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Non-archimedean Analysis and GAGA

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Zonglin Jiang

Committee in charge:

Professor Kiran S. Kedlaya, Chair
Professor Kenneth A. Intriligator
Professor Cristian D. Popescu
Professor Claus M. Sorensen
Professor Alexander Vardy

2019

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The dissertation of Zonglin Jiang is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California San Diego

2019

DEDICATION

To my teachers

EPIGRAPH

Of course all life is a process of breaking down.

— F. Scott Fitzgerald, " The Crack-Up "

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ACKNOWLEDGEMENTS

I would like to thank my advisor Professor Kiran S. Kedlaya first, for teaching me so much—from arithmetic geometry to programming in Python/Sage, for his generous financial support, for the countless rides he gave me and lunch he bought me and fun facts he told me. He is not just my academic advisor but a true mentor.

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VITA

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ABSTRACT OF THE DISSERTATION

Non-archimedean Analysis and GAGA

by

Zonglin Jiang

Doctor of Philosophy in Mathematics

University of California San Diego, 2019

Professor Kiran S. Kedlaya, Chair

We explore the non-archimedean analysis over various types of topological rings, in particular the relationship of topology and norms over such rings, which are motivated by the recent development in p-adic geometry and p-adic Hodge theory. We also make a conjecture about a GAGA theorem over products of Fargues-Fontaine curves.

Chapter 1

An Overview of p-adic Geometry

In this chapter, we briefly review the history of p-adic geometry through different constructions of spectrums of different types of affinoid rings. In 2007, the Arizona Winter School chose p-adic geometry as its topic. Brian Conrad sketched a few approaches to non-archimedean geometry, from Tate's rigid analytic spaces to Berkovich's spaces and Raynaud's approach based on formal schemes. Exactly a decade later in 2017, the Arizona Winter School ran the topic of perfectoid spaces. Since the introduction of perfectoid spaces, Roland Huber's adic spaces have become more popular as the foundation of p-adic geometry. While the story is still going on, we briefly review the history of the development.

1.1 Tate's Work

Tate's rigid analytic geometry is the first theory of p-adic analytic geometry. According to [10], it was so named because "... *one develops everything 'rigidly' by imitating the theory of schemes in algebraic geometry but using rings of convergent power series instead of polynomials.*"

1.1.1 Rigid Analytic Geometry in a Nutshell

It's well-known that $\mathbb{Q}_p$ (and its finite extensions) is totally disconnected, which poses some difficulty in creating a geometric theory over it. The naive idea would be to define a p-adic manifold using charts and analytic functions (functions with power series expansions) as in differential geometry or complex geometry. This approach has been successful in developing the theory of p-adic Lie groups, as in classical Lie theory, the extra analytic group structure allows us to focus on a neighborhood of the identity to construct the Lie algebra which carries the most important information about the space/group. Details can be found in [16].

More generally, the problems are overcome by Tate who introduced the rigid analytic spaces. Just as algebraic geometry is built upon polynomial rings, rigid analytic geometry is built over some power series rings.

Definition 1.1.1. *An absolute value on a field k is a function $|\cdot| : k \to \mathbb{R}^{\geq 0}$, such that for any $a, b \in k$,*

1. $|a| = 0$ *iff* $a = 0$
2. $|a - b| \leq |a| + |b|$
3. $|ab| = |a||b|$.

An absolute value is called nonarchimedean if the following strong triangle inequality holds

$$|a - b| \leq \max\{|a|, |b|\}$$

An absolute value is called nontrivial if $\exists a \in k$, $|a| \neq 0, 1$.

An absolute value is complete if the metric space (k, d) is complete (i.e. every Cauchy sequence converges), where the metric on k is defined by $d(a, b) := |a - b|$.

A field equipped with a complete nonarchimedean absolute value is called a non-archimedean field.

It can be shown that a field equipped with a complete archimedean absolute value must be isomorphic to $\mathbb{R}$ or $\mathbb{C}$, but there are many more non-isomorphic non-archimedean fields, such as $\mathbb{Q}_p$ for every prime number p and their finite extensions.

Definition 1.1.2. *Let k be a non-archimedean field. The n-variable Tate algebra over k is the subring* $k\{X_1, \cdots, X_n\} := \{\sum_{I \in \mathbb{N}^n} a_I X^I | a_I \in k, |a_I| \to 0$ *as* $|I| \to \infty\}$ *of the formal power series ring over k, with topology induced by the Gauss norm* $\|\sum a_I X^I\| = \max_I |a_I|$.

Geometrically, this is the ring of power series that converge in the disk

$$\{(a_1, a_2, \cdots, a_n) \in k^n | \forall i, |a_i| \leq 1\}$$

It can be proved that $k\{X_1, \cdots, X_n\}$ is a Banach algebra over k. Departing from algebraic geometry, we shall use the maximal spectrum of $k\{X_1, \cdots, X_n\}$ (and its quotients) to represent the underlying set of the geometry to be built.

Definition 1.1.3. *A k-affinoid algebra A is a topological k-algebra which is isomorphic to* $k\{X_1, \cdots, X_n\}/I$ *for some positive integer n and an ideal I of* $k\{X_1, \cdots, X_n\}$ *(which can be proved to be always closed). Let* $M(A)$ *be the set of maximal ideals of A, and if* $\phi : A \to B$ *is a map between affinoid algebras,* $M(\phi) : M(B) \to M(A)$ *is the obvious map of their maximal spectrum:* $\mathfrak{m} \mapsto \phi^{-1}(\mathfrak{m})$ *(it can be proved to be well-defined, unlike the maximal spectrum for general commutative rings).*

$M(A)$ is equipped with a Grothendieck topology.

Definition 1.1.4. *Let A be an affinoid k-algebra, a subset* $U \subset M(A)$ *is called an affinoid domain if there is a map* $i : A \to B$ *between k-affinoid algebras, such that* $M(i)$ *sends* $M(B)$ *to U, and i is* *initial among such datum. That is, given diagram of affinoid k-algebras*

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & B \\ & \searrow^{\phi} & \downarrow ? \\ & & C \end{array}$$

there is a unique map from B to C to make this diagram commute iff $M(\phi)$ sends $M(C)$ to U.

By some standard argument in category theory, the $i : A \to B$ is uniquely determined by U up to isomorphism, and we shall write B as $\mathcal{O}_{M(A)}(U)$ as it would help build the structure sheaf on $M(A)$ later. It's easy to see that if U and V are affinoid subdomains, so is $U \cap V$ with $\mathcal{O}_{M(A)}(U \cap V) \simeq \mathcal{O}_{M(A)}(U)\widehat{\otimes}\mathcal{O}_{M(A)}(V)$.

Definition 1.1.5. *A is an affinoid k-algebra. A subset $U \subset M(A)$ is called admissible open if it has a set-theoretic covering by affinoid subdomains $U_i \subset M(A)$, such that for any map of k-affinoid algebras $\phi : A \to B$ with $M(\phi)(M(B)) \subset U$, $M(\phi)(M(B))$ can be covered by finitely many U_i's.*

A collection $\{U_i\}$ of admissible open subsets of $M(A)$ is an admissible cover of its union U, if for any $\phi : A \to B$ with $M(\phi)(M(B)) \subset U$, the covering $\{M(\phi)^{-1}(U_i)\}$ of $M(B)$ has a refinement by a covering of finitely many affinoid subdomains.

The Tate topology (as a Grothendieck topology) on $M(A)$ has admissible open subsets as objects (and inclusion as morphisms) and admissible covers as covers.

Theorem 1.1.6 (Tate's acyclic theorem)**.** *The assignment $U \mapsto \mathcal{O}_{M(A)}(U)$ extends uniquely to a sheaf with respect to the Tate topology.*

Definition 1.1.7. *A rigid analytic space over k is a pair $(X, \mathcal{O}_X)$ where X is a G-topologized space, $\mathcal{O}_X$ is a sheaf on X, such that it admits a covering $\{U_i\}$, with $(U_i, \mathcal{O}_{U_i})$ isomorphic to $M(A)$ for some affinoid algebra M_A.*

1.1.2 Tate's Uniformization Theorem

Rigid analytic spaces are developed by Tate to state a general uniformization theorem. Classicaly, the uniformization theorem over $\mathbb{C}$ claims that there are only three non-isomorphic simply connected Riemann surfaces: the complex plane $\mathbb{C}$, the open disck $\mathbb{D}$, and the Riemann sphere $\mathbb{CP}^1$. For elliptic curve E over $\mathbb{C}$, its universal cover is isomorphic to $\mathbb{C}$, and E is biholomorphic to $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, $\tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0$, and apply the exponential map, we get

$$\mathbb{C}/\Gamma \xrightarrow{\sim} \mathbb{C}^*/q^{\mathbb{Z}}$$

where $q = e^{2\pi i\tau}$. Tate found this makes sense even in the non-archimedean case.

Theorem 1.1.8 (Tate's uniformization theorem)**.** *Let K an algebraically closed fields with a complete non-archimedean absolute value, E be an elliptic curve over K, then*

$$E^{rig} \simeq (\mathbb{G}_m^{rig})/q^{\mathbb{Z}}$$

where E^{rig} is the rigid analytic space associated to E, $\mathbb{G}_m$ is the multiplicative group scheme over K, $q \in K^*$ satisfies $|q| < 1$. In particular,

$$E(K) \simeq K^*/q^{\mathbb{Z}}$$

(the starting point of Tate's investigation).

We shall say more about how to construct an analytic rigid space from an algebraic variety of finite type over k, when we talk about GAGA.

1.2 Berkovich's Gelfand Spectrum

Berkovich's journey started with non-archimedean analysis. Later, his theory was found to be useful in discussing the étale cohomology of rigid analytic spaces.

Definition 1.2.1. *A semi-norm on a commutative unital ring is a function* $|\cdot| : R \to \mathbb{R}^{\geq 0}$ *such that*

1. $|0| = 0$
2. $|a-b| \leq |a| + |b|$
3. $|ab| \leq |a||b|$.

   A semi-norm is a norm if $|a| \neq 0$ *whenever* $a \neq 0$.

   A semi-norm is non-archimedean if it satisfies the strong triangle inequality

4. $|a-b| \leq \max\{|a|,|b|\}$

   A semi-normed ring is Banach if it is complete (i.e. any Cauchy sequence with respect to the metric $d(a,b) := |a-b|$ *converges to a unique point, in particular the ring is separated or Hausdorff). Every normed ring has a unique completion.*

Remark 1.2.2. *Note that* $|0-b| \leq |0| + |b| = |b|$*, and thus by symmetry* $|-b| = |b|$*, and* $|a+b| \leq |a| + |-b| \leq |a| + |b|$*. Therefore the above version of the triangle inequality implies the usual one* $|a+b| \leq |a| + |b|$*. However, the implication is only one-way: Let* $R = \mathbb{Z}$*, and* $|a| = a$ *if* $a \geq 0$ *and* $|a| = -a+1$ *if* $a < 0$*, then this "norm" satisfies* $|a+b| \leq |a| + |b|$*, but not* $|a-b| \leq |a| + |b|$*.*

Remark 1.2.3. *If* R *is the zero ring, there is a unique (semi-)norm on it. If* R *is not the zero ring, it has at least one norm on it:* $|a| = 1$ *unless* $a = 0$*. If* R *is semi-normed such that there is* $a \in R$*,* $|a| \neq 0$*, we have* $|1 \cdot a| \leq |1||a|$ *and hence* $|1| \geq 1$*. In fact, it doesn't harm to assume* $|1| = 1$*, by using the operator norm* $|a|_{op} := \sup_{b \in R, |b| \neq 0} \frac{|ab|}{|b|}$*. The operator (semi-)norm is equivalent to the original (semi-)norm in the sense of the following definition.*

Definition 1.2.4. *Two semi-norms $|\cdot|_1, |\cdot|_2$ on a ring R are equivalent if there exist positive constants c_1, c_2 such that $|\cdot|_1 \leq c_1 |\cdot|_2$ and $|\cdot|_2 \leq c_2 |\cdot|_1$.*

Remark 1.2.5. *Two equivalent norms always induce the same topology, but non-equivalent norms may induce the same topology too. For example, the usual p-adic norm on $\mathbb{Q}_p$ $|a|_p := p^{-v_p(a)}$ induce the same topology as $|a|_{p,\varepsilon} := \varepsilon^{v_p(a)}$ for any $0 < \varepsilon < 1$, but they are not equivalent in the above sense. Here is another one: on $\mathbb{Z}$, the usual absolute value $|\cdot|_\infty$ ($|a|_\infty = a$ if $a \geq 0$ and $|a|_\infty = -a$ if $a < 0$) and the trivial absolute value ($|a| = 1$ if $a \neq 0$) both induce the discrete topology but they are not equivalent.*

Remark 1.2.6. *An archimedean norm can be equivalent to a non-archimedean one: let R be an arbitrary nonzero ring, and $A = R[[X]]$ be the power series ring. Let $0 < r < 1$, and define norm on A by $|\sum_i a_i X^i| = \sum_i |a_i| r^i$ where $|a_i| = 1$ whenever $a_i \neq 0$. This norm doesn't satisfy the strong triangle inequality, but it is equivalent to a norm which does: $|\sum_i a_i X^i|' = r^{\min\{i | a_i \neq 0\}}$, as it's clear that $|\cdot|' \leq |\cdot| \leq \frac{1}{1-r}|\cdot|'$.*

Definition 1.2.7. *Let $(R, |\cdot|)$ be a Banach ring. The Gelfand-Berkovich spectrum $\mathcal{M}(R)$ is the set of all multiplicative seminorms $|\cdot|'$ on R bounded above by $|\cdot|$, that is*

$$\mathcal{M}(R) := \{|\cdot|' : R \to \mathbb{R}^{\geq 0} | |0|' = 0, |1|' = 1, |ab|' = |a|'|b|', |a-b|' \leq |a|' + |b|', |a|' \leq |a|\}$$

Its topology is induced by its embedding into the compact Hausdorff space $\prod_{a \in R}[0, |a|]$ (equivalent to the coarsest topology that makes the evaluation map $|\cdot|' \mapsto |a|'$ continuous for any $a \in R$). In particular, $\mathcal{M}(R)$ is compact and Hausdorff.

