigcup_{k=1,n} H_k$ or the elements of the uniform discretization grid, are then denoted as $\left[h_o^l, h_o^u\right] \ \ o = 1, N$.

Let $\left[h_{o,k,p}^l, h_{o,k,p}^u\right] \triangleq \left[h_o^l, h_o^u\right] \cap \left[h_k\left(x_{k,p}\right), h_k\left(x_{k,p+1}\right)\right] \forall o = 1, N; \forall k = 1, n; \forall p = 1, N_k - 1$. For each nonempty $\left[h_{o,k,p}^l, h_{o,k,p}^u\right]$, an inverse function $h_{k,p}^{-1}(\cdot)$ can be defined that maps $\left[h_{o,k,p}^l, h_{o,k,p}^u\right]$ to $\left[x_{o,k,p}^l, x_{o,k,p}^u\right] \triangleq \left[h_{k,p}^{-1}\left(h_{o,k,p}^l\right), h_{k,p}^{-1}\left(h_{o,k,p}^u\right)\right] \subset \left[x_{k,p}, x_{k,p+1}\right]$, so that

$\left[h_k\left(x_{o,k,p}^l\right), h_k\left(x_{o,k,p}^u\right)\right] = \left[h_{o,k,p}^l, h_{o,k,p}^u\right]$. This inverse function $h_{k,p}^{-1}(\cdot)$ exists, since the function $h_k(\cdot)$ is monotonous over $\left[x_{k,p}, x_{k,p+1}\right] \forall k = 1, n; \forall p = 1, N_k - 1$.

The following sets are then identified:

$$\left[h_k\left(x_k^l\right) - h_o^u, h_k\left(x_k^l\right) - h_o^l\right] \cap [0, +\infty) \;\; \forall o = 1, N; \;\; \forall k \in G_L^+$$

$$\left[h_k\left(x_k^l\right) - h_o^u, h_k\left(x_k^l\right) - h_o^l\right] \cap (-\infty, 0] \;\; \forall o = 1, N; \;\; \forall k \in G_L^-$$

$$\left[-h_k\left(x_k^u\right) + h_o^l, -h_k\left(x_k^u\right) + h_o^u\right] \cap [0, +\infty) \;\; \forall o = 1, N; \;\; \forall k \in G_U^+$$

$$\left[-h_k\left(x_k^u\right) + h_o^l, -h_k\left(x_k^u\right) + h_o^u\right] \cap (-\infty, 0] \;\; \forall o = 1, N; \;\; \forall k \in G_U^-$$

For all (o, k) pairs considered in 7 for which some of the intersection sets 7a,b,c,d are empty, the variable x_k cannot equal x_k^l if the sets 7a,b are empty, and cannot equal x_k^u if the sets 7c,d are empty. This implies that combinations which set x_k equal to its lower bound x_k^l (7a,b are empty), or x_k equal to its upper bound x_k^u (7c,d are empty respectively) need not be considered. For all other (o, k) pairs all interior, lower, upper bound combinations must be considered.

Following the screening process outlined in 8, consider an interval $\left[h_o^l, h_o^u\right]$ $o = 1, N$ and a combination for which some x_k variables are interior $\left(x_k^l < x_k < x_k^u\right)$, some are at their lower bound $x_k = x_k^l$, and the remaining are at their upper bound $x_k = x_k^u$. For those x_k variables at their bounds, the functions $f_k(\cdot)$, $g_k(\cdot)$ are evaluated at these bounds. For the remaining x_k that lie in their domain interior, consider all (o, k, p) triplets for which $\left[h_{o,k,p}^l, h_{o,k,p}^u\right]$ is not an empty set. For each interval $\left[h_o^l, h_o^u\right]$ $o = 1, N$, and using all feasible (o, k, p) triplets identified above, all possible feasible combinations $\left[x_{o,k,p}^l, x_{o,k,p}^u\right]$ $\forall k = 1, n; \forall p = 1, N_k - 1$ are considered for a given $\left[h_o^l, h_o^u\right]$ $o = 1, N$. Let one of these combinations be indicated as $\left(o, k, p(k)\right)_{k=1}^{n}$. Then one can evaluate intervals in which the

quantities $\sum_{i=1}^{n} f_i(x_i), \ \prod_{i=1}^{n} g_i(x_i)$ belong. Since all $h_k(\cdot), f_k(\cdot), g_k(\cdot)$ functions are monotonous over the corresponding kth domains $\left[x_{o,k,p(k)}^l, x_{o,k,p(k)}^u\right]$ these intervals are:

$$\left[\sum_{k=1}^{n} \min\left(f_k\left(x_{o,k,p(k)}^l\right), f_k\left(x_{o,k,p(k)}^u\right)\right), \sum_{k=1}^{n} \max\left(f_k\left(x_{o,k,p(k)}^l\right), f_k\left(x_{o,k,p(k)}^u\right)\right)\right], \text{ and}$$

$$\left[\prod_{k=1}^{n} \min\left(g_k\left(x_{o,k,p(k)}^l\right), g_k\left(x_{o,k,p(k)}^u\right)\right), \prod_{k=1}^{n} \max\left(g_k\left(x_{o,k,p(k)}^l\right), g_k\left(x_{o,k,p(k)}^u\right)\right)\right]$$

These intervals are then used to create boxes in a diagram with x-axis the parameter C, and y-axis the candidate optimum objective function ν. These boxes are used as follows to identify lower bound and upper bound envelopes on the global optimum, as a function of C.

4. CASE STUDY

Consider the following minimization problem:

$$\nu = \inf_{\{x_j\}_1^4 \in \mathbb{R}^4} \left[\sum_{j=1}^{4} f_j\left(x_j\right)\right]$$

$$s.t. \ \prod_{j=1}^{4} g_j\left(x_j\right) = C > 0$$

$$x_j \in \Omega_j \quad \forall j = 1, 4$$

Where $\Omega_j \triangleq \{x_j \in \sim : -\infty < x_j^l \le x_j \le x_j^u < \infty\} \ \forall j = 1,4$

$$f_1(x_1) = (\sin(x_1) - 5)^2 + 3 \quad g_1(x_1) = (x_1 + 5)(e^{0.5x_1} + 1)$$
$$f_2(x_2) = 2\sin(x_2^2) \quad g_2(x_2) = (x_2 + 2)(2e^{0.1x_2} + 3)$$
$$f_3(x_3) = 4x_3^3 \quad g_3(x_3) = (2 + 0.1e^{-0.1x_3})(x_3 + 7)$$
$$f_4(x_4) = 4\cos(e^{x_4}) \quad g_4(x_4) = (0.5x_4 + 7)(e^{0.2x_4} + 2)$$
$$x_1^l - x_1 = 0.0 - x_1 \le 0 \ (\mu_1),\ x_2^l - x_2 = 0.0 - x_2 \le 0 \ (\mu_2),\ x_3^l - x_3 = 0.0 - x_3 \le 0 \ (\mu_3),$$
$$x_4^l - x_4 = 0.0 - x_4 \le 0 \ (\mu_4)$$
$$x_1 - x_1^u = x_1 - 3.0 \le 0 \ (\nu_1),\ x_2 - x_2^u = x_2 - 6.0 \le 0 \ (\nu_2),\ x_3 - x_3^u = x_3 - 1.0 \le 0 \ (\nu_3),$$
$$x_4 - x_4^u = x_4 - 3.0 \le 0 \ (\nu_4)$$

The behavior of the f_k, g_k functions is illustrated in the figures 1 and 2 below:

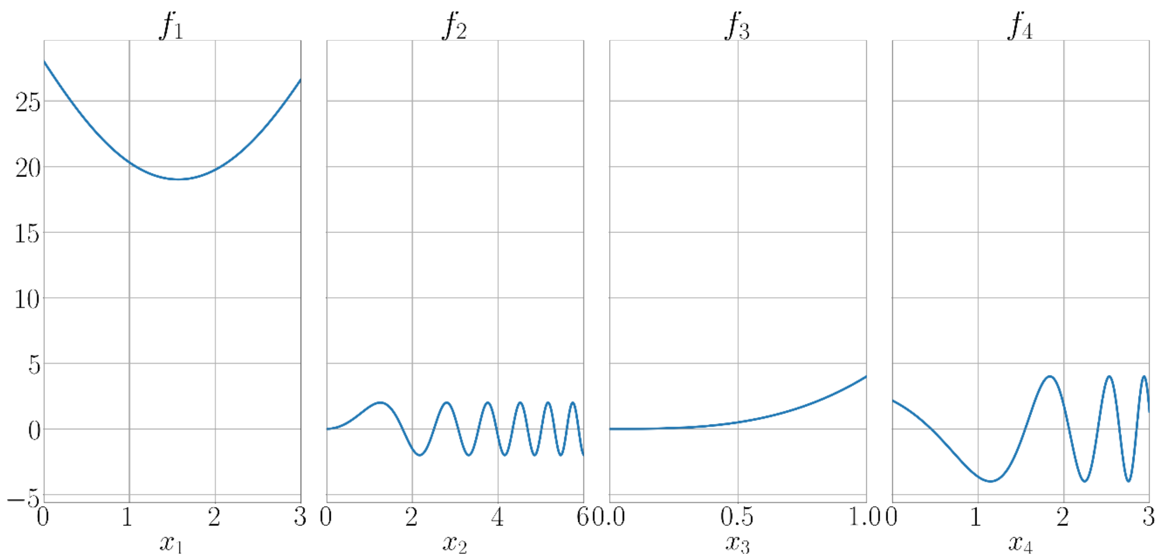


Figure 1

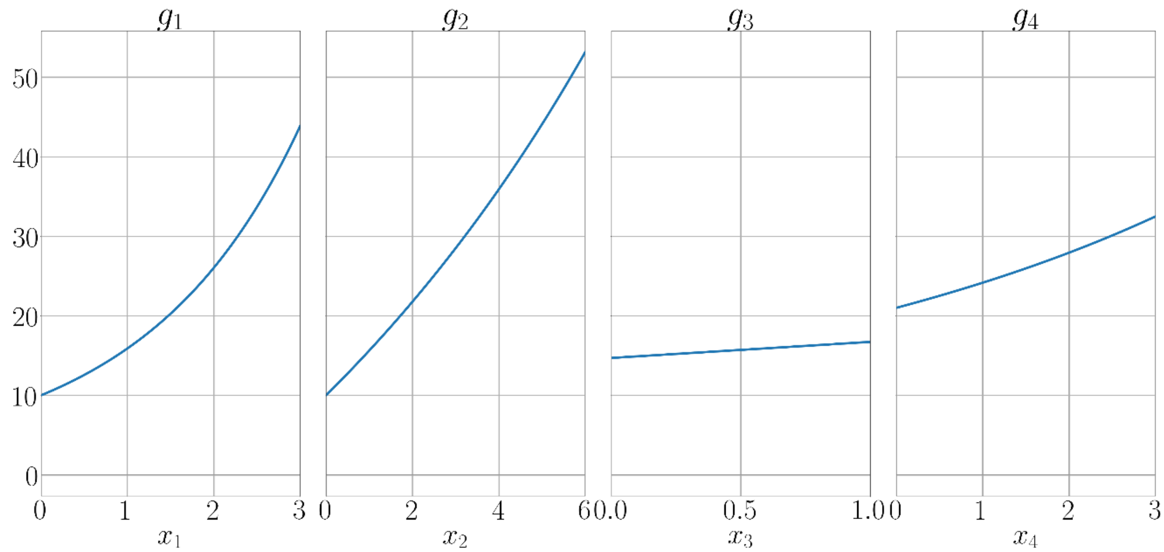


Figure 2

The h_k functions can then be readily evaluated as:

