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By

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Abstract

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In this thesis we explore two interesting relationships between string theory and quantum field theory.

Firstly, we develop an equivalence between two Hilbert spaces: (i) the space of states of $U(1)^n$ Chern-Simons theory with a certain class of tridiagonal matrices of coupling constants (with corners) on T^2; and (ii) the space of ground states of strings on an associated mapping torus with T^2 fiber. The equivalence is deduced by studying the space of ground states of $\mathrm{SL}(2,\mathbb{Z})$-twisted circle compactifications of $U(1)$ gauge theory, connected with a Janus configuration, and further compactified on T^2. The equality of dimensions of the two Hilbert spaces (i) and (ii) is equivalent to a known identity on determinants of tridiagonal matrices with corners. The equivalence of operator algebras acting on the two Hilbert spaces follows from a relation between the Smith normal form of the Chern-Simons coupling constant matrix and the isometry group of the mapping torus, as well as the torsion part of its first homology group.

Secondly, the Fractional Quantum Hall Effect appears as part of the low-energy description of the Coulomb branch of the A_1 $(2,0)$-theory formulated on $(S^1 \times \mathbb{R}^2)/\mathbb{Z}_k$, where the generator of $\mathbb{Z}_k$ acts as a combination of translation on S^1 and rotation by $2\pi/k$ on $\mathbb{R}^2$. At low-energy the configuration is described in terms of a 4+1D Super-Yang-Mills theory on a cone $(\mathbb{R}^2/\mathbb{Z}_k)$ with additional 2+1D degrees of freedom at the tip of the cone. Fractionally charged quasi-particles have a natural description in terms of BPS strings of the $(2,0)$-theory. We analyze the large k limit, where a smooth cigar-geometry provides an alternative description. In this framework a W-boson can be modeled as a bound state of k quasi-particles. The W-boson becomes a Q-ball, and it can be described by a soliton solution of BPS monopole equations on a certain auxiliary curved space. We show that axisymmetric solutions of these equations correspond to singular maps from AdS_3 to AdS_2, and we present some numerical results.

Dedicated to my Mother and Christina.

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Chapter 1

Introduction

Here we give a brief introduction to the main concepts used in this thesis.

1.1 Relativistic Quantum Mechanics

In order to describe the high energy behavior of the fundamental particles that make up the energy and matter in the universe, one must construct mathematical models for them that incorporate both special relativity and quantum mechanics.

In classical mechanics the equations of motion that govern the behavior of particles making up a system can be found by extremizing the action

$$S = \int dt\ L(x_i, \dot{x}_i, t),$$

which yields the Euler-Lagrange equations for the particle coordinates, $x_i(t)$.

In quantum mechanics a particle's position and momentum no longer commute but satisfy the canonical commutation relations

$$[x(t)_i, p(t)_j] = i\delta_{ij} \qquad p_i = \frac{\partial L}{\partial \dot{x}_i}.$$

The behavior of x_i and p_i can be found by making the Legendre transformation to the Hamiltonian picture

$$H(x_i, p_i, t) \equiv \sum_i p_i \dot{x}_i - L(x_i, \dot{x}_i, t)$$

and using the Heisenberg equations of motion, or the Feynman Path Integral. However, we see that time and space are on unequal footing since time is simply a parameter, while space is an operator, which makes it difficult to incorporate special relativity. There are two ways of alleviating this problem:

1. **Quantum Field Theory (QFT)**

   Demote space to a parameter and quantize fields $\phi(\mathbf{x}, t)$, governed by the action

   $$S = \int d^4x \ \mathcal{L}(\phi(\mathbf{x}, t), \partial_\mu \phi(\mathbf{x}, t), x^\mu), \qquad \pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}},$$

   by imposing the commutation relations

   $$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}).$$

   The Fourier modes of the field in momentum space can be interpreted as particle creation and annihilation operators that create one-particle momentum eigenstates.

   The advantage of this method is that it is fairly straightforward to deal with processes that do not conserve particle number. QFT has been successfully used to describe the relativistic theory of the strong, weak and electromagnetic interactions, but runs into major difficulties in attempts to describe quantum gravity.

2. **String Theory**

   Promote time to an operator. Now we have four operators, $X^\mu(\tau)$, labeling the particle's trajectory in space-time, parameterized by a coordinate τ. Our system is governed by an action for the one dimensional fields, $X^\mu(\tau)$,

   $$S = \int d\tau \ \mathcal{L}(X^\mu, \dot{X}^\mu, \tau).$$

   For example, the simplest Poincare-invariant action that does not depend on the parameter τ is proportional to the length of the world-line traced out by the $X^\mu(\tau)$. The Lagrangian is

   $$\mathcal{L} \sim \sqrt{-\dot{X}^\mu \dot{X}_\mu},$$

   and we can quantize this system by imposing the commutation relations

   $$[X^\mu(\tau), P^\nu(\tau)] = i\eta^{\mu\nu}, \qquad P^\nu = \frac{\partial L}{\partial \dot{X}^\nu}.$$

   The main difference between this approach and the previous one is that now the Fourier modes of our one dimensional space-time fields are operators that create and destroy single particle modes rather than creating and destroying particles, which makes it more difficult to deal with processes that do not conserve particle number.

   The advantage of this method is that it is relatively straightforward to write down a relativistic quantum theory of extended objects. We can quantize one-dimensional strings by using two dimensional fields, $X^\mu(\tau, \sigma)$, hence the name string theory. We can also quantize higher dimensional objects, known as p-branes, through the use of higher dimensional fields that are functions p spacial dimensions and one time dimension.

While all of the forces other than gravity can be straightforwardly quantized using QFT, the graviton appears in the spectrum of closed string states and this eventually allows for the construction of a consistent theory of quantum gravity. String theory also appears to have all of the necessary ingredients to describe a relativistic quantum theory of the strong weak and electromagnetic forces, but the exact nature of how this comes about is not yet known.

1.2 Chern-Simons Theory and $1/k$ Charged Particles

In this thesis we frequently refer to the 2+1D quantum field theory with action

$$S = \frac{k}{4\pi} \int A \wedge dA,$$

known as abelian Chern-Simons theory. A is one-form abelian gauge field, and k is an integer known as the Chern-Simons level.

$1/k$ charged particles arise in the Fractionally Charged Quantum Hall Effect in which electrons confined to a 2D spacial surface combine with flux quanta of the magnetic field to produce fractionally charged quasiparticles. The effective QFT description of this system in terms of a Chern-Simons theory gives us a simple way of seeing how this happens.

Let J^μ be the electromagnetic current of the 2+1D electrons. By charge conservation we have

$$\partial_\mu J^\mu = 0$$

Since the topology of flat 2+1D space is trivial we can write J^μ globally as the curl of a 3-vector

$$J^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$$

We notice immediately that the current is invariant under the gauge transformation $a_\mu \to a_\mu + \partial_\mu \Lambda$. The effective local Lagrangian description of this system is given to leading order by a Chern-Simons term:

$$\mathcal{L} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \cdots$$

The dots indicate the kinetic term for a as well as any other terms, however, the Chern-Simons term is the only gauge invariant, dimension 3 or less, local operator that we can write down, and thus the only relevant one at long distances. Let us now couple our electron system to an external electromagnetic field via the interaction $J^\mu A_\mu$. If we substitute our formula for J^μ as a function of a and integrate by parts (dropping a surface term) our Lagrangian becomes

$$\mathcal{L} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda$$

The quasiparticles are defined as the entities that couple to the gauge potential through the interaction $a_\mu j^\mu$, where j^μ is the quasiparticle current. Including quasiparticles our Lagrangian becomes

$$\mathcal{L} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + a_\mu j^\mu - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu A_\lambda$$

Now we make the change of variables $\tilde{j}^\mu = j^\mu - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$, use Loretz gauge for a^μ, and integrate out a^μ to get

$$\mathcal{L} = \frac{\pi}{k}\tilde{j}^\mu\left(\frac{\epsilon_{\mu\nu\lambda}\partial^\nu}{\partial^2}\right)\tilde{j^\lambda} = \frac{1}{4\pi k}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{1}{k}A_\mu j^\mu + \frac{\pi}{k}j^\mu\left(\frac{\epsilon_{\mu\nu\lambda}\partial^\nu}{\partial^2}\right)j^\lambda$$

The second term tells us that our quasiparticles have electric charge $\frac{1}{k}$.

1.3 Supersymmetry and the (2,0) Theory

Supersymmetry is a symmetry between bosons and fermions. The canonical example of a supersymmetric quantum field theory is the Wess-Zumino model, whose action is given by

$$S = \int d^4x(-\partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi).$$

This action describes a massless complex scalar and a massless 2-component left-handed Weyl spinor. This action is invariant under the transformations

$$\delta\phi = \epsilon\psi \qquad \delta\phi^* = \epsilon^\dagger\phi^\dagger$$
$$\delta\psi_\alpha = i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu\phi \qquad \delta\psi^\dagger_{\dot{\alpha}} = -i(\epsilon\sigma^\mu)_{\dot{\alpha}}\partial_\mu\phi^*$$

where ϵ^α is a two component infinitesimal Weyl spinor. The massless fermion and massless complex scalar are said to form a lefthanded chiral supermultiplet and by Noether's theorem there are two conserved supercharges associated with this symmetry.

There are many other supersymmetric quantum field theories in various dimensions with more supercharges and more complicated supermultiplets. One of the most interesting is the (2,0) Theory, which is a six dimensional supersymmetric quantum field theory with 16 supercharges. The (2,0) Theory is quite mysterious at the quantum level because it contains a tensor multiplet that has 5 scalars, 4 left-handed Weyl spinors, and a two-form gauge field, B, with self-dual field strength $H = dB$; $H = *H$. It is difficult to quantize B using standard methods because it is very difficult to find a Lorentz covariant Lagrangian for B. The most natural Lagrangian,

$$H \wedge *H = H \wedge H = -H \wedge H = 0$$

in six dimensions. In 1996 Pasti, Sorokin and Tonin managed to write down a rather complicated Lorentz covariant Lagrangian [1], however, this only works for an abelian gauge field. There is no known way to quantize the non-abelian (2,0) Theory, and thus anything one can learn about it is of great interest.

1.4 Compactification

Lower dimensional theories can be constructed from higher dimensional theories by placing the higher dimensional theories on a space-time in which one or more of the spacial dimensions form a compact subspace. At energies that are too small to resolve the compact subspace, the higher dimensional theory behaves like a lower dimensional one.

The canonical example is that of a D+1 dimensional massless scalar field on a space in which one of the spacial dimensions is a circle of length L. The equation of motion of the field in D+1 dimensions is

$$\Box^2_{D+1}\phi(\mathbf{x}, y) = 0$$

where $\mathbf{x}$ labels the non-compact directions, and y the circular one. Since the field is now periodic in y we can write it as a Fourier series

$$\phi(\mathbf{x}, y) = \sum_n \phi_n(\mathbf{x}, y) e^{\frac{iny}{L}}.$$

The equation of motion is now

$$\Box^2_D \phi_n(\mathbf{x}) - \frac{n^2}{L^2}\phi_n = 0,$$

and we see that the D+1 dimensional theory of a single massless scalar field behaves like a D dimensional field theory of infinitely many massive scalar fields.

1.5 Superstring Theory and M-Theory

By combining the bosonic $X^\mu(\tau, \sigma)$ coordinates with fermionic coordinates, $\psi^\mu(\tau, \sigma)$, that are related to the bosonic ones by supersymmetry, one can construct a theory of superstrings. The latter is the simplest way of incorporating fermions into string theory. Superstring theories have string modes that can be grouped into various supersymmetry multiplets such as the gravity multiplet (which contains the graviton among other things) as well as tensor and/or vector multiplets that contain gauge bosons.

There are five distinct superstring theories that are consistent quantum theories of gravity in 10 space-time dimensions: type I, type IIA, type IIB, $E_8 \times E_8$ Heterotic and SO(32) Heterotic. Type I consists of unoriented open and closed superstrings, has 16 supercharges, and is consistent only when the gauge group is SO(32). The two Heterotic superstring theories consist of closed oriented superstrings, have 16 supercharges, and have gauge groups of either SO(32) or $E_8 \times E_8$, hence the names. The type II string theories are theories of oriented closed superstrings, with 32 supercharges, and they also contain oriented open superstrings that can end on p-branes called Dp-branes. Type IIA contains only stable Dp-branes of even dimensionality while IIB contains only stable Dp-branes of odd dimensionality. The ends of the open strings in the type II theories can be charged under various gauge groups, which provides a simple way of constructing non-abelian gauge theories from them.

These theories can all be related to one another through a series of dualities and are all believed to derive from a unique theory in 11 space-time dimensions known as M-theory. Unfortunately, M-theory does not contain fundamental strings, which are required for a perturbative description, hence very little is known about the full theory. M-theory does however contain M2 branes and M5 branes, and the field theory that describes the world volume of an M5 brane is the (2,0) Theory. Non-abelian forms of the (2,0) theory can be realized as the world volumes of configurations of M5 branes.

It is hoped that one day we will be able to determine how our four dimensional world arises from compactifications of M-theory and its lower dimensional descendants. In the meantime, however, one can use the relationships between the various superstring theories and M-theory to learn new things about the field theories that are associated with them.

1.6 Relationships between String and Field Theory

In this thesis we explore two interesting relationships between string theory and field theory. In chapter 2 we develop an equivalence between the space of states of $U(1)^n$ Chern-Simons theory with a certain class of tridiagonal matrices of coupling constants (with corners) on T^2 and the space of ground states of strings on an associated mapping torus with T^2 fiber. In chapter 3 the Fractional Quantum Hall Effect, which can be derived from a Chern-Simons theory, appears as part of the low-energy description of the Coulomb branch of the A_1 $(2,0)$-theory formulated on $(S^1 \times \mathbb{R}^2)/\mathbb{Z}_k$, which is derived from M-theory.

Chapter 2

Janus configurations with $\mathrm{SL}(2,\mathbb{Z})$-duality twists, Strings on Mapping Tori, and a Tridiagonal Determinant Formula

We develop an equivalence between two Hilbert spaces: (i) the space of states of $U(1)^n$ Chern-Simons theory with a certain class of tridiagonal matrices of coupling constants (with corners) on T^2; and (ii) the space of ground states of strings on an associated mapping torus with T^2 fiber. The equivalence is deduced by studying the space of ground states of $\mathrm{SL}(2,\mathbb{Z})$-twisted circle compactifications of $U(1)$ gauge theory, connected with a Janus configuration, and further compactified on T^2. The equality of dimensions of the two Hilbert spaces (i) and (ii) is equivalent to a known identity on determinants of tridiagonal matrices with corners. The equivalence of operator algebras acting on the two Hilbert spaces follows from a relation between the Smith normal form of the Chern-Simons coupling constant matrix and the isometry group of the mapping torus, as well as the torsion part of its first homology group.

2.1 Introduction and summary of results

Our goal is to develop tools for studying circle compactifications of $\mathcal{N}=4$ Super-Yang-Mills theory on S^1 with a general $\mathrm{SL}(2,\mathbb{Z})$-duality twist (also known as a "duality wall") inserted at a point on S^1. The low-energy limit of such compactifications encodes information about the operator that realizes the $\mathrm{SL}(2,\mathbb{Z})$-duality, and can potentially teach us new facts about S-duality itself. Some previous works on duality walls and related compactifications include [2]-[11].

In this paper we consider only the abelian gauge group $G=U(1)$, leaving the extension to nonabelian groups for a separate publication [12]. We focus on the Hilbert space of ground states of the system and study it in two equivalent ways: (i) directly in field theory; and (ii)

via a dual type-IIA string theory system (extending the techniques developed in [27]). As we will show, the equivalence of these two descriptions implies the equivalence of:

(i) the Hilbert space of ground states of $U(1)^n$ Chern-Simons theory with action

$$L = \tfrac{1}{4\pi}\sum_{i=1}^{n} \mathbf{k}_i A_i \wedge dA_i - \tfrac{1}{2\pi}\sum_{i=1}^{n-1} A_i \wedge dA_{i+1} - \tfrac{1}{2\pi} A_1 \wedge dA_n \,,$$

on T^2, and

(ii) the Hilbert space of ground states of strings of winding number $w = 1$ on a certain target space that contains the *mapping torus* with T^2 fiber:

$$\mathbf{M}_3 \equiv \frac{I \times T^2}{(0, v) \sim (1, f(v))} \,, \qquad (v \in T^2) \,,$$

where $I = [0, 1]$ is the unit interval, and f is a large diffeomorphism of T^2 corresponding to the $\mathrm{SL}(2, \mathbb{Z})$ matrix

$$W \equiv \begin{pmatrix} \mathbf{k}_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \mathbf{k}_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{k}_1 & -1 \\ 1 & 0 \end{pmatrix} . \tag{2.1}$$

We will explain the construction of these Hilbert spaces in detail below.

An immediate consequence of the proposed equivalence of Hilbert spaces (i) and (ii) is the identity

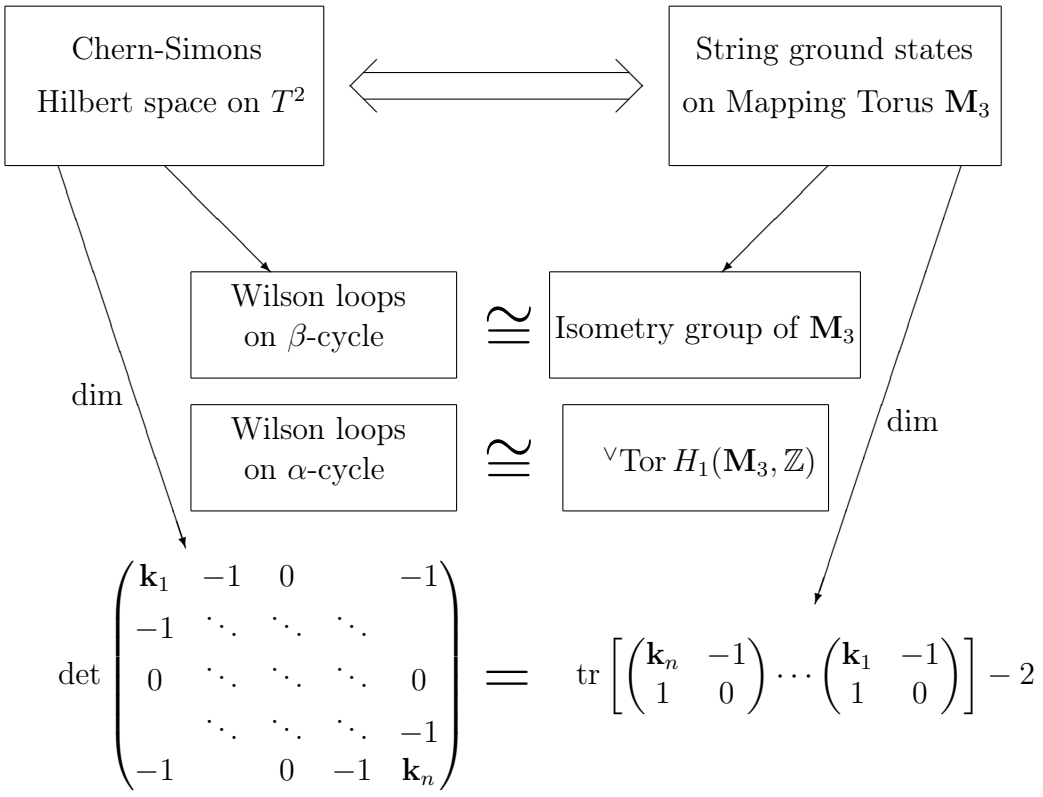
$$\det \begin{pmatrix} \mathbf{k}_1 & -1 & 0 & & -1 \\ -1 & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & -1 \\ -1 & & 0 & -1 & \mathbf{k}_n \end{pmatrix} = \mathrm{tr}\left[\begin{pmatrix} \mathbf{k}_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \mathbf{k}_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{k}_1 & -1 \\ 1 & 0 \end{pmatrix} \right] - 2 \,. \tag{2.2}$$

which follows from the equality of dimensions of the Hilbert spaces above. This is a known identity (see for instance [13]), and we will present a proof in Appendix A, for completeness.[1]

Moreover, equivalence of the operator algebras of the systems associated with (i) and (ii) allows us to make a stronger statement. The operator algebra of (i) is generated by Wilson loops along two fundamental cycles of T^2, and keeping only one of these cycles gives a maximal finite abelian subgroup. Let $\Lambda \subseteq \mathbb{Z}^n$ be the sublattice of $\mathbb{Z}^n$ generated by the columns of the Chern-Simons coupling constant matrix, which appears on the LHS of

[1] The continuum limit of (2.2) with $n \to \infty$ and $\mathbf{k}_i \to 2 + \frac{1}{n^2} V(\frac{i}{n})$ might be more familiar. It leads to a variant of the Gelfand-Yaglom theorem [14] with a periodic potential: $\det[-d^2/dx^2 + V(x)] = \mathrm{tr}\left[P \exp \oint \begin{pmatrix} \sqrt{V} + \frac{V'}{2V} & \frac{V'}{2V} \\ -\sqrt{V} & -\sqrt{V} \end{pmatrix} dx \right] - 2$ (up to a renormalization-dependent multiplicative constant).

