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Multisymplectic Geometry in General Relativity and other Classical Field Theories on Manifolds with Boundaries: A Deobfuscating Role

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Multisymplectic Geometry in General Relativity and other Classical Field Theories on Manifolds with Boundaries: A Deobfuscating Role

by

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Committee in charge:

Professor Nicolai Reshetikhin, Chair
Professor Richard Borcherds
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Abstract

Multisymplectic Geometry in General Relativity and other Classical Field Theories on Manifolds with Boundaries: A Deobfuscating Role

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In Chapter 2, the multisymplectic formalism of field theories developed over the last fifty years is extended to deal with manifolds that have boundaries. In particular, a multisymplectic framework for first-order covariant Hamiltonian field theories on manifolds with boundaries is developed. This work is a geometric fulfillment of Fock's formulation of field theories as it appears in recent work by Cattaneo, Mnev and Reshetikhin [**Ca14**]. This framework leads to a geometric understanding of conventional choices for boundary conditions and relates them to the moment map of the gauge group of the theory.

It is also shown that the natural interpretation of the Euler-Lagrange equations as an evolution system near the boundary leads to a presymplectic Hamiltonian system in an extended phase space containing the natural configuration and momenta fields at the boundary together with extra degrees of freedom corresponding to the transversal components at the boundary of the momenta fields of the theory. The consistency conditions for evolution at the boundary are analyzed and the reduced phase space of the system is shown to be a symplectic manifold with a distinguished isotropic submanifold corresponding to the boundary data of the solutions of Euler-Lagrange equations. This setting makes it possible to define well-posed boundary conditions, and provides the adequate setting for the canonical quantization of the system.

The notions of the theory are tested against three significant examples: scalar fields, Poisson σ-model and Yang-Mills theories.

In Chapter 3, inspired by problems encountered in the geometrical treatment of Yang-Mills theories and Palatini's gravity, a covariant formulation of Hamiltonian dynamical systems as a Hamiltonian field theory of dimension 1+0 on a manifold with

boundary is developed. After a precise statement of Hamilton's variational principle in this context, the geometrical properties of the space of solutions of the Euler-Lagrange equations of the theory are analyzed. A sufficient condition is obtained that guarantees that the set of solutions of the Euler-Lagrange equations at the boundary of the manifold, fill a Lagrangian submanifold of the space of fields at the boundary. Finally a theory of constraints is introduced that mimics the constraints arising in Palatini's gravity.

In Chapter 4, a covariant Hamiltonian description of Palatini's gravity on manifolds with boundary is presented. Palatini's gravity appears as a gauge theory satisfying a constraint in a certain topological limit. This approach allows the consideration of non-trivial topological situations.

The multisymplectic framework for first-order covariant Hamiltonian field theories on manifolds with boundary, developed in Chapter 2, enables analysis of the system at the boundary. The reduced phase space of the system is determined to be a symplectic manifold with a distinguished isotropic submanifold corresponding to the boundary data of the solutions of the Euler-Lagrange equations.

for my mother and for my boys and especially for Joshua
in memory of my father and my wonderful grandfather

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Chapter 1

Introduction

The GiMmsy papers of the 1990's, as they are often referred to, [**Go98**],[**Go04**], publicized the use of multisymplectic geometry to describe Lagrangian and Hamiltonian first-order covariant classical field theories. GiMmsy is an acronym for the authors of that project. Centered around Jerry Marsden, the authors were Gotay, Isenberg, Montgomery and Sniatycki and Yasskin. Their work built on many contributions in the history of constructing a geometrical picture for field theories, including Dirac's theory of constraints. We refer the reader to the comprehensive texts [**Go98**] and [**Bi11**]for details on this history.

The GiMmsy papers dealt with field theories only on manifolds that have no boundaries and that have the simple Cartesian structure $T \times S$ where T is time and S is space. In Chapter 2 of this dissertation, which is based on joint work with Alberto Ibort published in the Journal of Geometric Mechanics [**Ib15**], we extend multisymplectic geometry to manifolds with boundaries to describe first-order Hamiltonian field theories on such manifolds. No other restrictions are placed on the manifolds. The spacetime need not be foliated by Cauchy surfaces, it need not be globally hyperbolic.

Immediate fruits of our multisymplectic formulation include first a formula for the action and for the differential of the action functional, valid for any classical field theory. In the work of Cattaneo, Mnev and Reshetikhin [**Ca11**], [**Ca14**], and in many other works, for each classical field theory considered, the authors have to come up with a different action functional. To do so they need to decide what the momenta of the theory should be. In our work, having one expression for the action functional that describes all classical theories eliminates the need to choose what the momenta fields should be for each physical theory under examination: our multisymplecic formalism does it for us. This turns out to very important in helping us clarify the meaning of the Palatini action in Chapter 4.

Our analysis of the role of boundary terms in the computation of the critical points of the action functional allows a natural relation to emerge between the action functional and the canonical symplectic structure on the space of fields at the boundary. It is precisely this relation that allows a better understanding of the role of boundary conditions. In addition, the canonical 1-form on the space of fields at the boundary is shown to be directly related to the charges of the gauge symmetries of the theory. The condition that boundary values of the field lie in a Lagrangian submanifold is shown to be equivalent to the condition that boundary values of the field lie in the 0-level set of the moment map on $T^*\mathcal{F}_{\partial M}$, the cotangent space of the fields at the boundary, generated by the action of the group of automorphisms of the fiber bundle of the fields over the manifold, induced at the boundary, lifted to the cotangent bundle.

We also show that the natural way to interpret Euler-Lagrange equations as an evolution system near the boundary is as a presymplectic system in an extended phase space containing the natural configuration and momenta fields at the boundary together with extra degrees of freedom corresponding to the transversal component at the boundary of the momenta fields of the theory. The consistency conditions at the boundary are analyzed and the reduced phase space of the system is determined to be a symplectic manifold with a distinguished isotropic submanifold corresponding to the boundary data of solutions of Euler-Lagrange equations. We work out in detail three examples: scalar fields, Poisson-sigma model and Yang-Mills theories.

After the geometrical analysis of the theory has been performed, the space of quantum states of the theory would be obtained, in the best possible situations, by canonical or geometrical quantization of a reduced symplectic manifold of fields at the boundary that would describe its "true" degrees of freedom. The propagator of the theory would be obtained by quantizing a Lagrangian submanifold of the reduced phase space of the theory provided by the specification of admissible boundary conditions. The latter should preserve the fundamental symmetries of the theory, in the sense that the charges associated to them should be preserved. The resulting overall picture, as described for instance in the case of Chern-Simons theory [**At90**], is that the functor defining a quantum field theory is obtained by geometric quantization of the quasi-category of Lagrangian submanifolds associated to admissible boundary conditions at the boundaries of spacetimes and their corresponding fields.

The level of rigor throughout this dissertation is that of standard differential geometry. When dealing with finite-dimensional objects, they will be smooth differentiable manifolds, locally trivial bundles etc. However, when dealing with infinite-dimensional spaces we will assume,as customary, that the rules of global differential calculus apply and we will use them freely without providing constructions that will lead to bona fide Banach manifolds of maps and sections. Also, the notation of variational differentials and derivatives will be used for clarity without attempting to discuss the classes of spaces of generalized functions needed to justify their use.

In Chapter 3, which is based on published work joint with Alberto Ibort [**Ib16**], we analyze the theory of Hamiltonian dynamical systems using the multisymplectic approach developed in Chapter 2. We view these theories as first-order covariant Hamiltonian field theories on a manifold of dimension 1+0 with boundary. The reason for our interest in such simple field theories is twofold. First, it allows us to study in a simple setting two features which are relevant to two fundamental examples of first-order covariant Hamiltonian field theories, Yang-Mills and especially Palatini gravity. These features are the introduction of constraints and the 'topological phases' of the theory. In Palatini gravity, a constraint in the momenta fields of the theory must be introduced to recover the equations of motion of Palatini's gravity. This constraint relates the momenta to the vierbein fields e_μ of the standard treatment.

The second reason for our interest in such a simple theory as Hamiltonian dynamics, is that it allows us to test some common assumptions about field theories, among them the role of boundary conditions and the geometry of the restriction to the boundary of the space of solutions of the Euler-Lagrange equations. The fact that the basic space of the theory is one-dimensional and that the Euler-Lagrange equations of the theory are the standard Hamilton's equations, removes most of the analytic difficulties arising in higher order theories.

We formulate a mathematically precise expression for Hamilton's variational principle, this was previously absent in the literature. We define what we call a locally Dirichlet condition and prove several theorems giving different sufficient conditions under which $\pi(EL)$, the space of solutions of the Euler-Lagrange equations restricted to the boundary, is a Lagrangian submanifold. Then through examples we show that none of these conditions is necessary. In this way we show that for even so simple a theory as Hamiltonian dynamics, giving necessary and sufficient conditions under which $\pi(EL)$ is a Lagrangian submanifold is not a simple problem.

In Chapter 4 we turn to general relativity.Our understanding of the Hamiltonian structure of gravity has taken half a century. The initial difficulties faced by Dirac and Bergmann [Be58],[Be81], were slowly resolved through the work of Arnowitt, Deser and Misner [Ar62], all the way to Ashtekar's formulation [As87]. At least part of the motivation has been to place the theory of gravity on grounds that will make it suitable for a canonical quantization scheme.

In [Ro06], C. Rovelli illustrated a simple Hamiltonian formulation of general relativity which is manifestly 4d generally covariant and that drops the reference to the underlying space-time in Palatini's formulation of gravity. Rovelli's proposal is highly geometrical and constructs its space as the $4+16+24$ dimensional space $\widetilde{\mathcal{C}}$

with local coordinates $(x^\mu, e^I_\mu, A^{IJ}_\mu)$. In a further effort at extracting the geometrical essence of such space, the variables x^μ are dropped (accounting for the invariance of the theory under global diffeomorphisms) and we are led to a 40 dimensional space $\mathcal{C}$ [Ro01]. The disappearance of the spacetime manifold M and its coordinates x^μ, which survive only as arbitrary parameters on the 'gauge orbits' of the canonical geometrical structure defined on it, generalizes the disappearance of the time coordinate in the ADM formalism and is analogous to the disappearance of the Lagrangian evolution parameter in the Hamiltonian theory of a free particle [Ro01]. It simply means that the general relativistic space- time coordinates are not directly related to observations.

Our program in Chapter 4 is similar to Rovelli's but our inspiration is the geometrical foundations of covariant first-order Hamiltonian field theories on manifolds with boundary developed in Chapter 2. There the role of a covariant phase space for a first order Hamiltonian theory modelled on the affine dual space of the first jet bundle of the bundle defining the fields of the theory is assessed and the crucial role played by boundaries as determining symplectic spaces of fields defining the classical counterpart of the quantum states of the theory is stressed in accordance with the point of view expressed in [Sc51].

Actually a generally covariant notion of instantaneous state, or evolution of states and observables in time, make little physical sense. They are always referred to an initial data space-like surface that in the picture presented here, corresponds to the boundary of the space-times of events. Such notion does not really conflict with diffeomorphism invariance because a diffeomorphism of a smooth manifold with smooth boundary restricts to a diffeomorphism of the boundary. Thus, providing that the notion of boundary of a spacetime is incorporated in the basic description of the theory, we may still consider diffeomorphism invariance as a fundamental notion without contradicting it.

The covariant phase space of the theory carries a natural multisymplectic structure which is the exterior differential of a canonical m-form Θ defined on it. This geometrical structure has been considered in various guises in the various variational formulations of field theories, however its first use in the present setting is to help to identify the nature of the different fields of the theory. We show how the vierbein fields e^I_μ correspond to an algebraic constraint imposed on the momenta fields of the theory. Therefore we come to recognize that in Palatini gravity, a constraint in the momenta fields of the theory must be introduced to recover the equations of motion. We show that the corresponding action is invariant under the group of all automorphisms of the geometrical structure and that it induces the corresponding reduction on the space of gauge fields at the boundary. This reduction process is interpreted as the appropriate setting for the 'elimination' of the space-time M, i.e., the space of physical classical solutions of the theory in the bulk is the moduli space of the space of solutions of the Euler-Lagrange equations with respect to the group of automorphisms whereas, the phase space of physical degrees of freedom of the theory, associated to its boundary, is the reduced symplectic manifold of fields at the boundary.

Chapter 2

Covariant Hamiltonian field theories on manifolds with boundary: Yang-Mills theories

2.1 Introduction

Multisymplectic geometry provides a convenient framework for describing first-order classical field theories both in the Lagrangian and Hamiltonian formalism. (See for example [**Go98**],[**Go04**]). However, the field theories so described are only over spacetimes that do not have boundaries. In this chapter, based on joint published work with Alberto Ibort, [**Ib15**], we extend the multisymplectic formalism to deal with manifolds that have boundaries. We develop a multisymplectic framework for first-order covariant Hamiltonian field theories on manifolds with boundaries.

We are then led to a formula for the differential of the action functional,valid for any first-order Hamiltonian field theory, and a natural identification of the momenta of the theory. The explicit identification of the boundary term in the differential of the action with the pull-back of the canonical one-form on the cotangent bundle of boundary values of the fields of the theory, shows immediately that for all classical theories the space $\pi(\mathcal{EL})$ of boundary values of the solutions of the Euler-Lagrange equations is an isotropic submanifold.

The description of the theory in the bulk, while following along the lines already established in the literature, also includes analysis of the role of boundary terms in the computation of the critical points of the action functional. Thus a natural relation emerges between the action functional and the canonical symplectic structure on the space of fields at the boundary and it is precisely this relation that allows a better understanding of the role of boundary conditions. In addition, the canonical 1-form on the space of fields at the boundary can be directly related to the charges of the gauge symmetries of the theory.

We describe the natural presymplectic structure inherited by the space of all fields of the theory at the boundary and the Hamiltonian structure of the corresponding evolution near the boundary. The consistency conditions of such evolution are studied and the corresponding Hamiltonian dynamics in the reduced symplectic manifold of fields at the boundary is obtained.

The chapter is organized as follows: Section 2.2 is devoted to summarizing the basic geometric notions underlying the theory. The multisymplectic formalism is briefly reviewed, the action principle and the fundamental formula exhibiting the differential of the action functional of the theory is presented and proved. The role of symmetries, moment maps at the boundary and boundary conditions are elucidated. Section 2.3 presents the evolution formulation of the theory near the boundary. The presymplectic picture of the system is established and the subsequent constraints analysis is laid out. Its relation with reduction with respect to the moment map at the boundary is pointed out. Real scalar fields and the Poisson σ-model are analyzed to illustrate the theory. Finally, Section 2.4 focuses on the study of Yang-Mills theories on manifolds with boundary as first-order Hamiltonian field theories in the multisymplectic framework and the Hamiltonian reduced phase space of the theory is described.

2.2 The multisymplectic formalism for first order covariant Hamiltonian field theories on manifolds with boundary

The setting: the multisymplectic formalism

The geometry of Lagrangian and Hamiltonian field theories has been examined in the literature from varying perspectives. For our purposes here we single out for summary the Hamiltonian multisymplectic description of field theories on manifolds without boundary found in [**Ca91**]. Everything in this section will apply also to manifolds possessing boundaries. In the next section we will consider only manifolds having boundaries and we will extend the multisymplectic formalism to deal with Hamiltonian field theories over such manifolds.

A manifold M will model the space or spacetime at each point of which the classical field under discussion assumes a value. We will therefore take M to be an oriented $m = 1 + d$ dimensional smooth manifold. In most situations M is either Riemannian or Lorentzian and time-oriented. We will denote the metric on M by η. In either case we will denote by vol_M the volume form defined by the metric η on M. In an arbitrary local chart x^μ, $\mu = 0, 1, \ldots, d$, this volume form takes the form $\mathrm{vol}_M = \sqrt{|\eta|}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \cdots \wedge \mathrm{d}x^d$. Notice however that the only structure on M required to provide the kinematic setting of the theory will be a volume form vol_M and, unless specified otherwise, local coordinates will be chosen such that $\mathrm{vol}_M = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \cdots \wedge \mathrm{d}x^d$.

The fundamental geometric structure of a given theory will be provided by a locally trivial fiber bundle over M, $\pi\colon E \to M$. Local coordinates adapted to the fibration will be denoted as (x^μ, u^a), $a = 1, \ldots, r$, where r is the dimension of the standard fiber.

Let J^1E denote the first jet bundle of the bundle E, i.e. at each point $(x, u) \in E$, the fiber of J^1E consists of the set of equivalence classes of germs of sections of $\pi\colon E \to M$. If we let π_1^0 be the projection map, $\pi_1^0\colon J^1E \to E$, then (J^1E, π_1^0, E) is an affine bundle over E modelled on the vector bundle $VE \otimes \pi^*(T^*M)$ over E. (See [**Sa89**], and [**Gr15**] for details on affine geometry and the construction of the various affine bundles naturally associated to $E \to M$.) Given adapted local coordinates $(x^\mu; u^a)$ for the bundle $\pi\colon E \to M$, we denote by $(x^\mu, u^a; u^a_\mu)$ an adapted local chart for J^1E.

Let $\bigwedge^m(E)$ denote the bundle of m-forms on E and let $\bigwedge^m_k(E)$ denote the subbundle of $\bigwedge^m(E)$ consisting of all m-forms which vanish when $k+1$ of their arguments are vertical. In particular, elements of $\bigwedge^m_1(E)$ are called semi-basic forms and have the form $\rho^\mu_a\,\mathrm{d}u^a \wedge \pi^*\mathrm{vol}_\mu + \rho_0\pi^*\mathrm{vol}_M$ where $\mathrm{vol}_\mu = \frac{\partial}{\partial x^\mu} \lrcorner \mathrm{vol}_M$. Given the above bundle coordinates $(x^\mu; u^a)$ for E, we have adapted coordinates $(x^\mu, u^a; \rho_0, \rho^\mu_a)$ on $\bigwedge^m_1(E)$. Elements of the subbundle $\bigwedge^m_0(E)$ of the bundle $\bigwedge^m_1(E)$, appropriately named basic m-forms, have the form $\rho_0\pi^*\mathrm{vol}_M$.

The affine dual bundle to J^1E is the space of affine maps along the fibers of J^1E, denoted by $\mathrm{Aff}(J^1E, \mathbb{R})$. The choice of the volume form vol_M on the base manifold M allows the identification $\mathrm{Aff}(J^1E) \cong \bigwedge^m_1(E)$ by means of the assignment of the affine map defined in the local coordinates above by $u^a_\mu \mapsto \rho^\mu_a u^a_\mu + \rho_0$ to the element $\rho^\mu_a \mathrm{d}u^a \wedge \pi^*\mathrm{vol}_\mu + \rho_0\mathrm{vol}_M$ of $\bigwedge^m_1(E)$. This assignment is easily seen to be a vector bundle isomorphism. Note that this isomorphism also maps each constant affine map $u^a_\mu \mapsto \rho_0$ to the corresponding element $\rho_0\,\pi^*\mathrm{vol}_M$ of the subbundle $\bigwedge^m_0(E)$ of $\bigwedge^m_1(E)$. We therefore have the further identification $\mathrm{Aff}(J^1E)/\mathrm{R} \cong \bigwedge^m_1(E)/\bigwedge^m_0(E)$.

We will define the covariant phase space of the theory $P(E)$ as the quotient of the bundle of affine maps on J^1E mod out the constant ones, i.e. $P(E) := \mathrm{Aff}(J^1E, \mathbb{R})/\mathbb{R}$. It is easy to see that $\mathrm{Aff}(J^1E, \mathbb{R})/\mathbb{R}$ can be identified with the vector bundle $\pi^*(TM) \otimes V(E)^*$ over E. From our definition of $P(E)$ and the identifications in the previous paragraph, it is clear that $P(E) \cong \bigwedge_1^m(E)/\bigwedge_0^m(E)$. We have also the following short exact sequence:

$$0 \to \bigwedge\nolimits_0^m(E) \hookrightarrow \bigwedge\nolimits_1^m(E) \xrightarrow{\mu} P(E) \to 0$$

which shows that $\bigwedge_1^m(E)$ is a real line bundle over $P(E)$. The previous identification of the covariant phase space depended on the choice of a volume form vol_M and with respect to the choice of volume form the map $TM \to \bigwedge^{m-1}(T^*M)$ given by $\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} \lrcorner \mathrm{vol}_M$ is a bundle isomorphism. By means of this isomorphism it follows easily that the bundle $\pi^*(TM) \otimes V(E)^*$ over E is isomorphic to the bundle $\pi^*(\bigwedge^{m-1}(T^*M)) \otimes V(E)^*$. Thus it is clear that $P(E) \cong \pi^*(\bigwedge^{m-1}(T^*M)) \otimes V(E)^*$. Local coordinates can be introduced in $P(E)$ taking advantage of these identifications. Thus if we denote by $\rho_a^\mu \mathrm{d}u^a \wedge \mathrm{vol}_\mu + \bigwedge_0^m(E)$ a class in the quotient space $P(E)$, local adapted coordinates will be given by $(x^\mu, u^a; \rho_a^\mu)$.

The bundle $\bigwedge_1^m(E)$ carries a canonical m–form which may be defined by a generalization of the definition of the canonical 1-form on the cotangent bundle of a manifold. Let $\nu\colon \bigwedge_1^m(E) \to E$ be the canonical projection, then the canonical m-form Θ is defined by $\Theta_\varpi(U_1, U_2, \ldots, U_m) = \varpi(\nu_* U_1, \ldots, \nu_* U_m)$, where $\varpi \in \bigwedge_1^m(E)$ and $U_i \in T_\varpi(\bigwedge_1^m(E))$. With respect to the local coordinates $(x^\mu, u^a; \rho_0, \rho_a^\mu)$ above we have the local expression

$$\Theta = \rho_a^\mu \, \mathrm{d}u^a \wedge \mathrm{vol}_\mu + \rho_0 \, \mathrm{vol}_M \, .$$

The $(m+1)$-form $\Omega = \mathrm{d}\Theta$ defines a multisymplectic structure on the manifold $\bigwedge_1^m(E)$, i.e. $(\bigwedge_1^m(E), \Omega)$ is a multisymplectic manifold. There is some variation in the literature on the definition of multisymplectic manifold. For us, following [**Go98**] and [**Ca91**], a multisymplectic manifold is a pair (X, Ω) where X is a manifold of some dimension m and Ω is a d-form on X, $d \geq 2$, and Ω is closed and nondegenerate. By nondegenerate we mean that if $i_v \Omega = 0$ then $v = 0$.

We will refer to $\bigwedge_1^m(E)$ by $M(E)$ to emphasize that it is a multisymplectic manifold. We will denote the projection $M(E) \to E$ by ν, while the projection $M(E) \to P(E)$ will be denoted by μ. Thus $\nu = \tau_1^0 \circ \mu$, with $\tau_1^0\colon P(E) \to E$ the canonical projection (see figure 1).

A Hamiltonian H on $P(E)$ is a section of μ. Thus in local coordinates

$$H(x^\mu, u^a; \rho_a^\mu) = \rho_a^\mu \, \mathrm{d}u^a \wedge \mathrm{vol}_\mu - \mathbf{H}(x^\mu, u^a, \rho_a^\mu)\mathrm{vol}_M \, ,$$

where $\mathbf{H}$ is here a locally defined real-valued function also called the Hamiltonian function of the theory and the sign has been chosen for convenience.

We can use the Hamiltonian section H to define an m-form on $P(E)$ by pulling back the canonical m-form Θ from $M(E)$. We call the form so obtained the Hamiltonian m-form associated with H and denote it by Θ_H. Thus if we write the section defined by H in local coordinates as $\rho_0 = -\mathbf{H}(x^\mu, u^a, \rho_a^\mu)$, then

$$\Theta_H = \rho_a^\mu \, \mathrm{d}u^a \wedge \mathrm{vol}_\mu - \mathbf{H}(x^\mu, u^a, \rho_a^\mu) \, \mathrm{vol}_M \tag{2.2.1}$$

which shows that the minus sign in front of the Hamiltonian is chosen to be in keeping with the traditional conventions in mechanics for the integrand of the action over the manifold. When the form Θ_H is pulled back to the manifold M along a section of the canonical bundle projecction $\tau_1 \colon P(E) \to M$ as described in Section 2.2, the integrand of the action over M will have a form reminiscent of that of mechanics with a minus sign in front of the Hamiltonian. See equation (2.2.3). In what follows, we will use the same notation H both for the section and the real-valued function $\mathbf{H}$ defined by a Hamiltonian.

In what follows a first-order covariant Hamiltonian field theory (or a Hamiltonian field theory for short) will be defined as a pair $(P(E), H)$ where $P(E)$ is the covariant phase space defined by a bundle $\pi \colon E \to M$ and H is a Hamiltonian section as defined above. Notice that $P(E)$ always carries a canonical m-form Θ_H and a multisymplectic structure $\Omega_H = \mathrm{d}\Theta_H$.

The action and the variational principle

Sections and fields over manifolds with boundary

From here on we will assume that the manifold M has a smooth boundary ∂M. The orientation chosen on ∂M is consistent with the orientation on M. Everything in the last section applies. The presence of boundaries, apart from being a natural ingredient in any attempt at constructing a field theory, will enable us to enlarge the use to which the multisymplectic formalism can be applied, starting with the statement and proof of Lemma 2.1 describing the exterior differential of the action functional.

Let $(P(E), H)$ be a Hamiltonian field theory. The fields χ of the theory consist of the class of sections of the bundle $\tau_1 \colon P(E) \to M$ that can be factorized through sections P of the bundle $\tau_0^1 \colon P(E) \to E$ and Φ of $\pi \colon E \to M$, that is $\chi = P \circ \Phi$ (see Figure 2.1). The sections Φ will be called the configuration fields or just the configurations, and the sections P, the momenta fields of the theory. Notice that $\tau_0^1 \circ \chi = \Phi$ if χ is a field. In local adapted coordinates $(x^\mu; u^a; \rho_a^\mu)$ for $P(E)$ it is clear that we may write $u^a = \Phi^a(x)$ and $\rho_a^\mu = P_a^\mu(\Phi(x))$ for the local expression of the section $\chi = P \circ \Phi$. We will often denote the section $\chi = P \circ \Phi$ as a pair (Φ, P) to emphasize the different roles of the configurations and momenta content of the field.

$$
\begin{tikzcd}
 & M(E) \arrow[d, "\mu"'] & \Theta \\
 i^*(P(E)) \arrow[r] \arrow[d] & P(E) \arrow[u, bend right, "H"'] \arrow[d, "\tau_1^0"] \arrow[dd, "\tau_1"', shift right=3] & \Theta_H = \text{-}H^*\Theta \\
 E_{\partial M} = i^*E \arrow[r] \arrow[u, bend right, "{(p,\beta)}"'] \arrow[d, "\pi_{\partial M}"'] & E \arrow[u, bend right, "P"'] \arrow[d, "\pi"] & \\
 \partial M \arrow[r, "i"] \arrow[u, bend right, "\varphi"'] & M \arrow[u, bend right, "\Phi"'] \arrow[uu, bend right=60, "\chi"'] &
\end{tikzcd}
$$

Figure 2.1: Bundles, sections and fields: configurations and momenta

We will denote by $\mathcal{F}_M$ the space of sections Φ of the bundle $\pi \colon E \to M$, that is $\Phi \in \mathcal{F}_M$, and we will denote by $\mathcal{F}_{P(E)}$ the space of fields $\chi = P \circ \Phi$ (which we will denote by $\chi = (\Phi, P)$) . Thus $\mathcal{F}_{P(E)}$ will denote the space of fields of the theory, configurations and momenta, in the first-order covariant Hamiltonian formalism.

The equations of motion of the theory will be defined by means of a variational principle, i.e. they will be related to the critical points of a local action functional S on $\mathcal{F}_{P(E)}$. Such action will be simply given by,

$$
S(\chi) = \int_M \chi^* \Theta_H \,, \tag{2.2.2}
$$

or in a more explicit notation using the local representation of the section χ described above,

$$
S(\Phi, P) = \int_M \left(P_a^\mu(x) \partial_\mu \Phi^a(x) - H(x, \Phi(x), P(x)) \right) \mathrm{vol}_M, \tag{2.2.3}
$$

where $P_a^\mu(x)$ is shorthand for $P_a^\mu(\Phi(x))$. Of course, as is usual in the derivations of equations of motion via variational principles, we assume that the integral in Eq. (4.2.4) is well defined. It is also assumed that the 'differential' symbol in equation (2.5) below, defined in terms of directional derivatives, is well defined and that the same is true for any other similar integrals in this dissertation.

Lemma 2.2.1. *With the above notations we obtain,*

$$\mathrm{d}S(\chi)(U) = \int_M \chi^* \left(i_{\widetilde{U}} \mathrm{d}\Theta_H \right) + \int_{\partial M} (\chi \circ i)^* \left(i_{\widetilde{U}} \Theta_H \right) , \tag{2.2.4}$$

where U is a vector field on $P(E)$ along the section χ, $\widetilde{U}$ is any extension of U to a tubular neighborhood of the range of χ and $i \colon \partial M \to M$ is the canonical embedding.

Proof. If χ is a field, that is, a section of $P(E) \to M$, then we denote by $T_\chi \mathcal{F}_{P(E)}$ the tangent bundle to the space of fields $\mathcal{F}_{P(E)}$ at χ. Tangent vectors U at χ, i.e. $U \in T_\chi \mathcal{F}_{P(E)}$, are just vertical vector fields U on $P(E)$ with respect to the projection τ_1 along the map χ. In other words, they are maps $U \colon M \to TP(E)$ satisfying $\tau_{P(E)} \circ U = \chi$, where $\tau_{P(E)} \colon TP(E) \to P(E)$ denotes the canonical tangent bundle projection and such that $\tau_{1*} U = 0$. The reason for the latter being that a local curve defining the tangent vector U, by definition consists of a family of sections of χ_λ of the projection map τ_1. Hence it defines a vertical vector field along χ.

Thus if $U \in T_\chi \mathcal{F}_{P(E)}$ with $U(x) \in T_{\chi(x)} P(E)$, consider a curve $\chi_\lambda(x) := \chi(\lambda, x) \colon (-\epsilon, \epsilon) \times M \to P(E)$ such that $\chi(0, x) = \chi(x)$ and

$$U(\chi(x)) = \left.\frac{\partial}{\partial \lambda}\right|_{\lambda = 0} \chi(\lambda, x) \, .$$

We can extend the vector field U to a tubular neighborhood T_χ of the image $\chi(M)$ of χ in $P(E)$ and we will denote it by $\tilde{U}$. Consider the local flow φ_λ of $\tilde{U}$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi_\lambda = \tilde{U} \circ \varphi_\lambda \, ,$$

or in other words, we denote the integral curves of $\tilde{U}$ by $\varphi_\lambda(\xi)$, $\xi \in T_\chi \subset P(E)$. If $\xi = \chi(x)$ we then have, $\varphi_\lambda(\xi) = \varphi_\lambda(\chi(x)) = \chi(\lambda, x) = \chi_\lambda(x)$, i.e. $\varphi_\lambda \circ \chi = \chi_\lambda$. We thus obtain,

$$\begin{aligned}
\mathrm{d}S(\chi)(U) &= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} S(\chi_\lambda) = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \int_M \chi_\lambda^* \Theta_H = \\
&= \int_M \chi^* \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} \varphi_\lambda^* \Theta_H = \int_M \chi^* (\mathcal{L}_{\tilde{U}} \Theta_H) = \\
&= \int_M \chi^* \mathrm{d}(i_{\tilde{U}} \Theta_H) + \int_M \chi^* i_{\tilde{U}} \mathrm{d}\Theta_H \, .
\end{aligned} \tag{2.2.5}$$

Applying Stokes' theorem to the first term in eq. (2.2.5) then yields eq. (4.2.7). □

The cotangent bundle of fields at the boundary

The boundary term contribution to $\mathrm{d}S$ in eq. (4.2.7), $\int_{\partial M}(\chi \circ i)^* \,(i_{\tilde{U}}\Theta_H)$, suggests that there is a family of fields at the boundary that play a special role. Actually, we notice that the field $\tilde{U}$ being vertical with respect to the projection $\tau_1 \colon P(E) \to M$ has the local form $\tilde{U} = A^a \,\partial/\partial u^a + B^\mu_a \,\partial/\partial \rho^\mu_a$. Hence we obtain for the boundary term,

$$\int_{\partial M} (\chi \circ i)^* \left(i_{\tilde{U}}\Theta_H\right) = \int_{\partial M} (\chi \circ i)^* \rho^\mu_a \, A^a \,\mathrm{vol}_\mu = \int_{\partial M} i^*(P^\mu_a \, A^a \,\mathrm{vol}_\mu) \tag{2.2.6}$$

for $\chi = (\Phi, P)$.