Remark 1.2.8. *The condition $|a|' \leq |a|$ can be weakened to be $\exists c > 0$ such that $|a|' \leq c|a|$. Indeed, if there is such c, for any positive integer n,*

$$|a^n|' = (|a|')^n \leq c|a^n| \leq c|a|^n, |a|' \leq c^{1/n}|a|$$

Let $n \to \infty$, we have $|a|' \leq |a|$. In particular, we know that if $|\cdot|, |\cdot|'$ are equivalent norms on R, then $\mathcal{M}(R, |\cdot|) = \mathcal{M}(R, |\cdot|')$.

Remark 1.2.9. *However, the Gelfand-Berkovich spectrum depends on the specific equivalence class of norms on R, not just the topology. For example, let $R = \mathbb{Q}_p$, and v_p be the usual p-adic valuation that takes values in $\mathbb{Z}$. Given real numbers $c > 1$, we define $|x|_c = c^{-v_p(x)}$. In particular, $|\cdot|_p$ is the usual p-adic absolute value on $\mathbb{Q}_p$. Pick $1 < a < b$, and let $|x| := \max\{|x|_a, |x|_b\}$, then $(R, |\cdot|)$ is homeomorphic to $(R, |\cdot|_p)$. While $\mathcal{M}(R, |\cdot|_p)$ has a unique point, $\mathcal{M}(R, |\cdot|)$ is homeomorphic to $[a, b]$. This kind of exotic phenomenon won't happen if $(R, |\cdot|_R)$ is a Banach algebra over a non-archimedean field $(F, |\cdot|_F)$ such that $|ax|_A = |a|_F |x|_R$ for any $a \in F, x \in R$.*

Remark 1.2.10. *We have seen that a non-archimedean norm can be equivalent to an archimedean one. However, note that $\mathbb{Z} \cdot 1$ is norm bounded is a property that only depends on the equivalence class of norms. And in this case it can be shown that if $|\cdot| \in \mathcal{M}(R)$, $|\cdot|$ is non-archimedean. Indeed if $|a \cdot 1| \leq c$ for any $a \in \mathbb{Z}$, then*

$$|(a-b)^n| \leq \sum_i c|a|^i |b|^{n-i} \leq cn(\max\{|a|, |b|\})^n$$

$$|a-b| \leq \sqrt[n]{cn} \max\{|a|, |b|\}$$

and we get the desired inequality by letting $n \to \infty$.

Remark 1.2.11. *The Gelfand-Berkovich spectrum allows both archimedean and non-archimedean semi-norms. In particular, if R is a commutative Banach algebra over $\mathbb{C}$, it can be used to recover the Gelfand transform theory over $\mathbb{C}$. Thus the Berkovich approach has the potential to unify the archimedean and non-archimedean cases, but we would focus on the non-archimedean rings soon.*

Here are some classical examples, taken from [3].

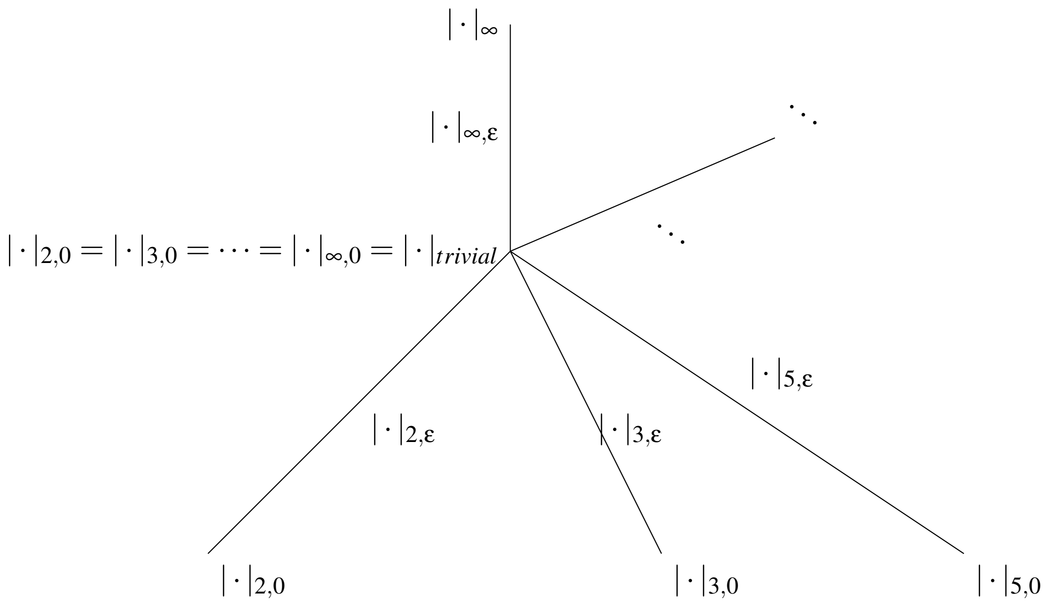

Figure 1.1: The visualization of Berkovich-Gelfand spectrum of $(\mathbb{Z}, |\cdot|_\infty)$

Example 1.2.12. *$\mathcal{M}(\mathbb{Z}, |\cdot|_\infty)$. Any multiplicative norm on $\mathbb{Z}$ can be extended to one on $\mathbb{Q}$, and thus by Ostrowski's theorem, such a norm must be $|\cdot|_{\infty,\varepsilon} = |\cdot|_\infty^\varepsilon$ for some $0 \leq \varepsilon \leq 1$, or $|\cdot|_{p,\varepsilon} = \varepsilon^{v_p(\cdot)}$ for $0 \leq \varepsilon \leq 1$. (When $\varepsilon = 0$, all these norms are simply the trivial norm.) The kernel of a multiplicative semi-norm must be a prime ideal, thus a semi-norm with a nontrivial kernel must be the pull-back of a multiplicative norm on $\mathbb{Z}/p\mathbb{Z}$ which must be trivial, and this norm may be denoted as $|\cdot|_{p,1}$. Hence we have the visualization of $\mathcal{M}(\mathbb{Z}, |\cdot|_\infty)$, as in Figure 1.1.*

That is, each place of $\mathbb{Q}$ occupies a branch which is homeomorphic to $[0,1]$ through the map $|\cdot|_{p,\varepsilon} \mapsto \varepsilon$ (where p is allowed to be ∞). A neighborhood of $|\cdot|_{trivial}$ contains all but finitely many branches. By suitably picking the length and position of each branch, $\mathcal{M}(\mathbb{Z}, |\cdot|_\infty)$ can be indeed topologically embedded into $\mathbb{R}^2$.

Example 1.2.13. *$\mathcal{M}(\mathbb{Z}, |\cdot|_{trivial})$ obviously embeds into $\mathcal{M}(\mathbb{Z}, |\cdot|_{trivial})$, and it's precisely the above tree-like graph without the $[|\cdot|_\infty, |\cdot|_{\infty,0})$ branch. As both $|\cdot|_\infty$ and $|\cdot|_{trivial}$ induce the discrete topology on $\mathbb{Z}$, we have another example of homeomorphic Banach rings with different Gelfand-Berkovich spectrum.*

Theorem 1.2.14 (Berkovich)**.** $\mathcal{M}((R,|\cdot|) = \emptyset$ *iff R is the zero ring.*

Proof. We only prove this when $|\cdot|$ is non-archimedean. If R is the zero ring, the conditions $|0| = 0$ and $|1| = 1$ are inconsistent, thus $\mathcal{M}(R) = \emptyset$. If R is not the zero ring, consider the set $S := \{|\cdot|' : R \to \mathbb{R}^{\geq 0} | |0|' = 0, |1|' = |1|, |a-b|' \leq \max(|a|', |b|'), |ab|' \leq |a|'|b|', |a|' \leq |a|\}$. This is a nonempty set, as $|\cdot|$ is in it. Given $|\cdot|_1, |\cdot|_2$ from S, we define $|\cdot|_1 \leq |\cdot|_2$ iff $|a|_1 \leq |a|_2$ for all $a \in R$. Then by Zorn's lemma there exists a minimal element $|\cdot|'$. It's enough to show $|\cdot|'$ is multiplicative, i.e. $|ab|' = |a|'|b|'$ for any $a, b \in R$.

To show this, we first observe that $|\cdot|'$ must be already power-multiplicative (i.e. $|a^n|' = (|a|')^n$ for $a \in R$), for otherwise $|a|'_{sp} := \lim_{n\to\infty} (|a^n|')^{1/n}$ will be a semi-norm strictly smaller than $|\cdot|'$. Now if $|b| = 0$, there must be $|ab|' \leq |a|'|b|' = 0$, thus $|ab|' = 0 = |a|'|b|'$, otherwise $|a|_b := \lim_{n\to\infty} \frac{|b^n a|'}{|b^n|'}$ defines a power-multiplicative norm on R bounded above by $|\cdot|'$, thus $|a|_b = |a|'$ by minimality of $|\cdot|'$. If $|ab|' < \varepsilon |a|'|b|'$ for some $\varepsilon \in (0,1)$, then $|ab^n|' \leq |ab|'|b^{n-1}|' \leq \varepsilon |a|'|b|'(|b|')^{n-1} = \varepsilon |a|'|b^n|'$, thus $\frac{|ab^n|'}{|b^n|'} \leq \varepsilon |a|'$, and after taking limit, $|a|_b \leq \varepsilon |a|'$, a contradiction. □

There is no surprise that Zorn's lemma was used in the above proof, as in algebraic geometry, to show $\mathrm{Spec}(R)$ is non-empty whenever R is not the zero ring, we need Zorn's lemma to construct a maximal ideal of R.

The above proof has used the following two lemmas.

Lemma 1.2.15. *If $|\cdot|$ is a non-archimedean submultiplicative seminorm on R, then $|a|_{sp} := \lim_{n\to\infty} |a^n|^{\frac{1}{n}} = \inf_n |a^n|^{\frac{1}{n}}$ exists and defines a power-multiplicative semi-norm on R (often called the spectral seminorm).*

Proof. If a is nilpotent, the limit obviously exists and equal to the infimum. Otherwise let $r = \inf_n |a^n|^{\frac{1}{n}}$. For any $\varepsilon > 0$, pick m such that $|a^m|^{1/m} < r + \varepsilon$, then for any n, rewrite $n = mk_n + l_n$ where $0 \leq l_n < m$, now $|a^n| \leq |a^m|^{k_n}|a^{l_n}|$, $|a^n|^{1/n} \leq |a^m|^{k_n/n}|a^{l_n}|^{1/n} \leq (r+\varepsilon)^{mk_n/n}|a^{l_n}|^{1/n} = (r+\varepsilon)^{1-\frac{l_n}{n}}|a^{l_n}|^{1/n} \to (r+\varepsilon)$, thus the limit superior is at most $r+\varepsilon$.