(2.2). Then, the abelian group generated by the maximal commuting set of Wilson loops is isomorphic to $\mathbb{Z}^n/\Lambda$. The operator algebra of (ii), on the other hand, is constructed by combining the isometry group of $\mathbf{M}_3$ with the group of operators that measure the various components of string winding number in $\mathbf{M}_3$. The latter is captured algebraically by the Pontryagin dual ${}^\vee(\cdots)$ of the torsion part Tor of the homology group $H_1(\mathbf{M}_3, \mathbb{Z})$. (The terms will be explained in more detail in §2.4.3.) Thus, ${}^\vee\,\mathrm{Tor}\, H_1(\mathbf{M}_3, \mathbb{Z})$ as well as the isometry group are both equivalent to $\mathbb{Z}^n/\Lambda$. Together, ${}^\vee\,\mathrm{Tor}\, H_1(\mathbf{M}_3, \mathbb{Z})$ and $\mathrm{Isom}(\mathbf{M}_3)$ generate a noncommutative (but reducible) group that is equivalent to the operator algebra of the Wilson loops of the Chern-Simons system in (i). The subgroup ${}^\vee\,\mathrm{Tor}\, H_1(\mathbf{M}_3, \mathbb{Z})$ corresponds to the group generated by the Wilson loops along one fixed cycle of T^2 (let us call it "the α-cycle") and $\mathrm{Isom}(\mathbf{M}_3)$ corresponds to the group generated by the Wilson loops along another cycle (call it "the β-cycle"), where α and β generate $H_1(T^2, \mathbb{Z})$. The situation is summarized in the following diagram:



We will now present a detailed account of the statements made above. In §2.2 we construct the $\mathrm{SL}(2,\mathbb{Z})$-twist from the QFT perspective, and in §2.3 we take its low-energy limit and make connection with $U(1)^n$ Chern-Simons theory, leading to Hilbert space (i). In §2.4 we describe the dual construction of type-IIA strings on $\mathbf{M}_3$. In §2.5 we develop the "dictionary" that translates between the states and operators of (i) and (ii). We conclude in §2.6 with a brief summary of what we have found so far and a preview of the nonabelian case.

2.2 The SL(2, ℤ)-twist

Our starting point is a free 3+1D $U(1)$ gauge theory with action

$$I = \frac{1}{4g_{\rm ym}^2}\int F \wedge^* F + \frac{\theta}{2\pi}\int F \wedge F,$$

where $F = dA$ is the field strength. As usual, we define the complex coupling constant

$$\tau \equiv \frac{4\pi i}{g_{\rm ym}^2} + \frac{\theta}{2\pi} \equiv \tau_1 + i\tau_2.$$

The $\mathrm{SL}(2,\mathbb{Z})$ group of dualities is generated by S and T that act as $\tau \to -1/\tau$ and $\tau \to \tau+1$, respectively.

Let the space-time coordinates be $x_0, \ldots, x_3$. We wish to compactify direction x_3 on a circle (so that $0 \le x_3 \le 2\pi$ is a periodic coordinate), but allow τ to vary as a function of x_3 in such a way that

$$\tau(0) = \frac{\mathbf{a}\tau(2\pi) + \mathbf{b}}{\mathbf{c}\tau(2\pi) + \mathbf{d}}\,,$$

where $W \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ defines an electric/magnetic duality transformation. Such a compactification contains two ingredients:

- The variable coupling constant τ; and
- The "duality-twist" at $x_3 = 0 \sim 2\pi$.

We will discuss the ingredients separately, starting from the duality-twist.

The duality-twist can be described concretely in terms of an abelian Chern-Simons theory as follows. Represent the $\mathrm{SL}(2,\mathbb{Z})$ matrix in terms of the generators S and T (nonuniquely) as

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = T^{k_1} S T^{k_2} S \cdots T^{k_n} S\,, \tag{2.3}$$

where $\mathbf{k}_1, \ldots, \mathbf{k}_n$ are integers, some of which may be zero. To understand how each of the operators T and S act separately, we pretend that x_3 is a time-direction and impose the temporal gauge condition $A_3 = 0$. At any given x_3 the wave-function is formally $\Psi(\mathcal{A})$, where $\mathcal{A}$ is the gauge field 1-form on the three-dimensional space parameterized by x_0, x_1, x_2. The action of the generators S and T on the wave-functions is then given by (see for instance [15, 16]):

$$S : \Psi(\mathcal{A}) \to \int e^{-\frac{i}{2\pi}\int \mathcal{A}\wedge d\mathcal{A}'}\Psi(\mathcal{A}')\mathcal{D}\mathcal{A}'\,, \qquad T : \Psi(\mathcal{A}) \to e^{\frac{i}{4\pi}\int \mathcal{A}\wedge d\mathcal{A}}\Psi(\mathcal{A})\,.$$

It is now clear how to incorporate the duality twist by combining these two elements to realize the $\mathrm{SL}(2,\mathbb{Z})$ transformation (2.3). We have to add to the action a Chern-Simons term at $x_3 = 0$ with additional auxiliary fields $A_1, \ldots, A_{n+1}$ and with action

$$I_{CS} = \tfrac{1}{4\pi} \sum_{i=1}^{n} k_i A_i \wedge dA_i - \tfrac{1}{2\pi} \sum_{i=1}^{n} A_i \wedge dA_{i+1} \,, \tag{2.4}$$

and then set

$$A_1 = A|_{x_3=0} \,, \qquad A_{n+1} = A|_{x_3=2\pi} \,.$$

The second ingredient is the varying coupling constant $\tau(x_3)$. Systems with such a varying τ are known as *Janus configurations* [17]. They have supersymmetric extensions [18]-[52] where the Lagrangian of $N = 4$ Super-Yang-Mills with variable τ is modified so as to preserve 8 supercharges. In such configurations the function $\tau(x_3)$ traces a geodesic in the hyperbolic upper-half τ-plane, namely, a half-circle centered on the real axis [52]. In this model, the surviving supersymmetry is described by parameters that vary as a function of x_3, so that in general the supercharges at $x_3 = 0$ are not equal to those at $x_3 = 2\pi$. This might have been a problem for us, since we need to continuously connect $x_3 = 0$ to $x_3 = 2\pi$ to form a consistent supersymmetric theory, but luckily, we also have the $\mathrm{SL}(2,\mathbb{Z})$-twist, and as shown in [21], in $N = 4$ Super-Yang-Mills (with a fixed coupling constant τ), the $\mathrm{SL}(2,\mathbb{Z})$ duality transformations do not commute with the supercharges. Following the action of duality, the SUSY generators pick up a known phase. But as it turns out, this phase exactly matches the phase difference from 0 to 2π in the Janus configuration. Therefore, we can combine the two separate ingredients and close the supersymmetric Janus configuration on the segment $[0, 2\pi]$ with an $\mathrm{SL}(2,\mathbb{Z})$ duality twist that connects 0 to 2π. We describe this construction in more detail in Appendix B.

The details of the supersymmetric action, however, will not play an important role in what follows, so we will just assume supersymmetry and proceed. Thanks to mass terms that appear in the Janus configuration (which are needed to close the SUSY algebra [52]), at low-energy the superpartners of the gauge fields are all massive (see Appendix B), with masses of the order of the Kaluza-Klein scale, and we can ignore them. We will therefore proceed with a discussion of only the free $U(1)$ gauge fields.

2.3 The Low-energy limit and Chern-Simons theory

At low-energy we have to set $A_1 = A_{n+1}$ in (2.4), since the dependence of A on x_3 is suppressed. Then, the low-energy system is described by a 2+1D Chern-Simons action with gauge group $U(1)^n$ and action

$$I = \tfrac{1}{4\pi} \sum_{i,j=1}^{n} K_{ij} A_i \wedge dA_j \,,$$

with coupling-constant matrix that is given by

$$K \equiv \begin{pmatrix} \mathbf{k}_1 & -1 & 0 & & -1 \\ -1 & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & -1 \\ -1 & & 0 & -1 & \mathbf{k}_n \end{pmatrix} . \tag{2.5}$$

We now make directions x_1, x_2 periodic, so that the theory is compactified on T^2, leaving only time uncompactified. The dimension of the resulting Hilbert space of states of this compactified Chern-Simons theory is $|\det K|$.

Next, we pick two fundamental cycles whose equivalence classes generate $H_1(T^2, \mathbb{Z})$. Let α be the cycle along a straight line from $(0,0)$ to $(1,0)$, and let β be a similar cycle from $(0,0)$ to $(0,1)$, in (x_1, x_2) coordinates. We define $2n$ Wilson loop operators:

$$U_j \equiv \exp\left(i \oint_\alpha A_j \right) , \qquad V_j \equiv \exp\left(i \oint_\beta A_j \right) , \qquad j = 1, \ldots, n.$$

They are unitary operators with commutation relations given by

$$U_i U_j = U_j U_i \,, \qquad V_i V_j = V_j V_i \,, \qquad U_i V_j = e^{2\pi i (K^{-1})_{ij}} V_j U_i \,.$$

[$(K^{-1})_{ij}$ is the i,j element of the matrix K^{-1}.] In particular, for any $j = 1, \ldots, n$ the operator $\prod_{i=1}^n U_i^{K_{ij}}$ commutes with all $2n$ operators, and hence is a central element. In an irreducible representation, it can be set to the identity. The U_i's therefore generate a finite abelian group, which we denote by $\mathbf{g}$. Similarly, we denote by $\mathcal{G}_\beta$ the finite abelian group generated by the V_i's. Both groups are isomorphic and can be described as follows. Let $\Lambda \subseteq \mathbb{Z}^n$ be the sublattice of $\mathbb{Z}^n$ generated by the columns of the matrix K. Then, $\mathbb{Z}^n/\Lambda$ is a finite abelian group and $\mathbf{g} \cong \mathcal{G}_\beta \cong \mathbb{Z}^n/\Lambda$, since an element of $\mathbb{Z}^n$ represents the powers of a monomial in the U_i's (or V_i's), and an element in Λ corresponds to a monomial that is a central element. We therefore map

$$\mathbf{g} \ni \prod_{i=1}^n U_i^{N_i} \mapsto (N_1, N_2, \ldots, N_n) \in \mathbb{Z}^n \pmod{\Lambda} , \tag{2.6}$$

and similarly

$$\mathcal{G}_\beta \ni \prod_{i=1}^n V_i^{M_i} \mapsto (M_1, M_2, \ldots, M_n) \in \mathbb{Z}^n \pmod{\Lambda} . \tag{2.7}$$

We denote the operator in $\mathbf{g}$ that corresponds to $\mathfrak{v} \in \mathbb{Z}^n/\Lambda$ by $\mathcal{O}_\alpha(\mathfrak{v})$, and similarly we define $\mathcal{O}_\beta(\mathfrak{v}) \in \mathcal{G}_\beta$ to be the operator in $\mathcal{G}_\beta$ that corresponds to $\mathfrak{v}$. For $\mathfrak{u}, \mathfrak{v} \in \mathbb{Z}^n/\Lambda$ we define

$$\chi(\mathfrak{u}, \mathfrak{v}) \equiv e^{2\pi i \sum_{i,j} (K^{-1})_{ij} N_i M_j} , \qquad (\mathfrak{u}, \mathfrak{v} \in \mathbb{Z}^n/\Lambda). \tag{2.8}$$

The definition is independent of the particular representatives $(N_1, \ldots, N_n)$ or $(M_1, \ldots, M_n)$ in $\mathbb{Z}^n/\Lambda$. The commutation relations can then be written as

$$\mathcal{O}_\alpha(\mathfrak{u})\mathcal{O}_\beta(\mathfrak{v}) = \chi(\mathfrak{u},\mathfrak{v})\mathcal{O}_\beta(\mathfrak{v})\mathcal{O}_\alpha(\mathfrak{u})\,. \tag{2.9}$$

We recall that for any nonsingular matrix of integers $K \in \mathrm{GL}(n,\mathbb{Z})$, one can find matrices $P, Q \in \mathrm{SL}(n,\mathbb{Z})$ such that

$$PKQ = \mathrm{diag}(d_1, d_2, \ldots, d_n) \tag{2.10}$$

is a diagonal matrix, $d_1, \ldots, d_n$ are positive integers, and d_i divides d_{i+1} for $i = 1, \ldots, n-1$. The integers $d_1, \ldots, d_n$ are unique, and we have

$$\mathbb{Z}^n/\Lambda \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_n}\,,$$

where $\mathbb{Z}_d$ is the cyclic group of d elements. The matrix on the RHS of (2.10) is known as the *Smith normal form* of K. The integer d_j is the greatest common divisor of all $j \times j$ minors of K. For K of the form (2.5), the minor that is made of rows $2, \ldots, n-1$ and columns $1, \ldots, n-2$ is $(-1)^{n-2}$, so it follows that $d_{n-2} = 1$ and therefore also $d_1 = \cdots d_{n-2} = 1$. We conclude that

$$\mathbf{g} \cong \mathcal{G}_\beta \cong \mathbb{Z}_{d_{n-1}} \oplus \mathbb{Z}_{d_n}\,.$$

2.4 Strings on a mapping torus

The system we studied in §2.2 has a dual description as the Hilbert space of ground states of strings of winding number $w = 1$ (around a 1-cycle to be defined below) on a certain type-IIA background. We will begin by describing the background geometry and then explain in §2.5 why its space of ground states is isomorphic to the space of ground states of the $\mathrm{SL}(2,\mathbb{Z})$-twisted compactification of §2.2.

Set

$$W = \begin{pmatrix} \mathbf{k}_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \mathbf{k}_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{k}_1 & -1 \\ 1 & 0 \end{pmatrix} = T^{\mathbf{k}_n} S \cdots T^{\mathbf{k}_2} S T^{\mathbf{k}_1} S \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,. \tag{2.11}$$

We will assume that $|\operatorname{tr} W| > 2$ so that W is a hyperbolic element of $\mathrm{SL}(2,\mathbb{Z})$. (The case of elliptic elements with $|\operatorname{tr} W| < 2$ was covered in [27], and parabolic elements with $|\operatorname{tr} W| = 2$ are conjugate to $\pm T^{\mathbf{k}}$ for some $\mathbf{k} \neq 0$, and since they do not involve S, they are elementary.)

Let $0 \leq \eta \leq 2\pi$ denote the coordinate on the interval $I = [0, 2\pi]$ and let (ξ_1, ξ_2) denote the coordinates of a point on T^2. The coordinates ξ_1 and ξ_2 take values in $\mathbb{R}/\mathbb{Z}$ (so they are periodic with period 1). We impose the identification

$$(\xi_1, \xi_2, \eta) \sim (\mathbf{d}\xi_1 + \mathbf{b}\xi_2, \mathbf{c}\xi_1 + \mathbf{a}\xi_2, \eta + 2\pi). \tag{2.12}$$

The metric is

$$ds^2 = R^2 d\eta^2 + (\frac{4\pi^2\rho^2}{\tau_2})|d\xi_1 + \tau(\eta) d\xi_2|^2$$

where R and ρ are constants, and $\tau = \tau_1 + i\tau_2$ is a function of η (with real and imaginary parts denoted by τ_1 and τ_2) such that

$$\tau(\eta + 2\pi) = \frac{\mathbf{a}\tau(\eta) + \mathbf{b}}{\mathbf{c}\tau(\eta) + \mathbf{d}}\,,$$

thus allowing for a continuous metric.