We will assume now and in what follows, that there exists a collar $U_\epsilon \subset M$ around the boundary such that $U_\epsilon \cong (-\epsilon, 0] \times \partial M$. We choose local coordinates (x^0, x^k) on the collar such that $x^0 = t \in (-\epsilon, 0]$ and x^k, $k = 1, \ldots, d$, define local coordinates for ∂M. In these coordinates $\mathrm{vol}_{U_\epsilon} = \mathrm{d}t \wedge \mathrm{vol}_{\partial M}$ with $\mathrm{vol}_{\partial M}$ a volume form on ∂M. Under these assumptions the r.h.s. of eq. (2.2.6) becomes,

$$\int_{\partial M} i^*(P^\mu_a \, A^a \,\mathrm{vol}_\mu) = \int_{\partial M} p_a \, A^a \,\mathrm{vol}_{\partial M} \,, \tag{2.2.7}$$

where $p_a = P^0_a \circ i$ is the restriction to ∂M of the zeroth component of the momenta field P^μ_a in a local coordinate chart of the collar U_ϵ with the previous form.

Consider the space of fields at the boundary obtained by restricting the zeroth component of sections χ to ∂M, that is, fields of the form (see Figure 2.1):

$$\varphi^a = \Phi^a \circ i \,, \qquad p_a = P^0_a \circ i \,.$$

Notice that the fields φ^a are nothing but sections of the bundle i^*E, the pull-back along i of the bundle E, while the space of fields p_a can be thought of as 1-semibasic d-forms on $i^*E \to \partial M$. This statement is made precise in the following:

Lemma 2.2.2. *Suppose we are given a collar around* ∂M, $U_\epsilon \cong (-\epsilon, 0] \times \partial M$, *and a volume form* $\mathrm{vol}_{\partial M}$ *on* ∂M *such that* $\mathrm{vol}_{U_\epsilon} = \mathrm{d}t \wedge \mathrm{vol}_{\partial M}$ *with* t *the normal coordinate in* U_ϵ. *Then the pull-back bundle* $i^*(P(E))$ *is a bundle over the pull-back bundle* i^*E *and decomposes as* $i^*P(E) \cong (\bigwedge^{m-1}(T^*\partial M) \otimes V^*(i^*E)) \oplus (\bigwedge^{m-2}(T^*\partial M) \otimes V^*(i^*E))$. *If* $i^*\zeta \in i^*P(E)$, *we will denote by* p *and* β *the components of the above decomposition, that is,* $i^*\zeta = p + \beta$.

Proof. By definition of pull-back, the fiber over a point $x \in \partial M$ of the bundle i^*E, consists of all elements in E_x. The pull-back bundle $i^*P(E)$ is a bundle over i^*E, the fiber over $(x, u) \in i^*E$ is $T_xM \otimes VE^*_u$. Using the volume form vol_M, we identify this fiber with $\bigwedge^{m-1}(T_xM) \otimes VE_u^*$ by contracting elements $\varpi = v \otimes \alpha \in T_xM \otimes VE_u^*$ with $\mathrm{vol}_M(x)$. The collar neighborhood $i_\epsilon \colon U_\epsilon \to M$ introduces a normal coordinate $t \in (-\epsilon, 0]$ such that $\mathrm{vol}_{U_\epsilon} = i_\epsilon^* \mathrm{vol}_M = \mathrm{d}t \wedge \mathrm{vol}_{\partial M}$. Notice that such factorization depends on the choice of the collar. An element chosen in the class of semibasic one-forms on U_ϵ representing the element ϖ at (x, u) has the form $\alpha = \rho_a^0 \mathrm{d}u^a \wedge \mathrm{vol}_{\partial M} + \rho_a^k \mathrm{d}u^a \wedge \mathrm{d}t \wedge i_{\partial/\partial x^k} \mathrm{vol}_{\partial M} + \rho_0 \mathrm{d}t \wedge \mathrm{vol}_{\partial M}$. Then $p = i_\epsilon^* \varpi$, and in local coordinates $p_a = \rho_a^0$. Finally, define $\beta = i_\epsilon^*(i_{\partial/\partial t}\alpha)$. Thus in local coordinates $\beta_a^k = \rho_a^k$, $k = 1, \ldots, d$. □

If we denote by $\mathcal{F}_{\partial M}$ the space of configurations of the theory, φ^a, i.e. $\mathcal{F}_{\partial M} = \Gamma(i^*E)$, then the space of momenta of the theory p_a can be identified with the space of sections of the bundle $\bigwedge_1^m(i^*E) \to i^*E$, according to Lemma 2.2.2. Therefore the space of fields (φ^a, p_a) can be identified with the cotangent bundle $T^*\mathcal{F}_{\partial M}$ over $\mathcal{F}_{\partial M}$ in a natural way, i.e. each field p_a can be considered as the covector at φ^a that maps the tangent vector $\delta\varphi$ to $\mathcal{F}_{\partial M}$ at φ into the number $\langle p, \delta\varphi \rangle$ given by,

$$\langle p, \delta\varphi \rangle = \int_{\partial M} p_a(x) \delta\varphi^a(x) \, \mathrm{vol}_{\partial M} \,. \tag{2.2.8}$$

Notice that the tangent vector $\delta\varphi$ at φ is a vertical vector field on i^*E along φ, and the section p is a 1-semibasic m-form on i^*E (Lemma 2.2.2). Hence the contraction of p with $\delta\varphi$ is an $(m-1)$-form along φ, and its pull-back $\varphi^*\langle p, \delta\varphi \rangle$ along φ is an $(m-1)$-form on ∂M whose integral defines the pairing above, Eq. (4.2.9).

Viewing the cotangent bundle $T^*\mathcal{F}_{\partial M}$ as double sections (φ, p) of the bundle $\bigwedge_1^m(i^*E) \to i^*E \to \partial M$ described by Lemma 2.2.2, the canonical 1-form α on $T^*\mathcal{F}_{\partial M}$ can be expressed as,

$$\alpha_{(\varphi,p)}(U) = \int_{\partial M} p_a(x) \delta\varphi^a(x) \, \mathrm{vol}_{\partial M} \tag{2.2.9}$$

where U is a tangent vector to $T^*\mathcal{F}_{\partial M}$ at (φ, p), that is, a vector field on the space of 1-semibasic forms on i^*E along the section (φ^a, p_a) and therefore of the form $U = \delta\varphi^a \, \partial/\partial u^a + \delta p_a \, \partial/\partial \rho_a$.

Finally, notice that the pull-back to the boundary map i^*, defines a natural map from the space of fields in the bulk, $\mathcal{F}_{P(E)}$, into the phase space of fields at the boundary $T^*\mathcal{F}_{\partial M}$. Such map will be denoted by Π in what follows, that is,

$$\Pi \colon \mathcal{F}_{P(E)} \to T^*\mathcal{F}_{\partial M} \,, \qquad \Pi(\Phi, P) = (\varphi, p), \quad \varphi = \Phi \circ i, \, p_a = P_a^0 \circ i \,.$$

With the notations above, by comparing the expression for the boundary term given by eq. (2.2.7), and the expression for the canonical 1-form α, eq. (4.2.10), we obtain,

$$\int_{\partial M} (\chi \circ i)^* \left(i_{\tilde{U}} \Theta_H \right) = (\Pi^* \alpha)_\chi(U) \,.$$

In words, the boundary term in eq. (4.2.7) is just the pull-back of the canonical 1-form α at the boundary along the projection map Π.

In what follows it will be customary to use the variational derivative notation when dealing with spaces of fields. For instance, if $F(\varphi, p)$ is a differentiable function defined on $T^*\mathcal{F}_{\partial M}$ we will denote by $\delta F/\delta\varphi^a$ and $\delta F/\delta p_a$ functions (provided that they exist) such that

$$\mathrm{d}F_{(\varphi,p)}(\delta\varphi^a, \delta p_a) = \int_{\partial M} \left(\frac{\delta F}{\delta \varphi^a} \delta\varphi^a + \frac{\delta F}{\delta p_a} \delta p_a \right) \mathrm{vol}_{\partial M} \,, \tag{2.2.10}$$

with $U = (\delta\varphi^a, \delta p_a)$ a tangent vector at (φ, p). We also use an extended Einstein's summation convention such that integral signs will be omitted when dealing with variational differentials. For instance,

$$\delta F = \frac{\delta F}{\delta \varphi^a} \delta\varphi^a + \frac{\delta F}{\delta p_a} \delta p_a \,, \tag{2.2.11}$$

will be the notation that will replace $\mathrm{d}F$ in Eq. (2.2.10). Also in this vein we will write,

$$\alpha = p_a \, \delta\varphi^a \,,$$

and the canonical symplectic structure $\omega_{\partial M} = -\mathrm{d}\alpha$ on $T^*\mathcal{F}_{\partial M}$ will be written as,

$$\omega_{\partial M} = \delta\varphi^a \wedge \delta p_a \,,$$

by which we mean

$$\omega_{\partial M}((\delta_1\varphi^a, \delta_1 p_a), (\delta_2\varphi^a, \delta_2 p_a)) = \int_{\partial M} (\delta_1\varphi^a(x)\delta_2 p_a(x) - \delta_2\varphi^a(x)\delta_1 p_a(x)) \, \mathrm{vol}_{\partial M} \,,$$

where $(\delta_1\varphi^a, \delta_1 p_a), (\delta_2\varphi^a, \delta_2 p_a)$ are two tangent vectors at (φ, p).

Euler-Lagrange's equations and Hamilton's equations

We now examine the contribution from the first term in $\mathrm{d}S$, eq.(2.2.4). Notice that such a term can be thought of as a 1-form on the space of fields on the bulk, $\mathcal{F}_{P(E)}$. We will call it the Euler-Lagrange 1-form and denote it by EL. Thus with the notation of Lemma 2.2.1,

$$\mathrm{EL}_\chi(U) = \int_M \chi^* \left(i_{\tilde{U}} \mathrm{d}\Theta_H \right) \,.$$

A double section $\chi = (\Phi, P)$ of $P(E) \to E \to M$ will be said to satisfy the Euler-Lagrange equations determined by the first-order Hamiltonian field theory defined by H, if $\mathrm{EL}_\chi = 0$, that is, if χ is a zero of the Euler-Lagrange 1-form EL on $\mathcal{F}_{P(E)}$. Notice that this is equivalent to

$$\chi^*(i_{\tilde{U}} d\Theta_H) = 0\,, \tag{2.2.12}$$

for all vector fields $\tilde{U}$ on a tubular neighborhood of the image of χ in $P(E)$. The set of all such solutions of Euler-Lagrange equations will be denoted by $\mathcal{EL}_M$ or just $\mathcal{EL}$ for short.

In local coordinates x^μ such that the volume element takes the form $\mathrm{vol_M} = dx^0 \wedge \cdots \wedge dx^d$, and for natural local coordinates (x^μ, u^a, ρ^μ_a) on $P(E)$, using eqn (4.2.3) we have,

$$\begin{aligned} i_{\partial/\partial\rho^\mu_a} \mathrm{d}\Theta_H &= -\frac{\partial H}{\partial \rho^\mu_a} d^m x + \mathrm{d}u^a \wedge \mathrm{d}^{m-1} x_\mu \\ i_{\partial/\partial u^a} \mathrm{d}\Theta_H &= -\frac{\partial H}{\partial u^a} d^m x - d\rho^\mu_a \wedge \mathrm{d}^{m-1} x_\mu . \end{aligned}$$

Applying Eq. (3.2.7) to these last two equations we obtain the Hamilton equations for the field in the bulk:

$$\frac{\partial u^a}{\partial x^\mu} = \frac{\partial H}{\partial \rho^\mu_a}\,; \qquad \frac{\partial \rho^\mu_a}{\partial x^\mu} = -\frac{\partial H}{\partial u^a}\,, \tag{2.2.13}$$

where a summation on μ is understood in the last equation. Note that had we not changed to normal coordinates on M, the volume form would not have the above simple form and therefore there would be related extra terms in the previous expressions and in Eqs. (2.2.13).

These Hamilton equations are often described as being covariant. This term must be treated with caution in this context. Clearly, by writing the equations in the invariant form $\chi^*(i_{\tilde{U}}\mathrm{d}\Theta_H) = 0$ we have shown that they are in a sense covariant. However, it is important to remember that in general the function H is only locally defined. In other words, there is in general no true 'Hamiltonian function', and the local representative H transforms in a non-trivial way under coordinate transformations. When $M(E)$ is a trivial bundle over $P(E)$, so that there is a predetermined global section, then the Hamiltonian section may be represented by a global function and no problem arises. This occurs for instance when E is trivial over M. In general however, there is no preferred section of $M(E)$ over $P(E)$ to relate the Hamiltonian section to, and in order to write the Hamilton equations in manifestly covariant form one must introduce a connection.

The fundamental formula

Thus we have obtained the formula that relates the differential of the action with a 1-form on a space of fields on the bulk manifold and a 1-form on a space of fields at the boundary:

$$\mathrm{d}S_\chi = \mathrm{EL}_\chi + \Pi^*\alpha_\chi\,, \qquad \chi \in \mathcal{F}_{P(E)}\,. \tag{2.2.14}$$

In the previous equation EL_χ denotes the Euler-Lagrange 1-form on the space of fields $\chi = (\Phi, P)$ with local expression (using variational derivatives):

$$\mathrm{EL}_\chi = \left(\frac{\partial \Phi^a}{\partial x^\mu} - \frac{\partial H}{\partial P^\mu_a}\right)\delta P^\mu_a - \left(\frac{\partial P^\mu_a}{\partial x^\mu} + \frac{\partial H}{\partial \Phi^a}\right)\delta\Phi^a\,, \tag{2.2.15}$$

or, more explicitly:

$$\mathrm{EL}_\chi(\delta\Phi, \delta P) = \int_M \left[\left(\frac{\partial \Phi^a}{\partial x^\mu} - \frac{\partial H}{\partial P^\mu_a}\right)\delta P^\mu_a - \left(\frac{\partial P^\mu_a}{\partial x^\mu} + \frac{\partial H}{\partial \Phi^a}\right)\delta\Phi^a\right] \mathrm{vol}_M\,.$$

In what follows we will denote by $(P(E), \Theta_H)$ the covariant Hamiltonian field theory with bundle structure $\pi\colon E \to M$ defined over the m-dimensional manifold with boundary M, Hamiltonian function H and canonical m-form Θ_H.

We will say that the action S is regular if the set of solutions of Euler-Lagrange equations $\mathcal{EL}_M$ is a submanifold of $\mathcal{F}_{P(E)}$. Thus we will also assume when needed that the action S is regular (even though this must be proved case by case) and that the projection $\Pi(\mathcal{EL})$ to the space of fields at the boundary $T^*\mathcal{F}_{\partial M}$ is a smooth manifold too.

This has the immediate implication that the projection of $\mathcal{EL}$ to the boundary ∂M is an isotropic submanifold:

Proposition 2.2.3. *Let $(P(E), \Theta_H)$ be a first order Hamiltonian field theory on the manifold M with boundary, with regular action S and such that $\Pi(\mathcal{EL}) \subset T^*\mathcal{F}_{\partial M}$ is a smooth submanifold. Then $\Pi(\mathcal{EL}) \subset T^*\mathcal{F}_{\partial M}$ is an isotropic submanifold.*

Proof. Along the submanifold $\mathcal{EL} \subset T^*\mathcal{F}_{\partial M}$ we have,

$$\mathrm{d}S\mid_{\mathcal{EL}} = \Pi^*\alpha \mid_{\mathcal{EL}}\,.$$

Therefore $\mathrm{d}(\Pi^*\alpha) = \mathrm{d}^2 S = 0$ along $\mathcal{EL}$, and $\mathrm{d}(\Pi^*\alpha) = \Pi^*\mathrm{d}\alpha$ along $\mathcal{EL}$. But Π being a submersion then implies that $\mathrm{d}\alpha = 0$ along $\Pi(\mathcal{EL})$. □

In many cases $\Pi(\mathcal{EL})$ is not only isotropic but Lagrangian. We will come back to the analysis of this in later sections.

Symmetries and the algebra of currents

Without attempting a comprehensive description of the theory of symmetry for covariant Hamiltonian field theories, we will describe some basic elements needed in what follows (see details in [**Ca91**]). Recall from Sect. 2.2, $(M(E), \Omega)$ is a multisymplectic manifold with $(m+1)$-dimensional multisymplectic form $\Omega = d\Theta$, where dim $M = m$. Canonical transformations in the multisymplectic framework for Hamiltonian field theories are diffeomorphisms $\Psi \colon M(E) \to M(E)$ such that $\Psi^*\Omega = \Omega$. Notice that if Ψ is a diffeomorphism such that $\Psi^*\Theta = \Theta$, then Ψ is a canonical transformation.

A distinguished class of canonical transformations is provided by those transformations Ψ induced by diffeomorphisms $\psi_E \colon E \to E$, i.e. $\Psi(\varpi) = (\psi_E^{-1})^*\varpi$, $\varpi \in M(E)$. If the diffeomorphism ψ_E is a bundle isomorphism, there will exist another diffeomorphism $\psi_M \colon M \to M$ such that $\pi \circ \psi_E = \psi_M \circ \pi$. Under such circumstances it is clear that the induced map $(\psi_E^{-1})^* \colon \bigwedge^m(E) \to \bigwedge^m(E)$ preserves both $\bigwedge_1^m(E)$ and $\bigwedge_0^1(E)$, thus the map $\Psi = (\psi_E^{-1})^* \colon M(E) \to M(E)$ induces a natural map $\psi_* \colon P(E) \to P(E)$ such that $\mu \circ (\psi_E^{-1})^* = \psi_* \circ \mu$. Canonical transformations induced from bundle isomorphisms will be called covariant canonical maps.

Given a one-parameter group of canonical transformations Ψ_t, its infinitesimal generator U satisfies

$$\mathcal{L}_U \Omega = 0\,.$$

Vector fields U on $M(E)$ satisfying the previous condition will be called (locally) Hamiltonian vector fields. Locally Hamiltonian vector fields U for which there exists an $(m-1)$-form f on $M(E)$ (we are assuming that Ω is a $(m+1)$-form) such that

$$i_U \Omega = \mathrm{d}f\ ,$$

will be called, in analogy with mechanical systems, (globally) Hamiltonian vector fields. The class $\mathbf{f} = \{f + \beta \mid \mathrm{d}\beta = 0,\, \beta \in \Omega^{m-1}(M(E))\}$ determined by the $(m-1)$-form f is called the Hamiltonian form of the vector field U and such a vector field will be denoted as $U_{\mathbf{f}}$.

The Lie bracket of vector fields induces a Lie algebra structure on the space of Hamiltonian vector fields that we denote as $\mathrm{Ham}(M(E), \Omega)$. Notice that Hamiltonian vector fields whose flows Ψ_t are defined by covariant canonical transformations are globally Hamiltonian because $\mathcal{L}_U\Theta = 0$, and therefore $i_U \mathrm{d}\Theta = -\mathrm{d} i_U \Theta$. The Hamiltonian form associated to U is the class containing the $(m-1)$-form $f = i_U\Theta$.

The space of Hamiltonian forms, denoted in what follows by $\mathcal{H}(M(E))$, carries a canonical bracket defined by

$$\{\mathbf{f}, \mathbf{f}'\} = i_{U_f} i_{U_{f'}} \Omega + Z^{m-1}(M(E))\,,$$

where $Z^{m-1}(M(E))$ denotes the space of closed $(m-1)$-forms on $M(E)$. The various spaces introduced so far are related by the short exact sequence:

$$0 \to H^{m-1}(M(E)) \to \mathcal{H}(M(E)) \to \mathrm{Ham}(M(E), \Omega) \to 0\,.$$

Let G be a Lie group acting on E by bundle isomorphisms and $\psi_g\colon E \to E$, the diffeomorphism defined by the group element $g \in G$. This action induces an action on the multisymplectic manifold $(M(E), \Omega)$ by canonical transformations. Given an element $\xi \in \mathfrak{g}$, where now and in what follows $\mathfrak{g}$ denotes the Lie algebra of the Lie group G, we will denote by $\xi_{M(E)}$ and ξ_E the corresponding vector fields defined by the previous actions on $M(E)$ and E, respectively. The vector fields $\xi_{M(E)}$ are Hamiltonian with Hamiltonian forms $\mathbf{J}_\xi$, that is,

$$i_{\xi_{M(E)}}\Omega = \mathrm{d}J_\xi\,, \tag{2.2.16}$$

with $J_\xi = i_{\xi_{M(E)}}\Theta$. It is easy to check that

$$\{\mathbf{J}_\xi, \mathbf{J}_\zeta\} = \mathbf{J}_{[\xi,\zeta]} + c(\xi,\zeta)$$

where $c(\xi,\zeta)$ is a cohomology class of order $m-1$. The bilinear map $c(\cdot,\cdot)$ defines an element in $H^2(\mathfrak{g}, H^{m-1}(M(E)))$ (see [**Ca91**]). In what follows we will assume that the group action is such that the cohomology class c vanishes. Such actions are called strongly Hamiltonian (or just Hamiltonian, for short).

So far our discussion has not involved a particular theory, that is, a Hamiltonian H. Let $(P(E), \Theta_H)$ be a covariant Hamiltonian field theory and G a Lie group acting on $\mathcal{F}_{P(E)}$. Among all possible actions of groups on the space of double sections $\mathcal{F}_{P(E)}$ those that arise from an action on $P(E)$ by covariant canonical transformations are of particular significance. Let G be a group acting on E by bundle isomorphisms. Let $\psi_*(g)$ denote the covariant diffeomorphism on $P(E)$ defined by the group element g. Then the transformed section χ^g is given by $\chi^g(x) = \psi_*(g)(\chi(\psi_M(g^{-1})x))$ where $\psi_M(g)$ is the diffeomorphism on M defined by the action of the group. We will often consider only bundle automorphisms over the identity, in which case $\chi^g(x) = \psi_*(g)(\chi(x))$. Such bundle isomorphisms will be called gauge transformations and the corresponding group of all gauge transformations will be called the gauge group of the theory and denoted by $\mathcal{G}(E)$, or just $\mathcal{G}$ for short, in what follows.

The group G will be said to be a symmetry of the theory if $S(\chi^g) = S(\chi)$ for all $\chi \in \mathcal{F}_{P(E)}$, $g \in G$. Notice that, in general, an action of G on $M(E)$ by bundle isomorphisms will leave Θ invariant and will pass to the quotient space $P(E)$, however it doesn't have to preserve Θ_H. Hence, it is obvious that a group G acting on $P(E)$ by covariant transformations will be a symmetry group of the Hamiltonian field theory defined by H iff $g^*\Theta_H = \Theta_H + \beta_g$, where now for ease of notation, we indicate the diffeomorphism $\psi_*(g)$ simply by g, and β_g is a closed m-form on M. In what follows we will assume that the group G acts on E and its induced action on $P(E)$ preserves the m-form Θ_H, that is $\beta_g = 0$ for all g.

Because the action of the group G preserves the m-form Θ_H, the group acts by canonical transformation on the manifold $(P(E), d\Theta_H)$ with Hamiltonian forms $\mathbf{J}_\xi$ given by (the equivalence class determined by the m-forms):

$$J_\xi = i_{\xi_{P(E)}}\Theta_H \,.$$

Theorem 2.2.4 (Noether's theorem)**.** *Let G be a Lie group acting on E which is a symmetry group of the Hamiltonian field theory $(P(E), \Theta_H)$ and such that it preserves the m-form Θ_H. If $\chi \in \mathcal{EL}$ is a solution of the Euler-Lagrange equations of the theory, then the $(m-1)$-form $\chi^* J_\xi$ on M is closed.*

Proof. Because χ is a solution of Euler-Lagrange equations, recalling eq. (3.2.7) we have

$$0 = \chi^*(i_{\xi_{P(E)}}\Omega_H) = \chi^* \mathrm{d}J_\xi = \mathrm{d}(\chi^* J_\xi)\,.$$

□

The de Rham cohomology classes determined by the closed $(m-1)$-forms $\chi^* J_\xi$ on M will be called currents and denoted by $\mathbf{J}_\xi[\chi]$. Using the Poisson bracket $\{\cdot,\cdot\}$ defined on the space of Hamiltonian forms $\mathcal{H}(P(E))$ we define a Lie bracket in the space of currents $\mathbf{J}_\xi[\chi] \in H^{m-1}(M)$ by

$$\{\mathbf{J}_\xi[\chi], \mathbf{J}_\zeta[\chi]\} = \chi^*\{\mathbf{J}_\xi, \mathbf{J}_\zeta\} = \mathbf{J}_{[\xi,\zeta]}[\chi]\,.$$

By Stokes' theorem, the $(m-1)$-forms $i^*(\mathbf{J}_\xi[\chi])$ on ∂M satisfy

$$\int_{\partial M} i^* \mathbf{J}_\xi[\chi] = 0\,. \tag{2.2.17}$$

We will refer to the quantity $Q\colon \mathcal{F}_{P(E)} \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$, defined by

$$\langle Q(\chi), \xi\rangle = \int_{\partial M} i^* \mathbf{J}_\xi[\chi]\,, \qquad \forall \xi \in \mathfrak{g}\,, \tag{2.2.18}$$

as the charge defined by the symmetry group. Notice that the pairing $\langle\cdot,\cdot\rangle$ on the left hand side of Eq. (2.2.18) is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. As a consequence of Noether's theorem we get $Q\mid_{\mathcal{EL}} = 0$.

The moment map at the boundary

Suppose that there is an action of a Lie group G on the bundle E that leaves invariant the restriction of the bundle E to the boundary, that is, the transformations Ψ_g defined by the elements of the group $g \in G$ restrict to the bundles $i^*(P(E))$ and $E_{\partial M} := i^*E$ (see Figure 2.1). We will denote such restriction as $\Psi_g \mid_{\partial M} = g_{\partial M}$.

Two elements $g, g' \in G$ will induce the same transformation on the bundle $E_{\partial M}$ if there exists an element h such that $g' = gh$ and $h_{\partial M} = \mathrm{id}_{E_{\partial M}}$. If we consider now the group $\mathcal{G}$ of all gauge transformations, then the set of group elements that restrict to the identity at the boundary is a normal subgroup of $\mathcal{G}$ which we will denote by $\mathcal{G}_0$. The induced action of $\mathcal{G}$ at the boundary is the action of the group $\mathcal{G}_{\partial M} = \mathcal{G}/\mathcal{G}_0$ which is the group of gauge transformations of the bundle $E_{\partial M} = i^*E$.

In particular the group $\mathcal{G}$ induces an action on $\mathcal{F}_{\partial M}$ by

$$g \cdot \varphi(x) = \psi_E(g)(\Phi(g^{-1}x)) = g_{\partial M}(\varphi(g^{-1}x)) \,, \qquad \forall x \in \partial M,\, g \in \mathcal{G} \,,$$

and similarly for the momenta field p.

Proposition 2.2.5. *Let $\mathcal{G}_{\partial M}$ denote the gauge group at the boundary, that is, the group whose elements are the transformations induced at the boundary by gauge transformations of E. Then the action of $\mathcal{G}_{\partial M}$ in the space of fields at the boundary is strongly Hamiltonian with moment map*

$$\mathcal{J} \colon T^*\mathcal{F}_{\partial M} \to \mathfrak{g}^*_{\partial M}$$

given by,

$$\langle \mathcal{J}(\varphi, p), \xi \rangle = \langle Q(\chi), \xi \rangle \qquad \forall \xi \in \mathfrak{g}_{\partial M} \,,$$

where $\Pi(\chi) = (\varphi, p)$, and $\mathfrak{g}_{\partial M}$, $\mathfrak{g}^*_{\partial M}$ denote respectively the Lie algebra and the dual of the Lie algebra of the group $\mathcal{G}_{\partial M}$. In other words, the projection map Π composed with the moment map at the boundary $\mathcal{J}$ is the charge Q of the symmetry group.

Proof. The action of the group $\mathcal{G}_{\partial M}$ on $T^*\mathcal{F}_{\partial M}$ is by cotangent liftings, thus its moment map $\mathcal{J}$ takes the particularly simple form,

$$\langle \mathcal{J}(\varphi, p), \xi \rangle = \langle p, \xi_{\mathcal{F}_{\partial M}}(\varphi) \rangle \,,$$

where $\xi_{\mathcal{F}_{\partial M}}$ denotes the infinitesimal generator defined by the action of $\mathcal{G}_{\partial M}$ on $\mathcal{F}_{\partial M}$. Such generator, because the action is by gauge transformations, i.e. bundle isomorphisms over the identity, has the explicit expression:

$$\xi_{\mathcal{F}_{\partial M}} = \xi \circ \varphi \frac{\delta}{\delta \varphi} \,,$$

where ξ is the infinitesimal generator of the action of $\mathcal{G}_{\partial M}$ on $E_{\partial M}$. Notice that the Lie algebra $\mathfrak{g}$ of the group of gauge transformations is precisely the algebra of vertical vector fields on E (similarly for $\mathfrak{g}_{\partial M}$ and $E_{\partial M}$). Hence $\xi \in \mathfrak{g}_{\partial M}$ is just a vertical vector field and the infinitesimal generator $\xi_{\mathcal{F}_{\partial M}}$ at φ, which is just a tangent vector to $\mathcal{F}_{\partial M}$ at φ, is the vector field along φ given by the composition $\xi \circ \varphi$.

However because the action of $\mathcal{G}_{\partial M}$ on $E_{\partial M}$ is exactly the action of $\mathcal{G}$ on $E\mid_{\partial M}$, ξ can be considered as an element on $\mathfrak{g}$ and (recalling the definition of the pairing in $T^*\mathcal{F}_{\partial M}$, eq. (4.2.9), and the discussion at the end of Sect. 4.2 on the conventions with variational derivatives) we get,

$$\langle p, \xi_{\mathcal{F}_{\partial M}}(\varphi)\rangle = \int_{\partial M} p_a\, \xi^a(\varphi(x))\, \mathrm{vol}_{\partial M} \tag{2.2.19}$$

$$= \int_{\partial M} i^* \left(\chi^* \left(i_{\xi_{P(E)}} \Theta_H\right)\right) = \int_{\partial M} i^* \mathbf{J}_\xi[\chi] = \langle Q(\chi), \xi\rangle\,, \tag{2.2.20}$$

with $\Pi(\chi) = (\varphi, p)$. In the previous computations we have used that $i_{\xi_{P(E)}}\Theta_H = \rho^\mu_a\, \xi^a(x, \rho^a)\, \mathrm{vol}_\mu$, therefore $\chi^*\left(i_{\xi_{P(E)}}\Theta_H\right) = P_a(\Phi(x))^\mu\, \xi^a(\Phi(x))\, \mathrm{vol}_\mu$ and thus

$$i^* \left(\chi^* \left(i_{\xi_{P(E)}} \Theta_H\right)\right) = p^a(\varphi(x))\, \xi^a(\varphi(x))\, \mathrm{vol}_{\partial M}\,.$$

Notice the particularly simple form that the currents take in this situation $J_\xi[\chi] = p_a \xi^a(\varphi)$. □

Thus Noether's Theorem, which implies that $Q|_{\mathcal{EL}} = 0$, together with Prop. 2.2.5, imply that for any $\chi = (\Phi, P) \in \mathcal{EL}$, $(\varphi, p) \in \mathcal{J}^{-1}(\mathbf{0})$ with $(\varphi, p) = \Pi(\chi)$.

The main, and arguably the most significant example of symmetries is provided by theories such that the symmetry group is the full group of automorphisms of the bundle $\pi\colon E \to M$. We are interested in particular in the normal subgroup of bundle automorphisms over the identity map, i.e. diffeomorphisms $\psi_E\colon E \to E$ preserving the structure of the bundle and such that $\pi \circ \psi_E = \psi_E$. As indicated above, such group will be called the gauge group of the theory (or the group of gauge transformations of the theory) and we will denote it by $\mathcal{G}(E)$ (or just $\mathcal{G}$ if there is no risk of confusion). In such case eqs. (2.2.19)-(2.2.20) imply the following:

Corollary 2.2.6. *With the above notations, for the group of gauge transformations we obtain,*

$$Q = \Pi^* \alpha\,,$$

where α is the canonical 1-form on $T^*\mathcal{F}_{\partial M}$, in the sense that for any $\xi \in \mathfrak{g}$ and $\chi \in J^1\mathcal{F}^*$,

$$\langle Q(\chi), \xi\rangle = \alpha_{\Pi(\chi)}(\xi_{\mathcal{F}_{\partial M}}) = \Pi^*\alpha_\chi(\xi_{P(E)})\,.$$

Boundary conditions

Because of the boundary term $\Pi^*\alpha$ arising in the computation of the critical points of the action S, the propagator of the corresponding quantum theory will be affected by such contributions and the theory could fail to be unitary [**As15**]. One way to avoid this problem is by selecting a subspace of the space of fields $\mathcal{F}_{P(E)}$ such that $\Pi^*\alpha$ will vanish identically when restricted to it (see for instance an analysis of this situation in Quantum Mechanics in [**As05**].)