To show it's a power-multiplicative semi-norm, the only hard part is to show the strong triangle inequality holds. $|(a-b)^n|^{1/n} \le (\max_{i+j=n}|a^i b^j|)^{1/n} \le (|a^{i_n}||b^{j_n}|)^{1/n}$. Since $0 \le \frac{i_n}{n} \le 1$, there is a subsequence n_k, such that $\frac{i_{n_k}}{n_k}$ converges, say to α, then

$$|a-b|_{sp} \le \lim_{k\to\infty}(|a^{i_{n_k}}|^{1/i_{n_k}})^{i_{n_k}/n_k}(|b^{j_{n_k}}|^{1/j_{n_k}})^{j_{n_k}/n_k} = |a|_{sp}^{\alpha}|b|_{sp}^{1-\alpha} \le \max\{|a|_{sp},|b|_{sp}\}$$

□

In fact, it can be shown that $|a|_{sp} = \max_{|\cdot|\in\mathcal{M}(R)}\{|a|\}$. Since we won't need this, the proof is omitted.

Lemma 1.2.16. *If $|\cdot|$ is a non-archimedean power-multiplicative semi-norm on R, let $c \in R$ such that $|c| \neq 0$, then for any $x \in R$, the limit $|x|_c := \lim_{n\to\infty}\frac{|xc^n|}{|c^n|}$ exists. And $|\cdot|_c$ defines a non-archimedean power-multiplicative semi-norm on R.*

Proof. See [4, Section 1.3.2, Proposition 2]. □

In fact, we discovered the alternative proof of Berkovich's theorem when reading [4]. The usual proof goes through a polynomial ring construction after the Zorn's lemma step (See [3],[13]).

1.3 Huber's Adic Spaces

Huber's approach is based on adic spectrum which contains significantly more points than Berkovich's approach. It was motivated by the classical result:

Theorem 1.3.1. *Let X be a compact Riemann surface (over $\mathbb{C}$), and $\mathbb{C}(X)$ be the function field of X (the field of meromorphic functions on X), then there is one-to-one correspondence between the (closed) points on X and valuation rings of $\mathbb{C}(X)$ which contains $\mathbb{C}$ but not $\mathbb{C}(X)$ itself.*

Proof. See [9, Corollary 5.8.2] □

This result was extended by Zariski to Zariski-Riemann spaces over a general field. This suggests to use valuations as points, instead of prime ideals. In order to do this, we need some preparation.

Definition 1.3.2. *A subset S of a topological ring R is called (topologically) bounded if for any neighborhood U of 0, there is a neighborhood V of 0, such that $V \cdot S \subset U$.*

An element a of R is power-bounded if $\{a^n | n \in \mathbb{N}\}$ is bounded. The set of power-bounded elements of R is denoted as R°, and $R^{\circ\circ}$ is often used for the set of topological nilpotent elements in the literature.

Remark 1.3.3. *Usually we want to define $V \cdot S = \{vs | v \in V, s \in S\}$ to include say Banach algebras over $\mathbb{C}$ in the above definition, but since we will work exclusively with nonarchimedean rings (aka topological rings that admits a neighborhood base of 0 consisting of additive subgroups, such as a nonarchimedean Banach ring), it doesn't hurt to use $V \cdot S$ as the group generated by the set $\{vs | v \in V, s \in S\}$.*

Definition 1.3.4. *A topological ring R is called adic if there is an ideal I such that $\{I^n | n \in \mathbb{N}\}$ forms a neighborhoods base at 0. Any ideal of R with such a property is called an ideal of definition of R.*

An f-adic ring R (or a Huber ring) is a topological ring such that it contains an open subring which is adic and admits a finitely generated ideal of definition. Any open subring that allows an ideal of definition is called a ring of definition. (It can be shown that an open subring is a ring of definition iff it's bounded.)

An f-adic ring is called Tate if it contains a topological nilpotent unit.

A ring of integral elements of an f-adic ring R is a subring which is open, integrally closed in R, consists of only power-bounded elements.

An affinoid ring (or a Huber pair) is a pair (R, R^+) where R is a f-adic ring and R^+ is a ring of integral elements of R.

Remark 1.3.5. *Noetherian adic rings are f-adic or Huber rings. They play the role of "affinoid rings" in the theory of formal schemes, as developed in [7, Section II.9].*

Definition 1.3.6. *Let R be a Huber ring, the n-variable Tate algebra over R is*

$$R\{X_1,\cdots,X_n\} = \{\sum_{I\in\mathbb{N}^n} a_I X^I | a_I \in \widehat{R},\ and\ a_I \to 0,\ as\ |I| \to \infty\}$$

To define the topology on the Tate algebra, when $\{U\}$ is a neighborhood base of 0 in $\widehat{R}$, the following form a neighborhood base of 0 in the Tate algebra.

$$U\{X_1,\cdots,X_n\} = \{\sum_{I\in\mathbb{N}^n} a_I X^I | a_I \in U,\ and\ a_I \to 0,\ as\ |I| \to \infty\}$$

Definition 1.3.7. *Let $(\Gamma,\cdot)$ be a totally ordered abelian group (with multiplication as its operation, and identity element being 1), $\Gamma\cup\{0\}$ be the extended ordered monoid with the extra definition that*

$$0\cdot\gamma = 0, 0 \leq \gamma$$

for all $\gamma\in\Gamma\cup\{0\}$. A valuation on a ring R with values in Γ is a function:

$$v : R \to \Gamma\cup\{0\}$$

such that for all $a,b \in R$,

1. $v(0) = 0, v(1) = 1$.

2. $v(a-b) \leq \max\{v(a), v(b)\}$.

3. $v(ab) = v(a)v(b)$.

Remark 1.3.8. *In commutative algebra, the value group of a valuation usually has "$+$" as its operation, and the corresponding extended monoid would be $\Gamma\cup\{-\infty\}$. But in p-adic geometry,* *as we have got used to using absolute values since Berkovich, it's more natural to use the multiplicative notation.*

Definition 1.3.9. *A valuation v on a topological ring R is continuous if for every $\gamma \in v(R) \setminus \{0\}$, $\{a \in R | v(a) < \gamma\}$ is a neighborhood of 0. Two valuations v, v' are equivalent iff they define the same order relation on A. The set of equivalence classes of valuations is denoted $Spv(R)$.*

It's more natural to think of a valuation as an order relation on R which satisfies certain properties, but more convenient to work with a value group. If $v(R)$ is trivial, then it's continuous iff $v^{-1}\{0\}$ is an open ideal of R. Otherwise $v(R)$ is continuous iff $\{a \in R | v(a) \leq \gamma\}$ is open for all $\gamma \in v(R) \setminus \{0\}$. The "only" part is clear, for the "if" part, note that $\{a \in R | v(a) < \gamma\} = \cup_{\delta<\gamma}\{a \in R | v(a) \leq \delta\}$.

An archimedean equivalence class of a totally ordered abelian group Γ is a class defined by the relation $a \sim b$ if $\exists m, n \in \mathbb{Z} \setminus \{0\}, a^n > b, b^n > a$. The archimedean classes are naturally ordered. The Hahn embedding theorem states that every totally ordered abelian group can be embedded into $(\mathbb{R}^+)^\Omega = \{f : \Omega \to R^+ |$ The support of f is well-ordered$\}$, where Ω is the archimedean classes of Γ. Traditionally, a valuation is called rank-1 if it has only one archimedean equivalence class. And from Hahn's theorem, the rank of any valuation is a well-defined cardinal. Using this language, we may say that the Berkovich-Gelfand spectrum have only seen the rank-1 valuations, thus must be incomplete.

Here is an example of higher rank valuation, taken from [18],

Example 1.3.10. *Let $R = \mathbb{Q}_p\{T\}$, the Tate algebra with the Gauss norm topology. Let $\Gamma = R^+ \times \gamma^{\mathbb{Z}}$, where $(a, \gamma^m) < (b, \gamma^n)$ iff either $a < b$ or $a = b, m > n$. Basically we are adding an infinitesimal neighborhood $\gamma^{\mathbb{Z}}$ around each point on the half line $\mathbb{R}^+$. And let $v(\sum_{i=0}^{\infty} a_i T^i) = \sup_i(|a_i|, \gamma^i)$.*

There is a similar valuation on $\mathbb{Q}_p\{T\}$ with γ infinitesimally greater than 1, that is to add an infinitesimal neighborhood $\mathbb{R}^+$ around each $\gamma^n, n \in \mathbb{Z}$, but morally this should not happen, as the valuation of T is supposed to be at most 1. The second component of a Huber pair can be used to rule these valuations out.

Definition 1.3.11. *Let (R,R^+) be a Huber pair, its adic spectrum is defined by*

$$Spa(R,R^+) := \{v \in Spv(R) | v \text{ is continuous and } v(a) \leq 1, \forall a \in R^+\}$$

A rational subspace of $Spa(R,R^+)$ is one of the form

$$U(\frac{f_1,\cdots,f_n}{g}) = \{v \in Spa(R,R^+) | v(f_i) \leq v(g) \neq 0, \forall i = 1,2,\cdots,n\}$$

$$(\text{or intuitively, } v(\frac{f_i}{g}) \leq 1, \text{ hence the name rational subspace})$$

if $f_1,\cdots,f_n$ generate an open ideal in R.

The topology of $Spa(R,R^+)$ is the one generated by the rational subspaces. Note that the rational subspaces are closed under finite intersection as

$$U(\frac{f_1,\cdots,f_n}{g_1}) \cap U(\frac{h_1,\cdots,h_m}{g_2})$$

$$= U(\frac{\{f_i g_2 | i = 1,\cdots,n\} \cup \{h_i g_1 | j = 1,\cdots,m\} \cup \{f_i h_j | i = 1,\cdots,n; j = 1,\cdots,m\}}{g_1 g_2})$$

(For the concern of inequalities, it's unnecessary to add $\{f_i h_j | i,j\}$ to the list of numerators, and they are present to guarantee the numerators generate an open ideal.)

Remark 1.3.12. *A Huber ring isn't necessarily complete, but this would not affect the theory of adic spaces. In fact, any continuous valuation on a Huber ring can be uniquely extended to its completion. And for the structure pre-sheaf to be defined soon, we always take the completion.*

Given $|\cdot| \in \text{Spa}(R,R^+)$, $\ker|\cdot| := \{a \in R \| a| = 0\}$ is a prime ideal of R, thus there is a natural (continuous) map from $\text{Spa}(R,R^+)$ to $\text{Spec}(R)$. This map is important in the comparison of analytic geometry based on adic spaces and algebraic geometry based on schemes. If we pick a suitable norm on R (which is always possible, see Section 2.1), there is also a natural map from Gelfand-Berkovich spectrum $\mathcal{M}(R)$ to $\mathrm{Spa}(R,R^+)$: If $v \in \mathcal{M}(R)$, as a valuation, it is obviously continuous, and for any power-bounded element $a \in R$, $\{v(a)^n = v(a^n) | n \in \mathbb{Z}\}$ must be bounded in $\mathbb{R}$, hence $v(a) \leq 1$. However, this map isn't necessarily injective (and it's not continuous in general). For example, let $R = R^+ = \mathbb{Z}_p$. Pick the p-adic norm $|a| = p^{-v_p(a)}$ on R, then the Gelfand-Berkovich spectrum of R is isomorphic to $[1/p, 1]$, but $\mathrm{Spa}(R,R^+)$ consists of only two points, as $\mathbb{Z}_p$ is discrete valuation ring. From the map $\mathcal{M}(R) \to \mathrm{Spa}(R,R^+)$, it immediately follows that $\mathrm{Spa}(R,R^+)$ is not empty unless R is the zero ring.

Is it possible to further enlarge the Huber adic spectrum? There are good reasons to believe no. To explain this, let's recall a theorem of M. Hochster and the definition of spectral spaces,

Theorem 1.3.13 (Hochster)**.** *The following conditions on a topological space X are equivalent:*

- *It's sober (meaning every irreducible closed subset is the closure of a unique point called the generic point), quasi-compact and has a basis consisting of quasi-compact open subsets, and any finite intersection of quasi-compact open subsets is still quasi-compact (and of course open)*
- *There is a commutative ring R such that X is homeomorphic to Spec(R).*
- *X is an inverse limit of finite T_0 spaces.*

Definition 1.3.14. *A topological space that satisfies the above equivalent conditions is called spectral.*

From the above theorem, we may consider a spectral space "complete" with no further points due. And in this sense we expect $\mathrm{Spa}(R,R^+)$ to have already seen all points underlying the geometry of R.

Theorem 1.3.15 (Huber). *$Spa(R,R^+)$ is quasicompact and the rational subspaces form a topological basis consisting of quasicompact open subsets.*

The spectrum functor that sends a commutative ring to a locally ringed space can be defined as the right adjoint functor to the functor of taking global sections of a locally ringed space. A similar statement holds, but not in the generality we have been working with. Next we will define a structure pre-sheaf on $\mathrm{Spa}(R,R^+)$, similar to how we define them in rigid analytic spaces, but unfortunately it fails to be a sheaf in general. We will further discuss this issue once we work with the suitable generality.

The following definition is analogous to the one about the maximal spectrum of the classical Tate algebras.