2.4.1 The number of fixed points

We will need the number of fixed points of the $\mathrm{SL}(2,\mathbb{Z})$ action on T^2, i.e., the number of solutions to:

$$(\xi_1, \xi_2) = (\mathbf{d}\xi_1 + \mathbf{b}\xi_2, \mathbf{c}\xi_1 + \mathbf{a}\xi_2) \pmod{\mathbb{Z}^2}\,.$$

Let $f : T^2 \to T^2$ be the map given by

$$f : (\xi_1, \xi_2) \to (\mathbf{d}\xi_1 + \mathbf{b}\xi_2, \mathbf{c}\xi_1 + \mathbf{a}\xi_2)\,. \tag{2.13}$$

The Lefschetz fixed-point formula states that

$$\sum_{\text{fixed point } p} i(p) = \sum_{j=0}^{2} (-1)^j \operatorname{tr}(f_* | H_j(T^2, \mathbb{Z})) = 2 - \operatorname{tr} W = 2 - \mathbf{a} - \mathbf{d}.$$

The index $i(p)$ of a fixed point is given by [22]:

$$i(p) = \operatorname{sgn}\det(\mathcal{J}(p) - \mathbf{I}) = \operatorname{sgn}\det(W - \mathbf{I})\,,$$

where $\mathcal{J}(p)$ is the Jacobian matrix of the map f at p. In our case, $i(p)$ is either $+1$ or -1 for all p, and therefore the number of fixed points is

$$|2 - \operatorname{tr} W| = |\det(W - \mathbf{I})| = |2 - \mathbf{a} - \mathbf{d}|\,.$$

2.4.2 Isometries

Let $v_1, v_2 \in \mathbb{R}/\mathbb{Z}$ be constants and consider the map

$$(\xi_1, \xi_2, \eta) \mapsto (\xi_1 + v_1, \xi_2 + v_2, \eta). \tag{2.14}$$

It defines an isometry of $\mathbf{M}_3$ if

$$\begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} \equiv \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} \qquad (\mathrm{mod}\ \mathbb{Z})\,. \tag{2.15}$$

Set

$$H \equiv W - \mathbf{I} = \begin{pmatrix} \mathbf{a} - 1 & \mathbf{c} \\ \mathbf{b} & \mathbf{d} - 1 \end{pmatrix}\,, \qquad \mathbf{v} \equiv \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}\,. \tag{2.16}$$

Then, the isometries are given by $\mathbf{v} = H^{-1}\begin{pmatrix} n_2 \\ n_1 \end{pmatrix}$ for some $n_1, n_2 \in \mathbb{Z}$. The set of vectors $\mathbf{v}$ that give rise to isometries therefore live on a lattice $\widetilde{\Lambda}$ generated by the columns of H^{-1}. Since $H \in \mathrm{GL}(2,\mathbb{Z})$ we have $\mathbb{Z}^2 \subseteq \widetilde{\Lambda}$, and since the isometries that correspond to $\mathbf{v} \in \mathbb{Z}^2$ are trivial, the group of isometries of type (2.14) is isomorphic to $\widetilde{\Lambda}/\mathbb{Z}^2$. Changing basis to $\mathbf{u} \equiv H\mathbf{v}$, we can replace $\mathbf{v} \in \widetilde{\Lambda}/\mathbb{Z}^2$ with $\mathbf{u} \in \mathbb{Z}^2/\Lambda'$, where $\Lambda' \subseteq \mathbb{Z}^2$ is the sublattice generated by the columns of H, and the group $\mathcal{G}_{\text{iso}}$ of isometries of type (2.14) is therefore

$$\mathcal{G}_{\text{iso}} \cong \widetilde{\Lambda}/\mathbb{Z}^2 \cong \mathbb{Z}^2/\Lambda' \,. \tag{2.17}$$

Its order is

$$|\mathcal{G}_{\text{iso}}| = |\det H| = |2 - \mathbf{a} - \mathbf{d}|. \tag{2.18}$$

2.4.3 Homology quantum numbers

To proceed we also need the homology group $H_1(\mathbf{M}_3,\mathbb{Z})$. Let γ be the cycle defined by a straight line from $(0,0,0)$ to $(0,0,2\pi)$, in terms of (ξ_1,ξ_2,η) coordinates. Let α' be the cycle from $(0,0,0)$ to $(1,0,0)$ and let β' be the cycle from $(0,0,0)$ to $(0,1,0)$. The homology group $H_1(\mathbf{M}_3,\mathbb{Z})$ is generated by the equivalence classes $[\alpha']$, $[\beta']$ and $[\gamma]$, subject to the relations

$$[\alpha'] = \mathbf{d}[\alpha'] + \mathbf{c}[\beta'] \,, \qquad [\beta'] = \mathbf{b}[\alpha'] + \mathbf{a}[\beta']. \tag{2.19}$$

Now suppose that $(c_1\, c_2)$ is a linear combination of the columns of H [defined in (2.16)] with integer coefficients. Then the relations (2.19) imply that $c_1[\alpha'] + c_2[\beta']$ is zero in $H_1(\mathbf{M}_3,\mathbb{Z})$. With $\Lambda' \subset \mathbb{Z}^2$ being the sublattice generated by the columns of H, as defined in §2.4.2, it follows that

$$H_1(\mathbf{M}_3,\mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}^2/\Lambda'), \tag{2.20}$$

where the $\mathbb{Z}$ factor is generated by $[\gamma]$ and $(\mathbb{Z}^2/\Lambda')$ is generated by $[\alpha']$ and $[\beta']$. In particular, the torsion part is

$$\mathrm{Tor}\, H_1(\mathbf{M}_3,\mathbb{Z}) \cong \mathbb{Z}^2/\Lambda' \,. \tag{2.21}$$

Denote the Smith normal form [see (2.10)] of the matrix H by $\begin{pmatrix} d_1' & \\ & d_2' \end{pmatrix}$. We prove in Appendix A that $d_{n-1} = d_1'$ and $d_n = d_2'$, where d_{n-1} and d_n were defined in (2.10). Thus, combining (2.17) and (2.20) we have

$$\mathbb{Z}^2/\Lambda' \cong \mathcal{G}_{\text{iso}} \cong \mathrm{Tor}\, H_1(\mathbf{M}_3,\mathbb{Z}) \cong \mathbb{Z}_{d_{n-1}} \oplus \mathbb{Z}_{d_n} \,.$$

The physical meaning of these results will become clear soon.

2.4.4 The Hilbert space of states

As we have seen in §2.4.2, the Hilbert space of string ground states has a basis of states of the form $|\mathbf{v}'\rangle$ with $\mathbf{v}' \in \widetilde{\Lambda}/\mathbb{Z}^2$. In this state, the string is at (ξ_1, ξ_2) coordinates given by $\mathbf{v}'$. According to (2.17), an element $\mathbf{v} \in \widetilde{\Lambda}/\mathbb{Z}^2$ defines an isometry, which we denote by $\mathcal{Y}(\mathbf{v})$, that acts as

$$\mathcal{Y}(\mathbf{v})|\mathbf{v}'\rangle = |\mathbf{v}+\mathbf{v}'\rangle\,, \qquad \mathbf{v},\mathbf{v}' \in \widetilde{\Lambda}/\mathbb{Z}^2\,.$$

Given the string state $|\mathbf{v}'\rangle$, we can ask what is the element in $H_1(\mathbf{M}_3,\mathbb{Z})$ that represents the corresponding 1-cycle. The answer is $[\gamma] + N_1'[\alpha'] + N_2'[\beta']$, where the torsion part $N_1'[\alpha'] + N_2'[\beta']$ is mapped under (2.21) to $\mathbf{v}'$. To see this, note that for $0 \le t \le 1$, the loops C_t in $\mathbf{M}_3$ that are given by

$$\begin{cases} (4\pi s, tv_1', tv_2') & \text{for } 0 \le s \le \frac{1}{2} \\ (2\pi, tv_1' + (2s-1)t[(\mathbf{d}-1)v_1' + \mathbf{b}v_2'], tv_2' + (2s-1)t[\mathbf{c}v_1' + (\mathbf{a}-1)v_2']) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

[which go along direction η at a constant (ξ_1,ξ_2) given by $t\mathbf{v}'$, and then connect $t\mathbf{v}'$ to its $\mathrm{SL}(2,\mathbb{Z})$ image $tW\mathbf{v}'$] are homotopic to the loop corresponding to string state $|0\rangle$. Setting $t=1$ we find that C_1 breaks into two closed loops, one corresponding to string state $|\mathbf{v}'\rangle$, and the other is a closed loop in the T^2 fiber above $\eta = 0$, which corresponds to the homology element

$$((\mathbf{d}-1)v_1' + \mathbf{b}v_2')[\alpha'] + (\mathbf{c}v_1' + (\mathbf{a}-1)v_2')[\beta']\,,$$

and this is precisely the element corresponding to $H\mathbf{v}' \in \mathbb{Z}^2/\Lambda' \cong \mathrm{Tor}\, H_1(\mathbf{M}_3,\mathbb{Z})$, as defined in §2.4.3.

We now wish to use the torsion part of the homology to define a unitary operator $\mathcal{R}(\tilde{\mathbf{u}})$ for every $\tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda'$. This operator will measure a component of the charge associated with the homology class of the string. For this purpose we need to construct the Pontryagin dual group ${}^\vee\mathrm{Tor}\, H_1(\mathbf{M}_3,\mathbb{Z})$, which is defined as the group of characters of $\mathrm{Tor}\, H_1(\mathbf{M}_3,\mathbb{Z})$ (i.e., homomorphisms from $\mathrm{Tor}\, H_1(\mathbf{M}_3,\mathbb{Z})$ to $\mathbb{R}/\mathbb{Z}$). The dual group is isomorphic to $\mathbb{Z}^2/\Lambda'$, but not canonically. In our construction $\tilde{\mathbf{u}}$ is naturally an element of the dual group and not the group itself. We define $\mathcal{R}(\tilde{\mathbf{u}})$ as follows. For

$$\tilde{\mathbf{u}} = (M_1', M_2') \in \mathbb{Z}^2/\Lambda'\,, \qquad \mathbf{v} = (N_1', N_2') \in \mathbb{Z}^2/\Lambda'\,,$$

we define the phase

$$\varphi(\tilde{\mathbf{u}},\mathbf{v}) \equiv e^{2\pi i (H^{-1})_{ij} N_i' M_j'}\,, \qquad \tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda'\,, \quad \mathbf{v} \in \mathbb{Z}^2/\Lambda'\,. \tag{2.22}$$

This definition is independent of the representatives (N_1', N_2') and (M_1', M_2') of $\mathbf{v}$ and $\tilde{\mathbf{u}}$, and it corresponds to the character of $\mathrm{Tor}\, H_1(\mathbf{M}_3,\mathbb{Z})$ associated with $\tilde{\mathbf{u}}$. We then define the operator $\mathcal{R}(\tilde{\mathbf{u}})$ to be diagonal in the basis $|\mathbf{v}\rangle$ and act as:

$$\mathcal{R}(\tilde{\mathbf{u}})|\mathbf{v}\rangle = \varphi(\tilde{\mathbf{u}},\mathbf{v})|\mathbf{v}\rangle\,, \qquad \tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda'\,, \quad \mathbf{v} \in \mathbb{Z}^2/\Lambda'\,.$$

From the discussion above about the homology of the string state, and from the linearity of the phase of $\varphi(\tilde{\mathbf{u}},\mathbf{v})$ in $\tilde{\mathbf{u}}$ and $\mathbf{v}$, it follows that

$$\mathcal{R}(\tilde{\mathbf{u}})\mathcal{Y}(\mathbf{v}) = \varphi(\tilde{\mathbf{u}},\mathbf{v})\mathcal{Y}(\mathbf{v})\mathcal{R}(\tilde{\mathbf{u}})\,. \tag{2.23}$$

2.5 Duality between strings on $\mathbf{M}_3$ and the compactified SL$(2,\mathbb{Z})$-twisted $U(1)$ gauge theory

We can now connect the string theory model of §2.4 with the field theory model of §2.3. We claim that the Hilbert space of ground states of a compactification of a $U(1)$ gauge theory on S^1 with an SL$(2,\mathbb{Z})$ twist and string ground states on $\mathbf{M}_3$ are dual. This is demonstrated along the same lines as in [27]. We realize the (supersymmetric extension of the) $U(1)$ gauge theory on a D3-brane along directions x_1, x_2, x_3. We compactify direction x_3 on a circle with a Janus-like configuration and SL$(2,\mathbb{Z})$-twisted boundary conditions. We assume that the Janus configuration can be lifted to type-IIB, perhaps with additional fluxes, but we will not worry about the details of the lift. We then compactify (x_1, x_2) on T^2 and perform T-duality on direction 1, followed by a lift from type-IIA to M-theory (producing a new circle along direction 10), followed by reduction to type-IIA along direction 2. This combined U-duality transformation transforms the SL$(2,\mathbb{Z})$-twist to the geometrical transformation (2.12). It also transforms some of the charges of the type-IIB system to the following charges of the type-IIA system:

$$D3_{123} \to F1_3\,, \quad F1_1 \to P_1\,, \quad F1_2 \to F1_{10}\,, \quad D1_1 \to F1_1\,, \quad D1_2 \to P_{10}\,. \tag{2.24}$$

where P_j is Kaluza-Klein momentum along direction j, D$p_{j_1\ldots j_r}$ is a Dp-brane wrapped along directions $j_1, \ldots, j_r$, and F1_j is a fundamental string along direction j.

Now suppose we take the limit that all directions of $\mathbf{M}_3$ are large. The dual geometry has a Hilbert space of ground states which corresponds to classical configurations of strings of minimal length that wind once around the x_3 circle. This means that the projection of their $H_1(\mathbf{M}_3,\mathbb{Z})$ homology class on the $\mathbb{Z}$ factor of (2.20) is required to be the generator $[\gamma]$. The torsion part of their homology is unrestricted. The string configurations of minimal length must have constant (x_1, x_2) which in particular means that (x_1, x_2) is invariant under the SL$(2,\mathbb{Z})$ twist, i.e.,

$$\begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \equiv \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \pmod{\mathbb{Z}}\,.$$

But this is precisely the same equation as (2.15), and indeed when the isometry that corresponds to a vector $\mathbf{v} \in \widetilde{\Lambda}/\mathbb{Z}^2$ acts on the solution with $(x_1, x_2) = (0, 0)$ it converts it to the solution with $(x_1, x_2) = (v_1, v_2)$. The dimension of the Hilbert space of ground states of the type-IIA system is therefore the order of $\mathcal{G}_{\rm iso}$, which is given by (2.18). This is also the number of fixed points of the W action on T^2, as we have seen in §2.4.1. Since the number of ground states of the Chern-Simons theory is $|\det K|$, we conclude from the duality of the Chern-Simons theory and string theory that

$$|\det K| = |\mathcal{G}_{\rm iso}| = |2 - \mathbf{a} - \mathbf{b}|\,.$$

This is the physical explanation we are giving to (2.2).

2.5.1 Isomorphism of operator algebras

Going one step beyond the equality of dimensions of the Hilbert spaces, we would like to match the operator algebras of the string and field theory systems. Starting with the field theory side, realized on a D3-brane in type-IIB, consider a process whereby a fundamental string that winds once around the β-cycle of T^2 is absorbed by the D3-brane at some time t. This process is described in the field theory by inserting a Wilson loop operator V_1 at time t into the matrix element that calculates the amplitude. On the type-IIA string side, the charge $F1_2$ that was absorbed is mapped by (2.24) to winding number along the α' cycle (denoted by $F1_{10}$). The operator that correpsonds to V_1 on the string side must therefore increase the homology class of the string state by $[\alpha']$. Since the state $|\mathbf{v}\rangle$, for $\mathbf{v} = (N_1', N_2')$, has homology class $[\gamma] + N_1'[\alpha'] + N_2'[\beta']$, it follows that the isometry operator $\mathcal{Y}(\mathbf{v}')$ with $\mathbf{v}' = (1,0)$ does what we want. We therefore propose to identify

$$V_1 \to \mathcal{Y}(\mathbf{v}')\,, \qquad \text{for } \mathbf{v}' = (1,0).$$

By extension, we propose to map the abelian subgroup $\mathcal{G}_\beta$ generated by the Wilson loops $V_1, \ldots, V_n$ with the isometry group generated by $\mathcal{Y}(\mathbf{v}')$ for $\mathbf{v}' \in \mathbb{Z}^2/\Lambda'$.

Next, on the type-IIB side, consider a process whereby a fundamental string that winds once around the α-cycle of T^2 is absorbed by the D3-brane. This process is described in the field theory by inserting a Wilson loop operator U_1 into the matrix element that calculates the amplitude. On the type-IIA string side, the charge $F1_1$ that was absorbed is mapped by (2.24) to momentum along the β' cycle (denoted by $P1_1$). The operator that correpsonds to U_1 on the string side must therefore increase the momentum along the $[\alpha']$ cycle by one unit. We claim that this operator is $\mathcal{R}(\tilde{\mathbf{u}})$ for $\tilde{\mathbf{u}} = (1,0)$. To see this we note that, by definition of "momentum", an operator $\mathcal{X}$ that increases the momentum by M_1' units along the $[\alpha']$ cycle and M_2' units along the $[\beta']$ cycle must have the following commutation relations with the translational isometries $\mathcal{Y}(\mathbf{v}')$:

$$\mathcal{Y}(\mathbf{v}')^{-1}\mathcal{X}\mathcal{Y}(\mathbf{v}') = \varphi(\tilde{\mathbf{u}}, \mathbf{v}')\mathcal{X}\,, \qquad \tilde{\mathbf{u}} = (M_1', M_2') \in \mathbb{Z}^2/\Lambda'.$$

But given (2.23), this means that up to an unimportant central element, we can identify $\mathcal{X} = \mathcal{R}(\tilde{\mathbf{u}})$, as claimed. So, we have

$$U_1 \to \mathcal{Y}(\tilde{\mathbf{u}})\,, \qquad \text{for } \tilde{\mathbf{u}} = (1,0),$$

and by extension, we propose to map the abelian subgroup $\mathbf{g}$ generated by the Wilson loops $U_1, \ldots, U_n$ with the subgroup generated by $\mathcal{R}(\tilde{\mathbf{u}})$ for $\tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda'$.

In particular, $\mathbf{g} \cong \mathcal{G}_\beta \cong \mathbb{Z}^n/\Lambda$ implies that $(\mathbb{Z}^2/\Lambda') \cong (\mathbb{Z}^n/\Lambda)$. This is equivalent to requiring that the Smith normal form of H is

$$P'HQ' = \mathrm{diag}(d_{n-1}, d_n)$$

where d_{n-1} and d_n are the same last two entries in the Smith normal form of K. We provide an elementary proof of this fact in Appendix A.

Since the Smith normal forms of H and K are equal, the abelian groups $\mathbb{Z}^n/\Lambda$ and $\mathbb{Z}^2/\Lambda'$ are equivalent, and it is also not hard to see that under this equivalence χ that was defined in (2.8) is mapped to φ defined in (2.22). We have the mapping

$$\mathcal{O}_\alpha(\mathfrak{v}) \to \mathcal{Y}(\mathbf{v}')\,, \qquad \mathfrak{v} \in \mathbb{Z}^n/\Lambda\,, \qquad \mathbf{v}' \in \mathbb{Z}^2/\Lambda'$$

and

$$\mathcal{O}_\beta(\mathfrak{u}) \to \mathcal{R}(\tilde{\mathbf{u}})\,, \qquad \mathfrak{u} \in \mathbb{Z}^n/\Lambda\,, \qquad \tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda'.$$

The commutation relations (2.9) are then mapped to (2.23).

2.6 Discussion

We have argued that a duality between $U(1)^n$ Chern-Simons theory on T^2 with coupling constant matrix (2.5) and string configurations on a mapping torus provide a geometrical realization to the algebra of Wilson loop operators in the Chern-Simons theory. Wilson loop operators along one cycle of T^2 correspond to isometries that act as translations along the fiber of the mapping torus, while Wilson loop operators along the other cycle correspond to discrete charges that can be constructed from the homology class of the string.

These ideas have an obvious extension to the case of $U(N)$ gauge group with $N > 1$, where $\mathrm{SL}(2,\mathbb{Z})$-duality is poorly understood. The techniques presented in this paper can be extended to construct the algebra of Wilson loop operators. The Hilbert space on the string theory side is constructed from string configurations on a mapping torus whose $H_1(\mathbf{M}_3,\mathbb{Z})$ class maps to N under the projection map $\mathbf{M}_3 \to S^1$. In other words, the homology class projects to $N[\gamma]$ when the torsion part is ignored. Such configurations could be either a single-particle string state wound N times, or a multi-particle string state. A string state with r strings with winding numbers $N_1, \ldots, N_r$ is described by a partition $N = N_1 + \cdots N_r$, and the j^{th} single-particle string state is described by an unordered set of N_j points on T^2 that is invariant, as a set, under the action of f in (2.13). The counterparts of the Wilson loops on the string theory side can then be constructed from operations on these sets. A more complete account of the nonabelian case will be reported elsewhere [12].

It is interesting to note that some similar ingredients to the ones that appear in this work also appeared in [31] in the study of vacua of compactifications of the free $(2,0)$ theory on Lens spaces. More specifically, a Chern-Simons theory with a tridiagonal coupling constant matrix and the torsion part of the first homology group played a role there as well. It would be interesting to further explore the connection between these two problems.

