The Arithmetic of Graph Polynomials

by

Maryam Farahmand

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Committee in charge:

Professor Matthias Beck, Co-chair
Professor Mark Haiman, Co-chair
Professor Cathryn Carson
Professor Fraydoun Rezakhanlou

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Abstract

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We investigate three graph polynomials including antimagic, super edge-magic, and chromatic polynomials. The Antimagic Graph Conjecture asserts that every connected graph $G = (V, E)$ except K_2 admits an edge labeling such that each label $1, 2, \dots, |E|$ is used exactly once and the sums of the labels of the edges incident to each vertex are distinct. We introduce partially magic labelings where the vertex sums are the same in a subset of V. By using the quasi-polynomial structure of the partially magic labeling counting function, we show that every bipartite graph satisfies a relaxed version of the Antimagic Graph Conjecture (that is, repetition of labels are allowed).

A total labeling $f : E \cup V \to \mathbb{Z}_{\geq 0}$ of a graph $G = (V, E)$ is called *super edge-magic* if each vertex label is in $\{1, \dots, |V|\}$ and the sum of the edge label plus labels of its two ends is the same for all edges of G. We prove that the counting function of super edge-magic labelings of every tree is a polynomial. This helps us to show that every tree admits a relaxed super edge magic labeling which is the relaxed version of *Super Edge-Magic Tree Conjecture*. Moreover, we show that every tree with one extra edge that makes a unique even cycle admits a super edge-magic labeling. On the other hand, a *harmonious labeling* is an injective function $L : V \to \{0, 1, \dots, |E| - 1\}$ such that the induced edge labels $L^*(e) \equiv L(u) + L(v) \pmod{|E|}$ for every edge $e = \{u, v\} \in E$, are distinct. The *Harmonious Tree Conjecture* indicates that every tree admits a harmonious labeling. We use our results on super edge-magic labelings to prove that every tree admits a relaxed version of the Harmonious Tree Conjecture.

Lastly, we extend classic order polynomials to a two variable version and we drive a reciprocity theorem for strict and weak order polynomials, reminiscent of Stanley's reciprocity theorem. We also study the *bivariate chromatic polynomial* which counts the number of vertex coloring of G with $1 \leq x$ colors allowing the same colors for adjacent vertices if the color is $\geq y$ $(1 \leq y \leq x)$. We decompose bivariate chromatic polynomial into bivariate order polynomials and we find a reciprocity relation linking bivariate chromatic polynomials to acyclic orientations.

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Chapter 1

Introduction

In this thesis, we explore polynomials concerning two main streams in graph theory, that is, graph labelings and graph colorings.

In 1967, Rosa [46] introduced graph labelings (or what he has called *valuations*) in an attempt to solve a famous conjecture by Ringel [44]. The conjecture claimed that the complete graph K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges.

Graph labelings have many applications in fields such as circuit design, communication networks, coding theory, crystallography, astronomy, and data base management (see, for example, [59]). There are many types of graph labelings: graceful labelings, harmonious labelings, magic labelings, antimagic labelings, etc. [28].

In Chapter 3, we work on a long-standing and still-wide-open conjecture, the *Antimagic Graph Conjecture* which asserts that every connected graph $G = (V, E)$ except K_2 admits an *antimagic labeling*, that is, an edge labeling such that each label $1, 2, \ldots, |E|$ is used exactly once and the sums of the labels of the edges incident to each vertex are distinct [31]. We will prove that bipartite graphs satisfy a relaxed version (i.e., label repetition allowed) of the conjecture (Theorem 3.1.5). The complete graph K_n for $n \geq 3$ [31] and regular graphs [14] are among those graphs that admit antimagic labelings. The Antimagic Graph Conjecture is one of the most famous conjectures in the theory of *labeled graphs* that has been studied excessively (see, e.g., [35, 2, 14, 26, 17, 3, 21, 9, 18]). On the other extreme, an edge labeling is *magic* if the sums of the labels on all edges incident to each vertex are the same. K_n for $n = 2$ and $n \geq 5$ and the bipartite graph $K_{n,n}$ for $n \geq 3$ are among those graphs that admit magic labelings [55].

We approach antimagic labelings by introducing *partially magic labelings*, where "magic occurs" just in a subset of V (Definition 3.1.6). The partially magic labelings have a semigroup structure which allows us to use techniques from commutative algebra (Section 3.2).

We generalize Stanley's theorem (Theorem 3.1.3) about the magic labeling counting function to the associated counting function of partially magic labelings and prove that it is a quasi-polynomial of period at most 2, and in the case that G is a bipartite graph it is a polynomial (Theorem 3.1.7). In the process of the proof we use the fact that the partially magic labelings are in bijection with lattice points in a rational polytope defined by the vertex sum equations in the definition of partially magic labelings (Section 3.2). Therefore, we can employ the rich techniques of polyhedral geometry, specifically, Ehrhart theory [5]. This work has been published in [4].

In Chapter 4 we study harmonious and super edge-magic labelings. A *harmonious labeling*, introduced by Graham and Sloane in the 1980s [30], is a vertex labeling such that each vertex label $0, \ldots, |E|-1$ is used exactly once and the induced edge labels $L^*(e) \equiv L(u) + L(v) \pmod{|E|}$ are unique for every edge $e = \{u, v\} \in E$. It has been conjectured for more than three decades [30] that every tree admits a harmonious labeling (Conjecture 2.1.3). The complete graph K_n for $n \leq 4$, the cycle C_n for odd n, and the complete bipartite graph $K_{m,n}$, when m or n is equal to 1, are among graphs that admit harmonious labelings [30]. Various other classes of graphs have been shown to be harmonious [19, 25, 16].

A total labeling $f : E \cup V \to \mathbb{Z}_{\geq 0}$ is called *edge-magic* if there exists a constant c (called *magic constant* or *valance of* f) such that $f(u)+f(v)+f(e) = c$ for all edges $e = \{u, v\} \in E$. In addition, if f is an edge-magic total labeling such that $f(V) = \{1, \ldots, |V|\}$, then f is called *super edge-magic*. Centeneo et.al. [24, Theorem 17] showed that if a tree G admits a *super-edge magic labeling*, then it is harmonious. Super edge-magic labelings have a semigroup structure (Section 4.2) like partially magic labelings, and we apply our techniques to show that every tree has a relaxed super edge-magic labeling (that is, label repetition is allowed) (Theorem 4.1.4). Moreover, we will show that a tree with one extra edge that makes a unique even cycle has also a super edge-magic labeling (Theorem 4.1.5). Then we will use Centeno's theorem (Theorem 4.1.3) to attack the Harmonious Tree Conjecture (Conjecture 2.1.3). Specifically, we prove that every tree admits a relaxed harmonious labeling (that is, repetition of labels are allowed). Furthermore, every tree with extra edge that makes a unique even cycle admits relaxed harmonious labeling (Corollary 4.1.6).

Chapter 5 concerns *ordered structures* and *graph coloring*. A partially ordered set or *poset* is a set P along with a binary relation $\preceq$ that is reflexive, antisymmetric, and transitive. A map $\varphi : P \to \{1, \ldots, n\}$ is called *order preserving* if for any two elements $a, b \in P$, the relation $a \preceq b \longrightarrow \varphi(a) \leq \varphi(b)$ holds. The map φ is called *strictly order preserving* if $a \prec b \longrightarrow \varphi(a) < \varphi(b)$ holds. The polynomials $\Omega_P(n)$ and $\Omega_P^\circ(n)$ count the number of order preserving maps and strictly order preserving maps, respectively. Stanley in [52] found a formula to compute $\Omega_P(n)$ (Theorem 2.3.2). He also found in [49] a reciprocity relation between $\Omega_P(n)$ and $\Omega_P^\circ(n)$ (Theorem 2.3.3).

In Section 5.2 we introduce bivariate order polynomials which are an extension of order polynomials into two variables. A *bicolored poset* is a poset such that elements are colored by two colors, celeste and silver. A map $\varphi : P \to \{1, \ldots, y, \ldots, x\}$ is called *an order preserving* (x, y)*-map* if it is a classic order preserving map and additionally for every celeste element c, the relation $\varphi(c) \geq y$ holds. The map φ is called a *strictly order preserving* (x, y)*-map* if it is strictly order preserving map and for every celeste element c, the relation $\varphi(c) > y$ holds. The polynomials $\Omega_P(x, y)$ and $\Omega_P^\circ(x, y)$ count the number of order preserving (x, y)-maps and strictly order preserving (x, y)-maps, respectively. We find formulas to compute both $\Omega_P(x, y)$ and $\Omega_P^\circ(x, y)$ (Theorems 5.2.8 and 5.2.10). We generalize Stanley's reciprocity theorem (Theorem 2.3.3) and we find a reciprocity relation between $\Omega_P(x, y)$ and $\Omega_P^\circ(x, y)$ (Theorem 5.2.12).

Graph coloring is an assignment of colors to the elements of graphs (vertices, edges, or both) subject to some specific constraints. The *chromatic polynomial* of a graph G in k counts the number of vertex colorings of G using at most k colors such that adjacent vertices receive different colors. The chromatic polynomial is in fact a polynomial in k (Theorem 2.2.1). Stanley in [50] showed that the chromatic polynomials can be decomposed into order polynomials (Theorem 2.3.4).

In [20], Dohman et.al. introduced the bivariate chromatic polynomial which is a generalization of the chromatic polynomial into two variables. The *bivariate chromatic polynomial* counts the number of coloring of vertices of a graph with colors $\{1, \ldots, y, \ldots, x\}$ such that adjacent vertices get different colors from $\{1, \ldots, y\}$ and they can be colored with the same colors if the colors are from $\{y + 1, \ldots, x\}$. The bivariate chromatic polynomials were the motivation for the bivariate order polynomials defined in Section 5.2.

We decompose bivariate chromatic polynomials into bivariate order polynomials (Theorem 5.3.1) which is a generalization of Theorem 2.3.4. We also find a reciprocity relation linking bivariate chromatic polynomials to acyclic orientations (Theorem 5.4.1). Chapter 5 is joint work with Gina Karunaratne and Sandra Zuniga Ruiz.

Chapter 2

Background

2.1 Graph Labelings

Let G be a finite unoriented graph without loops and multiple edges. We shall denote the set of vertices of G by V, the set of edges by E, the number of vertices by p, and the number of edges by q. A ***labeling*** of G is a map that assigns a set of numbers called ***labels*** to the graph elements and have some properties depending on the type of labeling we are considering. The graph elements can be the vertices alone (***vertex labeling***), or the edges alone (***edge labeling***) or both vertices and edges (***total labeling***) and the set of numbers can be positive integers, nonnegative integers, or integers modulo q.

Example 2.1.1. Let G be the graph of an octahedron and $L : V \to \{0, \ldots, 12\}$ be a vertex labeling of G such that for every edge $\{u, v\} \in E$, the value $|L(v) - L(u)|$ is distinct. Such a labeling is called a ***graceful labeling*** and was introduced by Rosa [46]. Figure 2.1 shows a graceful labeling of the graph of an octahedron.

We are studying an ***edge labeling*** of G which is an assignment $L : E \to \mathbb{Z}_{\geq 0}$. A ***magic*** labeling is an injective function $L : E \to \mathbb{Z}_{\geq 0}$ such that the sum of all the edge labels incident with any vertex is the same. If the function is not injective, we call it ***relaxed magic*** labeling. On the other hand, an edge labeling is called ***antimagic*** where the vertex sums are distinct. G is antimagic if it admits and antimagic labeling. Figure 2.2 presents a magic labeling on wheel W_5 and a magic labeling on K_5.

The following has been conjectured for more than two decades:

Conjecture 2.1.2 (Antimagic Graph Conjecture [31]). *Every connected graph except for K_2 admits an antimagic labeling.*

Surprisingly this conjecture is still open even for ***trees***, i.e., connected graphs without cycles, though it has been proven that trees without vertices of degree 2 are antimagic [35]. Moreover, the validity of the Antimagic Graph Conjecture was proved in [2] for connected graphs

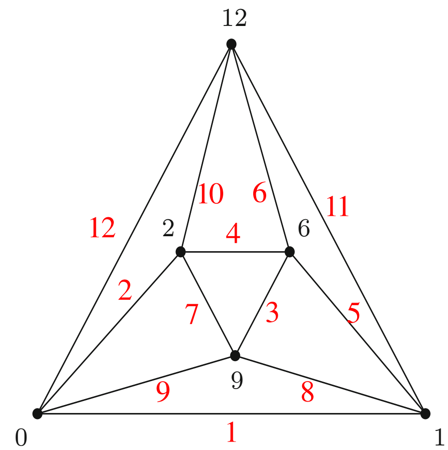

Figure 2.1: A graceful labeling on the graph of an octahedron.

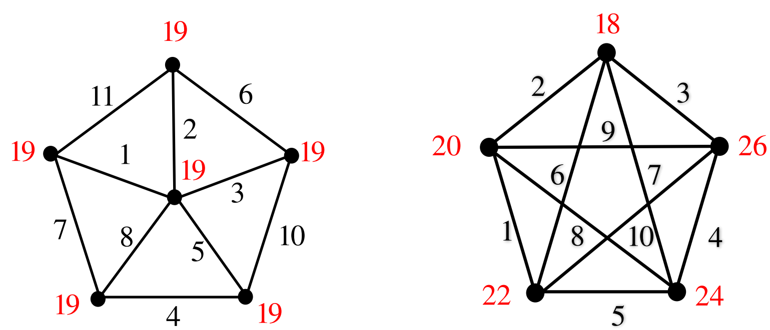

Figure 2.2: A magic labeling on the wheel W_5 and an antimagic labeling on K_5.

with minimum degree $\geq c \log |V|$ (where c is a universal constant), for connected graphs with maximum degree $\geq |V| - 2$, and in [21] for graphs with average degree at least a universal constant. We also know that connected k-regular graphs with $k \geq 2$ are antimagic [18, 9, 14]. Furthermore, all Cartesian products of regular graphs of positive degree are antimagic [17], as are joins of complete graphs [3]. For more related results, see [28].

A ***harmonious labeling*** is a type of vertex labelings defined as an injective function $L : V \to \{0, 1, \ldots, q-1\}$ such that the induced edge labels $L^*(e) \equiv L(u) + L(v) \pmod q$ for every edge $e = \{u, v\} \in E$, are distinct [30]. For trees exactly one vertex label may be repeated. The following has been conjectured for more than thirty years:

Conjecture 2.1.3 (**Harmonious Tree Conjecture** [30]). *Every tree admits a harmonious labeling.*

Determining whether a graph has a harmonious labeling is a hard problem and, in fact, was shown to be NP-complete [37]. It has been checked by computer that the conjecture is valid for any tree with ≤ 26 vertices [1]. It has also been shown that paths and caterpillars (that is, trees with the property that the removal of its endpoints leaves a path) are harmonious (i.e., admits a harmonious labeling) [30]. Moreover, Graham and Sloane proved that the complete graph K_n is harmonious if and only if $n \leq 4$, the cycle C_n is harmonious if and only if n is odd, and the complete bipartite graph $K_{m,n}$ is harmonious if and only if m or n is equal to 1, that is, when it is a star [30]. Different classes of graphs have been shown to be harmonious [19, 25, 16].