Definition 1.3.16. *An affinoid subdomain of $Spa(A,A^+)$ is a quasicompact open subset U such that there is an affinoid homomorphism $\phi:(A,A^+)\to(B,B^+)$ which is initial among morphisms $\psi:(A,A^+)\to(C,C^+)$ where (C,C^+) is a complete Huber rings for which $\psi^*(Spa(C,C^+))\subset U$. In particular, the map ϕ is canonically associated with U, and will be called the affinoid localization.*

It's difficult to characterize affinoid subdomains, but it can be shown that all rational subspaces are affinoid subdomains.

Theorem 1.3.17. *Let $U=\{v\in Spa(A,A^+):v(f_i)\le v(g)\neq 0, i=1,\cdots,n\}$, then*

- *The subspace U is an affinoid subdomain certified by the rational localization*

$$\phi:(A,A^+)\to(B,B^+)$$

where B is

$$A\{T_1,\cdots,T_n\}/\overline{\{gT_1-f_1,\cdots,gT_n-f_n\}}$$

equipped with the quotient topology, and B^+ is the topological closure of the integral closure of the image of $A^+[T_1, \cdots, T_n]$ in B.

- *The map $\phi^* : Spa(B,B^+) \to Spa(A,A^+)$ induces a homeomorphism $Spa(B,B^+) \simeq U$, and moreover, the rational subspaces of $Spa(B,B^+)$ correspond to the rational subspaces of $Spa(A,A^+)$ contained in U.*

Proof. See [14, Lemma 2.4.13]. □

Definition 1.3.18. *A rational covering (resp. affinoid covering) of $Spa(A,A^+)$ is a finite collection $\{U_i\}$ of rational (resp. affinoid) subdomains of $Spa(A,A^+)$ such that $\cup_i U_i = Spa(A,A^+)$.*

Definition 1.3.19. *The structure presheaf $\mathcal{O}$ on $Spa(A,A^+)$ is defined as*

$$(\mathcal{O}(U), \mathcal{O}^+(U)) = \varprojlim_{Spa(B,B^+)\subset U} (B,B^+)$$

where B is a rational localization of A.

In general, the presheaf is not a sheaf, and when it is, the pair (A,A^+) is called sheafy. This in fact on depends on A.

Definition 1.3.20. *An adic space is a triple $(X, \mathcal{O}_X, (v_x)_{x\in X})$ that is locally isomorphic to $Spa(A,A^+)$ for sheafy (A,A^+). Here $(X, \mathcal{O}_X)$ is a locally ringed space, and v_x is a valuation on the stalk $\mathcal{O}_{X,x}$.*

Theorem 1.3.21 (Huber)**.** *Let (A,A^+) and (B,B^+) be sheafy Huber pairs with B complete. The natural map $Hom((A,A^+),(B,B^+)) \to Hom(Spa(B,B^+), Spa(A,A^+))$ is a bijection.*

Here a morphism between the two huber pairs is a continuous ring homomorphism with $f(A^+) \subset f(B^+)$, and the homomorphism between the adic spectrum is a continuous homomorphism of adic spaces.

Huber also proved a rigid-adic comparison theorem.

Theorem 1.3.22 (Huber). *Let k be an non-archimedean field, there is fully faithful functor:*

$$r : \{\textit{rigid analytic varieties over } k\} \to \{\textit{adic spaces over}\}k\}$$

which sends $\mathcal{M}(R)$ to $Spa(R, R^{\circ})$ for any classical Tate algebra over k. Moreover, the functor is an equivalence between the quasi-separated rigid analytic varieties over k and quasi-separated adic spaces locally of finite type over k.

Compared with rigid analytic spaces, Huber's adic spaces are genuine locally ringed spaces, which makes it easier to work with, and also easier to be compared with schemes.

1.4 GAGA: From Complex Geometry to Rigid One

Let X be a scheme of finite type over $\mathbb{C}$, we can associate to it a complex analytic space X^{an} as follows. First cover X with affine open subsets $\mathrm{Spec}(A_i)$, and since X is of finite type, each A_i is isomorphic to $\mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$. The common zero locus of $f_1, \cdots, f_m$ form a complex analytic subspace of $\mathbb{C}^n$, and the gluing of $\mathrm{Spec}(A_i)$ through X can be used to glue the zero loci. We can prove that this construction is independent of the choices, and is functorial.

Moreover, there is a canonical map $X^{an} \to X$ as locally ringed spaces.

In general, the resulting space X^{an} is only a complex analytic one, not necessarily a complex manifold, and X^{an} is a manifold iff X is smooth as a scheme. In his paper "GAGA" [15], Serre proved the following comparison theorem between algebraic geometry and analytic geometry:

Theorem 1.4.1. *Let X be a projective scheme over $\mathbb{C}$. Then the functor $X \mapsto X^{an}$ induces an equivalence of categories from the category of coherent sheaves on X to the category of coherent analytic sheaves on X^{an}. Furthermore, for every coherent sheaf $\mathcal{F}$ on X, the natural maps $H^i(X, \mathcal{F}) \to H^i(X^{an}, \mathcal{F}^{an})$ are isomorphisms, for all i, where $\mathcal{F}^{an}$ is the pullback of $\mathcal{F}$ along the* *canonical map* $X^{an} \to X$.

Similarly, there is a GAGA theorem between algebraic geometry and rigid analytic geometry. Let k be a field equipped with a complete nontrivial nonarchimedean absolute value. Given a scheme X that is of locally finite type over k, we can define a rigid analytification X^{rig}, similar to how we associate a complex analytic space to a scheme of locally finite type over $\mathbb{C}$, and a canonical homomorphism of locally G-ringed spaces $(X^{rig}, \mathcal{O}_{X^{rig}}) \to (X, \mathcal{O}_X)$. X^{rig} can also be characterized by the following universal property: Given any rigid analytic space $(Y, \mathcal{O}_Y)$ over k and a morphism of locally G-ringed k-spaces $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, f uniquely factors through the canonical morphism from $(X^{rig}, \mathcal{O}_{X^{rig}})$ to $(X, \mathcal{O}_X)$. We can also pull back the coherent sheave $\mathcal{F}$ on X along the canonical map to get a coherent sheaf $\mathcal{F}^{rig}$ on X^{rig}. The analytification construction is functorial, and it also satisfies a GAGA theorem similar to Serre's.

Theorem 1.4.2. *Let X be a proper scheme over k, then the category of coherent sheaves over X is equivalent to the category of coherent sheaves over X^{rig}, and for any coherent sheaf $\mathcal{F}$, there is a canonical isomorphism between $H^i(X, \mathcal{F})$ and $H^i(X^{rig}, \mathcal{F}^{rig})$ for $i \geq 0$.*

1.5 Conclusion

Although there are various comparison theorems, rigid analytic spaces, formal schemes, Berkovich spaces and adic spaces are modeled over different types of rings, and hence enjoy different generalities. It seems they all work well with analytic spaces that come from algebraic geometry (algebraic varieties of finite type over a non-archimedean field).

For example, we still don't know how to characterize sheafy rings. It took Grothendieck and Serre several years to realize "*The best category of commutative rings is the category of all commutative rings*". Will we ever find the best category of topological commutative rings? In the next chapter, we'll give some clue to this question.

Chapter 2

Some Non-archimedean Analysis

Non-archimedean analysis over a non-archimedean field can be developed in a similar fashion to functional analysis over $\mathbb{R}$ or $\mathbb{C}$, most proofs can be easily adapted, but the lack of a base field makes life much harder. For example, a continuous map between Banach spaces over general Banach rings isn't necessarily bounded, which makes the relationship between topology and norm more mysterious.

In this chapter, we assume all the (semi-)norms are non-archimedean, although some of the results apply to archimedean case too (and occasionally we will use the usual triangle inequality instead of the strong one to make this explicit). We also assume Huber rings are Hausdorff (but not necessarily complete).

2.1 Huber Rings versus Normed Rings

While Huber rings form the foundation of adic spaces, it's usually more convenient to work with normed rings which is central to Berkovich's theory. The following makes it possible to translate results about Banach rings to Huber rings, such as to prove (R, R^+) is nonempty by the Gelfand-Berkovich spectrum is nonempty.

Proposition 2.1.1. *A Huber ring R admits a norm which induces its original topology.*

Proof. Let (R_0, I) be a couple of definition, i.e. R_0 is an open subring, I is a (finitely generated) ideal of R_0, such that I^n form a base of neighborhoods at 0, with the convention $I^0 = R_0$. Define a norm $\|a\| := \inf\{e^{-n} | n \in \mathbb{Z}^{\geq 0}, a \in I^n\}$ if $a \in R_0$, or $\|a\| := \inf\{e^n | n \in \mathbb{Z}^{\geq 0}, aI^n \subset I^0\}$ if $a \notin R_0$. Note that, since R_0 is open and the multiplication by a map is continuous on R, $\{n | n \in \mathbb{Z}^{\geq 0}, aI^n \subset I^0\}$ is nonempty, thus the norm is always well-defined.

$\|0\| = 0$ is obvious, and if $\|a\| = 0$, then $a \in I^n$ for any n. By R is complete, in particular separated (Hausdorff), $\cap_n I^n = \{0\}$, thus $a = 0$. Given $a, b \in R_0$, if $\|a\| = e^{-n}, \|b\| = e^{-m}$, then $ab \in I^{n+m}$, $\|a+b\| \leq e^{-(n+m)} = \|a\|\|b\|$, and $a+b \in I^{\min\{m,n\}}, \|a+b\| \leq e^{-\min\{m,n\}} = \max\{\|a\|, \|b\|\}$. If $a \in R_0, b \notin R_0$ and $\|a\| = e^{-n}, \|b\| = e^m$, then $a+b \notin R_0$ and $(a+b)I^m \subset R_0 + R_0 \subset R_0$, thus $\|a+b\| \leq e^m = \|b\| = \max\{\|a\|, \|b\|\}$. If $ab \in R_0$ and $m \geq n$, $\|ab\| \leq 1 \leq \|a\|\|b\|$, if $ab \in R_0$ and $m < n$, then $ab \in bI^n \subset bI^m I^{n-m} \subset I^0 I^{n-m} = I^{n-m}$, $\|ab\| \leq e^{m-n} = \|a\|\|b\|$. Finally if $a, b \notin R_0$ and $\|a\| = e^n, \|b\| = e^m$, then if $a+b \in R_0, \|a+b\| \leq 1 \leq \max\{\|a\|, \|b\|\}$, and if $ab \in R_0, \|ab\| \leq 1 \leq \|a\|\|b\|$; if $a+b \notin R_0$, $(a+b)I^{\max\{m,n\}} \subset aI^n + bI^m \subset R_0$, $\|a+b\| \leq e^{\max\{m,n\}} = \max\{\|a\|, \|b\|\}$, if $ab \notin R_0$, $abI^{m+n} = (aI^m)(b^n) \subset R_0 R_0 = R_0$, $\|ab\| \leq e^{m+n} \leq \|a\|\|b\|$.

Given any $r > 0$, pick non-negative integer m such that $e^{-m} < r$, then the ball $\{x \in R | \|x\| < r\}$ contains I_n, thus is open in R with the original topology. On the other hand, if $\|x\| < e^{-n}$, then $x \in xI^0 \subset I^n$, thus I^n contains the ball $\{x \in R | \|x\| < e^{-n}\}$, and hence is open in R with the norm topology. □

Remark 2.1.2. *The above proof is essentially the same as the sketch from [13, Remark 1.5.4].*

However, a normed ring isn't necessarily Huber.

Example 2.1.3. *Let R be an arbitrary nonzero ring, and A be the R-algebra generated by a countable family of variables $x_1, x_2, \cdots$ satisfying the relations $x_i x_j = 0$ for any $i, j \geq 1$. Using the convention $x_0 = 1$, define norm on A by $|\sum_{i=0}^{\infty} a_i x_i| = 2^{-\min\{i | a_i \neq 0\}}$. Then A is a Banach ring, but not Huber.*

The following concept is defined by Kedlaya in [13].

Definition 2.1.4. *A topological ring is analytic if it has no proper open ideal.*

In [14, Remark 2.3.9], it was stated as unknown whether an analytic Banach ring has to be Tate, and the first counter-example shows up in [13, Example 1.5.7].

Proposition 2.1.5. *An analytic normed ring is a Huber ring.*

Proof. Let R be an analytic normed ring, then $R_0 := \{a \in R | \|a\| \leq 1\}$ is an open subring. Since R is analytic, the R-ideal generated by $I' = \{a \in R \|a\| < \frac{1}{2}\}$ is R itself. Thus there are $a_1, \cdots, a_n \in I', b_1, \cdots, b_n \in R_0$ such that $\sum_{i=1}^n a_i b_i = 1$. We claim $I = \sum_{i=1}^n R_0 a_i \subset I'$ is an ideal of definition for R_0. Pick U to be a neighborhood of 0 such that $b_i U \subset R_0$ for all i, then for any $x \in U$, $x = (\sum_{i=1}^n a_i b_i) x \in \sum_{i=1}^n R_0 a_i \in I$, thus $U \subset I$. Hence I is open, and similarly we can show I^n is open in R.