Chapter 3

Fractional Quantum Hall Effect, Quasi-Particles, and the $(2,0)$-Theory

The Fractional Quantum Hall Effect appears as part of the low-energy description of the Coulomb branch of the A_1 $(2,0)$-theory formulated on $(S^1 \times \mathbb{R}^2)/\mathbb{Z}_k$, where the generator of $\mathbb{Z}_k$ acts as a combination of translation on S^1 and rotation by $2\pi/k$ on $\mathbb{R}^2$. At low-energy the configuration is described in terms of a 4+1D Super-Yang-Mills theory on a cone $(\mathbb{R}^2/\mathbb{Z}_k)$ with additional 2+1D degrees of freedom at the tip of the cone. Fractionally charged quasi-particles have a natural description in terms of BPS strings of the $(2,0)$-theory. We analyze the large k limit, where a smooth cigar-geometry provides an alternative description. In this framework a W-boson can be modeled as a bound state of k quasi-particles. The W-boson becomes a Q-ball, and it can be described by a soliton solution of BPS monopole equations on a certain auxiliary curved space. We show that axisymmetric solutions of these equations correspond to singular maps from AdS_3 to AdS_2, and we present some numerical results.

3.1 Introduction

The Fractional Quantum Hall Effect (FQHE) with filling-factor $1/\mathbf{k}$ ($\mathbf{k} \in \mathbb{Z}$) appears in 2+1D condensed matter systems whose low-energy effective degrees of freedom can be described by the Chern-Simons action

$$I = \frac{\mathbf{k}}{4\pi} \int \mathfrak{a} \wedge d\mathfrak{a} + \frac{1}{2\pi} \int \mathbf{A} \wedge d\mathfrak{a}\,. \tag{3.1}$$

Here, $\mathbf{A}$ is the electromagnetic gauge field, and $\mathfrak{a}$ is a 2+1D $U(1)$ gauge field that describes the low-energy internal degrees of freedom of the system. It is related to the electromagnetic current by $j = {}^*d\mathfrak{a}$. Excited levels of the system may include quasi-particle excitations that are charged under the gauge symmetry associated with $\mathfrak{a}$. Such quasi-particles with one unit of $\mathfrak{a}$-charge will have $1/\mathbf{k}$ electromagnetic charge.

The FQHE appears in appropriately constructed 2+1D condensed-matter systems with strongly correlated electrons in a strong magnetic field. The (Laughlin) wave-functions of

the low-energy states are then holomorphic in the position of the electrons (up to a common Gaussian factor). One of the remarkable features of this system is that there is a dual description of the low-energy spectrum in which the quasi-particles are fundamental and the electrons can be viewed as bound states of the quasi-particles.

The goal of this paper is to gain a better understanding of this dual description of an integrally charged particle as a bound state of quasi-particles using a particularly intuitive string-theoretic model of the FQHE. Over the past two decades several realizations of FQHE in string theory have been constructed [50]-[55]. Generally speaking, these realizations construct the Chern-Simons action (3.1) as low-energy effective description of a $(d+2)$-dimensional brane compactified on a d-dimensional compact space, possibly in the presence of suitable fluxes, to yield the requisite $2+1$D effective description. We will use a realization in terms of a compactification of the 5+1D $(2,0)$-theory. Our system is a special case of a general class of $2+1$D theories obtained from the $(2,0)$-theory by taking three of the dimensions to be a nontrivial manifold. A beautiful framework for understanding such compactifications has been developed in [29]-[31]. We will focus on a particular aspect of the system which is the dynamics of the quasi-particles that in the condensed-matter system can arise from impurities. The $(2,0)$-theory allows for a simple geometrical realization of the quasi-particles and their relationship with the integrally charged particles. In our construction, the geometry of the extra dimensions will have long 1-cycles and short 1-cycles, the short ones being $1/\mathbf{k}$ the size of the long ones. The quasi-particles will be realized as BPS strings of the $(2,0)$ theory wound around short 1-cycles, while the integrally charged particles (the "electrons") will be realized as strings wound around long 1-cycles. The construction in terms of D3-branes ending on (p,q) 5-branes [54] is dual to ours.

We are especially interested in the limit $\mathbf{k} \gg 1$, where the filling fraction becomes extremely small. This is the strong-coupling limit of the condensed-matter system, and as we will see, our model has a dual description where quasi-particles are elementary and the integrally charged particles can be described as classical solitons, or rather Q-balls, in terms of the fundamental quasi-particle fields. We will show that solutions to the equations describing these solitons correspond to harmonic maps from AdS_3 to AdS_2.

The paper is organized as follows. In §3.2 we describe the $(2,0)$ compactification. In §3.3 we study the quasi-particles which are BPS strings and calculate their quantum numbers. In §3.4 we study the large $\mathbf{k}$ limit and write down the semiclassical action of the system. In §3.5 we develop the differential equations that describe the integrally charged particles as solitons of the fundamental quasi-particle fields in the large $\mathbf{k}$ limit. We show that they can be mapped to the equations describing a magnetic monopole on a 3D space with metric $ds^2 = x_3^2(dx_1^2+dx_2^2+dx_3^2)$. In §3.6 we analyze the soliton equations in more detail. They are not integrable in the standard sense, and we were unable to solve them in closed form, but we were able to make several observations: (i) using a rather complicated transformation we can recast the equations in terms of a single "potential" function; (ii) we present an expansion up to second order in an asymptotic corner of moduli space; (iii) we plot a numerical solution; and (iv) we propose a connection with Sine-Gordon equation in a corner of moduli space.

3.2 The $(2,0)$ theory on $(\mathbb{R}^2 \times S^1)/\mathbb{Z}_\mathbf{k}$

Our starting point is the 5+1D A_1 $(2,0)$-theory on $\mathbb{R}^{2,1} \times \mathbf{M}_3$, where $\mathbb{R}^{2,1}$ is 2+1D Minkowski space and $\mathbf{M}_3 \simeq (\mathbb{R}^2 \times S^1)/\mathbb{Z}_\mathbf{k}$ is the flat, noncompact, smooth three-dimensional manifold defined as the quotient of $\mathbb{R}^2 \times S^1$ by the isometry that acts as a simultaneous rotation of $\mathbb{R}^2$ by an angle $2\pi/\mathbf{k}$, and a translation of S^1 by $1/\mathbf{k}$ of its circumference. The A_1 $(2,0)$-theory is the low-energy limit of either type-IIB on $\mathbb{R}^4/\mathbb{Z}^2$ [25] or of 2 M5-branes [26] (after decoupling of the center of mass variables). We are interested in the low-energy description of the Coulomb branch of the theory, and in particular in the low-energy degrees of freedom that are localized near the origin of $\mathbb{R}^2$. The fractional quantum Hall effect, as we shall see, naturally appears in this context.

3.2.1 The geometry

The space $\mathbf{M}_3$ can be constructed as a quotient of $\mathbb{R}^3$ as follows. We parameterize $\mathbb{R}^3$ by x_3, x_4, x_5 and set $\mathbf{z} \equiv x_4 + ix_5$. Then, $\mathbf{M}_3$ is defined by the equivalence relation

$$(x_3,\, \mathbf{z}) \sim (x_3 + 2\pi R,\, \mathbf{z}e^{-2\pi i/\mathbf{k}})\,, \qquad \text{[defining relation of } \mathbf{M}_3\text{]} \tag{3.2}$$

where R is a constant parameter that sets the scale, and $\mathbf{k} > 1$ is an integer. The Euclidean metric on $\mathbf{M}_3$ is given by

$$ds^2 = dx_3^2 + dx_4^2 + dx_5^2 = dx_3^2 + |d\mathbf{z}|^2\,.$$

For future reference we define the $(2\mathbf{k})^{th}$ root of unity:

$$\omega \equiv e^{\pi i/\mathbf{k}}\,. \tag{3.3}$$

We also set

$$\mathbf{z} = re^{i\theta}\,,$$

so that (3.2) can be written as

$$(x_3, r, \theta) \sim (x_3 + 2\pi R, r, \theta - \tfrac{2\pi}{\mathbf{k}})\,. \tag{3.4}$$

The $\mathbf{z} = 0$ locus [i.e., the set of points $(x_3, 0)$ with arbitrary x_3] forms an S^1 of radius R that we will call *the minicircle* and denote by S^1_m. The space $\mathbf{M}_3 \setminus S^1_m$ (which is $\mathbf{M}_3$ with the minicircle excluded) is a circle-bundle over a cone:

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathbf{M}_3 \\ & & \downarrow \\ & & \mathbb{C}/\mathbb{Z}_\mathbf{k} \end{array} \tag{3.5}$$

The cone $\mathbb{C}/\mathbb{Z}_\mathbf{k}$ is parameterized by $\mathbf{z}$, subject to the equivalence relation $\mathbf{z} \sim \omega^2\mathbf{z}$. In polar coordinates the cone is parameterized by (r, θ) with $0 < r < \infty$ and $0 \le \theta < 2\pi/\mathbf{k}$. ($\theta$ is

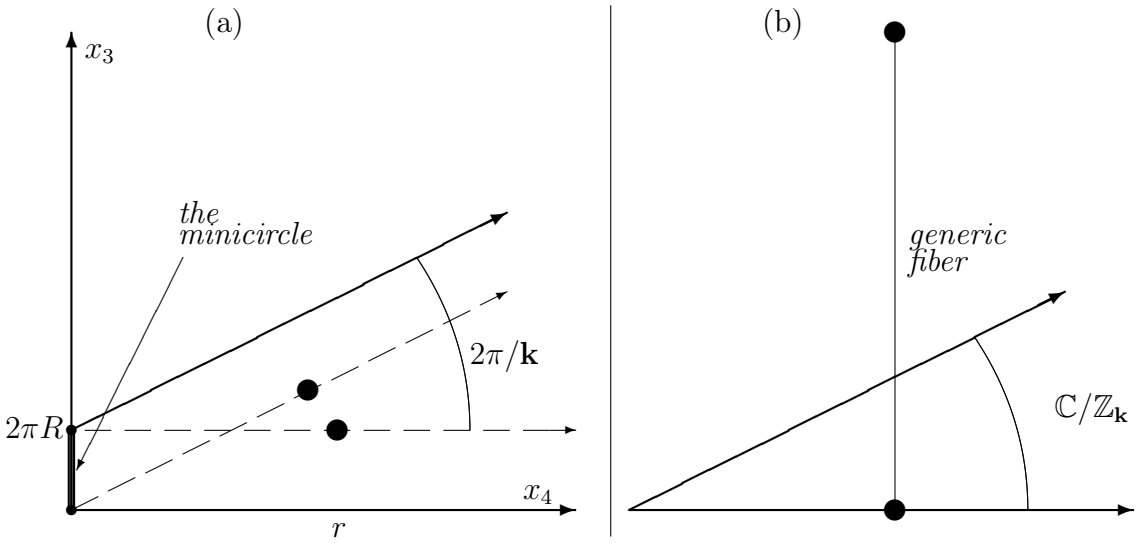


Figure 3.1: (a) The geometry of $\mathbf{M}_3 \simeq (\mathbb{C} \times S^1)/\mathbb{Z}_{\mathbf{k}}$: in the coordinate system $(x_4 + ix_5, x_3)$, the point $(r, 0)$ is identified with $(re^{-2\pi i/\mathbf{k}}, 2\pi R)$ and $(r, 2\pi \mathbf{k} R)$; The large dots indicate equivalent points; (b) The fibration $\mathbf{M}_3 \to \mathbb{C}/\mathbb{Z}_{\mathbf{k}}$ with the generic fiber that is of radius $\mathbf{k}R$.

understood to have period $2\pi/\mathbf{k}$ when describing the cone.) The projection $\mathbf{M}_3 \to \mathbb{C}/\mathbb{Z}_{\mathbf{k}}$ is given by $(x_3, \mathbf{z}) \mapsto \mathbf{z}$. For a given $\mathbf{z} \neq 0$, the fiber S^1 of the fibration (3.5) over $\mathbf{z} \simeq \omega^2 \mathbf{z}$ is given by all points $(x_3, \mathbf{z})$ with $0 \leq x_3 < 2\pi \mathbf{k} R$. The equivalence (3.2) then implies $(x_3 + 2\pi \mathbf{k} R, \mathbf{z}) \sim (x_3, \mathbf{z})$, and so this S^1 has radius $\mathbf{k}R$.

In order to preserve half of the 16 supersymmetries we augment (3.2) by an appropriate R-symmetry twist as follows. Let $\mathrm{Spin}(5) \simeq Sp(2)$ be the R-symmetry of the $(2, 0)$-theory. In the M5-brane realization of the $(2, 0)$-theory [26], Spin(5) is the group of rotations (acting on spinors) in the five directions transverse to the M5-branes, which we take to be $6, \ldots, 10$. We now split them into the subsets $6, 7$ and $8, 9, 10$. This corresponds to the rotation subgroup $[\mathrm{Spin}(3) \times \mathrm{Spin}(2)]/\mathbb{Z}_2 \subset \mathrm{Spin}(5)$. Let $\gamma \in \mathrm{Spin}(5)$ correspond to a $2\pi/\mathbf{k}$ rotation in the $6, 7$ plane. We then augment the RHS of the geometrical identification (3.2) by an R-symmetry transformation γ. The setting now preserves 8 supersymmetries.

We now go to the Coulomb branch of the $(2, 0)$-theory by separating the two M5-branes of §3.2.1 in the M-theory direction x_{10}. This breaks Spin(3) to an $SO(2)$ subgroup (corresponding to rotations in directions $8, 9$) which we denote by $SO(2)_r$. On the Coulomb branch of the $(2, 0)$-theory there is a BPS string whose tension we denote by $\tilde{V}$.

At energies $E \ll 1/\mathbf{k}R$, sufficiently far away from S^1_m, the dynamics of the $(2, 0)$-theory on $\mathbb{R}^{2,1} \times \mathbf{M}_3$ reduces to $SU(2)$ 4+1D Super-Yang-Mills theory on $\mathbb{R}^{2,1} \times (\mathbb{C}/\mathbb{Z}_k)$. The coupling constant is given by

$$\frac{4\pi}{g^2_{\mathrm{ym}}} = \frac{1}{\mathbf{k}R}\,. \tag{3.6}$$

All fields are functions of the coordinates $(x_0, x_1, x_2, r, \theta)$, but the periodicity $\theta \sim \theta + 2\pi/\mathbf{k}$

is modified in two ways:

- The shift by $2\pi R$ in x_3, expressed in (3.4), implies that as we cross the $\theta = 2\pi/\mathbf{k}$ ray a translation by $2\pi R$ in x_3 is needed in order to patch smoothly with the $\theta = 0$ ray. Since x_3-momentum corresponds to conserved instanton charge in the low-energy SYM, we find that we have to add to the standard SYM action an additional term

$$\frac{1}{16\mathbf{k}\pi}\int_{\theta=0} \mathrm{tr}(F \wedge F)\,, \tag{3.7}$$

  where the integral is performed on the ray at $\theta = 0$.

- the R-symmetry twist γ introduces phases in the relation between values of fields at $\theta = 0$ and at $\theta = 2\pi/\mathbf{k}$. Of the five (gauge group adjoint-valued) scalar fields $\Phi^6, \ldots, \Phi^{10}$ (corresponding to M5-brane fluctuations in directions $6, \ldots, 10$) the last three $\Phi^8, \Phi^9, \Phi^{10}$ are neutral under γ and hence periodic in θ, while the combination $Z \equiv \Phi^6 + i\Phi^7$ satisfies

$$Z(x_0, x_1, x_2, r, \theta + \tfrac{2\pi}{\mathbf{k}}) = \omega^2 \Omega^{-1} Z(x_0, x_1, x_2, r, \theta)\Omega\,. \tag{3.8}$$

  where we have included an arbitrary gauge transformation $\Omega(x_0, x_1, x_2, r) \in SU(n)$. The gluinos have similar boundary conditions with appropriate $\exp(\pm\pi/\mathbf{k})$ phases.

At the origin $\mathbf{z} = 0$, which is the tip of the cone $\mathbb{C}/\mathbb{Z}_k$, boundary conditions need to be specified and additional 2+1D degrees of freedom need to be added. These degrees of freedom and their interactions with the bulk SYM fields are the main focus of this paper and will be discussed in §3.2.4. But at this point we can make a quick observation. When a BPS string of the $(2, 0)$-theory wraps the S^1 of (3.5) we get the W-boson of the effective 4+1D SYM. The circle has radius $\mathbf{k}R$ and so the mass of the W-boson is $2\pi\mathbf{k}R\tilde{V}$. On the other hand, the BPS string can also wrap the minicircle S^1_m whose radius is only R. The resulting particle in 2+1D has mass $2\pi R\tilde{V}$ which is $1/\mathbf{k}$ of the mass of the W-boson. Its charge is also $1/\mathbf{k}$ of the charge of the W-boson. This is our first hint that we are dealing with a system that exhibits a Fractional Quantum Hall Effect. We will soon see that indeed a BPS string that wraps S^1_m can be identified with a *quasiparticle* of FQHE.

3.2.2 Symmetries

Now, let us discuss the symmetries of the theory at a generic point on the Coulomb branch. The continuous isometries of $\mathbf{M}_3$ are generated by translations of x_3 and rotations of the $\mathbf{z}$-plane. We denote the latter by $SO(2)_{\mathbf{z}}$ and normalize the respective charge so that $d\mathbf{z}$ has charge $+1$. The isometry group of $\mathbf{M}_3$ also contains a discrete $\mathbb{Z}_2$ factor generated by the orientation-preserving isometry

$$(x_3, \mathbf{z}) \mapsto (-x_3, \overline{\mathbf{z}}).$$

This by itself does not preserve our setting because it converts the R-symmetry twist γ to γ^{-1}. To cure this problem, we introduce an extra reflection $x_7 \to -x_7$, and finally, in order to preserve parity we also introduce another reflection, say, $x_9 \to -x_9$. Altogether, we define the discrete symmetry $\mathbb{Z}_2'$ to be generated by

$$(x_0, x_1, x_2, x_3, \mathbf{z}, x_6, x_7, x_8, x_9, x_{10}) \mapsto (x_0, x_1, x_2, -x_3, \overline{\mathbf{z}}, x_6, -x_7, x_8, -x_9, x_{10})\,. \qquad [\mathbb{Z}_2'] \tag{3.9}$$

Next, the $SO(2)$ subgroup of the R-symmetry that corresponds to rotations in the $6-7$ plane will be referred to as $SO(2)_\gamma$ and normalized so that $\Phi^6 + i\Phi^7$ has charge $+1$. The $SU(2) = \mathrm{Spin}(3)$ subgroup of the R-symmetry that corresponds to rotations in the $8, 9, 10$ directions will be referred to as $SU(2)_R$. For future reference we also denote the $SO(2)$ subgroup of rotations in the $8, 9$ plane by $SO(2)_r$.

The parity symmetry of M-theory [28], which acts as reflection on an odd number of dimensions combined with a reversal of the 3-form gauge field ($C_3 \to -C_3$) can also be used to construct a symmetry of our background. We define $\mathbb{Z}_2''$ as the discrete symmetry generated by the reflection that acts as

$$x_{10} \to -x_{10}, \quad C_3 \to -C_3\,. \qquad [\mathbb{Z}_2'']$$

This symmetry preserves the M5-brane configuration and the twist. We summarize the symmetries in the following table:

$SO(2)_{\mathbf{z}}$	rotations of the $\mathbf{z}$ ($x_4 - x_5$) plane;
$SO(2)_\gamma$	rotations of the $x_6 - x_7$ plane;
$SU(2)_R$	rotations of the x_8, x_9, x_{10} plane;
$SO(2)_r$	rotations of the x_8, x_9 plane;
$\mathbb{Z}_2'$	reflection in directions x_3, x_5, x_7, x_{10};
$\mathbb{Z}_2''$	reflection in direction x_{10} (and $C_3 \to -C_3$);

We denote the conserved charges associated with $SO(2)_{\mathbf{z}}$, $SO(2)_\gamma$, and $SO(2)_r$ by $q_{\mathbf{z}}$, q_γ, and q_r, respectively. These are the spins in the $4-5$, $6-7$, and $8-9$ planes. The supersymmetry generators are also charged under these groups, and the background preserves those supercharges for which $q_{\mathbf{z}} + q_\gamma = 0$. These observations will become useful in §3.3, where we will study the quantum numbers of the quasi-particles.