Graham and Sloan obtained that almost all graphs are not harmonious [30]. In fact, a necessary condition for a graph to admit a harmonious labeling is the following: for a harmonious graph G with an even number of edges q, if the degree of every vertex is divisible by 2^α, then q is divisible by $2^{\alpha+1}$ [30, Theorem 11]. Consequently, if $q \equiv 2 \pmod 4$ and each vertex has even degree, then G is not harmonious. Furthermore, for a harmonious graph with even q, there exists a partition of V into two sets A and B such that the number of edges joining the vertices of A and B is $\frac{q}{2}$ [39]. Liu and Zhang [38] have generalized this condition: if a harmonious graph has degree sequence $d_1, d_2, \ldots, d_n$, then $\gcd(d_1, d_2, \ldots, d_n, q)$ divides $\frac{q(q-1)}{2}$. They have also proved that every graph is a subgraph of a harmonious graph. More generally, it has been shown that any given set of graphs $G_1, G_2, \ldots, G_t$ can be embedded in a harmonious or graceful graph (i.e., a graph that admits a vertex labeling with labels between 0 and q such that no two vertices share a label, each edge label is the absolute difference of its end points, and they are all distinct) [48].

Another type of edge labelings is *edge-magic labeling*. The injection function $f : V \cup E \to \{1, \ldots, p+q\}$ is called an ***edge-magic*** labeling if there exists a constant c (called the *magic constant* or *valance of* f) such that $f(u) + f(v) + f(e) = c$ for all edges $e = \{u, v\} \in E$ [36]. In addition, if f is an edge-magic labeling such that $f(V) = \{1, \ldots, p\}$ and $f(E) = \{p+1, \ldots, p+q\}$, then f is called ***super edge-magic*** [45]. Figure 2.3 shows an edge-magic labeling with valance 29 on the Peterson graph. Super edge-magic labelings were introduced by Ringel et. al. in 1989 [45] (Wallis [60, 41] calls these labelings *strongly edge-magic*). In order to approach the Harmonious Tree Conjecture (Conjecture 2.1.3), we are going to apply techniques from super edge-magic labelings.

Ringel et. al. 1989 in [45] showed the following: C_n is super edge-magic if and only if n is odd; caterpillars are super edge-magic; $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$; and K_n is super edge-magic if and only if $n = 1, 2$, or 3. They also proved that if a graph with p vertices and q edges is super edge-magic, then $q \leq 2p - 3$. The following is the main conjecture on edge-magic labelings by Ringel et. al.

Conjecture 2.1.4 (Super Edge-Magic Tree Conjecture [45]). *Every tree admits a super edge-magic labeling.*

It has been verified for trees with up to 17 vertices with a computer [42]. In [27], the authors

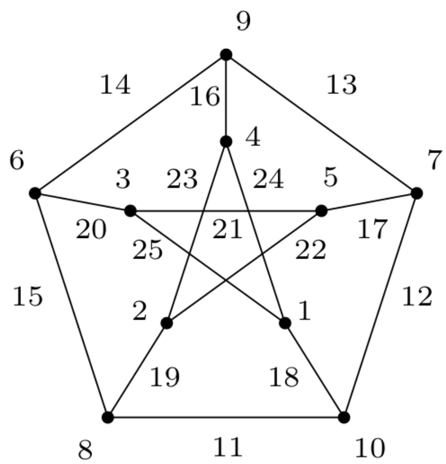

Figure 2.3: An edge-magic labeling with valence 29 on the Peterson graph.

showed that if T is a tree of order $p \geq 2$ that has diameter greater than or equal to $p-5$, then T has a super edge-magic labeling. Ichishima et. al. have shown that any tree of order p is contained in a tree of order at most $2p-3$ has a super edge-magic labeling [34].

The reader refers to the comprehensive survey on graph labelings by Gallian in [28].

2.2 Chromatic Polynomials

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E that has no loops and multiple edges. We denote the number of vertices by p and the number of edges by q.

A ***(vertex) k-coloring*** of G is an assignment $\varphi : V \to \{1, \ldots, k\}$, where we think of $\{1, \ldots, k\}$ as the set of available colors. It is called ***proper*** if for adjacent vertices $u, v \in V$, the relation $\varphi(v) \neq \varphi(u)$ holds. Otherwise, we say that φ is ***improper***. The function $P_G(k)$ counts the number of proper k-coloring of G.

Theorem 2.2.1. *[29, 12] For any graph G, $P_G(k)$ is a polynomial in k of degree p.*

Based on this theorem, the function $P_G(k)$ is called ***chromatic polynomial*** of G. Theorem 2.2.1 can be proven by induction on the number of edges and the fact that chromatic polynomials satisfy the ***Fundamental Reduction Theorem***, or ***deletion-contraction recurrence***:

Theorem 2.2.2. *[43] For a graph G and $e \in E$,*

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

where $G - e$ denotes the graph obtained by removing the edge e, and G/e is obtained by merging the two end vertices of e and removing all resulting loops and duplicated edges.

Example 2.2.3. [43] The chromatic polynomial of any tree with p vertices is $k(k-1)^{p-1}$ and for any p-gon is $(k-1)^p + (-1)^p(k-1)$.

The next theorem includes some properties of chromatic polynomials.

Theorem 2.2.4. *[43] Let G be a graph with p vertices. Then*

1. *$P_G(k)$ is a monic polynomial of degree p;*
2. *$P_G(k)$ has no constant term;*
3. *the terms in $P_G(k)$ alternate in sign;*
4. *the absolute value of the second coefficient of $P_G(k)$ is the number of edges in G;*
5. *G is a tree if and only if $P_G(k) = k(k-1)^{p-1}$.*

To study chromatic polynomials further, we refer to [43, 13, 11].

2.3 Ordered Structures

A partially ordered set or ***poset*** is a set P along with a binary relation $\preceq$ that is reflexive ($a \preceq a$), antisymmetric ($a \preceq b$ and $b \preceq a$ means $a = b$), and transitive ($a \preceq b$ and $b \preceq c$ means $a \preceq c$). A ***chain*** (or ***linearly ordered set***) is a poset in which any two elements are comparable. The set $\{1, \ldots, n\}$ with the natural order is a chain and we denote it by $[n]$. We call $\varphi : P \to [n]$ a ***(weakly) order-preserving map*** if for $a, b \in P$

$$a \preceq b \qquad \longrightarrow \qquad \varphi(a) \leq \varphi(b).$$

Similarly, a map φ is ***strictly order preserving*** if

$$a \prec b \qquad \longrightarrow \qquad \varphi(a) < \varphi(b).$$

The functions $\Omega_P(n)$ and $\Omega_P^\circ(n)$ count the number of order preserving maps and strictly order preserving maps, respectively.

Theorem 2.3.1. *[49] For a finite poset $(P, \preceq)$ the functions $\Omega_P(n)$ and $\Omega_P^\circ(n)$ are polynomials in n with rational coefficients.*

Let S be a set of n elements. A ***permutation*** of S is defined as a linear ordering $(\omega_1, \ldots, \omega_n)$ of the elements of S. We can think of ω as a word $\omega_1 \cdots \omega_n$ in the alphabet S. In general if $S = \{x_1, \ldots, x_n\}$, the word ω corresponds to the bijection $\omega : S \to S$ given by $\omega(x_i) = \omega_i$. Among many statistics associated with permutations, we will study descents and ascents. A

descent of the word $\omega = \omega_1 \cdots \omega_n$ is $1 \leq i \leq n-1$ with $\omega_i > \omega_{i+1}$. The ***descent set*** of ω is defined as

$$D(\omega) = \{i : \omega_i > \omega_{i+1}\}$$

Similarly, an ***ascent*** of the word $\omega = \omega_1 \cdots \omega_n$ is $1 \leq i \leq n-1$ with $\omega_i < \omega_{i+1}$. The ***ascent set*** of ω is defined as

$$A(\omega) = \{i : \omega_i < \omega_{i+1}\}.$$

The number of descents of ω is denoted by $\mathrm{des}(\omega) = |D(\omega)|$ and the number of ascents of ω is denoted by $\mathrm{asc}(\omega) = |A(\omega)|$. In particular, $\mathrm{des}(\omega) + \mathrm{asc}(\omega) = n - 1$. For further details, see [52] and [54].

Let P be a poset with n elements. A ***natural labeling*** of P is a bijection $\omega : P \to [n]$ where $a \prec b$ in P implies $\omega(a) < \omega(b)$. Similarly, a ***natural reverse labeling*** of P is where $a \prec b$ in P implies $\omega(a) > \omega(b)$. A ***linear extension*** L is a chain that refines P so that $a \preceq b$ in P gives $a \preceq b$ in L. We find all possible linear extensions of P by considering two cases between each pair of incomparable elements: $a \prec b$ or $b \prec a$ (e.g., Figure 2.4).

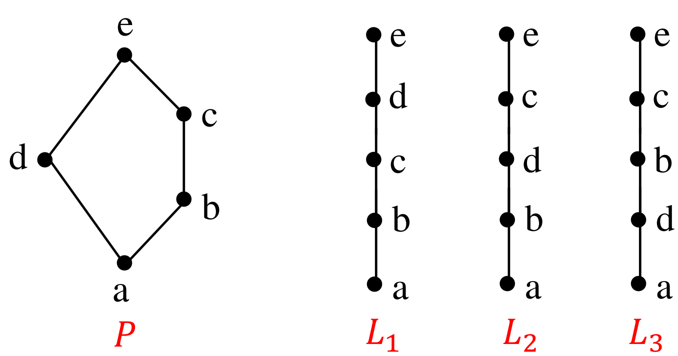

Figure 2.4: A poset P and its linear extensions L_1, L_2, and L_3.

Theorem 2.3.2. *[52] Let P be a poset with n elements and a fixed natural labeling ω. Then*

$$\Omega_P(x) = \sum_L \binom{x - \mathrm{des}(\omega_L)}{n}$$

where the sum is over all possible linear extensions L of P and ω_L is the word associated with the natural labeling ω.

The following is the relation between order preserving maps and strictly order preserving maps:

Theorem 2.3.3 (Order Polynomial Reciprocity Theorem [49]). *Let P be a finite poset. Then*

$$\Omega_P(n) = (-1)^{|P|}\, \Omega^{\circ}_P(-n).$$

An ***orientation*** of any graph G is an assignment of a direction to each edge $\{u, v\} \in E$ denoted by $u \to v$ or $v \to u$. An orientation of G is ***acyclic*** if it has no coherently directed cycle. There is a natural way that any acyclic orientation on a graph G gives rise to a poset. Given an orientation σ of G, let P be a poset with elements that are vertices of G. For any $a, b \in P$, define $a \preceq b$ if and only if there is a coherently oriented path from a to b in σ.

There is a connection between the graph colorings and the ordered structures. It says that a chromatic polynomial can be decomposed into order polynomials that come from acyclic orientations of a graph.

Theorem 2.3.4 (**Chromatic Polynomial Decomposition Theorem** [50]). *Let G be a graph. Then*

$$P_G(x) = \sum_{\sigma} \Omega^{\circ}_{\sigma}(x),$$

where the sum is over all possible acyclic orientations σ of G.

2.4 Hyperplane Arrangements

This section presents background information on the geometric approach of our work. Let $\mathcal{K}$ be a field. An (affine) ***hyperplane*** is a set of the form

$$\mathcal{H} := \{x \in \mathcal{K}^d : a \cdot x = b\}$$

where a is a fixed nonzero vector in $\mathcal{K}^d$, b is a displacement vector in $\mathcal{K}$ and $a \cdot x$ is the standard dot product. A ***hyperplane arrangement*** $\mathcal{A}$ is a finite set of hyperplanes in a finite-dimensional vector space $\mathcal{K}^d$. The arrangement has combinatorial properties when $\mathcal{K} = \mathbb{R}$. It divides $\mathbb{R}^d$ into regions. A ***region*** of $\mathcal{A}$ is a maximal connected component of $\mathbb{R}^d - \cup_{\mathcal{H} \in \mathcal{A}} \mathcal{H}$.

Example 2.4.1. The d-dimensional ***real braid arrangement*** is the arrangement $\mathcal{B} = \{x_j = x_k : 1 \leq j < k \leq d\}$ with $\binom{d}{2}$ hyperplanes.

Let us assume that G is a simple graph on the vertex set $[p]$ with edges E. To each edge $\{i, j\} \in E$ we associate a hyperplane $H_{ij} := \{x \in \mathbb{R}^p : x_i = x_j\}$. The ***graphical arrangement*** $\mathcal{A}_G$ in $\mathbb{R}^p$ is then

$$\{H_{ij} : \{i, j\} \in E\}.$$

The graphical arrangement is a sub-arrangement of the braid arrangement $\mathcal{B}$ introduced in Example 2.4.1. If $G = \mathcal{K}_p$, the complete graph on p vertices, then $\mathcal{A}_{\mathcal{K}_p}$ is the full braid arrangement.

Each region of the hyperplane arrangement $\mathcal{A}_G$ corresponds to an acyclic orientation of G: each region of the $\mathcal{A}_G$ is determined by some inequalities $x_i < x_j$ or $x_i > x_j$ where i and j

come from edge $\{i, j\} \in E$. Naturally, $x_i < x_j$ implies that we can orient the edge $\{i, j\} \in E$ from $i \to j$ and $x_i > x_j$ implies that we can orient the edge $\{i, j\} \in E$ from $j \to i$. This can be generalized by the following proposition.

Proposition 2.4.2. *[51] Let G be a simple graph and $\mathcal{A}_G$ be the graphical arrangement of G. The regions of $\mathcal{A}_G$ are in one-to-one correspondence with the acyclic orientations of G.*

A ***flat*** of $G = (V, E)$ is a set of edges $F \subseteq E$ such that for any edge $e \in E \setminus F$, the number of connected components of the graph $G_F := (V, F)$ is strictly larger than that of $G_{F*} := (V, F \cup \{e\})$. On the other hand, a ***flat*** of the graphical arrangement $\mathcal{A}_G$ corresponding to $F \subseteq E$ is

$$\mathcal{F} = \cap_{ij \in F} H_{ij} = \{x \in \mathbb{R}^p : x_i = x_j \text{ for all } ij \in F\}.$$

Particularly, we can think of a flat in the following different ways. First, all possible intersections of the arrangement are the possible flats. Because the graphical arrangement comes from a graph G these hyperplane intersections come from the possible contractions of G.

Proposition 2.4.3. *[6] The set of contractions of G are in bijection with the set of flats of the graphical arrangement $\mathcal{A}_G$.*

Secondly, we can also think of a flat through transitivity. Suppose that we have the hyperplanes $x_1 = x_2, x_2 = x_3, \ldots, x_{k-1} = x_k$. Then it must be true that $x_1 = x_k$ since the equality is transitive. Transitivity gives us the tools to be able to decompose the bivariate chromatic polynomial in Chapter 4. See [51, 6] for more interesting results.