Moreover, we would like to choose a maximal subspace with this property. Then these two requirements will amount to choosing boundary conditions determined by a maximal submanifold $\mathcal{L} \subset T^*\mathcal{F}_{\partial M}$ such that $\alpha \mid_{\mathcal{L}} = 0$, that is, $\mathcal{L}$ is a special Lagrangian submanifold of the cotangent bundle $T^*\mathcal{F}_{\partial M}$.

In general we will consider not just a single boundary condition but a family of them defining a Lagrangian fibration of $T^*\mathcal{F}_{\partial M}$. An example of such a choice is the Lagrangian fibration $\mathcal{L}$ corresponding to the vertical fibration of $T^*\mathcal{F}_{\partial M}$. For $\varphi \in \mathcal{F}_{\partial M}$, the subspace of fields defined by the leaf $\mathcal{L}_\varphi$, $\varphi \in \mathcal{F}_{\partial M}$ is just the subspace of fields $\chi = (\Phi, P)$ such that $\Phi \mid_{\partial M} = \varphi$.

Another argument justifying the use of special Lagrangian submanifolds of $T^*\mathcal{F}_{\partial M}$ as boundary conditions, relies just on the structure of the classical theory and its symmetries and not on its eventual quantization. Recall from Cor. 2.2.6, that if a theory $(P(E), \Theta_H)$ has the group of gauge transformations $\mathcal{G}$ of the bundle E as a symmetry group, then $Q = \Pi^*\alpha$. Therefore the admissible fields of the theory - not necessarily solutions of Euler-Lagrange equations - are those such that the charge Q is preserved along the boundary, that is, those that lie on a special Lagrangian submanifold $\mathcal{L} \subset T^*\mathcal{F}_{\partial M}$.

We will say that a (classical field) theory is Dirichlet if, for any $\varphi \in \mathcal{F}_{\partial M}$, there exists a unique solution $\chi = (\Phi, P)$ of the Euler-Lagrange equations, i.e. $\chi \in \mathcal{EL}$ such that $\Phi|_{\partial M} = \varphi$.

Theorem 2.2.7. *Let S be a regular action defined by a first order Hamiltonian field theory $(P(E), \Theta_H)$. If the theory is Dirichlet then $\Pi(\mathcal{EL})$ is a Lagrangian submanifold of $T^*\mathcal{F}_{\partial M}$.*

Proof. Recall the discussion in Sec. 2.2. If the action is regular, i.e. if the solutions of the Euler-Lagrange equations $\mathcal{EL}$ define a submanifold of $\mathcal{F}_{P(E)}$, then from the fundamental relation eq. (4.2.8) we get Prop. 2.2.3, that $\Pi(\mathcal{EL})$ is an isotropic submanifold of $T^*\mathcal{F}_{\partial M}$.

Let the functional W denote the composition $W = S \circ D$ where $S : \mathcal{F}_{P(E)} \to \mathbb{R}$ is the action of our theory defined by H, i.e. $S(\chi) = \int_M \chi^*\Theta_H$ and $D : \mathcal{F}_{\partial M} \to \mathcal{F}_{P(E)}$ is the map that assigns to any boundary data φ the unique solution (Φ, P) of the

Euler-Lagrange equations such that $\Phi|_{\partial M} = \varphi$. Thus

$$W(\varphi) = \int_M \chi^* \Theta_H = S(\chi)\,,$$

for any $\varphi \in \mathcal{F}_{\partial M}$. By Eq. (3.2.9) and since $D(\phi) = (\Phi, P) = \chi \in \mathcal{EL}$ it follows that $\mathrm{EL}_\chi = 0$. A simple computation then shows that

$$\mathrm{d}W(\varphi)(\delta\varphi) = p_a \delta\varphi^a\,,$$

where $p_a = P^0_a\ |_{\partial M}$. Thus the graph of the 1-form $\mathrm{d}W = \Pi(\mathcal{EL})$, i.e. W is a generating function for $\Pi(\mathcal{EL})$ and therefore $\Pi(\mathcal{EL})$ is a Lagrangian submanifold of $\mathcal{T}^*\mathcal{F}_{\partial M}$.

$\square$

The Dirichlet condition can be weakened and a corresponding proof for a natural extension of Theorem 2.2.7 can be provided. See our work on Hamiltonian dynamics [**Ib16**].

2.3 The presymplectic formalism at the boundary

The evolution picture near the boundary

We discuss in what follows the evolution picture of the system near the boundary. As discussed in Section 2.2, we assume that there exists a collar $U_\epsilon \cong (-\epsilon, 0] \times \partial M$ of the boundary ∂M with adapted coordinates $(t; x^1, \ldots, x^d)$, where $t = x^0$ and where x^i, $i = 1, \ldots, x^d$ define a local chart in ∂M. The normal coordinate t can be used as an evolution parameter in the collar. We assume again that the volume form in the collar is of the form $\mathrm{vol}_{U_\epsilon} = dt \wedge \mathrm{vol}_{\partial M}$.

If M happens to be a globally hyperbolic space-time $M \cong [t_0, t_1] \times \Sigma$ where Σ is a Cauchy surface, $[t_0, t_1] \subset \mathbb{R}$ denotes a finite interval in the real line, and the metric has the form $-dt^2 + g_{\partial M}$ where $g_{\partial M}$ is a fixed Riemannian metric on ∂M. Then t represents a time evolution parameter throughout the manifold and the volume element has the form $\mathrm{vol}_M = dt \wedge \mathrm{vol}_{\partial M}$. Here however, all we need to assume is that our manifold has a collar at the boundary as described above.

Restricting the action S of the theory to fields defined on U_ϵ, i.e. sections of the pull-back of the bundles E and $P(E)$ to U_ϵ, we obtain,

$$S_\epsilon(\chi) = \int_{U_\epsilon} \chi^* \Theta_H = \int_{-\epsilon}^{0} \mathrm{d}t \int_{\partial M} \mathrm{vol}_{\partial M} \left[P_a^0 \partial_0 \Phi^a + P_a^k \partial_k \Phi^a - H(\Phi^a, P_a^0, P_a^k) \right] . \tag{2.3.1}$$

Defining the fields at the boundary as discussed in Lemma 2.2.2,

$$\varphi^a = \Phi^a|_{\partial M} , \qquad p_a = P_a^0|_{\partial M} , \qquad \beta_a^k = P_a^k|_{\partial M} ,$$

we can rewrite (2.3.1) as

$$S_\epsilon(\chi) = \int_{-\epsilon}^{0} \mathrm{d}t \int_{\partial M} \mathrm{vol}_{\partial M} [p_a \dot{\varphi}^a + \beta_a^k \partial_k \varphi^a - H(\varphi^a, p_a, \beta_a^k)] .$$

Letting $\langle p, \dot{\varphi} \rangle = \int_{\partial M} p_a \dot{\varphi}^a \, \mathrm{vol}_{\partial M}$ denote, as in (4.2.9), the natural pairing and similarly,

$$\langle \beta, \mathrm{d}_{\partial M} \varphi \rangle = \int_{\partial M} \beta_a^k \partial_k \varphi^a \, \mathrm{vol}_{\partial M},$$

we can define a density function $\mathcal{L}$ as,

$$\mathcal{L}(\varphi, \dot{\varphi}, p, \dot{p}, \beta, \dot{\beta}) = \langle p, \dot{\varphi} \rangle + \langle \beta, \mathrm{d}_{\partial M} \varphi \rangle - \int_{\partial M} H(\varphi^a, p_a, \beta_a^k) \, \mathrm{vol}_{\partial M} , \tag{2.3.2}$$

and then

$$S_\epsilon(\chi) = \int_{-\epsilon}^{0} \mathrm{d}t \; \mathcal{L}(\varphi, \dot{\varphi}, p, \dot{p}, \beta, \dot{\beta})\,.$$

Notice again that because of the existence of the collar U_ϵ near the boundary and the assumed form of $\mathrm{vol}_{U_\epsilon}$, the elements in the bundle $i^*P(E)$ have the form $\rho_a^0 \mathrm{d}u^a \wedge \mathrm{vol}_{\partial M} + \rho_a^k \mathrm{d}u^a \wedge \mathrm{d}t \wedge i_{\partial/\partial x^k} \mathrm{vol}_{\partial M}$ and as discussed in Lemma 2.2.2, the bundle $i^*P(E)$ over i^*E is isomorphic to the product $\bigwedge_1^m(i^*E) \times B$, where $B = \bigwedge_1^{m-1}(i^*E)$. The space of double sections (φ, p) of the bundle $\bigwedge_1^m(i^*E) \to i^*E \to \partial M$ correspond to the cotangent bundle $T^*\mathcal{F}_{\partial M}$ and the double sections (φ, β) of the bundle $B \to i^*E \to \partial M$ correspond to a new space of fields at the boundary denoted by $\mathcal{B}$.

We will introduce now the total space of fields at the boundary $\mathcal{M}$ which is the space of double sections of the iterated bundle $i^*P(E) \to i^*E \to \partial M$. Following the previous remarks it is obvious that $\mathcal{M}$ has the form,

$$\mathcal{M} = \mathcal{T}^*\mathcal{F}_{\partial M} \times_{\mathcal{F}_{\partial M}} \mathcal{B} = \{(\varphi, p, \beta)\}\,.$$

Thus the density function $\mathcal{L}$, Eq. (2.3.2), is defined on the tangent space $T\mathcal{M}$ to the total space of fields at the boundary and could be called accordingly the boundary Lagrangian of the theory.

Consider the action $A = \int_{-\epsilon}^{0} \mathcal{L}\; \mathrm{d}t$ defined on the space of curves $\sigma\colon (-\epsilon, 0] \to \mathcal{M}$. If we compute $\mathrm{d}A$ we obtain a bulk term, that is, an integral on $(-\epsilon, 0]$ and a term evaluated at $\partial[-\epsilon, 0] = \{-\epsilon, 0\}$. Setting the bulk term equal to zero, we obtain the Euler-Lagrange equations of this system considered as a Lagrangian system on the space $\mathcal{M}$ with Lagrangian function $\mathcal{L}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \mathcal{L}}{\delta \dot{\varphi}^a} = \frac{\delta \mathcal{L}}{\delta \varphi^a}\,, \tag{2.3.3}$$

which becomes,

$$\dot{p}_a = -\partial_k \beta_a^k - \frac{\partial H}{\partial \varphi^a}\,. \tag{2.3.4}$$

Similarly, we get for the fields p and β:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \mathcal{L}}{\delta \dot{p}_a} = \frac{\delta \mathcal{L}}{\delta p_a}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \mathcal{L}}{\delta \dot{\beta}_a^k} = \frac{\delta \mathcal{L}}{\delta \beta_a^k}$$

that become respectively,

$$\dot{\varphi}_a = \frac{\partial H}{\partial p_a}\,, \tag{2.3.5}$$

and the constraint equation:

$$\mathrm{d}_{\partial M}\varphi - \frac{\partial H}{\partial \beta_a^k} = 0\,. \tag{2.3.6}$$

Thus Euler-Lagrange equations in a collar U_ϵ near the boundary, can be understood as a system of evolution equations on $T^*\mathcal{F}_{\partial M}$ depending on the variables β_a^k, together with a constraint condition on the extended space $\mathcal{M}$. The analysis of these equations, Eqs. (2.3.4), (2.3.5) and (2.3.6), is best understood in a presymplectic framework.

The presymplectic picture at the boundary and constraints analysis

We will introduce now a presymplectic framework on $\mathcal{M}$ that will be helpful in the study of Eqs. (2.3.4)-(2.3.6).

Let $\varrho : \mathcal{M} \longrightarrow \mathcal{T}^*\mathcal{F}_{\partial M}$ denote the canonical projection $\varrho(\varphi, p, \beta) = (\varphi, p)$. (See Figure 2.2.) Let Ω denote the pull-back of the canonical symplectic form $\omega_{\partial M}$ on $\mathcal{T}^*\mathcal{F}_{\partial M}$ to $\mathcal{M}$, i.e. let $\Omega = \varrho^*\omega_{\partial M}$. Note that the form Ω is closed but degenerate, that is, it defines a presymplectic structure on $\mathcal{M}$. An easy computation shows that the characteristic distribution $\mathcal{K}$ of Ω, is given by

$$\mathcal{K} = \ker\Omega = \mathrm{span}\left\{\frac{\delta}{\delta\beta_a^k}\right\}\,.$$

Let us consider the function defined on $\mathcal{M}$,

$$\mathcal{H}(\varphi, p, \beta) = -\langle \beta, d_{\partial M}\varphi\rangle + \int_{\partial M} H(\varphi^a, p_a, \beta_a^k)\,\mathrm{vol}_{\partial M}\,.$$

We will refer to $\mathcal{H}$ as the boundary Hamiltonian of the theory. Thus $\mathcal{L}$ can be rewritten as

$$\mathcal{L}(\varphi, \dot\varphi, p, \dot p, \beta, \dot\beta) = \langle p, \dot\varphi\rangle - \mathcal{H}(\varphi, p, \beta)$$

and

$$S_\epsilon(\varphi, p, \beta) = \int_{-\epsilon}^{0} [\langle p, \dot\varphi\rangle - \mathcal{H}(\varphi, p, \beta)]\mathrm{d}t\,, \tag{2.3.7}$$

and therefore the Euler-Lagrange equations (2.3.5), (2.3.4) and (2.3.6) can be written as

$$\dot\varphi^a = \frac{\delta\mathcal{H}}{\delta p_a}\,, \qquad \dot p_a = -\frac{\delta\mathcal{H}}{\delta\varphi^a}\,, \tag{2.3.8}$$

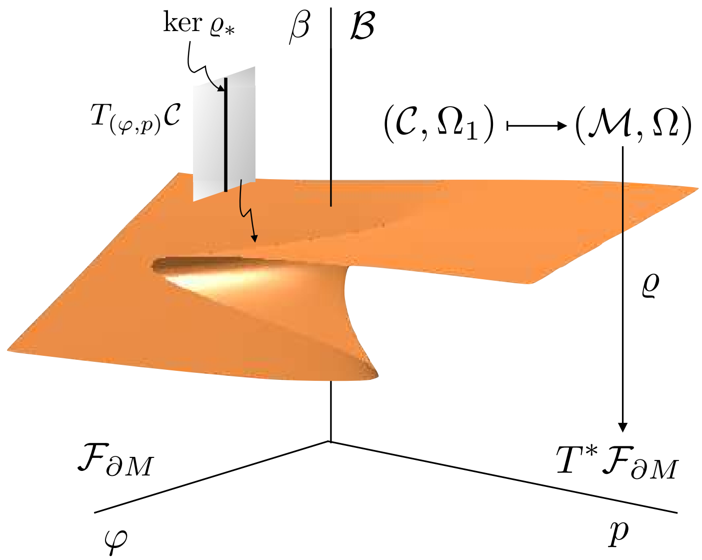

Figure 2.2: The space of fields at the boundary $\mathcal{M}$ and its relevant structures.

and

$$0 = \frac{\delta \mathcal{H}}{\delta \beta_a^k} \,. \tag{2.3.9}$$

Now it is easy to prove the following:

Theorem 2.3.1. *The solutions to the equations of motion defined by the Lagrangian $\mathcal{L}$ over a collar U_ϵ at the boundary, ϵ small enough, are in one-to-one correspondence with the integral curves of the presymplectic system $(\mathcal{M}, \Omega, \mathcal{H})$, i.e. with the integral curves of the vector field Γ on $\mathcal{M}$ satisfying*

$$i_\Gamma \Omega = \mathrm{d}\mathcal{H} \,. \tag{2.3.10}$$

Proof. Let $\Gamma = A^a \frac{\delta}{\delta \varphi^a} + B^a \frac{\delta}{\delta p^a} + C^a \frac{\delta}{\delta \beta_a^k}$ be a vector field on $\mathcal{M}$. (Notice that we are using an extension of the functional derivative notation introduced in Section 2.2 on the space of fields $\mathcal{M}$.) Then because $\Omega = \delta\varphi^a \wedge \delta p_a$, we get from $i_\Gamma\Omega = \mathrm{d}\mathcal{H}$ that,

$$A^a = \frac{\delta \mathcal{H}}{\delta p_a}, \qquad B^a = -\frac{\delta \mathcal{H}}{\delta \varphi^a}\,, \qquad 0 = \frac{\delta \mathcal{H}}{\delta \beta_a^k}\,.$$

Thus Γ satisfies Eq. (2.3.10) iff

$$\dot{\varphi}^a = \frac{\delta \mathcal{H}}{\delta p_a}, \qquad \dot{p}_a = -\frac{\delta \mathcal{H}}{\delta \varphi^a}\,, \quad \text{and} \quad 0 = \frac{\delta \mathcal{H}}{\delta \beta_a^k}\,.$$

□

Let us denote by $\mathcal{C}$ the submanifold of the space of fields $\mathcal{M} = T^*\mathcal{F}_{\partial M} \times \mathcal{B}$ defined by eq. (3.9). It is clear that the restriction of the solutions of the Euler-Lagrange equations on M to the boundary ∂M, are contained in $\mathcal{C}$; i.e. $\Pi(\mathcal{EL}) \subset \mathcal{C}$.

Given initial data φ, p and fixing β, existence and uniqueness theorems for initial value problems when applied to the initial value problem above, would show the existence of solutions for small intervals of time, i.e. in a collar near the boundary.

However, the constraint condition given by eq. (2.3.9), satisfied automatically by critical points of S_ϵ on U_ϵ, must be satisfied along the integral curves of the system, that is, for all t in the neighborhood U_ϵ of ∂M. This implies that consistency conditions on the evolution must be imposed. Such consistency conditions are just that the constraint condition eq. (2.3.9) is preserved under the evolution defined by eqs. (2.3.8). This is the typical situation that we will find in the analysis of dynamical problems with constraints and that we are going to summarily analyze in what follows.

The Presymplectic Constraints Algorithm (PCA)

Let i denote the canonical immersion $\mathcal{C} = \{(\varphi, p, \beta) | \frac{\delta \mathcal{H}}{\delta \beta} = 0\} \to \mathcal{M}$ and consider the pull-back of Ω to $\mathcal{C}$, i.e. $\Omega_1 = i^*\Omega$. Clearly then, $\ker \Omega_1 = \ker \varrho_* \cap T\mathcal{C}$. But $\mathcal{C}$ is defined as the zeros of the function $\delta\mathcal{H}/\delta\beta$. Therefore if $\delta^2\mathcal{H}/\delta^2\beta$ is nondegenerate (notice that the operator $\delta^2\mathcal{H}/\delta\beta^i_a\delta\beta^j_b$ becomes the matrix $\partial^2 H/\partial\beta^i_a\partial\beta^j_b$), by an appropriate extension of the Implicit Function Theorem, we could solve β as a function of φ and p. In such case, locally, $\mathcal{C}$ would be the graph of a function $F\colon T^*\mathcal{F}_{\partial M} \to \mathcal{B}$, say $\beta = F(\varphi, p)$. This is precisely the situation we will see in the simple example of scalar fields in the next section. Collecting the above yields:

Proposition 2.3.2. *The submanifold $(\mathcal{C}, \Omega_1)$ of $(\mathcal{M}, \Omega, \mathcal{H})$ is symplectic iff H is regular, i.e. $\partial^2 H/\partial\beta^i_a\partial\beta^j_b$ is non-degenerate. In such case the projection ϱ restricted to $\mathcal{C}$, which we denote by ϱ_C, is a local symplectic diffeomorphism and therefore $\varrho_C^*\omega_{\partial M} = \Omega_1$.*

When the situation is not as described above, and β is not a function of φ and p, then $(\mathcal{C}, \Omega_1)$ is indeed a presymplectic submanifold of $\mathcal{M}$ and $i_\Gamma\Omega = d\mathcal{H}$ will not hold necessarily at every point in $\mathcal{C}$. In this case we would apply Gotay's Presymplectic Constraints Algorithm [Go78], to obtain the maximal submanifold of $\mathcal{C}$ for which $i_\Gamma\Omega = d\mathcal{H}$ is consistent and that can be summarized as follows.

Consider a presymplectic system $(\mathcal{M}, \Omega, \mathcal{H})$ where $\mathcal{M} = T^*\mathcal{F}_{\partial M} \times \mathcal{B}$ and, Ω and $\mathcal{H}$ are as defined above. Let $\mathcal{M}_0 = \mathcal{M}$, $\Omega_0 = \Omega$, $\mathcal{K}_0 = \ker \Omega_0$, and $\mathcal{H}_0 = \mathcal{H}$. We define the primary constraint submanifold $\mathcal{M}_1$ as the submanifold defined by the consistency condition for the equation $i_\Gamma \Omega_0 = \mathrm{d}\mathcal{H}_0$, i.e.,

$$\mathcal{M}_1 = \{\chi \in \mathcal{M}_0 \mid \langle Z_0(\chi), \mathrm{d}\mathcal{H}_0(\chi)\rangle = 0,\ \forall Z_0 \in \mathcal{K}_0\}\,.$$

Thus $\mathcal{M}_1 = \mathcal{C}$. Denote by $i_1 \colon \mathcal{M}_1 \to \mathcal{M}_0$ the canonical immersion. Let $\Omega_1 = i_1^* \Omega_0$, $\mathcal{K}_1 = \ker \Omega_1$, and $\mathcal{H}_1 = i_1^* \mathcal{H}_0$. We now define recursively the $(k+1)$-th constraint submanifold as the consistency condition for the equation $i_\Gamma \Omega_k = \mathrm{d}\mathcal{H}_k$, that is,

$$\mathcal{M}_{k+1} = \{\chi \in \mathcal{M}_k \mid \langle Z_k(\chi), \mathrm{d}\mathcal{H}_k(\chi)\rangle = 0,\ \forall Z_k \in \mathcal{K}_k\} \qquad k \geq 1\,,$$

and $i_{k+1} \colon \mathcal{M}_{k+1} \to \mathcal{M}_k$ is the canonical embbeding (assuming that $\mathcal{M}_{l+1}$ is a regular submanifold of $\mathcal{M}_k$), and $\Omega_{k+1} = i_{k+1}^* \Omega_k$, $\mathcal{K}_{k+1} = \ker \Omega_{k+1}$ and $\mathcal{H}_{k+1} = i_{k+1}^* \mathcal{H}_k$.

The algorithm stabilizes if there is an integer $r > 0$ such that $\mathcal{M}_r = \mathcal{M}_{r+1}$. We refer to this $\mathcal{M}_r$ as the final constraints submanifold and we denote it by $\mathcal{M}_\infty$. Letting $i_\infty \colon \mathcal{M}_\infty \to \mathcal{M}_0$ denote the canonical immersion, we define,

$$\Omega_\infty = i_\infty^* \Omega_0, \qquad \mathcal{K}_\infty = \ker \Omega_\infty\,, \qquad \mathcal{H}_\infty = i_\infty^* \mathcal{H}_0\,.$$

Notice that the presymplectic system $(\mathcal{M}_\infty, \Omega_\infty, \mathcal{H}_\infty)$ is always consistent, that is, the dynamical equations defined by $i_\Gamma \Omega_\infty = d\mathcal{H}_\infty$ will always have solutions on $\mathcal{M}_\infty$. The solutions will not be unique if $\mathcal{K}_\infty \neq 0$, hence the integrable distribution $\mathcal{K}_\infty$ will be called the "gauge" distribution of the system, and its sections (that will necessarily close a Lie algebra), the "gauge" algebra of the system.

In the particular theories considered in this work we found that $\mathcal{M}_\infty = \mathcal{M}_1 = \mathcal{C}$ and we do not needed to go beyond the first step of the algorithm to obtain the final constraints submanifold.

The quotient space $\mathcal{R} = \mathcal{M}_\infty / \mathcal{K}_\infty$, provided it is a smooth manifold, inherits a canonical symplectic structure ω_∞ such that $\pi_\infty^* \omega_\infty = \Omega_\infty$, where $\pi_\infty \colon \mathcal{M}_\infty \to \mathcal{R}$ is the canonical projection. We will refer to it as the reduced phase space of the theory. Notice that the Hamiltonian $\mathcal{H}_\infty$ also passes to the quotient and we will denote its projection by h_∞ i.e., $\pi_\infty^* h_\infty = \mathcal{H}_\infty$.

Thus the Hamiltonian system $(\mathcal{R}, \omega_\infty, h_\infty)$ will provide the canonical picture of the theory at the boundary and its quantization will describe the states and dynamics of the theory with respect to observers sitting at the boundary ∂M.

Of course all the previous constructions depend on the boundary ∂M of the manifold M. For instance, if we assume that M is a globally hyperbolic spacetime of the form $M \cong [t_0, t_1] \times \Sigma$, then $\partial M = \{t_0\} \times \Sigma \cup \{t_1\} \times \Sigma$. But if we use a different

Cauchy surface Σ', the boundary of our spacetime will vary and we will get a new reduced phase space $(\mathcal{R}', \omega', h')$ for the theory. However in this case it is easy to show that there is a canonical symplectic diffeomorphism $S\colon \mathcal{R} \to \mathcal{R}'$ such that $h = S^* h'$. (Recall that in such case there will exist a canonical diffeomorphism $\Sigma \to \Sigma'$ that will eventually induce the map S above.)

Recall that $\Pi(\mathcal{EL}) \subset \mathcal{C}$. We easily show then that after the reduction to $\mathcal{R}$, the reduced submanifold of boundary values of Euler-Lagrange solutions of the theory, $\widetilde{\Pi}(\mathcal{EL})$ is an isotropic submanifold, now of the reduced phase space.

Theorem 2.3.3. *The reduction $\widetilde{\Pi}(\mathcal{EL})$ of the submanifold of Euler-Lagrange fields of the theory is an isotropic submanifold of the reduced phase space $\mathcal{R}$ of the theory.*

Proof. It is clear that $\Pi(\mathcal{EL}) \subset \Pi(\mathcal{EL}_\epsilon) \subset \mathcal{M}_\infty$ where $\mathcal{EL}_\epsilon = \mathcal{EL}_{U_\epsilon}$ are the critical points of the action S_ϵ, i.e. solutions of the Euler-Lagrange equations of the theory on U_ϵ.

The reduction $\widetilde{\Pi}(\mathcal{EL}) = \Pi(\mathcal{EL})/(\mathcal{K}_\infty \cap T\,\Pi(\mathcal{EL}))$ of the isotropic submanifold $\Pi(\mathcal{EL})$ to the reduced phase space $\mathcal{R} = \mathcal{M}_\infty/\mathcal{K}_\infty$ is isotropic because $\pi_\infty^* \omega_\infty = \Omega_\infty$. Hence $\pi_\infty^*(\omega_\infty \mid_{\widetilde{\Pi}(\mathcal{EL})}) = (\pi_\infty^* \omega_\infty) \mid_{\Pi(\mathcal{EL})} = \varrho^* \mathrm{d}\alpha \mid_{\Pi(\mathcal{EL})} = 0$. □

Reduction at the boundary and gauge symmetries

If our theory $(P(E), \Theta_H)$ has $\mathcal{G}$ as a covariant symmetry group, then because of Noether's theorem, Thm. 2.2.4, and Eq. (2.2.17), we have that $J_\xi[\chi]$, with $\Pi(\chi) = (\varphi, p)$ a closed $(m-1)$-form. Hence $\int_{\partial M} i^* J_\xi[\chi] = 0$, and so

$$\langle \mathcal{J}(\varphi, p), \xi \rangle = \int_{\partial M} i^* J_\xi[\chi] = 0\,.$$

Then $\mathcal{J}(\Pi(\chi)) = 0$, and therefore,

$$\Pi(\mathcal{EL}) \subset \mathcal{J}^{-1}(\mathbf{0})\,.$$

There is a natural reduction of the theory at the boundary defined by the covariant symmetry $\mathcal{G}$ for the following reason: Provided that the value $\mathbf{0}$ of the moment map $\mathcal{J}$ is weakly regular, the submanifold $\mathcal{J}^{-1}(\mathbf{0}) \subset T^*\mathcal{F}_{\partial M}$ is a coisotropic submanifold and the characteristic distribution $\ker i_0^* \omega_{\partial M}$ of the pull-back of the canonical symplectic form on $T^*\mathcal{F}_{\partial M}$ to it, is the distribution defined by the orbits of the group $\mathcal{G}_{\partial M}$. From Prop. 2.2.5, $\mathcal{J}$ is the moment map of the canonical lifting of the action of the group $\mathcal{G}_{\partial M}$ on $\mathcal{F}_{\partial M}$.

From the above and by Thm 2.3.1, Lemma 2.3.4 follows easily.

Lemma 2.3.4.

$$\mathcal{J}^{-1}(\mathbf{0}) \subset \varrho(\mathcal{M}_\infty)\,.$$

Proof. If $\mathcal{G}$ is a symmetry group of the Hamiltonian H of the theory, then it is clear that $\mathcal{G}_{\partial M}$ is a symmetry group of the function $\mathcal{H}$, with the canonical action of $\mathcal{G}_{\partial M}$ on the total space of fields at the boundary $\mathcal{M}$.

Then if $\zeta \in \mathcal{M}_\infty$, there exists Γ at ζ such that $i_\Gamma \Omega_\infty = \mathrm{d}\mathcal{H}_\infty$ and the integral curve γ of Γ passing through ζ lies in $\mathcal{M}_\infty$. But $\varrho(\gamma) \subset \mathcal{J}^{-1}(\mathbf{0})$, because it is the projection of an integral curve of a solution of Euler-Lagrange equations in U_ϵ. But because the Hamiltonian $\mathcal{H}$ is invariant, the trajectory must lie in a level surface of the moment map $\mathcal{J}$. Hence $\mathcal{J}^{-1}(\mathbf{0}) \subset \varrho(\mathcal{M}_\infty)$. □

Because, $\mathcal{R} = \mathcal{M}_\infty/\mathcal{K}_\infty$ and $\ker \varrho_* \cap T\mathcal{M}_\infty \subset \mathcal{K}_\infty$, we get that $\mathcal{M}_\infty/\mathcal{K}_\infty \cong \varrho(\mathcal{M}_\infty)/\varrho_*(\mathcal{K}_\infty)$. Now if we are in the situation where $\varrho(\mathcal{M}_\infty) = \mathcal{J}^{-1}(\mathbf{0})$, then $\mathcal{R} \cong \varrho(\mathcal{M}_\infty)/\varrho_*(\mathcal{K}_\infty) = \mathcal{J}^{-1}(\mathbf{0})/\ker \omega_{\partial M}\mid_{\mathcal{J}^{-1}(\mathbf{0})}$. Hence because of the standard Marsden-Weinstein reduction theorem the reduced phase space of the theory is obtained simply as,

$$\mathcal{R} \cong \mathcal{J}^{-1}(\mathbf{0})/\mathcal{G}_{\partial M}\,. \tag{2.3.11}$$

A simple example: the scalar field

We will consider the simple example of a real scalar field on a globally hyperbolic spacetime (M, η) of dimension $m = 1 + d$ with boundary ∂M a Cauchy surface and hence $M \cong (-\infty, a] \times \partial M$. The configuration fields of the system are sections of the (real) line bundle $\pi \colon E \to M$, where π is projection onto the first factor. Bundle coordinates will have the form (x^μ, u), $\mu = 0, 1, ..., d$.

If the bundle $E \to M$ were trivial, $E \cong M \times \mathbb{R}$, the first jet bundle J^1E would be the affine bundle $J^1E \cong T^*M \times \mathbb{R} \to E$ with bundle coordinates $(x^\mu, u; u_\mu)$, $\mu = 0, 1, ..., d$. The covariant phase space $P(E)$, in such case, would be isomorphic to $TM \times \mathbb{R}$ with bundle coordinates $(x^\mu, u; \rho^\mu)$.

As explained in Section 2.2, by using the volume form $\mathrm{vol}_M = \sqrt{|\eta|}\, \mathrm{d}^m x$ defined by the metric η (in arbitrary local coordinates x^μ), elements in $P(E)$ can be identified with semi-basic m-forms on E, $w \in \bigwedge_1^m(E)$, $w = \rho^\mu \mathrm{d}u \wedge \mathrm{vol}^d_\mu + \rho_0 \mathrm{vol}_M$, $\mathrm{vol}^d_\mu = i_{\partial/\partial x^\mu} \mathrm{vol}_M$, after we mod out basic m-forms, $\rho_0 \mathrm{vol}_M$.

The space of fields in the bulk, $\mathcal{F}_{P(E)} = \{\chi = (\Phi, P)\}$, consists of double sections of the iterated bundle $P(E) \to E \to M$, $\Phi \colon M \to E$, $u = \Phi(x)$, and $P \colon E \to P(E)$, $\rho = P(u)$. When E is a trivial bundle the configuration fields are maps $\Phi \colon M \to \mathbb{R}$ and the momenta fields are $(m-1)$-forms, $P = P^\mu(x) \mathrm{vol}^d_\mu$.