3.2.3 Relation to D3-(p, q)5-brane systems

As we have seen in §3.2.1, following dimensional reduction on the S^1 fiber of (3.5), we get a low-energy description in terms of 4+1D SYM on the cone $\mathbb{C}/\mathbb{Z}_{\mathbf{k}}$, interacting with additional (as yet unknown, but to be described below) degrees of freedom at the tip of the cone (at $x_4 = x_5 = 0$). These additional degrees of freedom are three-dimensional and can be expressed in terms of $SU(2)$ Chern-Simons theory coupled to the IR limit of a $U(1)$ gauge theory with two charged hypermultiplets (with $\mathcal{N} = 4$ supersymmetry in $2+1$D).

The latter is the self-mirror theory introduced in [56], and named $T(SU(2))$ by Gaiotto and Witten [54]. The arguments leading to the identification of the degrees of freedom at the tip of the cone were presented, in a somewhat different but related context, in [27]. The idea is to relate the local degrees of freedom of M-theory on the geometry of §3.2.1 to those of a (p, q) 5-brane of type-IIB, as originally done in [34], and then map our two M5-branes to two D3-branes, to obtain the problem of two D3-branes ending on a (p, q) 5-brane. This is precisely the problem that was solved in [54] in terms of $T(SU(2))$, which thus also furnishes the solution to our problem. On the Coulomb branch, the gauge part of the system reduces to $U(1)$ Chern-Simons theory interacting with $T(U(1))$, which reproduces (3.1). Although the details of the argument will not be needed for the rest of this paper, we will review them below for completeness. More details can be found in [27].

Our geometry in directions $3, \ldots, 7$ is of the form $(S^1 \times \mathbb{C}^2)/\mathbb{Z}_{\mathbf{k}}$, and leads to a $(1, \mathbf{k})$ 5-brane. This was demonstrated in [34] by replacing $\mathbb{C}^2$ with a Taub-NUT space, whose metric can be written as

$$ds^2 = \left(1 + \frac{\tilde{R}}{2\tilde{r}}\right)^{-1} (dy + \cos\tilde{\theta}\, d\tilde{\phi})^2 + \left(1 + \frac{\tilde{R}}{2\tilde{r}}\right) [d\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\phi}^2)]\,, \tag{3.10}$$

where y is a periodic coordinate with range $0 \leq y < 2\pi$. We then add an additional S^1, parameterized by x_3 as in (3.2). The plane $\mathbb{C}$ that appears in (3.2) is now embedded in the $\mathbb{C}^2$ tangent space of the Taub-NUT space at the origin $\tilde{r} = 0$, and is recovered in the limit $\tilde{R} \to \infty$. In that limit, and with a change of variables $\tilde{r} = \tilde{R}r^2$, we can identify the $\mathbb{C}$ plane of (3.2) as a plane at constant $(\tilde{\theta}, \tilde{\phi})$ (say $\tilde{\theta} = \pi/2$ and $\tilde{\phi} = 0$), and the $\mathbf{z} \equiv x_4 + ix_5$ coordinate of (3.2) is identified with

$$\mathbf{z} = re^{iy} = \sqrt{\tilde{R}\tilde{r}}\, e^{iy}.$$

In this limit $\tilde{R} \to \infty$, the x_6, x_7 plane is identified with a transverse plane to the $\mathbf{z}$-plane, which we can take to be given by $\tilde{\phi} = \pm\pi/2$ and $y = 0$. We now return to the finite $\tilde{R}$ geometry, and impose the $\mathbb{Z}_{\mathbf{k}}$ equivalence of (3.2) by setting

$$(x_3, y, \tilde{r}, \tilde{\theta}, \tilde{\phi}) \sim (x_3 + 2\pi R, y - \tfrac{2\pi}{\mathbf{k}}, \tilde{r}, \tilde{\theta}, \tilde{\phi})\,.$$

We then wrap two M5-branes on the $(\tilde{\theta} = \frac{\pi}{2}, \tilde{\phi} = 0)$ subspace of this 5-dimensional geometry. In the limit $\tilde{R} \to \infty$ this reproduces the setting of §3.2.1.

The technique that Witten employed in [34] is to convert the Taub-NUT geometry to a D6-brane by reduction on the y-circle from M-theory to type-IIA, and then apply T-duality on the x_3-circle to get type-IIB with a complex string coupling constant of the form

$$\tau_{\rm IIB} = \frac{2\pi i}{g_{\rm IIB}} - \frac{1}{\mathbf{k}}\,,$$

which turns out to be strongly coupled ($g_{\rm IIB} \to \infty$) in the limit $\tilde{R} \to \infty$, but can, in turn, be converted to weak coupling with an $\mathrm{SL}(2, \mathbb{Z})$ transformation

$$\tau_{\rm IIB} \to \tau'_{\rm IIB} = \frac{\tau_{\rm IIB}}{\mathbf{k}\tau_{\rm IIB} + 1} = \tfrac{1}{\mathbf{k}} + \frac{ig_{\rm IIB}}{2\pi\mathbf{k}^2} \to \tfrac{1}{\mathbf{k}} + i\infty\,.$$

As explained in [34], the combined transformations convert the Taub-NUT geometry to a 5-brane of (p, q)-type $(1, \mathbf{k})$ [where $\mathbf{k}$ is the NS5-charge and 1 is the D5-charge]. It also converts the M5-branes to D3-branes. The boundary degrees of freedom where the two D3-branes end on the $(1, \mathbf{k})$ 5-brane were found in [54] as follows. Let A denote the boundary $2+1$D value of the $SU(2)$ gauge field of the D3-branes (with the superpartners left implicit). One then notes that

$$\begin{pmatrix} 1 \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{k} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Thus a $(1, \mathbf{k})$ 5-brane can be obtained from an NS5-brane by a combining the two transformations: $\tau \to \tau + \mathbf{k}$, followed by $\tau \to -1/\tau$. Each transformation can be implemented on the boundary conditions. The $\tau \to \tau + \mathbf{k}$ transformation introduces a level-$\mathbf{k}$ Chern-Simons theory expressed in terms of an ancillary $SU(2)$ gauge field that we denote by A', and the $\tau \to -1/\tau$ (S-duality) transformation introduces $2+1$D degrees of freedom, named $T(SU(2))$ by Gaoitto and Witten, that couple to both the A and A' gauge fields. $T(SU(2))$ was identified with the Intriligator-Seiberg theory [56] that is defined as the low-energy limit of $\mathcal{N} = 4$ $U(1)$ gauge theory coupled to two hypermultiplets. The theory has a classical $SU(2)$ flavor symmetry (which will ultimately couple to, say, the gauge field A), and it also has a $U(1)$ global symmetry under which only magnetic operators are charged, and this symmetry is enhanced to $SU(2)$ in the (strongly coupled) low-energy limit. This $SU(2)$ is then coupled to A'. It is also not hard to check that A is the $r \to 0$ limit of the $4+1$D gauge field on the cone. To see this, consider the T^2 formed by varying (x_3, y) for fixed r, θ, and ϕ. The $\mathrm{SL}(2, \mathbb{Z})$ transformation $\begin{pmatrix} 1 & 0 \\ \mathbf{k} & 1 \end{pmatrix}$ converts 1-cycle from $(0, 0)$ to $(2\pi R, -2\pi/\mathbf{k})$ into the 1-cycle from $(0, 0)$ to $(2\pi \mathbf{k} R, 0)$, and this is precisely the 1-cycle used in the reduction from the $(2, 0)$-theory to 4+1D SYM.

3.2.4 Appearance of the fractional quantum Hall effect

On the Coulomb branch the $SU(2)$ gauge group of 4+1D SYM is broken to $U(1)$. At energies below the breaking scale, the $SU(2)$ gauge fields A and A' reduce to $U(1)$ gauge fields which we denote by $\mathbf{A}$ and $\mathfrak{a}$. The theory $T(SU(2))$ reduces to $T(U(1))$ which is described by the action [54] $(1/2\pi) \int \mathbf{A} \wedge d\mathfrak{a}$. The total gauge part of the action at the tip of the cone is therefore given by (3.1). As we have already seen, the BPS strings that wrap the minicircle S^1_m have fractional charge $1/\mathbf{k}$ under the bulk $\mathbf{A}$ which we have now identified as the unbroken $U(1)$ gauge field of the bulk 4+1D SYM. If we move such a string away from the tip, adiabatically, we will get a string that, in the (x_3, y) coordinates of §3.2.3, wraps the 1-cycle from $(0, 0)$ to $(2\pi R, -2\pi/\mathbf{k})$. This implies that it has one unit of charge under $\mathfrak{a}$, which lends credence to the proposal of identifying such a string with a quasi-particle of FQHE. The quasi-particle is confined to $\mathbb{R}^{2,1}$, because everywhere else a wound string is longer than the BPS bound $2\pi R$.

Following the breaking of $SU(2)$ to $U(1)$, the bulk 4+1D W-boson gets a mass. The W-boson corresponds to a $(2, 0)$-string wound around the S^1 fiber of (3.5), and the homotopy

class of the bulk S^1 fiber is $\mathbf{k}$ times the homotopy class of the minicircle S^1_m. It is therefore clear that, in principle, we should be able to design a process in which a bulk W-boson reaches the tip of the cone and breaks-up into $\mathbf{k}$ strings that wrap the minicircle:

$$W \longrightarrow \mathbf{k} \text{ quasi-particles.} \tag{3.11}$$

Alternatively, it should be possible to describe the W-boson as a bound state of $\mathbf{k}$ quasi-particles. In §3.4-§3.6, we will show how this works in the limit of large $\mathbf{k}$. Before we proceed to this analysis, which is the main focus of our paper, let us compute the spin quantum numbers of the quasi-particles.

3.3 Quasi-particles

The quasi-particle is obtained by wrapping the $(2,0)$ BPS string on the minicircle S^1_m. Its quantum numbers can be deduced by quantization of the zero-modes of the low-energy fermions that live on the BPS string of the $(2,0)$-theory. Let us begin by reviewing the low-energy fermionic degrees of freedom on a BPS string. We assume that the M5-branes are in directions $0,\ldots,5$, separated in direction 10, and the BPS string is in direction x_3. We first ignore the equivalence (3.2) and the R-symmetry twist. For simplicity we will now refer to rotation groups as $SO(m)$ instead of $\mathrm{Spin}(m)$. Thus, the VEV breaks the R-symmetry to $SO(4)_R \subset SO(5)_R$, and the presence of the string breaks the Lorentz group down to $SO(1,1)\times SO(4)$. We will denote the last factor by $SO(4)_T$, and we will describe representations of $SO(1,1)\times SO(4)_T \times SO(4)_R$ as $(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r_4})_\mathbf{s}$, where $(\mathbf{r_1},\mathbf{r_2})$ is a representation of $SO(4)_T \sim SU(2)\times SU(2)$, $(\mathbf{r_3},\mathbf{r_4})$ is a representation of $SO(4)_R \sim SU(2)\times SU(2)$, and $\mathbf{s}$ is an $SO(1,1)$ charge (spin). The representation of the unbroken supersymmetry charges is the same as the supersymmetry that is preserved by an M2-brane ending on an M5-brane. If the M2-brane is in directions $0,3,10$ and the M5-brane is in directions $0,1,2,3,4,5$ then a preserved SUSY parameter ϵ satisfies

$$\epsilon = \Gamma^{0123456789\natural}\epsilon = \Gamma^{03\natural}\epsilon = \Gamma^{012345}\epsilon, \tag{3.12}$$

where we denote $\natural \equiv 10$, to avoid ambiguity. The SUSY parameter therefore transforms as

$$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})_{+\frac{\mathbf{1}}{\mathbf{2}}} \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})_{-\frac{\mathbf{1}}{\mathbf{2}}}\,.$$

On the worldsheet of the BPS string there are 4 scalars X^A $(A=1,2,4,5)$ that correspond to translations of the string in transverse directions. These are in the representation $(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})_\mathbf{0}$. In addition, there are fermions in

$$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})_{+\frac{\mathbf{1}}{\mathbf{2}}} \oplus (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})_{-\frac{\mathbf{1}}{\mathbf{2}}}\,. \tag{3.13}$$

Now, consider this theory on $\mathbb{R}^{2,1}\times \mathbf{M}_3$ and let the BPS string be at rest at $x_1 = x_2 = 0$. It thus breaks the Lorentz group $SO(2,1)$ to the rotation group $SO(2)$ in the $x_1 - x_2$ plane,

which we denote by $SO(2)_J$. The representations appearing in the brackets of (3.13) refer to $SO(4)_T \times SO(4)_R$, but in our setting, according to the discussion above, we have to reduce $SO(4)_T \to SO(2)_J \times SO(2)_{\mathbf{z}}$ and $SO(4)_R \to SO(2)_\gamma \times SO(2)_r$. Thus, denoting representations as

$$(q_J, q_{\mathbf{z}}, q_\gamma, q_r)_s \,, \tag{3.14}$$

we decompose the left-moving spinors of (3.13) as

$$(+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2})_{+\frac{\mathbf{1}}{\mathbf{2}}} \oplus (+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2})_{+\frac{\mathbf{1}}{\mathbf{2}}} \oplus (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2})_{+\frac{\mathbf{1}}{\mathbf{2}}} \oplus (-\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2})_{+\frac{\mathbf{1}}{\mathbf{2}}} \tag{3.15}$$

and the right-movers as

$$(+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2})_{-\frac{\mathbf{1}}{\mathbf{2}}} \oplus (+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})_{-\frac{\mathbf{1}}{\mathbf{2}}} \oplus (-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2})_{-\frac{\mathbf{1}}{\mathbf{2}}} \oplus (-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})_{-\frac{\mathbf{1}}{\mathbf{2}}} \tag{3.16}$$

These modes can be described by fermionic fields on the string worldsheet, which are functions of (x_0, x_3). To get the quantum numbers of the lowest-energy multiplet we need to find the zero-modes of these fermionic fields. The twisted boundary conditions such as (3.8) also introduce twists on some of the modes (3.15)-(3.16). On a field $\psi(x_0, x_3)$ with charges $q_{\mathbf{z}}$ and q_γ, these boundary conditions are

$$\psi(x_0, x_3 + 2\pi R) = \omega^{2(q_{\mathbf{z}}+q_\gamma)}\psi(x_0, x_3).$$

The only zero modes are therefore of those modes with $q_{\mathbf{z}} + q_\gamma = 0$. These have quantum numbers

$$(+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2})_{+\frac{\mathbf{1}}{\mathbf{2}}} \oplus (-\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2})_{+\frac{\mathbf{1}}{\mathbf{2}}} \oplus (+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2})_{-\frac{\mathbf{1}}{\mathbf{2}}} \oplus (-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})_{-\frac{\mathbf{1}}{\mathbf{2}}} \tag{3.17}$$

Quantizing these modes gives a multiplet with quantum numbers

$$\begin{array}{llll} (q_J^{(0)} - \frac{1}{2}, & q_{\mathbf{z}}^{(0)}, & q_\gamma^{(0)}, & q_r^{(0)} - \frac{1}{2}), \\ (q_J^{(0)}, & q_{\mathbf{z}}^{(0)} - \frac{1}{2}, & q_\gamma^{(0)} + \frac{1}{2}, & q_r^{(0)}), \end{array} \qquad \begin{array}{llll} (q_J^{(0)}, & q_{\mathbf{z}}^{(0)} + \frac{1}{2}, & q_\gamma^{(0)} - \frac{1}{2}, & q_r^{(0)}), \\ (q_J^{(0)} + \frac{1}{2}, & q_{\mathbf{z}}^{(0)}, & q_\gamma^{(0)}, & q_r^{(0)} + \frac{1}{2}), \end{array} \tag{3.18}$$

where the charges $q_J{}^{(0)}$, $q_{\mathbf{z}}{}^{(0)}$, $q_\gamma{}^{(0)}$, $q_r{}^{(0)}$ still need to be determined. To determine them, consider the discrete symmetry $\mathbb{Z}_2'$, defined in §3.2.2. It preserves the setting and the BPS particle but does not commute with all the charges q_J, $q_{\mathbf{z}}$, q_γ, q_r . It acts on the charges as follows:

$$q_J \to q_J \,, \quad q_{\mathbf{z}} \to -q_{\mathbf{z}} \,, \quad q_\gamma \to -q_\gamma \,, \quad q_r \to q_r \,. \qquad \text{[generator of } \mathbb{Z}_2'\text{]}$$

The constants $q_J{}^{(0)}$, $q_{\mathbf{z}}{}^{(0)}$, $q_\gamma{}^{(0)}$, $q_r{}^{(0)}$ must therefore be chosen so that the charges (3.18) will be invariant, as a set, under $\mathbb{Z}_2'$. In other words, $\mathbb{Z}_2'$ is allowed to permute the states in (3.18), but must convert an existing state to an existing state. This is only possible if both

${q_{\mathbf{z}}}^{(0)}$ and ${q_\gamma}^{(0)}$ vanish. The BPS states are therefore in a multiplet with quantum numbers given by:

$$({q_J}^{(0)}-\tfrac{1}{2},0,0,{q_r}^{(0)}-\tfrac{1}{2})\oplus({q_J}^{(0)},+\tfrac{1}{2},-\tfrac{1}{2},{q_r}^{(0)})\oplus({q_J}^{(0)},-\tfrac{1}{2},+\tfrac{1}{2},{q_r}^{(0)})\oplus({q_J}^{(0)}+\tfrac{1}{2},0,0,{q_r}^{(0)}+\tfrac{1}{2})\,.$$

Spin-statistics requires ${q_J}^{(0)}$ and ${q_r}^{(0)}$ to be half-integral. To determine them we note that the setting of (3.4) can be defined for any value of $\mathbf{k}$, not necessarily an integer (as suggested in [34]). And so we can determine the quantum numbers from the limit $\mathbf{k}\to\infty$ at which the multiplet must become part of the multiplet of the wrapped string of the $(2,0)$-theory. This determines the charges up to an overall sign (which can be determined arbitrarily and flipped with a parity transformation). So we pick ${q_J}^{(0)}=-{q_r}^{(0)}=\frac{1}{2}$ and find the following multiplet structure:

$$(0,0,0,-1)\oplus(+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})\oplus(+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2})\oplus(1,0,0,0)\,. \tag{3.19}$$

As a corollary, we can immediately restrict the types of processes described in (3.11). Let us write down the q_J, $q_{\mathbf{z}}$, q_γ, and q_r quantum numbers of the W-boson supermultiplet. The bosons (vectors and scalars) are in

$$(\pm1,0,0,0)\oplus(0,\pm1,0,0)\oplus(0,0,\pm1,0)\oplus(0,0,0,\pm1)\,. \tag{3.20}$$

and the gluinos are ingeom

$$(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2})\,. \qquad \text{[even number of } (-\tfrac{1}{2})\text{'s]} \tag{3.21}$$

Starting with a W-boson with charges $(1,0,0,0)$, a process such as

$$W\text{-boson}\longrightarrow\mathbf{k}\text{ quasi-particles.} \tag{3.22}$$

can only produce $\mathbf{k}$ quasi-particles of charge $(1,0,0,0)$, and $(\mathbf{k}-1)$ units of orbital angular momentum need to convert into spin. We therefore expect that if the W-boson's velocity u in the x_1-x_2 plane is small, the amplitude will be suppressed as $u^{\mathbf{k}-1}$.