2.5 Ehrhart Theory

Let $\{v_1, v_2, \ldots, v_n\}$ be a finite set of points in $\mathbb{R}^d$. The ***convex hull*** C of these points is the smallest convex set containing them. That is,

$$C = \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R}_{\geq 0} \text{ for all } i \text{ and} \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

A ***convex polytope*** $\mathcal{P}$ is the convex hull of finitely many points in $\mathbb{R}^d$. This definition is called the ***vertex description*** of $\mathcal{P}$, and we use the notation

$$\mathcal{P} = \operatorname{conv}\{v_1, \ldots, v_n\}.$$

Polytopes can be described as the bounded intersection of finitely many half-spaces and hyperplanes. This is called ***hyperplane description*** of a polytope which is equivalent to the vertex description of the polytope. The equivalence is highly nontrivial and we refer to [5, 62].

The ***dimension*** of a polytope $\mathcal{P}$ is the dimension of the affine space spanned by $\mathcal{P}$:

$$\operatorname{span} \mathcal{P} := \{x + \lambda(y - x) : x, y \in \mathcal{P}, \lambda \in \mathbb{R}\}.$$

We note that if $\mathcal{P} \subset \mathbb{R}^d$, then it does not necessarily have dimension d.

Example 2.5.1. Let $e_i \in \mathbb{R}^{d+1}$ for $1 \leq i \leq d+1$ be standard basis vectors. The ***standard d-simplex*** is a d-dimensional polytope defined as

$$\Delta_d = \operatorname{conv}\{e_1, \ldots, e_{d+1}\} = \left\{ x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i = 1, x_i \geq 0 \right\}.$$

Example 2.5.2. The ***unit d-cube*** is defined as

$$\square_d = \operatorname{conv}\{(x_1, \ldots, x_d) : \text{all}\, x_i = 0 \,\text{or}\, 1\} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1 \,\text{for all}\, 1 \leq i \leq d\}.$$

We say that the hyperplane $\mathcal{H} = \{x \in \mathbb{R}^d : a.x = b\}$ is a ***supporting hyperplane*** of $\mathcal{P}$, if $\mathcal{P}$ lies entirely on one side of $\mathcal{H}$. A ***face*** of $\mathcal{P}$ is the set of the form $\mathcal{P} \cap \mathcal{H}$, where $\mathcal{H}$ is a supporting hyperplane of $\mathcal{P}$. $\mathcal{P}$ and $\varnothing$ are defined to be faces of $\mathcal{P}$. For a convex polytope $\mathcal{P} \subset \mathbb{R}^d$, the $(d-1)$-dimensional faces are called ***facets***, 1-dimensional faces are called ***edges***, and 0-dimensional faces are called ***vertices*** of $\mathcal{P}$.

A convex polytope $\mathcal{P}$ is called ***lattice*** if all of its vertices have integer coordinates, and $\mathcal{P}$ is called ***rational*** if all of its vertices have rational coordinates.

Let $\mathcal{P} \subset \mathbb{R}^d$ be a d-dimensional lattice polytope. Then $k\mathcal{P} = \{(kx_1, \ldots, kx_d) : (x_1, \ldots, x_d) \in \mathcal{P}\}$ is the k^{th} dilation of the polytope $\mathcal{P}$. The ***lattice point enumerator*** for the k^{th} dilate of $\mathcal{P}$ is

$$E_{\mathcal{P}}(k) := \#(k\mathcal{P} \cap \mathbb{Z}^d)$$

for $k \in \mathbb{Z}_{>0}$. Here is the fundamental theorem concerning lattice point enumeration in integral convex polytopes.

Theorem 2.5.3 (Ehrhart's Theorem [22]). *Let $\mathcal{P}$ be an integral convex polytope of dimensional d. Then $E_{\mathcal{P}}(k)$ is a polynomial in k of degree d with rational coefficients.*

We recall that a ***quasi-polynomial*** is a function $f : \mathbb{Z} \to \mathbb{C}$ of the form $f(k) = c_d(k)\, k^d + \cdots + c_1(k)\, k + c_0(k)$ where $c_0(k), \ldots, c_d(k)$ are periodic functions in k and the ***period*** of f is the least common multiple of the periods of $c_0(k), \ldots, c_d(k)$. For every quasi-polynomial f there exist an integer $N > 0$ (namely, the least common multiple of the periods of $c_0(k), \ldots, c_d(k)$) and polynomials $p_0, \ldots, p_{N-1}$ (which are called the ***constituents*** of f) such that $f(k) = p_i(k)$ when $k \equiv i \pmod N$. The following theorem says that $E_{\mathcal{P}}(k)$ is a quasi-polynomial if $\mathcal{P}$ is a rational polytope.

Theorem 2.5.4 (Ehrhart's Theorem fo Rational Polytopes [22]). *Let $\mathcal{P}$ be a rational convex polytope of dimensional d. Then $E_{\mathcal{P}}(k)$ is a quasi-polynomial in k of degree d. Its period divides the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$.*

Based on Theorem 2.5.3, we call $E_{\mathcal{P}}(k)$ the ***Ehrhart polynomial***. The ***interior*** of $\mathcal{P}$, denoted by $\mathcal{P}^\circ$, is defined as the set of all points $x \in \mathcal{P}$ such that for some $\epsilon > 0$, the ϵ-ball $B_\epsilon(x)$ around x is contained in $\mathcal{P}$. Similarly, the ***relative interior*** is the set of all points $x \in P$ such that for some $\epsilon > 0$, the intersection $B_\epsilon(x) \cap \text{aff}(P)$ is contained in P [33].

Theorem 2.5.5 (**Ehrhart–Macdonald Reciprocity Theorem** [22, 40])**.** *For any d-dimensional rational polytope $\mathcal{P}$,*

$$E_{P^\circ}(k) = (-1)^d E_P(-k).$$

Example 2.5.6. Consider the unit d-cube in Example 2.5.2. For $d = 1$, the unit 1-cube is $[0, 1]$ and the k^{th} dilate of 1-cube is $[0, k]$. Therefore $E_{\square_1}(k) = (k + 1)$. In general,

$$E_{\square_d}(k) := \#([0, k]^d \cap \mathbb{Z}^d) = (k + 1)^d.$$

Similarly, the interior of 1-cube is $(0, 1)$ and its k^{th} dilate is $(0, k)$. Thus $E_{\square_1^\circ}(k) = (k - 1)$ and in general,

$$E_{\square_d^\circ}(k) := \#((0, k)^d \cap \mathbb{Z}^d) = (k - 1)^d.$$

Notice that evaluation of $(k + 1)^d$ at negative integers is equal to $(-1)^d(k - 1)^d$, confirming the Ehrhart–Macdonald Reciprocity Theorem.

For more details and results on polytopes, we refer to [5, 33, 62].

Chapter 3

Partially Magic Labeling

3.1 Introduction

Graph theory is abundant with fascinating open problems. In this chapter we propose a new *ansatz* to the *Antimagic Graph Conjecture* (Conjecture 2.1.2). Our approach generalizes Stanley's enumeration results for magic labelings of a graph [53] to partially magic labelings, with which we analyze the structure of antimagic labelings of graphs.

Our work concerns *magic* and *antimagic* labelings introduced in section 2.1, for which there are various (conflicting) definitions in the literature; thus we start by carefully introducing our terminology. Let $G = (V, E)$ be a graph with vertex set V and edge set E.

Definition 3.1.1. Let $L : E \to \mathbb{Z}_{\geq 0}$ be an edge labeling. We call L an ***$[a, b]$-labeling*** when each edge label is in $\{a, \ldots, b\} \subset \mathbb{Z}$, and we call L ***injective*** if no label repetition is allowed.

Definition 3.1.2. An $[a, b]$-labeling is called ***magic*** (respectively ***antimagic***) if all the vertex sums, i.e., the sums of the labels of the edges incident to each vertex, are equal (respectively distinct); here the label of a loop at a given vertex is counted only once.

The ***index*** of a magic labeling is the constant vertex sum. Magic labelings are also called ***regularisable*** [10], while antimagic $[0, \infty)$-labelings are also called ***irregular*** [15]. Graphs are commonly called ***magic*** (respectively ***antimagic***) if they admit an injective magic (respectively antimagic) $[1, |E|]$-labeling.

Our first motivation is the well-known, decades-old conjecture [31] that says that every connected graph except for K_2 is antimagic (the Antimagic Graph Conjecture, Conjecture 2.1.2). Our second motivation is the following result.

Theorem 3.1.3 (Stanley [53])**.** *Let $H_G(r)$ be the number of magic $[0, \infty)$-labelings of G of index r. Then there exist polynomials $P_G(r)$ and $Q_G(r)$ such that $H_G(r) = P_G(r) + (-1)^r Q_G(r)$. Moreover, if the graph obtained by removing all loops from G is bipartite, then $Q_G(r) = 0$, i.e., $H_G(r)$ is a polynomial of r.*

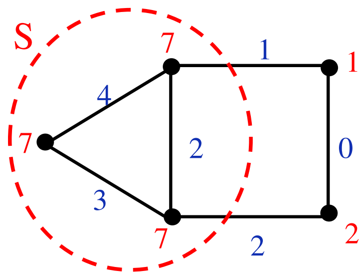

Figure 3.1: A partially magic labeling of a graph $G = (V, E)$ over $S \subseteq E$.

Theorem 3.1.3 can be rephrased in the language of quasi-polynomials (defined in Section 2.5). It says that $H_G(r)$ is a quasi-polynomial of period at most 2.

We emphasize that Theorem 3.1.3 is *not* about injective magic labelings; in fact, for this more restrictive notion, one still obtains quasi-polynomial counting functions, but there does not seem to be any bound on the periods [7]. Our first main result is as follows.

Theorem 3.1.4. *The number $A_G(k)$ of antimagic $[1, k]$-labelings of G is a quasi-polynomial in k of period at most 2. Moreover, if G minus its loops is bipartite, then $A_G(k)$ is a polynomial in k.*

We remark that antimagic counting functions of the flavor of $A_G(k)$ already surfaced in [8]. Theorem 3.1.4 implies that for bipartite graphs we have a chance of using the polynomial structure of $A_G(k)$ to say something about the antimagic character of G, in a non-injective sense. Borrowing a leaf from the notion of relaxed graceful labelings [47, 58], we call a graph ***relaxed antimagic*** if it admits an antimagic $[1, |E|]$-labeling.

Theorem 3.1.5. *Suppose G has no K_2 component and at most one isolated vertex. Then G admits an antimagic $[1, 2|E|]$-labeling. If, in addition, G is bipartite, then G is relaxed antimagic.*

We approach antimagic labelings by introducing a new twist on magic labelings which might be of independent interest. Fix a subset S of vertices of G.

Definition 3.1.6. A labeling of G is ***partially magic over S*** if the vertex sums are equal for all vertices of S (Figure 3.1).

Theorem 3.1.7. *Let G be a finite graph and $S \subseteq V$. The number $M_S(k)$ of partially magic $[0, k]$-labelings over S is a quasi-polynomial in k with period at most 2. Moreover, if G minus its loops is bipartite, then $M_S(k)$ is a polynomial in k.*

In order to prove Theorem 3.1.7, we will follow Stanley's lead in [53] and use linear Diophantine homogeneous equation and Ehrhart quasi-polynomials to describe partially magic labelings of the graph G; Section 3.2 contains a proof of Theorem 3.1.7. In Section 3.3 we prove Theorems 3.1.4 and 3.1.5. We conclude in Section 3.4 with some comments on a directed version of the Antimagic Graph Conjecture, as well as open problems.

3.2 Enumerating Partially Magic Labeling

Given a finite graph $G = (V, E)$ and a subset $S \subseteq V$, we introduce a variable x_e for each edge e and let $\{v_1, \ldots, v_s\}$ be the set of all vertices of S, where $|S| = s$. In this setup, a partially magic $[0, k]$-labeling over S corresponds to an integer solution of the system of equations and inequalities

$$\sum_{e \text{ incident to } v_j} x_e = \sum_{e \text{ incident to } v_{j+1}} x_e \quad (1 \le j \le s-1) \quad \text{and} \quad 0 \le x_e \le k\,. \tag{3.1}$$

Define Φ as the set of all pairs (L, k) where $L \in \mathbb{Z}_{\ge 0}^{|E|}$ is a partially magic $[0, k]$-labeling; that is, (L, k) is a solution to (3.1). If L is a partially magic $[0, k]$-labeling and L' is a partially magic $[0, k']$-labeling, then $L + L'$ is a partially magic $[0, k + k']$-labeling. Thus Φ is a semigroup with identity $\mathbf{0}$. This is also evident from (3.1).

For the next step, we will use the language of generating functions, encoding all partially magic $[0, k]$-labelings as monomials. Let $q = |E|$ and define

$$F(Z) = F(z_1, \ldots, z_q, z_{q+1}) := \sum_{(L,k) \in \Phi} z_1^{L(e_1)} \cdots z_q^{L(e_q)} z_{q+1}^k\,. \tag{3.2}$$

Note that if we substitute $z_1 = \cdots = z_q = 1$ in $F(Z)$, we enumerate all partially magic $[0, k]$-labelings:

$$F(\mathbf{1}, z_{q+1}) = \sum_{(L,k) \in \Phi} z_{q+1}^k = \sum_{k \ge 0} M_S(k)\, z_{q+1}^k, \tag{3.3}$$

where we abbreviated $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{Z}^q$.

Definition 3.2.1. We call a nonzero element $L = (L_1, \ldots, L_q, k) \in \Phi$ ***fundamental*** if it cannot be written as the sum of two nonzero elements of Φ; furthermore, L is ***completely fundamental*** if no positive integer multiple of it can be written as the sum of nonzero, nonparallel elements of Φ (i.e., they are not scalar multiple of each other).

In other words, a completely fundamental element $(L, k) \in \Phi$ is a nonnegative integer vector such that for each positive integer n, if $nL = \alpha + \beta$ for some $(\alpha, a), (\beta, b) \in \Phi$, then $\alpha = jL$ and $\beta = (n - j)L$ for some nonnegative integer j. Note that by taking $n = 1$ in the above definition, we see that every completely fundamental element is fundamental. Also note that any fundamental element $(L_1, \ldots, L_q, k) \in \Phi$ necessarily satisfies $k = \max\{L_1, \ldots, L_q\}$.

Now we focus on the generating function (3.2) and employ [52, Theorem 4.5.11], which says in our case that the generation function $F(Z)$ can be written as a rational function with denominator

$$D(Z) \ := \prod_{(L,k)\in \mathrm{CF}(\Phi)} \left(1 - Z^{(L,k)}\right), \tag{3.4}$$

where $\mathrm{CF}(\Phi)$ is the set of completely fundamental elements of Φ and we used the monomial notation $Z^{(L,k)} := z_1^{L(e_1)} z_2^{L(e_2)} \cdots z_q^{L(e_q)} z_{q+1}^k$. To make use of (3.4), we need to know some information about completely fundamental solutions to (3.1). To this extent, we borrow the following lemmas from magic $[0, \infty)$-labelings [53], i.e., the case $S = V$:

Lemma 3.2.2. *Every completely fundamental magic $[0, \infty)$-labeling of G has index 1 or 2. More precisely, if L is any magic $[0, \infty)$-labeling of G, then $2L$ is a sum of magic $[0, \infty)$-labelings of index 2.*

Lemma 3.2.3. *The following conditions are equivalent:*

1. *Every completely fundamental magic $[0, \infty)$-labeling of G has index 1.*
2. *If G' is any spanning subgraph of G such that every connected component of G' is a loop, an edge, or a cycle of length ≥ 3, then every one of these cycles of length greater than or equal to 3 must have even length.*

Lemma 3.2.2 implies that every completely fundamental magic $[0, \infty)$-labelings has index 1 or 2 and therefore, it cannot have a label ≥ 3 (because labels are nonnegative). By the same reasoning, if G satisfies the condition (2) in Lemma 3.2.3, every completely fundamental magic $[0, \infty)$-labelings of it has index 1 and so cannot have labels ≥ 2. We now give the analogous result for *partially* magic labelings:

Lemma 3.2.4. *Let $k \in \mathbb{Z}_{\geq 3}$. Every completely fundamental partially magic $[0, k]$-labeling of G over S has labels 0, 1, or 2.*

Proof. If $S = V$, then every completely fundamental partially magic $[0, k]$-labeling over S is a completely fundamental magic $[0, k]$-labeling over G. By Lemma 3.2.2, it has index 1 or 2 and so the labels are among 0, 1, or 2.