$$h_1(x_1) = \frac{2\cos(x_1)}{\frac{e^{\frac{x_1}{2}}}{2}(x_1+5) + e^{\frac{x_1}{2}} + 1}(x_1+5)\left(e^{\frac{x_1}{2}}+1\right)(\sin(x_1)-5)$$

$$h_2(x_2) = \frac{4x_2(x_2+2)\left(2e^{0.1x_2}+3\right)\cos\left(x_2^2\right)}{0.2(x_2+2)e^{0.1x_2}+2e^{0.1x_2}+3}$$

$$h_3(x_3) = \frac{12x_3^2\left(2+0.1e^{-0.1x_3}\right)(x_3+7)}{-0.01(x_3+7)e^{-0.1x_3}+2+0.1e^{-0.1x_3}}$$

$$h_4(x_4) = -\frac{4(0.5x_4+7)\left(e^{0.2x_4}+2\right)e^{x_4}\sin\left(e^{x_4}\right)}{0.2(0.5x_4+7)e^{0.2x_4}+0.5e^{0.2x_4}+1.0}$$

As shown in Figure 3 below, the h_k functions possess many critical points, all of which are identified, using the modified Brent algorithm [7], thus enabling the evaluation of the finite cardinality N_k sets $S_k \; \forall k = 1,4$ as:

$S_1 = \{0.0, 1.57, 3.0\}$

$$S_2 = \begin{cases} 0.0, 0.85, \; 1.25, \; 1.83, \; 2.17, \; 2.53, \; 2.8, \; 3.08, \; 3.32, \; 3.55, \; 3.76, \; 3.97, \; 4.16, \; 4.35, \; 4.52, \\ 4.69, \; 4.85, \; 5.02, \; 5.17, \; 5.32, \; 5.46, \; 5.61, \; 5.74, \; 5.88, 6.0 \end{cases}$$

$S_3 = \{0.0, 1.0\}$

$S_4 = \{0.0, 0.7, \; 1.14, \; 1.59, \; 1.84, \; 2.08, \; 2.24, \; 2.41, \; 2.53, \; 2.65, \; 2.75, \; 2.85, \; 2.94, 3.0\}$

$N_1 = 3, N_2 = 25, N_3 = 2, N_4 = 14$

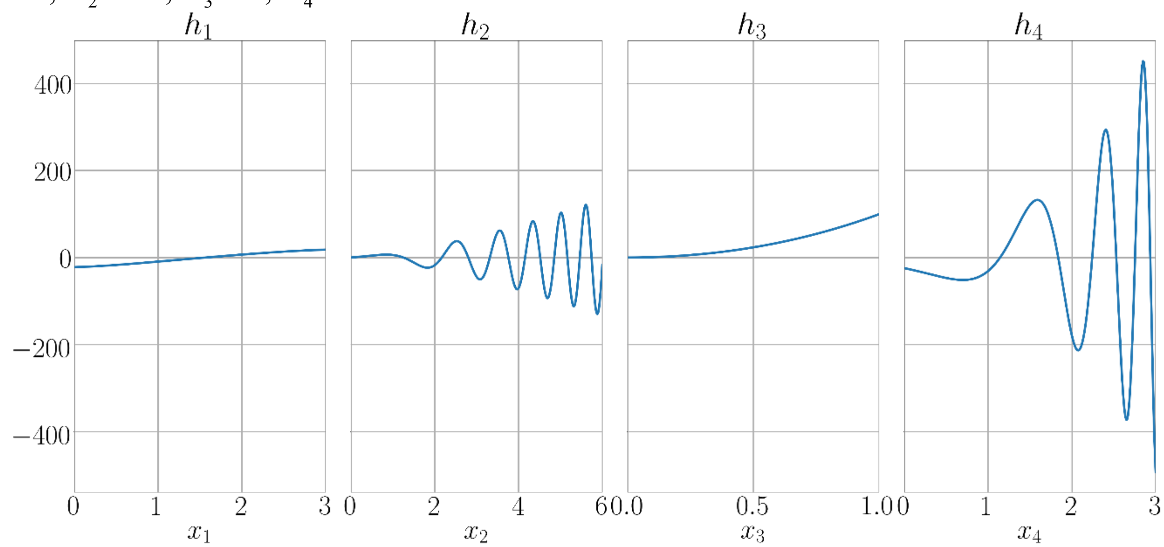


Figure 3

Then the values of the $g_k, \dot{g}_k, h_k, \frac{g_k}{\dot{g}_k}$ functions are evaluated at the corresponding interval edges:

$$g_1\left(x_1^l = 0.0\right) = 10.0,\; g_2\left(x_2^l = 0.0\right) = 10.0,\; g_3\left(x_3^l = 0.0\right) = 14.7,\; g_4\left(x_4^l = 0.0\right) = 21.0$$

$$g_1\left(x_1^u = 3.0\right) = 43.85,\; g_2\left(x_2^u = 6.0\right) = 53.15,\; g_3\left(x_3^u = 1.0\right) = 16.72,\; g_4\left(x_4^u = 3.0\right) = 32.49$$

$$\dot{g}_1\left(x_1^l = 0.0\right) = 4.5,\; \dot{g}_2\left(x_2^l = 0.0\right) = 5.4,\; \dot{g}_3\left(x_3^l = 0.0\right) = 2.03,\; \dot{g}_4\left(x_4^l = 0.0\right) = 2.9$$

$$\dot{g}_1\left(x_1^u = 3.0\right) = 23.41,\; \dot{g}_2\left(x_2^u = 6.0\right) = 9.56,\; \dot{g}_3\left(x_3^u = 1.0\right) = 2.02,\; \dot{g}_4\left(x_4^u = 3.0\right) = 5.01$$

$$h_1\left(x_1^l = 0.0\right) = -22.22,\; h_2\left(x_2^l = 0.0\right) = 0.0,\; h_3\left(x_3^l = 0.0\right) = 0.0,\; h_4\left(x_4^l = 0.0\right) = -24.37$$

$$h_1\left(x_1^u = 3.0\right) = 18.02,\; h_2\left(x_2^u = 6.0\right) = -17.08,\; h_3\left(x_3^u = 1.0\right) = 99.44,\; h_4\left(x_4^u = 3.0\right) = -492.19$$

$$\frac{g_1\left(x_1^l = 0.0\right)}{\dot{g}_1\left(x_1^l = 0.0\right)} = 2.22,\; \frac{g_2\left(x_2^l = 0.0\right)}{\dot{g}_2\left(x_2^l = 0.0\right)} = 1.85,\; \frac{g_3\left(x_3^l = 0.0\right)}{\dot{g}_3\left(x_3^l = 0.0\right)} = 7.24,\; \frac{g_4\left(x_4^l = 0.0\right)}{\dot{g}_4\left(x_4^l = 0.0\right)} = 7.24$$

$$\frac{g_1\left(x_1^u = 3.0\right)}{\dot{g}_1\left(x_1^u = 3.0\right)} = 1.87,\; \frac{g_2\left(x_2^u = 6.0\right)}{\dot{g}_2\left(x_2^u = 6.0\right)} = 5.56,\; \frac{g_3\left(x_3^u = 1.0\right)}{\dot{g}_3\left(x_3^u = 1.0\right)} = 8.29,\; \frac{g_4\left(x_4^u = 3.0\right)}{\dot{g}_4\left(x_4^u = 3.0\right)} = 6.49$$

The following sets are then identified:

$$G_L^+ \triangleq \{1,2,3,4\},\; G_L^- \triangleq \varnothing,\; G_U^+ \triangleq \{1,2,3,4\},\; G_U^- \triangleq \varnothing$$

$$H_k \triangleq \{ y \in \sim : y = h_k(x),\ x \in S_k \}$$

$$H_1 = \{-22.22, -5.7E-12, 18.02\}$$

$$H_2 = \begin{Bmatrix} 0.0, 6.49,\ -1.45E-09,\ -23.44,\ -1.62E-08,\ 37.65,\ -4.70E-08,\ -50.32, \\ -2.56E-08,\ 62.01,\ -9.22E-08,\ -72.98,\ -2.58E-09,\ 83.39,\ 9.21E-10, \\ -93.34,\ 1.44E-08,\ 102.89,\ 4.51E-08,\ -112.09,\ -7.66E-09,\ 120.99,\ 2.32E-07, \\ -129.6, -17.08 \end{Bmatrix}$$

$$H_3 = \{0.0, 99.44\}$$

$$H_4 = \begin{Bmatrix} -24.37, -51.6,\ -7.06E-09,\ 132.26,\ 1.58E-07,\ -213.39,\ 1.98E-07,\ 293.65, \\ 6.07E-12,\ -373.09,\ 1.26E-06,\ 451.85,\ -6.83E-07, -492.19 \end{Bmatrix}$$

$$sp\left[\bigcup_{k=1,n} H_k \right] = [-492.19, 451.85]$$

$$sp[H_1] = [-22.22, 18.02], sp[H_2] = [-129.60, 120.99], sp[H_3] = [0.00, 99.44],$$
$$sp[H_4] = [-492.19, 451.85]$$

A uniform discretization grid of $span\left[\bigcup_{k=1,n} H_k \right]$ with three interior grid points, then yields the following $N = 45$ subintervals $\left[h_o^l, h_o^u \right]$ $o = 1,45$:

$\left[-492.19,-373.09\right],\left[-373.09,-256.18\right],\left[-256.18,-213.39\right],\left[-213.39,-129.6\right],$
$\left[-129.6,-112.09\right],\left[-112.09,-93.339\right],\left[-93.339,-72.98\right],\left[-72.98,-51.595\right],$
$\left[-51.595,-50.316\right],\left[-50.316,-24.374\right],\left[-24.374,-23.443\right],\left[-23.443,-22.222\right],$
$\left[-22.222,-20.17\right],\left[-20.17,-17.076\right],\left[-17.076,-6.8273E-07\right],$
$\left[-6.8273E-07,-9.219E-08\right],\left[-9.219E-08,-4.7006E-08\right],$
$\left[-4.7006E-08,-2.5594E-08\right],\left[-2.5594E-08,-1.6171E-08\right],$
$\left[-1.6171E-08,-7.6625E-09\right],\left[-7.6625E-09,-7.0556E-09\right],$
$\left[-7.0556E-09,-2.5833E-09\right],\left[-2.5833E-09,-1.4481E-09\right],$
$\left[-1.4481E-09,-5.6963E-12\right],\left[-5.6963E-12,0\right],\left[0,6.0693E-12\right],$
$\left[6.0693E-12,9.2098E-10\right],\left[9.2098E-10,1.4415E-08\right],\left[1.4415E-08,4.5093E-08\right],$
$\left[4.5093E-08,1.5777E-07\right],\left[1.5777E-07,1.9763E-07\right],\left[1.9763E-07,2.3203E-07\right],$
$\left[2.3203E-07,1.2639E-06\right],\left[1.2639E-06,6.4911\right],\left[6.4911,18.023\right],\left[18.023,37.649\right],$
$\left[37.649,62.005\right],\left[62.005,83.393\right],\left[83.393,99.443\right],\left[99.443,102.89\right],\left[102.89,120.99\right],$
$\left[120.99,132.26\right],\left[132.26,215.84\right],\left[215.84,293.65\right],\left[293.65,451.85\right]$

Since $G_L^- = \varnothing, G_U^- = \varnothing$ sets 7b, 7d need not be considered, while 7a, 7c must be considered $\forall k = 1,4$. For algorithm illustration purposes, the subinterval $\left[h_{39}^l, h_{39}^u\right] = \left[83.393, 99.443\right]$ is then considered, and the following sets are identified below, and illustrated in Figure 4:

a.

$$\left[h_1\left(x_1^l\right)-h_{39}^u, h_1\left(x_1^l\right)-h_{39}^l\right]\cap\left[0,+\infty\right)=\left[-22.22-99.443,-22.22-83.393\right]\cap\left[0,+\infty\right)=\varnothing$$

$$\left[h_2\left(x_2^l\right)-h_{39}^u, h_2\left(x_2^l\right)-h_{39}^l\right]\cap\left[0,+\infty\right)=\left[0.0-99.443,0.0-83.393\right]\cap\left[0,+\infty\right)=\varnothing$$

$$\left[h_3\left(x_3^l\right)-h_{39}^u, h_3\left(x_3^l\right)-h_{39}^l\right]\cap\left[0,+\infty\right)=\left[0.0-99.443,0.0-83.393\right]\cap\left[0,+\infty\right)=\varnothing$$

$$\left[h_4\left(x_4^l\right)-h_{39}^u, h_4\left(x_4^l\right)-h_{39}^l\right]\cap\left[0,+\infty\right)=\left[-24.37-99.443,-24.37-83.393\right]\cap\left[0,+\infty\right)=\varnothing$$

c.

$$\left\{\left[-h_1\left(x_1^u\right)+h_{39}^l,-h_1\left(x_1^u\right)+h_{39}^u\right]\cap[0,+\infty)= \atop [-18.02+83.393,-18.02+99.443]\cap[0,+\infty)=[65.373,\,81.423]\right\}$$

$$\left\{\left[-h_2\left(x_2^u\right)+h_{39}^l,-h_2\left(x_2^u\right)+h_{39}^u\right]\cap[0,+\infty)= \atop [17.08+83.393,17.08+99.443]\cap[0,+\infty)=[100.473,\,116.523]\right\}$$

$$\left\{\left[-h_3\left(x_3^u\right)+h_{39}^l,-h_3\left(x_3^u\right)+h_{39}^u\right]\cap[0,+\infty)= \atop [-99.44+83.393,-99.44+99.443]\cap[0,+\infty)=[0,\,0]\right\}$$

$$\left\{\left[-h_4\left(x_4^u\right)+h_{39}^l,-h_4\left(x_4^u\right)+h_{39}^u\right]\cap[0,+\infty)= \atop [492.19+83.393,492.19+99.443]\cap[0,+\infty)=[575.583,\,591.633]\right\}$$

$$\left[h_{39,1,1}^l,h_{39,1,1}^u\right]=\left[h_{39,1,2}^l,h_{39,1,2}^u\right]=\varnothing$$

$$\left[h_{39,2,1-16}^l,h_{39,2,1-16}^u\right]=\left[h_{39,2,19-20}^l,h_{39,2,19-20}^u\right]=\left[h_{39,2,23-24}^l,h_{39,2,23-24}^u\right]=\varnothing$$

$$\left[h_{39,4,1-2}^l,h_{39,4,1-2}^u\right]=\left[h_{39,4,5-6}^l,h_{39,4,5-6}^u\right]=\left[h_{39,4,9-10}^l,h_{39,4,9-10}^u\right]=\left[h_{39,4,13}^l,h_{39,4,13}^u\right]=\varnothing$$

$$\left[h_{39,2,17}^l,h_{39,2,17}^u\right]=\left[h_{39,2,18}^l,h_{39,2,18}^u\right]=\left[h_{39,2,21}^l,h_{39,2,21}^u\right]=\left[h_{39,2,22}^l,h_{39,2,22}^u\right]=[83.393,99.443],$$

$$\left[x_{39,2,17}^l=4.953043,x_{39,2,17}^u=4.99017\right],\left[x_{39,2,18}^l=5.077702,\,x_{39,2,18}^u=5.041858\right],$$

$$\left[x_{39,2,21}^l=5.533877,\,x_{39,2,21}^u=5.552516\right],\left[x_{39,2,22}^l=5.678342,\,x_{39,2,22}^u=5.660538\right]$$

$$\left[h_{39,3,1}^l=83.393,h_{39,3,1}^u=99.443\right],\left[x_{39,3,1}^l=0.920378,\,x_{39,3,1}^u=1.0\right],$$

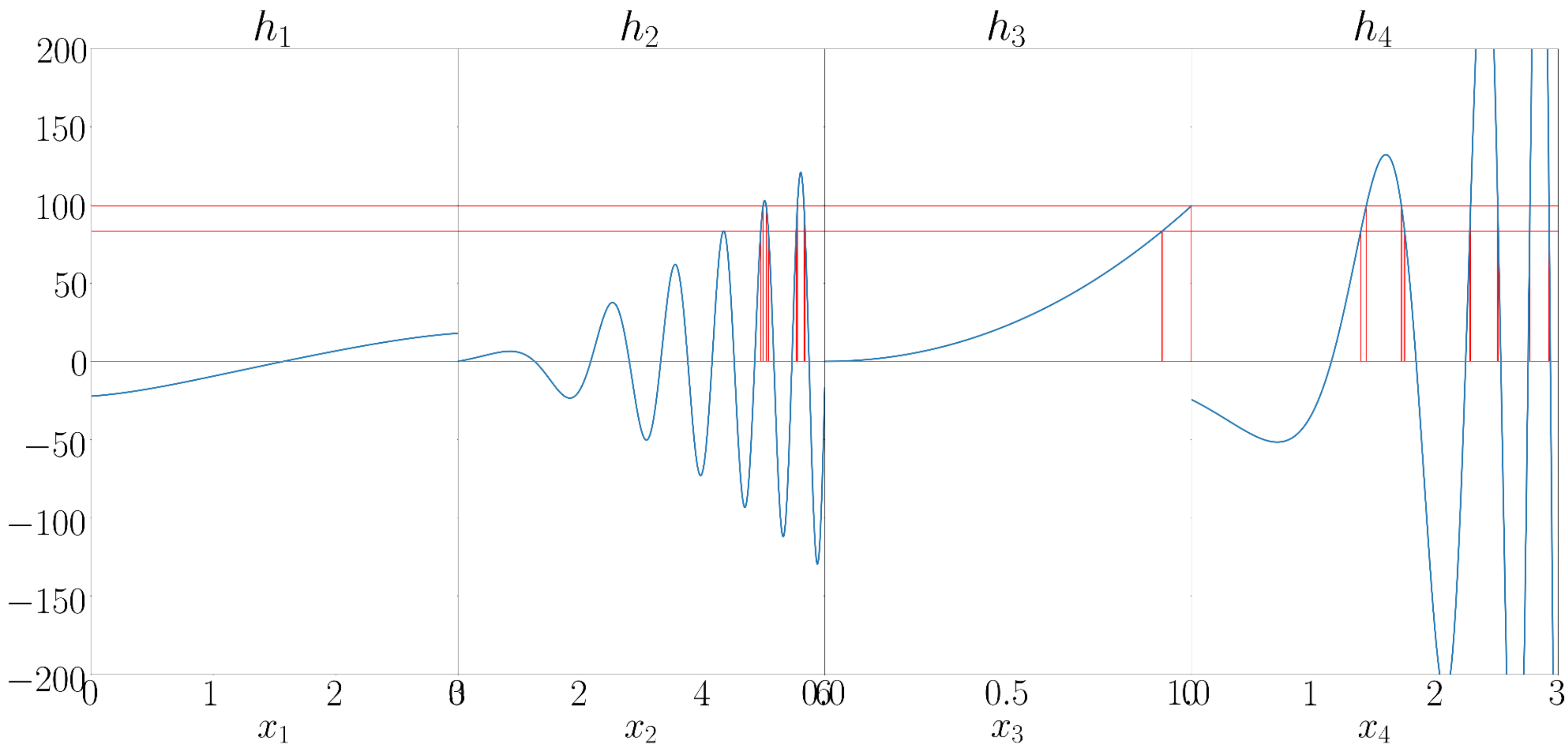
$$\left[h_{39,4,3}^l,h_{39,4,3}^u\right]=\left[h_{39,4,4}^l,h_{39,4,4}^u\right]=\left[h_{39,4,7}^l,h_{39,4,7}^u\right]=\left[h_{39,4,8}^l,h_{39,4,8}^u\right]=$$

$$=\left[h_{39,4,11}^l,h_{39,4,11}^u\right]=\left[h_{39,4,12}^l,h_{39,4,12}^u\right]=[83.393,99.443]$$

$$\left[x_{39,4,3}^l=1.385189,\,x_{39,4,3}^u=1.430038\right],\left[x_{39,4,4}^l=1.743979,\,x_{39,4,4}^u=1.717725\right],$$

$$\left[x_{39,4,7}^l=2.277307,\,x_{39,4,7}^u=2.283769\right],\left[x_{39,4,8}^l=2.51022,\,x_{39,4,8}^u==2.505935\right],$$

$$\left[x_{39,4,11}^l=2.766909,\,x_{39,4,11}^u=2.769351\right],\left[x_{39,4,12}^l=2.9273,\,x_{39,4,12}^u=2.925479\right],$$



h_1
h_2
h_3
h_4
200
150
100
50
0
−50
−100
−150
−200
0
1
2
3
0
2
4
6
0.0
0.5
1.0
0
1
2
3
x_1
x_2
x_3
x_4