The process (3.22) also suggests that the W boson can be viewed as a bound state of $\mathbf{k}$ quasi-particles. This is similar to the well-known result in FQHE theory that the electron can be regarded as a bound state of $\mathbf{k}$ fractionally charged edge-states. The edge-states are the low-energy excitations of the Chern-Simons theory that reside on the boundary, or on impurities in the bulk. In the analogy the FQHE system, our quasi-particles correspond to external impurities that couple to the Chern-Simons theory gauge field.

Our goal is to develop a concrete description of the W-boson as a composite of $\mathbf{k}$ quasi-particles. For this purpose we will first need to switch to a dual formulation of the low-energy theory whereby the quasi-particles are fundamental.

3.4 The large k limit

A weakly-coupled dual formulation of our system can be constructed in the limit $\mathbf{k} \to \infty$. In FQHE terminology, this is the *small filling fraction* regime which in ordinary systems corresponds to very strong interactions. More insight can be gained in this limit by choosing a different fibration structure for $\mathbf{M}_3$ than the one represented in (3.5). While (3.5) is convenient to work with because the fibers are of constant size and are geodesics, the fibration was singular at the origin $\mathbf{z} = 0$ — indeed the tip of the cone is singular, and the fiber over $\mathbf{z} = 0$ is smaller by a factor of $\mathbf{k}$ from the generic one.

Instead, in this section we will represent $\mathbf{M}_3$ as a smooth fibration in another way. The base is the well-known cigar geometry and the fiber corresponds to a loop at constant $|\mathbf{z}|$. (See also [58][59] for other uses of this technique.) We will then reduce the $(2,0)$-theory to $4+1$D SYM along this fiber. The fiber's size varies and the base's geometry is curved, but nevertheless this representation is very useful, as we shall see momentarily.

3.4.1 Cigar geometry

To arrive at the the alternative fibration we change variables on $\mathbf{M}_3$ from $(x_3, \mathbf{z})$ to x_3 and

$$\tilde{\mathbf{z}} \equiv \exp\left(-\frac{ix_3}{\mathbf{k}R}\right)\mathbf{z} \equiv re^{i\bar{\theta}}\,. \tag{3.23}$$

We then write the metric as

$$ds^2 = dx_3^2 + |d\mathbf{z}|^2 = \widetilde{\alpha}(dx_3 + \frac{r^2}{\mathbf{k}R\widetilde{\alpha}}d\bar{\theta})^2 + dr^2 + \widetilde{\alpha}^{-1}r^2d\bar{\theta}^2\,, \qquad (\widetilde{\alpha} \equiv 1 + \frac{r^2}{\mathbf{k}^2R^2}) \tag{3.24}$$

This metric describes a circle fibration over a cigar-like base with metric

$$ds_B^2 = dr^2 + \widetilde{\alpha}^{-1}r^2d\bar{\theta}^2 = dr^2 + (\frac{\mathbf{k}^2R^2r^2}{\mathbf{k}^2R^2 + r^2})d\bar{\theta}^2\,. \tag{3.25}$$

We denote the cigar space by Υ. Note that the cigar-metric is smooth everywhere and for $r \gg \mathbf{k}R$ it behaves like a cylinder $\mathbb{R}_+ \times S^1$, where S^1 has radius $\mathbf{k}R$. The "global angular form" of the circle fibration is

$$\chi \equiv dx_3 + \frac{r^2}{\mathbf{k}R\widetilde{\alpha}}d\bar{\theta} \equiv dx_3 + R\mathfrak{a}\,, \tag{3.26}$$

where we have defined the 1-form

$$\mathfrak{a} \equiv \frac{r^2}{\mathbf{k}R^2\widetilde{\alpha}}d\bar{\theta} = (\frac{\mathbf{k}r^2}{\mathbf{k}^2R^2 + r^2})d\bar{\theta}\,. \tag{3.27}$$

In this context, $\mathfrak{a}$ is a $U(1)$ gauge field on the cigar with associated field-strength

$$d\chi \equiv d\mathfrak{a} = \frac{2\mathbf{k}^3R^2r}{(\mathbf{k}^2R^2 + r^2)^2}dr \wedge d\bar{\theta}\,.$$

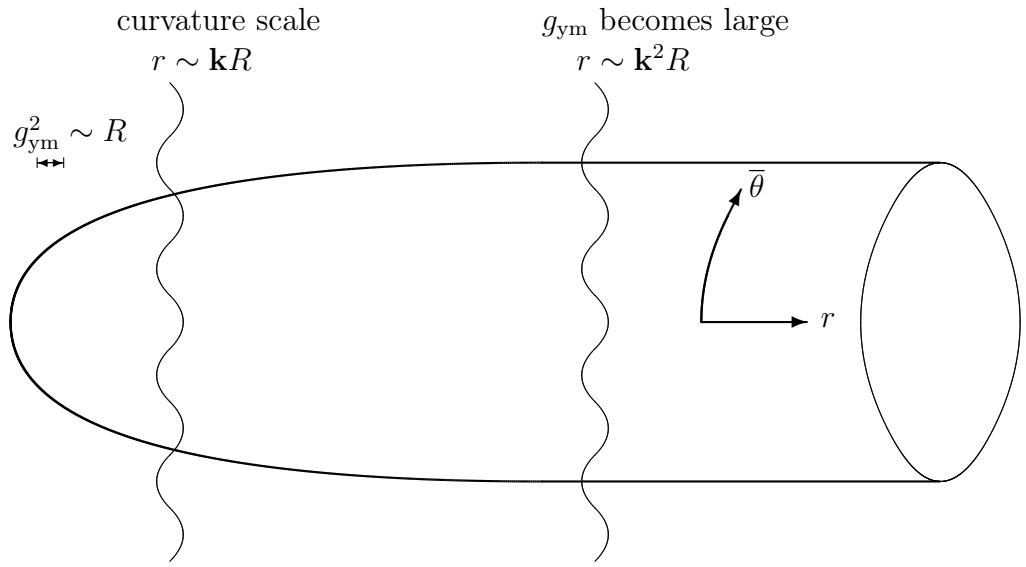


Figure 3.2: The cigar geometry with the typical scales indicated. The curvature of the cigar sets the length scale $\mathbf{k}R$, and the $4+1$D SYM coupling constant sets the length scale $g^2_{\rm ym}$.

The total magnetic flux of this gauge field is $\int_B d\mathfrak{a} = 2\pi\mathbf{k}$.

An anti-self-dual field $H = -{}^*H$ on $\mathbf{M}_3 \times \mathbb{R}^{2,1}$ can be reduced along the fibers of the circle fibration (3.24) to obtain a $4+1$D gauge field strength f on $\Upsilon \times \mathbb{R}^{2,1}$ as follows:

$$H = \left(dx_3 + \frac{r^2}{\mathbf{k}R\widetilde{\alpha}} d\overline{\theta}\right) \wedge f - \widetilde{\alpha}^{-\frac{1}{2}} ({}^*f). \tag{3.28}$$

Here ${}^*f$ is the 4+1D Hodge dual of the 2-form f on $\Upsilon \times \mathbb{R}^{2,1}$. The coupling constant of the effective $4+1$D super Yang-Mills theory for f is

$$g^2_{\rm ym} = \widetilde{\alpha}^{1/2} R = (1 + \frac{r^2}{\mathbf{k}^2 R^2})^{\frac{1}{2}} R\,. \tag{3.29}$$

The coupling constant $g^2_{\rm ym}$ has dimensions of length and can be compared to the length scale set by the order of magnitude of the curvature of the cigar metric at the origin – this length-scale is $\mathbf{k}R$. For $r \sim \mathbf{k}R$ we find $g^2_{\rm ym} \ll \mathbf{k}R$, and so the Yang-Mills theory is weakly coupled on length scales of the order of the curvature. The Yang-Mills theory becomes strongly coupled only when the two scales become comparable, which happens for $r \sim \mathbf{k}^2 R$, and therefore for large $\mathbf{k}$ our low-energy semi-classical $4+1$D SYM approximation is valid for $r \ll \mathbf{k}^2 R$. The various length scales are depicted in Figure 3.2.

3.4.2 Equations of motion

The bosonic fields of maximally supersymmetric 4+1D SYM are the gauge field and 5 adjoint scalars. The scalars correspond to the relative motion of the M5-branes in directions $x_6, \ldots, x_{10}$. We will be interested in supersymmetric solutions where only the scalar

corresponding to direction x_{10} can be nonzero. We will therefore ignore the remaining 4 scalars, as well as the fermions, and denote the single adjoint-valued relevant scalar by Φ. The boundary conditions at infinity are

$$\Phi \to \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} \qquad \text{(up to a gauge transformation)},$$

where $v \equiv 2\pi\tilde{V}$, where $\tilde{V}$ is the tension of the BPS string defined in §3.2.1.

We set polar coordinates in the $x_1 - x_2$ plane by

$$\rho \equiv \sqrt{x_1^2 + x_2^2}\,, \qquad x_1 + ix_2 = \rho e^{i\varphi}\,. \tag{3.30}$$

The SYM theory is therefore formulated on a space with 4+1D metric

$$ds^2 = -dt^2 + dr^2 + \widetilde{\alpha}^{-1} r^2 d\overline{\theta}^2 + d\rho^2 + \rho^2 d\varphi^2\,.$$

The action contains three terms,

$$I_{\text{bosonic}} = I_\Phi + I_{\text{YM}} + I_\theta\,, \tag{3.31}$$

where I_Φ is the action of the scalar field, I_{YM} is the standard Yang-Mills action with variable coupling constant, and I_θ is the 4+1D θ-term that arises due to the nonzero connection $\mathfrak{a}$ (see (3.27)). We will only consider $\overline{\theta}$-independent field configurations. For such configurations the explicit expressions for the terms in the action are

$$I_\Phi = \tfrac{1}{2} R\,\mathrm{tr}\int \big[(D_0\Phi)^2 - (D_\rho\Phi)^2 - \tfrac{1}{\rho^2}(D_\varphi\Phi)^2 - (D_r\Phi)^2\big] r\rho dr d\rho d\varphi dt\,, \tag{3.32}$$

$$I_{\text{YM}} = \tfrac{1}{2} R^{-1}\,\mathrm{tr}\int \tfrac{1}{\widetilde{\alpha}}\big(F_{0r}^2 + F_{0\rho}^2 - F_{r\rho}^2 - \tfrac{1}{\rho^2}F_{r\varphi}^2 - \tfrac{1}{\rho^2}F_{\rho\varphi}^2\big) r\rho dr d\rho d\varphi dt\,, \tag{3.33}$$

$$I_\theta = \mathrm{tr}\int \tfrac{r^2}{\mathbf{k}R^2\widetilde{\alpha}}\big(F_{0r}F_{\rho\varphi} - F_{0\rho}F_{r\varphi} + F_{0\varphi}F_{r\rho}\big) dr d\rho d\varphi dt\,, \tag{3.34}$$

where D_μ is the covariant derivative of an adjoint-valued field. The equations of motion are:

$$0 = D^\beta F_{\alpha\beta} + D_r F_{\alpha r} - \tfrac{1}{r}F_{\alpha r} - i\tfrac{r^2}{\mathbf{k}^2R^2}[D_\alpha\Phi, \Phi]\,, \tag{3.35}$$

$$0 = D^\beta F_{r\beta} - i\tfrac{r^2}{\mathbf{k}^2R^2}[D_r\Phi, \Phi]\,, \tag{3.36}$$

$$0 = D^\alpha D_\alpha\Phi + D_r D_r\Phi + \tfrac{1}{r}D_r\Phi\,. \tag{3.37}$$

where $\alpha, \beta = 0, 1, 2$.

3.5 Integrally charged particles as bound states of quasi-particles

We now have two alternative descriptions of the low-energy limit in terms of 4+1D SYM. In the first description, studied in §3.2, the 4+1D SYM theory is formulated on a cone, with

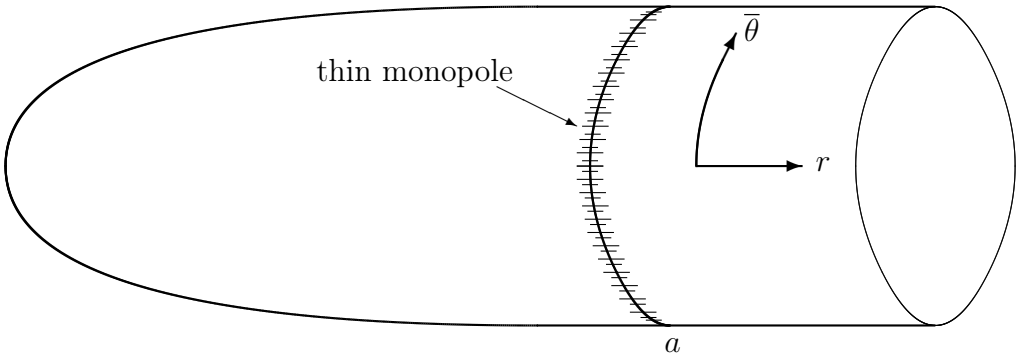


Figure 3.3: In the limit $va^2 \gg 1$ the soliton is approximately described by the Prasad-Sommerfield solution (of width $1/va$) near $r = a$ and $\rho = 0$. Note that $\rho = \sqrt{x_1^2 + x_2^2}$ and the directions x_1, x_2 are not drawn since they are perpendicular to the r, θ directions.

extra degrees of freedom at the tip. In the second description, studied in §3.4, the 4+1D SYM theory is formulated on a cigar geometry. The latter description is most suitable in the large $\mathbf{k}$ limit, because the strongly-coupled region is pushed to $r = \infty$ (see Figure 3.2). The quasi-particles that we studied in §3.3 are the fundamental fields of 4+1D SYM in the cigar-setting. We have seen that $\mathbf{k}$ quasi-particles can form a bound state that is free to move into the bulk of the cone. Let us now identify this state in the cone-setting.

From the perspective of the $(2, 0)$-theory, the bound state is a string wrapped on the fiber of (3.5). Let us consider such a string wrapped on the fiber at the cone base point given by coordinates $r = a$ and $\theta = x_1 = x_2 = 0$, with variable x_3. In the cigar variables, this reduces to a string at fixed $r = a$ and $x_1 = x_2 = 0$ but variable $\overline{\theta}$. Recall that on the Coulomb branch of $SU(2)$ 4+1D SYM, the monopole is a 1+1D object – a monopole-string. The bound state of $\mathbf{k}$ quasi-particles is therefore associated with a monopole-string wrapped around the $\overline{\theta}$-circle of the cigar at $r = a$, as depicted in Figure 3.3. Thanks to the θ-term (3.34), the monopole-string gains $\mathbf{k}$ units of charge, as required.

In flat space, a monopole-string is described by the Prasad-Sommerfield solution [36]. In our case, the Prasad-Sommerfield solution is a good approximation if the thickness of the monopole is small compared to the typical scale $\mathbf{k}R$ over which the coupling constant varies, and also small compared to a. In this case, setting

$$\mathbf{w} \equiv \sqrt{(r-a)^2 + \rho^2}\,,$$

we find the gauge invariant magnitude of the scalar field near the core $r = a$ to be given by

$$|\Phi| \equiv \sqrt{\mathrm{tr}(\Phi^2)/2} = \tilde{v}\coth(\tilde{v}\mathbf{w}) - \frac{1}{\mathbf{w}}\,, \tag{3.38}$$

where

$$\tilde{v} = (1 + \tfrac{a^2}{\mathbf{k}^2R^2})^{1/2}Rv$$

is the effective VEV of the normalized scalar field $\widetilde{\alpha}^{1/2}R\Phi$ at the core ($r = a$) of the monopole. The "thickness" of the Prasad-Sommerfield solution is of the order of $1/\tilde{v}$, and the condition that the monopole should be "thin" becomes

$$a \gg \frac{1}{Rv}\,.$$

If this condition is not met, the Prasad-Sommerfield solution does not provide a good approximation for the particle that corresponds to a $(2,0)$-string wrapped on the generic fiber (of size $\mathbf{k}R$) of (3.5). Nevertheless, this is a BPS state with charge $\mathbf{k}$, which can be described in the large $\mathbf{k}$ limit by a soliton solution to the equations of motion (3.35)-(3.37). The solution describes a Q-ball, and we expect the position a to be a modulus of the solution. In the next section we will derive the BPS equations that this soliton satisfies.

3.5.1 BPS equations

As we will derive in §3.5.2, the BPS equations that describe static solutions that preserve the same amount of supersymmetry as a $(2,0)$-string wrapped on a fiber of (3.5) are given by:

$$D_r\Phi = \frac{\mathbf{k}R}{r}F_{12} = F_{0r}\,, \qquad D_1\Phi = \frac{\mathbf{k}R}{r}F_{2r} = F_{01}\,, \qquad D_2\Phi = -\frac{\mathbf{k}R}{r}F_{1r} = F_{02}\,. \tag{3.39}$$

These equations imply the equations of motion (3.35)-(3.37). Assuming that A_r, A_1, A_2 are time independent, we find $D_\mu\Phi = F_{0\mu} = -D_\mu A_0$ (for $\mu = 1,2,r$), which is solved by $\Phi = -A_0$. So the equations are reduced to

$$D_r\Phi = \frac{\mathbf{k}R}{r}F_{12}\,, \qquad D_1\Phi = \frac{\mathbf{k}R}{r}F_{2r}\,, \qquad D_2\Phi = -\frac{\mathbf{k}R}{r}F_{1r}\,, \qquad \Phi = -A_0\,. \tag{3.40}$$

The nonzero A_0 is consistent with a Q-ball [63]. It can be gauged away at the expense of creating time-varying phases for the other fields, but we will not do so. We can rewrite the first three equations of (3.40) as the Prasad-Sommerfield [36] equations

$$D_i\widetilde{\Phi} = B_i \tag{3.41}$$

where

$$\widetilde{\Phi} \equiv \frac{1}{\mathbf{k}R}\Phi\,, \qquad B_i \equiv \frac{1}{2\sqrt{g}}g_{ij}\epsilon^{jkl}F_{kl}\,, \tag{3.42}$$

are defined on a 3D auxiliary space $\mathcal{W}$ parameterized by x_1, x_2, r, with metric g_{ij} given by

$$ds^2 = g_{ij}dx^i dx^j = r^2(dr^2 + dx_1^2 + dx_2^2) = r^2(dr^2 + d\rho^2 + \rho^2 d\varphi^2). \tag{3.43}$$

In §3.6.2 we will show that the problem of finding an axisymmetric (φ-independent) BPS soliton can be converted to the problem of finding a harmonic map from the AdS_3 space with metric

$$ds^2 = \frac{1}{r^2}(dr^2 + d\rho^2 + \rho^2 d\varphi^2)$$

to AdS_2, with a certain singular behavior along a Dirac-like string at $\rho = 0$ and $0 < r < a$.