Suppose that $S \subsetneq V$ and let L be a partially magic $[0, k]$-labeling of G over S that has a label ≥ 3 on the edge e incident to vertices u and v. We will show that L is not completely fundamental. There are three cases:
Case 1: $u, v \notin S$, that is, e is not incident to any vertex in S. We can write L as the sum of L' and L'', where all the labels of L' are zero except for e with $L'(e) = 1$ and all the labels of L'' are the same as L except e with $L''(e) = L(e) - 1$; see Figure 3.2. Since L' and

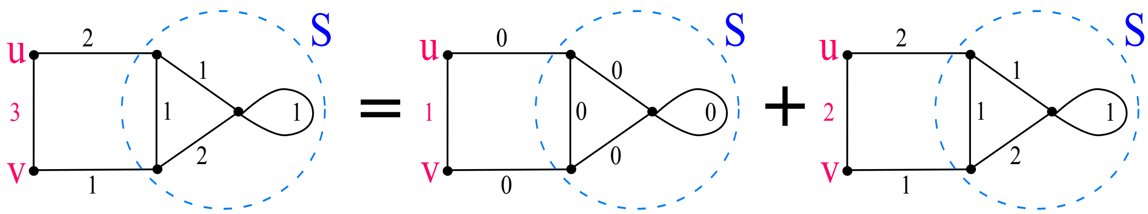

Figure 3.2: A non-completely fundamental partially magic $[0, k]$-labeling in Case 1.

L'' are both partially magic $[0, k]$-labelings over S, then by definition L is not a completely fundamental partially magic $[0, k]$-labeling over S.

*C*ase 2: $u \notin S$ and $v \in S$. Let G_S be the graph with vertex set S obtained from G by removing all the edges of G that are not incident to some vertex of S and making loops out of those edges that are incident to both S and $V \setminus S$. Now define a labeling L_S over G_S such that all the edges that are incident to S get the same labels as L and all the new loops get the labels of L that were on the original edges, as in Figure 3.3.

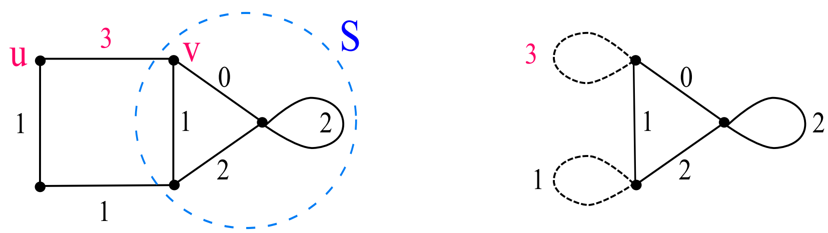

Figure 3.3: A graph G_S and magic $[0, k]$-labeling L_S in Case 2.

Since L is partially magic over S, L_S is a magic $[0, k]$-labeling of G_S. However, $L_S(e) = L(e) \geq 3$ and so S has a vertex with sum ≥ 3. Therefore, by Lemma 3.2.2, L_S is not a completely fundamental magic $[0, k]$-labeling of G_S and so there exist magic $[0, k]$-labelings L^i_S of index 2 such that $2L_S = \sum L^i_S$, as in Figure 3.4. Now we extend each magic $[0, k]$-

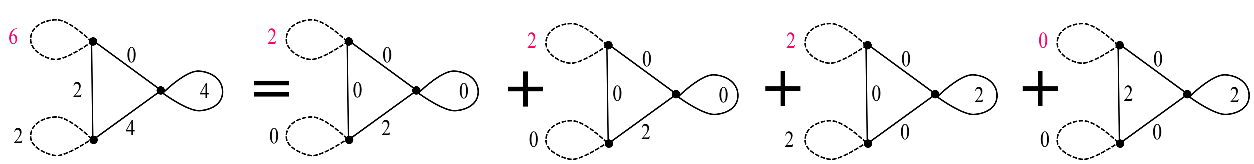

Figure 3.4: A graph G_S and the magic $[0, k]$-labelings L^i_S where $2L_S = \sum_{i=0}^{4} L^i_S$.

labeling L_S^i to a partially magic $[0,k]$-labeling L^i over G as follows, for $i = 1,2$ we define

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or } e \text{ is incident to vertices of } S \text{ and } V \setminus S, \\ L(e) & \text{if } e \text{ is not incident to } S. \end{cases}$$

For $i \geq 3$, the extensions are

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or } e \text{ is incident to vertices of } S \text{ and } V \setminus S, \\ 0 & \text{if } e \text{ is not incident to } S. \end{cases}$$

Therefore $2L(e) = \sum L^i(e)$ for all $e \in E$; see Figure 3.5. By definition, L^i is nonzero partially magic $[0,k]$-labeling of G over S with labels among $0,1,2$, for every $i > 1$. This proves that L is not a completely fundamental partially magic $[0,k]$-labeling.

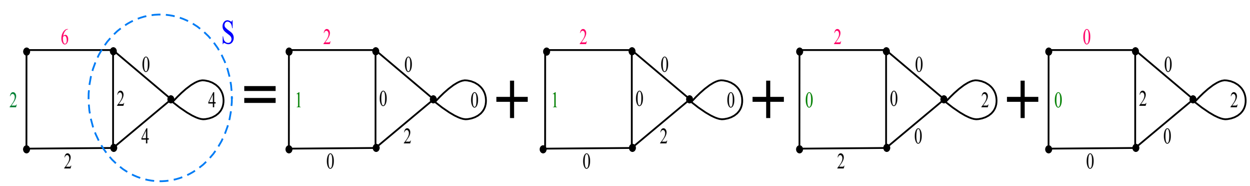

Figure 3.5: A non-completely fundamental partially magic $[0,k]$-labeling L with $2L = \sum_{i=0}^{4} L^i$.

*C*ase 3: $u, v \in S$. The argument is similar to Case 2, by constructing the graph G_S with the labeling L_S. Since L is partially magic $[0,k]$-labeling over S, L_S is a magic $[0,k]$-labeling of G_S. However, it is not completely fundamental because it has an edge e with label $L_S(e) = L(e) \geq 3$. So there exist magic $[0,k]$-labelings L_S^i of index 2, such that $2L_S = \sum L_S^i$. We extend each L_S^i to a labeling L^i over G as follows, for $i = 1,2$ we define

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or } e \text{ is incident to vertices of } S \text{ and } V \setminus S, \\ L(e) & \text{if } e \text{ is not incident to } S. \end{cases}$$

For $i \geq 3$, we extend the labeling L_S^i to L^i over G as follows:

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or } e \text{ is incident to vertices of } S \text{ and } V \setminus S, \\ 0 & \text{if } e \text{ is not incident to } S. \end{cases}$$

By definition, $2L = \sum L^i$ where each L^i is a partially magic $[0,k]$-labeling over S and has labels $0, 1$, or 2. Therefore, L is not a completely fundamental partially magic $[0,k]$-labeling of G over S. $\square$

Proof of Theorem 3.1.7. By (4.2) and (3.4), the function

$$F(\mathbf{1}, z) \;=\; \sum_{k\geq 0} M_S(k)\, z^k$$

is a rational function with denominator

$$D(\mathbf{1}, z) \;=\; \prod_{(L,k)\in \mathrm{CF}(\Phi)} \left(1 - \mathbf{1}^L z^k\right) \tag{3.5}$$

where $\mathrm{CF}(\Phi)$ is the set of completely fundamental elements of Φ. According to Lemma 3.2.4, every completely fundamental element of Φ has labels 1 or 2. Therefore

$$\sum_{k\geq 0} M_S(k)\, z^k \;=\; \frac{h(z)}{(1-z)^a(1-z^2)^b} \tag{3.6}$$

for some nonnegative integers a and b, and some polynomial $h(z)$. Basic results on rational generating functions (see, e.g., [52]) imply that $M_S(k)$ is a quasi-polynomial in k with period at most 2. ☐

Remark 3.2.5. Let G be a bipartite graph and $k \in \mathbb{Z}_{\geq 0}$. Then every completely fundamental partially magic $[0,k]$-labeling of G over S has labels 0 or 1.

Proof. If $S = V$, then every completely fundamental partially magic $[0,k]$-labeling over S is a completely fundamental magic $[0,k]$-labeling over G. By Lemma 3.2.3, it has index 1 and so the labels are 0 or 1.

Suppose that $S \subsetneq V$ and let L be a partially magic $[0,k]$-labeling of G over S that has a label ≥ 2 on the edge e incident to vertices u and v. We will show that L is not completely fundamental. There are three cases:

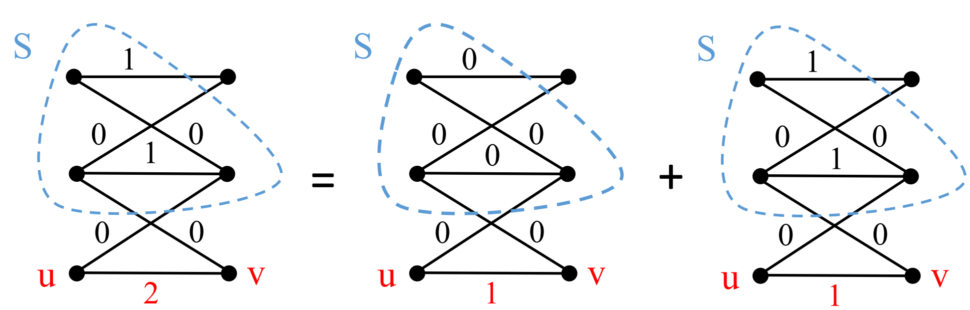

Figure 3.6: A non-completely fundamental partially magic $[0,k]$-labeling in Case 1.

Case 1: $u, v \notin S$, that is, e is not incident to any vertex in S. We can write L as the sum of L' and L'', where all the labels of L' are zero except for e with $L'(e) = 1$ and all the labels of L'' are the same as L except e with $L''(e) = L(e) - 1$; see Figure 3.6. Since L' and L'' are both partially magic $[0, k]$-labelings over S, then by definition L is not a completely fundamental partially magic $[0, k]$-labeling over S.

Case 2: $u \notin S$ and $v \in S$. Let G_S be the graph with vertex set S obtained from G by removing all the edges of G that are not incident to some vertex of S and making loops out of those edges that are incident to both S and $V \setminus S$. Now define a labeling L_S over G_S such that all the edges that are incident to S get the same labels as L and all the new loops get the labels of L that were on the original edges, as in Figure 3.7.

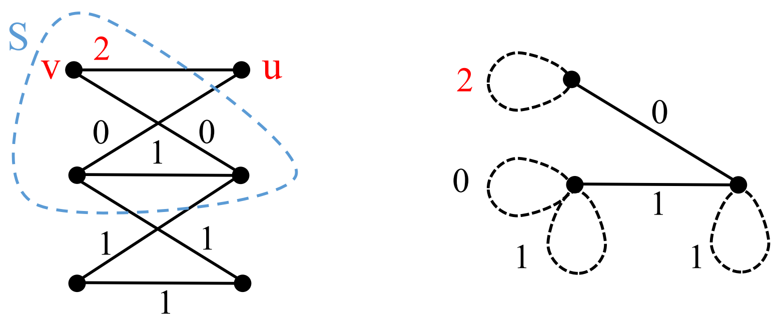

Figure 3.7: A graph G_S and magic $[0, k]$-labeling L_S in Case 2.

Since L is partially magic over S, L_S is a magic $[0, k]$-labeling of G_S. However, $L_S(e) = L(e) \geq 2$ and so S has a vertex with sum ≥ 2. Therefore, by Lemma 3.2.3, L_S is not a completely fundamental magic $[0, k]$-labeling of G_S and so there exist magic $[0, k]$-labelings L_S^i of index 1 such that $2L_S = \sum L_S^i$, as in Figure 3.8. Now we extend each magic $[0, k]$-

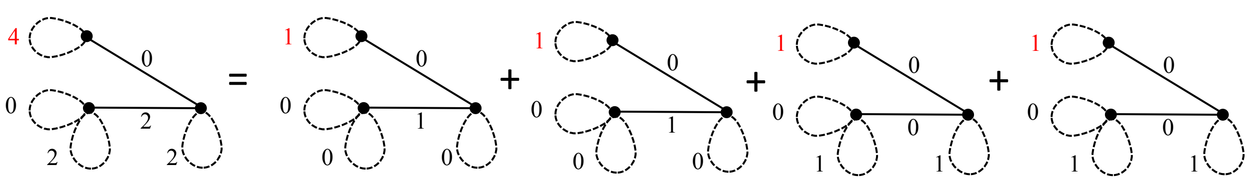

Figure 3.8: A graph G_S and the magic $[0, k]$-labelings L_S^i where $2L_S = \sum_{i=0}^{4} L_S^i$.

labeling L_S^i to a partially magic $[0, k]$-labeling L^i over G as follows, for $i = 1, 2$ we define

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or } e \text{ is incident to vertices of } S \text{ and } V \setminus S, \\ L(e) & \text{if } e \text{ is not incident to } S. \end{cases}$$

For $i \geq 3$, the extensions are

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or } e \text{ is incident to vertices of } S \text{ and } V \setminus S, \\ 0 & \text{if } e \text{ is not incident to } S. \end{cases}$$

Therefore $2L(e) = \sum L^i(e)$ for all $e \in E$; see Figure 3.9. By definition, L^i is nonzero partially magic $[0, k]$-labeling of G over S with labels among $0, 1$, for every $i > 1$. This proves that L is not a completely fundamental partially magic $[0, k]$-labeling.

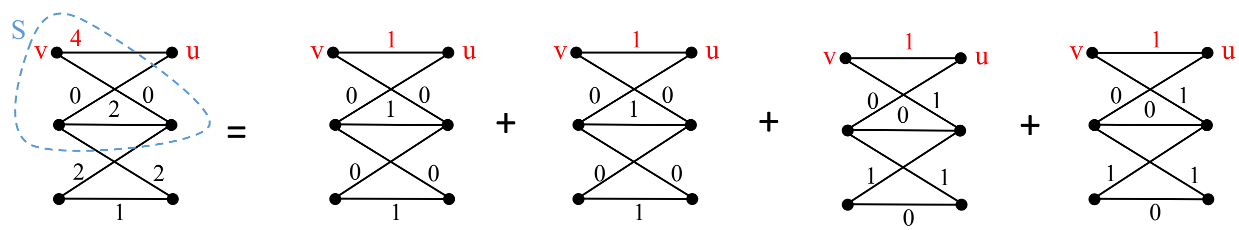

Figure 3.9: A non-completely fundamental partially magic $[0, k]$-labeling L with $2L = \sum_{i=0}^{4} L^i$.