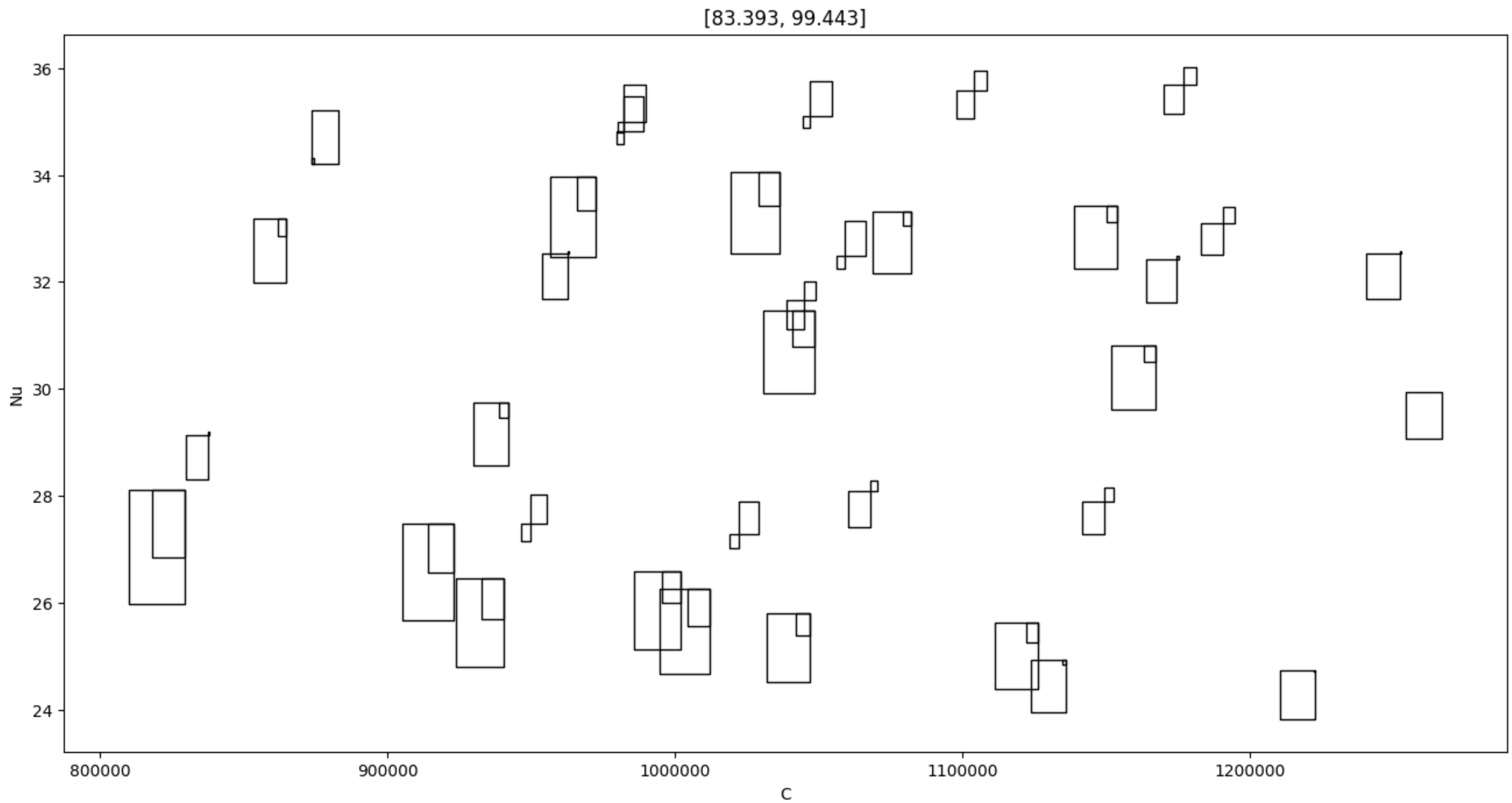
Figure 4

Next, all feasible lower bound, interval interior, upper bound, x_j, $j = 1,4$ variable combinations corresponding to $\left[h_{39}^l, h_{39}^u\right] = [83.393, 99.443]$ are considered, and the corresponding objective function $\sum_{j=1}^{4} f_j\left(x_j\right)$, and parameter $C = \prod_{j=1}^{4} g_j\left(x_j\right)$ intervals are shown in the Table below and illustrated in Figures 5-9 as boxes with C as the x-axis coordinate.

Combinations															
x_1	x_2	x_3	x_4	x_1^l	x_1^u	x_2^l	x_2^u	x_3^l	x_3^u	x_4^l	x_4^u	OF^l	OF^u	C^l	C^u
N_D	N_I	N_I	N_I	3	3	4.953043	4.99017	0.920378	1	1.385189	1.430038	25.96992	28.11663	810080.6	829267.6
N_D	N_I	N_I	N_I	3	3	4.953043	4.99017	0.920378	1	1.743979	1.717725	31.98021	33.18087	853483.3	864720.5
N_D	N_I	N_I	N_I	3	3	4.953043	4.99017	0.920378	1	2.277307	2.283769	24.80803	26.4495	923590.2	940262.4
N_D	N_I	N_I	N_I	3	3	4.953043	4.99017	0.920378	1	2.51022	2.505935	32.46471	33.9586	956499.5	972195.3
N_D	N_I	N_I	N_I	3	3	4.953043	4.99017	0.920378	1	2.766909	2.769351	24.67871	26.26587	994518.3	1011869
N_D	N_I	N_I	N_I	3	3	4.953043	4.99017	0.920378	1	2.9273	2.925479	32.53854	34.06603	1019255	1036361
N_D	N_I	N_I	N_I	3	3	5.077702	5.041858	0.920378	1	1.385189	1.430038	28.31027	29.1417	830008.4	837665.2
N_D	N_I	N_I	N_I	3	3	5.077702	5.041858	0.920378	1	1.743979	1.717725	34.32055	34.20594	874478.8	873477.1
N_D	N_I	N_I	N_I	3	3	5.077702	5.041858	0.920378	1	2.277307	2.283769	27.14837	27.47457	946310.3	949784
N_D	N_I	N_I	N_I	3	3	5.077702	5.041858	0.920378	1	2.51022	2.505935	34.80506	34.98367	980029.2	982040.2
N_D	N_I	N_I	N_I	3	3	5.077702	5.041858	0.920378	1	2.766909	2.769351	27.01906	27.29094	1018983	1022116
N_D	N_I	N_I	N_I	3	3	5.077702	5.041858	0.920378	1	2.9273	2.925479	34.87889	35.0911	1044328	1046856
N_D	N_I	N_I	N_I	3	3	5.533877	5.552516	0.920378	1	1.385189	1.430038	25.67569	27.46919	905176.6	923106.2
N_D	N_I	N_I	N_I	3	3	5.533877	5.552516	0.920378	1	1.743979	1.717725	31.68597	32.53343	953674.3	962570.9
N_D	N_I	N_I	N_I	3	3	5.533877	5.552516	0.920378	1	2.277307	2.283769	24.51379	25.80207	1032011	1046661
N_D	N_I	N_I	N_I	3	3	5.533877	5.552516	0.920378	1	2.51022	2.505935	32.17048	33.31117	1068784	1082207
N_D	N_I	N_I	N_I	3	3	5.533877	5.552516	0.920378	1	2.766909	2.769351	24.38448	25.61844	1111266	1126371
N_D	N_I	N_I	N_I	3	3	5.533877	5.552516	0.920378	1	2.9273	2.925479	32.24431	33.4186	1138906	1153634
N_D	N_I	N_I	N_I	3	3	5.678342	5.660538	0.920378	1	1.385189	1.430038	28.57206	29.74585	929741.2	941773.8
N_D	N_I	N_I	N_I	3	3	5.678342	5.660538	0.920378	1	1.743979	1.717725	34.58234	34.81009	979555.1	982036.5
N_D	N_I	N_I	N_I	3	3	5.678342	5.660538	0.920378	1	2.277307	2.283769	27.41017	28.07873	1060018	1067827
N_D	N_I	N_I	N_I	3	3	5.678342	5.660538	0.920378	1	2.51022	2.505935	35.06685	35.58782	1097788	1104092
N_D	N_I	N_I	N_I	3	3	5.678342	5.660538	0.920378	1	2.766909	2.769351	27.28085	27.89509	1141423	1149149
N_D	N_I	N_I	N_I	3	3	5.678342	5.660538	0.920378	1	2.9273	2.925479	35.14068	35.69525	1169814	1176963
N_D	N_I	N_I	N_D	3	3	4.953043	4.99017	0.920378	1	3	3	29.91223	31.46531	1030725	1048317
N_D	N_I	N_I	N_D	3	3	5.077702	5.041858	0.920378	1	3	3	32.25257	32.49038	1056081	1058933
N_D	N_I	N_I	N_D	3	3	5.533877	5.552516	0.920378	1	3	3	29.61799	30.81788	1151723	1166943
N_D	N_I	N_I	N_D	3	3	5.678342	5.660538	0.920378	1	3	3	32.51437	33.09453	1182978	1190542
N_D	N_I	N_D	N_I	3	3	4.953043	4.99017	1	1	1.385189	1.430038	26.85133	28.11663	817941.1	829267.6