□

Therefore, analytic Huber rings and analytic normed rings form the same subcategory of topological rings.

Let R be a normed ring, a norm bounded subset must be bounded, but a bounded subset of R is not necessarily norm bounded.

Example 2.1.6. $(\mathbb{Z}[x], |\cdot|)$ *where* $|\sum_{i=0}^n a_i x^i| = \max_{0 \leq i \leq n, a_i \neq 0} 2^i$*, then* $\mathbb{Z}[x]$ *is discrete, thus any subset is bounded, but* $\{x^n\}_{i=0}^{\infty}$ *is not norm bounded.*

When R is analytic, a topologically bounded subset must be norm bounded.

Proposition 2.1.7. *Let* $(R, |\cdot|)$ *be an analytic normed ring, then a (topologically) bounded subset of* R *must be norm bounded.*

Proof. Let S be a bounded subset of R, then there exists a neighborhood V of 0 such that $V \cdot S$ is contained in the closed unit ball. By R is analytic, there are finitely many elements $a_1, \cdots, a_n \in R$

and $b_1, \cdots, b_n \in V$ such that $\sum_{i=1}^n a_i b_i = 1$. Thus for any $x \in S$, $|x| = |\sum_{i=1}^n a_i b_i x| \leq \sum_{i=1}^n |a_i||b_i x| \leq \sum_{i=1}^n |a_i|$. (The proof doesn't use the completeness of R, hence holds for any analytic normed ring.) □

In particular, $x \in R^\circ$ iff $\{x^n\}_{n=1}^\infty$ is norm bounded.

Definition 2.1.8. *A Huber ring R is uniform if R° is bounded.*

Example 2.1.9. *Let $R = \mathbb{Q}_p[\varepsilon]/(\varepsilon^2)$ with norm $|a + b\varepsilon| = \max\{|a|_p, |b|_p\}$. This is a Tate ring, but $R^0 = \mathbb{Z}_p \oplus \mathbb{Q}_p\varepsilon$ is not norm bounded (hence not topologically bounded).*

As one can see from the example, the existence of nilpotent elements could be the key reason of a Huber ring not to be uniform.

Proposition 2.1.10. *If R is an analytic Huber ring, and Rx is bounded for $x \in R$, then $x = 0$. If R is further uniform, then R is reduced.*

Proof. For any ideal of definition I, pick $a_1, \cdots, a_n \in I$, and $b_1 \cdots, b_n \in R$ such that

$$\sum_{i=1}^n a_i b_i = 1$$

then

$$x = \sum_{i=1}^n a_i b_i x \in IRx$$

This is true for any ideal of definition, hence $x \in \cap_{i=1}^\infty I^n Rx$. As Rx is bounded,

$$\cap_{n=1}^\infty I^n Rx = \{0\}$$

If R is uniform, and x is a nilpotent, then $Rx \subset R^0$ is bounded, thus $x = 0$.

□

The above statement can be improved,

Proposition 2.1.11. *In a uniform analytic normed ring R, $|x|_{sp} = 0$ iff $x = 0$. In other words, the spectral semi-norm on R is a norm.*

Proof. If $|x|_{sp} = 0$, for any $y \in R$, $|xy|_{sp} \leq |x|_{sp}|y|_{sp} = 0$. Thus for sufficiently large n, $|(xy)^n| \leq (\frac{1}{2})^n$, $xy \in R^0$. Hence $Rx \subset R^0$ is bounded, $x = 0$ by the last proposition. □

Classically, a Banach algebra (over $\mathbb{C}$ or $\mathbb{R}$) is called uniform if its norm is power-multiplicative. This is adopted for Banach rings by Berkovich, see [3, Section 1.3]. Now we explore the relationship of the uniform Banach rings in this sense and uniform Huber rings.

Proposition 2.1.12. *The following properties of a norm on a ring R are equivalent:*

- *The norm on R is equivalent to some power-multiplicative norm.*
- *The norm on R is equivalent to its spectral seminorm.*
- *There is $c > 0$ such that $|x^2| \geq c|x|^2$ for all $x \in R$. (One gets another equivalent condition by replacing 2 with any larger integer.)*

The conditions are stated in [14, Section 2.8] (where the the Banach rings are assumed to be Tate), but the proof is missed there.

Proof. If $|\cdot| \leqslant c|\cdot|'$ where $c > 0$ and $|\cdot|'$ is power-multiplicative, then $|x^n| \leqslant c|x^n|' = c(|x|')^n$, $|x^n|^{1/n} \leqslant c^{1/n}|x|'$ and let $n \to \infty$, $|x|_{sp} = |x|'$. This shows that if the norm on R is equivalent to some power-multiplicative norm, the power-multiplicative one must be exactly the spectral norm on R.

To show the last two are equivalent, if $|\cdot|$ is equivalent to its spectral seminorm, then there is $c > 0$ such that $|\cdot|_{sp} \geq c|\cdot|$. Then by $|x|_{sp} = \inf_n |x^n|^{1/n}$, $|x^2|^{1/2} \geq c|x|$, $|x^2| \geq c^2|x|^2$. If $|x^2| \geq c|x|^2$ for all x, then by induction $|x^{2^n}| \geq c^{1+2+\cdots+2^{n-1}}|x|^{2^n} = c^{2^n-1}|x|^{2^n}$, $|x^{2^n}|^{1/2^n} \geq c^{1-1/2^n}|x|$, and let $n \to \infty$, $|x|_{sp} \geq c|x|$. As $|x|_{sp} \leq |x|$ always holds, $|\cdot|$ is equivalent to $|\cdot|_{sp}$. If 2 is replaced by any larger integer, the same argument works.

□

Problem 2.1.13. *If an analytic normed ring satisfies the above equivalent properties, it is uniform as a Huber ring.*

Proof. If R satisfies the equivalent conditions, we may assume the norm on R is power-multiplicative. And if R is analytic, then $x \in R^\circ$ iff there is $c > 0$, for all n, $|x^n| = |x|^n < c$. This holds iff $|x| \leq 1$, thus R° is norm bounded (hence bounded). □

Proposition 2.1.14. *If R is a uniform analytic normed ring, then the spectral norm induces the same topology as the norm. In particular, any uniform Huber ring admits a power-multiplicative norm that induces its topology.*

Proof. It suffices to show any neighborhood U of 0 contains a spectral ball $\{x||x|_{sp} < \varepsilon\}$ for some $\varepsilon > 0$. Pick a small enough ideal of definition $I \subset U$ for R^0, as usual, pick $a_i \in I, b_i \in R$ such that $\sum_i a_i b_i = 1$, we claim $\{x||b_i x|_{sp} < 1, \forall i\} \subset I$. Indeed, if $|b_i x|_{sp} < 1$, then $b_i x \in R^0$, and $x = \sum_i a_i b_i x \in IR^0 = I$. Now pick a small enough ε such that $\varepsilon |b_i|_{sp} < 1$ for all i, then for any element in the spectral ball $\{x||x|_{sp} < \varepsilon\}$, and any i, $|b_i x|_{sp} < 1$. So $\{x||x|_{sp} < \varepsilon\} \subset \{x||b_i x|_{sp} < 1, \forall i\} \subset U$. □

Corollary 2.1.15. *For an analytic nomred ring R, an element $a \in R$ is a topological nilpotent iff $|a|_{sp} < 1$. Similarly, an element $a \in R$ is power-bounded iff $|a|_{sp} \leq 1$.*

Proof. Using the last proposition, a is a topological nilpotent element iff $|a^n|_{sp} \to 0$ when $n \to \infty$, and since $|a|_{sp}^n = |a|_{sp}^n$, the convergence is equivalent to $|a| < 1$. □

The following is an alternative proof of Proposition 2.1.14 for uniform Tate rings through a concrete construction. It's the same construction that was used to promote a Huber ring to a normed ring, but the structure of the norm and the ring are better analyzed.

Proposition 2.1.16. *If R is a uniform Tate ring, it admits a power-multiplicative norm that induces its topology.*

Proof. Pick a topological nilpotent unit t in R, then $R = R^0[\frac{1}{t}]$. For every element $x \in R \setminus \{0\}$, it can be uniquely written as $x = t^n a$ where $n \in \mathbb{Z}$ and $a \in R^0 \setminus tR^0$. Indeed, as $t^n \to 0$ when $n \to \infty$, $t^n x \in R^0$ for sufficiently large n. On the other hand, if for every $N > 0$, there is still $n_N \geq N$ such that $t^{-n_N} x \in R^0$, then $x \in t^{n_N} R^0 \to 0$ when $N \to \infty$ because R^0 is bounded, thus $x = 0$. Therefore $\{n \in \mathbb{Z} | t^n x \in R^0\}$ is a non-empty set that is bounded below, hence there is a minimal element n, and then $x = t^{-n}(t^n x)$ where $t^n x \in R \setminus tR^0$. If $t^n a = t^m b$ for $n \geq m$ and $a, b \in R^0 \setminus tR^0$, then $t^{n-m} a = b \notin tR^0$, $n - m = 0, a = b$.

Now we define $|t^n a| = 2^{-n}$. We want to show this norm satisfies $|x^2| \geq c|x|^2$ for some $c > 0$. We show that if $a \in R^0 \setminus tR^0$, then $a^2 \in R^0 \setminus t^2 R^0$. If $a^2 = t^2 b \in t^2 R^0$, then $b = (\frac{a}{t})^2$ is power-bounded and hence so is $\frac{a}{t}$, that is $\frac{a}{t} \in R^0$, $a \in tR^0$, a contradiction. Therefore $|(t^n a)^2| = |t^{2n} a^2| \geq 2^{-2n-1} = \frac{1}{2} |t^n a|^2$. □

It's not true that any norm on a uniform Tate ring which induces its prescribed topology is equivalent to its spectral norm. Here is an example,

Example 2.1.17. *Let k be a non-discrete non-archimedean field with multiplicative norm $|\cdot|$, define the new norm $|x|' = |x|$ if $|x| \leq 1$ and $|x|' = |x|(1 + \log|x|)$ if $|x| > 1$. The strong triangle inequality can be checked using the function $r \mapsto r(1 + \log r)$ increases for $r > 1$. To show it's submultiplicative, if $|x| \geq |y| > 1$, then $|xy|' = |x|(1 + \log(xy)) \leq |xy|(1 + \log|x| + \log|y|) \leq |x|(1 + \log|x|)|y|(1 + \log|y|) = |x|'|y|'$, and if $|x| > 1, |y| \leq 1, |xy| \leq 1$, then $|xy|' = |x||y| \leq |x|'|y| = |x|'|y|'$, if $|x| > 1, |y| \leq 1, |xy| > 1$, then $|xy|' = |x||y|(1 + \log|xy|) \leq |x|(1 + \log|x|)|y| = |x|'|y|'$.*

As $\frac{|x|'}{|x|} = 1 + \log|x| \to \infty$ as $|x| \to \infty$, $|x|'$ is not equivalent to $|\cdot|$, but $|x|'_{sp} = |x|$.

However, if there is a topological nilpotent unit t with $|t^{-1}||t| = 1$, then $|x| = |t^{-n} t^n x| \leq |t^{-1}|^n |t^n x| = |t|^{-n} |t^n x|$, $|t^n x| = |t|^n |x|$ for any integer n, and pick $C_1, C_2 > 0$ such that $C_1 \leq |R^0 \setminus tR^0| \leq C_2$, then $|t^n a| = |t|^n |a| \in [C_1 |t|^n, C_2 |t|^n]$ whenever $a \in R^0 \setminus tR^0$. Thus the norm is equivalent to the new norm $|t^n a|' := |t|^n$ which is shown to be equivalent to its spectral norm through the argument in the proof of Proposition 2.1.16. In particular, if R is a uniform normed ring over a non-archimedean field F with a F-homogeneous norm (i.e. $|ax| = |a||x|, \forall a \in F, x \in R$), then the norm on R is equivalent to its spectral norm.