Case 3: $u, v \in S$. The argument is similar to Case 2, by constructing the graph G_S with the labeling L_S. Since L is partially magic $[0, k]$-labeling over S, L_S is a magic $[0, k]$-labeling of G_S. However, it is not completely fundamental because it has an edge e with label $L_S(e) = L(e) \geq 2$. So there exist magic $[0, k]$-labelings L_S^i of index 1, such that $2L_S = \sum L_S^i$. We extend each L_S^i to a labeling L^i over G as follows, for $i = 1, 2$ we define

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or } e \text{ is incident to vertices of } S \text{ and } V \setminus S, \\ L(e) & \text{if } e \text{ is not incident to } S. \end{cases}$$

For $i \geq 3$, we extend the labeling L_S^i to L^i over G as follows:

$$L^i(e) := \begin{cases} L_S^i(e) & \text{if } e \text{ is incident to vertices of } S \text{ or } e \text{ is incident to vertices of } S \text{ and } V \setminus S, \\ 0 & \text{if } e \text{ is not incident to } S. \end{cases}$$

By definition, $2L = \sum L^i$ where each L^i is a partially magic $[0, k]$-labeling over S and has labels 0 or 1. Therefore, L is not a completely fundamental partially magic $[0, k]$-labeling of G over S. $\square$

The equations in (3.1) together with $x_e \geq 0$ describe a pointed rational cone, and adding the inequalities $x_e \leq 1$ gives a rational polytope $\mathcal{P}_S$. Our reason for considering $\mathcal{P}_S$ is the structural results of Ehrhart and Macdonald described in Section 2.5. A partially magic $[0, k]$-labeling of a graph G with labels among $\{0, 1, \ldots, k\}$ (which is a solution of (3.1)) is therefore an integer lattice point in the k-dilation of $\mathcal{P}_S$, i.e.,

$$M_S(k) = E_{\mathcal{P}_S}(k) \,.$$

Let $M_S^\circ(k)$ be the number of partially magic $[1, k]$-labelings of a graph G over a subset S of vertices of G. Thus $M_S^\circ(k) = E_{\mathcal{P}_S^\circ}(k+1)$, where $\mathcal{P}_S^\circ$ is the relative interior of the polytope $\mathcal{P}_S$.

Ehrhart's theorem (Theorem 2.5.3) implies that $E_{\mathcal{P}_S}(t)$ is a quasi-polynomial in t of degree $\dim \mathcal{P}_S$, and the Ehrhart–Macdonald reciprocity theorem for rational polytopes (Theorem 2.5.5) gives the algebraic relation $(-1)^{\dim \mathcal{P}} E_{\mathcal{P}}(-t) = E_{\mathcal{P}^\circ}(t)$, which implies for us:

Corollary 3.2.6. *Let $G = (V, E)$ be a graph and $S \subseteq V$. Then $M_S^\circ(k) = \pm\, M_S(-k-1)$. In particular, $M_S(k)$ and $M_S^\circ(k)$ are quasi-polynomials with the same period.*

3.3 Enumerating Antimagic Labeling

By definition of a partially magic labeling of a graph G over a subset $S \subseteq V$, all the vertices of S have the same vertex sum. By letting S range over all subsets of V of size ≥ 2, we can write the number $A_G(k)$ of antimagic $[1, k]$-labelings as an inclusion-exclusion combination of the number of partially magic $[1, k]$-labelings:

$$A_G(k) \;= k^{|E|} - \sum_{\substack{S \subseteq V \\ |S| \geq 2}} c_S\, M_S^\circ(k) \tag{3.7}$$

for some $c_S \in \mathbb{Z}$. Thus Theorem 3.1.7 and Corollary 3.2.6 imply Theorem 3.1.4.

Example 3.3.1. The number of antimagic $[1, k]$-labelings for the bipartite graph $K_{3,1}$ is $A_{K_{3,1}}(k) = k^3 - 3k^2 + k$.

In preparation for our proof of Theorem 3.1.5, we give a few basic properties of $A_G(k)$.

Lemma 3.3.2. *The constant term of the quasi-polynomial $A_G(k)$ equals $A_G(0) = 0$.*

Proof. By definition $\mathcal{P}_S \subseteq [0, 1]^E$, and so $M_S^\circ(0) = E_{\mathcal{P}_S^\circ}(1) = 0$. The statement follows now from (3.7). □

Lemma 3.3.3. *The quasi-polynomial $A_G(k)$ is constant zero if and only if G has a K_2 component or more than one isolated vertex.*

Proof. If G has a K_2 component or more than one isolated vertex, then clearly there is no antimagic labeling and so $A_G(k) = 0$.

Conversely, suppose G has no K_2 component and at most one isolated vertex. Then assigning the edges with distinct powers of 2 yields an antimagic labeling (by the uniqueness of binary expansions), and so $A_G(k)$ is not constant zero.

We label the edges by $e_0, \ldots, e_{q-1}$ and assign 2^i to the edge e_i for $0 \leq i \leq q-1$. Then the vertex sum of each vertex can be express in the binary form $(a_{q-1}, a_{q-2} \ldots, a_0)$ where $a_i = 1$ if the edge e_i is incident to v, and $a_i = 0$ otherwise. Then the vertex sums are distinct since no two vertices are adjacent to the same set of edges and binary expansion of every integer is unique. Therefore, we found an injective antimagic $[1, 2^{q-1}]$-labeling and so $A_G(k)$ is not constant zero. [15] □

Proof of Theorem 3.1.5. By Theorem 3.1.4 and Lemma 3.3.3, we know that $A_G(k)$ is a nonzero quasi-polynomial in k of period ≤ 2. By definition and Lemma 3.3.2, $A_G(k+1) \geq A_G(k)$ for $k \geq 0$. So both even and odd constituents of $A_G(k)$ are polynomials in k with degree at most $|E|$, which can have at most $|E|$ integer roots. By Lemma 3.3.2, one of the roots is 0, and consequently $A_G(2|E|) > 0$.

The second statement can be proven similarly to the first statement. Namely, $A_G(k)$ is a weakly increasing polynomial in k with maximum degree $|E|$ (by Theorem 3.1.4 and definition) and $A_G(0) = 0$ (by Lemma 3.3.2). Thus $A_G(|E|) > 0$, i.e., G has an antimagic $[1, |E|]$-labeling. □

3.4 Directed Graphs

Among the more recent results on antimagic graphs are some for ***directed graphs*** (for which one of the endpoints of each edge e is designated to be the head, the other the tail of e); given an edge labeling of a directed graph, we denote the ***oriented sum*** $s(v)$ at the vertex v to be the sum of the labels of all edges oriented away from v minus the sum of the labels of all edges oriented towards v. Such a labeling is ***antimagic*** if each label is a distinct element of $\{1, 2, \ldots, |E|\}$ and the oriented sums $s(v)$ are pairwise distinct.

It is known that every directed graph whose underlying undirected graph is dense (in the sense that the minimum degree at least $C \log |V|$ for some absolute constant $C > 0$) is antimagic, and that almost every regular graph admits an orientation that is antimagic [32].

Hefetz, Mütze, and Schwartz suggest in [32] a directed version of the Antimagic Graph Conjecture; the two natural exceptions are the complete graph K_3 on three vertices with an edge orientation that makes an oriented cycle, and $K_{1,2}$, the bipartite graph on the vertex partition $\{v_1\}$ and $\{v_2, v_3\}$ where the orientations are from v_2 to v_1 and v_1 to v_3.

Conjecture 3.4.1 (**Directed Antimagic Graph Conjecture** [32])**.** *Every connected directed graph except for the directed graphs K_3 and $K_{1,2}$ admits an antimagic labeling.*

It is tempting to adjust our techniques to the directed settings, but there seem to be road blocks. For starters, no directed graph has a ***magic labeling***, i.e., all sums $s(v)$ are equal. To see this, let A be the square matrix with A_{ij} the oriented sum of the vertex v_i using the labels of all edges between v_i and v_j. Now if L is a magic labeling with index r, the sum of each row of A equals r, and so r is an eigenvalue of A (with eigenvector $[1, 1, \ldots, 1]$). However, A is by construction a skew matrix, and so it cannot have a real eigenvalue.

At any rate, a directed graph will have *partially* magic labelings, defined analogously to the undirected case, and so we can enumerate antimagic labelings according to the directed analogue of (3.7). To assert the existence of an antimagic labeling, one would like to bound the period of the antimagic quasi-polynomial, as in Theorem 3.1.7. However, this does not seem possible. Namely, if the subset $S \subset V$ includes a directed path $\cdots \to v_1 \to v_2 \to \cdots \to v_s \to \cdots$ such that the vertices $v_2, \ldots, v_{s-1}$ are not adjacent to any other vertices, then a completely fundamental partially magic labeling L_S with index ≥ 1 implies that the label on each edge of the path is greater than that on the previous one. Thus, contrary to the situation in Lemma 3.2.4, the upper bound for the labels in L_S can be arbitrarily large. Consequently, the periods of the partial-magic quasi-polynomials, and thus those of the antimagic quasi-polynomials, can be arbitrarily large.

The papers [8, 32] give several further open problems on antimagic graphs, some of which could be tackled with the methods presented here. We close with an open problem about a natural extension of our antimagic counting function. Namely, it follows from [8] that the number of antimagic labelings of a given graph G with *distinct* labels between 1 and k is a quasi-polynomial in k. Can anything substantial be said about its period? It is unclear to us whether the methods presented here are of any help, however, any positive result would open the door to applying these ideas once more towards the Antimagic Graph Conjecture.

Chapter 4

Super Edge-Magic Labeling

4.1 Introduction

In this chapter, we will study *harmonious* labelings which were described in section 2.1. Our ultimate goal is to attack the Harmonious Tree Conjecture (Conjecture 2.1.3) for which we will use results from *super edge-magic labelings* which were also described in section 2.1. We begin by introducing our terminology.

Definition 4.1.1. We call an assignment $L : V \cup E \to \mathbb{Z}_{\geq 0}$ an ***(a, b)-total labeling*** of the graph G when $L(V \cup E) \subseteq \{a, \ldots, b\}$.

The term total labeling was introduced by Wallis in [60]. Now let us define our terminology for super edge-magic labeling.

Definition 4.1.2. An (a, b)-total labeling is called ***edge-magic*** if there exists a constant $c \in \mathbb{Z}_{\geq 0}$, which we call the ***magic invariant***, such that $L(u) + L(v) + L(e) = c$ for every $e = \{u, v\} \in E$. In addition, if L is an edge-magic $(a, b + c)$-total labeling $(a < b \leq c)$ such that $L(V) \subseteq \{a, \ldots, b\}$ and $L(E) \subseteq \{b + 1, \ldots, b + c\}$, then L is called ***super edge-magic (a, b, c)-total labeling***.

Graphs are commonly called edge-magic (as defined in section 2.1) if they admit an injective edge-magic $(1, p + q)$-total labeling. Moreover, they are called super edge-magic labeling (as defined in section 2.1) if they admit an injective super edge-magic $(1, p, q)$-total labeling.

The connection between super edge-magic labelings and harmonious labelings is coming from the following theorem:

Theorem 4.1.3. *[23] If a tree T admits a super edge-magic labeling, then it admits a harmonious labeling. More generally, if a graph G with $q \geq p$ admits a super edge-magic labeling, then G admits a harmonious labeling.*

Based on this theorem, we can use super edge-magic labelings on trees to approach the Harmonious Tree Conjecture (Conjecture 2.1.3). Our main results in this chapter are as follows.

Theorem 4.1.4. *Suppose T is a tree. Then T admits a super edge-magic $(1, p, p)$-total labeling.*

Theorem 4.1.5. *Suppose T_c is a tree with one extra edge that makes a unique even cycle, then it admits a super edge-magic $(1, p, p)$-total labeling.*

The Super Edge-Magic Tree Conjecture (Conjecture 2.1.4) asserts that every tree admits a super edge-magic labeling. Therefore, by Theorem 4.1.4, we show that every tree satisfies a relaxed version (where repetition of labels are allowed) of this conjecture.

Borrowing a leaf from the notion of relaxed graceful labelings [47, 58], we call a graph ***relaxed harmonious*** if it admits a harmonious labeling where repetition of labels are allowed.

Corollary 4.1.6. *Every tree admits a relaxed harmonious labeling in which one more label $2p$ is allowed. Furthermore, every tree with an extra edge that makes a unique even cycle admits relaxed harmonious labeling.*

Proof. The result follows from Theorem 4.1.3, Theorem 4.1.4, and Theorem 4.1.5 □

Corollary 4.1.6 indicates that two classes of graphs, i.e., trees and tress with an extra edge that makes a unique even cycle, satisfy the relaxed version of the Harmonious Tree Conjecture (Conjecture 2.1.3).

In Section 4.2, we use the language of generating functions and [52, Theorem 4.5.11] to prove Theorem 4.1.4 and Theorem 4.1.5. Moreover, we employ Theorem 4.1.3 to show that trees and trees with an extra edge that makes a unique even cycle admits relaxed version of the Harmonious Tree Conjecture (Conjecture 2.1.3).

4.2 Enumerating Super Edge-Magic Labeling

Define Φ to be the set of all tuples (L, k) where L is an edge-magic $(0, k)$-total labeling. If $L \in \Phi$ is an edge-magic $(0, k)$-total labeling and $L' \in \Phi$ is an edge-magic $(0, k')$-total labeling, then $L + L'$ is an edge-magic $(0, k + k')$-total labeling. Thus Φ is a semigroup with identity $\mathbf{0}$.

For the next step, we will use the language of generating functions over the monoid Φ. We write every edge-magic $(0, k)$-total labeling as a monomial to encode the information contained by the labelings. Define the generation function

$$F(X, Y, z) = F(x_1, \ldots, x_p, y_1, \ldots, y_q, z) = \sum_{(L,k)\in\Phi} x_1^{L(v_1)} \cdots\ x_p^{L(v_p)}\ y_1^{L(e_1)} \cdots\ y_q^{L(e_q)}\ z^k, \quad (4.1)$$

where the sum ranges over all edge-magic $(0, k)$-total labelings L of G.

Let $M_G(k)$ be the number of all edge-magic $(0, k)$-total labelings. Note that if we plug in $\mathbf{1} = (1, \ldots, 1)$ for $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ in the generating function F, we can count all the edge-magic $(0, k)$-total labelings:

$$F(\mathbf{1}, z) = \sum_{(L,k)\in\Phi} 1^{L(v_1)} \cdots\ 1^{L(v_p)}\ 1^{L(e_1)} \cdots\ 1^{L(e_q)}\ z^k = \sum_{k\geq 0} M_G(k)\ z^k. \tag{4.2}$$

Similar to the procedure described in Section 3.2, we now employ [52, Theorem 4.5.11] to analyze the structure of our generating function.
Therefore, we need to know some information about completely fundamental elements of Φ. As we will discuss later in this section, for a tree T the set of all super edge-magic $(1, p, p)$-total labelings and the set of all edge-magic $(1, p)$-total labelings are in bijection. Therefore, we focus on trees and we try to find a description of the generating function $F(\mathbf{1}, z)$ for a tree T. The next lemma describes the fundamental elements of Φ for bipartite graphs. Note that a tree is a bipartite graph.