We will conclude with a formula for the energy. For a static configuration, the energy is minus the integral of the Lagrangian that appears in the action (3.31). If we set $A_0 = -\Phi$ the energy for static and φ-independent configurations can be written as

$$\begin{aligned}\mathcal{E}_{\text{static}} = \pi R \,\mathrm{tr} \int \tfrac{1}{\widetilde{\alpha}}\big[F_{r\rho}^2 + \tfrac{1}{\rho^2}(F_{r\varphi} - \tfrac{r\rho}{\mathbf{k}R}D_\rho\Phi)^2 + \tfrac{1}{\rho^2}(F_{\rho\varphi} - \tfrac{r\rho}{\mathbf{k}R}D_r\Phi)^2\big] r\rho dr d\rho \\ + \pi \,\mathrm{tr} \int \big[\partial_\rho\big(\tfrac{r^2}{\mathbf{k}\widetilde{\alpha}}F_{r\varphi}\Phi\big) + \partial_r\big(\tfrac{r^2}{\mathbf{k}\widetilde{\alpha}}F_{\rho\varphi}\Phi\big)\big] dr d\rho\end{aligned} \tag{3.44}$$

The term on the RHS vanishes when the BPS equations (3.40) are satisfied, and the second line is a total derivative.

3.5.2 Derivation of the BPS equations

In this subsection we will explain how (3.39) were derived. (The rest of the paper does not rely on this subsection, and it can be skipped.) We wish to find the equations that describe the W-boson from (3.22) in terms of the low-energy fields of 4+1D SYM on $\Upsilon \times \mathbb{R}^{2,1}$, where $\mathbb{R}^{2,1}$ corresponds to directions $0, 1, 2$, and Υ was defined in §3.4.1. Let us first discuss the equations on the Coulomb branch of the $(2,0)$-theory. The contents of the low-energy theory is a free tensor multiplet with 2-form field B, field strength $H = dB$, five scalar fields $\Phi^6, \ldots, \Phi^{10}$, and chiral fermionic fields ψ in the representation of $\mathbf{4} \times \mathbf{4}$ of $SO(5,1) \times SO(5)$. We assume

$$\Phi^6 = \Phi^7 = \Phi^8 = \Phi^9 = 0$$

and only allow $\Phi^{10} \equiv \phi$ to be nonzero. The BPS equations are derived from the SUSY transformation of the fermions. Let ϵ be a constant SUSY parameter, which we represent as a 32-component spinor on which the 10+1D Dirac matrices Γ^I ($I = 0, \ldots, 10$) can act. The BPS conditions on ϵ are:

- Invariance of ϵ under simultaneous rotations by $2\pi/\mathbf{k}$ in the planes $4-5$ and $6-7$;
- Invariance of an M5-brane in direction $0, \ldots, 5$ under a SUSY transformation of 10+1D SUGRA with parameter ϵ; and
- Invariance of an M2-brane in directions $0, 3, 10$ under a SUSY transformation of 10+1D SUGRA with parameter ϵ.

Therefore, the equations are (we set $10 \equiv \natural$ in Dirac matrices):

$$\epsilon = \Gamma^{012345}\epsilon = \Gamma^{03\natural}\epsilon = \Gamma^{4567}\epsilon\,. \tag{3.45}$$

To get the BPS equations we require that the fermions ψ of the tensor multiplet of the $(2,0)$-theory be invariant under any SUSY transformation with a parameter ϵ that satisfies (3.45):

$$0 = \delta\psi = (H_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma} - \partial_\mu\phi\Gamma^{\mu\natural})\epsilon\,. \tag{3.46}$$

There are four linearly independent solutions to (3.45), and substituting these into (3.46) we find the BPS equations in the form

$$H_{03\mu} = \partial_\mu\phi\,, \qquad H_{0ij} = 0\,, \qquad (i,j = 1,2,4,5). \tag{3.47}$$

The other components of H are determined by anti-self-duality $H = -{}^*H$. We now convert the BPS equations (3.47) to $\Upsilon \times \mathbb{R}^{2,1}$ using (3.28) and the change of variables (3.23). To avoid ambiguity, we momentarily denote by x_3' and θ' the coordinates before the change of variables, so that the change of variables is given by

$$x_3 = x_3'\,, \qquad \overline{\theta} = \theta' - \frac{x_3'}{\mathbf{k}R}\,.$$

We then find:

$$0 = H_{03'r} - \partial_r\phi = H_{03'\theta'} - \partial_{\theta'}\phi = \partial_{3'}\phi = \partial_0\phi\,, \qquad 0 = H_{03'i} - \partial_i\phi\,, \qquad (i = 1,2)\,, \tag{3.48}$$

and

$$0 = H_{012} = H_{0ir} = H_{0i\theta'} = H_{0i\bar\theta} = H_{0r\theta'} = H_{0r\bar\theta}\,, \qquad (i = 1,2)\,.$$

The dual relations are

$$0 = H_{3'r\theta'} = H_{3'i\theta'} = H_{3'ir} = H_{3'12}\,, \qquad (i = 1,2)\,,$$

which are transformed in the $x_3, \overline{\theta}$ coordinates to

$$0 = H_{3r\bar\theta} = H_{3i\bar\theta} = H_{3ir} - \frac{1}{\mathbf{k}R}H_{\bar\theta ir} = H_{312} - \frac{1}{\mathbf{k}R}H_{\bar\theta 12}\,, \qquad (i = 1,2)\,. \tag{3.49}$$

Next we use the anti-self-duality conditions

$$H_{03'r} = \frac{1}{r}H_{\theta'12} = \frac{1}{r}H_{\bar\theta 12}\,, \qquad H_{03'1} = \frac{1}{r}H_{r\theta'2} = \frac{1}{r}H_{r\bar\theta 2}\,, \qquad H_{03'2} = -\frac{1}{r}H_{r\theta'1} = -\frac{1}{r}H_{r\bar\theta 1}\,,$$

and the relations (3.49) to write

$$H_{03'r} = \frac{1}{r}H_{\bar\theta 12} = \frac{\mathbf{k}R}{r}H_{312}\,, \quad H_{03'1} = \frac{1}{r}H_{r\bar\theta 2} = \frac{\mathbf{k}R}{r}H_{32r}\,, \quad H_{03'2} = -\frac{1}{r}H_{r\bar\theta 1} = -\frac{\mathbf{k}R}{r}H_{31r}\,. \tag{3.50}$$

Combining with (3.48), we end up with the BPS equations

$$\partial_r\phi = H_{03'r} = \frac{\mathbf{k}R}{r}H_{312}\,, \qquad \partial_1\phi = H_{03'1} = \frac{\mathbf{k}R}{r}H_{32r}\,, \qquad \partial_2\phi = H_{03'2} = -\frac{\mathbf{k}R}{r}H_{31r}\,, \tag{3.51}$$

and further combining with (3.28) we have

$$\partial_r\phi = \frac{\mathbf{k}R}{r}f_{12}\,, \qquad \partial_1\phi = \frac{\mathbf{k}R}{r}f_{2r}\,, \qquad \partial_2\phi = -\frac{\mathbf{k}R}{r}f_{1r}\,. \tag{3.52}$$

Altogether, we have

$$\partial_r\phi = \frac{\mathbf{k}R}{r}f_{12} = f_{0r}\,, \qquad \partial_1\phi = \frac{\mathbf{k}R}{r}f_{2r} = f_{01}\,, \qquad \partial_2\phi = -\frac{\mathbf{k}R}{r}f_{1r} = f_{02}\,. \tag{3.53}$$

The equations (3.39) are the nonabelian extension of (3.53), and the fact that they imply the equations of motion (3.35)-(3.37) shows that no additional terms are needed.

3.6 Analysis of the BPS equations

We are looking for a solution to the BPS equations (3.40) that describes the $(2,0)$-string wrapped on the fiber of (3.5). We can assume that the string is at the origin of the $x_1 - x_2$ coordinate system, and the solution will therefore be axisymmetric (independent of φ). We expect the solution to have a modulus a corresponding to the position of the string in the $x_4 - x_5$ plane. Technically, these solutions could only describe a string that is "smeared" along the angular coordinate θ of the $x_4 - x_5$ plane, so only the r coordinate of the string is fixed. The soliton is spread out in the r direction as well, but we can expect its core to be around $r \sim a$, and we will see that a can be defined via the boundary conditions at $r \to \infty$ or $\rho \to \infty$. More generally, we can look for axisymmetric solutions that describe several $(2,0)$-strings centered at different r locations, but all at $\rho = 0$. This will be described by a more complicated axisymmetric solution of (3.40). To proceed, we will treat the BPS equations as Bogomolnyi-Prasad-Sommerfield monopole equations (3.41) on a curved space (3.43).

3.6.1 Manton gauge

The Bogomolnyi monopole equations on $\mathbb{R}^3$ have the renowned Prasad-Sommerfield solution [36] for one $SU(2)$ monopole, and the general solution was given by Nahm [72]. It was given a string-theoretic interpretation in [39]. The extension to hyperbolic space is known [43], but we are unaware of an extension of Nahm's technique to the space given by the metric (3.43), and so we will proceed using other means. We adopt a remarkable ansatz developed in [45] for axially symmetric solutions. Adapted from $\mathbb{R}^3$ to our metric (3.43) we make the Ansatz:

$$\Phi = \tfrac{1}{2}(\Phi_1\sigma_1 + \Phi_2\sigma_2)\,, \qquad A = -[(\eta_1\sigma_1 + \eta_2\sigma_2)d\varphi + W_2\sigma_3 d\rho + W_1\sigma_3 dr], \tag{3.54}$$

The BPS equations then reduce to

$$\partial_\rho \Phi_1 - W_2 \Phi_2 = -\tfrac{1}{r\rho}(\partial_r \eta_1 - W_1 \eta_2)\,, \tag{3.55}$$
$$\partial_\rho \Phi_2 + W_2 \Phi_1 = -\tfrac{1}{r\rho}(\partial_r \eta_2 + W_1 \eta_1)\,, \tag{3.56}$$
$$\eta_2 \Phi_1 - \eta_1 \Phi_2 = \tfrac{\rho}{r}(\partial_\rho W_1 - \partial_r W_2)\,, \tag{3.57}$$
$$\partial_r \Phi_1 - W_1 \Phi_2 = \tfrac{1}{r\rho}(\partial_\rho \eta_1 - W_2 \eta_2)\,, \tag{3.58}$$
$$\partial_r \Phi_2 + W_1 \Phi_1 = \tfrac{1}{r\rho}(\partial_\rho \eta_2 + W_2 \eta_1)\,, \tag{3.59}$$

Next, we adapt to our metric the technique developed in [41], solving (3.55)-(3.57) by setting

$$\Phi_1 = -\frac{1}{r} f^{-1} \partial_r \chi\,, \quad \Phi_2 = \frac{1}{r} f^{-1} \partial_r f\,, \quad \eta_1 = \rho f^{-1} \partial_\rho \chi\,, \quad \eta_2 = -\rho f^{-1} \partial_\rho f\,, \tag{3.60}$$

and

$$W_1 = -f^{-1} \partial_r \chi\,, \quad W_2 = -f^{-1} \partial_\rho \chi\,. \tag{3.61}$$

where f and χ are as yet undetermined real functions of r and ρ.

3.6.2 Harmonic maps from AdS_3 to AdS_2

We plug the ansatz (3.60)-(3.61) into (3.58)-(3.59) and get:

$$0 = f\chi_{rr} + f\chi_{\rho\rho} - 2 f_r \chi_r - 2 f_\rho \chi_\rho + \tfrac{1}{\rho} f \chi_\rho - \tfrac{1}{r} f \chi_r\,, \tag{3.62}$$
$$0 = f_r^2 + f_\rho^2 - \chi_r^2 - \chi_\rho^2 - f f_{rr} - f f_{\rho\rho} + \tfrac{1}{r} f f_r - \tfrac{1}{\rho} f f_\rho\,, \tag{3.63}$$

where subscripts $(\cdots)_r$ and $(\cdots)_\rho$ denote derivatives with respect to r and ρ, respectively.

The equations (3.62)-(3.63) can be derived from the action

$$I = \int \frac{\rho}{r f^2} (f_r^2 + f_\rho^2 + \chi_r^2 + \chi_\rho^2) dr d\rho\,. \tag{3.64}$$

We give a simple geometrical interpretation to the equations of motion (3.62)-(3.63) by considering an auxiliary AdS_3 space parameterized by (r, ρ, φ) with metric

$$ds^2 = \frac{1}{r^2}(dr^2 + d\rho^2 + \rho^2 d\varphi^2)\,.$$

We can then interpret the function $f(r,\rho)$ and $\chi(r,\rho)$ as describing an axisymmetric map from AdS_3 to the two-dimensional (f,χ) "target-space." If we further endow this target-space with the AdS_2 metric

$$ds^2 = \frac{1}{f^2}(df^2 + d\chi^2)\,, \tag{3.65}$$

it is then easy to see that the equations of motion derived from (3.64) describe harmonic maps

$$(f, \chi) : AdS_3 \mapsto AdS_2\,. \tag{3.66}$$

The connection between AdS_2 (the "pseudosphere") and axisymmetric solutions to monopole equations on $\mathbb{R}^3$ was first noted in [41]. The harmonic map (3.66) is required to have a singularity along a Dirac-like string. To see this we first need to discuss the asymptotic behavior of the maps far away from the core, where the solution reduces to a $U(1)$ monopole.

3.6.3 The abelian solution

We can find a special solution to (3.62)-(3.63) by setting $\chi = 0$. The remaining equation (3.63) then states that $\log f$ is a harmonic map on AdS_3. Alternatively, the solution describes a $U(1)$ monpole on the (x_1, x_2, r) space with metric (3.43), centered at $(0, 0, a)$, which becomes a singular point. But it is easiest to construct the solution starting with 5+1D. In the abelian limit, the fields of the $(2, 0)$ theory that are relevant to our problem reduce to a free anti-self-dual 3-form field $H = -{}^*H$ and a free scalar field ϕ. We start by solving (3.47) on $\mathbb{R}^{5,1}$, which in particular implies that ϕ is harmonic. Consider a solution that describes the H and ϕ fields that emanate from a $(2, 0)$-string centered at

$$(x_1, x_2, x_4, x_5) = (0, 0, a\cos\theta, a\sin\theta)\,.$$

The scalar field is given by

$$\phi = v + \frac{1}{x_1^2 + x_2^2 + (x_4 - a\cos\theta)^2 + (x_5 - a\sin\theta)^2}\,. \tag{3.67}$$

But the solution that we need must be indepedent of θ. We can, however, obtain it from (3.67) by "smearing":

$$\phi(x_1, x_2, r) = v + \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\rho^2 + (r\cos\theta - a)^2 + r^2\sin^2\theta} = \frac{1}{\sqrt{(\rho^2 + r^2 + a^2)^2 - 4a^2r^2}} \tag{3.68}$$

and it is not hard to check that

$$A = \Big(\frac{\rho^2 + a^2 - r^2}{2\sqrt{(\rho^2 + r^2 + a^2)^2 - 4a^2r^2}} - 1\Big)\frac{x_2dx_1 - x_1dx_2}{\rho^2}\,. \tag{3.69}$$

The solution corresponds to

$$f = \exp\int \phi(r, \rho)rdr = e^{-\frac{1}{2}vr^2}\left(\rho^2 + r^2 - a^2 + \sqrt{(\rho^2 + r^2 + a^2)^2 - 4a^2r^2}\right). \tag{3.70}$$

To explore the Dirac string singularity, it is convenient to use instead of the Poincaré coordinates on AdS_3, a coordinate system with the point $r = a$ at the origin, via the coordinate transformation,

$$\frac{\rho}{r} = \sinh\chi\,\sin\alpha\,, \quad \frac{\rho^2 + r^2 - a^2}{2ar} = \sinh\chi\,\cos\alpha\,, \qquad ds^2 = d\chi^2 + \sinh^2\chi(d\alpha^2 + \sin^2\alpha\, d\varphi^2),$$

with the inverse transformation

$$r = a\frac{\cosh\chi + \sinh\chi\,\cos\alpha}{1+\sinh^2\chi\,\sin^2\alpha}\,, \qquad \rho = a\frac{\cosh\chi + \sinh\chi\,\cos\alpha}{1+\sinh^2\chi\,\sin^2\alpha}\sinh\chi\,\sin\alpha\,.$$

In these coordinates we have, up to an unimportant constant,

$$\begin{aligned}\log f = -\tfrac{1}{2}va^2\left(\frac{\cosh\chi + \sinh\chi\,\cos\alpha}{1+\sinh^2\chi\,\sin^2\alpha}\right)^2 + \log\left(\frac{\cosh\chi + \sinh\chi\,\cos\alpha}{1+\sinh^2\chi\,\sin^2\alpha}\right)\\ + \log\sinh\chi + \log(1+\cos\alpha)\,.\end{aligned} \tag{3.71}$$

The singularity in the last term at $\alpha = \pi$ represents the Dirac string. The abelian solution must describe the asymptotic behavior of the nonabelian solution when either $r \to \infty$ or $\rho \to \infty$ (or both).

3.6.4 Comments on (lack of) integrability

The classic BPS equations for monopoles on $\mathbb{R}^3$ admit the well-known Nahm solutions [72], which also have a nice string-theoretic interpretation [39]. The rich properties of these solutions essentially stem from an underlying integrable structure. One way to describe the structure is to map a solution of the BPS equations to a holomorphic vector bundle over minitwistor space [37, 38]. (Minitwistor space is the space of oriented straight lines on $\mathbb{R}^3$ and it has a complex structure.) The BPS equations arise as the integrability condition for an auxiliary set of equations in terms of an auxiliary 2-component field ψ, that require ψ's gauge-covariant derivative along a line in $\mathbb{R}^3$ to be related to multiplication by the scalar field $\widetilde{\Phi}$, and also require ψ to be holomorphic in the directions transverse to the line. This technique can be extended to other metrics, such as AdS_3 (whose corresponding minitwistor space also possesses a complex structure and is equivalent to $\mathbb{CP}^1 \times \mathbb{CP}^1$). But this technique fails for the metric (3.43), whose space of geodesics is not complex, and the monopole equations (3.41) cannot be expressed as the integrability condition for an auxiliary system of linear differential equations, at least not in an obvious way.

Another way to see where integrability fails is to focus on axially-symmetric solutions as in [41]. Defining the symmetric $\mathrm{SL}(2,\mathbb{R})$ matrix

$$\mathcal{G} \equiv \frac{1}{f}\begin{pmatrix} 1 & -\chi \\ -\chi & (f^2+\chi^2)\end{pmatrix}\,,$$

the equations of motion (3.62)-(3.63) can then be recast as

$$0 = \nabla^\alpha(\nabla_\alpha \mathcal{G}\mathcal{G}^{-1})\,, \tag{3.72}$$

where the covariant derivatives are with respect to another auxiliary metric,

$$ds^2 = dr^2 + d\rho^2 + (\frac{\rho^2}{r^2})d\varphi^2\,, \tag{3.73}$$

and $\mathcal{G}(r,\rho)$ is, of course, assumed to be independent of φ. It is possible [41] to recast axially symmetric solutions of the BPS equations on $\mathbb{R}^3$ in the form (3.72) – the metric in that case would be the Euclidean metric

$$ds^2 = dr^2 + d\rho^2 + \rho^2 d\varphi^2 \,,$$

and the connection with the σ-model (3.72) leads to an integrable structure. To describe the integrable structure we switch to complex coordinates,

$$\xi \equiv r + i\rho \,, \qquad \overline{\xi} \equiv r - i\rho \,,$$

and write (3.72) as the integrability condition for a system of first order linear differential equations for a two-component field $\Psi(\xi,\overline{\xi})$:

$$\Psi_\xi = \frac{1}{1+\gamma}\mathcal{G}_\xi\mathcal{G}^{-1}\Psi \,, \qquad \Psi_{\overline{\xi}} = \frac{1}{1-\gamma}\mathcal{G}_{\overline{\xi}}\mathcal{G}^{-1}\Psi \,,$$

where $(\cdots)_\xi$ and $(\cdots)_{\overline{\xi}}$ are derivatives with respect to ξ and $\overline{\xi}$, and the function $\gamma(\xi,\overline{\xi})$ has to be suitably chosen (so that the integrability condition $(\Psi_\xi)_{\overline{\xi}} = (\Psi_\xi)_{\overline{\xi}}$ will be automatically satisfied). There are, in fact, infinitely many choices for the function γ, but it has to be a solution of

$$\gamma_\xi = \frac{\gamma}{\xi-\overline{\xi}}\left(\frac{1+\gamma}{1-\gamma}\right) \,, \qquad \gamma_{\overline{\xi}} = -\frac{\gamma}{\xi-\overline{\xi}}\left(\frac{1-\gamma}{1+\gamma}\right) \,,$$

which are compatible (see [42] for review). This construction is easy to extend to any metric of the form

$$ds^2 = dr^2 + d\rho^2 + \Lambda(r,\rho)^2 d\varphi^2 \,,$$

as long as $\Lambda(r,\rho)$ is harmonic (in the metric $dr^2+d\rho^2$). In our case $\Lambda = \rho/r$ is not harmonic, so the standard integrability structure is not present.