Remark 2.1.18. *Recall the famous but not presently relevant conjecture of Kaplansky that any norm on $C(X)$, the complex algebra of complex continuous functions on a compact Hausdorff space X, is equivalent to the uniform norm $|f| := \max_{x\in X} |f(x)|$. This turns out to be independent of ZFC.*

2.2 Open Mapping Theorem for Analytic Banach Rings

The classical open mapping theorem in functional analysis says,

Theorem 2.2.1. *Let $k = \mathbb{R}$ or $\mathbb{C}$, M, N be k-Banach spaces, and $f : M \to N$ be a surjective continuous k-linear map, then f is an open map, i.e. it sends open subsets of M to an open subset of N. In particular, N is homeomorphic to $M/Ker(f)$ with the quotient topology.*

The same is true when k is a field with a nontrivial nonarchimedean absolute value (and a k-Banach space is just a k-linear space with a complete norm satisfying the strong triangle inequality).

Henkel generalized the theorem in [8].

Theorem 2.2.2. *Let R be a topological ring that has a sequence of units converge to 0 (e.g. R is a Tate ring in the sense of Huber). Let M and N be topological R-modules that have a countable fundamental systems of open neighborhoods of 0. Let M be complete, then the following are equivalent:*

1. *N is complete and u is surjective.*

2. *N is complete and $u(M)$ is open in N.*

3. *$u(M) \subset N$ is not meagre, that is $u(M)$ is not a countable union of nowhere dense subsets.*

4. *For all neighborhoods V of 0 in M, $\overline{u(V)}$ is a neighborhood of 0 in N.*

5. *u is open.*

Note that the theorem only requires M to be complete, not R itself. And R doesn't even have to be metrizable or first-countable, e.g. $\prod_{i\in I} F_i$ equipped with the product topology, where F_i is a field with a nontrivial absolute value, and I is a uncountable set.

In preparing his Arizona winter school notes [13], Kedlaya proposed the definition of analytic rings. He conjectured analytic Banach rings are precisely the Banach rings for which open mapping theorem holds. We have constructed the following example to confirm that open mapping theorem fails for non-analytic rings before Arizona Winter School 2017.

Example 2.2.3. *Let R be a non-analytic ring, then it contains a proper open ideal I. Let R/I be the topological R-module with discrete topology, and $M = \prod_{i\in\mathbb{N}} R/I$ with the discrete topology, $N = \prod_{i\in\mathbb{N}} R/I$ with the product topology, then the set-theoretic identity map from M to N is continuous and surjective, but not open.*

In collaboration with Atticus Christensen and Anthony Wang, we finished the proof of the open mapping theorem, during the Arizona Winter School 2017.

Theorem 2.2.4. *Let R be an analytic Banach ring, and M,N be first countable, complete topological R-modules, and $f : M \to N$ is a surjective continuous R-morphism. Then f is an open mapping, i.e it sends open subsets of M to open subsets of N.*

In fact, we proved the slightly more general version that can be seen as a direct generalization of Henkel's version.

Theorem 2.2.5. *If R is a topological ring for which $\exists m \in \mathbb{Z}^{n>0}$, $I_n \subset R$ for $n = 1,2,\cdots$, such that $|I_n| \le m$, and I_n generates the unit ideal, $I_n \to 0$ as $n \to 0$, then the open mapping theorem holds for R.*

Note that when $m = 1$, we get Henkel's version.

The following two classical results are needed in the proof.

Lemma 2.2.6 (Baire Category Theorem)**.** *A complete metric space is not meagre.*

Lemma 2.2.7 (The Birkhoff-Kakutani theorem)**.** *A topological group is metrizable iff it's first countable and Hausdorff. Moreover, if G is metrizable, it admits a left-invariant metric.*

Proof of theorem. It's enough to show f sends a neighborhood of 0 in M to a neighborhood of 0 in N. As in the proof of the classical open mapping theorem, we prove this in two steps: First for any neighborhood U of 0 in M, $\overline{f(U)}$ is a neighborhood of 0 in N. Then we use an approximation argument to get rid of the closure, so to show $f(U)$ is a neighborhood of 0 in N.

Fix m and $I_n = \{a_{n,1}, \cdots, a_{n,m}\}$ as in the statement of the theorem, and a neighborhood U of 0. Here if $|I_n| < m$, we assume $a_{n,i} = 0$ for $|I_n| < i \leq m$. Consider the sequence of sets $W_{U,n} = \{x \in M | a_{n,i}x \in U, \forall i = 1, 2, \cdots, m\}$. As $I_n \to 0$, $\cup_{n=1}^{\infty} W_{U,n} = M$, and since f is surjective, $\cup_{n=1}^{\infty} f(W_{U,n}) = N$. By the Birkhoff-Katutani theorem and the Baire category theorem, M is not meagre, therefore there is n such that $\overline{f(W_{U,n})} \subset N$ contains an interior point. If necessary, we may pick U' such that $U' = -U' \subset U, U' + U' \subset U$, and run the same argument for U' instead of U, we can conclude that $\overline{f(W_{U,n})} \supset \overline{f(W_{U',n})} + \overline{f(W_{U',n})}$ must be a neighborhood of 0 for some n. Moreover $f(W_{U,n}) = \{f(x) | \forall i, a_{n,i}x \in U\} \subset \{f(x) | \forall i, a_{n,i}f(x) \in f(U)\} \subset \{y \in N | \forall i, a_{n,i}y \in f(U)\}$, and after taking closure $\overline{f(W_{U,n})} \subset \overline{\{y \in N | \forall i, a_{n,i}y \in f(U)\}} \subset \{y \in N | \forall i, a_{n,i}y \in \overline{f(U)}\}$, hence $\{y \in N | \forall i, a_{n,i}y \in \overline{f(U)}\}$ is a neighborhood of 0 in N.

Now pick a neighborhood U' of 0 in M, such that $\underbrace{U' + \cdots + U'}_{m \text{ copies}} \subset U$, by the above argument there is an n such that $V = \{y \in N | \forall i, a_{n,i}y \in \overline{f(U')}\}$ is a neighborhood of 0 in N. Pick $b_1, \cdots, b_m$ such that $\sum_{i=1}^{m} a_{n,i}b_i = 1$, then $V' = \{y \in N | \forall i, b_i x \in V\}$ is also a neighborhood of 0 in N. And finally $V' = (\sum_{i=1}^{m} a_{n,i}b_i)V' \subset \sum_{i=1}^{m} a_{n,i}b_iV' \subset \sum_{i=1}^{m} a_{n,i}V \subset \sum_{i=1}^{m} \overline{f(U')} \subset \overline{f(U)}$, therefore $\overline{f(U)}$ is a neighborhood of 0 in N.

As for the second step, let's first pick translation invariant metrics on M and N, and

$B_r^M(0)$(resp. $B_r^N(0)$) be the open ball of radius r around 0 in M (resp. N). For any $r > 0$, by the first step, $\overline{f(B_r^N(0))}$ contains $B_{s_r}^N(0)$ for some $s_r > 0$. Given a neighborhood U of 0 in M, pick $r > 0$ such that $B_r^M(0) \subset U$, . Now for any $y \in B_{s_{r/2}}^N(0)$, pick $x_1 \in B_{r/2}^M(0)$ such that $|y - f(x_1)| < \min\{\frac{1}{4}, s_{r/4}\}$, and inductively we pick $x_n \in B_{r/2^n}^M(0)$, such that $|y - \sum_{i=1}^{n-1} f(x_i) - f(x_n)| < \min\{\frac{1}{2^{n+1}}, s_{r/2^{n+1}}\}$. Note that $\sum_{i=1}^n x_i$ is a Cauchy sequence, and $|\sum_{i=1}^\infty x_i| \leq \sum_{i=1} |x_i| < r$, thus $\sum_{i=0}^\infty x_i$ is in $B_r^M(0)$, and $B_{s_{r/2}}^N(0) \subset f(B_r^M(0))$, $f(B_r^M(0))$ is a neighborhood of 0 in N.

□

From the proof, it seems more natural to consider the open mapping theorem as a result about topological abelian groups with certain family of operators acting on them, instead of topological modules over topological rings, as the full ring isn't used.

We left the following questions unanswered. These questions don't seem to matter in p-adic geometry, but only for the concern of generality.

Question 2.2.8. *Does open mapping theorem applies to all analytic topological rings?*

Specifically, if R admits a sequence of subsets I_n such that $I_n \to 0$ and I_n generates the unit ideal, that is, if we can drop the condition that the sizes of I_n is bounded, can we adapt the above proof to show the open mapping theorem for R? Note that any analytic and first-countable topological rings have the above property.

Question 2.2.9. *Can we characterize rings for which open mapping theorem holds for finitely generated modules?*

For example, any finite ring satisfies the open mapping theorem for finitely generated modules, and thus the above question might be ill-posed.

Question 2.2.10. *What about complete modules that are not first countable?*

There are obvious difficulties, such as Baire category arguments would fail, however we expect certain version of the theorem can be generalized. To define completeness of a topological group that is not first-countable, we need to use Cauchy net instead of Cauchy sequences. And the uniform structure can be described by a family of semi-norms.

2.3 Consequences of the Open Mapping Theorem

The open mapping theorem combined with the example we gave earlier lead to following characterization of the analytic Banach rings.

Theorem 2.3.1. *For a Huber ring R, the following are equivalent:*

- *R is analytic*
- *Any ideal of definition of R generates the unit ideal in R.*
- *For any nontrivial ideal I, the quotient topology on R/I is not discrete.*
- *The only discrete topological R-module is the zero one.*
- *R satisfies the open mapping theorem statement.*
- *All points of $Spa(R, R^+)$ are analytic (for whichever $R^+ \subset R^0$).*

Also, the open mapping theorem implies that for any Banach analytic ring R, there is a unique topology on a finite free module M over R that makes M a topological module over R.

2.4 Banach Fields versus Non-archimedean Fields

During the early days of functional analysis, the Scottish Book was formed by some Polish mathematicians to document problems for people to solve collaboratively. The idea was most likely suggested by Stefan Banach. As p-adic geometry evolves, many questions with a strong flavor of non-archimedean analysis show up and Kedlaya started listing them in his new Scottish book [11] in 2015. The first question is the following,

Question 2.4.1. *Let K be a nonarchimedean commutative Banach ring whose underlying ring is a field. Suppose in addition that K is uniform, i.e., its norm is equivalent to a power-multiplicative norm. Is K necessarily a nonarchimedean field, that is, is the topology on K defined by some multiplicative norm?*

In [12], Kedlaya answered the question positively when K is a Banach algebra over a non-discretely valued (non-archimedean) field F. Using various results from [12], we prove.

Theorem 2.4.2. *If A is a uniform Banach field over a non-discrete non-archimedean field, but not itself a non-archimedean field. Then there exists a power-multiplicative norm on A that induces its topology, and a subfield $F \subset A$, such that F is non-archimedean with the inherited norm, and for any $x \in A \setminus F$, x is transcendental over F, and the Tate algebra $T = F\{\rho^{-1}x\}$ ($\rho = |x|$) with the Gauss norm isometrically embeds into A.*

Proof. All cited results refer to the ones in [12].

Consider the set of (non-discrete, complete) non-archimedean subfields. By the given condition, this set is non-empty. Using Zorn's lemma, pick a maximal non-archimedean subfield F. Using lemma 2.9, we may assume in A, $|ab| = |a|_F |b|_A, \forall a \in F, b \in A$, and by A is uniform, may further assume $|\cdot|$ is power-multiplicative. If there is an algebraic element x over F in $A \setminus F$, the norm on F can be uniquely extended to $F[x]$, contradicting the maximality of F. Thus any element in $A \setminus F$ is transcendental over F. For any such x, we claim the norm on $F[x]$ in A is the Gauss norm $\|\sum_{i=0}^{n} a_i x^i\| = \max_i |a_i| \rho^i$, where $\rho = |x|$. If there is a unique i_0 achieving the maximum, then $\|\sum_{i=0}^{n} a_i x^i\| = |a_{i_0} x^{i_0}| = |a_{i_0}||x^{i_0}| = |a_{i_i}|\rho^{i_0}$. Otherwise we can disturb one such a_i to make it slightly larger, and use the norm is continuous to conclude. Thus $T = F\{\rho^{-1}x\}$ isometrically embeds into A. □

We don't know whether there is such an exotic example A, and we don't know if the condition that A contains a non-archimedean subfield can be removed. From the above argument, we may try to establish that the fraction field of $T = F\{\rho^{-1}x\}$ with norm defined by $|\frac{f}{g}| = \frac{|f|}{|g|}$

isometrically embeds into A, which would contradict the maximality of F, and thus rule out the existence of such an A. A thorough analysis of possible extension of the Gauss norms on $T = F\{\rho^{-1}x\}$ would be helpful.

Chapter 3

Perfectoid Rings and Perfectoid Spaces

Perfectoid spaces are introduced by Kedlaya-Liu [14] (based on earlier work of Kedlaya) and Peter Scholze [17] independently. Since then, the concept has been generalized a few times. In this chapter, we review the basics of the theory.