Lemma 4.2.1. *Let G be a bipartite graph. Every completely fundamental edge-magic $(0, k)$-total labeling of G has $k = 1$.*

Proof. We will prove the lemma by induction on the magic invariant. Let L be an edge-magic $(0, k)$-total labeling with the magic invariant c. If $c = 0$, then all the labels of L are 0. If $c = 1$, then for every edge $\{u, v\} \in E$, one of the labels $L(u)$, $L(v)$, or $L(e)$ is 1 and the others are 0. L can not be decomposed further and so it is a completely fundamental edge-magic $(0, 1)$-total labeling.

Assume that we can decompose every edge-magic $(0, k)$-total labeling with magic invariant $\leq n$ as linear combination of edge-magic $(0, 1)$-total labelings. Now let L be an edge-magic $(0, k)$-total labeling with magic invariant $n + 1$. We will show that we can write L as the linear combination of edge-magic $(0, 1)$-total labelings.

Take the minimum vertex label m_v of L. If $m_v \geq 1$, subtract it from all the vertex labels and so the magic invariant reduces by $2m_v$. Similarly, take the minimum edge label m_e of L. If $m_e \geq 1$, subtract it from all the edge labels and so the magic invariant reduces by m_e. By induction, we can decompose the resulting labeling into completely fundamental edge-magic $(0, 1)$-total labelings. Therefore, we may assume that minimum vertex and edge labels of L are 0. Now assume that the vertices $w_1, \ldots, w_n \subseteq V$ have maximum vertex label $m > 0$. There are two cases:

*C*ase 1: If none of $w_1, \ldots, w_n$ are adjacent to each other, subtract 1 from $w_1, \ldots, w_n$ and all the edges that are not incident with $w_1, \ldots, w_n$. The labels of the other vertices and edges remain the same. Note that since the magic invariant is fixed, the labels of the edges incident with $w_1, \ldots, w_n$ have smaller values than edges not incident with $w_1, \ldots, w_n$. Since the magic invariant of L is $n+1$, the new edge-magic labeling L' has magic invariant $< n+1$. By induction, we can decompose L' into completely fundamental edge-magic $(0,1)$-total labelings $L_1, \ldots, L_r$. Let L'' be an edge labeling on G with 1 on the vertices $w_1, \ldots, w_n$ and 1 on the edges not incident with $w_1, \ldots, w_n$. So we have:

$$L = L'' + \sum_{i=0}^{i=r} a_i L_i.$$

*C*ase 2: If the graph induced by $w_1, \ldots, w_n$ forms a forest F that contains at least one edge, let $W_1, \ldots, W_s$ be the connected components of F that are not isolated vertices. By definition, each W_i is bipartite. Therefore for every $1 \le i \le s$, there exist nonempty disjoint sets A_i and B_i such that $W_i = A_i \cup B_i$ and every edge of W_i connects a vertex of A_i to a vertex of B_i. Now subtract 1 from all the isolated vertices of F, all the vertices of A_i for every $1 \le i \le s$, and all the edges of G that are not incident with the vertices whose labels were reduced. The labels of the other vertices and edges remain the same. Note that there are three kinds of edges in G:

1. The edge e_1 is adjacent with two vertices of F, so the label of e_1 is $L(e_1) = c - m - m$.
2. The edge e_2 is adjacent with a vertex in F and a vertex $v \notin F$, so the label of e_2 is $L(e_2) = c - m - L(v)$.
3. The edge e_3 is adjacent with the vertices $u, v \notin F$, so the label of e_3 is $L(e_3) = c - L(u) - L(v)$.

Clearly, $L(e_1)$, $L(e_2) < L(e_3)$. Therefore, $L(e) \neq 0$ for any edge e that is not incident with vertices of F, that is, we can safely subtract 1 from these edges. Thus, the new edge-magic labeling L' has magic invariant $< n+1$. The induction would be similar to the Case 1, where can decompose L' into completely fundamental edge-magic $(0,1)$-total labelings.

In both cases, take L'' to be a labeling on G with 1 on the vertices and edges whose labels were reduced, and 0 on other vertices and edges. Therefore, $L = L' + L''$. Based on the induction step, L is a linear combination of completely fundamental edge-magic $(0,1)$-total labelings. □

Theorem 4.2.2. *Let G be a bipartite graph. The number $M_G(k)$ of edge-magic $(0,k)$-total labelings is a polynomial in k with degree p.*

Proof. By (4.2) and [52, Theorem 4.5.11], the function

$$F(\mathbf{1}, z) \;=\; \sum_{k \ge 0} M_G(k)\, z^k$$

is a rational function with denominator

$$D(z) \;=\; \prod_{(L,k)\in \mathrm{CF}(\Phi)} \left(1 - z^k\right), \tag{4.3}$$

where $\mathrm{CF}(\Phi)$ is the set of completely fundamental elements of Φ. According to Lemma 4.2.1, every completely fundamental element of Φ has $k = 1$. Therefore,

$$\sum_{k\geq 0} M_G(k)\, z^k \;=\; \frac{h(z)}{(1-z)^a} \tag{4.4}$$

for some nonnegative integer a and some polynomial $h(z)$. Basic results on rational generating functions (see, e.g., [52]) imply that $M_G(k)$ is a polynomial in k. For the degree of this polynomial, we consider the system of equations coming from the definition of edge-magic $(0, k)$-total labelings. More specifically, we introduce a variable $0 \leq x_e \leq k$ for each edge $e \in E$ and a variable $0 \leq x_v \leq k$ for each $v \in V$. By definition, there is a homogeneous system of equations. The matrix of coefficients, $[\, I_{q\times q} \,|\, A_{q\times p} \,]$ where $A_{q\times p}$ is an integer matrix, has rank q. Therefore, the degree of the polynomial $M_G(k)$ is $p + q - q$. □

We now consider the edge-magic $(0, k)$-total labelings of a graph G geometrically. That is, every edge-magic $(0, k)$-total labeling with magic invariant c corresponds to an integer solution of the system of equations and inequalities

$$x_{e_i} + x_u + x_v \;= x_{e_{i+1}} + x_z + x_w \qquad \text{and} \qquad 0 \leq x_u, x_v, x_z, x_w, x_{e_i}, x_{e_{i+1}} \leq 1, \tag{4.5}$$

for every $1 \leq i \leq q - 1$, where $e_i = \{u, v\}$ and $e_{i+1} = \{z, w\}$ are in E. These equations describe a rational polytope $\mathcal{P}$. Similar to Section 3.2, an edge-magic $(0, k)$-total labelings of G is an integer lattice point in the k-dilation of $\mathcal{P}$, i.e.,

$$M_G(k) = E_{\mathcal{P}}(k).$$

Let $M_G^\circ(k)$ be the number of edge-magic $(1, k)$-total labelings on G. Thus $M_G^\circ(k) = E_{\mathcal{P}^\circ}(k + 1)$, where $\mathcal{P}^\circ$ is the relative interior of the polytope $\mathcal{P}$. By Ehrhart's theorem, $E_{\mathcal{P}}(t)$ is a quasi-polynomial in t of degree $\dim \mathcal{P}$ [5]. Also, the Ehrhart–Macdonald reciprocity theorem for rational polytopes [5] implies that:

Corollary 4.2.3. *Let G be a graph. Then $M_G^\circ(k) = \pm\, M_G(-k-1)$. In particular, If G is a bipartite graph $M_G(k)$ and $M_G^\circ(k)$ are both polynomials in k with the same degree.*

Let $T = (V, E)$ be a tree; so here $p = q + 1$. We remark that there is a bijection between the set of all super edge-magic $(1, p, p)$-total labelings and the set of all edge-magic $(1, p)$-total labelings of T. Namely, given a super edge-magic labeling $f : V \cup E \to \{1, \ldots, 2p\}$ with magic invariant c, define $g : V \cup E \to \mathbb{Z}_{\geq 0}$ by $g(v) = f(v)$ for all $v \in V$ and $g(e) = f(e) - p$ for all $e \in E$. The magic invariant of g is $g(u) + g(v) + g(e) = c - p$, then g is an edge-magic

$(1, p)$-total labeling.

Conversely, let $g : V \cup E \to \mathbb{Z}_{\geq 0}$ be an edge-magic $(1, p)$-total labeling with magic invariant c. For every $v \in V$ and $e \in E$ define $f(v) = g(v)$ and $f(e) = g(e) + p$. Then the magic invariant of f is $f(v) + f(u) + f(e) = c + p$. This proves that f is a super edge-magic $(1, p, p)$-labeling. Through this bijection we can extend our results for edge-magic labeling to super edge-magic labelings in trees.

Proof of Theorem 4.1.4. By Theorem 4.2.2, the number of edge-magic $(1, k)$-total labelings $M_T^\circ(k)$ is a polynomial of degree p which can have at most p integer roots. By definition $\mathcal{P} \subseteq [0, 1]^V$ and so $M_T^\circ(0) = L_{\mathcal{P}^\circ}(1) = 0$. Therefore one of the roots is 0. On the other hand, by definition $M_T^\circ(k) \leq M_T^\circ(k + 1)$ for $k \geq 0$ and consequently $M_T^\circ(p) > 0$. Since there is a bijection between the set of all super edge-magic $(1, p, p)$-total labelings and the set of all edge-magic $(1, p)$-total labelings, $M_T^\circ(p) > 0$ implies that T admits a super edge-magic $(1, p, p)$-total labeling. The second part of the statement follows similarly. □

Now let T_c be a tree with one extra edge that makes a unique even cycle, then $p = q$. We have the same bijection as for trees between the set for all super edge-magic $(1, p, p)$-total labelings and the set of all edge-magic $(1, p)$-total labelings. Moreover, the completely fundamental elements of the semigroup Φ have the same property as trees.

Lemma 4.2.4. *If T_c is a tree with one unique even cycle such that $p = q$, then every completely fundamental edge-magic $(0, k)$-total labeling of T_c has $k = 1$.*

Proof. This is similar to the proof of Lemma 4.2.1, since every even cycle is bipartite. □

Theorem 4.2.5. *Let T_c be a tree with one unique even cycle such that $p = q$. The number $M_{T_c}(k)$ of edge-magic $(0, k)$-total labelings of T_c is a polynomial in k with degree p.*

Proof. This is similar to the proof of Theorem 4.2.2. □

Theorem 4.2.5 implies that for a tree with one unique cycle T_c we have a chance of using the polynomial structure of $M_{T_c}(k)$ to say something about the edge-magic character of T_c, in a non-injective sense. Specifically, the proof of Theorem 4.1.5 is similar to the proof of Theorem 4.1.4.

Chapter 5

Bivariate Order and Chromatic Polynomials

5.1 Background

The chromatic polynomial was introduced by George David Birkhoff in 1912 [12], for planar graphs, in an attempt to prove the Four Color Theorem. The chromatic polynomial counts the number of vertex colorings of a graph using at most a given number of colors such that adjacent vertices receive different colors. Birkhoff was hoping that by studying roots of polynomials and applying tools from analysis and algebra, he can say something about the chromatic number, the minimum number of colors required to color vertices of a graph properly.

In 1932, the chromatic polynomial was generalized from planar graphs to general graphs by Hassler Whitney [61]. Later in 1967, Reed [43] studied the polynomials that can be chromatic polynomial of some graph and the concept of chromatically equivalent graphs, i.e., graphs that have the same chromatic polynomials, was introduced [43].

Tutte [57, 56] generalized the chromatic polynomial to the Tutte polynomial, which is a two variable function that contains information on how the graph is connected. By setting one of the variables equals to zero we get the chromatic polynomial or the flow polynomial. In 2003, Dohmen et.al. [20] proposed a different generalization and introduced a two-variable version of the classic chromatic polynomial by adding improper colors.

Let $G = (V, E)$ be a graph and $y \leq x$. A ***vertex*** (x, y)***-coloring*** of G is a map $\varphi : V \to [x]$. The map is called ***proper*** if for all $\{u, v\} \in E$ with $\varphi(u) \leq y$ and $\varphi(v) \leq y$ the relation $\varphi(u) \neq \varphi(v)$ holds. Consequently, adjacent vertices can be colored with the same color only if the color is $\geq y + 1$. The colors $\leq y$ are called *proper* and the colors $\geq y + 1$ are called *improper*. Dohmen et.al. [20] introduced the bivariate chromatic polynomial which is a generalization of classic chromatic polynomial into two variables by weakening the requirements and allowing improper colors.

By the following theorem, the function $P_G(x, y)$ that counts the number of proper vertex (x, y)-coloring of G is in fact a polynomial and thus is called the ***bivariate chromatic polynomial*** or ***generalized chromatic polynomial*** of G.

Theorem 5.1.1. *[20] Let G be a finite graph and $y \leq x$ positive integers. Then $P_G(x, y)$ is a polynomial in y and x with integral coefficients.*

Dohmen et.al. showed that $P_G(x, y)$ is closely related to Stanley's chromatic symmetric function and it generalizes the matching polynomial [20].

Let us go over some interesting properties of the two-variable chromatic polynomial. When the set of proper colors is empty, we get $P_G(x, 0) = x^p$, where $p = |V|$. Also, note that if the set of improper colors is empty, we get the one-variable chromatic polynomial, that is $P_G(y, y) = P_G(y)$. Therefore, the bivariate chromatic polynomial generalizes the classic chromatic polynomial. The following theorem shows that the $P_G(x, y)$ can be expressed in terms of bivariate chromatic polynomials of subgraphs of G:

Theorem 5.1.2. *[20] Let G be a graph and $y \leq x$. Then*

$$P_G(x, y) = \sum_{X \subseteq V} (x - y)^{|X|} P_{G-X}(x, y)$$

where $G - X$ is a subgraph of G obtained from G by removing all vertices of a subset $X \subseteq V$.

An ***independent vertex set*** of G is a subset of the vertices such that no two vertices in the subset represent an edge of G. Let a_i denote the number of independence vertex sets of cardinality $p - i$, where p is the number vertices of G. Then the ***independence polynomial*** of G is defined by $I_G(x) = \sum_{i=0}^{n} a_i x^i$. Using Theorem 5.1.2, we can show that the bivariate chromatic polynomial generalizes the independence polynomial:

Corollary 5.1.3. *[20]*

$$I_G(x) = P_G(x + 1, 1).$$

In particular, $P_G(2, 1)$ is the number of independent vertex sets of G.

The next theorem gives us a recursive formula via deletion-contraction for bivariate chromatic polynomial.

Theorem 5.1.4. *[20] Let e be an edge in G, and let v be the vertex to which e contracts in G/e. Then*

$$P_G(x, y) = P_{G-e}(x, y) - P_{G/e}(x, y) + xP_{G/e-v}(x, y).$$

Note that the bivariate chromatic polynomial is not an evaluation of Tutte polynomial [57, 56]. Namely, the Tutte polynomial for every tree with n vertices is $T_G(x, y) = x^{n-1}$. However, the bivariate chromatic polynomial of a path with 4 vertices is $P_{T_4}(x, y) = x^4 - 3x^2y + y^2 + 2xy - y$. For more details and properties of bivariate chromatic polynomials, we refer to [20].