The Hamiltonian H of the theory determines a section of the projection $M(E) \to P(E)$ by fixing the variable ρ_0 above, i.e. $\rho_0 = -H(x^\mu, u, \rho^\mu)$. One standard choice for H in such case is:

$$H(x^\mu, u; \rho^\mu) = \frac{1}{2}\eta_{\mu\nu}\rho^\mu\rho^\nu + V(u)\,,$$

with $V(u)$ a smooth function on $\mathbb{R}$. The particular instance of $V(u) = m^2u^2$ gives us the Klein-Gordon system.

The canonical m-form Θ in $\bigwedge_1^m(E)$ can be pulled back to $P(E)$ along H and takes the form,

$$\Theta_H = \rho^\mu du \wedge \mathrm{vol}_\mu^d - H(u)\mathrm{vol}_M\,.$$

With the above choice for H, the action functional of the theory becomes,

$$S(\Phi, P) = \int_M \left[P^\mu(x)\partial_\mu\Phi(x) - \frac{1}{2}\eta_{\mu\nu}P^\mu P^\nu - V(\Phi)\right]\sqrt{|\eta|}\,\mathrm{d}^m x\,. \tag{2.3.12}$$

The space of boundary fields $T^*\mathcal{F}_{\partial M} = \{(\varphi, p)\}$ is given by $\varphi = \Phi\mid_{\partial M}$, $p = P^0\mid_{\partial M}$. Computing the differential of the action we get,

$$\begin{aligned}\mathrm{d}S_{(\Phi,P)}(\delta\Phi, \delta P) &= \int_M [\delta P^\mu(\partial_\mu\Phi - \eta_{\mu\nu}P^\nu) + \delta\Phi(-\frac{1}{\sqrt{|\eta|}}\partial_\mu(P^\mu\sqrt{|\eta|})\\ &- V'(\Phi))]\sqrt{|\eta|}\,\mathrm{d}^m x + \int_{\partial M} p\delta\varphi\,\mathrm{vol}_{\partial M}\,,\end{aligned}$$

and the Euler-Lagrange equations of the theory are given by,

$$\frac{1}{\sqrt{|\eta|}}\partial_\mu(P^\mu\sqrt{|\eta|}) + V'(\Phi) = 0\,, \qquad \partial_\mu\Phi - \eta_{\mu\nu}P^\nu = 0\,. \tag{2.3.13}$$

From the second of the Euler equations we get, $P^\nu = \eta^{\mu\nu}\partial_\mu\Phi$, and substituting into the first we get

$$\frac{1}{\sqrt{|\eta|}}\partial_\mu(\sqrt{|\eta|}\eta^{\mu\nu}\partial_\nu\Phi) = -V'(\Phi)\,. \tag{2.3.14}$$

The first term is the Laplace-Beltrami operator of the metric η, i.e. the d'Alembertian in the case of the Minkowski metric.

Note that had we instead chosen normal local coordinates on M, the volume element in such charts would take the form

$$\mathrm{vol}_M = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d$$

and then equations (3.13) would just be Hamilton's equations:

$$\partial_\mu P^\mu = -\frac{\partial H}{\partial\Phi} \qquad \partial_\mu\Phi = \frac{\partial H}{\partial P^\mu}\,.$$

The evolution picture near the boundary

We consider a collar around the boundary $U_\epsilon = (-\epsilon, 0] \times \partial M$ with coordinates $t = x^0$ and x^i, $i = 1, \dots, d$. We assume that $\eta = -\mathrm{d}t^2 + \eta_{0i}(x)\mathrm{d}t \otimes \mathrm{d}x^i + g_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j$ and $g = g_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j$ defines a Riemannian metric on ∂M. Writing again the action functional S restricted to fields Φ, P defined on U_ϵ, we have

$$S_\epsilon(\Phi, P) = \int_{-\epsilon}^{0} \mathrm{d}t \int_{\partial M} \mathrm{vol}_{\partial M} \sqrt{|\eta|}(P^0 \partial_0 \Phi + P^i \partial_i \Phi - \frac{1}{2}\eta_{\mu\nu} P^\mu P^\nu - V(\Phi)) \, .$$

Consider the fields at the boundary φ and p defined above and $\beta^i = P^i \mid_{\partial M}$. Also, let $\Delta = \sqrt{|\eta|}/\sqrt{|g|}$. Then $\sqrt{|\eta|} d^m x = \Delta \, \mathrm{d}t \wedge \mathrm{vol}_{\partial M}$.

Therefore we can write,

$$\begin{aligned} S_\epsilon(\Phi, P) &= \int_{-\epsilon}^{0} \mathrm{d}t \int_{\partial M} \mathrm{vol}_{\partial M} \, \Delta \, [p\dot{\varphi} + \beta^i \partial_i \phi + \frac{1}{2}p^2 - \eta_{0i} p \beta^i - \frac{1}{2} g_{ij} \beta^i \beta^j - V(\phi)] \\ &= \int_{-\epsilon}^{0} \mathrm{d}t \;\; [\langle p, \dot{\varphi} \rangle - \mathcal{H}(\varphi, p, \beta)] \end{aligned}$$

where

$$\langle p, \dot{\varphi} \rangle = \int_{\partial M} p(x)\dot{\varphi}(x) \Delta \, \mathrm{vol}_{\partial M} \tag{2.3.15}$$

denotes the scalar product on functions on ∂M defined by the volume $\Delta \, \mathrm{vol}_{\partial M}$, and $\mathcal{H} \colon \mathcal{M} \to \mathbb{R}$ denotes the Hamiltonian function induced from the Hamiltonian H of the theory,

$$\mathcal{H}(\varphi, p, \beta) = -\langle \beta, d_{\partial M} \varphi \rangle - \frac{1}{2}\langle p, p \rangle + \langle p, \tilde{\beta} \rangle + \frac{1}{2}\langle \beta, \beta \rangle + \int_{\partial M} V(\varphi) \Delta \, \mathrm{vol}_{\partial M}$$

with $\tilde{\beta} = \eta(d/\mathrm{d}t, \beta) = \eta_{i0}\beta^i$, $\langle p, p \rangle$ and $\langle p, \tilde{\beta} \rangle$ defined as in eq. (2.3.15). The product $\langle \beta, \beta \rangle$ denotes the scalar product of vector fields defined by the metric g, i.e.

$$\langle \beta, \beta \rangle = \int_{\partial M} g_{ij} \beta^i(x) \beta^j(x) \Delta \, \mathrm{vol}_{\partial M} \, ,$$

and $\langle \beta, d_{\partial M} \varphi \rangle$ is the natural pairing between vector fields and 1-forms on ∂M, that is

$$\langle \beta, d_{\partial M} \rangle = \int_{\partial M} \beta^i(x) \partial_i \varphi(x) \Delta \, \mathrm{vol}_{\partial M} \, .$$

As in Section 2.3 we denote the space of all fields at the boundary, the dynamical fields φ, p and the fields β^i, as $\mathcal{M} = T^*\mathcal{F}_{\partial M} \times \mathcal{B} = \{(\varphi, p; \beta)\}$ and Hamilton's equations for $\mathcal{H}$ are given by,

$$\dot{\varphi} = \frac{\delta \mathcal{H}}{\delta p} = -p + \tilde{\beta}\,, \qquad \dot{p} = -\frac{\delta \mathcal{H}}{\delta \varphi} = -V'(\varphi) - \operatorname{div}\beta\,,$$

together with the constraint equation obtained from the variation of S_ϵ with respect to β,

$$0 = \frac{\delta \mathcal{H}}{\delta \beta^i} = -\partial_i \phi + \eta_{0i} p + g_{ij}\beta^j\,.$$

Thus we get,

$$\begin{aligned} \dot{p} &= -\operatorname{div}\beta - V'(\varphi) \\ \dot{\varphi} &= -p + \tilde{\beta} \end{aligned}$$

and the constraints equations,

$$-\mathrm{d}_{\partial M}\varphi + p^\flat + \beta^\flat = 0\,, \tag{2.3.16}$$

where $\beta^\flat = g(\beta, \cdot)$ is the 1-form associated to the vector β by the metric g, and $p^\flat$ is the 1-form associated to the vector $p\,\partial/\partial t$.

Let $\mathcal{C} = \{(\varphi, p, \beta) \in \mathcal{M} \mid \delta\mathcal{H}/\delta\beta = 0\}$, the submanifold of $\mathcal{M}$ defined by the constraints (2.3.16), and let $\varrho\colon \mathcal{M} \to T^*\mathcal{F}_{\partial M}$ denote the canonical projection. We can solve for β^i as a function of φ and p in the constraint equation (2.3.16), obtaining $\beta^j = g^{ij}(\partial_i\varphi - g_{0i}p)$ or more intrisically,

$$\beta = \mathrm{d}_{\partial M}\varphi^\sharp - p\frac{\partial}{\partial t}\,,$$

where $d_{\partial M}\varphi^\sharp$ is the vector field associated to the 1-form $d_{\partial M}\varphi$ by means of the metric g. Thus the restriction of ϱ to $\mathcal{C}$ is a diffeomorphism onto $T^*\mathcal{F}_{\partial M}$. If we denote by Ω the pull-back $\varrho^*\omega_{\partial M}$ to $\mathcal{M}$ of the canonical symplectic form on $T^*\mathcal{F}_{\partial M}$ and by $\Omega_{\mathcal{C}}$ its restriction to the submanifold $\mathcal{C}$, the restriction of the canonical projection $\varrho\colon \mathcal{M} \to T^*\mathcal{F}_{\partial M}$ to $\mathcal{C}$ provides a symplectic diffeomorphism $(\mathcal{C}, \Omega_{\mathcal{C}}) \cong (T^*\mathcal{F}_{\partial M}, \omega_{\partial M})$.

Moreover, the projection $\Pi(\mathcal{EL})$ of the space of solutions to the Euler-Lagrange equations (2.3.13) to the boundary, defines, wherever it is a smooth submanifold, an isotropic submanifold of $T^*\mathcal{F}_{\partial M}$, as shown in Thm. 2.2.7. $\Pi(\mathcal{EL})$ is not necessarily a Lagrangian submanifold because in general the Dirichlet problem defined by boundary conditions (φ, p) for Eq. (2.3.14) doesn't have a solution. The situation is different in the Euclidean case, i.e. if (M, η) is a Riemannian manifold the Laplace-Beltrame operator would be elliptic and the Dirichlet problem would always have a unique solution. In such case the space $\Pi(\mathcal{EL})$ would certainly be a Lagrangian submanifold of $T^*\mathcal{F}_{\partial M}$.

Another example: The Poisson σ-model

We will illustrate the previous ideas as they apply to the case of the Poisson σ-model. We note that the Poisson σ-model (PσM for short) was analyzed in depth by A. Cattaneo *et al* [**Ca00**] and provides a quantum field theory interpretation of Konsevitch's quantization of Poisson structures. We will just concentrate on its first order covariant Hamiltonian formalism along the lines described earlier in this chapter.

We will consider a Riemann surface Σ with smooth boundary $\partial\Sigma \neq \emptyset$. We may assume that Σ also carries a Lorentzian metric. This will not play a significant role in the discussion and we can stick to a Euclidean picture by selecting a Riemannian metric on Σ. Local coordinates on Σ will be denoted as always by x^μ, $\mu = 0, 1$.

Let (P, Λ) be a Poisson manifold with local coordinates u^a, $a = 1, \dots, r$. The Poisson tensor Λ will be expressed in local coordinates as

$$\Lambda = \Lambda^{ab}(y) \frac{\partial}{\partial u^a} \wedge \frac{\partial}{\partial u^b} \, ,$$

and it defines a Poisson bracket on functions f, g on P,

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) \, .$$

The bundle E of the theory, will be the trivial bundle $E = \Sigma \times P$ with projection π, the canonical projection onto the first factor. The first jet bundle J^1E is the affine bundle over E modeled on $VE \otimes T^*\Sigma$. In this case, because of the triviality of E, we have that $VE \cong TP$ and the affine bundle is trivial. Now the dual bundle $P(E)$ will be naturally identified with the vector bundle over E modeled on $T^*P \otimes T\Sigma$, that is, its sections will be vector fields on Σ with values on 1-forms on P. However as shown in the general case, we may use a volume form vol_Σ on Σ (for instance that provided by a Riemannian metric) to identify elements on $P(E)$ with 1-semibasic forms on E, i.e.

$$P = P_a^\mu \, \mathrm{d}u^a \wedge i_{\partial/\partial x^\mu} \mathrm{vol}_\Sigma \, ,$$

and the corresponding double sections $\chi = (\Phi, P)$ of $P(E) \to E \to \Sigma$, with 1-forms η on Σ with values on 1-forms on P along the map $\Phi \colon \Sigma \to P$, that is,

$$P \colon T\Sigma \to T^*P \, , \qquad \tau_P \circ P = \Phi \, .$$

The covariant Hamiltonian of the theory will be given by,

$$H(x, u; P) = \frac{1}{2} \Lambda^{ab}(u) (P_a^\mu, P_b^\nu) \epsilon_{\mu\nu}$$

with $\mathrm{vol}_\Sigma = \epsilon_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. The action of the theory is thus

$$S_P(\chi) = \int_\Sigma \chi^*\Theta_H = \int_\Sigma \left[P_a^\mu \partial_\mu \Phi^a - H\right] \mathrm{vol}_\Sigma. \tag{2.3.17}$$

Notice that $P_a = P_a^\mu \mathrm{d}x_\mu$ and that $\mathrm{d}x_\mu = i_{\partial/\partial x^\mu}\mathrm{vol}_\Sigma$ is a 1-form on Σ. $H\mathrm{vol}_\Sigma$ can be expressed as

$$H(x,u;P)\mathrm{vol}_\Sigma = \frac{1}{2}\Lambda^{ab}(u)(P_a \wedge P_b)$$

and the first term in the action becomes simply $P_a \wedge d\Phi^a$. Thus the action of the theory is simply given as

$$S_P(\Phi,P) = \int_\Sigma P_a(x)\mathrm{d}\Phi^a(x) - \frac{1}{2}\Lambda^{ab}(\Phi(x))(P_a(x) \wedge P_b(x)),$$

or more succinctly,

$$S_P(\Phi,P) = \int_\Sigma \langle P \wedge \mathrm{d}\Phi\rangle - \frac{1}{2}(\Lambda \circ \Phi)(P \wedge P),$$

where $\langle\cdot,\cdot\rangle$ now denotes the natural pairing between T^*P and TP.

To get the evolution picture of the theory near the boundary, we choose a collar $U_\epsilon \cong (-\epsilon, 0] \times \partial\Sigma$ around the boundary $\partial\Sigma$ and we expand the action S_P of the theory, eq. (2.3.17) restricted to fields defined on U_ϵ. We obtain,

$$S_{P,U_\epsilon} = \int_{-\epsilon} \mathrm{d}t \int_{\partial\Sigma} \mathrm{d}u \left[p_a\dot{\varphi}^a + \beta_a\acute{\varphi}^a - \Lambda^{ab}p_a\beta_b\right],$$

where the boundary fields p_a and β_a are defined as before,

$$p_a = P_a^0 \mid_{\partial\Sigma}, \qquad \beta_a = P_a^1 \mid_{\partial\Sigma}.$$

The volume form and the coordinate u along the boundary $\partial\Sigma$ have been chosen so that $\mathrm{vol}_\Sigma = \mathrm{d}t \wedge \mathrm{d}u$, and $\acute{\varphi}^a$ denotes $\partial\varphi^a/\partial u$.

As before, the cotangent bundle of boundary fields is $T^*\mathcal{F}_{\partial\Sigma}$ with the canonical form $\alpha = p_a\delta\varphi^a$. In order to analyze the consistency of the Hamiltonian theory at the boundary, we introduce the extended phase space $\mathcal{M} = T^*\mathcal{F}_{\partial\Sigma} \times \mathcal{B}$ with its presymplectic structure $\Omega = \delta\varphi^a \wedge \delta p_a$ and the boundary Hamiltonian

$$\mathcal{H}(\varphi,p,\beta) = -\beta_a\acute{\varphi}^a + \Lambda^{ab}(\varphi)p_a\beta_b.$$

Solving for the Euler-Lagrange equations we obtain two evolution equations,

$$\dot{\varphi}^a = \frac{\delta \mathcal{H}}{\delta p_a} = \Lambda^{ab}\beta_b \,, \qquad \dot{p_a} = -\frac{\delta \mathcal{H}}{\delta \varphi^a} = -\acute{\beta}_a - \frac{\partial \Lambda^{bc}}{\partial \xi^a} p_b \beta_c \,,$$

and one constraint equation equation,

$$0 = \frac{\delta \mathcal{H}}{\delta \beta_a} = -\acute{\varphi}^a - \Lambda^{ab}(\varphi) p_b \,. \tag{2.3.18}$$

Thus the first constraints submanifold $\mathcal{M}_1$ will be defined by eq. (2.3.18). Notice the constraint defining $\mathcal{M}_1$ does not depend on the fields β^a, thus $\mathcal{M}_1$ is a cylinder along the projection ϱ over its projection $W = \varrho(\mathcal{M}_1) \subset T^*\mathcal{F}_{\partial M}$.

Notice that $\Omega = \varrho^* \omega_{\partial M}$ is such that $\ker \Omega = \mathcal{K} = \{\delta / \delta \beta^a\}$. Thus, $\mathcal{K} \subset \ker \Omega_1$, where Ω_1 is the restriction of Ω to $\mathcal{M}_1$. It is easy to check that $\ker \Omega_1 = \mathcal{K} \oplus \ker \Omega_{\mathcal{C}}$, where $\Omega_{\mathcal{C}}$ is the pull-back of $\omega_{\partial M}$ to $\mathcal{C}$.

The submanifold $W \subset T^*\mathcal{F}_{\partial M}$ is defined by the constraint

$$\Psi^a(\varphi, p) = -\acute{\varphi}^a - \Lambda^{ab}(\varphi) p_b \,,$$

whose Hamiltonian vector field X_a, i.e. X_a such that

$$i_{X_a} \omega_{\partial M} = \mathrm{d}\Psi^a \,,$$

is given by

$$X_a(\varphi, p) = \Lambda^{ab}(\varphi) \frac{\delta}{\delta \varphi^b} - \left(\partial_u \delta_a^c - p_b \frac{\partial \Lambda^{ab}}{\partial \varphi^c} \right) \frac{\delta}{\delta p_c} \,.$$

A simple computation shows that

$$X_a(\Psi^b) \mid_{\mathcal{C}} = 0 \,.$$

Hence $TW^{\perp} \subset TW$ and consequently, not only W, but also $\mathcal{M}_1$ are coisotropic submanifolds.(In describing $\mathcal{M}_1$ as a coisotropic submanifold of the presymplectic manifold $\mathcal{M}$ we mean simply that $T\mathcal{M}_1^{\perp} \subset T\mathcal{M}_1$.)

The stability of the constraints shows that the PCA algorithm stops at $\mathcal{M}_1$. Then the reduced (or physical) phase space of the theory is

$$\mathcal{R} = \mathcal{M}_1 / \ker \Omega_1 \cong \mathcal{C} / \mathrm{span}\{X_a\} \,.$$

The reduced phase space is a symplectic manifold, that in this case happens to be finite-dimensional.

In some particular cases it can be computed explicitly (for instance $\Sigma = [0,1] \times [0,1]$ with appropriate boundary conditions). In some instances it happens to inherit a groupoid structure that becomes the symplectic groupoid integrating the Poisson manifold P [**Ca01**].

2.4 Yang-Mills theories on manifolds with boundary as a covariant Hamiltonian field theory

The multisymplectic setting for Yang-Mills theories

Recall from the introduction, (M, η) is an oriented smooth manifold of dimension $m = 1 + d$ with boundary $\partial M \neq \emptyset$. It carries either a Riemannian or a Lorentzian metric η, in the latter case of signature $(- + \cdots +)$ and such that the connected components of ∂M are space-like submanifolds, that is, the restriction $\eta_{\partial M}$ of the Lorentzian metric to them is a Riemannian metric.

Yang-Mills fields are principal connections A on some principal fiber bundle $\rho \colon P \to M$ with structural group G. For clarity in the exposition we are going to make the assumption that P is trivial (which is always true locally), i.e. $P \cong M \times G \to M$ where (again, for simplicity) G is a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$.

Under these assumptions, principal connections on P can be identified with $\mathfrak{g}$-valued 1-forms on M, i.e., with sections of the bundle $E = T^*M \otimes \mathfrak{g} \longrightarrow M$. Local bundle coordinates in the bundle $E \to M$ will be written as (x^μ, A^a_μ), $\mu = 1, \ldots, m$, $a = 1, \ldots, \dim \mathfrak{g}$, where $A = A^a_\mu \, \xi_a \in \mathfrak{g}$ with ξ_a a basis of the Lie algebra $\mathfrak{g}$. Thus a section of the bundle can be written as

$$A(x) = A^a_\mu(x) \, \mathrm{d}x^\mu \otimes \xi_a \, . \tag{2.4.1}$$

As discussed in Sect.2.2, in the covariant Hamiltonian formalism fields are sections from the space-time manifold into $P(E)$, the affine dual of the first jet bundle J^1E, J^1E is an affine bundle modeled on the vector bundle $\pi^*(T^*M) \otimes VE$ and $P(E)$ is the vector bundle $\pi^*(TM) \otimes VE^*$. In the case of Yang-Mills since $E = T^*M \otimes \mathfrak{g}$ it follows that $J^1E \cong \pi^*(T^*M) \otimes VE \cong T^*M \otimes T^*M \otimes \mathfrak{g}$. Since $E = T^*M \otimes \mathfrak{g}$ is a vector bundle, $VE^* \cong E \times_M E^*$ and therefore from the previous line $P(E) \cong E \times_M (E^* \otimes \pi^*TM) \cong (T^*M \otimes \mathfrak{g}) \times_M (TM \otimes TM \otimes \mathfrak{g}^*)$ with projection τ^1_0 the canonical projection on the first factor $T^*M \otimes \mathfrak{g}$. Elements in $P(E)$ can be written as pairs (A, P) where A now denotes the element $A = A^a_\mu \, \mathrm{d}x^\mu \otimes \xi^a$ and P is an element in $TM \otimes TM \otimes \mathfrak{g}^*$, that is, $P = P^{\mu\nu}_a \partial_\mu \otimes \partial_\nu \otimes \xi_a$. For reasons that will be clear later on we will restrict ourselves to the skewsymmetric part of the tensor product $TM \otimes TM$ along the fibers of $P(E)$, or in other words we will consider only elements $P = P^{\mu\nu}_a \partial_\mu \wedge \partial_\nu \otimes \xi_a$, and $P^{\mu\nu}_a = -P^{\nu\mu}_a$.

As discussed in Sect.2.2, using the volume form vol_M on M we may identify the fields of the theory $\mathcal{F}_{P(E)}$, that is sections χ of $P(E) \to M$ that factorize through a section A of $\pi \colon E \to M$ and a section $P \colon E \to P(E)$ of τ_0^1, with equivalence classes of 1-semibasic m-forms on E i.e.

$$P = P_a^{\mu\nu}(x, A)\,\mathrm{d}A_\mu^a \wedge \mathrm{d}^{m-1}x_\nu + \textstyle\bigwedge_0^m(E)$$

where $\mathrm{d}^{m-1}x_\nu = i_{\partial/\partial x^\nu}\mathrm{vol}_\eta$. Thus the fields of the theory in the multisymplectic picture are provided by sections $\chi = P \circ A$, often denoted simply by (A, P), of the double bundle $P(E) \to E \to M$. In terms of the natural fields A, P, we write the action functional following the general principle, eq. (2.2.3):

$$S_{\mathrm{YM}}(A, P) = \int_M P_a^{\mu\nu}\mathrm{d}A_\mu^a \wedge \mathrm{d}x_\nu^{m-1} - H(A, P)\mathrm{vol}_M\,. \tag{2.4.2}$$

The Hamiltonian function H of the theory is defined as[1],

$$H(A, P) = \frac{1}{2}\epsilon_{bc}^a P_a^{\mu\nu} A_\mu^b A_\nu^c + \frac{1}{4} P_a^{\mu\nu} P_{\mu\nu}^a\,, \tag{2.4.3}$$

where the indexes $\mu\nu$ (a) in $P_a^{\mu\nu}$ have been lowered (raised) with the aid of the Lorentzian metric η (the Killing-Cartan form on $\mathfrak{g}$, respect.). Expanding the right hand side of eq. (4.2.11), we get[2],

$$S_{\mathrm{YM}}(A, P) = -\int_M \frac{1}{2}\left[P_a^{\mu\nu}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon_{bc}^a A_\mu^b A_\nu^c) + \frac{1}{2} P_a^{\mu\nu} P_{\mu\nu}^a\right] \mathrm{vol}_M\,. \tag{2.4.4}$$

Notice that if A is given by eq. (4.2.1), then its curvature is given by,

$$\begin{aligned} F_A &= \mathrm{d}_A A = \mathrm{d}A + \frac{1}{2}[A \wedge A] = F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &= \frac{1}{2}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon_{bc}^a A_\mu^b A_\nu^c\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \xi_a\,. \end{aligned} \tag{2.4.5}$$

Thus the previous expression for the Yang-Mills action becomes,

$$S_{\mathrm{YM}}(A, P) = -\int_M \left[P_a^{\mu\nu} F_{\mu\nu}^a + \frac{1}{4} P_a^{\mu\nu} P_{\mu\nu}^a\right] \mathrm{vol}_M\,.$$

[1] Physical constants, like the coupling strength constant g, have been removed from the formalism as only the mathematical geometrical structure is discussed in this dissertation.

[2] The minus sign in front comes form the expansion of $P_a^{\mu\nu} dA_\mu^a \wedge \mathrm{d}x_\nu^{m-1}$ that gives $P_a^{\mu\nu}(\partial_\nu A_\mu^a - \partial_\mu A_\nu^a)\mathrm{vol}_M$.

The Euler-Lagrange equations of the theory are very easy to obtain from the previous expression, they are

$$\frac{1}{2}P^a_{\mu\nu} = -F^a_{\mu\nu}\,, \qquad \partial_\mu P^{\mu\nu}_a + \epsilon^c_{ab}A^b_\mu P^{\mu\nu}_c = 0\,. \tag{2.4.6}$$

(From here one can already glean the reason for the choice made above to restrict to the skew-symmetric part of the momenta fields P.)

The canonical formalism near the boundary

As discussed in Sect.2.3, in order to obtain an evolution description for Yang-Mills and to provide the ground for its canonical quantization we need to introduce a local time parameter near the boundary. In the case that M is a Lorenztian manifold it is customary to assume that M is globally hyperbolic (even if far less strict causality assumptions on M would suffice), therefore the time parameter can be chosen globally. However as we did before, we will only assume that a collar $i_\epsilon : U_\epsilon = (-\epsilon, 0] \times \partial M \to M$ can be chosen around the boundary such that $i_{\epsilon *}(\partial/\partial t)$ is time-like everywhere and such that a choice of a time parameter $t = x^0$ can be made near the boundary that can be used to describe the evolution of the system. The fields of the theory would then be considered as fields defined on a given spatial frame that evolve in time for $t \in (-\epsilon, 0]$.

The dynamics of such fields would be determined by the restriction of the Yang-Mills action (4.2.13) to the space of fields on U_ϵ,

$$S_{\mathrm{YM},U_\epsilon}(A,P) = -\int_{-\epsilon}^{0} \mathrm{d}t \int_{\partial M} \mathrm{vol}_{\partial M}\left[P^{\mu\nu}_a F^a_{\mu\nu} + \frac{1}{4}P^{\mu\nu}_a P^a_{\mu\nu}\right]\,, \tag{2.4.7}$$

where now we are assuming that the collar U_ϵ is such that $\mathrm{vol}_{U_\epsilon} = \mathrm{d}t \wedge \mathrm{vol}_{\partial M}$ where $\mathrm{vol}_{\partial M}$ is the canonical volume defined by the restriction of the metric η to the boundary[3].

[3]Notice that even if this assumption is not necessary for the developments that follow it helps us avoid having spurious factors appear in the formulas.

Expanding (2.4.7) we obtain,

$$
\begin{aligned}
S_{\mathrm{YM},U_\epsilon}(A,P) &= -\frac{1}{2}\int_{-\epsilon}^{0}\mathrm{d}t\int_{\partial M}\mathrm{vol}_{\partial M}\left[P_a^{\mu\nu}\left(\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+\epsilon_{bc}^a A_\mu^b A_\nu^c\right)+\frac{1}{2}P_a^{\mu\nu}P_{\mu\nu}^a\right]\\
&= -\frac{1}{2}\int_{-\epsilon}^{0}\mathrm{d}t\int_{\partial M}\mathrm{vol}_{\partial M}\left[P_a^{k0}\left(\partial_k A_0^a-\partial_0 A_k^a+\epsilon_{bc}^a A_k^b A_0^c\right)+\right.\\
&+ P_a^{0k}\left(\partial_0 A_k^a-\partial_k A_0^a+\epsilon_{bc}^a A_0^b A_k^c\right)+\\
&+ \left.P_a^{kj}\left(\partial_k A_j^a-\partial_j A_k^a+\epsilon_{bc}^a A_k^b A_j^c\right)+\frac{1}{2}P_a^{k0}P_{k0}^a+\frac{1}{2}P_a^{0k}P_{0k}^a+\frac{1}{2}P_a^{kj}P_{kj}^a\right]\\
&= \int_{-\epsilon}^{0}\mathrm{d}t\int_{\partial M}\mathrm{vol}_{\partial M}\left[P_a^{k0}\left(\partial_0 A_k^a-\partial_k A_0^a-\epsilon_{bc}^a A_k^b A_0^c\right)+\right.\\
&- \left.\frac{1}{2}P_a^{kj}\left(\partial_k A_j^a-\partial_j A_k^a+\epsilon_{bc}^a A_k^b A_j^c\right)-\frac{1}{2}P_a^{k0}P_{k0}^a-\frac{1}{4}P_a^{kj}P_{kj}^a\right].
\end{aligned}
$$

In the previous expressions ϵ_{bc}^a denote the structure constants of the Lie algebra $\mathfrak{g}$ with respect to the basis ξ_a, that is $[\xi_b,\xi_c]=\epsilon_{bc}^a\xi_a$. Notice that $\epsilon_{bc}^a A_0^b A_0^c=0$ because for fixed a, ϵ_{bc}^a is skew-symmetric. Moreover the indexes μ and a have been pushed down and up by using the metric η and the Killing-Cartan form $\langle\cdot,\cdot\rangle$ respectively.

In equation (4.2.13) we introduced the assumption that P is a bivector, i.e., $P_a^{\mu\nu}$ is skew symmetric in μ and ν. Therefore $P_a^{00}=0$, and also $P_a^{k0}P_{k0}^a=P_a^{0i}P_{0i}^a$, because $P^{k0}=-P^{0k}$, etc. This assumption will be justified later on (see Sect. 2.4)

The previous expression acquires a clearer structure by introducing the appropriate notations for the fields restricted at the boundary and assuming that they evolve in time t. Thus the pull-backs of the components of the fields A and P to the boundary will be denoted respectively as,

$$
\begin{array}{rcllll}
a_k^a &:=& A_k^a\mid_{\partial M}; & a=(a_k^a)\,, & a_0^a:=A_0^a\mid_{\partial M}; & a_0=(a_0^k)\,,\\
p_a^k &:=& P_a^{k0}\mid_{\partial M}; & p=(p_a^k)\,, & p_a^0:=P_a^{00}\mid_{\partial M}=0; & p_0=(p_a^0)=0\,,\\
\beta_a^{ki} &:=& P_a^{ki}\mid_{\partial M}; & \beta=(\beta_a^{ki})\,. & &
\end{array}
$$

Given two fields at the boundary, for instance p and a, we will denote as usual by $\langle p,a\rangle$ the following expression:

$$
\langle p,a\rangle=\int_{\partial M}p_a^\mu a_\mu^a\,\mathrm{vol}_{\partial M}\,,
$$

and the contraction of the inner (Lie algebra) indices by using the Killing-Cartan form and the integration over the boundary is understood.