N_D	N_I	N_D	N_I	3	3	4.953043	4.99017	1	1	1.743979	1.717725	32.86161	33.18087	861765	864720.5
N_D	N_I	N_D	N_I	3	3	4.953043	4.99017	1	1	2.277307	2.283769	25.68943	26.4495	932552.1	940262.4
N_D	N_I	N_D	N_I	3	3	4.953043	4.99017	1	1	2.51022	2.505935	33.34612	33.9586	965780.8	972195.3
N_D	N_I	N_D	N_I	3	3	4.953043	4.99017	1	1	2.766909	2.769351	25.56012	26.26587	1004168	1011869
N_D	N_I	N_D	N_I	3	3	4.953043	4.99017	1	1	2.9273	2.925479	33.41995	34.06603	1029145	1036361
N_D	N_I	N_D	N_I	3	3	5.077702	5.041858	1	1	1.385189	1.430038	29.19167	29.1417	838062.3	837665.2
N_D	N_I	N_D	N_I	3	3	5.077702	5.041858	1	1	1.743979	1.717725	35.20195	34.20594	882964.2	873477.1
N_D	N_I	N_D	N_I	3	3	5.077702	5.041858	1	1	2.277307	2.283769	28.02977	27.47457	955492.7	949784
N_D	N_I	N_D	N_I	3	3	5.077702	5.041858	1	1	2.51022	2.505935	35.68646	34.98367	989538.8	982040.2
N_D	N_I	N_D	N_I	3	3	5.077702	5.041858	1	1	2.766909	2.769351	27.90046	27.29094	1028871	1022116
N_D	N_I	N_D	N_I	3	3	5.077702	5.041858	1	1	2.9273	2.925479	35.76029	35.0911	1054462	1046856
N_D	N_I	N_D	N_I	3	3	5.533877	5.552516	1	1	1.385189	1.430038	26.55709	27.46919	913959.8	923106.2
N_D	N_I	N_D	N_I	3	3	5.533877	5.552516	1	1	1.743979	1.717725	32.56737	32.53343	962928.2	962570.9
N_D	N_I	N_D	N_I	3	3	5.533877	5.552516	1	1	2.277307	2.283769	25.39519	25.80207	1042025	1046661
N_D	N_I	N_D	N_I	3	3	5.533877	5.552516	1	1	2.51022	2.505935	33.05188	33.31117	1079155	1082207
N_D	N_I	N_D	N_I	3	3	5.533877	5.552516	1	1	2.766909	2.769351	25.26588	25.61844	1122049	1126371
N_D	N_I	N_D	N_I	3	3	5.533877	5.552516	1	1	2.9273	2.925479	33.12571	33.4186	1149957	1153634
N_D	N_I	N_D	N_I	3	3	5.678342	5.660538	1	1	1.385189	1.430038	29.45346	29.74585	938762.8	941773.8
N_D	N_I	N_D	N_I	3	3	5.678342	5.660538	1	1	1.743979	1.717725	35.46375	34.81009	989060.1	982036.5
N_D	N_I	N_D	N_I	3	3	5.678342	5.660538	1	1	2.277307	2.283769	28.29157	28.07873	1070304	1067827
N_D	N_I	N_D	N_I	3	3	5.678342	5.660538	1	1	2.51022	2.505935	35.94825	35.58782	1108441	1104092
N_D	N_I	N_D	N_I	3	3	5.678342	5.660538	1	1	2.766909	2.769351	28.16225	27.89509	1152499	1149149
N_D	N_I	N_D	N_I	3	3	5.678342	5.660538	1	1	2.9273	2.925479	36.02209	35.69525	1181165	1176963
N_D	N_I	N_D	N_D	3	3	4.953043	4.99017	1	1	3	3	30.79363	31.46531	1040727	1048317
N_D	N_I	N_D	N_D	3	3	5.077702	5.041858	1	1	3	3	33.13397	32.49038	1066329	1058933
N_D	N_I	N_D	N_D	3	3	5.533877	5.552516	1	1	3	3	30.4994	30.81788	1162899	1166943
N_D	N_I	N_D	N_D	3	3	5.678342	5.660538	1	1	3	3	33.39577	33.09453	1194457	1190542
N_D	N_D	N_I	N_I	3	3	6	6	0.920378	1	1.385189	1.430038	25.11583	26.59085	985804.7	1001847
N_D	N_D	N_I	N_I	3	3	6	6	0.920378	1	1.743979	1.717725	31.12611	31.65509	1038622	1044678
N_D	N_D	N_I	N_I	3	3	6	6	0.920378	1	2.277307	2.283769	23.95393	24.92373	1123937	1135941
N_D	N_D	N_I	N_I	3	3	6	6	0.920378	1	2.51022	2.505935	31.61062	32.43283	1163985	1174519
N_D	N_D	N_I	N_I	3	3	6	6	0.920378	1	2.766909	2.769351	23.82462	24.7401	1210251	1222450

N_D	N_D	N_I	N_I	3	3	6	6	0.920378	1	2.9273	2.925479	31.68445	32.54025	1240353	1252039
N_D	N_D	N_I	N_D	3	3	6	6	0.920378	1	3	3	29.05813	29.93954	1254312	1266483
N_D	N_D	N_D	N_I	3	3	6	6	1	1	1.385189	1.430038	25.99723	26.59085	995370.3	1001847
N_D	N_D	N_D	N_I	3	3	6	6	1	1	1.743979	1.717725	32.00751	31.65509	1048701	1044678
N_D	N_D	N_D	N_I	3	3	6	6	1	1	2.277307	2.283769	24.83533	24.92373	1134843	1135941
N_D	N_D	N_D	N_I	3	3	6	6	1	1	2.51022	2.505935	32.49202	32.43283	1175280	1174519
N_D	N_D	N_D	N_I	3	3	6	6	1	1	2.766909	2.769351	24.70602	24.7401	1221994	1222450
N_D	N_D	N_D	N_I	3	3	6	6	1	1	2.9273	2.925479	32.56585	32.54025	1252389	1252039



[83.393, 99.443]
36
34
32
30
28
26
24
Nu
800000
900000
1000000
1100000
1200000
C

Figure 5

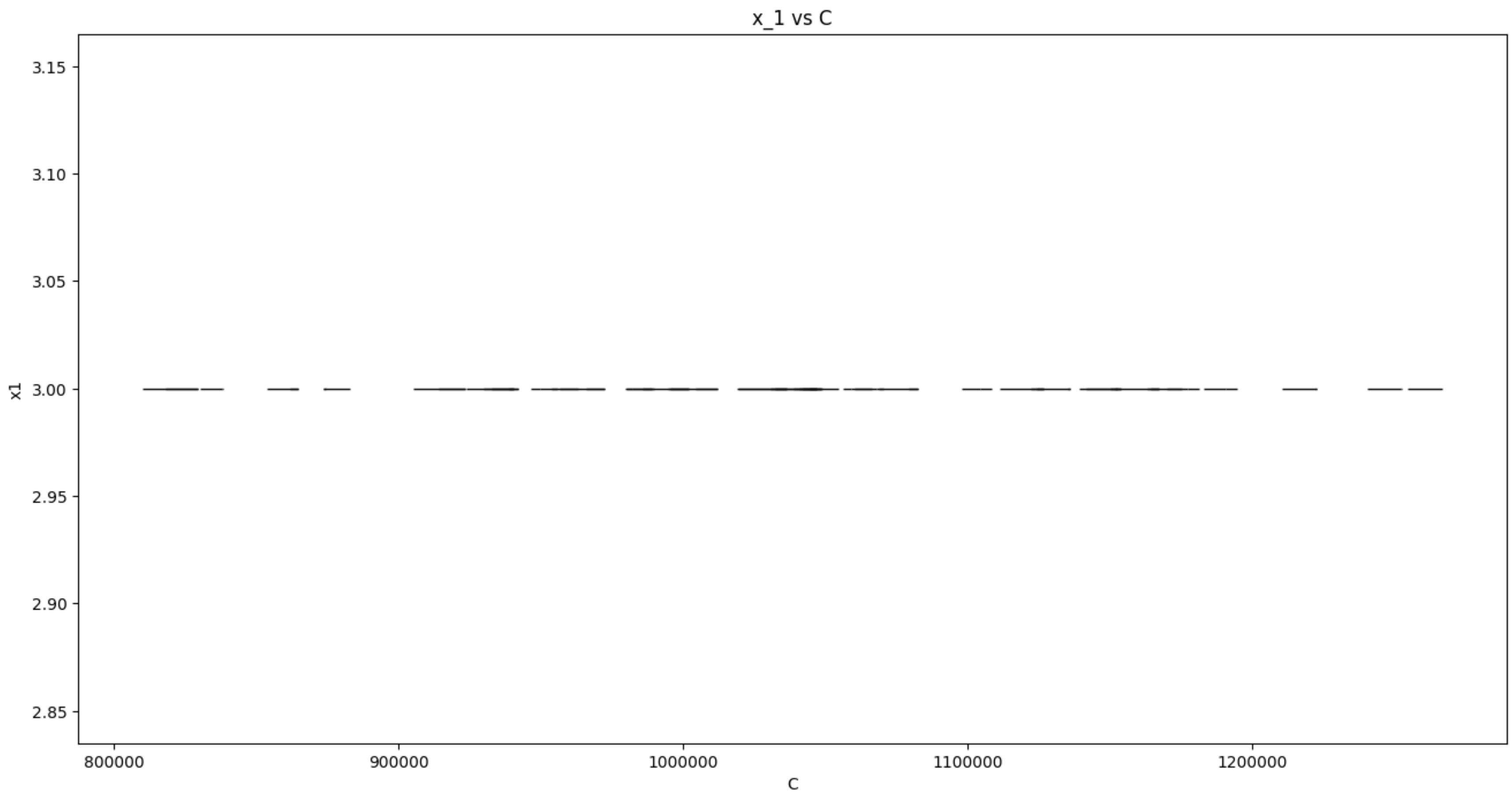

x_1 vs C
3.15
3.10
3.05
3.00
2.95
2.90
2.85
x1
800000
900000
1000000
1100000
1200000
C