In this chapter, we propose generalizations of two fundamental results on classic chromatic polynomial to bivariate chromatic polynomial. The first one is the Chromatic Polynomial Decomposition Theorem (Theorem 2.3.4) which connecting the graph coloring to posets.

The second result gives a combinatorial interpretation to the evaluation of chromatic polynomial at negative integers, which involves acyclic orientations of the graph defined in Section 2.3. Let α be an acyclic orientation and $\varphi : V \to \{1, \ldots, x\}$ be a (not necessarily proper) vertex x-coloring of G. φ is said to be ***compatible*** with α if $u \to v$ in the orientaion α implies $\varphi(u) \geq \varphi(v)$.

Theorem 5.1.5 (**Chromatic Polynomial Reciprocity Theorem** [50])**.** *Let G be a graph with p vertices. Then $(-1)^p P_G(-x)$ equals the number of pairs (α, φ) consisting of an acyclic orientation α of G and a compatible x-coloring φ. In particular, $(-1)^p P_G(-1)$ equals the number of acyclic orientations of G.*

In section 5.2, we introduce bicolored posets and we generalize order polynomials to two variables. Also, we will find a direct formula for computing bivariate order polynomials (which is, in fact, a generalization of Theorem 2.3.2). In section 5.3, we use our results for bivariate order polynomials to generalize the Chromatic Polynomial Decomposition Theorem (Theorem 2.3.4). Lastly, in section 5.4, we generalize the Chromatic Polynomial Reciprocity Theorem (Theorem 5.1.5).

5.2 Bivariate Order Polynomials

We begin this section by introducing bicolored posets. Bivariate order polynomials will be defined on bicolored posets.

Definition 5.2.1. The triple $(P, C, \preceq)$ is called a ***bicolored poset*** if $(P, \preceq)$ is a poset and P is a disjoint union of C and S, where we think of C as the set of *celeste* elements and S as the set of *silver* elements. A ***bicolored chain*** is a bicolored poset that is a chain. We will simply use the notation P for bicolored poset $(P, C, \preceq)$.

Definition 5.2.2. Let P be a bicolored poset and $y \leq x$ be positive integers. A map $\varphi : P \to [x]$ is a ***(weakly) order preserving*** (x, y)***-map*** if for all the celeste elements $c \in C$, the relation $\varphi(c) \geq y$ holds and for all $a, b \in P$,

$$a \preceq b \qquad \longrightarrow \qquad \varphi(a) \leq \varphi(b).$$

Similarly, a map φ is an ***strictly order preserving*** (x, y)***-map*** if for all the celeste elements $c \in C$, the relation $\varphi(c) > y$ holds and for $a, b \in P$,

$$a \prec b \qquad \longrightarrow \qquad \varphi(a) < \varphi(b).$$

The functions $\Omega_P(x, y)$ and $\Omega_P^\circ(x, y)$ count the number of order preserving (x, y)-maps and strictly order preserving (x, y)-maps, respectively. If $y = x$, then $\Omega_P(x, y)$ and $\Omega_P^\circ(x, y)$ are the usual order polynomials defined in Section 2.3.

Example 5.2.3. Consider the bicolored chain $P : \{a_1, a_2\}$, $a_1 \preceq a_2$, where a_2 is the celeste element. We would like to find all possible strictly order preserving (x, y)-maps $\varphi : P \to [x]$ such that $\varphi(a_2) > y$. The first case is

$$1 \leq \varphi(a_1) \leq y < \varphi(a_2) \leq x.$$

This gives us y choices for $\varphi(a_1)$ and $x - y$ choices for $\varphi(a_2)$. The second case is

$$1 \leq y < \varphi(a_1) < \varphi(a_2) \leq x$$

which gives us $\binom{x-y}{2}$ choices for $\varphi(a_1)$ and $\varphi(a_2)$. Therefore the bivariate order polynomial of P is $\Omega_P^\circ(x, y) = y(x - y) + \binom{x-y}{2} = \frac{1}{2}(x^2 - x - y^2 - y)$ which is indeed a polynomial.

Like for one-variable order polynomials, we want to show that the bivariate order polynomials, $\Omega_P(x, y)$ and $\Omega_P^\circ(x, y)$, are in fact polynomials. We will prove this in three steps:

1. we show polynomiality of bivariate order functions for a bicolored chain (Theorem 5.2.5),
2. we prove polynomiality of bivariate order polynomials for linear extensions of a bicolored poset (Theorem 5.2.8),
3. and finally, we show that the bivariate order polynomials of a bicolored poset can be decomposed into that of linear extensions (Theorem 5.2.10).

As a result of each of these steps, we will discover a reciprocity between bivariate order polynomials $\Omega_P(x, y)$ and $\Omega_P^\circ(x, y)$. We start with a practical lemma which we will be using in our proofs later.

Lemma 5.2.4. *[52] Consider the sets $A = \{x \in \mathbb{Z}^d : a \leq x_1 < \cdots < x_d \leq b\}$ and $B = \{x \in \mathbb{Z}^d : a \leq x_1 \leq \cdots \leq x_d \leq b\}$. The cardinality of these sets are $\binom{b-a+1}{d}$ and $\binom{b-a+d}{d}$, respectively.*

Theorem 5.2.5. *If $P = \{a_1 < \cdots < a_k < a_{k+1} < \cdots < a_n\}$ is a bicolored chain of length n and $a_{k+1} \in C$ is the minimal celeste element in P, then*

$$\Omega_P^\circ(x, y) = \sum_{i=0}^{k} \binom{y}{i} \binom{x-y}{n-i}$$

and

$$\Omega_P(x,y) = \sum_{i=0}^{k} \binom{y-2+i}{i}\binom{x-y+n-i}{n-i}.$$

Proof. First, we count the number of strictly order preserving maps $\varphi : P \to [x]$ with $\varphi(a_{k+1}) > y$. This gives us a set of inequalities for φ with $k+1$ cases for which the number of possible maps follow by Lemma 5.2.4. The first case is

$$1 \le \varphi(a_1) < \cdots < \varphi(a_k) \le y < \varphi(a_{k+1}) < \cdots < \varphi(a_n) \le x.$$

There are $\binom{y}{k}\binom{x-y}{n-k}$ possible maps φ. For the second case

$$1 \le \varphi(a_1) < \cdots < \varphi(a_{k-1}) \le y < \varphi(a_k) < \cdots < \varphi(a_n) \le x,$$

there are $\binom{y}{k-1}\binom{x-y}{n-k+1}$ possible maps φ. We repeat this process as above. In the last case

$$1 \le y < \varphi(a_1) < \cdots < \varphi(a_n) \le x,$$

there are $\binom{y}{0}\binom{x-y}{n}$ possible maps φ. The result will follow by adding up all the possible maps.

Now we count the number of weakly order preserving maps $\varphi : P \to [x]$ with $\varphi(a_{k+1}) \ge y$. There are $k+1$ cases. To avoid over counting, we set one inequality strict. For the first case,

$$1 \le \varphi(a_1) \le \cdots \le \varphi(a_k) < y \le \varphi(a_{k+1}) \le \cdots \le \varphi(a_n) \le x,$$

there are $\binom{y-1-1+k}{k}\binom{x-y+n-k}{n-k}$ possible maps. For the second case,

$$1 \le \varphi(a_1) \le \cdots \le \varphi(a_{k-1}) < y \le \varphi(a_k) \le \cdots \le \varphi(a_n) \le x,$$

There are $\binom{y-1-1+k-1}{k-1}\binom{x-y+n-k+1}{n-k+1}$ possible maps. We proceed until the last case

$$1 \le y \le \varphi(a_1) \le \cdots \le \varphi(a_n) \le x,$$

with $\binom{y-1+0}{0}\binom{x-y+n}{n}$ possible maps φ. The result will follow by adding up all the possible maps. $\square$

The following corollary gives us reciprocity between $\Omega_P^\circ(x,y)$ and $\Omega_P(x,y)$ for bicolored chains which will be generalized to bicolored posets in Theorem 5.2.12.

Corollary 5.2.6. *Let $P = \{a_1 < \cdots < a_k < a_{k+1} < \cdots < a_n\}$ be a bicolored chain of length n where $a_{k+1} \in C$ is the minimal celeste element in P. Then*

$$\Omega_P^\circ(-x,-y) = (-1)^n\, \Omega_P(x, y+1).$$

Proof. We will be using the identity $(-1)^d\binom{-n}{d} = \binom{n+d-1}{d}$. By Theorem 5.2.5, we have

$$\begin{aligned}\Omega^{\circ}_P(-x,-y) &= \sum_{i=0}^{k}\binom{-y}{i}\binom{-x+y}{n-i} \\ &= \sum_{i=0}^{k}(-1)^i\binom{y+i-1}{i}(-1)^{n-i}\binom{x-y+n-i-1}{n-i} \\ &= (-1)^n\Omega_P(x,y+1).\end{aligned}$$

□

The natural and reverse labelings and also linear extensions on the bicolored posets will be defined similar to those of posets in Section 2.3. For any given linear extension $L = \{a_1 < a_2 < \ldots < a_n\}$ on the bicolored poset P and the labeling $\omega : P \to [n]$, we associate the word $\omega_L = \omega_1 \cdots \omega_n$ where $\omega_i = \omega(a_i)$. An order preserving (x,y)-map φ of L is of ***type-ω_L*** if for $i \in \mathrm{Asc}(\omega_L)$ we have $\varphi(a_i) \le \varphi(a_{i+1})$, and for $i \in \mathrm{Des}(\omega_L)$ we have $\varphi(a_i) < \varphi(a_{i+1})$.

Let $\Omega_{\omega_L}(x,y)$ count the number of type-ω_L order preserving (x,y)-maps where $\varphi(c) \ge y$ for the minimal celeste element $c \in L$. Similarly, $\Omega^{\circ}_{\omega_L}(x,y)$ counts the number of type-ω_L strictly order preserving (x,y)-maps where $\varphi(c) > y$.

Example 5.2.7. Consider the poset on P and labeling ω in the Figure 5.3. For the linear extension L_1, L_2, and L_3 the associated words are $\omega_{L_1} = 12345$, $\omega_{L_2} = 12435$, and $\omega_{L_3} = 14235$, respectively.

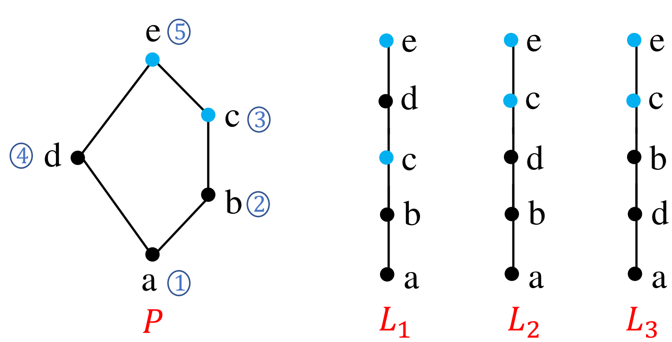

Figure 5.1: A bicolored poset P with labeling ω and its linear extensions L_1, L_2, and L_3.

We will compute a type-ω_{L_3} order preserving (x,y)-map φ and the rest can be computed similarly. The word $\omega_{L_3} = 14235$ has a descent at $i = 2$. Thus, a type-14235 order preserving (x,y)-map satisfies the set of inequalities

$$1 \le \varphi(a) \le \varphi(d) < \varphi(b) \le \varphi(c) \le \varphi(e) \le x \tag{5.1}$$

and $\varphi(c), \varphi(e) > y$. We are able to remove equalities by defining a bijection $\varphi \to \bar{\varphi}$ as following:

$$\begin{aligned}
\bar{\varphi}(a) &= \varphi(a),\\
\bar{\varphi}(d) &= \varphi(d) + \operatorname{asc}(14) = \varphi(d) + 1,\\
\bar{\varphi}(b) &= \varphi(b) + \operatorname{asc}(142) = \varphi(b) + 1,\\
\bar{\varphi}(c) &= \varphi(c) + \operatorname{asc}(1423) = \varphi(c) + 2,\\
\bar{\varphi}(e) &= \varphi(e) + \operatorname{asc}(14235) = \varphi(e) + 3.
\end{aligned}$$

Therefore, (5.1) becomes

$$1 \leq \bar{\varphi}(a) < \bar{\varphi}(d) < \bar{\varphi}(b) < \bar{\varphi}(c) < \bar{\varphi}(e) \leq x + 3.$$

By definition $\varphi(c) > y$, so $\bar{\varphi}(c) > y + 2$. By Theorem 5.2.5 with $x - 3$ and $y - 2$,

$$\Omega^{\circ}_{L_3}(x, y) = \sum_{i=0}^{2} \binom{y-2}{i} \binom{x-3-y+2}{5-i}.$$

The proof of the next theorem is coming from the idea of this example.

Theorem 5.2.8. *Consider the linear extension $L = \{a_1 < \cdots < a_n\}$ of a bicolored poset P with the fixed labeling ω and the associated word $\omega_L = \omega(a_1) \cdots \omega(a_n)$. Let a_{k+1} be the minimal celeste element in L and let $\tilde{w}_L = \omega(a_1) \cdots \omega(a_{k+1})$. Then*

$$\Omega^{\circ}_{\omega_L}(x, y) = \sum_{i=0}^{k} \binom{y - \operatorname{asc}(\tilde{\omega}_L)}{i} \binom{x - y + \operatorname{asc}(\tilde{\omega}_L) - \operatorname{asc}(\omega_L)}{n-i}$$

and

$$\Omega_{\omega_L}(x, y) = \sum_{i=0}^{k} \binom{y - \operatorname{des}(\tilde{\omega}_L) - 2 + i}{i} \binom{x - y + \operatorname{des}(\tilde{\omega}_L) - \operatorname{des}(\omega_L) + n - i}{n-i}.$$

Proof. We want to count the number of (x, y) order preserving maps $\varphi : L \to [x]$ of type-ω_L. The map φ gives the set of inequalities $\varphi(a_{k+1}) > y$, $\varphi(a_i) \leq \varphi(a_{i+1})$ if i is an ascent of ω_L, and $\varphi(a_i) < \varphi(a_{i+1})$ if i is a descent of ω_L. We eliminate weak inequalities by creating a bijection $\varphi \to \bar{\varphi}$ defined by $\bar{\varphi}(a_1) = \varphi(a_1)$ and

$$\bar{\varphi}(a_i) = \varphi(a_i) + \operatorname{asc}\left(\omega(a_1) \cdots \omega(a_j)\right) \quad \text{for } 2 \leq i \leq n.$$

If $i \in \operatorname{Des}(\omega_L)$, then $\operatorname{asc}\left(\omega(a_1) \cdots \omega(a_{i+1})\right) - \operatorname{asc}\left(\omega(a_1) \cdots \omega(a_i)\right) = 0$. Thus,

$$\varphi(a_i) < \varphi(a_{i+1}) \qquad \longrightarrow \qquad \bar{\varphi}(a_i) < \bar{\varphi}(a_{i+1}).$$

If $i \in \operatorname{Asc}(\omega_L)$, then $\operatorname{asc}\left(\omega(a_1) \cdots \omega(a_{i+1})\right) - \operatorname{asc}\left(\omega(a_1) \cdots \omega(a_i)\right) = 1$. Thus,

$$\varphi(a_i) \leq \varphi(a_{i+1}) \qquad \longrightarrow \qquad \bar{\varphi}(a_i) < \bar{\varphi}(a_{i+1}).$$

Hence, $\bar{\varphi}$ is a strictly increasing function with $\bar{\varphi}(a_{k+1}) > y + \mathrm{asc}(\tilde{\omega}_L)$ and

$$1 \leq \bar{\varphi}(a_1) < \cdots < \bar{\varphi}(a_n) \leq x + \mathrm{asc}(\omega_L).$$

Let $X = x + \mathrm{asc}(\omega_L)$ and $Y = y + \mathrm{asc}(\tilde{\omega}_L)$. Therefore, by Theorem 5.2.5 with $X - \mathrm{asc}(\omega_L)$ and $Y - \mathrm{asc}(\tilde{\omega}_L)$, the total number of type-ω_L order preserving (x, y)-maps follows.