Introducing the notations and observations above in the expression for $S_{\mathrm{YM},U_\epsilon}$ we obtain,

$$\begin{aligned} S_{\mathrm{YM},U_\epsilon}(A,P) &= \int_{-\epsilon}^{0} \mathrm{d}t \int_{\partial M} \mathrm{vol}_{\partial M} \left[p_a^k \left(\dot{a}_k^a - \partial_k a_0^a - \epsilon_{bc}^a a_k^b a_0^c \right) + \right. \\ &- \frac{1}{2}\beta_a^{ki} \left(\partial_k a_i^a - \partial_i a_k^a + \epsilon_{bc}^a a_k^b a_i^c \right) - \frac{1}{4}\beta_a^{ki}\beta_{ki}^a - \frac{1}{2}p_a^k p_k^a \left. \right] \\ &= \int_{-\epsilon}^{0} \mathrm{d}t\, \mathcal{L}(a, \dot{a}, a_0, \dot{a}_0, p, \dot{p}, \beta, \dot{\beta}) \end{aligned} \tag{2.4.8}$$

where now $\mathcal{L}$ denotes the boundary Lagrangian, Eq. (2.3.2), and depends on the restrictions to the boundary of the fields of the theory. Collecting terms and simplifying we can then write $\mathcal{L}$ as,

$$\mathcal{L}(a, \dot{a}, a_0, \dot{a}_0, p, \dot{p}, \beta, \dot{\beta}) = \langle p, \dot{a} - \mathrm{d}_a a_0 \rangle - \langle \beta, F_a \rangle - \frac{1}{2}\langle p, p \rangle - \frac{1}{4}\langle \beta, \beta \rangle\,. \tag{2.4.9}$$

Now we can find the Euler-Lagrange equations corresponding to the Lagrangian function $\mathcal{L}$ as an infinite-dimensional mechanical system defined on the configuration space $P(E) = \{a, a_0, p, \beta\}$. Notice that the fields a, p are 1-forms on ∂M with values in the Lie algebra $\mathfrak{g}$, while the field a_0 is a function on ∂M with values in $\mathfrak{g}$, and the field β is a 2-form on ∂M with values in $\mathfrak{g}$ too. Thus the configuration space is the space of sections of the bundle $(T^*M \oplus T^*M \oplus \Lambda^2(T^*M) \oplus \mathbb{R}) \otimes \mathfrak{g}$.

Euler-Lagrange equations will have the form:

$$\frac{d}{\mathrm{d}t}\frac{\delta \mathcal{L}}{\delta \dot{\chi}} = \frac{\delta \mathcal{L}}{\delta \chi}\,,$$

where $\chi \in P(E)$ and $\delta/\delta\chi$ denotes the variational derivative of the functional $\mathcal{L}$.

Thus for $\chi = p$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{p}} = 0, \quad \text{hence} \quad 0 = \frac{\delta \mathcal{L}}{\delta p} = -p + \dot{a} - \mathrm{d}_a a_0\,,$$

and thus,

$$\dot{a} = p + \mathrm{d}_a a_0\,. \tag{2.4.10}$$

This equation corresponds to the Legendre transformation of the velocity and agrees with the standard minimal coupling definition of the momenta $p = \dot{a} - \mathrm{d}_a a_0$.

For $\chi = \beta$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{\beta}} = 0, \quad \text{thus} \quad 0 = \frac{\delta \mathcal{L}}{\delta \beta} = -F_a - \frac{1}{2}\beta$$

and consequently,

$$\beta = -2F_a\,. \tag{2.4.11}$$

For $\chi = a$ we obtain,

$$\frac{\delta\mathcal{L}}{\delta\dot{a}} = p, \quad \text{hence} \quad \dot{p} = \frac{d}{\mathrm{d}t}\frac{\delta\mathcal{L}}{\delta\dot{a}} = \frac{\delta\mathcal{L}}{\delta a} = \mathrm{d}_a^*\beta + [p, a_0].$$

Thus we get the equation determining the evolution of the momenta field (the Yang-Mills electric field) p:

$$\dot{p} = \mathrm{d}_a^*\beta + [p, a_0]\,. \tag{2.4.12}$$

Finally for $\chi = a_0$ we obtain,

$$\frac{\delta\mathcal{L}}{\delta\dot{a}_0} = 0, \quad \text{and therefore,} \quad \frac{\delta\mathcal{L}}{\delta a_0} = \mathrm{d}_a^* p\,.$$

Thus we obtain,

$$\mathrm{d}_a^* p = 0 \tag{2.4.13}$$

that must be interpreted as Yang-Mills Gauss law (in the absence of charges). Thus we have two evolution equations, (2.4.10) and (4.3.6), and two constraint equations (2.4.11) and (2.4.13).

Notice that the field a_0 is undetermined. This fact, clearly a consequence of the gauge invariance of the theory, will be interpreted in the next section.

We will study the consistency of the previous equations in the following section.

The Legendre transform

The Legendre transform in the bulk

So far we have presented a covariant Hamiltonian theory,the equation following (4.5), whose Euler-Lagrange equations are equivalent to Yang-Mills equations. However it is not automatically true that such theory is equivalent to the standard Yang-Mills theory. The standard Yang-Mills theory is a Lagrangian theory determined by a Lagrangian density which is nothing but the square norm of the curvature F_A of the connection 1-form A, and its action the L^2 norm of F_A, i.e.

$$S = -\frac{1}{4}\int_M \mathrm{Tr}\,(F_A \wedge \star F_A) = \int_M L_{\mathrm{YM}}(A)\mathrm{vol}_M\,. \tag{2.4.14}$$

Standard quantum field theories describing gauge interactions use exactly this Lagrangian description (and provide accurate results). Thus if we will assume that the correct Yang-Mills theory is provided by the action above, eq. (2.4.14), then we would like to relate the covariant Hamiltonian picture above to this Lagrangian picture.

For this task we have to introduce the natural extension of Legendre transform to the setting of covariant first order Lagrangian field theories. The Legendre transform is defined [**Ca91**] as the bundle map $\mathcal{F}L_{YM}\colon J^1E \to P(E)$, given by $\mathcal{F}L_{YM}(x^\mu, A^a_\mu; A^a_{\mu\nu}) = (x^\mu, A^a_\mu; P^{\mu\nu}_a)$ where

$$P^{\mu\nu}_a = \frac{\partial L_{\mathrm{YM}}}{\partial A^a_{\mu\nu}}$$

and $L_{\mathrm{YM}} = -\frac{1}{4}\mathrm{Tr}\,(F_A \wedge \star F_A)$. Recall that $\alpha \wedge \star\beta = (\alpha, \beta)_\eta \mathrm{vol}_M,\ \alpha, \beta,\ k$-forms, where $(\cdot,\cdot)_\eta$ denotes the inner product on k-forms. Thus we will write $\alpha \wedge \star\beta = \alpha_{\mu_1\cdots\mu_k}\beta^{\mu_1\cdots\mu_k}\mathrm{vol}_M$ where we have raised the indexes by using the $\eta^{\mu\nu}$. Hence,

$$L_{YM} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\,. \tag{2.4.15}$$

Hence in bundle coordinates $(x^\mu, A^a_\mu; A^a_{\mu\nu})$, we have,

$$F_{\mu\nu} = \frac{1}{2}\left(A^a_{\nu\mu} - A^a_{\mu\nu} + \epsilon^a_{bc}A^b_\mu A^c_\nu\right)\,. \tag{2.4.16}$$

Thus

$$P^{\mu\nu}_a = F^{\mu\nu}_a\,.$$

Notice that on the graph of the Legendre map, the Yang-Mills action in the Hamiltonian first order formalism, eq. (4.2.13), is just, up to a coefficient, the previous action eq. (2.4.14).

It was assumed that the momenta fields $P^{\mu\nu}_a$ are skew-symmetric in the indices μ and ν. From the definition of the momenta fields as sections of the bundle $P(E)$ there is no restriction on them. However because Yang-Mills theories are Lagrangian theories, the Legendre transform selects a subspace on the space of momenta that corresponds to fields P which are skew-symmetric on the indices μ, ν.

The presymplectic formalism: Yang-Mills at the boundary and reduction

As discussed in general in section 3.2, we define the extended Hamiltonian $\mathcal{H}$, so that $\mathcal{L} = \langle p, \dot{a}\rangle - \mathcal{H}$. Thus

$$\mathcal{H}(a, \beta) = \langle p, \mathrm{d}_a a_0\rangle + \frac{1}{2}\langle p, p\rangle + \langle \beta, F_a + \frac{1}{2}\beta\rangle\,. \tag{2.4.17}$$

Thus the Euler-Lagrange equations can be rewritten as

$$\dot{a} = \frac{\delta \mathcal{H}}{\delta p}; \quad \dot{p} = -\frac{\delta \mathcal{H}}{\delta a}\,, \tag{2.4.18}$$

$$\frac{\delta \mathcal{H}}{\delta a_0} = 0 \tag{2.4.19}$$

$$\frac{\delta \mathcal{H}}{\delta \beta} = 0\,. \tag{2.4.20}$$

We denote again by $\varrho \colon \mathcal{M} \to T^*\mathcal{F}_{\partial M}$ the canonical projection $\varrho(a, a_0, p, \beta) = (a, a_0, p)$. Let $\omega_{\partial M}$ denote the form on the cotangent bundle $T^*\mathcal{F}_{\partial M}$,

$$\omega_{\partial M} = \delta a \wedge \delta p.$$

We will denote again by Ω the pull-back of this form to $\mathcal{M}$ along ϱ, i.e., $\Omega = \varrho^*\omega_{\partial M}$. Clearly, $\ker \Omega = \mathrm{span}\{\delta/\delta\beta, \delta/\delta a_0\}$, and we have the particular form that Thm. 2.3.1 takes here.

Theorem 2.4.1. *The solution to the equation of motion defined by the Lagrangian L_{YM}, i.e. the Yang-Mills equations, are in one-to-one correspondence with the integral curves of the presymplectic system $(\mathcal{M}, \Omega, \mathcal{H})$, i.e. with the integral curves of the vector field Γ on $\mathcal{M}$ such that $i_\Gamma \Omega = \mathrm{d}\mathcal{H}$.*

The primary constraint submanifold $\mathcal{M}_1$ is defined by the two constraint equations,

$$\mathcal{M}_1 = \{(a, a_0, p, \beta) | \mathcal{F}_a + \beta = 0, d_a^* p = 0\}\,.$$

Since β is just a function of a, we have that $\mathcal{M}_1 \cong \{(a, a_0, p) | d_a^* p = 0\}$ and $\ker \Omega|_{\mathcal{M}_1} = \mathrm{span}\{\frac{\partial}{\partial a_0}\}$. Thus $\mathcal{M}_2 = \mathcal{M}_1/(\ker \Omega | \mathcal{M}_1) \cong \{(a, p) | \mathrm{d}_a^* p = 0\}$.

Gauge transformations: symmetry and reduction

The group of gauge transformations $\mathcal{G}$, i.e, the group of automorphisms of the principal bundle P over the identity, is a fundamental symmetry of the theory. Notice that the action S_{YM} is invariant under the action of $\mathcal{G}$ (however it is not true that H is $\mathcal{G}$-invariant).

The quotient of the group of gauge transformations by the normal subgroup of identity gauge transformations at the boundary defines the group of gauge transformations at the boundary $\mathcal{G}_{\partial M}$, and it constitutes a symmetry group of the theory at the boundary, i.e. it is a symmetry group both of the boundary Lagrangian $\mathcal{L}$ and of the presymplectic system $(\mathcal{M}, \Omega, \mathcal{H})$. We may take advantage of this symmetry to provide an alternative description of the constraints found in the previous section.

Proposition 2.4.2. *With the notations above, $\mathcal{J}(a,p) = \mathrm{d}_a^* p$.*

Proof. The moment map $\mathcal{J}\colon T^*\mathcal{F}_{\partial M} \to \mathfrak{g}^*_{\partial M}$ is given by,

$$\langle \mathcal{J}(a,p), \xi \rangle = \langle p, \xi_{\mathcal{F}_{\partial M}} \rangle = \langle p, \mathrm{d}_a \xi \rangle \, ,$$

because the gauge transformation $g_s = \exp s\xi$ acts in a as $a \mapsto g_s \cdot a = g_s^{-1} a g_s + g_s^{-1} \mathrm{d} g_s$ and the induced tangent vector is given by,

$$\xi_{\mathcal{A}_{\partial M}}(a) = \frac{\mathrm{d}}{\mathrm{d}s} g_s \cdot a \mid_{s=0} = \mathrm{d}_a \xi \, .$$

□

Let $\mathcal{A}_{\partial M}$ denote the space of connections a defined on the boundary ∂M. The constraint submanifold $\mathcal{M}_1$ projected to the space $T^*\mathcal{A}_{\partial M}$, by means of the projection map $(a, a_0, p) \mapsto (a, p)$, is such that $\mathcal{C} = \mathcal{J}^{-1}(\mathbf{0})$. This is exactly the situation depicted in Sect.2.3. Hence the standard Marsden-Weinstein reduction, eq. (2.3.11), will give the reduced phase space,

$$\mathcal{R}_{\mathrm{YM}} = \mathcal{J}^{-1}(\mathbf{0}) / \mathcal{G}_{\partial M}$$

and its Hamiltonian,

$$h([a],[p]) = \frac{1}{2}\langle p, p \rangle - \frac{1}{2}\langle F_a, F_a \rangle \, ,$$

where $[a]$ and $[p]$ denote equivalence classes of connections and momenta with respect to the action of the gauge group $\mathcal{G}_{\partial M}$. Notice that both terms in the Hamiltonian function h are $\mathcal{G}_{\partial M}$-invariant, and the Hamiltonian system h defined on the reduced phase space $\mathcal{R}_{\mathrm{YM}}$ has the structure of an infinite-dimensional mechanical system with potential function $V([a]) = \frac{1}{2}||F_a||^2$.

The reduction of the boundary values of solutions of Yang-Mills equation in the bulk is of course, an isotropic submanifold of the reduced space. In the case where M is Riemannian, an existence and uniqueness theorem for solutions of Yang-Mills equations on manifolds with boundary can be proved and hence this submanifold, following the proof of Theorem 2.7, is a Lagrangian submanifold.

2.5 Conclusions and discussion

It has been shown that the multisymplectic geometry of the covariant phase space $P(E)$ provides a convenient framework to study first order covariant Hamiltonian field theories on manifolds with boundaries. In particular it induces a natural presymplectic structure on the total space of fields at the boundary whose reduction provides the symplectic phase space of the theory. The solution of the Euler-Lagrange equations on the bulk induce an isotropic submanifold in the reduced symplectic phase space at the boundary. Provided that the boundary conditions are well-posed, this submanifold is in fact Lagrangian.

The gauge symmetries of the theory fit nicely into the picture and the symplectic reduction of the theory at the boundary induced by the moment map, i.e., by the conserved charges of the theory, is in perfect agreement with the presymplectic analysis of the theory. Various instances are discussed illustrating the main features of the theoretical framework: the real scalar field, the Poisson σ-model and Yang-Mills theories. Each of them allows as to stress different aspects of the theory. The regular situation for the scalar field, the coisotropic structure at the boundary in the case of the Poisson σ-model and the reduction using the moment map at the boundary in the case of Yang-Mills theories.

The theory presented in this work is particularly well suited for describing Palatini's gravity. C. Rovelli's [**Ro04**], [**Ro06**], can be read in part as seeking and arguing for precisely such a theory. We interpret Rovelli's canonical form Θ_H as alluding to a multisymplectic structure in the bulk. We take up Palatini gravity in Chapter 4.

Chapter 3

Covariant Hamiltonian dynamics with constraints

3.1 Introduction

In this chapter, based on joint published work with Alberto Ibort [**Ib16a**], we analyze in depth the theory of Hamiltonian dynamical systems, viewing these systems as first-order covariant Hamiltonian field theories on a manifold of dimension 1+0 with boundary. These field theories present a simple setting for studying features relevant to more complex examples of first-order covariant Hamiltonian field theories like Yang-Mills theories and Palatini gravity. Such features include the introduction of constraints and the 'topological phases' of the theory, that is, the limit of the gauge theory where the kinetic term of the theory disappears. On these simple theories we can also test some common assumptions about first-order covariant Hamiltonian field theories, in particular the role of boundary conditions and the geometry of the restriction to the boundary of the space of solutions of the Euler-Lagrange equations.

In this chapter, the simple theory of Hamiltonian dynamics interpreted as covariant first-order Hamiltonian field theories on manifolds with boundary in dimension 1+0 is subjected to a similar analysis as that undertaken in chapter 2. The fact that the basic space of the theory is one-dimensional and that the Euler-Lagrange equations of the theory are the standard Hamilton's equations, removes most of the analytic difficulties arising in higher-order theories. It is possible to show that under appropriate conditions, the spaces of fields have specific geometric structures. Most notably it is shown that for theories named locally Dirichlet, the space of solutions of the Euler-Lagrange equations restricted to the boundary, $\pi(\mathcal{EL})$, is a Lagrangian submanifold. Through examples however we find that this condition is not necessary. In fact we find that for even so simple a theory as Hamiltonian dynamics, it is not easy to give necessary and sufficient conditions for which $\pi(\mathcal{EL})$ is a Lagrangian submanifold. A locally defined generating function is constructed.

In Section 3.2 Hamiltonian dynamics is formulated as a first-order Hamiltonian field theory on a manifold with boundary. We review material covered in sections 2.1 - 2.3 of Chapter 2 and apply it to the case of Hamiltonian dynamics. In Section 3.3 the problem of determining under what conditions the space of solutions of Euler-Lagrange equations at the boundary is Lagrangian is addressed. A solution is provided by explicitly constructing locally defined generating functions and a number of detailed examples are presented. Finally, in Section 3.4 the introduction of constraints in the theory is analyzed, in particular constraints affecting the momenta of the theory. A reduction theory for these is obtained.

3.2 The geometry of the covariant phase space for Hamiltonian dynamics

We quickly review some basic notions and notations for first-order covariant Hamiltonian field theories. We refer the reader to Chapter 2 for full expositions.

The covariant phase space of first order Hamiltonian field theories

The fundamental geometric structure of a first-order covariant Hamiltonian field theory is provided by a fiber bundle $\pi\colon E \to M$ with M an $m = (1+d)$-dimensional orientable smooth manifold with smooth boundary $\partial M \neq \emptyset$ and local coordinates adapted to the fibration (x^μ, u^a), $a = 1, \ldots, r$, where r is the dimension of the standard fiber. Because M is orientable we will assume that a given volume form vol_M is selected. Notice that it is always possible to chose local coordinates x^μ such that $\mathrm{vol}_M = dx^0 \wedge dx^1 \wedge \cdots \wedge dx^d$.

In this setting, the formalism for Hamiltonian dynamics as a covariant Hamiltonian field theory will be provided by the bundle $\pi\colon E = Q \times [0,1] \to [0,1]$, where π denotes the standard projection on the second factor, Q is an r-dimensional smooth manifold without boundary, and the spacetime of dimension $m = 1 + 0$, $M = [0,1] \subset \mathbb{R}$, with standard metric $\eta = -dt^2$. The volume form is given by $\mathrm{vol}_M = dt$. A standard bundle chart will have the form (t, u^a) where u^a, $a = 1, \ldots, r$, is a local chart for Q.

We will denote by $\pi_1^0 \colon J^1E \to E$ the affine 1-jet bundle of the bundle $E \stackrel{\pi}{\to} M$. The elements of J^1E are equivalence classes of germs of sections ϕ of π. If $(x^\mu; u^a)$, $\mu = 0, \ldots, d$ is a bundle chart for the bundle $\pi \colon E \to M$, then we will denote by $(x^\mu, u^a; u^a_\mu)$ a local chart for the jet bundle J^1E.

The affine dual of J^1E is the vector bundle over E whose fiber at $\xi = (x, u)$ is the linear space of affine maps $\mathrm{Aff}(J^1E_\xi, \mathbb{R})$. The vector bundle $\mathrm{Aff}(J^1E, \mathbb{R})$, possesses a natural subbundle defined by constant functions along the fibers of $J^1E \to E$, that we will denote again, with an abuse of notation, as $\mathbb{R}$. The quotient bundle $\mathrm{Aff}(J^1E, \mathbb{R})/\mathbb{R}$ will be called the covariant phase space bundle of the theory and will be denoted by $P(E)$.

Local coordinates on $P(E)$ can be introduced as follows: Affine maps on the fibers of J^1E have the form $u^a_\mu \mapsto \rho_0 + \rho^\mu_a u^a_\mu$ where u^a_μ are natural coordinates on the fiber over the point ξ in E with coordinates (x^μ, u^a). Thus a bundle chart for the bundle $\tau_1^0 \colon P(E) \to E$ is given by $(x^\mu, u^a; \rho^\mu_a)$.

The choice of a distinguished volume form vol_M in M allows us to identify the fibers of $P(E)$ with a subspace of m-forms on E as follows: The map $u^a_\mu \to \rho^\mu_a u^a_\mu$ corresponds to the m-form $\rho^\mu_a du^a \wedge \mathrm{vol}_\mu$ where vol_μ stands for $i_{\partial/\partial x^\mu} \mathrm{vol}_M$. Let $\bigwedge^m(E)$ denote the bundle of m-forms on E. Let $\bigwedge^m_k(E)$ be the subbundle of $\bigwedge^m(E)$ consisting of those m-forms which vanish when k of their arguments are vertical. So in our local coordinates, elements of $\bigwedge^m_1(E)$, i.e. m-form on E that vanish when two of their arguments are vertical, commonly called semi-basic 1-forms, have the form $\rho^\mu_a du^a \wedge \mathrm{vol}_\mu + \rho_0 \mathrm{vol}_M$, and elements of $\bigwedge^m_0(E)$, i.e. basic m-forms, have the form $p_0 \mathrm{vol}_M$. $\bigwedge^m_1 E$ is a real line bundle over $P(E)$ and, for each point $\zeta = (x, u, p) \in P(E)$, the fiber is the quotient $\bigwedge^m_1(E)_\zeta / \bigwedge^m_0(E)_\zeta$

In the case of Hamiltonian dynamics it is clear that the first jet bundle J^1E is canonically isomorphic to $TQ \times [0,1]$ and that $P(E)$ is isomorphic to $T^*Q \times [0,1]$. Bundle coordinates on $T^*Q \times [0,1]$ will be denoted as (t, u^a, p_a) and the projection $\tau_1^0 \colon P(E) \to E$ above, becomes $\tau_1^0(t, u^a, p_a) = (t, u^a)$

Notice that the space of 1-semi-horizontal forms on $E = Q \times [0,1] \to [0,1]$ is just the space of 1-forms on $Q \times [0,1]$, that is $\bigwedge^m_1 E = T^*(Q \times [0,1])$ and the projection map $\mu \colon \bigwedge^m_1 E \to P(E)$ is just the projection $\mu \colon T^*(Q \times [0,1]) \to T^*Q \times [0,1]$, given as $\mu(t, u^a; p_0, p_a) = (t, u^a, p_a)$ where $p_a du^a + p_0 dt$ is a generic element in $T^*(Q \times [0,1])$. The canonical 1-form Θ on $(\bigwedge^m_1 E) \cong T^*(Q \times [0,1])$ is just the canonical Liouville 1-form on the cotangent bundle $T^*(Q \times [0,1])$, and has the expression:

$$\Theta = p_a du^a + p_0 dt \,.$$

We define a Hamiltonian H on $P(E)$ to be a section of μ, and we can use a Hamiltonian section $\rho = -H(x^\mu, u^a, \rho^\mu_a)$ to define an m-form on $P(E)$ by pulling back the canonical m-form Θ from $\bigwedge_1^m E$. We call the form so obtained the Hamiltonian m-form associated with H and denote it by Θ_H. Thus

$$\Theta_H = \rho_a^\mu du^a \wedge \mathrm{vol}_\mu - H(x^\mu, u^a, \rho_a^\mu)\,\mathrm{vol}_M. \tag{3.2.1}$$

and the pair (J^1E^*, Θ_H) will be called the Hamiltonian covariant phase space of the theory.

In the simple example in dimension $1+0$ we are considering, a Hamiltonian, i.e. a section of μ, can be identified with a map $H\colon T^*Q \times I \to \mathbb{R}$, and $p_0 = -H(t, u^a, p_a)$, because the bundle defined by the projection μ is trivial. The pull-back of the canoncial 1-form Θ to $J^1E^* \cong T^*Q \times [0,1]$ becomes the standard 1-form of Hamiltonian dynamics:

$$\Theta_H = p_a du^a - H(t, u, p)dt\,.$$

The action and the variational principle

The fields of the theory are double sections of the bundle $P(E)$, that is, sections $\chi\colon M \to J^1E^*$ of the bundle structure over M such that both $\Phi = \tau_1^0 \circ \chi\colon M \to E$ and $P = \chi \circ \pi\colon E \to J^1E^*$ are sections of $\pi\colon E \to M$ and $\tau_1^0\colon P(E) \to E$ respectively. Notice that in such a case $P \circ \Phi = \chi$. We will denote such a section χ by (Φ, P) to indicate the double bundle structure of $P(E)$.

In the case of Hamiltonian dynamics viewed as a field theory over $M = [0,1] \subset \mathrm{R}$, double sections of the bundle $P(E) = T^*Q \times [0,1] \to E = Q \times [0,1] \to [0,1]$, have the form $\chi = P \circ \Phi$, with $\Phi(t) = (u(t), t)$ and $P(u, p, t) = (u, p(t), t)$, i.e. therefore $\chi(t) = (u(t), p(t), t)$. Thus the space of fields of the theory can be described equivalently as the space of smooth curves on T^*Q.

We will denote by $\mathcal{F}_M(E)$, or just $\mathcal{F}_M$ if there is no risk of confusion, the sections $\Gamma(E)$ of the bundle E, that is $\Phi \in \mathcal{F}_M$, and by $J^1\mathcal{F}_M^*$ the double sections $\chi = (\Phi, P)$. Thus $J^1\mathcal{F}_M^*$ represents the space of fields of the theory in the first-order covariant Hamiltonian formalism. In our particular instance $\mathcal{F}_M \cong C^\infty([0,1], Q)$ and $J^1\mathcal{F}_M^* \cong C^\infty([0,1], T^*Q)$.

The equations of motion of the theory will be defined by means of a variational principle, i.e., they will be characterized as the critical points of an action functional S on $J^1\mathcal{F}_M^*$. Such action will be simply given by:

$$S(\chi) = \int_M \chi^*\Theta_H\,, \tag{3.2.2}$$

that in our case of Hamiltonian dynamics viewed as a field theory on $M = [0,1] \subset \mathrm{R}$ becomes,

$$S(\chi) = \int_0^1 \left(p_a(t)\dot{u}^a(t) - H(t, u(t), p(t))\right) dt\,, \tag{3.2.3}$$

which is just the standard functional in Hamilton's variational principle.

A simple computation leads to,

$$\mathrm{d}S(\chi)(U) = \int_M \chi^* \left(i_{\widetilde{U}} d\Theta_H\right) + \int_{\partial M} (\chi \circ i)^* \left(i_{\widetilde{U}}\Theta_H\right)\,, \tag{3.2.4}$$

where U is a vector field on J^1E^* along the section χ, $\widetilde{U}$ is any extension of U to tubular neighborhood of the graph of χ, and $i\colon \partial M \to M$ is the canonical embedding.

The cotangent bundle of fields at the boundary

Consider a collar around the boundary $U_\epsilon \cong (-\epsilon, 0] \times \partial M$, and local coordinates $x^0 = t$, x^k, $k = 1, \ldots, d$, such that $\mathrm{vol}_M = dt \wedge \mathrm{vol}_{\partial M}$. In the theory we are considering, we have $\partial M = \partial[0,1] = \{1, 0\}$.

The fields at the boundary are obtained by restricting the zeroth component of sections χ to ∂M, that is, fields of the form:

$$\varphi^a = \Phi^a \circ i\,, \qquad p_a = P_a^0 \circ i\,.$$

In the case of Hamiltonian dynamics viewed as a field theory on $M = [0,1] \subset \mathrm{R}$ the fields at the boundary are just $\varphi^a = u^a \mid_{\{1,0\}} = (u^a(1), u^a(0))$ and $p_a = P_a^0 \mid_{\{1,0\}} = (p_a(1), p_a(0))$.

If we denote by $\mathcal{F}_{\partial M}$ the space of fields at the boundary φ^a, then the space of fields (φ^a, p_a) can be identified with the cotangent bundle $T^*\mathcal{F}_{\partial M}$ over $\mathcal{F}_{\partial M}$ in a natural way, i.e. each field p_a can be considered as the covector at φ^a that maps the tangent vector $\delta\varphi^a$ at φ^a into the number $\langle p, \delta\varphi\rangle$ given by:

$$\langle p, \delta\varphi\rangle = \int_{\partial M} p_a(x)\delta\varphi^a(x)\mathrm{vol}_{\partial M}\,. \tag{3.2.5}$$

The canonical 1-form α on $T^*\mathcal{F}_{\partial M}$ will have the expression:

$$\alpha_{(\varphi,p)}(U) = \int_{\partial M} p_a(x)\delta\varphi^a(x)\,\mathrm{vol}_{\partial M} \tag{3.2.6}$$

with U a tangent vector to $T^*\mathcal{F}_{\partial M}$ at (φ, p), that is, a vector field on the space of 1-semi-horizontal forms on i^*E along the section (φ^a, p_a), hence $U = \delta\varphi^a\,\partial/\partial u^a + \delta p_a\,\partial/\partial\rho_a$.

In the case of Hamiltonian dynamics viewed as a field theory, it is clear that $\mathcal{F}_{\partial M}$ is identified with $Q \times Q$ and $T^*\mathcal{F}_{\partial M}$ is just $T^*Q \times T^*Q$ and the canonical 1-form α will be

$$\alpha = \mathrm{pr}_2^*\theta - \mathrm{pr}_1^*\theta = p_a(1)du^a(1) - p_a(0)du^a(0)\,,$$

where θ is the canonical Liouville 1-form on T^*Q and $\mathrm{pr}_{1,2} : \overline{T^*Q} \times T^*Q \to T^*Q$ denote the canonical projections onto the first and second factor respectively. In what follows, we will denote by $\overline{T^*Q} \times T^*Q$ the manifold $T^*Q \times T^*Q$ equipped with the 1-form α above and the corresponding symplectic structure, $\omega = \mathrm{pr}_2^* d\theta - \mathrm{pr}_1^* d\theta = d\alpha$.

Finally, notice that the 'pull-back to the boundary' map, defines a natural map from the space of fields in the bulk, $J^1\mathcal{F}_M^*$ into the phase space of fields at the boundary $T^*\mathcal{F}_{\partial M}$ that will be denoted by Π, that is:

$$\Pi\colon J^1\mathcal{F}_M^* \to T^*\mathcal{F}_{\partial M}\,, \qquad \Pi(\Phi, P) = (\varphi, p), \quad \varphi = \Phi \circ i,\, p_a = P_a^0 \circ i\,.$$

that in the case of Hamiltonian dynamics viewed as a field theory on $M = [0,1] \subset \mathrm{R}$ becomes $\Pi(\chi) = (u(1), p(1); u(0), p(0))$.

With the notations above, by comparing the expression for the boundary term in Eq.(3.2.4) and the expression for the canonical 1-form α, Eq.(3.2.6), we get:

$$\int_{\partial M} (\chi \circ i)^* \left(i_{\tilde{U}}\Theta_H\right) = (\Pi^*\alpha)_\chi(U)\,.$$

or in other words, the boundary term in Eq.(3.2.4) is just the pull-back of the canonical 1-form α at the boundary along the projection map Π.

Euler-Lagrange's equations and Hamilton's equations

Referring to eqn (3.2.4), let $\mathrm{EL}_\chi(U) = \int_M \chi^*\left(i_{\tilde{U}} d\Theta_H\right)$. Note that EL_χ is a 1-form on the space of fields on the bulk, $J^1\mathcal{F}_\mathcal{M}^*$. A double section χ of $P(E) \to E \to M$ will be said to satisfy the Euler-Lagrange equations determined by the first-order Hamiltonian field theory with Hamiltonian H if $\mathrm{EL}_\chi = 0$, that is if χ is a zero of the Euler-Lagrange 1-form EL on $J^1\mathcal{F}_M^*$. Notice that this is equivalent to

$$\chi^*(i_{\tilde{U}}\mathrm{d}\Theta_H) = 0\,, \tag{3.2.7}$$

for all vector fields $\tilde{U}$ on (a tubular neighborhood of the range of χ in) $P(E)$. In the case of Hamiltonian dynamics viewed as a field theory on a manifold of dimension 1+0 ,(3.2.7) becomes the standard Hamilton's equations for the Hamiltonian function H:

$$\dot{u}^a = \frac{\partial H}{\partial p_a}\,, \qquad \dot{p}_a = -\frac{\partial H}{\partial u^a} \tag{3.2.8}$$

Similarly, the previous expression for EL_χ can be written explicitly as:

$$\mathrm{EL}_\chi(\delta u, \delta p) = \int_0^1 \left[\left(\dot{u}^a - \frac{\partial H}{\partial p_a} \right) \delta p_a + \left(\dot{p}_a + \frac{\partial H}{\partial u^a} \right) \delta u^a \right] dt \,. \tag{3.2.9}$$

We have obtained in this way the fundamental formula that relates the differential of the action with a 1-form on the space of fields on the bulk manifold and a 1-form on the space of fields at the boundary.

$$\mathrm{d}S_\chi = \mathrm{EL}_\chi + \Pi^* \alpha_\chi \,, \qquad \chi \in J^1\mathcal{F}_M^* \,. \tag{3.2.10}$$

We will denote by $\mathcal{EL}$ the set of solutions of the Euler-Lagrange equations of our theory. The set $\mathcal{EL}$ is the set of zeros of the 1-form EL and the zeros of a 1-form form a submanifold under exactly the same conditions as do the zeros of a function. We can easily prove the following with complete generality, for any first-order covariant Hamiltonian field theory. (See Proposition 1, in Chapter 2, section 2.3):

Theorem 3.2.1. *Assuming that $\mathcal{EL}$ of our first-order covariant Hamiltonian field theory is a submanifold of $J^1\mathcal{F}_M^*$, the restriction $\Pi(\mathcal{EL}) \subset T^*\mathcal{F}_{\partial M}$ of $\mathcal{EL}$ to the boundary ∂M is an isotropic submanifold of $T^*\mathcal{F}_{\partial M}$.*

Hamilton's variational principle

A precise statement of Hamilton's Principle can be given as follows:

The trajectories of a classical dynamical system with configuration space Q, Hamiltonian function $H \colon T^*Q \times [0,1] \to \mathbb{R}$ and endpoints $u_0, u_1 \in Q$ are the critical points of the action functional S, Eq. (3.2.2), restricted to the space of curves in T^*Q with fixed endpoints u_0, u_1.