Figure 6

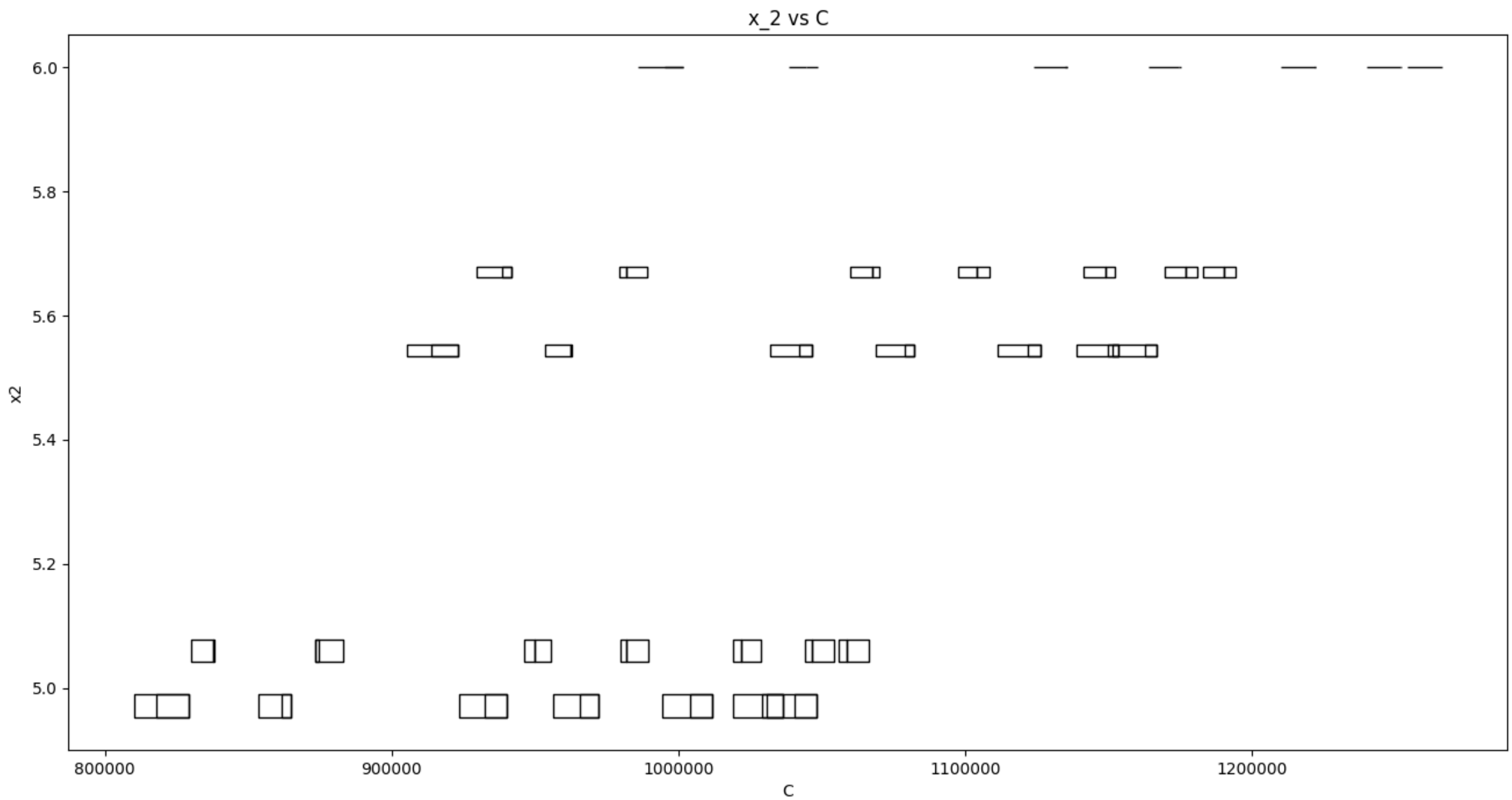

x_2 vs C
6.0
5.8
5.6
5.4
5.2
5.0
x2
800000
900000
1000000
1100000
1200000
C

Figure 7

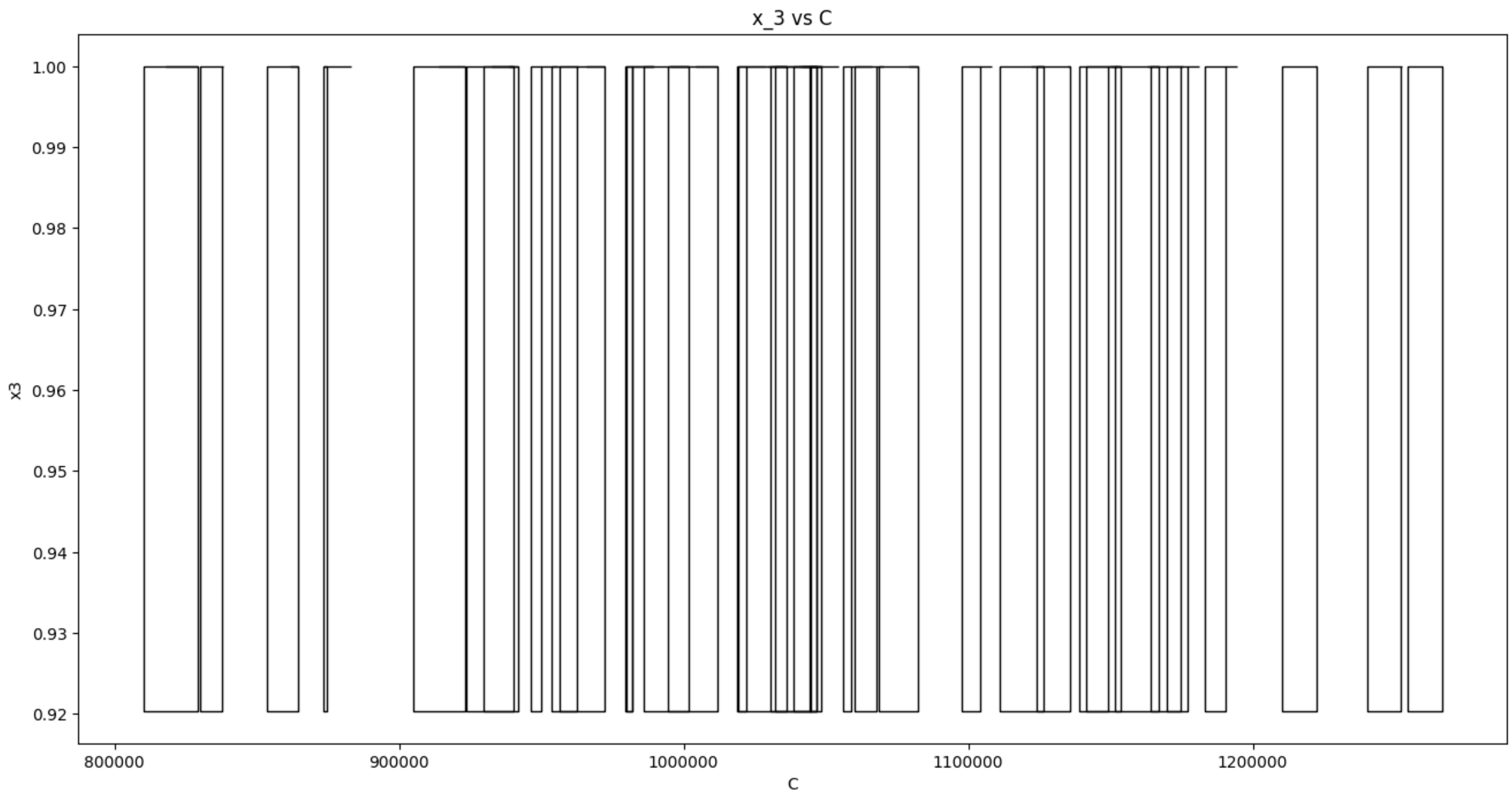
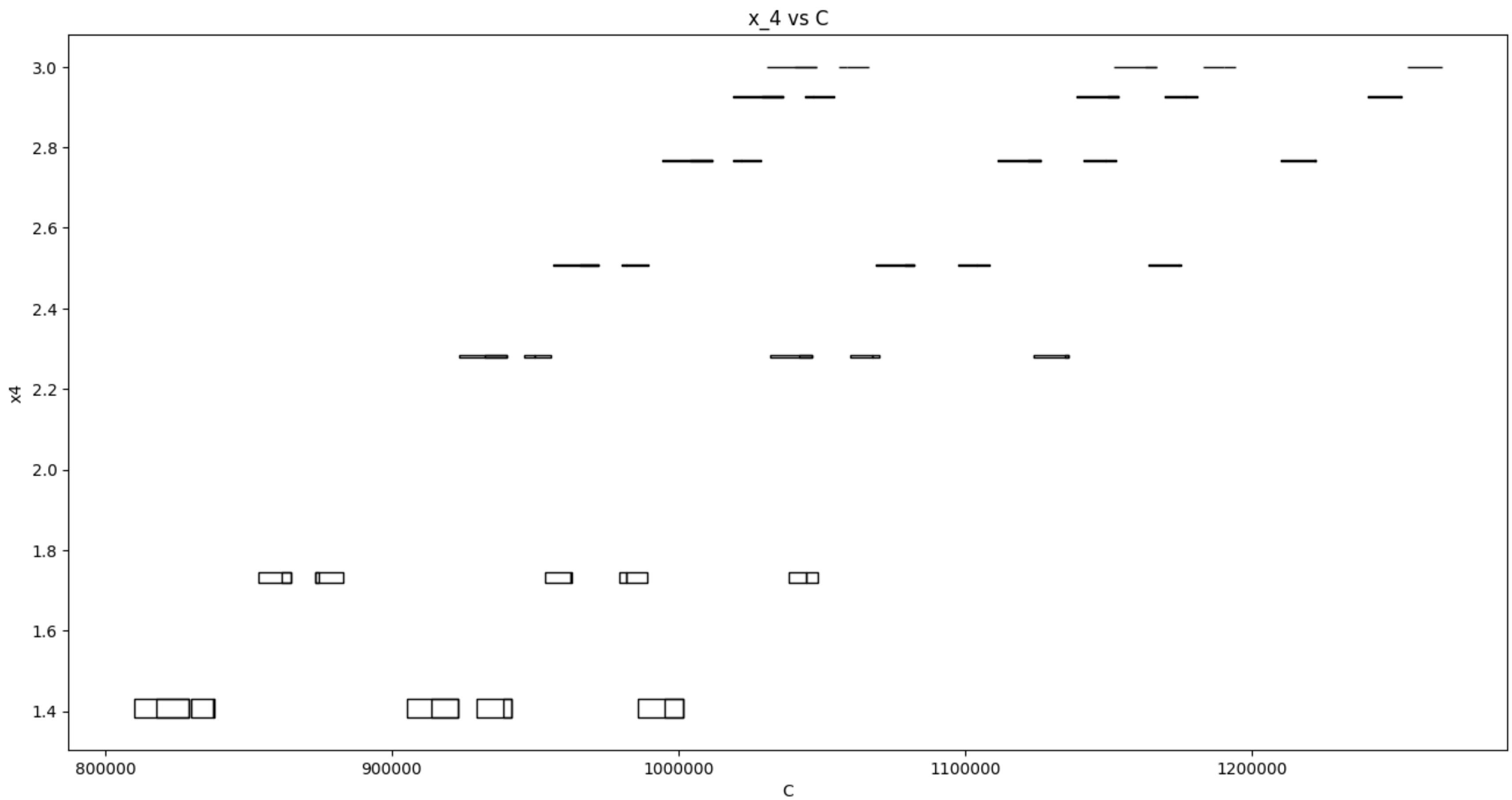

x_3 vs C
1.00
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0.92
x3
800000
900000
1000000
1100000
1200000
C

Figure 8


x_4 vs C
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
x4
800000
900000
1000000
1100000
1200000
C
Figure 9

Having illustrated the algorithm's implementation for the single interval $\left[h_{39}^{l},h_{39}^{u}\right]=\left[83.393,99.443\right]$, the optimization problem's global optimum and the corresponding values of the x_j, $j=1,4$ variables at the global optimum are then identified as follows:

The interval $sp\left[\underset{k=1,n}{\cup} H_k\right]$ is uniformly discretized with a 10,000 point grid. The algorithm described above for $\left[h_{39}^{l},h_{39}^{u}\right]=\left[83.393,99.443\right]$ is applied to all resulting sub-intervals, with edges either the elements of $\underset{k=1,n}{\cup} H_k$ or the elements of the uniform discretization grid, yielding numerous boxes in the $\left(C,\sum_{j=1}^{4} f_j\left(x_j\right)\right)$ plane. The union of boxes, whose lower edge contributes to the lower boundary of the union of all created boxes in the $\left(C,\sum_{j=1}^{4} f_j\left(x_j\right)\right)$ plane, then contains the graph of the optimum objective function value $v\left(C\right)$ as a function of C, between the minimum and maximum values of C equal to 30879.97 and 1266455.28 respectively. Figure 10 illustrates the union of boxes that contains the graph $\left(C,v\left(C\right)\right)$, while Figures 11, 12, 13, 14 illustrate the corresponding unions of boxes containing the graphs $\left(C,x_j\left(C\right)\right)$ $j=1,4$ of the optimum variable values $x_j\left(C\right)$ as functions of C.