3.1 Motivation

The following classical theorem suggests a deep connection between fields of mixed characteristics and fields of characteristic p.

Theorem 3.1.1 (Fontaine-Wintenberger)**.** *Let* $L := \mathbb{Q}_p[p^{1/p^\infty}] := \cup_{n=1}^{\infty}\mathbb{Q}_p[p^{1/p^n}]$*, and* $L^\flat := \mathbb{F}_p((t))[t^{1/p^\infty}] := \cup_{n=1}^{\infty}\mathbb{F}_p((t))[t^{1/p^n}]$*, the absolute Galois groups of* L *and* $L^\flat$ *are (canonically) isomorphic to each other as profinite groups.*

The original proof depends on higher ramification theory of local fields. Using different methods (functorial constructions which was already known to Fontaine), this result was generalized by the perfectoid correspondence.

3.2 Perfectoid Fields and Rings

From now on, let's fix a prime number p.

Definition 3.2.1. *Let K be a non-archimedean field, i.e. K is complete with respect to a multiplicative nonarchimedean absolute value $|\cdot|$. Let $\mathfrak{o}_K := \{x \in K : |x| \leq 1\} = K^\circ$, which is a complete valuation ring with the unique maximal ideal $\mathfrak{m}_K = \{x \in K : |x| < 1\}$.*

A non-archimedean field K is perfectoid if it's not discretely valued, $p \in \mathfrak{m}_K$, and the Frobenius map $\varphi : \mathfrak{o}_K/(p) \to \mathfrak{o}_K/(p), a \mapsto a^p$ is surjective.

Example 3.2.2. $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$, $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, $\widehat{\mathbb{Q}_p(\mu_\infty)}$ *are all perfectoid fields. But any finite extension of $\mathbb{Q}_p$ is discretely valued, thus not perfectoid.*

Definition 3.2.3. *Let (R, R^+) be Huber pair such that R is uniform and analytic. By uniformity, R^+ is a ring of definition of R. (R, R^+) is perfectoid if there exists an ideal of definition I in R^+, such that $p \in I^p$ and the Frobenius map $\varphi : R^+/I \to R^+/I^p$ is surjective (but not necessarily injective). It turns out that the definition only depends on R, and we will also call such a ring perfectoid.*

Example 3.2.4. *If (R, R^+) is a perfectoid Huber pair, then*

$$(R\{T_1^{p^{-\infty}}, \cdots, T_n^{p^{\infty}}\}, R^+\{T_1^{p^{-\infty}}, \cdots, T_n^{p^{-\infty}}\})$$

is a perfectoid Huber pair too.

In Scholze's original work [17], he didn't define perfectoid rings, but only perfectoid algebras over a perfectoid field. Fontaine removed the ground field, and defined perfectoid rings in the above manner, but required the Huber ring to be Tate. In his Arizona Winter School notes [13], Kedlaya allowed the ring to be analytic.

It's natural to ask the relationship between perfectoid fields and perfectoid rings as we define them. This is answered by Kedlaya in [12].

Theorem 3.2.5. *A perfectoid ring whose underlying ring is actually a field is a perfectoid field, i.e. its topology can be induced by a multiplicative norm.*

3.3 The Ring of Witt Vectors

In this section, we recall the construction of the ring of Witt vectors. The construction captures the essential idea of how to construct $\mathbb{Z}_p$ from the residue field $\mathbb{F}_p$. To develop the general theory, we need the following lemma.

Lemma 3.3.1. *In any ring R, for $n \geq 1$, if $x \equiv y$ mod p^n, then $x^p \equiv y^p$ mod p^{n+1}.*

Proof. $x^p - y^p = (x-y)(x^{p-1} + xy^{p-2} + \cdots + y^{p-1})$. If $x \equiv y \bmod p^n$, then in particular $x \equiv y \bmod p$, $x^{p-1} + xy^{p-2} + \cdots + y^{p-1} \equiv px^{p-1} \equiv 0 \bmod p$. □

Intuitively, if R is p-adically complete (in particular separated), in order to approximate an element x, we may find the residue class of of a p^n-th root of x (if it exists) for $n = 1, 2, \cdots$. To make this work, we are led to the definition of p-strict rings, but before we do it, we want to give an explicit construction of the Witt rings.

Lemma 3.3.2. *For any $Q \in \mathbb{Z}[X_0, \cdots, X_n; Y_0, \cdots, Y_n]$ and $i \in \mathbb{N}$*

$$(Q(X_0^p, \cdots, X_n^p; Y_0^p, \cdots, Y_n^p))^{p^i} \equiv Q(X_0, \cdots, X_n; Y_0, \cdots, Y_n)^{p^{i+1}} \mod p^{i+1}$$

Proof. Inductively apply the last lemma. □

Proposition 3.3.3. *Let $X_0, X_1, \cdots$ be an infinite sequence of variables, and*

$$W_n = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n, n \geq 0$$

Then there exist unique polynomials $S_0, S_1, \cdots; P_0, P_1, \cdots \in \mathbb{Z}[X_0, X_1, \cdots; Y_0, Y_1, \cdots]$ *such that*

$$W_n(S_0, S_1, \cdots) = W_n(X_0, X_1, \cdots) + W_n(Y_0, Y_1, \cdots)$$

$$W_n(P_0, P_1, \cdots) = W_n(X_0, X_1, \cdots) \cdot W_n(Y_0, Y_1, \cdots)$$

Proof. We show the existence and uniqueness of $S_0, S_1, \cdots, S_n, \cdots$ by induction on n. Here S_i is a polynomial of $X_0, \cdots, X_i$ and $Y_0, \cdots, Y_i$. (The same method with minor modification can be used to show the existence and uniqueness of $P_0, P_1, \cdots$. This part will be omitted here.)

It's clear that $S_0 = X_0 + Y_0$ is the only choice such that $W_0(X_0) + W_0(Y_0) = W_0(S_0)$ since $W_0(Z) = Z$ for any Z. So the base case is established. Suppose that we have shown there exists unique $S_0, \cdots, S_{n-1}$ such that for any $i = 0, \cdots, n-1$

$$W_i(S_0, \cdots, S_i) = W_i(X_0, \cdots, X_i) + W_i(Y_0, \cdots, Y_i)$$

If S_n satisfying $W_n(S_0, \cdots, S_n) = W_n(X_0, \cdots, X_n) + W_n(Y_0, \cdots, Y_n)$ exists, then S_n must be $\frac{1}{p^n}(W_n(X_0, \cdots, X_n) + W_n(Y_0, \cdots, Y_n) - W_n(S_0, S_1, \cdots, S_{n-1}, 0))$ and thus is unique. The question now is if the formula gives a polynomial with only integer coefficients.

To show the existence, first by induction hypothesis,

$$\begin{aligned}
&W_n(X_0, \cdots, X_n) + W_n(Y_0, \cdots, Y_n) \\
=&W_{n-1}(X_0^p, \cdots, X_{n-1}^p) + W_{n-1}(Y_0^p, \cdots, Y_{n-1}^p) + p^n(X_n + Y_n) \\
=&W_{n-1}(S_0(X_0^p; Y_0^p), S_1(X_0^p, X_1^p; Y_0^p, Y_1^p), \cdots) + p^n(X_n + Y_n)
\end{aligned}$$

Apply the lemma to each individual term of

$$W_{n-1}(S_0(X_0^p; Y_0^p), S_1(X_0^p, X_1^p; Y_0^p, Y_1^p), \cdots)$$

$$= S_0(X_0^p;Y_0^p)^{p^{n-1}} + p(S_1(X_0^p,X_1^p;Y_0^p,Y_1^p))^{p^{n-2}} + \cdots$$

We draw the conclusion,

$$W_{n-1}(S_0(X_0^p;Y_0^p), S_1(X_0^p,X_1^p;Y_0^p,Y_1^p),\cdots) \equiv W_n(S_0,S_1,\cdots,S_{n-1},0) \mod p^n$$

$$W_n(X_0,\cdots,X_n) + W_n(Y_0,\cdots,Y_n) \equiv W_n(S_0,S_1,\cdots,S_{n-1},0) \mod p^n$$

Thus,

$$S_n := \frac{1}{p^n}(W_n(S_0,S_1,\cdots,S_{n-1},0) - W_n(X_0,\cdots,X_n) - W_n(Y_0,\cdots,Y_n))$$
$$\in \mathbb{Z}[X_0,\cdots,X_n;Y_0,\cdots,Y_n].$$

□

Theorem 3.3.4. *R be a commutative ring with unity, let $W(R) := \prod_{i=0}^{\infty} R$, and for*

$$a = (a_0,a_1,\cdots,), b = (b_0,b_1,\cdots) \in W(R)$$

$$a + b = (S_0(a,b), S_1(a,b),\cdots)$$

define

$$a \cdot b = (P_0(a,b), P_1(a,b),\cdots)$$

then $(W(R),+,\cdot)$ is a commutative ring with unity. This is the ring of Witt vectors.

Proof. Let B be the ring $\{a = (a_i)_{i=0}^{\infty} : a_i \in A\}$ with componentwise addition and multiplication. Define $\rho : W(A) \to B$ by $\rho((a_i)) = (W_i(a_0,\cdots,a_i))$. By the last theorem, we know that the image of ρ forms a subring of B. Let's call this subring C.

Assume p is not a zero divisor in A, then we claim ρ is injective. In fact, we can recover a

from $\rho(a)$ by the recursive relation:

$$a_0 = (\rho(a))(0), p^n a_n = (\rho(a))(n) - (a_0^{p^n} + p a_1^{p^{n-1}} + \cdots + p^{n-1} a_{n-1}^p)$$

From here, the map ρ induces a ring structure on the set $W(A)$ through the ring structure on C, i.e. $a +_\rho b := \rho^{-1}(\rho(a) + \rho(b))$ and $a \cdot_\rho b = \rho^{-1}(\rho(a) \cdot \rho(b))$. And these ring operations are exactly given by $a +_\rho b = (S_0(a,b), S_1(a,b), \cdots)$ and $a \cdot_\rho b = (P_0(a,b), P_1(a,b), \cdots)$.

When p is a zero divisor in A, A is always a quotient ring of a ring A' in which p is not a zero divisor. (In fact, A' can be always taken as $\mathbb{Z}[X_I]$ for some index set I.) We have the addition and multiplication defined on $W(A')$ satisfy all the axioms of a commutative ring as shown above. These properties easily pass to $W(A)$.

□

Since the addition and multiplication on $W(A)$ is defined through polynomials of integer coefficients, the construction is functorial, thus we have a functor from the category of commutative rings to itself. In general, this functor doesn't behave well, but when restricted to a special class of rings, it induces an equivalence of categories.

Definition 3.3.5. *A strict p-ring is a p-torsion free, p-adically complete ring R such that $R/(p)$ is perfect (i.e the Frobenius map $x \mapsto x^p$ is a bijection).*

Theorem 3.3.6. *The functor that sends a ring R to $R/(p)$ induces an equivalence of categories from the category of strict p-rings to the category of perfect rings of characteristic p, with a quasi-inverse given by the ring of Witt vectors.*

Proof. See [14]. □

To be more explicit, there is a unique multiplicative lifting of the map $R \to R/(p)$, denoted by $\bar{x} \mapsto [\bar{x}]$, which is called the Teichmüller lifting, and each element of R can be uniquely written as $\sum_{i=0}^{\infty} [\bar{x}_n] p^n$.

3.4 Tilting

Definition 3.4.1. *Let (R,R^+) be a perfectoid pair, the tilt of R is a perfectoid pair $(R^b,R^{\flat+})$ such that*

- *As a multiplicative topological monoid, R^b is isomorphic to $\varprojlim_{x\mapsto x^p} R$,*

  To be more explicit,

  The underlying set of $R^\flat$ is $\{(a_n)_n = (\cdots,a_n,a_{n-1},\cdots,a_0)|\forall n, a_n \in R, a_{n+1}^p = x_n\}$,

  The multiplication is defined by $(a_n)_n(b_n)_n = (a_nb_n)_n$

  The topology on $R^\flat$ is the inverse limit topology.

- *The addition is defined by $(a_n)+(b_n) = (\lim_{m\to\infty}(a_{n+m}+b_{n+m})^{p^m})_n$.*

- $(R^\flat)^+ = \varprojlim_{x\mapsto x^p} R^+$.

It can be proved that with the above datum, $R^\flat$ is indeed a perfectoid ring. Moreover, $R^{\flat+} = \varprojlim_{x\mapsto x^p} R^+/I$ for any ideal of definition I satisfies $p \in I^p$. This observation is useful in showing $R^\flat$ is complete among other things.

Given $x = (a_n)_n \in R^\flat$, a_0 is often denoted as $\sharp(x)$ or $x^\sharp$.