In other words, let $u_0, u_1 \in Q$ be two points in the configuration space of the system. Consider now the space of maps:

$$\Omega_{u_0,u_1} = \{\chi = (u,p) \colon [0,1] \to T^*Q \mid u(0) = u_0, \;\; u(1) = u_1\} = \Pi^{-1}(T^*_{u_0}Q \times T^*_{u_1}Q) \,,$$

This is the space of all curves $\chi : [0,1] \to T^*Q, \chi(t) = (u(t), p(t))$ with local endpoints $u_0, u_1 : u(0) = u_0, u(1) = u_1$ and free values for the momenta endpoints.Then the trajectories of the system are the critical points of S restricted to Ω_{u_0,u_1}, i.e. the zeros of $\mathrm{d}(S \mid_{\Omega_{u_0,u_1}})$.

Notice that because Π is a submersion, Ω_{u_0,u_1} is a regular submanifold. Moreover $T^*_{u_0}Q \times T^*_{u_1}Q$ is a Lagrangian submanifold of $T^*\mathcal{F}_{\partial M} = \overline{T^*Q} \times T^*Q$ such that, not only $d\alpha$ but also α vanishes on it. Then, by Eq.(10) we obtain,

$$\mathrm{d}(S\mid_{\Omega_{u_0,u_1}})_\chi = \mathrm{EL}_\chi\,. \tag{3.2.11}$$

Thus, trajectories of the system are solutions of Euler-Lagrange equations with the given boundary conditions. The space of solutions of Euler-Lagrange equations $\mathcal{EL}$ is the union for all $u_0, u_1 \in Q$ of the corresponding spaces of solutions of Hamilton's equations.

If we consider now, instead of the Lagrangian submanifold $T^*_{u_0}Q \times T^*_{u_1}Q$, an arbitrary Lagrangian submanifold $\mathcal{L} \subset \overline{T^*Q} \times T^*Q$, we may modify Hamilton's principle and state that the trajectories of the classical dynamical system with Hamiltonian function $H\colon T^*Q \times [0,1] \to \mathbb{R}$ are the the critical points χ of the action functional S, Eq. (3.2.3), such that $\Pi(\chi) \in \mathcal{L}$. If we denote now by $\Omega_{\mathcal{L}}$ the space of curves such that their endpoints lie in $\mathcal{L}$, i.e.,

$$\Omega_{\mathcal{L}} = \Pi^{-1}(\mathcal{L})\,,$$

then because $\mathcal{L}$ is Lagrangian, $\alpha\mid_{\mathcal{L}}$ is closed, thus, locally there exists a function F on $\mathcal{L}$ such that $\alpha\mid_{\mathcal{L}}= \mathrm{d}F$. Hence, instead of Eq. (3.2.11), we get:

$$\mathrm{d}(S\mid_{\Omega_{\mathcal{L}}})_\chi = \mathrm{EL}_\chi + \Pi^*(\mathrm{d}F)_\chi\,,$$

and now, critical points of the action restricted to $\Omega_{\mathcal{L}}$ are zeros of the modified Euler-Lagrange form $\mathrm{EL} + \Pi^*(\mathrm{d}F)$.

3.3 Hamilton's generating function

In the previous section we have shown that for regular theories the space $\Pi(\mathcal{EL})$ is an isotropic submanifold of the space of fields at the boundary or, in terms of the particular instance of Hamiltonian dynamics, that the trace at the boundary of actual solutions of Hamilton's equations in the full interval determines an isotropic submanifold in $\overline{T^*Q} \times T^*Q$. In many instances it is true that such submanifold is maximal, i.e, it is a Lagrangian submanifold. In this section we will discuss two different approaches to reach this conclussion and, along the way, we will establish sufficient conditions that will guarantee the desired result.

Hamilton's generating function

Now we are ready to explore under which circumstances $\Pi(\mathcal{EL})$ is Lagrangian.

Theorem 3.3.1. *Consider Hamiltonian dynamics as a Hamiltonian field theory on a manifold M of dimension 1+0 with boundary. If the flow of the dynamics given by Hamilton's equations (3.2.8) exists for all* $t \in [0,1]$ *then* $\Pi(\mathcal{EL})$ *is a Lagrangian submanifold of* $J^1\mathcal{F}_{\partial M}$.

Proof. Consider the Hamiltonian vector field X_H defined by the Hamiltonian function H and its local flow φ_t. Now suppose that the flow is globally defined $\varphi_t\colon T^*Q \to T^*Q$ for all $t \in [0,1]$. This implies that the integral curves $\chi(t) = (u(t), p(t))$ of Hamilton's equations are defined for all t and for all initial data $(u_0, p_0) \in T^*Q$. Because of uniqueness and regular dependence on initial conditions of solutions of ode's the map φ_t is bijective and smooth. Moreover because X_H is Hamiltonian, its flow consists of symplectic diffeomorphisms,i.e. symplectomorphisms. Therefore $\varphi_1\colon T^*Q \to T^*Q$ is a symplectomorphism. We conclude by noticing that

$$\mathrm{graph}(\varphi_1) = \Pi(\mathcal{EL})\,,$$

where $\mathrm{graph}(\varphi_1) = \{(u_0, p_0; u_1, p_1) \in \overline{T^*Q} \times T^*Q \mid (u_1, p_1) = \varphi_1(u_0, p_0)\}$. Hence because the graph of a diffeomorphism between symplectic manifolds is symplectic iff its graph is a Lagrangian submanifold with respect to the difference of the pull-backs of the corresponding symplectic forms to the product manifold, we reach the conclusion, $\Pi(\mathcal{EL})$ is a Lagrangian submanifold of $\overline{T^*Q} \times T^*Q$. □

Notice that the key assumption in the previous proof is that solutions to Hamilton's equations exist for all $t \in [0,1]$. The existence theorem for solutions of ode's guarantees the existence of solutions for times small enough, but not necessarily larger than 1. If the space where the equations are defined were compact without boundary then the flow would exist for all t, however T^*Q is not compact.

The following is a familiar example of a Hamiltonian system which satisfies the conditions of Theorem 3.1. It therefore follows for this theory that $\Pi(\mathcal{EL})$ is Lagrangian: Free particle. In this case $Q = \mathbb{R}$ and the Hamiltonian is given by $H(u,p) = p^2/2m$. Hamilton's equations of motion are $\dot{u} = p/m$, $\dot{p} = 0$, whose solutions have the form $p(t) = p_0 = \text{constant}$, $u(t) = u_0 + (p_0/m)t$. The flow $\varphi_t\colon T^*\mathbb{R} \to T^*\mathbb{R}$ of the Hamiltonian vector field $X_H = p/m\partial/\partial u$ is defined for all t:

$$(u(t), p(t)) = \varphi_t(u_0, p_0) = (u_0 + p_0/mt, p_0)\,.$$

The Lagrangian submanifold $\Pi() \subset T^*\mathbb{R} \times T^*\mathbb{R} \cong T^*(\mathbb{R} \times \mathbb{R})$ is just the plane $p_1 = p_0 = m(u_1 - u_0)$. The following is an example of a Hamiltonian theory for which the flow under the Hamiltonian vector field is not defined for all $t \in [0,1]$. Therefore Theorem 3.1 cannot be applied to this theory to prove that $\Pi(\mathcal{EL})$ is a Lagrangian submanifold:

Quartic potential. As in the previous example $Q = \mathbb{R}$ but the Hamiltonian is given by $H(u,p) = p^2/2m - mu^4/4$. Hamilton's equations of motion are now $\dot{u} = p/m$, $\dot{p} = mu^3$. Notice that then $\ddot{u} = u^3$. The general solution can be found easily by noticing that H is a constant of the motion. Thus $p^2/2m - mu^4/4 = E =$ constant. Then $\dot{u} = \pm\sqrt{\frac{2E}{m} + u^4/2}$ and the general solution $u(t)$ with initial data u_0 is given by the elliptic integral:

$$t = \int_{u_0}^{u} \frac{du}{\pm\sqrt{2E/m + u^4/2}}\,.$$

Notice that for such solution $p_0 = m\dot{u}(0) = \pm\sqrt{2mE + m^2u_0^4/2}$.

Choosing for instance $E = 0$, we get $p_0 = mu_0^2/2$, and $\dot{u} = \pm u^2/\sqrt{2}$, whose solutions are given by:

$$u(t) = \frac{2u_0}{2 \mp u_0 t}\,.$$

Clearly, if $2/u_0 < 1$, then the solution $u(t)$ explodes before reaching $t = 1$ and the flow φ_t doesn't exist for all $t \in [0,1]$.

Theorem 3.1, though powerful, cannot be extended to higher-dimensional field theories because the notion of flow cannot be extended in a natural way beyond the $1+0$ dimensional theory. There is however and alternative idea that can be used in any dimension.

If the flow φ_1 exists globally, the Lagrangian submanifold $\Pi(\mathcal{EL}) \subset \overline{T^*Q} \times T^*Q$ is transverse to the projection onto the first factor of $\overline{T^*Q} \times T^*Q$, however it doesn't have to be transverse to the canonical projection onto $Q \times Q$ obtained by identifying $\overline{T^*Q} \times T^*Q$ with $T^*(Q \times Q)$. If $\Pi(\mathcal{EL})$ were transverse to the canonical cotangent bundle projection $\pi_{Q\times Q} \colon T^*(Q \times Q) \to Q \times Q$, then it would define the graph of a closed 1-form on $Q \times Q$, also $\Pi(\mathcal{EL})$ would be transverse to the fibers of $\pi_{Q\times Q}$. Hence if $\Pi(\mathcal{EL}) \cap \pi_{Q\times Q}^{-1}(u_0,u_1) \neq \emptyset$, any $\chi \in \Pi(\mathcal{EL}) \cap \pi_{Q\times Q}^{-1}(u_0,u_1)$ will be a solution of Hamilton's equations and it will be defined for all $t \in [0,1]$ with endpoints u_0, u_1. Moreover, in this case, because $\Pi(\mathcal{EL})$ will define the graph of a closed 1-form on $Q \times Q$, there will exist a neighborhood V of $(u_0,u_1) = \pi_{Q\times Q}(\chi)$ and a function W defined on V such that $\Pi(\mathcal{EL}) \cap \pi_{Q\times Q}^{-1}(V) = \mathrm{graph}(dW)$. Such local function is called a generating function of the Lagrangian submanifold.

Actually, to prove that a submanifold is Lagrangian it suffices to do it locally, then the picture described in the previous paragraph holds locally. Moreover it suffices to assume that there exist solutions of Hamilton's equations on an open neighborhood of (u_0, u_1) to guarantee it as shown in the following lemma.

Lemma 3.3.2. *Let $H: T^*Q \to \mathbb{R}$ be a Hamiltonian function. Given two points u_0, u_1 in Q. Suppose there exists an open neighborhood V of (u_0, u_1) such that for any $(u_0', u_1') \in V$ there exist solutions of Hamilton's equations $\chi(t) = (u(t), p(t))$, $t \in [0,1]$, for which $u(0) = u_0'$, $u(1) = u_1'$. Then there exists an open neighborhood U of $(u_0, p_0) \in T^*Q$ with $p_0 = p(0)$ such that $\Pi(\mathcal{EL}_U)$ is transversal to the canonical projection $\pi_{Q\times Q}$, where $\mathcal{EL}_U$ denotes the space of solutions of Hamilton's equations with initial data lying in U.*

Proof. Consider a solution of Hamilton's equations $\chi(t) = (u(t), p(t))$ such that $u(0) = u_0$ and $u(1) = u_1$. Then if we denote by $p_0 = p(0) \in T^*_{u_0}Q$, clearly there exists a flow box around the point (u_0, p_0) such that the flow φ_t of the Hamiltonian vector field X_H defined by the Hamiltonian function H is defined for all $t \in [0, 1+\epsilon)$. Notice that because of the existence and uniqueness theorem for initial value problems for ode's we can extend the solution $\chi(t)$ arriving at time 1 to u_1 for some time $0 < t < 2\epsilon$, then because of the regular dependence of solutions on initial conditions, we can choose an open neighborhood U of (u_0, p_0) small enough such that the flow for all points in U is defined for all $t \in [0, 1+\epsilon)$.

Hence, because the flow φ_t consists of local symplectic diffeomorphisms the manifold $\varphi_1(T^*_{u_0}Q)$ is a regular Lagrangian submanifold. Moreover the space $\Pi(\mathcal{EL}_U)$ is transversal to the canonical projection $\pi_{Q\times Q}$. We argue by contradiction, if this were not so, then the projection $\pi_{Q\times Q}$ restricted to $\Pi(\mathcal{EL}_U)$ will not be a submersion. But notice that $\Pi(\mathcal{EL}_U)$ is just the graph of the diffeomorphism $\varphi_1: U \to T^*Q$, hence $\pi_{Q\times Q}$ restricted to $\Pi(\mathcal{EL}_U)$ will not be open. Then shrinking U if necessary, this contradicts the existence of a neighborhood of (u_0, u_1) all of whose points lie in $\pi_{Q\times Q}\mid_{\Pi(\mathcal{EL}_U)}$. □

Definition 3.3.3 (Dirichlet's assumption)**.** *We will say that a theory is Dirichlet if for any boundary data φ, there exists a unique solution of Euler-Lagrange equations $\chi = (\Phi, P) \in \mathcal{EL}$ such that $\Phi\mid_{\partial M} = \varphi$.*

Thus a covariant Hamiltonian dynamics with configuration space Q will be Dirichlet, or satisfy Dirichlet's assumption, if for each pair $(u_0, u_1) \in Q \times Q$, there exists a unique solution $(u(t), p(t))$ on the interval $[0, 1]$ of Hamilton's equations such that $u(0) = u_0$ and $u(1) = u_1$. Clearly, Examples 3.3 and 3.3 are not Dirichlet, but Example 3.2, the free particle, is Dirichlet. For Dirichlet theories it is easy to prove the following theorem.

Theorem 3.3.4. *Suppose our Hamiltonian dynamical system is Dirichlet. The space* $\Pi()$ *of solutions of Euler-Lagrange equations at the boundary is a Lagrangian submanifold with generating function given by Hamilton's principal function* W*.*

For Dirichlet theories the proof is based on the construction of an explicit generating function for $\Pi(\mathcal{EL})$. In the case of Hamiltonian dynamics such generating function is just Hamilton's principal function defined as follows:

$$W(u_0, u_1) = \int_0^1 \chi^* \Theta_H = \int_0^1 (p_a(t)\dot{u}^a(t) - H(t, u(t), p(t))dt \,,$$

where $\chi(t) = (u(t), p(t))$ is the unique solution to Hamilton's equations with endpoints u_0, u_1. The notion of Hamilton's principal function can be easily extended to any Dirichlet theory as follows:

$$W(\varphi) = \int_M \chi^* \Theta_H \,, \tag{3.3.1}$$

where $\chi \in \mathcal{EL}$ denotes the unique solution with boundary data φ.

The Dirichlet condition is too strong, it implies that the generating function W in Eq. (3.3.1) is defined globally. The following example of the planar pendulum does not have a unique solution but rather it has two solutions joining each pair of endpoints. Thus it is not Dirichlet and we cannot apply Theorem 3.6 to show that $\Pi(EL)$ is a Lagrangian submanifold of $T^*Q \times T^*Q$.

The planar pendulum. Consider now the system with $Q = S^1$, $T^*Q \cong S^1 \times \mathbb{R} = \{(\theta, p) \mid 0 \leq \theta < 2\pi, p \in \mathbb{R}\}$, and Hamiltonian function $H(\theta, p) = p^2/2m - k\cos\theta$, where m and k are positive constants. The phase space is the cylinder over the circle S^1. Given two angles θ_0, θ_1, there are always infinitely many trajectories joining them with time 1. (In particular, for $\theta_0 = \theta_1$, one can realize any winding number $n \in \mathbb{Z}$ of a trajectory.) Moreover they are separated in the space of curves $C^\infty([0,1], S^1 \times \mathbb{R})$.

We can relax the Dirichlet assumption, allowing Hamilton's Principal function to be defined locally:

Definition 3.3.5. *Let* $\tau \colon T^*\mathcal{F}_{\partial M} \to \mathcal{F}_{\partial M}$ *denote the canonical projection map. We will say that a theory is locally Dirichlet if* τ *restricted to* $\Pi(\mathcal{EL}))$ *is open and if given any boundary data* $\varphi \in \tau(\Pi(\mathcal{EL}))$ *the solutions* $\chi = (\Phi, P) \in \mathcal{EL}$ *of the Euler-Lagrange equations such that* $\tau(\chi) = \Phi\mid_{\partial M} = \varphi$*, are isolated.*

Notice that Example 3.3, the planar pendulum, is locally Dirichlet. The following Hamiltonian system however, is not locally Dirichlet:

Geodesic flow on the sphere. The system we consider now is the geodesic flow on the sphere S^2 for the metric induced by the Euclidean metric in $\mathbb{R}^3$. In this case

$Q = S^2$, $T^*Q = T^*S^2$ and the Hamiltonian function is given by $H(\mathbf{u}, \mathbf{p}) = \frac{1}{2}\mathbf{p} \cdot \mathbf{p}$, where $\mathbf{u} \in S^2$ is a unitary vector in $\mathbb{R}^3$, i.e., $\mathbf{u} \cdot \mathbf{u} = 1$, and $\mathbf{p} \in \mathbb{R}^3$ satisfies that $\mathbf{u} \cdot \mathbf{p} = 0$. (Notice that we are identifying T^*S^2 with TS^2 by using the metric on S^2.)

The space of solutions of Euler-Lagrange equations are just the integral curves χ of the Hamiltonian vector field defined by H. The projection $u(t)$ of the integral curves χ are just maximal geodesics in S^2, that is, maximal circles in S^2. This theory is not locally Dirichlet because if any two points $\mathbf{u}_0$, $\mathbf{u}_1$ in the sphere can be joined by a maximal circle, in the case that $\mathbf{u}_0$, $\mathbf{u}_1$ are antipodal, that is $\mathbf{u}_1 = -\mathbf{u}_0$, then there is family of maximal circles passing through them. Actually this family is parametrized by any circle in the sphere transverse to them, and these solutions are not isolated.

Note however that in this case there is an obvious symmetry of the theory that is responsible for this phenomena and that by reducing the theory, such ambiguity will be removed.

Now, for locally Dirichlet theories a theorem stating that $\Pi(\mathcal{EL})$is Lagrangian can be easily proved.

Theorem 3.3.6. *Let (J^1E^*, H) be a covariant Hamiltonian first order field theory that is locally Dirichlet, i.e., such that for any boundary data φ, the solutions of Euler-Lagrange equations $\chi = (\Phi, P) \in \mathcal{EL}$ such that $\Phi \mid_{\partial M} = \varphi$ are isolated, then $\Pi(\mathcal{EL})$ is a Lagrangian submanifold of $T^*\mathcal{F}_{\partial M}$.*

Proof. We will write the proof in the particular instance of Hamiltonian dynamics we have been discussing so far, however the proof in the general situation is a straight forward extension of this case.

The proof amounts to showing that Hamilton's principal function exists locally. Let $(u_0, u_1) \in Q \times Q$ and let $\chi(t) = (u(t), p(t))$ be an integral curve of Hamilton's equations such that $u(0) = u_0$ and $u(1) = u_1$. Let V be an open neighborhood of χ in $\mathcal{EL}$ that doesn't contain another solution with the same boundary data. (We use initial data to provide a topology for $\mathcal{EL}$, i.e. an open neighborhood of χ is the set of all solutions with initial data in an open neighborhood V of (u_0, u_1).) Then, because of Lemma 3.3.2, there is an open neighborhood $V' \subset V$ such that every $\chi' \in V'$ is separated from any other solution of Hamilton's equations with the same boundary data as χ'. Now, define the function $W \colon V' \to \mathbb{R}$ as:

$$W(u_0', u_1') = \int_{[0,1]} \chi'\Theta_H = \int_0^1 (p_a'(t)\dot{u}'^a(t) - H(t, u'(t), p'(t))\, dt$$

where $\chi'(t) = (u'(t), p'(t))$ is the only solution in V' with boundary conditions (u_0', u_1'). Then, W is a generating function for $\Pi(V')$, $V' \subset \mathcal{EL}$ is open, $\alpha \mid_{\Pi(V')} = dW$

and $\Pi(V')$ is Lagrangian. But $\Pi(V')$ is an open subset of $\Pi(\mathcal{EL})$, for any (u_0, u_1), hence $\Pi(\mathcal{EL})$ is Lagrangian. $\square$

The following in an example of a Hamiltonian dynamical system which is neither Dirichlet nor locally Dirichlet but for which $\Pi(\mathcal{EL})$ is a Lagrangian submanifold of $\overline{T^*Q} \times T^*Q$. Thus neither the Dirichlet or the locally Dirichlet properties are necessary and Theorem 3.10 can only be said to be providing sufficient conditions for $\Pi(\mathcal{EL})$ to be Lagrangian.

Consider the example of the Hamiltonian system $H(u,p) = p_a X^a(u)$ where $X = X^a(u)\frac{\partial}{\partial u^a}$ is a vector field on Q. Hamilton's equations of motion are given by,

$$\dot{u}^a = X^a(u)\,, \qquad \dot{p}_a = -p_b \frac{\partial X^b}{\partial u^a}\,. \tag{3.3.2}$$

and the flow φ_t of the system is the cotangent lifting of the flow φ_t^X of the vector field X i.e. $\varphi_t = (T\varphi_{-t}^X)^*$. Thus if X is complete so is X_H and the flow φ_1 exists. X_H is the complete lifting to T^*Q of the vector field X. Therefore by Theorem 3.1, $\Pi(\mathcal{EL})$ is Lagrangian. Notice however that only points u_0, u_1 such that $u_1 = \varphi_1^X(u_0)$ are joined by solutions of Hamilton's equations. Thus $\Pi_{Q\times Q}(\Pi(\mathcal{EL})) = \text{graph}(\varphi_1^X)$ and the theory is neither Dirichlet nor locally Dirichlet.

Regular theories and topological phases

Consider as before a Hamiltonian theory $(T^*Q \times [0,1], H)$, but now we assume that the Hamiltonian function has the form:

$$H_\lambda(u,p) = \frac{\lambda}{2} p_a p^a + p_a X^a(u) \qquad \lambda \geq 0\,.$$

This Hamiltonian has the typical form of the Hamiltonian in many field theories. It has a kinetic term $\frac{\lambda}{2} p_a p^a$ that makes it regular (that is, there exists a well defined invertible Legendre transform) and a linear term $p_a X^a(u)$, where $X = X^a(u)\partial/\partial u^a$ is a vector field on Q. The solutions of the Euler-Lagrange equations of the theory are the integral curves of the Hamiltonian vector field X_{H_λ}, i.e., solutions of the system of differential equations:

$$\dot{u}^a = \lambda p^a + X^a(u)\,, \qquad \dot{p}_a = -p_b \frac{\partial X^b}{\partial u^a}\,. \tag{3.3.3}$$

These equations can be written as a system of second order differential equations on u^a. Differentiating the first equation in (3.3.3) we get:

$$\ddot{u}^a = \lambda \dot{p}^a + \frac{\partial X^a}{\partial u^b}\dot{u}^b = X_b \frac{\partial X^b}{\partial u^a} + \frac{\partial X^a}{\partial u^b}\dot{u}^b - \frac{\partial X^b}{\partial u^a}\dot{u}^b\,.$$

These equations are the Euler-Lagrange equations defined by the Lagrangian $L\colon TQ \to \mathbb{R}$,

$$L(u,\dot u) = \frac{1}{2\lambda}(\dot u^a - X^a(u))(\dot u_a - X_a(u))$$

i.e., critical points of:

$$S = \frac{1}{2\lambda}\int_0^1 ||\dot u - X(u)||^2 dt\,,$$

which is a shifted version of the standard equation for geodesics.

Now the limit $\lambda \to 0$ gives the Hamiltonian $H_0 = p_a X^a(u)$ discussed in Sect. 3.3, Ex. 3.3. The equations of motion are given by Eqs. (3.3.2). The theory is shown there in Example 3.11 to not be Dirichlet nor locally Dirichlet. The theory for $\lambda \neq 0$ however is Dirichlet: Notice that the geodesic flow is complete on a complete manifold.

We must recall that the limit $\lambda \to 0$ of the action functional S_λ in the Hamiltonian formalism given by,

$$S_\lambda = \int_0^1 p_a du^a - H_\lambda(u,p)\,dt = \int_0^1 \left(p_a \dot u^a - \frac{\lambda}{2} p_a p^a - p_a X^a(u)\right) dt$$

becomes

$$S_0 = \int_0^1 p_a(\dot u^a - X^a(u))dt\,.$$

which represents a "topological phase" of the system in the sense that it no longer depends on a metric and the symmetry group is larger that the group of isometries of the metric used to construct H_λ, $\lambda \neq 0$. Actually, the group of symmetries of the theory is the group of diffeomorphisms of Q commuting with the flow of X.

This situation corresponds to what happens in the case of the Hamiltonian formulation of Yang-Mills theories [**Ib15**]. The action of the theory can be written as:

$$S_{YM,\lambda}(A,P) = -\int_M \left(P_a^{\mu\nu} F_{\mu\nu}^a + \frac{\lambda}{4} P_{\mu\nu}^a P_a^{\mu\nu}\right) \mathrm{vol}_M$$

and we see that if we take the limit $\lambda \to 0$ in the Yang-Mills action above, we get:

$$S_{YM,0}(A,P) = \int_M P_a^{\mu\nu} F_{\mu\nu}^a \mathrm{vol}_M$$

whose equations of motion are given by:

$$F_A = 0\,, \qquad d_A^* P = 0\,.$$

Thus the moduli space of solutions of Euler-Lagrange equations is given by:

$$\mathcal{M} = \{F_A = 0, d_A^* P = 0\}/\mathcal{G}_M \,,$$

where $\mathcal{G}_M$ denotes the group of gauge transformations of the theory.

Dynamics at the boundary

We discuss briefly the relation between the Euler-Lagrange equations of the theory and the dynamics near the boundary. We have been making implicit use of this relation, so it bears spelling out.

Hamiltonian dynamics on T^*Q considered as a field theory in dimension $1+0$ is defined on the space of 'fields' $C^\infty([0,1]; T^*Q)$ and has canonical 1-form Θ_H defined on $J^1E^* \cong T^*Q \times [0,1]$ (recall Sects. 3.2). From this point of view solutions of Euler-Lagrange equations of the theory are functions $\chi\colon [0,1] \to T^*Q$ such that (see Sect. 3.2, Eq. (3.2.7):

$$\chi^*(i_Z d\Theta_H) = 0\,, \qquad \forall Z \in \mathfrak{X}(T^*Q)\,. \tag{3.3.4}$$

The tangent vector $(\dot{u}, \dot{p})$ to the curve $\chi(t)$ then must lie in the kernel of $d\Theta_H$ and after a trivial computation we get $\dot{u}^a = \partial H/\partial p_a$, $\dot{p}_a = -\partial H/\partial u^a$. Hence the space $\mathcal{EL}_M$ of solutions of Euler-Lagrange equations of the theory are functions χ on $[0,1]$ satisfying the standard Hamilton's equations that in covariant form are given by Eq. (3.3.4). A Hamiltonian dynamics interpretation as an initial value problem can always be achieved near the boundary. In this situation $\partial([0,1]) = \{0,1\}$, and a collar of the boundary U_ϵ has the form, $U_\epsilon = [0,\epsilon) \cup (1-\epsilon, 1]$, $\epsilon < 1/2$. For ϵ small enough, the equations of motion derived from the restriction of the action functional S (Sect. 3.2), Eq. (4.2.4)), i.e., Hamilton's equations, always have a unique solution as an initial value problem for any initial data $(u_0, p_0; u_1, p_1) \in \overline{T^*Q} \times T^*Q$, and $t \in U_\epsilon$, because of the existence and uniqueness theorem for ode's. Thus the Hamiltonian dynamical evolution interpretation of field theories near the boundary corresponds exactly with the Hamiltonian mechanical picture of the theory, while the field theoretic description corresponds to a picture in the bulk where fields, i.e., curves $\chi(t)$ are defined for all $t \in [0,1]$.

Hamiltonian dynamics, considered as a field theory, doesn't lead to constraints and there is no need to proceed to reduce the theory at the boundary. The canonical description at the boundary, contrary to what happens in general in higher dimensions, is always defined on all $T^*Q \times T^*Q \cong T^*\mathcal{F}_{\partial M}$.

3.4 Constraints

The Euler-Lagrange equations of a theory with constraints

We will now introduce constraints on the covariant phase space $J^1E^* \cong T^*Q \times [0,1]$ of Hamiltonian dynamics. As explained in the introduction, the geometry of constrained Hamiltonian dynamics should give us insight into the geometry of Palatini's gravity. The latter is obtained as the limit $\lambda \to 0$ of a Yang-Mills theory together with an explicit constraint on the momenta fields of the theory.

Let $K \times Q \times [0,1] \to [0,1]$ be the trivial bundle where K, for instance, is a Lie group. Guided by the situation we find in Palatini's gravity, we assume that there is a bundle map $\Sigma : K \times Q \times [0,1] \to J^1E^* \cong T^*Q \times [0,1]$ given by $\Sigma(k,u,t) = (u,p,t)$ with $p = \sigma(k)$ where $\sigma : K \to pr_2(T^*Q)$. Thus the momenta p are restricted by the (in general, non-linear) map σ, that is, they lie in the image of σ and we denote the image of Σ as N which is a submanifold of $T^*Q \times [0,1]$.

We will introduce the constraint in the variational principle by means of a Lagrange multiplier. We have to restrict the action functional $S(\chi)$ that was defined on $J^1\mathcal{F}^* \cong C^\infty([0,1], T^*Q)$, to the submanifold $\mathcal{N} \subset C^\infty([0,1], T^*Q)$ defined as:

$$\mathcal{N} = \{\chi\colon [0,1] \to T^*Q \mid \chi(t) = (u(t), p(t))\,, p(t) \in \mathrm{Im}(\Sigma)\}\,.$$

The space of sections $\mathcal{S}$ of the (trivial) bundle $K \times Q \times [0,1] \to [0,1]$ can be identified with the space of maps $C^\infty([0,1], K \times Q)$. Then the bundle map Σ induces a map (denoted with the same symbol) $\Sigma\colon \mathcal{S} \to J^1\mathcal{F}^*$, given by $(e(t), u(t)) \mapsto \chi(t) = (u(t), p(t) = \Sigma(e(t))$. With these notations we also have that the submanifold $\mathcal{N}$ can be identified with the image of the map Σ, that is, $\mathcal{N} = \{\chi\colon [0,1] \to T^*Q \mid \chi(t) = \Sigma(e(t), u(t)),\ (e(t), u(t)) \in \mathcal{S}\}$.