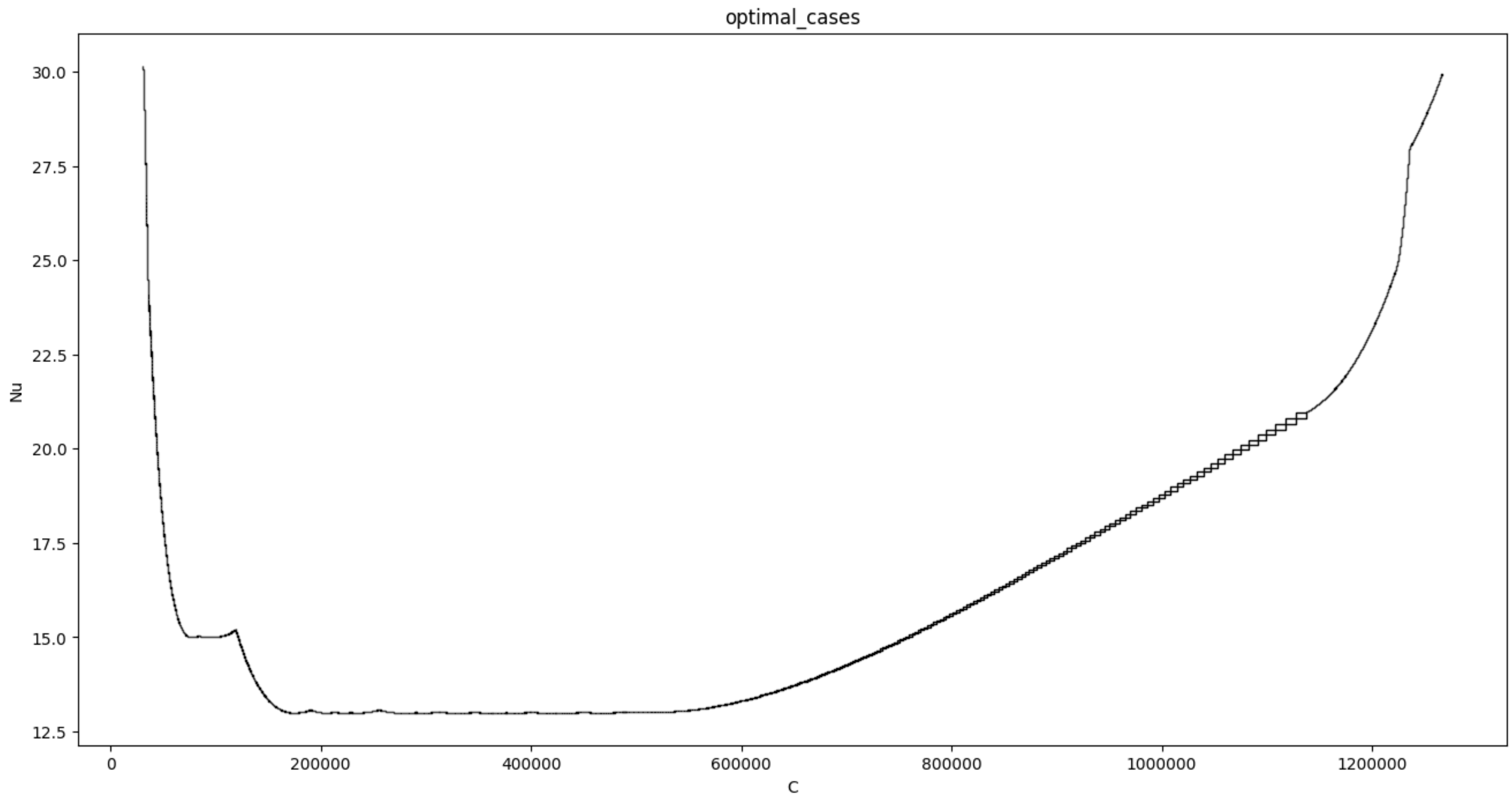

optimal_cases
Nu
C
30.0
27.5
25.0
22.5
20.0
17.5
15.0
12.5
0
200000
400000
600000
800000
1000000
1200000

Figure 10

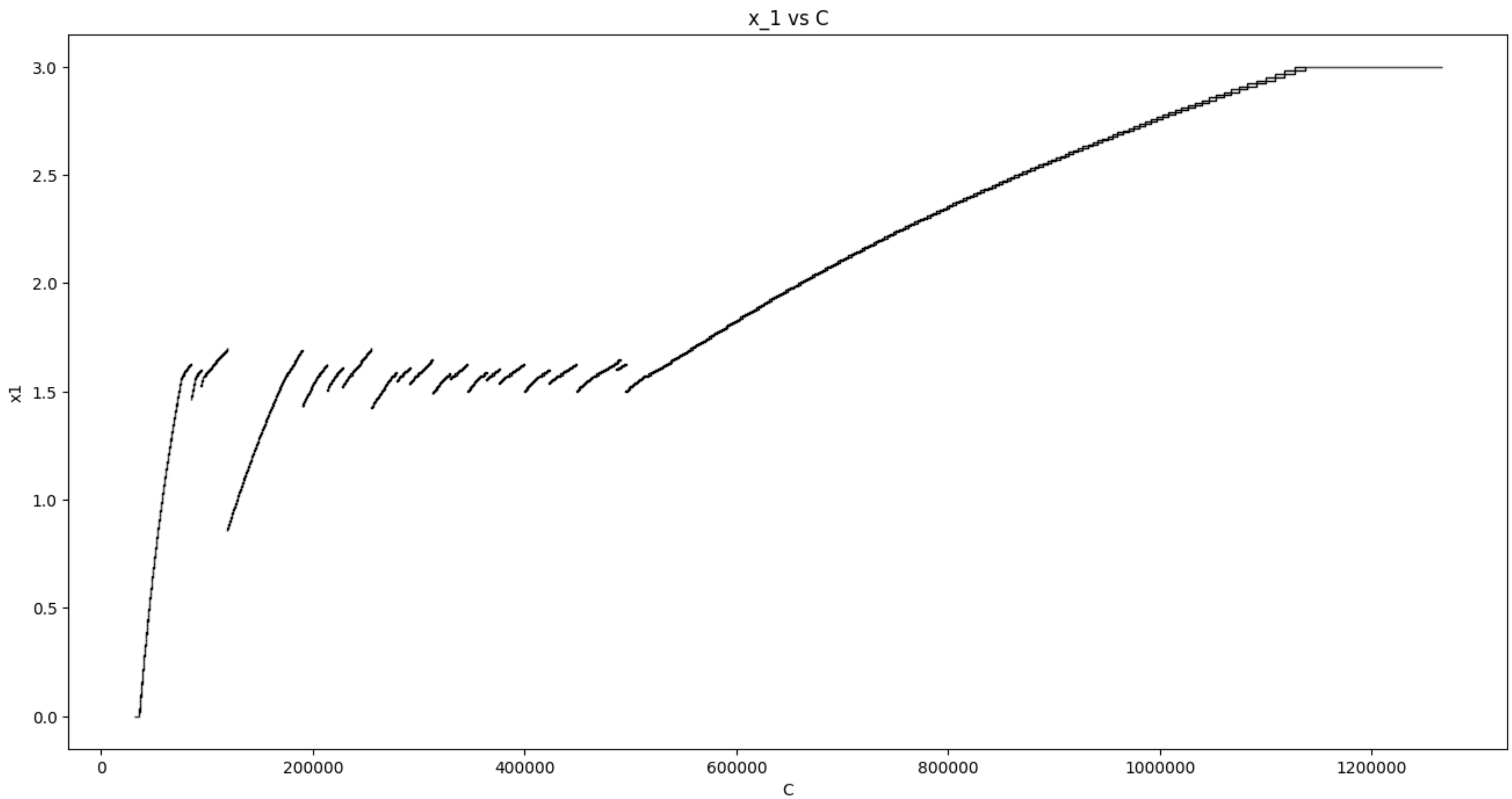
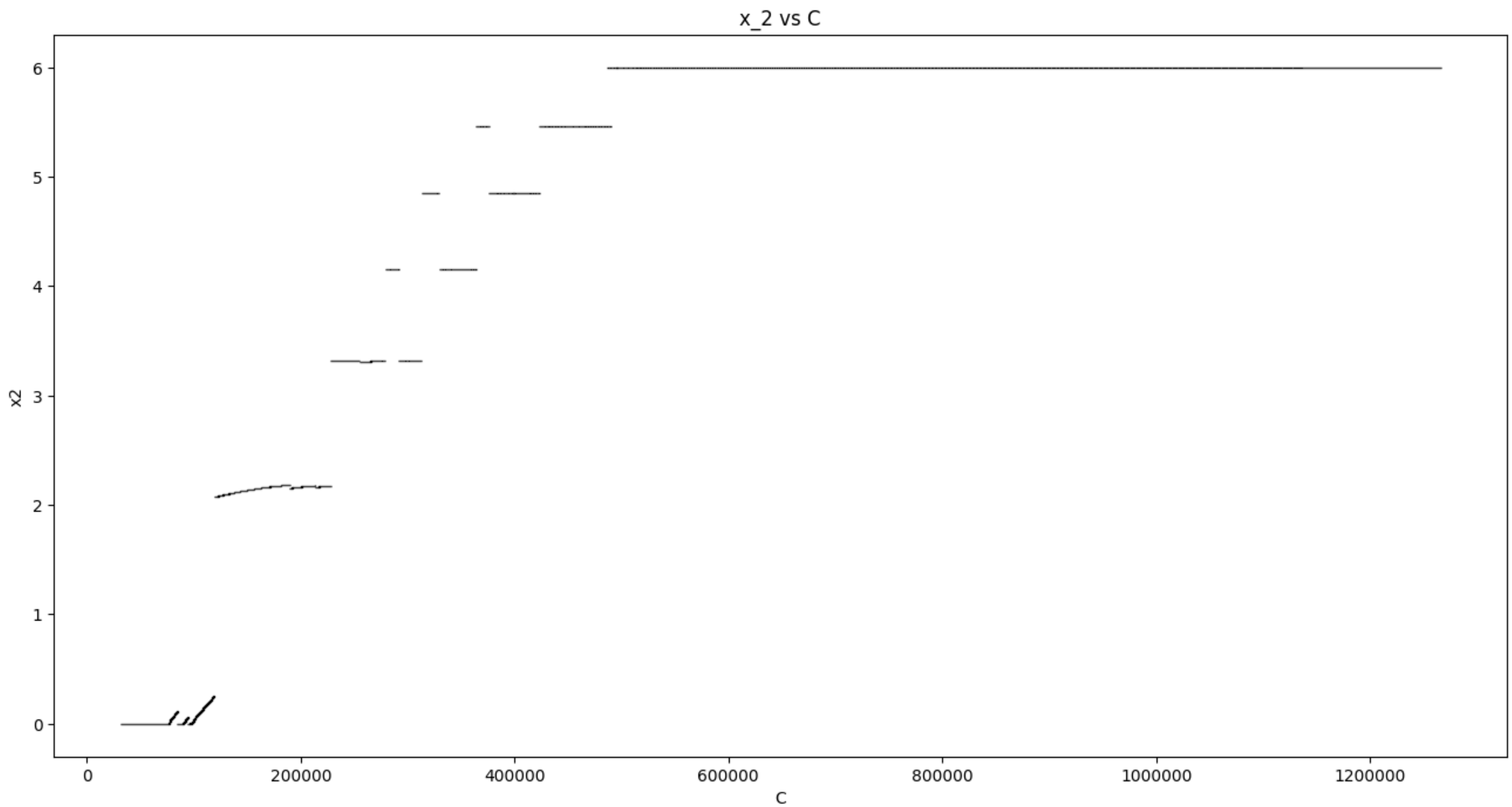