Definition 3.4.2. *Let (R,R^+) be a perfectoid pair of characteristic p. An element $x = \sum_{n=0}\infty[\bar{x}_n]p^n$ is primitive if x_0 is a topological nilpotent and x_1 is a unit in R^+. Equivalently, $x = [\bar{x}_0]+px_1$ where x_1 is a unit in $W(R^+)$. An ideal of $W(R^+)$ is primitive if it's generated by a primitive element. Note that a primitive element multiplied by a unit in $W(R^+)$ is still primitive, thus any generator of a primitive ideal is a primitive element.*

The following theorem is due to Kedlaya-Liu [14], Scholze[17], and Kedlaya [13] for various generalities.

Theorem 3.4.3 (Perfectoid correspondence)**.** *The following functor*

$$(R, R^+, I) \mapsto (A := W^\flat(R)/IW^\flat(R), A^+ := W(R^+)/I)$$

is an equivalence of categories. Here (R, R^+) is a perfectoid pair of characterisitc p, and I is a primitive ideal of $W(R^+)$. A quasi-inverse functor is given by the tilting construction.

Proof. See [13, Theorem 2.3.9] □

3.5 Sheafiness

A Huber ring is strongly noetherian if for any n, the n-variable Tate algebra over it is noetherian. Huber proved that strongly noetherian rings are sheafy, but perfectoid rings are essentially never noetherian.

Theorem 3.5.1. *Any noetherian perfectoid ring is a finite direct product of perfectoid fields.*

Proof. See [13, Corollary 2.9.3] □

Scholze [17] proved perfectoid algebras over a perfectoid field is sheafy, using almost ring theory. Later, a more general criteria was established in [6] for Tate rings, and The Tate condition is removed in subsequent works. See [13, Theorem 1.2.13].

Definition 3.5.2. *(R, R^+) is stably uniform if for every rational localization $(R, R^+) \to (S, S^+)$, S is uniform. (This depends on only R, not R^+.)*

Theorem 3.5.3. *Let (R, R^+) be a stably uniform analytic ring, then $Spa(R, R^+)$ is an adic space, in other words, the presheaf $\mathcal{O}_X$ on X is a sheaf of complete topological rings.*

It can be shown that if (A, A^+) is a perfectoid pair, then for any rational localization $(A, A^+) \to (B, B^+)$, (B, B^+) is also perfectoid and in particular uniform. Therefore, perfectoid rings are sheafy. With the help of this, we can define perfectoid spaces.

Definition 3.5.4. *A perfectoid space is an adic space which are locally adic spectrums of perfectoid Huber pairs.*

And the tilting construction glues, so we have this global result.

Theorem 3.5.5. *Let (R,R^+) be a perfectoid pair, then $v \mapsto v \circ \sharp$ defines a bijection*

$$Spa(A,A^+) \simeq Spa(A^\flat, A^{\flat +})$$

which identifies rational subspaces on both sides; in particular, this map is a homeomorphism.

Chapter 4

Fargues-Fontaine Curves and GAGA

The Fargues-Fontaine curve, also known as the fundamental curve of p-adic Hodge theory, was first defined by Fargues and Fontaine as a scheme (over an algebraically closed non-archimedean field). They also classified all the vector bundles over the scheme. After the introduction of perfectoid spaces, the adic version of the curve was defined, and it was soon proved that there is a GAGA principle that allows one transfer vector bundles on the schematic curve to the adic curve and vice versa.

4.1 The Fargues-Fontaine Curves

Definition 4.1.1. *Let (R,R^+) be a perfectoid pair of characteristic p, and set*

$$S = Spa(R,R^+),\ A_{inf} := W(R^+)$$

$$Y_S = Spa(A_{inf},A_{inf}) \setminus \{v|v(p) \neq 0 \text{ and } v([\bar{x}]) \neq 0 \text{ for some topological nilpotent } \bar{x} \in R^+\}$$

Then the Frobenius action on R^+ induces a properly discontinuous action on Y_S, and define $X_S := Y_S/\phi^{\mathbb{Z}}$ to be the adic Fargues-Fontaine curve over S.

Definition 4.1.2. *Let $P_S := \oplus_{n=0}^{\infty} P_{S,n}$ be a graded ring, where $P_{S,n} := H^0(X_S, \mathcal{O}(n))$. Then the scheme Proj$(P_S)$ is called the schematic Fargues-Fontaine curve over S.*

4.2 The GAGA Theorem

Once we have both versions of the curve, it's natural to ask whether a GAGA theorem holds for them. Because Prof(P_S) is not finitely generated, the classical non-archimedean GAGA theorems don't apply to it. But it turns out GAGA indeed holds in this case.

Theorem 4.2.1 (Kedlaya-Liu, Fargues when R is a field)**.** *The morphism $X_S \to Proj(X_S)$ satisfies,*

- *If R is Tate, the pull-back of vector bundles along the morphism from Proj(X_S) to X_S defines an equivalence of categories.*
- *If $(R, R^+) = (F, \mathfrak{o}_F)$ where F is a perfectoid field of characteristic p, then even the pull-back of coherent sheaves is an equivalence of categories.*
- *In both cases above, the pullback preserves sheaf cohomology.*

Proof. See [13, Theorem 3.1.17]. □

This theorem is not a special case of the GAGA types of results we have reviewed earlier, because the curve X_S is not of finite type. It would be nice to understand why GAGA holds for this particular types of "curves", and for it, it may be useful to answer the following question,

Question 4.2.2. *Is there a way to characterize Fargues-Fontaine curves among the adic spaces? Or in parallel, when a scheme is a schematic Fargues-Fontaine curve?*

Here are some properties that the schematic Fargues-Fontaine curves are known to satisfy: They are connected, separated, and noetherian schemes, regular of dimension 1, whose Picard group is isomorphic to $\mathbb{Z}$, and the residue fields of any closed point is algebraically closed. Note that these properties can be stated without any explicit reference to the prime number p.

When R is a perfectoid field, the vector bundles over X_S is fully classified, similar to the classification of vector bundles over compact Riemann surfaces.

Definition 4.2.3. *For each rational number* $d = \frac{r}{s}$ *where* $\gcd(r,s) = 1, s > 0$*, we define a vector bundle* $\mathcal{O}(d)$ *over* X_S*. This would be the push-forward of the line bundle* $\mathcal{O}(r)$ *on the s-fold cover of* X_S*.*

Now we can state the following theorem. To make it easier, we assume R is algebraically closed.

Theorem 4.2.4 (Kedlaya, Fargues-Fontaine)**.** *If* R *is an algebraically closed perfectoid field of characteristic* p*, then every vector bundle over* X_S *is isomorphic to a direct sum of vector bundles of the form* $\mathcal{O}(d_i)$ *for some* $d_i \in \mathbb{Q}$*.*

To continue our discussion on how to characterize the Fargues-Fontaine curves, we may add the classification of vector bundles to the list of properties they need to satisfy. Again, the classification doesn't depend on any particular perfectoid fields or even the characteristic p, so it naturally leads to the question whether this shall hold for some "absolute Fargues-Fontaine curve" without any base field. This was already explored in [1], where the author didn't define this geometric object, but classified the vector bundles over this hypothetical object.

4.3 Product of Fargues-Fontaine Curves

The arithmetic aspect of p-adic Hodge theory concerns the continuous p-adic representation of $\mathrm{Gal}(\overline{K}/K)$ for a non-archimedean field K over $\mathbb{Q}_p$. If K is perfectoid, then the category of continuous $\mathbb{Q}_p$ representations of $\mathrm{Gal}(\overline{K}/K)$ can be fully faithfully mapped to the category of vector bundles on the curve $X_{\mathrm{Spa}(K,K^0)}$. This is the modern reformulation of the theory of

(ϕ,Γ)-modules. In [5], a systematic development of the multivariate (ϕ,Γ)-modules for studying representations of products of Galois groups of p-adic fields is initiated.

Given two perfectoid fields K_1, K_2 of characteristic p, if they have the same residue field κ, then we can take the product of the Fargues-Fontaine curves over $\mathrm{Frac}(W(\kappa))$. This product shall capture properties of both "K_1-geometry" and "K_2-geometry", rather than just $W(\kappa)$-geometry.

In the next, we will explore some aspects of this product of curves.

4.4 Toward GAGA on the Surfaces

For simplicity, we shall assume $K = K_1 = K_2$ is algebraically closed, and let C be the Fargues-Fontaine curve $X_{\mathrm{Spa}(K,K^{\circ})}$, and $Z^{an} = C \times_{\mathrm{Frac}(W(\mathfrak{o}_K/\mathfrak{m}_K))} C$ and Z be the product of the schematic curves over the same base field.

We raise the following questions about these objects.

Question 4.4.1. *Is GAGA true for the map $Z^{an} \to Z$?*

Question 4.4.2. *Can the vector bundles over Z^{an}(or Z) be classified as in the single curve case? In particular, is every bundle of the form $V_1 \boxtimes V_2$ where V_1, V_2 are bundles over a single curve? Is the Picard group of Z^{an} isomorphic to $\mathbb{Z} \times \mathbb{Z}$?*

Question 4.4.3. *Is the product of ample bundles still ample? In particular, is $\mathcal{O}(n) \boxtimes \mathcal{O}(m)$ ample?*

One possible approach to these questions will be to split the bundles over Z into product of bundles over single copies of the curve. However, the splitting even if exists may not be canonical. If we push forward the bundle over the surfaces to a single copy of the curves, then the bundles are not finitely generated any more, thus we can't apply the classification of vector bundles over a single curve.

We sketch a strategy for classifying line bundles. Given a line bundle on $C \times C$, restrict the bundle to $C \times \{x\}$ where x is a point on the second copy of the curve. The restriction of a line bundle is still a line bundle, then we can use the divisor associated to the bundle to define the degree of the bundle. If we can show the degree doesn't depend on x, then we have a well-defined degree. Do the same thing for the other copy of curves. Now we have a well-defined a map from Picard to $\mathbb{Z} \times \mathbb{Z}$, and obviously this map is surjective, we just have to show the degree zero bundles are trivial. This approach came from the difficulty of gluing the local decomposition of bundles over the surface into box product of bundles over curves. At the moment, we couldn't carry this approach to a rigorous proof.

Bibliography

[1] Johannes Anschütz, G-bundles on the absolute Fargues-Fontaine curve, available at https://arxiv.org/abs/1606.01029.

[2] M. Baker, B. Conrad, S. Dasgupta, K. S. Kedlaya, J. Teitelbaum, *p-adic Geometry: Lectures from the 2007 Arizona Winter School*, American Mathematical Society Providence, Rhode Island, 2008.

[3] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Surveys and Monographs 33, Amer. Math. Soc., Providence, 1990.

[4] S. Bosch, U. Güntzer, R. Remmert, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Springer-Verlag Berlin Heidelberg, 1984.

[5] A. Carter, K. S. Kedlaya, G. Zábrádi, Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate (ϕ,Γ)-modules, available at https://arxiv.org/abs/1808.03964.

[6] K. Buzzard, A. Verberkmoes, Stably uniform affinoids are sheafy, available at https://arxiv.org/abs/1404.7020v2

[7] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag New York Inc., 1977.

[8] T. Henkel, An Open Mapping Theorem for rings which have a zero sequence of units, available at https://arxiv.org/abs/1407.5647v2.

[9] J. Jost, *Compact Riemann Surfaces*, Third Edition, Springer Berlin Heidelberg New York, 2000.

[10] K. S. Kedlaya, *p-adic Differential Equations*, Cambridge Studies in Advanced Math. 125, Cambridge Univ. Press, Cambridge, 2010.

[11] K. S. Kedlaya, *The Nonarchimedean Scottish Book*, available at https://scripts.mit.edu/~kedlaya/wiki/index.php?title=The_Nonarchimedean_Scottish_Book

[12] K. S. Kedlaya, On Commutative Nonarchimedean Banach fields, available at https://arxiv.org/abs/1602.09004v6

[13] K. S. Kedlaya, *Sheaves, Stacks, and Shtukas*, lecture notes for the 2017 Arizona Winter School, revised version available at https://kskedlaya.org/papers/aws-notes.pdf

[14] K. S. Kedlaya, R. Liu, *Relative p-adic Hodge theory: Foundations*, Paris : Société mathématique de France, 2015

[15] J.-P. Serre, Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier* 6 (1956), 1-42.

[16] P. Schneider, *p-adic Lie groups*, Springer-Verlag Berlin Heidelberg, 2011.

[17] P. Scholze, Perfectoid Spaces, *Pulb. Math. IHÉS* 116 (2012), 245-313.

[18] J. Weinstein, Adic Spaces, Arizona Winter School 2017 lecture notes available at http://swc.math.arizona.edu/aws/2017/2017WeinsteinNotes.pdf.