A natural extension of Lagrange's multipliers theorem will allow us to obtain the critical points of $S\mid_{\mathcal{N}}$ in terms of the critical points of the extended functional:

$$\begin{aligned}\mathbb{S}(\chi, \Lambda, e) &= S(\chi) + \langle \Lambda, p - \Sigma(e)\rangle \\ &= \int_0^1 (p_a(t)\dot{u}^a(t) - H(t, u(t), p(t))dt + \int_0^1 \Lambda^a(t)(p_a(t) - \Sigma_a(e(t))) \end{aligned} \qquad (3.4.1)$$

Notice that the Lagrange multiplier Λ lies in the dual of J^1E^*, i.e., in J^1E (see discussion below). A trivial computation of $\mathrm{d}\mathbb{S}$, shows:

$$\begin{aligned} \mathrm{d}\mathbb{S}_{(\chi,\Lambda,e)}(\delta\chi,\delta\Lambda,\delta e) &= \int_0^1 \delta p_a(\dot{u}^a - \frac{\partial H}{\partial p_a} + \Lambda)^a dt - \int_0^1 \delta u^a(\dot{p}_a + \frac{\partial H}{\partial u^a})dt \\ &+ \int_0^1 \delta\Lambda^a(p_a - \Sigma_a(e))dt - \int_0^1 \delta e\frac{\partial \Sigma_a}{\partial e}dt + \text{boundary terms} \end{aligned} \tag{3.4.2}$$

Thus, from $\delta\mathbb{S}_{u,p,\Lambda,e)}(\delta u,\delta p,\delta\Lambda,\delta e) = 0$ for all $\delta u,\delta p,\delta\Lambda,\delta e$, we get the new Euler-Lagrange equations of the theory:

$$\dot{u}^a = \frac{\partial H}{\partial p_a} - \Lambda^a\,, \qquad \dot{p}_a = -\frac{\partial H}{\partial u^a}\,, \tag{3.4.3}$$

together with the constraints given by the restriction to the submanifold $\mathcal{N}$:

$$p_a = \Sigma_a(e)\,, \tag{3.4.4}$$

and the new constraint:

$$\Lambda^a\frac{\partial \Sigma_a}{\partial e} = 0\,. \tag{3.4.5}$$

So we see that we have obtained the former equations of motion, where only the equation for $\dot{u}^a$ changes with the addition of the Lagrange multiplier $-\Lambda^a$, and the constraints imposed by the submanifold $\mathcal{N}$ automatically implemented. The Lagrange multiplier must in turn satisfy the constraint imposed by Λ^a, i.e., $\Lambda^a\partial\Sigma_a/\partial e = 0$.

We will analyze first the meaning of the new constraint Eq. (3.4.5). Clearly the map $\Sigma_*\colon TF \to T(J^1E^*)$ can be restricted, for any u and t (because it is time and u-independent) to a map $\Sigma_{(u,t)}*\colon T_eK \to T_{(u,p)}(T^*Q)$, where $p = \Sigma(e)$. Clearly $\Sigma_{(u,t)*}$ maps T_eK into the tangent space to the submanifold N at the point $(u,\Sigma(e))$. That is, tangent vectors to N have the form $V = \Sigma_{(u,t)*}(e)(W)$ with $W \in T_eK$.

Given a manifold M and a subspace $S \subset T_xM$, $x \in M$, we will denote by S^0 the annihilator of S, that is, $S^0 = \{\alpha \in T_x^*M \mid \alpha(V) = 0\,, \forall V \in T_xM\}$. In the previous setting $S = T_{(u,p)}N \subset T_{(u,p)}(T^*Q)$. Then because $N = \cup_{u\in Q}\Sigma_{(u,t)}(K)$, then $T_{u,\Sigma(e)}N = T_uQ \oplus \Sigma_{(u,t)*}T_eK$. But now $S^0 \subset T^*(T^*Q)$ and, at the point $(u,\Sigma(e))$ we can identify $T^*_{(u,\Sigma(e)}(T^*Q)$ with $T_u^*Q \oplus T^*_{\Sigma(e)}(T_u^*Q)$, hence $T_u^*Q \oplus T_uQ$. Then the vectors in $(T_{u,p}N)^0$ will have the form $(0,\Lambda)$ with $\Lambda = \Lambda^a\partial/\partial u^a \in T_uQ$ satisfying that $\langle\Lambda, V\rangle = 0$ for all $V = \Sigma_{(u,t)*}(W)$, $W \in T_eK$. Or, in other words, $\Lambda^a\partial\Sigma_a/\partial e = 0$ using a local chart in T^*Q adapted to the map Σ.

Thus we have shown that the new constraint (3.4.5) is just the condition that the Lagrange multiplier Λ lies in the annihilator or the polar to the tangent space to the constraint submanifold. For this reason we will call this constraint the polar constraint. Hence the term added to the equation for $\dot{u}$ in (3.4.3) means that because of the constraints in phase space, there is an extra freedom in the equations of motion: any vector in the polar space to the constraint submanifold can be added to the equations of motion.

Once we understand the modified Euler-Lagrange equations of the theory, it remains to determine the implications near the boundary.

Constraints and the dynamics at the boundary

Near the boundary, as discussed in the last section, the dynamics is described by the vector field on T^*Q given by:

$$X = \left(\frac{\partial H}{\partial p_a} - \Lambda^a \right) \frac{\partial}{\partial u^a} - \frac{\partial H}{\partial u^a} \frac{\partial}{\partial p_a} \,, \tag{3.4.6}$$

together with the constraints (now understood as defining a submanifold at the boundary) given by Eqs. (3.4.4), (3.4.5). Thus the first task is to check that they are consistent.

If we denote by $\mathcal{M}_0 = T^*Q \oplus_Q TQ \times K$ the extended phase space with elements (u, p, Λ, e), the vector field X defines a dynamical system on the submanifold $\mathcal{M}_1$ defined by the constraints above, that is,

$$\mathcal{M}_1 = \{(u, p, \Lambda, e) \in \mathcal{M}_0 \mid p = \Sigma(e),\, \Lambda \in (TN)^0\} \,.$$

This system has a presymplectic picture as in the standard discussion for Hamiltonian field theories at the boundary [**Ib15**]. Consider the presymplectic form Ω_0 on $\mathcal{M}_0$ obtained as the pull-back of the canonical 2–form ω on T^*Q along the canonical projection from $\mathcal{M}_0$ onto T^*Q, i.e., $\Omega_0 = du^a \wedge dp_a$. Define now the Hamiltonian function on $\mathcal{M}_0$:

$$\mathcal{H}_0 = H(u, p) + \Lambda^a (p_a - \Sigma_a(e)) \,.$$

The presymplectic system $(\mathcal{M}_0, \Omega_0, \mathcal{H}_0)$ is equivalent to the constrained dynamical field X above. We will analyze it using Gotay's constraints algorithm for presymplectic systems [**Go78**]. If we consider the solutions to the vector fields Γ satisfying the equation,

$$i_\Gamma \Omega_0 = d\mathcal{H}_0 \,, \tag{3.4.7}$$

we find easily that they are integral curves of vector fields of the form given by Eq. (3.4.6) together with the constraint defined by the map Σ and the polar constraint.

Actually, Eq. (3.4.7), would have a solution iff $i_Z d\mathcal{H}_0 = 0$ for all $Z \in \ker \Omega_0$. But $\ker \Omega_0 = \mathrm{span}\{\partial/\partial\Lambda^a, \partial/\partial e\}$, hence we get:

$$
\begin{aligned}
0 &= i_{\partial/\partial\Lambda^a} d\mathcal{H}_0 = \frac{\partial \mathcal{H}_0}{\partial \Lambda^a} \quad \text{iff} \quad p_a = \Sigma_a(e)\,, \\
0 &= i_{\partial/\partial e} d\mathcal{H}_0 = \frac{\partial \mathcal{H}_0}{\partial e} \quad \text{iff} \quad \Lambda^a \frac{\partial \Sigma_a}{\partial e} = 0\,.
\end{aligned}
$$

Hence the submanifold $\mathcal{M}_1$ is just the primary constraints submanifold of the theory.

Consider now the restriction of the dynamics to $\mathcal{M}_1$, that is, we are looking for a vector field on $\mathcal{M}_1$ of the form,

$$
\Gamma = \left(\frac{\partial H}{\partial p_a} - \Lambda^a \right) \frac{\partial}{\partial u^a} - \frac{\partial H}{\partial u^a} \frac{\partial}{\partial p_a} + C^a \frac{\partial}{\partial \Lambda^a} + D \frac{\partial}{\partial e}\,,
$$

such that

$$
i_\Gamma \Omega_1 = d\mathcal{H}_1,
$$

where Ω_1 is the restriction of Ω_0 to $\mathcal{M}_1$ and likewise for $\mathcal{H}_1$.

Computing the derivative of the constraint functions $\phi_a = p_a - \Sigma_a(e)$ with respect to the vector field Γ we get,

$$
\Gamma(\phi_a) = \dot{p}_a - \frac{\partial \Sigma_a}{\partial e} \dot{e} = -\frac{\partial H}{\partial u^a} - D \frac{\partial \Sigma_a}{\partial e} = 0\,,
$$

on $\mathcal{M}_1$. However on $\mathcal{M}_1$, contracting the last expression with Λ^a and using the polar constraint we get that along $\mathcal{M}_1$,

$$
\Lambda^a \frac{\partial H}{\partial u^a} = 0\,, \qquad \forall \Lambda \in (TN)^0\,. \tag{3.4.8}
$$

We conclude that $\frac{\partial H}{\partial u^a} \in TN$. But if $\frac{\partial H}{\partial u}$ is a vector in the tangent space to the constraint manifold, then there exists a vector D tangent to K (perhaps not unique) such that $\partial H/\partial u^a = -\partial \Sigma_a/\partial e$ and consequently the constraints ϕ_a are stable.

Moreover, computing the evolution of the polar constraint $\psi = \Lambda^a \partial \Sigma_a/\partial e$ with respect to the dynamics, we get that on $\mathcal{M}_1$

$$
\Gamma(\psi) = \dot{\Lambda}^a \frac{\partial \Sigma_a}{\partial e} + \Lambda \frac{\partial^2 \Sigma_a}{\partial e^2} \dot{e} = C^a \frac{\partial \Sigma_a}{\partial e} + D \Lambda^a \frac{\partial^2 \Sigma_a}{\partial e^2} = 0\,.
$$

Then given D, the vector C^a is determined (not uniquely) from the last condition and there are no further constraints. The constraints algorithm ends here and $\mathcal{M}_1$ is the final constraints submanifold.

The reduced space of the system $\mathcal{R}$ is obtained by quotienting $\mathcal{M}_1$ by the null directions Ω_1. Denoting by $K_1 = \ker\Omega_1$ the integrable characteristic distribution of Ω_1,we then have that

$$\mathcal{R} = \mathcal{M}_1/\mathcal{K}_1\,,$$

where $\mathcal{K}_1$ denotes the leaves of the distribution K_1.

Finally, because

$$\Omega_1 = \Omega_0\mid_{\mathcal{M}_1} = du^a \wedge d\Sigma_a = \frac{\partial \Sigma_a}{\partial e} du^a \wedge de\ ,$$

$K_1 = \ker\Omega_1$ is spanned by the vector fields $\partial/\partial\Lambda^a$ and $\Lambda^a\partial/\partial u^a$. Now notice that because $[\partial/\partial\Lambda^a, \Lambda^b\partial/\partial u^b] = \partial/\partial u^a$, we can compute the quotient in stages. Quotienting first by the distribution generated by $\partial/\partial\Lambda^a$ we get,

$$\mathcal{M}_1/\mathrm{span}\{\partial/\partial\Lambda^a\} \cong N\,,$$

and, finally,

$$\mathcal{R} = N/TN^0\,,$$

where now $TN^0 \subset TN$ is the tautological distribution associated to the constraint $p = \Sigma(e)$. Notice that because of Eq. (3.4.8), the Hamiltonian function H descends to the quotient and defines a Hamiltonian system on the reduced space $\mathcal{R}$.

Chapter 4

On a covariant Hamiltonian description of Palatini's gravity on manifolds with boundary

4.1 Introduction

Our understanding of the Hamiltonian structure of Gravity has taken half a century. The initial difficulties faced by Dirac and Bergmann [Be58],[Be81], were slowly resolved through the work of Arnowitt, Deser and Misner [Ar62], all the way to Ashtekar's formulation [As87]. At least part of the motivation has been to place the theory of gravity on grounds that will make it suitable for a canonical quantization scheme.

In [Ro06], C. Rovelli illustrated a simple Hamiltonian formulation of General Relativity which is manifestly 4D generally covariant and that drops the reference to the underlying spacetime in Palatini's formulation of gravity. Rovelli's proposal is highly geometric and constructs its space as the $4 + 16 + 24$ dimensional space $\widetilde{\mathcal{C}}$ with local coordinates $(x^\mu, e^I_\mu, A^{IJ}_\mu)$. In a further effort at extracting the geometric essence of such space, the variables x^μ are dropped (accounting by the invariance of the theory under global diffeomorphisms) and we are led to a 40 dimensional space $\mathcal{C}$ [Ro01]. The disappearance of the spacetime manifold M and its coordinates x^μ, which survive only as arbitrary parameters on the 'gauge orbits' of the canonical geometrical structure defined on it, generalizes the disappearance of the time coordinate in the ADM formalism and is analogous to the disappearance of the Lagrangian evolution parameter in the Hamiltonian theory of a free particle [Ro01]. It simply means that the general relativistic spacetime coordinates are not directly related to observations.

This chapter is based on joint work with Alberto Ibort, [**Ib16b**]. Our program here is similar to Rovelli's but our inspiration is the geometric foundations of covariant first-order Hamiltonian field theories on manifolds with boundary, laid out in Chapter 2. There the role of a covariant phase space for a first-order Hamiltonian theory modeled on the affine dual space of the first jet bundle of the bundle defining the fields of the theory is assessed and the crucial role played by boundaries as determining symplectic spaces of fields defining the classical counterpart of the quantum states of the theory is stressed in accordance with the point of view expressed in [Sc51].

Actually a generally covariant notion of instantaneous state, or evolution of states and observables in time, make little physical sense. They are always referred to an initial data space-like surface that in the picture presented here, corresponds to the boundary of the spacetimes of events. Such notion does not really conflict with diffeomorphism invariance because a diffeomorphism of a smooth manifold with smooth boundary restricts to a diffeomorphism of the boundary. Thus providing that the notion of boundary of a spacetime is incorporated in the basic description of the theory, we may still consider diffeomorphism invariance as a fundamental notion without contradicting it.

The covariant phase space of the theory carries a natural multisymplectic structure which is the exterior differential of a canonical m-form Θ defined on it. This geometric structure has been considered in various guises in the various variational formulations of field theories, however its first use in the present setting is to assist in identifying the nature of the different fields of the theory. Thus it will be show how the vierbein fields e_μ^I correspond to an algebraic constraint imposed on the momenta fields of the theory. The corresponding action will be seen to be invariant under the group of all automorphisms of the geometric structure and it will induce the corresponding reduction on the space of gauge fields at the boundary. This reduction process is interpreted as the appropriate setting for the 'elimination' of the spacetime M. The space of physical classical solutions of the theory in the bulk is the moduli space of the space of solutions of the Euler-Lagrange equations with respect to the group of automorphisms. The phase space of physical degrees of freedom of the theory, associated to its boundary, is the reduced symplectic manifold of fields at the boundary.

4.2 The geometry of the covariant phase space for Yang-Mills theories

As discussed in the introduction, our approach to Palatini's gravity will be to consider it as a constrained first-order covariant Hamiltonian field theory on a manifold with boundary obtained as a topological phase of a gauge theory. We will first briefly review the geometric setting for covariant first-order Hamiltonian Yang-Mills theories and the topological phase that will interest us.

A brief account of the multisymplectic formalism for first order covariant Hamiltonian Yang-Mills theories on manifolds with boundary

We briefly review some of the basic notions and notations for first-order covariant Hamiltonian field theories. For a fuller exposition we refer the reader to Chapter 2 of this dissertation.

The covariant phase space of Yang-Mills theories

The fundamental geometric structure of a given first order Hamiltonian theory will be provided by a fiber bundle $\pi\colon E \to M$ with M an $m = (1+d)$-dimensional orientable smooth manifold with smooth boundary $\partial M \neq \emptyset$ and local coordinates adapted to the fibration (x^μ, u^a), $a = 1, \ldots, r$, where r is the dimension of the standard fiber. Because M is orientable we will assume that a given volume form vol_M is selected. Notice that it is always possible to chose local coordinates x^μ such that $\mathrm{vol}_M = dx^0 \wedge dx^1 \wedge \cdots \wedge dx^d$.

Yang-Mills fields are principal connections A on some principal fiber bundle $P \to M$ with structural group G. For clarity in the exposition we are going to make the assumption that P is trivial (which is always true locally), i.e. $P \cong M \times G \to M$ where (again, for simplicity) G is a Lie group with Lie algebra $\mathfrak{g}$. Under these assumptions, principal connections on P can be identified with $\mathfrak{g}$-valued 1-forms on M, i.e. with sections of the bundle $E = T^*M \otimes \mathfrak{g} \longrightarrow M$. Local bundle coordinates in the bundle $E \to M$ will be written as (x^μ, A^a_μ), $\mu = 1, \ldots, m$, $a = 1, \ldots, \dim \mathfrak{g}$, where $A = A^a_\mu \, \xi_a \in \mathfrak{g}$ with ξ_a a basis of the Lie algebra $\mathfrak{g}$. Thus a section of the bundle can be written as

$$A(x) = A^a_\mu(x) dx^\mu \otimes \xi_a \, . \tag{4.2.1}$$

We will denote by $\pi^0_1\colon J^1E \to E$ the affine 1-jet bundle of the bundle $E \xrightarrow{\pi} M$. The elements of J^1E are equivalence classes of germs of sections ϕ of π, i.e. two sections ϕ, ϕ' at $x \in M$ are equivalent or represent the same germ if $\phi(x) = \phi'(x)$ and $d\phi(x) = d\phi'(x)$. If $(x^\mu; u^a)$, $\mu = 0, \ldots, d$ is a bundle chart for the bundle $\pi\colon E \to M$, then we will denote by $(x^\mu, u^a; u^a_\mu)$ a local chart for the jet bundle J^1E. Thus in the case of Yang-Mills, local coordinates on J^1E will be denoted by (x, A^a, A^a_μ).

The affine dual of J^1E is the vector bundle over E whose fiber at $\xi = (x, u)$ is the linear space of affine maps $\mathrm{Aff}(J^1E_\xi, \mathbb{R})$. The vector bundle $\mathrm{Aff}(J^1E, \mathbb{R})$, possesses a natural subbundle defined by constant functions along the fibers of $J^1E \to E$, that we will denote again, with an abuse of notation, as $\mathbb{R}$. The quotient bundle $\mathrm{Aff}(J^1E, \mathbb{R})/\mathbb{R}$ will be called the covariant phase space bundle of the theory and will be denoted by $P(E)$.

Local coordinates on $P(E)$ can be introduced as follows: Affine maps on the fibers of J^1E have the form $u^a_\mu \mapsto \rho_0 + \rho^\mu_a u^a_\mu$ where u^a_μ are natural coordinates on the fiber over the point ξ in E with coordinates (x^μ, u^a). Thus a bundle chart for the bundle $\tau^0_1\colon P(E) \to E$ is given by $(x^\mu, u^a; \rho^\mu_a)$.

The choice of a distinguished volume form vol_M in M allows us to identify the fibers of $P(E)$ with a subspace of m-forms on E as follows: The map $u^a_\mu \to \rho^\mu_a u^a_\mu$ corresponds to the m-form $\rho^\mu_a du^a \wedge \mathrm{vol}_\mu$ where vol_μ stands for $i_{\partial/\partial x^\mu}\mathrm{vol}_M$. Let $\bigwedge^m(E)$ denote the bundle of m-forms on E. Let $\bigwedge^m_k(E)$ be the subbundle of $\bigwedge^m(E)$ consisting of those m-forms which vanish when k of their arguments are vertical. So in our local coordinates, elements of $\bigwedge^m_1(E)$, i.e. m-form on E that vanish when two of their arguments are vertical, commonly called semi-basic 1-forms, have the form $\rho^\mu_a du^a \wedge \mathrm{vol}_\mu + \rho_0 \mathrm{vol}_M$, and elements of $\bigwedge^m_0(E)$, i.e. basic m-forms, have the form $p_0\mathrm{vol}_M$. $\bigwedge^m_1 E$ is a real line bundle over $P(E)$ and, for each point $\zeta = (x, u, p) \in P(E)$, the fiber is the quotient $\bigwedge^m_1(E)_\zeta / \bigwedge^m_0(E)_\zeta$.

In the case of Yang-Mills, elements of $P(E)$ have the form $P = P^{\mu\nu}_a dA^a_\mu \wedge d^{m-1}x_\nu$.

The $(m+1)$-form $\Omega = d\Theta$ where $\Theta = \rho^\mu_a\, du^a \wedge \mathrm{vol}_\mu + \rho\,\mathrm{vol}_M$, defines a multisymplectic structure on the manifold $\bigwedge^m_1(E)$ i.e. $(\bigwedge^m_1(E), \Omega)$ is a multisymplectic manifold.

A Hamiltonian H on $P(E)$ is a section of μ. Thus in local coordinates

$$H(\rho^\mu_a\, du^a \wedge \mathrm{vol}_\mu) = \rho^\mu_a du^a \wedge \mathrm{vol}_\mu - \mathbf{H}(x^\mu, u^a, \rho^\mu_a)\mathrm{vol}_M\,,$$

where $\mathbf{H}$ is here a real-valued function.

We can use the Hamiltonian section H to define an m-form on $P(E)$ by pulling back the canonical m-form Θ from $\wedge^m_1(E)$. We call the form so obtained the Hamiltonian m-form associated with H and denote it by Θ_H. Thus if we write the section defined in local coordinates $(x^\mu, u^a; \rho, \rho^\nu_a)$ as

$$\rho = -\mathbf{H}(x^\mu, u^a, \rho^\mu_a)\,, \tag{4.2.2}$$

then

$$\Theta_H = \rho_a^\mu \, du^a \wedge \mathrm{vol}_\mu - \mathbf{H}(x^\mu, u^a, \rho_a^\mu) \, \mathrm{vol}_M \, . \qquad (4.2.3)$$

In (4.2.3) the minus sign in front of the Hamiltonian is chosen to be in keeping with the traditional conventions in mechanics for the integrand of the action over the manifold: $pdq - Hdt$. When the form Θ_H is pulled back to the manifold M the integrand of the action over M will have a form reminiscent of that of mechanics, with a minus sign in front of the Hamiltonian. See equation (4.2.5).

The action and the variational principle

The fields χ of the theory in the Hamiltonian formalism constitute a class of sections of the bundle $\tau_1 : P(E) \to M$. The sections that will be used to describe the classical fields in the Hamiltonian formalism are those sections $\chi\colon M \to P(E)$,i.e. $\tau_1 \circ \chi = \mathrm{id}_M$, such that $\chi = P \circ \Phi$ where $\phi\colon M \to E$ is a section of $\pi : E \to M$, i.e. $\pi \circ \Phi = \mathrm{id}_M$, and $P\colon E \to P(E)$, is a section of $\tau_1^0\colon P(E) \to E$, i.e. $\tau_1^0 \circ P = \mathrm{id}_P$. The sections Φ will be called the configurations and the sections P the momenta of the theory. In other words $u^a = \Phi^a(x)$ and $\rho_a^\mu = P_a^\mu(\Phi(x))$ will provide local expression for the section $\chi = P \circ \Phi$. We will denote such a section χ by (Φ, P) to indicate the iterated bundle structure of $P(E)$ and we will refer to χ as a double section[1].

We will denote by $\mathcal{F}_M$ the space of sections Φ of the bundle $\pi\colon E \to M$, that is $\Phi \in \mathcal{F}_M$, and we will denote by $\mathcal{F}_{P(E)}$ the space of double sections $\chi = (\Phi, P)$. Thus $\mathcal{F}_{P(E)}$ represents the space of fields of the theory in the first-order covariant Hamiltonian formalism.

Thus the fields of the theory in the multisymplectic picture for Yang-Mills theories are provided by sections (A, P) of the double bundle $P(E) \to E \to M$.

The equations of motion of the theory will be defined by means of a variational principle, i.e. they will be characterized as the critical points of an action functional S on $\mathcal{F}_{P(E)}$. Such action will be given simply by

$$S(\chi) = \int_M \chi^* \Theta_H . \qquad (4.2.4)$$

In the case of Yang-Mills theories, the action in a first-order covariant Hamiltonian formulation of the theory is given by,

$$S_{\mathrm{YM}}(A, P) = \int_M P_a^{\mu\nu} dA_\mu^a \wedge dx_\nu^{m-1} - H_\lambda(A, P)\mathrm{vol}_M \qquad (4.2.5)$$

[1]It can also be said that χ is a section of $P(E)$ along Φ.

with Hamiltonian function,

$$H_\lambda(A,P) = \frac{1}{2}\epsilon^a_{bc} P^{\mu\nu}_a A^b_\mu A^c_\nu + \frac{\lambda}{4} P^{\mu\nu}_a P^a_{\mu\nu} \tag{4.2.6}$$

for some $\lambda \geq 0$, where the indexes $\mu\nu$ (a) in $P^{\mu\nu}_a$ have been lowered (raised) with the aid of the Lorentzian metric η (the Killing-Cartan form on $\mathfrak{g}$, respect.).

Of course, as is usual in the derivations of equations of motion via variational principles, we assume that the integral in Eq. (2.4) is well defined. It is also assumed that the 'differential' symbol in equation (2.7) below, defined in terms of directional derivatives, is well defined and that the same is true for any other similar integrals that will appear in this work.

A simple computation leads to,

$$\mathrm{d}S(\chi)(U) = \int_M \chi^*\left(i_{\widetilde{U}} d\Theta_H\right) + \int_{\partial M} (\chi \circ i)^* \left(i_{\widetilde{U}}\Theta_H\right) \,, \tag{4.2.7}$$

where U is a vector field on $P(E)$ along the section χ, $\widetilde{U}$ is any extension of U to a tubular neighborhood of the image of χ, and $i\colon \partial M \to M$ is the canonical embedding.

We omit the derivation from (4.2.7) of the fundamental formula that relates the differential of the action with a 1-form on a space of fields on the bulk manifold and a 1-form on a space of fields at the boundary,

$$\mathrm{d}S_\chi = \mathrm{EL}_\chi + \Pi^*\alpha_\chi \,, \qquad \chi \in \mathcal{F}_{P(E)} \,. \tag{4.2.8}$$

Details can be found in Chapter 2 section 2.2

The cotangent bundle of fields at the boundary

Consider the space of fields at the boundary obtained by restricting the zeroth component of sections χ to ∂M, that is the fields of the form:

$$\varphi^a = \Phi^a \circ i \,, \qquad p_a = P^0_a \circ i \,.$$

The space of fields (φ^a, p_a) can be identified with the cotangent bundle $T^*\mathcal{F}_{\partial M}$ over $\mathcal{F}_{\partial M}$ in a natural way, i.e. each field p_a can be considered as the covector at φ^a that maps the tangent vector $\delta\varphi^a$ at φ^a into the number $\langle p, \delta\varphi\rangle$ given by,

$$\langle p, \delta\varphi\rangle = \int_{\partial M} p_a(x)\delta\varphi^a(x)\mathrm{vol}_{\partial M} \,. \tag{4.2.9}$$

Notice that the tangent vector $\delta\varphi$ at φ is a vertical vector field on E along φ, and the section p is a 1-semibasic m-form on E (Lemma 2.1). Hence the contraction of p with $\delta\varphi$ is an $(m-1)$-form along φ, and its pull-back $\varphi^*\langle p, \delta\varphi\rangle$ along φ is an $(m-1)$-form on ∂M whose integral defines the pairing above, Eq. (2.10).

Viewing the cotangent bundle $T^*\mathcal{F}_{\partial M}$ as double sections (φ, p) of the bundle $\bigwedge_1^m(i^*E)$, the canonical 1-form α on $T^*\mathcal{F}_{\partial M}$ can be expressed as,

$$\alpha_{(\varphi,p)}(U) = \int_{\partial M} p_a(x)\delta\varphi^a(x)\,\mathrm{vol}_{\partial M} \tag{4.2.10}$$

where U is a tangent vector to $T^*\mathcal{F}_{\partial M}$ at (φ, p), that is, a vector field on the space of 1-semibasic forms on i^*E along the section (φ^a, p_a), and therefore of the form $U = \delta\varphi^a\,\partial/\partial u^a + \delta p_a\,\partial/\partial\rho_a$.

Finally, notice that the pull-back to the boundary map i^*, defines a natural map from the space of fields in the bulk, $\mathcal{F}_{P(E)}$, into the phase space of fields at the boundary $T^*\mathcal{F}_{\partial M}$. Such map will be denoted by Π, that is,

$$\Pi\colon \mathcal{F}_{P(E)} \to T^*\mathcal{F}_{\partial M}\,, \qquad \Pi(\Phi, P) = (\varphi, p), \quad \varphi = \Phi\circ i,\ p_a = P_a^0\circ i\,.$$

We will write,

$$\alpha = p_a\,\delta\varphi^a\,,$$

and the canonical symplectic structure $\omega_{\partial M} = -d\alpha$ on $T^*\mathcal{F}_{\partial M}$ will be written as,

$$\omega_{\partial M} = \delta\varphi^a \wedge \delta p_a\,,$$

by which we mean

$$\omega_{\partial M}((\delta_1\varphi^a, \delta_1 p_a), (\delta_2\varphi^a, \delta_2 p_a)) = \int_{\partial M} (\delta_1\varphi^a(x)\delta_2 p_a(x) - \delta_2\varphi^a(x)\delta_1 p_a(x))\,\mathrm{vol}_{\partial M}\,,$$

where $(\delta_1\varphi^a, \delta_1 p_a), (\delta_2\varphi^a, \delta_2 p_a)$ are two tangent vectors at (φ^a, p_a).

The limit $\lambda \to 0$ of Yang-Mills theories

Recall equations (4.2.5) and (4.2.6) for the action of Yang-Mills theories in a first-order Hamiltonian formulation of the theory:

$$S_{\mathrm{YM},\lambda}(A, P) = \int_M P_a^{\mu\nu} dA_\mu^a \wedge \mathrm{d}x_\nu^{m-1} - H_\lambda(A, P)\mathrm{vol}_M \tag{4.2.11}$$

with Hamiltonian function,

$$H_\lambda(A, P) = \frac{1}{2}\epsilon_{bc}^a P_a^{\mu\nu} A_\mu^b A_\nu^c + \frac{\lambda}{4} P_a^{\mu\nu} P_{\mu\nu}^a \tag{4.2.12}$$

for some $\lambda \geq 0$, where the indexes $\mu\nu$ (a) in $P_a^{\mu\nu}$ have been lowered (raised) with the aid of the Lorentzian metric η (the Killing-Cartan form on $\mathfrak{g}$, respect.).

Plugging (4.2.12) into (4.2.11) and expanding the right hand side of (4.2.11), we obtain,

$$S_{\mathrm{YM},\lambda}(A,P) = -\int_M \frac{1}{2}\left[P_a^{\mu\nu}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon_{bc}^a A_\mu^b A_\nu^c) + \frac{\lambda}{2} P_a^{\mu\nu} P_{\mu\nu}^a\right] \mathrm{vol}_M \,. \quad (4.2.13)$$

Using that the curvature,

$$\begin{aligned} F_A &= d_A A = dA + \frac{1}{2}[A \wedge A] = F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &= \frac{1}{2}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon_{bc}^a A_\mu^b A_\nu^c\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \xi_a \end{aligned} \quad (4.2.14)$$

we can rewrite eqn (4.2.13) as,

$$S_{\mathrm{YM},\lambda}(A,P) = -\int_M \left[P_a^{\mu\nu} F_{\mu\nu}^a + \frac{\lambda}{4} P_a^{\mu\nu} P_{\mu\nu}^a\right] \mathrm{vol}_M \,.$$

(4.2.15)

This last expression is the action of the Yang-Mills theory for any given $\lambda \geq 0$. If we take its limit $\lambda \to 0$, we obtain,

$$S_{YM,0}(A,P) = -\int_M P_a^{\mu\nu} F_{\mu\nu}^a \mathrm{vol}_M \ , \quad (4.2.16)$$

whose equations of motion are given by,

$$F_A = 0\,, \qquad d_A^* P = 0\,.$$

Thus the moduli space of solutions of the Euler-Lagrange equations is given by,

$$\mathcal{M} = \{F_A = 0, d_A^* P = 0\}/\mathcal{G}_M \,,$$

where $\mathcal{G}_M$ denotes the group of gauge transformations of the theory.

4.3 Palatini's Gravity

Palatini's Yang-Mills

The primary fields of a theory of gravity *a la* Palatini will be given by principal connections A on a G-principal bundle over a smooth manifold M with G the Lorentz group $O(1,d)$, the group of isometries preserving the non-degenerate quadratic form Q of signature $- + \cdots +$, with $m = 1 + d = \dim M$.

The connections A can be considered as vertical equivariant 1-forms on a principal fiber bundle with structural group $O(1,d)$. The choice of the principal fiber bundle $P \to M$ determines a sector of a full theory of gravity where, in addition to the bundle P, we should consider all equivalence classes of principal $O(1,d)$ bundles over M. If we fix a topology on M, the corresponding family of classes of principal fiber bundles are in one-to-one correspondence with homotopy classes of maps $f\colon M \to B_{O(1.d)}$, where $B_{O(1.d)}$ is the universal classifying space of the Lorentz group and the principal fiber bundle corresponding to the map f is given by $P_f = f^* E_{O(1,d)}$, where $E_{O(1.d)} \to B_{O(1.d)}$ is the universal principal $O(1,d)$ bundle. Thus the fields corresponding to each equivalence class will define connected components in the space of all fields and we will focus on one of them.