x_1 vs C
3.0
2.5
2.0
1.5
1.0
0.5
0.0
x1
0
200000
400000
600000
800000
1000000
1200000
C

Figure 11


x_2 vs C
x2
C
0
1
2
3
4
5
6
0
200000
400000
600000
800000
1000000
1200000
Figure 12

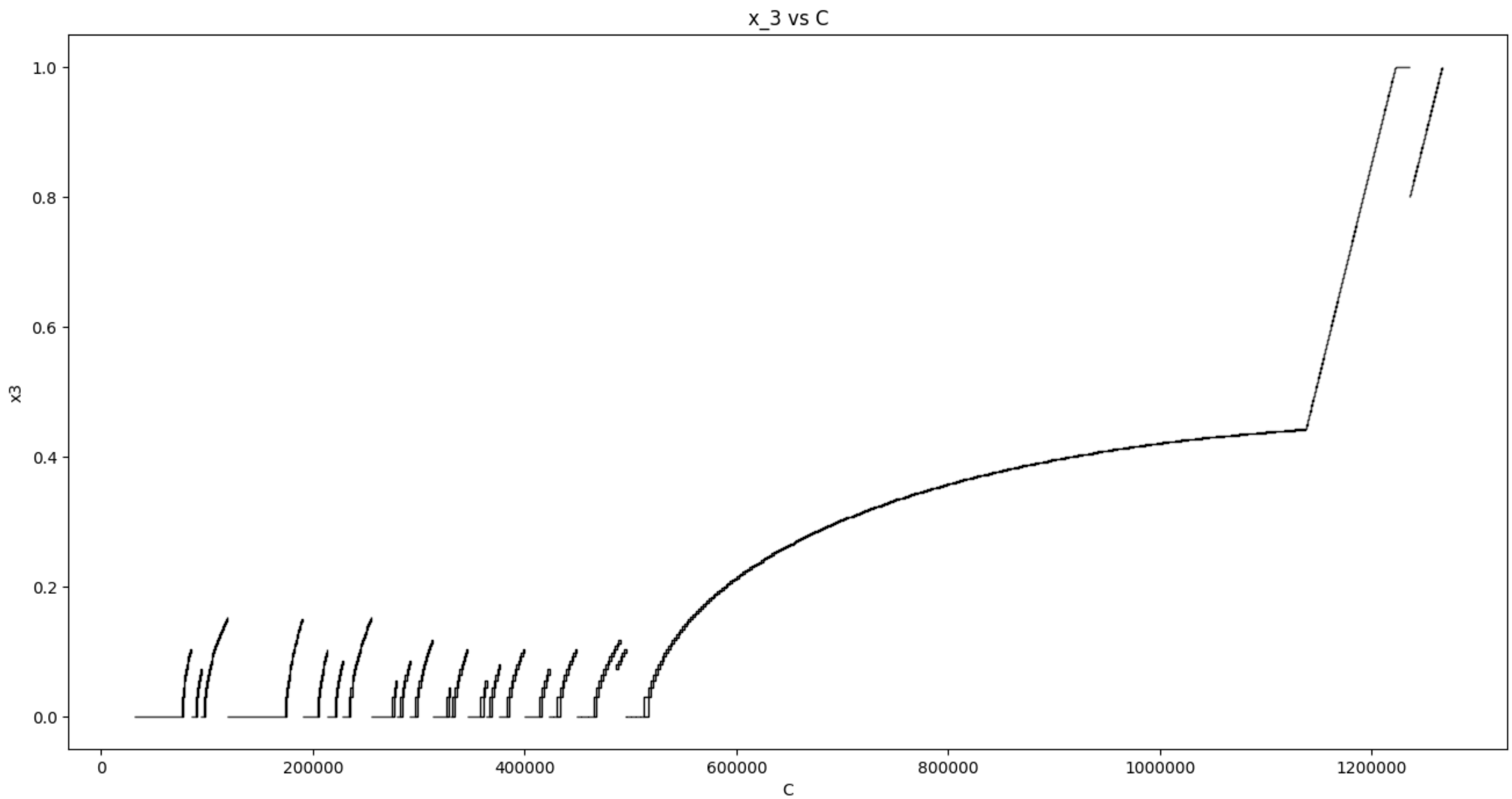

x_3 vs C
1.0
0.8
0.6
0.4
0.2
0.0
x3
0
200000
400000
600000
800000
1000000
1200000
C

Figure 13

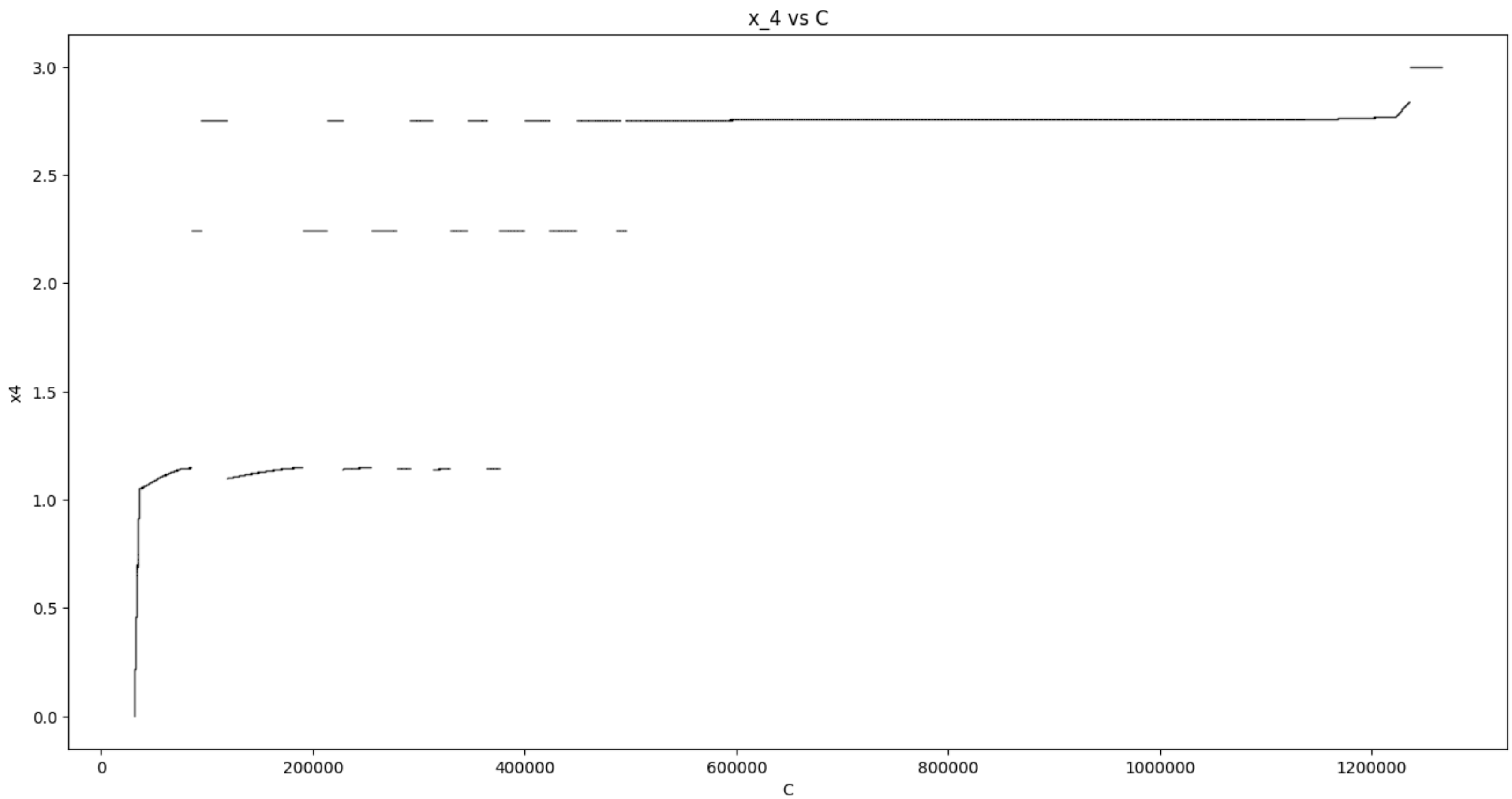

x_4 vs C
x4
C
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
200000
400000
600000
800000
1000000
1200000

Figure 14

5. CONCLUSIONS

In this work, a novel algorithm is proposed to identify the global optimum of a general class of sequential process, parametric optimization problems. The algorithm employs interval analysis and the problem's necessary conditions of optimality to identify, to any desired accuracy, sets containing the graph of the global optimum as a function of the optimization problem's single parameter. A case study is presented illustrating the algorithm's ability to correctly identify the global optimum as a function of the optimization problem's single parameter.

6. REFERENCES

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