The second part is similar to the first part except we need to replace all ascents by descents and again use Theorem 5.2.5. □

The ***inverse*** of $\omega = \omega_1 \cdots \omega_n$ is the word $\bar{\omega}$ defined by $\bar{\omega_j} \to \omega_{n+1-j}$. This changes all ascents to descents and vice versa in the original word ω. In particular, $\mathrm{asc}(\omega_L) = \mathrm{des}(\bar{\omega}_L)$ and $\mathrm{des}(\omega_L) = \mathrm{asc}(\bar{\omega}_L)$.

Corollary 5.2.9. *Consider the linear extension $L = \{a_1 < \cdots < a_n\}$ of a bicolored poset P with a fixed labeling ω and associated word $\omega_L = \omega(a_1) \cdots \omega(a_n)$. Then*

$$\Omega^{\circ}_{\omega_L}(-x, -y) = (-1)^n \; \Omega_{\bar{\omega}_L}(x, y + 1).$$

Proof. Let a_{k+1} be the minimal celeste element in L and let $\tilde{\omega}_L = \omega(a_1) \cdots \omega(a_{k+1})$. Since $\mathrm{asc}(\tilde{\omega}_L) = \mathrm{des}(\bar{\tilde{\omega}}_L)$, we see that

$$\begin{aligned}
\Omega^{\circ}_{\omega_L}(-x, -y) &= \sum_{i=0}^{k} \binom{-y + \mathrm{asc}(\tilde{\omega}_L)}{i} \binom{-x + y + \mathrm{asc}(\tilde{\omega}_L) - \mathrm{asc}(\omega_L)}{n - i} \\
&= \sum_{i=0}^{k} (-1)^n \binom{y - \mathrm{asc}(\tilde{\omega}_L) + i - 1}{i} \binom{x - y - \mathrm{asc}(\tilde{\omega}_L) + \mathrm{asc}(\omega_L) + n - i - 1}{n - i} \\
&= (-1)^n \sum_{i=0}^{k} \binom{y - \mathrm{des}(\bar{\tilde{\omega}}_L) + i - 1}{i} \binom{x - y - 1 - \mathrm{des}(\bar{\tilde{\omega}}_L) + \mathrm{des}(\bar{\omega}_L) + n - i}{n - i} \\
&= (-1)^n \, \Omega_{\bar{\omega}_L}(x, y + 1).
\end{aligned}$$

□

The following theorem shows that we can decompose bivariate order functions of a poset into those of its linear extensions.

Theorem 5.2.10. *Let P be a bicolored poset with a fixed reverse natural labeling ω. Then*

$$\Omega^{\circ}_P(x, y) = \sum_L \Omega^{\circ}_{\omega_L}(x, y).$$

If τ is a fixed natural labeling, then

$$\Omega_P(x, y) = \sum_L \Omega_{\tau_L}(x, y)$$

where both sums are over all possible linear extensions L of P.

Proof. We will show that there is a bijection between the set of strictly order preserving (x, y)-maps φ and the set of all the pairs of $(L, \hat{\varphi})$, where L is a linear extension and $\hat{\varphi}$ is a type-ω_L map on the bicolored poset P. Suppose φ is a strictly order preserving (x, y)-map. We will build a linear extension L as following. Let $a_i, a_j \in P$.

- If $\varphi(a_i) < \varphi(a_j)$, then let $a_i \prec a_j$ in L. Note that $\omega(a_i) > \omega(a_j)$ due to the natural reverse labeling.
- If $\varphi(a_i) = \varphi(a_j)$ with $\omega(a_i) < \omega(a_j)$, then let $a_i \prec a_j$ in L.
- If $\varphi(a_i) = \varphi(a_j)$ with $\omega(a_i) > \omega(a_j)$, then let $a_i \succ a_j$ in L.

This gives us the linear extension L with associated word ω_L. By construction, our strictly order preserving map φ is of type-ω_L.

Conversely, let L be a linear extension of P and let $\hat{\varphi}$ be a type-ω_L map. Type-ω_L maps agree with strictly order preserving maps between comparable elements in a poset by definition. The map $\hat{\varphi}$ may give us weak or strict inequalities between incomparable elements in P. Strictly order preserving maps place no restrictions on what happens between incomparable elements in the poset. Thus, $\hat{\varphi}$ is also a strictly order preserving map on P. Let L_1 and L_2 be two different linear extensions of P. Then there must be at least one pair $a_i, a_j \in P$ where $a_i \prec a_j$ in L_1 and $a_i \succ a_j$ in L_2. This means we have an ascent in one of the associated words from the linear extensions and a descent in the same spot in the other word. Hence, the type-ω_L maps are distinct.

The proof of the second part is similar to that of the first part with one major difference. Here we have $\tau(a_i) < \tau(a_j)$ when $a_i \prec a_j$ in the linear extension L due to the natural labeling. □

Based on Theorem 5.2.10, we can write the bivariate order polynomial of a bicolored poset as the combination of bivariate order polynomials of its linear extensions. Since by Theorem 5.2.8, the counting functions $\Omega_{\omega_L}(x, y)$ and $\Omega^{\circ}_{\omega_L}(x, y)$ are the sums of polynomials (product of binomial coefficients are polynomial), we have:

Theorem 5.2.11. *Let P be a bicolored poset. Then $\Omega_P(x, y)$ and $\Omega^{\circ}_P(x, y)$ are polynomials in x and y.*

Proof. Follows by Theorem 5.2.8 and 5.2.10. □

The following theorem is a generalization of the Order Polynomial Reciprocity Theorem (Theorem 2.3.3) to the bivariate version.

Theorem 5.2.12 (Bivariate Order Polynomial Reciprocity Theorem). *Let P be a bicolored poset with n elements. Then*

$$\Omega^{\circ}_P(-x, -y) = (-1)^n \; \Omega_P\,(x, y + 1).$$

Proof. Fix a natural reverse labeling ω of P. Then $\bar{\omega}$ is a natural labeling. Using Theorem 5.2.10 and Corollary 5.2.9,

$$\begin{aligned}\Omega_P^\circ(-x,-y) &= \sum_L \Omega_{\omega_L}^\circ(-x,-y)\\ &= (-1)^n \sum_L \Omega_{\bar{\omega}_L}(x,y+1)\\ &= (-1)^n \ \Omega_P\,(x,y+1).\end{aligned}$$

□

5.3 Bivariate Chromatic Polynomials

One of our goals is to find a relation between bivariate chromatic polynomials and order polynomials. More specifically, we would like to generalize the Chromatic Polynomial Decomposition Theorem (Theorem 5.1.5) to a bivariate version.

Given a graph G and a vertex (x, y)-coloring $\varphi : V \to [x]$ with $y \leq x$, let H be the graph obtaining from contracting all edges that are adjacent with vertices of the same colors $\geq y+1$. For any edge $e = (u, v)$ in H, we orient $u \to v$ if $\varphi(u) > \varphi(v)$. Therefore, φ induces an acyclic orientation σ on H. As described in Section 2.3, σ gives rise to a poset P. We color the contracted vertices of H celeste and the others silver. Therefore, P is a bicolored poset. Then, we compute bivariate order polynomial of P. Let us note that if we do not contract any vertices, then we have the one variable case of the chromatic polynomials. Therefore, we have the following theorem:

Theorem 5.3.1 (**Bivariate Chromatic Polynomial Decomposition Theorem**). *Let G be a finite graph and $y \leq x$ be positive integers. Let $C(H)$ be the set of celeste elements that come from the contracted vertices of H. Then,*

$$P_G(x,y) = \sum_{\substack{\text{flat}\\ H \text{ of } G}} \ \sum_{\substack{\text{acyclic}\\ \text{orientation}\\ \sigma \text{ of } H}} \Omega^\circ_{P(\sigma),\,C(H)}(x,y)$$

where $P(\sigma)$ is the poset corresponding to the acyclic orientation σ of H and $\Omega^\circ_{P(\sigma),C(H)}(x,y)$ counts the number of order preserving (x,y)-map of $P(\sigma)$ in which for all $c \in C(H)$, the relation $\varphi(c) > y$ holds.

Proof. Let φ be a proper (x, y)-coloring of G. Let $H = (V(H), E(H))$ be the graph obtained by contracting every edge whose endpoints have the same color. The coloring φ induces an acyclic orientation σ of H as following. For every $\{u, v\} \in E(H)$, if $\varphi(u) < \varphi(v)$ then we orient $v \to u$. We regard σ as a poset P (as we described in Section 2.3) and the contracted vertices of H as the celeste elements $C(H)$ of P. Therefore P is a bicolored poset and the

(x, y)-coloring φ can be thought as an order preserving (x, y)-map on P such that for all $c \in C(H)$, the relation $\varphi(c) > y$ holds.

Conversely, let H be a contracted subgraph of G with acyclic orientation σ and a strictly order preserving map φ of H. The strict order preserving (x, y)-map φ has an order on H and so the vertices of H gets different colors. We color the vertices of G corresponding to the contracted edges by the same color. Therefore, this coloring of G will become a generalized proper coloring of G. Thus, the result follows. $\square$

Example below illustrates the proof of Theorem 5.3.1.

Example 5.3.2. Consider the complete graph K_3 and decompose it into possible contractions. We denote v_{ij} as the resulting vertex after contracting the edge $v_i v_j$. We color the contracted vertices celeste and the others silver, Figure 5.3.2.

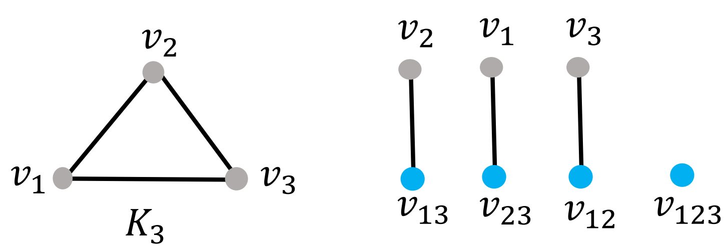

Figure 5.2: Contractions of K_3.

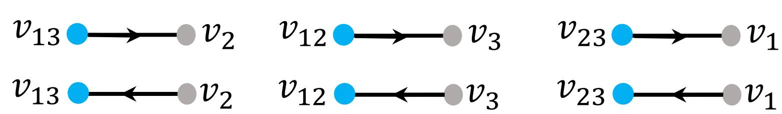

Figure 5.3: Acyclic orientations of contractions of K_3.

Then we find all the possible acyclic orientations of the contracted graphs, Figure 5.3.2. We regard each acyclic orientation as a bicolored poset. Then we compute the bivariate order polynomial of each poset as following:

No contractions: we have the one variable case, $P_{K_3}(x) = x^3 - 3x^2 + 2x$.
Contract one edge: we have two acyclic orientations that lead to two posets, P_1 and P_2.

$$\Omega^{\circ}_{P_1, v_{12}}(x, y) = \binom{x-y}{2} + (x-y)y \qquad \Omega^{\circ}_{P_2, v_{12}}(x, y) = \binom{x-y}{2}$$

Contract two edges: $\Omega^{\circ}_{P,v_{123}}(x,y) = x - y$.
Thus,

$$P_{K_3}(x,y) = P_{K_3}(x) + 3\Omega^{\circ}_{P_1,v_{12}}(x,y) + 3\Omega^{\circ}_{P_2,v_{12}}(x,y) + \Omega^{\circ}_{P,v_{123}}(x,y) = x^3 - 3xy + y.$$

5.4 Bivariate Graph Coloring Reciprocity

In this section, we will prove a generalization of the reciprocity theorem for chromatic polynomials (Theorem 5.1.5) to bivariate chromatic polynomials.

We recall that for a given acyclic orientation σ on a graph G and a map $\varphi : V \to [x]$, we say that σ is ***compatible*** with φ if $u \to v$ in the orientation σ implies $\varphi(u) \geq \varphi(v)$.

Theorem 5.4.1 (**Bivariate Chromatic Polynomial Reciprocity Theorem**). *Let $G = (V, E)$ be a graph. Then*

$$P_G(-x,-y) = \sum_{\substack{\text{flat} \\ H \text{ of } G}} (-1)^{|V(H)|}\, M_H(\sigma,\varphi),$$

where $M_H(\sigma,\varphi)$ is the number of pairs (σ,φ) consisting of an acyclic orientation σ of H and a proper coloring $\varphi : V(H) \to [x]$ compatible to σ, where $V(H)$ is the vertex set of H.

Proof. By Theorem 5.3.1,

$$P_G(-x,-y) = \sum_{\substack{\text{flat} \\ H \text{ of } G}} \sum_{\substack{\text{acyclic} \\ \text{orientation} \\ \sigma \text{ of } H}} \Omega^{\circ}_{P(\sigma),\, C(H)}(-x,-y).$$

Now by the Bivariate Order Polynomial Reciprocity Theorem (Theorem 5.2.12),

$$P_G(-x,-y) = \sum_{\substack{\text{flat} \\ H \text{ of } G}} \sum_{\substack{\text{acyclic} \\ \text{orientation} \\ \sigma \text{ of } H}} (-1)^{|V(H)|}\Omega_{P(\sigma),\, C(H)}(x,y+1),$$

where $\Omega_{P(\sigma),\, C(H)}(x,y+1)$ counts the number of order preserving maps $\varphi : P(\sigma) \to [x]$, subject to two conditions:

1. For any celeste element $c \in C(H)$, the relation $\varphi(c) \geq y+1$ holds,
2. φ is compatible with the acyclic orientation σ.

The second condition is because $P(\sigma)$ is the poset corresponding to the acyclic orientation σ of H. Let

$$M_H(\sigma,\varphi) := \sum_{\substack{\text{acyclic} \\ \text{orientation} \\ \sigma \text{ of } H}} \Omega_{P(\sigma),C(H)}(x,y+1).$$

Thus, $M_H(\sigma, \varphi)$ counts the number of pairs (σ, φ) where σ is an acyclic orientation of H and $\varphi : V(H) \to [x]$ is a proper coloring compatible to σ such that for $c \in C(H)$, the relation $\varphi(c) \geq y + 1$ holds. $\square$

Let us note that when there are no improper colors, we do not need to contract any edges of the graph. Therefore, the one-variable Chromatic Polynomial Reciprocity Theorem (Theorem 5.1.5) follows as a special case.

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