Palatini's constraint

Palatini's constraint determines a subbundle of the covariant phase space whose sections define a submanifold of the space of fields $J^1\mathcal{F}^*$ such that the restriction of the topological sector of $SO(1,3)$–Yang-Mills is equivalent to Palatini's action.

Consider the bundle $F = GL(\tau_m, TM) \subset \mathrm{Hom}(\tau_m, TM) \cong \tau_m^* \otimes TM$ over M whose fiber at $x \in M$ consists of invertible linear maps $e(x)$ from $\tau_m(x)$ to T_xM and where $\tau_m = M \times \mathbb{M}^m$ is the trivial bundle over M with fiber the m-dimensional Minkowski space $\mathbb{M}^m$ with metric $\eta = \mathrm{diag}(-, + \cdots, +)$. Notice that local cross sections of the bundle F can be thought as local frames on M, i.e. if U is an open set on M such that $TM\mid_U \cong U \times \mathbb{R}^m$ then a cross section $e\colon U \to \tau_m^* \otimes TM$ defines a map $e_x := e(x)\colon \mathbb{R}^m \to T_xM$ for each $x \in U$. They form a family of linearly independent vectors $e_I(x)$, $I = 0, 1, \ldots, d$, which are the images under e_x of the standard orthogonal basis u_i on $\mathbb{M}^m$, that is $\eta(u_0, u_0) = -1$, $\eta(u_k, u_k) = 1$, $k = 1, \ldots, d$. With a slight abuse of notation we will denote $e_x(u_I) = e_I(x)$. Global cross sections e are usually called vierbeins for an arbitrary dimension m, or tetrad fields if $m = 4$. In what follows we will not assume that there exist globally defined sections of F. Notice that given a local cross section e, it defines a Lorentz metric on U by means of $g_x(u,v) = \eta_x(e^{-1}(u), e^{-1}(v))$ for any $u, v \in T_xU$. The metric g

is Lorentz because clearly the vectors $e_I(x)$ determine an orhonormal basis for g at T_xM such that $g_x(e_I(x), e_J(x))$ is diagonal with diagonal $(-, +\dots, +)$.

Choosing local coordinates x^μ on U we will have that $e_I = e_I^\mu(x)\partial/\partial x^\mu$ will define a local vector field on U for each I. With this notation we may also write the local cross section e as $e = e_I \otimes u^I = e_I^\mu(x)\partial/\partial x^\mu \otimes u^I$ where u^I denotes the canonical dual basis of the standard orthogonal basis u_I.

Let us recall that we have a distinguished volume form vol_M on M, i.e. a global section of the determinant bundle $\det(M) = \Lambda^m(TM)$. Morevoer there is a canonical section of the bundle $\det(\tau_m) = \Lambda^m(\tau_m)$ given by $\mathrm{vol}_\eta = u^0 \wedge u^1 \wedge \cdots \wedge u^d$. Then a linear map $e_x\colon \tau_m(x) \to T_xM$ defines a pull-back $e^*(\mathrm{vol}_M) = \epsilon\mathrm{vol}_\eta$, in other words, $\epsilon(x)$ is the determinant of the map e_x. In local coordinates,

$$\epsilon(x) = \det(e_I^\mu(x))\,.$$

Consider the map $P\colon F \to P(E)$ defined by,

$$P(e) = \epsilon e \wedge e$$

where $e \wedge e$ is defined as the linear map from $\tau_m \wedge \tau_m$ to $T_xM \wedge T_xM$ given by $e \wedge e(u \wedge v) = e(u) \wedge e(v)$. Using the previous notation we may write:

$$P(e) = \epsilon e_I^\mu e_J^\nu \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} \otimes u^I \wedge u^J\,.$$

Notice that if we write the tensor $P(e)$ in the local basis $\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} \otimes u^I \wedge u^J$ as

$$P(e) = P_{IJ}^{\mu\nu} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} \otimes u^I \wedge u^J\,,$$

then

$$P_{IJ}^{\mu\nu} = \det(e_I^\mu)\, e_{[I}^{[\mu} e_{J]}^{\nu]}\,,$$

with $P_{IJ}^{\mu\nu} = -P_{IJ}^{\nu\mu} = -P_{JI}^{\mu\nu} = P_{JI}^{\nu\mu}$. We will sometimes use the notation $P_{IJ}^{\mu\nu} = \det(e_I^\mu)\, e_I^\mu \wedge e_J^\nu$ to indicate the skew symmetry in the pairs of indices IJ and $\mu\nu$.

Finally notice that $P(e)$ actually lies in $P(E)$ as the fiber of $P(E)$ at x is given by $T_xM \wedge T_xM \otimes \mathfrak{so}(1,d)$ and $\tau_m \wedge \tau_m \subset \mathfrak{so}(1,d)$.

The image of F under the map P will be called the Palatini subbundle of $P(E)$ and will be denoted simply by $P(F) \subset P(E)$. Double sections of this bundle are the fields of the theory we are interested in. Such space of sections will be denoted as $\mathcal{P} \subset J^1\mathcal{F}_M^*$. Notice that a double section (A, P) of $\mathcal{P}$ is a section of $P(E)$ such that locally there exists e such that $P = \epsilon\, e \wedge e$.

Hence the space of fields of the theory we are constructing can be considered as a submanifold of the space of fields $J^1\mathcal{F}_M^*$ defined by the range of the map P.

The action

The action of the topological phase of Yang-Mills given by Eq. (4.2.16) can be rewritten with the above notations as,

$$S_{YM,0} = -\int_M P^{\mu\nu}_{IJ} F^{IJ}_{\mu\nu} \mathrm{vol}_M \, , \tag{4.3.1}$$

with $(A, P) \in J^1\mathcal{F}^*_M$. If we restrict (A, P) to $\mathcal{P}$, the action becomes,

$$S_{YM,0} \mid_{\mathcal{P}} = -\int_M \epsilon \, e^\mu_I e^\nu_J F^{IJ}_{\mu\nu} \mathrm{vol}_M \, , \tag{4.3.2}$$

which is exactly Palatini's action for gravity.

The Euler-Lagrange equations of the theory can be obtained by standard methods by computing the differential of $S_{YM,0}$ restricted to $\mathcal{P}$ or alternatively, using an appropriate version of Lagrange's multipliers theorem to obtain the critical points of $S_{YM,0}$ restricted to $\mathcal{P}$. We will develop this point of view in the following section.

Critical points and Euler-Lagrange equations

Lagrange's multipliers theorem

We will discuss first the version of Lagrange's multipliers theorem suited to the problem at hand.

Theorem: Let $\mathcal{M}$ be an affine manifold and let $F\colon \mathcal{M} \to \mathbb{R}$ be a differentiable function. Let $\mathcal{D}$ be a smooth manifold and let $\Phi\colon \mathcal{D} \to \mathcal{M}$ be a smooth injective function. Let $\mathcal{N} = \{x \in \mathcal{M} \mid \exists e \in \mathcal{D}\, , x = \Phi(e)\}$.

$x \in \mathcal{N}$ is a critical point of $F \mid_{\mathcal{N}} \colon \mathcal{N} \to \mathbb{R}$ iff there exists $e \in \mathcal{D}$ and $\lambda \in \mathcal{M}^*$ such that (x, λ, e) is a critical point of the extended function $\mathbb{F}\colon \mathcal{M} \times \mathcal{M}^* \times \mathcal{D} \to \mathbb{R}$ given by:

$$\mathbb{F}(x, \lambda, e) = F(x) + \langle \lambda, x - \Phi(e) \rangle \, .$$

Proof:

Suppose $x \in \mathcal{N}$ is a critical point of $F \mid_{\mathcal{N}}$, i.e. $\mathrm{d}(F \mid_{\mathcal{N}})_x(\delta x) = 0$ for all $\delta x \in T_x\mathcal{N}$, or $\mathrm{d}F_x \in T_x\mathcal{N}^{0\,2}$. Since $\Phi\colon \mathcal{D} \to \mathcal{N}$ is bijective, there exists $e \in \mathcal{D}$ such that $\Phi(e) = x$ and for given $\delta e \in T_e\mathcal{D}$ there exists $\delta x \in T_{\Phi(e)}\mathcal{N}$ such that $\Phi_*(e)(\delta e) = \delta x$, where $\Phi_*(e)\colon T_e\mathcal{D} \to T_{\Phi(e)}\mathcal{N}$ denotes the tangent map to Φ at $e \in \mathcal{D}$. It therefore follows that since $\mathrm{d}(F \mid_{\mathcal{N}})_x(\delta x) = 0$ for all $\delta x \in T_x\mathcal{N}$, $(\mathrm{d}F)(\Phi_*(e)(\delta e)) = 0$ for any $\delta e \in T_e\mathcal{D}$.

Computing the differential of $\mathbb{F}$, we obtain,

$$\mathrm{d}\mathbb{F}_{(x,\lambda,e)}(\delta x, \delta\lambda, \delta e) = dF_x(\delta x) + \langle \delta\lambda, x - \Phi(e) \rangle + \langle \lambda, \delta x - \Phi_*(e)(\delta e) \rangle \, , \tag{4.3.3}$$

where $\delta e \in T_e\mathcal{D}$, $\delta x \in T_x\mathcal{M} \cong \mathcal{M}$, $\delta\lambda \in T_\lambda\mathcal{M}^* \cong T^*_\lambda\mathcal{M} \cong \mathcal{M}^*$. The notation $\langle \lambda, x\rangle$ denotes the natural pairing between $\mathcal{M}$ and its dual space $\mathcal{M}^*$.

For (x, λ, e) such that $x \in \mathcal{N}$ is a critical point of $\mathcal{F} \mid_{\mathcal{N}}$, $\Phi(e) = x$ and $\lambda = -\mathrm{d}F_x \in T_x\mathcal{N}^0 \subset T^*_x\mathcal{M} \cong \mathcal{M}^*$, it follows from (3.1) and from the prior statements that $\mathrm{d}\mathbb{F}_{(x,\lambda,e)}(\delta x, \delta\lambda, \lambda e) = 0$ for all $\delta x, \delta\lambda$ and δe. Thus (x, λ, e) is a critical point of $\mathbb{F}$.

Now we prove the other direction of the theorem. Let (x, λ, e) be a critical point of $\mathbb{F}$, i.e. $\mathrm{d}\mathbb{F}_{(x,\lambda,e)}(\delta x, \delta\lambda, \delta e) = 0$ for all $\delta x, \delta\lambda, \delta e$. In particular, fixing $\delta x = \delta e = 0$, for any $\delta\lambda$, since (x, λ, e) is a critical point of $\mathbb{F}$, $\mathrm{d}\mathbb{F}_{(x,\lambda,e)}(0, \delta\lambda, 0) = 0$. This implies by (3.1) that $x = \Phi(e)$, thus $x \in \mathcal{N}$. For any $\delta x \in T_x\mathcal{N}$, since $x = \Phi(e)$ and since $\Phi\colon \mathcal{D} \to \mathcal{N}$ is bijective, there exists $\delta e \in \mathcal{D}$ such that $\Phi_*(e)(\delta e) = \delta x$. So for our critical point (x, λ, e) of $\mathbb{F}$ and for any $\delta x \in T_x\mathcal{N}$, applying (3.1), we obtain $0 = \mathrm{d}\mathbb{F}_{(x,\lambda,e)}(\delta x, \delta\lambda, \delta e) = \mathrm{d}F_x(\delta x)$. Thus for any $\delta x \in T_x\mathcal{N}, \mathrm{d}F_x(\delta x) = 0$, i.e. x is a critical point of F.

Critical points

We apply Lagrange's multipliers theorem discussed in the previous section to the following setting. The affine manifold $\mathcal{M}$ is the space of fields $J^1\mathcal{F}^*_M$ in the covariant phase space. The manifold $\mathcal{D}$ is the manifold of vierbein fields, i.e, sections e of the bundle F discussed before. The submanifold $\mathcal{N}$ is the submanifold $\mathcal{P}$ defined by Palatini's constraints, i.e., we have the map $P\colon \mathcal{D} \to J^1\mathcal{F}^*$ given by $P(e) = \epsilon\, e \wedge e$. Then, finally, the function $F\colon \mathcal{M} \to \mathbb{R}$ is the topological Yang-Mills action functional $S_{YM,0}\colon J^1\mathcal{F}^*_M \to \mathbb{R}$.

Then we conclude that critical points of Palatini's action S_P are in correspondence with families of critical points of the extended action:

$$\mathbb{S}(A, P, \Lambda, e) = S_{YM,0}(A, P) + \langle \Lambda, P - \epsilon\, e \wedge e\rangle\,,$$

or more explicitly

$$\mathbb{S}(A, P, \Lambda, e) = \int_M -P^{\mu\nu}_{IJ} F^{IJ}_{\mu\nu} + \Lambda^{\mu\nu}_{IJ}\left(P^{IJ}_{\mu\nu} - \epsilon\, e^I_\mu e^J_\nu\right) \mathrm{vol}_M\,. \tag{4.3.4}$$

According to Lagrange's multipliers theorem, the critical points of $\mathbb{S}$ have the form (A, P, Λ, e) where (A, P) is a critical point of $S_{YM,0} \mid_{\mathcal{P}} = S_P$, for all Λ, i.e., $P = \epsilon\, e \wedge e$ for some vierbein field e and (A, e) is a critical point of

$$S_P = -\int_M \epsilon e^\mu_I e^J_\nu F^{IJ}_{\mu\nu}\, \mathrm{vol}_M\,.$$

Then standard arguments show that the Palatini connection is torsionless and that the metric with respect to the metric g_e defined by the vierbein field, that is A, is the Levi-Civita connection of the metric g_e. Moreover it satisfies Ricci's equation,

$$\mathrm{Ric}(A) = 0\,.$$

From Eq.(4.3.3) we also get that if (x, λ, e) is a critical point of $\mathbb{F}$, then at $x \in \mathcal{N}$

$$\mathrm{d}F_x(\delta x) = -\langle \lambda, \delta x - \Phi_*(e)\delta e\rangle\,,$$

and δx an arbitrary vector in $T_x\mathcal{M}$, that is not necessarily in $T_x\mathcal{N}$. This shows that if $(A, P = P(e), \Lambda, e)$ is a critical point of $\mathbb{S}$, then

$$\mathrm{d}S_P(A, P = P(e))(\delta A, \delta P) = -\langle \Lambda, \delta P - P_*(e)\delta e\rangle.$$

The canonical formalism near the boundary

In this section we apply the general theory of evolution near the boundary developed in Chapter 2, section 2.3, to the case of Palatini gravity. We refer the reader to the exposition there.

In order to obtain an evolution description for Palatini Gravity and to prepare the ground for canonical quantization, we need to introduce a local time parameter. We will only assume that a collar $U_\epsilon = (-\epsilon, 0] \times \partial M$ around the boundary can be chosen so that a choice of a time parameter $t = x^0$ can be made near the boundary that would be used to describe the evolution of the system. The fields of the theory would then be considered as fields defined on a given spatial frame that evolve in time for $t \in (-\epsilon, 0]$.

The dynamics of such fields would be determined by the restriction of the Palatini action above to the space of fields on U_ϵ. Expanding we obtain,

$$\begin{aligned}
&\mathrm{S}_{U_\epsilon}(A, P, \Lambda, e) = \int_{U_\epsilon}[-P^{\mu\nu}_{IJ}F^{IJ}_{\mu\nu} + \Lambda^{IJ}_{\mu\nu}(P^{IJ}_{\mu\nu} - \epsilon e^I_\mu e^J_\nu)]vol_M \\
&= \int_{U_\epsilon}[P^{\mu\nu}_{IJ}(-\tfrac{1}{2})(\partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu \\
&+ \epsilon^{IJ}_{KL,MN}A^{KL}_\mu A^{MN}_\nu) + \Lambda^{IJ}_{\mu\nu}(P^{IJ}_{\mu\nu} - \epsilon e^I_\mu e^J_\nu)]vol_M \\
&= \int_{-\epsilon}^0 dt \int_{\partial M} vol_{\partial M}[P^{k0}_{IJ}(\partial_0 A^{IJ}_k - \partial_k A^{IJ}_0 + \epsilon^{IJ}_{KL,MN}A^{KL}_0 A^{MN}_k) - \tfrac{1}{2}P^{kj}_{IJ}(\partial_k A^{IJ}_j - \partial_j A^{IJ}_k + \\
&\epsilon^{IJ}_{KL,MN}A^{KL}_k A^{MN}_j) + 2\Lambda^{IJ}_{k0}(P^{IJ}_{k0} - \epsilon e^I_k e^J_0) + \Lambda^{IJ}_{kj}(P^{IJ}_{kj} - \epsilon e^I_k e^J_j)].
\end{aligned}$$

In the previous expressions ϵ^a_{bc} denote the structure constants of the Lie algebra $\mathfrak{g}$ with respect to the basis ξ_a, that is $[\xi_b, \xi_c] = \epsilon^a_{bc}\xi_a$. Notice that $\epsilon^a_{bc}A^b_0 A^c_0 = 0$ because for fixed a, ϵ^a_{bc} is skew-symmetric. Moreover the indexes μ and a have been pushed down and up by using the metric η and the Killing-Cartan form $\langle\cdot,\cdot\rangle$ respectively.

In the last equation we used that P is a bivector, i.e. $P^{\mu\nu}_a$ is skew symmetric in μ and ν. Therefore $P^{00}_a = 0$ and also $P^{k0}_a P^a_{k0} = P^{0i}_a P^a_{0i}$ because $P^{k0} = -P^{0k}$, etc. The momenta fields are defined as sections of the bundle $P(E)$ and as such are unrestricted. However, because Yang-Mills theories are Lagrangian theories the Legendre transform selects a subspace of the space of momenta that corresponds to fields P skew-symmetric in the indices μ, ν.(For more details see Chapter 2, section 2.4.)

The previous expression acquires a clearer structure by introducing the appropriate notations for the fields restricted at the boundary and assuming that they evolve in time t. Thus the pull-backs of the components of the fields A and P to the boundary will be denoted respectively as,

$$\begin{array}{rcl} a^a_k &:=& A^a_k\mid_{\partial M}; \qquad a = (a^a_k)\,, \qquad a^a_0 := A^a_0\mid_{\partial M}; \qquad a_0 = (a^k_0)\,, \\ p^k_a &:=& P^{k0}_a\mid_{\partial M}; \qquad p = (p^k_a)\,, \qquad p^0_a := P^{00}_a\mid_{\partial M} = 0; \qquad p_0 = (p^0_a) = 0\,, \\ \beta^{ki}_a &:=& P^{ki}_a\mid_{\partial M}; \qquad \beta = (\beta^{ki}_a)\,. \end{array}$$

Given two fields at the boundary, for instance p and a, we will denote as usual by $\langle p, a\rangle$ the expression,

$$\langle p, a\rangle = \int_{\partial M} p^\mu_a a^a_\mu \,\mathrm{vol}_{\partial M}\,,$$

and the contraction of the inner (Lie algebra) indices by using the Killing-Cartan form and the integration over the boundary is understood.

Introducing the notations and observations above in the expression for S_{U_ϵ} we obtain,

$$\begin{aligned} S_{U_\epsilon}(A, P, \Lambda, e) &= \int_{-\epsilon}^0 dt \int_{\partial M} vol_{\partial M}[p^k_{IJ}(\partial_0 a^{IJ}_k - \partial_k a^{IJ}_0 + \epsilon^{IJ}_{KL,MN} a^{KL}_0 a^{MN}_k) \\ &\quad -\frac{1}{2}\beta^{kj}_{IJ}(\partial_k a^{IJ}_j - \partial_j a^{IJ}_k + \epsilon^{IJ}_{KL,MN} a^{KL}_k a^{MN}_j) + 2\Lambda^{k0}_{IJ}(p^{IJ}_k - \epsilon e^I_k e^J_0) + \Lambda^{kj}_{IJ}(\beta^{IJ}_{kj} - \epsilon e^I_k e^J_j)] \\ &= \int_{-\epsilon}^0 \mathrm{d}t\, \mathcal{L}(a, \dot a, a_0, \dot a_0, p, \dot p, \beta, \dot\beta, \Lambda, \dot\Lambda, \Lambda_0, \dot\Lambda_0, e, \dot e) \end{aligned}$$

where

$$\mathcal{L}(a, \dot a, a_0, \dot a_0, p, \dot p, \beta, \dot\beta, \Lambda, \dot\Lambda, \Lambda_0, \dot\Lambda_0, e, \dot e) = < p, \dot a - d_a a_0 + 2\Lambda_0 >$$

$$-<\beta, F_a - \Lambda> + <\Lambda_0, -2\epsilon e \wedge e_0> + <\Lambda, -\epsilon e \wedge e> .$$

Euler-Lagrange equations will have the form:

$$\frac{d}{\mathrm{d}t}\frac{\delta \mathcal{L}}{\delta \dot{\chi}} = \frac{\delta \mathcal{L}}{\delta \chi} ,$$

where $\chi \in P(E)$ and $\delta/\delta\chi$ denotes the variational derivative of the functional $\mathcal{L}$.

Thus for $\chi = p$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{p}} = 0, \quad \text{hence} \quad 0 = \frac{\delta \mathcal{L}}{\delta p} = \dot{a} - \mathrm{d}_a a_0 + 2\Lambda_0 ,$$

and thus,

$$\dot{a} = \mathrm{d}_a a_0 - 2\Lambda_0 . \tag{4.3.5}$$

For $\chi = a$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{a}} = p, \quad \text{hence} \quad \dot{p} = \frac{\delta \mathcal{L}}{\delta a} = d^*\beta + [p, a_0] ,$$

that is,

$$\dot{p} = d^*\beta + [p, a_0]. \tag{4.3.6}$$

For $\chi = a_0$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{a_0}} = 0, \quad \text{hence} \quad 0 = \frac{\delta \mathcal{L}}{\delta a_0} = \frac{\partial}{\partial a_0} <p, d_a a_0> = \frac{\partial}{\partial a_0} - <d_a^* p, a_0> = -d_a^* p ,$$

that is,

$$d_a^* p = 0. \tag{4.3.7}$$

For $\chi = \beta$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{\beta}} = 0, \quad \text{hence} \quad 0 = \frac{\delta \mathcal{L}}{\delta \beta} = -F_a + \Lambda,$$

that is,

$$F_a = \Lambda. \tag{4.3.8}$$

For $\chi = \Lambda$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{\Lambda}} = 0, \quad \text{hence} \quad 0 = \frac{\delta \mathcal{L}}{\delta \Lambda} = \beta - \epsilon e \wedge e,$$

that is,

$$\beta = \epsilon e \wedge e. \tag{4.3.9}$$

For $\chi = \Lambda_0$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{\Lambda_0}} = 0, \quad \text{hence} \quad 0 = \frac{\delta \mathcal{L}}{\delta \Lambda_0} = 2p - 2\epsilon e \wedge e_0,$$

that is,

$$\mathrm{p} = \epsilon e \wedge e_0. \tag{4.3.10}$$

For $\chi = e$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{e}} = 0, \quad \text{hence} \quad 0 = \frac{\delta \mathcal{L}}{\delta e} = -2\epsilon e_0 \Lambda_0 - 2\epsilon e \Lambda,$$

that is,

$$-e_0 \Lambda_0 = e\Lambda. \tag{4.3.11}$$

For $\chi = e_0$ we obtain,

$$\frac{\delta \mathcal{L}}{\delta \dot{e_0}} = 0, \quad \text{hence} \quad 0 = \frac{\delta \mathcal{L}}{\delta e_0} = 2\epsilon e \Lambda_0$$

i.e.

$$e\Lambda_0 = 0. \tag{4.3.12}$$

Thus solving for the Euler-Lagrange equations, we have obtained two evolution equations,(4.3.5) and (4.3.6) and six constraint equations (4.3.7) – (4.3.12).

The presymplectic formalism: Palatini at the boundary and reduction

In Chapter 2, section 2.3, we introduced a presymplectic framework on $\mathcal{M} = \mathcal{T}^*\mathcal{F}_{\partial M} \times_{\mathcal{F}_{\partial M}} \mathcal{B}$, the total space of fields at the boundary, for a general first-order covariant Hamiltonian field theory, and we carried out the constraints analysis. Here we apply that theory to the case of Palatini gravity. We refer the reader to that earlier material for background on what follows here.

As discussed in Chapter 2, section 2.3, we define the extended Hamiltonian, $\mathcal{H}$, so that $\mathcal{L} = \langle p, \dot{a}\rangle - \mathcal{H}$:

$$\mathcal{H}(a, a_0, p, \beta, \Lambda, \Lambda_0, e) =< p, -d_a a_0 + 2\Lambda_0 > - < \beta, F_a - \Lambda > +$$

¡$\Lambda_0, -2\epsilon e \wedge e_0 >$

$+ \langle \Lambda, -\epsilon e \wedge e\rangle$.(4.3.13)

Thus the Euler-Lagrange equations can be rewritten as

$$\dot{a} = \frac{\delta \mathcal{H}}{\delta p}; \quad \dot{p} = -\frac{\delta \mathcal{H}}{\delta a}\,, \tag{4.3.14}$$

$$\frac{\delta \mathcal{H}}{\delta a_0} = 0; \quad \frac{\delta \mathcal{H}}{\delta \beta} = 0; \quad \frac{\delta \mathcal{H}}{\delta \Lambda} = 0; \quad \frac{\delta \mathcal{H}}{\delta \Lambda_0} = 0; \quad \frac{\delta \mathcal{H}}{\delta e} = 0; \quad \frac{\delta \mathcal{H}}{\delta e_0} = 0. \tag{4.3.15}$$

We denote by $\varrho \colon \mathcal{M} \to T^*\mathcal{F}_{\partial M}$ the canonical projection $\varrho(a, a_0, p, \beta) = (a, a_0, p)$. Let $\omega_{\partial M}$ denote the form on the cotangent bundle $T^*\mathcal{F}_{\partial M}$,

$$\omega_{\partial M} = \delta a \wedge \delta p.$$

We will denote again by Ω the pull-back of this form to $\mathcal{M}$ along ϱ, i.e., $\Omega = \varrho^*\omega_{\partial M}$. Clearly, $\ker \Omega = \operatorname{span}\{\delta/\delta\beta, \delta/\delta a_0\}$, and we have the particular form that Thm 3.1 in Chapter 2, takes here.

Theorem 4.3.1. *The solutions to the equations of motion defined by the Palatini Lagrangian are in one-to-one correspondence with the integral curves of the presymplectic system $(\mathcal{M}, \Omega, \mathcal{H})$, i.e. with the integral curves of the vector field Γ on $\mathcal{M}$ such that $i_\Gamma \Omega = \mathrm{d}\mathcal{H}$.*

The primary constraint submanifold $\mathcal{M}_1$ is defined by the six constraint equations,

$$\mathcal{M}_1 = \{(a, a_0, p, \beta, \Lambda, e) | F_a = \Lambda, d_a^* p = 0, \beta = \epsilon e \wedge e, p = \epsilon e \wedge e_0, e_0\Lambda_0 = e\Lambda, e\Lambda_0 = 0\}\,.$$

Since $\Lambda = F_a$, and β is a just a function of e, we have that

$\mathcal{M}_1 \cong \{(a, a_0, p, e) | d_a^* p = 0, p = \epsilon e \wedge e_0, e_0 F_{a0} = eF_a, eF_{a0} = 0\}$

and $\quad \ker \Omega|_{\mathcal{M}_1} \supset \text{span}\{\frac{\partial}{\partial a_0}\}$.

Thus $\mathcal{M'}_1 = \mathcal{M}_1/(\ker \Omega|\mathcal{M}_1) \cong \{(a, p, e) | \mathrm{d}_a^* p = 0, p = \epsilon e \wedge e_0, e_0 F_{a0} = eF_a, eF_{a0} = 0\}$.

Gauge transformations: symmetry and reduction

The group of gauge transformations $\mathcal{G}$, i.e the group of automorphisms of the principal bundle P over the identity, is a fundamental symmetry of the theory. Notice that the Palatini action is invariant under the action of $\mathcal{G}$.

The quotient of the group of gauge transformations by the normal subgroup of identity gauge transformations at the boundary defines the group of gauge transformations at the boundary $\mathcal{G}_{\partial M}$, and it constitutes a symmetry group of the theory at the boundary. It is a symmetry group both of the boundary Lagrangian $\mathcal{L}$ and of the presymplectic system $(\mathcal{M}, \Omega, \mathcal{H})$. We may take advantage of this symmetry to provide an alternative description of the constraints found in the previous section.

Proposition 4.3.2. *The map $\mathcal{J} : T^*\mathcal{F}_{\partial M} \to \mathfrak{g}^*_{\partial M}$ given by $\mathcal{J}(a, p) = d_a^* p$ is the moment map of the action of the group $\mathcal{G}_{\partial M}$ on $T^*\mathcal{F}_{\partial M}$ where the action of $\mathcal{G}_{\partial M}$ on $T^*\mathcal{F}_{\partial M}$ is by cotangent liftings.*

Proof. The moment map $\mathcal{J} : T^*\mathcal{F}_{\partial M} \to \mathfrak{g}^*_{\partial M}$ is given by,

$$\langle \mathcal{J}(a, p), \xi \rangle = \langle p, \xi_{\mathcal{F}_{\partial M}} \rangle = \langle p, \mathrm{d}_a \xi \rangle = \langle -d_a^* p, \xi \rangle \,,$$

because the gauge transformation $g_s = \exp s\xi$ acts in a as $a \mapsto g_s \cdot a = g_s^{-1} a g_s + g_s^{-1} \mathrm{d} g_s$ and the induced tangent vector is given by,

$$\xi_{\mathcal{A}_{\partial M}}(a) = \frac{\mathrm{d}}{\mathrm{d}s} g_s \cdot a \mid_{s=0} = \mathrm{d}_a \xi \,.$$

□

By the standard Marsden-Weinstein reduction, $\mathcal{J}^{-1}(\mathbf{0}) = \{(a, p) \in \mathcal{T}^*\mathcal{F}_{\partial M} | d_a^* p = 0\}$ is a coisotropic submanifold of the symplectic manifold $\mathcal{T}^*\mathcal{F}_{\partial M}$ and $\mathcal{J}^{-1}(\mathbf{0})/\mathcal{G}_{\partial M}$ is symplectic. $\{(a, p) \in \mathcal{T}^*\mathcal{F}_{\partial M} | p = e \wedge e_0\}$ is easily seen to be a symplectic submanifold of $(\mathcal{T}^*\mathcal{F}_{\partial M}), \Omega), \Omega = \delta a \wedge \delta p$. $e_0 F_{a0} = eF_a$ and $eF_{a0} = 0$ are coisotropic submanifolds of $\mathcal{T}^*\mathcal{F}_{\partial M}$. This follows from the elementary observation that in a symplectic manifold a subspace defined by a function $\phi = 0$ is a coisoptropic submanifold of the symplectic manifold. Now we need to check that the intersection of the coisotropic submanifolds comprising $\mathcal{M}'_1$ is a coisotropic submanifold. But this follows easily from the fact that the kernel $\mathcal{J}^{-1}(\mathbf{0}) = \{(a,p) \in \mathcal{T}^*\mathcal{F}_{\partial M} | d_a^* p = 0\}$ is spanned by the action of the gauge group $\mathcal{G}_{\partial M}$ and from the observation that the action of $\mathcal{G}_{\partial M}$ leaves invariant the submanifolds $e_0 F_{a0} = eF_a$ and $eF_{a0} = 0$. ker $\mathcal{J}^{-1}(\mathbf{0})$ is tangent to $\{(a,e)|eF_{a0} = 0\}$ and to $\{(a,e)|e_0 F_{a0} = eF_a\}$ and is therefore contained in the tangent spaces of the two surfaces, and vice versa. Thus $\mathcal{M}'_1$ is a coisotropic. As described in section Chapter 2, section 2.3, the reduced space $\mathcal{R} = \mathcal{M}'_1/\mathcal{G}_{\partial M}$ is symplectic and $\widetilde{\Pi}(\mathcal{EL})$, the reduction of the submanifold of Euler-Lagrange fields of the theory, is an isotropic submanifold of the reduced phase space $\mathcal{R}$ of the theory.

4.4 Conclusions and discussion

Using multisymplectic geometry we have described a Hamiltonian formulation of Palatini's General Relativity that is simple. Unlike ADM it does not involve lapse and shift operators and it does not require for it's application the assumption that spacetime is topologically $R \times S$ where S is space. All we need to assume is that our spacetime manifold has a boundary and that the boundary has a collar.
After the presymplectic constraint analysis, the analysis in the collar provides consistent solutions of the initial value problem for General Relativity. Unlike ADM, we use a formalism that is canonical, i.e. at every step the fundamental structures are preserved, both when discussing the constraints introduced from the bulk Palatini constraint $P = e \wedge e$ and when reducing the system by using gauge invariance. The reduced phase space is determined to be a symplectic manifold with a distinguished isotropic submanifold corresponding to the boundary data of the solutions of the Euler-Lagrange equations.

In a following work we will apply our techniques to study Ashtekar gravity and to the corresponding quantum aspects.

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