One can also attempt to extend the technique of [39], to "probe" the solution with a string that extends in an extra dimension, say x_8. It is not hard to construct BPS string solutions that preserve some supersymmetry, compatible with that of the M5-branes and the twist. For example, in the M-theory variables we can take an M2-brane along a holomorphic curve given by

$$x_4 + ix_5 = C_0 e^{\frac{i}{\mathbf{k}R}(x_3+ix_8)} \,,$$

where C_0 is a constant. This would translate in type-IIA to a string whose x_8 coordinate varies logarithmically with r. However, this string does not preserve any common SUSY with the soliton. We were unable to find an exact solution to (3.41), and in fact, the appearance of polylogarithms in the expansion at large VEV (see Appendix D) suggests that even if a closed form exists, it is very complicated. We will therefore proceed to numerical analysis.

3.7 Numerical results

As a first step, we find it convenient to recast the equations in a different gauge. We begin by parameterizing the scalar field as:

$$\phi^\alpha = x_\alpha(\mathbf{f}+\mathbf{h})\,, \quad \phi^3 = \mathbf{g}\,, \tag{3.74}$$

and the gauge field as:

$$A^\alpha_\beta = x_{(\beta}\epsilon_{\alpha)\gamma}x^\gamma\mathbf{p} + \tfrac{1}{2}\epsilon_{\alpha\beta}\mathbf{q}\,, \quad A^\alpha_r = -r\epsilon_{\alpha\gamma}x_\gamma(\mathbf{f}-\mathbf{h})\,, \quad A^3_\beta = \epsilon_{\beta\gamma}x^\gamma\mathbf{v}\,, \quad A^3_r = 0\,. \tag{3.75}$$

with $\alpha,\beta,\gamma = 1,2$, $\epsilon_{\alpha\beta}$ being the anti-symmetric Levi-Civita symbol, and with $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{v}$ functions of (r,ρ) only. Next, we fix the gauge by setting $\mathbf{p}=0$. Defining

$$\mathbf{U} \equiv \rho^2\,, \qquad \mathbf{V} \equiv r^2,$$

the BPS equations (3.41) reduce (after rescaling ϕ by $\mathbf{k}R$) to:

$$0 = \mathbf{hv} - 2\tfrac{\partial \mathbf{h}}{\partial \mathbf{U}}\,, \tag{3.76}$$

$$0 = 2\mathbf{U}\tfrac{\partial \mathbf{f}}{\partial \mathbf{U}} + \mathbf{Uvf} + 2\mathbf{f} + \tfrac{\partial \mathbf{q}}{\partial \mathbf{V}} + \tfrac{1}{2}\mathbf{gq}\,, \tag{3.77}$$

$$0 = \mathbf{V}(\mathbf{h}-\mathbf{f})\mathbf{g} + \tfrac{1}{2}\mathbf{qv} + 2\mathbf{V}\tfrac{\partial \mathbf{f}}{\partial \mathbf{V}} + 2\mathbf{V}\tfrac{\partial \mathbf{h}}{\partial \mathbf{V}} - \tfrac{\partial \mathbf{q}}{\partial \mathbf{U}}\,, \tag{3.78}$$

$$0 = \tfrac{\partial \mathbf{v}}{\partial \mathbf{V}} - \tfrac{\partial \mathbf{g}}{\partial \mathbf{U}} + \tfrac{1}{2}\mathbf{hq}\,, \tag{3.79}$$

$$0 = \mathbf{UV}(\mathbf{f}^2-\mathbf{h}^2) + \tfrac{1}{4}\mathbf{q}^2 + 2\mathbf{v} + 2\mathbf{V}\tfrac{\partial \mathbf{g}}{\partial \mathbf{V}} + 2\mathbf{U}\tfrac{\partial \mathbf{v}}{\partial \mathbf{U}}\,. \tag{3.80}$$

Let us also set

$$\mathbf{Z} \equiv \tfrac{1}{2a}(\rho^2+r^2-a^2)\,, \qquad \mathbf{R} \equiv \sqrt{\rho^2+\mathbf{Z}^2} = \tfrac{1}{2a}\sqrt{(\rho^2+r^2-a^2)^2+4a^2\rho^2}\,. \tag{3.81}$$

The advantage of the ansatz (3.74)-(3.75) is that the abelian solution (3.68)-(3.69) can be written in the form:

$$\mathbf{f} = \frac{v}{2\mathbf{R}} - \frac{1}{a\mathbf{R}^2}\,, \quad \mathbf{g} = \frac{v\mathbf{Z}}{\mathbf{R}} - \frac{\mathbf{Z}}{a\mathbf{R}^2}\,, \quad \mathbf{h} = \frac{v}{2\mathbf{R}}\,, \quad \mathbf{q} = \frac{a^2+\mathbf{U}-\mathbf{V}}{a\mathbf{R}^2}\,, \quad \mathbf{v} = -\frac{1}{\mathbf{R}^2} - \frac{\mathbf{Z}}{a\mathbf{R}^2}\,. \tag{3.82}$$

which has no singularities except at $r=a$ (and in particular no Dirac string).

We require that at either limit $r\to\infty$ or $\rho\to\infty$ the solution should reduce to the abelian solution. At the tip $r=0$ the solution is required to be regular. This allows us to determine $\mathbf{q}$, $\mathbf{h}$, and $\mathbf{v}$ at the tip as follows. Setting $\mathbf{V}=0$ in (3.76), (3.78), and (3.80), we get the ordinary differential equations

$$\mathbf{hv} - 2\tfrac{\partial \mathbf{h}}{\partial \mathbf{U}} = \tfrac{1}{2}\mathbf{qv} - \tfrac{\partial \mathbf{q}}{\partial \mathbf{U}} = \tfrac{1}{4}\mathbf{q}^2 + 2\mathbf{v} + 2\mathbf{U}\tfrac{\partial \mathbf{v}}{\partial \mathbf{U}} = 0\,, \qquad (\mathbf{V}=0) \tag{3.83}$$

which we can solve uniquely, given the known boundary conditions at $\mathbf{U}\to\infty$, by expressing $\mathbf{q}$ and $\mathbf{h}$ in terms of the function $(1+\mathbf{Uv})$ and its derivatives, and changing variables to

$\log \mathbf{U}$. The result is that unique solution to (3.83) that satisfies the boundary conditions at $\mathbf{U} \to \infty$ is

$$\mathbf{q} = \frac{4a}{\mathbf{U}+a^2}\,, \qquad \mathbf{v} = -\frac{2}{\mathbf{U}+a^2}\,, \qquad \mathbf{h} = \frac{va}{\mathbf{U}+a^2}\,, \qquad (\mathbf{V}=0). \tag{3.84}$$

which is non other than the abelian solution (3.82) at $\mathbf{V}=0$.

We cannot determine $\mathbf{f}$ and $\mathbf{g}$ at $\mathbf{V}=0$ so easily, and our strategy will be to find an approximate solution to (3.76)-(3.80) by the variational method, minimizing the energy of the field configuration within a certain class of trial functions of $(\mathbf{U},\mathbf{V})$. For the energy we take the expression for the excess energy above the BPS bound for a static configuration of gauge field and minimally coupled adjoint scalar on a manifold given by the three dimensional metric (3.43):

$$\begin{aligned}\mathcal{E} &\equiv \frac{1}{2}\,\mathrm{tr}\int \sqrt{g}g^{ij}(D_i\widetilde{\Phi}-B_i)(D_j\widetilde{\Phi}-B_j)d^3x \\ &= \frac{1}{2}\,\mathrm{tr}\int \left[(rD_r\widetilde{\Phi}-F_{12})^2+(rD_1\widetilde{\Phi}-F_{2r})^2+(rD_2\widetilde{\Phi}-F_{r1})^2\right]\rho d\rho(\tfrac{dr}{r})\,,\end{aligned} \tag{3.85}$$

where B_i and $\widetilde{\Phi}$ were defined in (3.42), and the "tr" is in the fundamental representation. Note that $\mathcal{E}$ is different from the physical energy (3.44). The integrand in (3.85) is $\widetilde{\alpha}/r^2$ bigger than the integrand in the first term on the RHS of (3.44), but they are both minimized on the BPS configurations, and (3.85) gives more weight to the vicinity of $r=0$. We can rewrite $\mathcal{E}$ in terms of the right-hand-sides of (3.76)-(3.80),

$$\mathcal{X}_1 = \mathbf{hv} - 2\tfrac{\partial \mathbf{h}}{\partial \mathbf{U}}\,, \tag{3.86}$$

$$\mathcal{X}_2 = 2\mathbf{U}\tfrac{\partial \mathbf{f}}{\partial \mathbf{U}} + \mathbf{Uvf} + 2\mathbf{f} + \tfrac{\partial \mathbf{q}}{\partial \mathbf{V}} + \tfrac{1}{2}\mathbf{gq}\,, \tag{3.87}$$

$$\mathcal{X}_3 = \mathbf{V}(\mathbf{h}-\mathbf{f})\mathbf{g} + \tfrac{1}{2}\mathbf{qv} + 2\mathbf{V}\tfrac{\partial \mathbf{f}}{\partial \mathbf{V}} + 2\mathbf{V}\tfrac{\partial \mathbf{h}}{\partial \mathbf{V}} - \tfrac{\partial \mathbf{q}}{\partial \mathbf{U}}\,, \tag{3.88}$$

$$\mathcal{X}_4 = \tfrac{\partial \mathbf{v}}{\partial \mathbf{V}} - \tfrac{\partial \mathbf{g}}{\partial \mathbf{U}} + \tfrac{1}{2}\mathbf{hq}\,, \tag{3.89}$$

$$\mathcal{X}_5 = \mathbf{UV}(\mathbf{f}^2-\mathbf{h}^2) + \tfrac{1}{4}\mathbf{q}^2 + 2\mathbf{v} + 2\mathbf{V}\tfrac{\partial \mathbf{g}}{\partial \mathbf{V}} + 2\mathbf{U}\tfrac{\partial \mathbf{v}}{\partial \mathbf{U}}\,, \tag{3.90}$$

as

$$\mathcal{E} = \int \big(\frac{1}{8}\mathbf{U}^2\mathcal{X}_1^2 + \frac{1}{8}\mathcal{X}_2^2 + \frac{\mathcal{X}_3^2}{16\mathbf{V}} + \frac{1}{4}\mathbf{U}\mathcal{X}_4^2 + \frac{\mathcal{X}_5^2}{16\mathbf{V}}\big)d\mathbf{U}d\mathbf{V}\,. \tag{3.91}$$

We also note that the BPS bound on energy is given by

$$\begin{aligned}\mathcal{E}_{\rm BPS} &= \mathrm{tr}\int \sqrt{g}g^{ij}(B_jD_i\widetilde{\Phi})d^3x \\ &= \mathrm{tr}\int \left[F_{12}D_r\widetilde{\Phi} + F_{2r}D_1\widetilde{\Phi} + F_{r1}D_2\widetilde{\Phi}\right]\rho d\rho dr = \int d\lambda\,,\end{aligned} \tag{3.92}$$

where the 1-form λ is defined by

$$\begin{aligned}\lambda \equiv &\left[\tfrac{1}{8}\mathbf{Uqv}(\mathbf{f}+\mathbf{h}) + \tfrac{1}{16}\mathbf{q}^2\mathbf{g} + \tfrac{1}{2}\mathbf{vg} + \tfrac{1}{2}\mathbf{Ug}\frac{\partial \mathbf{v}}{\partial \mathbf{U}} - \tfrac{1}{4}\mathbf{U}(\mathbf{f}+\mathbf{h})\frac{\partial \mathbf{q}}{\partial \mathbf{U}}\right]d\mathbf{U} \\ &+ \left[\tfrac{1}{8}\mathbf{Ugq}(\mathbf{h}-\mathbf{f}) + \tfrac{1}{4}\mathbf{U}(\mathbf{h}^2-\mathbf{f}^2)(1+\mathbf{Uv}) - \tfrac{1}{4}\mathbf{U}(\mathbf{f}+\mathbf{h})\frac{\partial \mathbf{q}}{\partial \mathbf{V}} + \tfrac{1}{2}\mathbf{Ug}\frac{\partial \mathbf{v}}{\partial \mathbf{V}}\right]d\mathbf{V}\,.\end{aligned} \tag{3.93}$$

Requiring the asymptotic behavior for large $\mathbf{U}$ and $\mathbf{V}$ to be as in (3.82), we find

$$\mathcal{E}_{\rm BPS} = 2va^2\,.$$

We construct our trial functions by modifying the abelian solution (3.82). But first we need to smooth out the singularity of that solution at $\mathbf{V} = a^2$, while preserving the asymptotic behavior at large $\mathbf{U}$ and $\mathbf{V}$, as well as the behavior (3.84) at $\mathbf{V} = 0$. For this purpose we define:

$$\mathcal{R} \equiv \sqrt{\mathbf{U}+\mathbf{V}+a^2} = \sqrt{r^2+\rho^2+a^2} \tag{3.94}$$

and then define

$$\begin{aligned}
\tilde{\mathbf{f}} &\equiv \frac{av}{\mathcal{R}^2} + \frac{2a(va^2-2)}{\mathcal{R}^4} - \frac{2a^3v\mathbf{U}}{\mathcal{R}^6}\,,\\
\tilde{\mathbf{g}} &\equiv v - \frac{2}{\mathcal{R}^2} - \frac{2va^2\mathbf{U}}{\mathcal{R}^4}\,,\\
\tilde{\mathbf{h}} &\equiv v\left(\frac{a}{\mathcal{R}^2} + \frac{2a^3}{\mathcal{R}^4} - \frac{2a^3\mathbf{U}}{\mathcal{R}^6} - \frac{2a^5(a^2+\mathbf{U})}{\mathcal{R}^8}\right)\,,\\
\tilde{\mathbf{q}} &\equiv \frac{4a}{\mathcal{R}^2} - \frac{8a\mathbf{V}}{\mathcal{R}^4}\,,\\
\tilde{\mathbf{v}} &\equiv -\frac{2}{\mathcal{R}^2} - \frac{8a^2}{\mathcal{R}^4} + \frac{8a^2(a^2+\mathbf{U})}{\mathcal{R}^6}\,,
\end{aligned}$$

so that for fixed $\mathbf{U}$ and $\mathbf{V}\to\infty$ we have

$$\begin{aligned}
\tilde{\mathbf{f}} &= \frac{v}{2\mathbf{R}} - \frac{1}{a\mathbf{R}^2} + O\left(\frac{1}{\mathbf{V}^4}\right)\,,\\
\tilde{\mathbf{g}} &= \frac{v\mathbf{Z}}{\mathbf{R}} - \frac{\mathbf{Z}}{a\mathbf{R}^2} + O\left(\frac{1}{\mathbf{V}^3}\right)\,,\\
\tilde{\mathbf{h}} &= \frac{v}{2\mathbf{R}} + O\left(\frac{1}{\mathbf{V}^4}\right)\,,\\
\tilde{\mathbf{q}} &= \frac{a^2+\mathbf{U}-\mathbf{V}}{a\mathbf{R}^2} + O\left(\frac{1}{\mathbf{V}^3}\right)\,,\\
\tilde{\mathbf{v}} &= -\frac{1}{\mathbf{R}^2} - \frac{\mathbf{Z}}{a\mathbf{R}^2} + O\left(\frac{1}{\mathbf{V}^4}\right)\,,
\end{aligned}$$

and $\tilde{\mathbf{f}}$, $\tilde{\mathbf{g}}$, $\tilde{\mathbf{h}}$, $\tilde{\mathbf{v}}$, $\tilde{\mathbf{q}}$ are smooth everywhere. We also define

$$\mathcal{R}_b \equiv \sqrt{\mathbf{U}+\mathbf{V}+b^2} = \sqrt{r^2+\rho^2+b^2}\,,$$

where b is a parameter to be determined dynamically by the variational principle. We now pick a sufficiently large integer N (we chose $N = 20$ below), and construct trial functions in

the form:

$$\mathbf{f} = \tilde{\mathbf{f}} + \frac{1}{\mathcal{R}_b^{5+2N}} \sum_{n,m\geq 0}^{n+m\leq N} \mathbf{f}_{m,n}\mathbf{U}^m\mathbf{V}^n\,,$$
$$\mathbf{g} = \tilde{\mathbf{g}} + \frac{1}{\mathcal{R}_b^{4+2N}} \sum_{n,m\geq 0}^{n+m\leq N} \mathbf{g}_{m,n}\mathbf{U}^m\mathbf{V}^n\,,$$
$$\mathbf{h} = \tilde{\mathbf{h}} + \frac{\mathbf{V}}{\mathcal{R}_b^{5+2N}} \sum_{n,m\geq 0}^{n+m\leq N-1} \mathbf{h}_{m,n}\mathbf{U}^m\mathbf{V}^n\,,$$
$$\mathbf{q} = \tilde{\mathbf{q}} + \frac{\mathbf{V}}{\mathcal{R}_b^{4+2N}} \sum_{n,m\geq 0}^{n+m\leq N-1} \mathbf{q}_{m,n}\mathbf{U}^m\mathbf{V}^n\,,$$
$$\mathbf{q} = \tilde{\mathbf{v}} + \frac{\mathbf{V}}{\mathcal{R}_b^{4+2N}} \sum_{n,m\geq 0}^{n+m\leq N-1} \mathbf{v}_{m,n}\mathbf{U}^m\mathbf{V}^n\,,$$

where $\mathbf{f}_{m,n}$, $\mathbf{g}_{m,n}$, $\mathbf{h}_{m,n}$, $\mathbf{q}_{m,n}$, $\mathbf{v}_{m,n}$ are coefficients to be determined. These expressions are designed to preserve the boundary condition (3.84), as well as the asymptotic behavior for large $\mathbf{U}$ and $\mathbf{V}$. We then find the coefficients $\mathbf{f}_{m,n}$, $\mathbf{g}_{m,n}$, $\mathbf{h}_{m,n}$, $\mathbf{q}_{m,n}$, $\mathbf{v}_{m,n}$ that minimize $\mathcal{E}$, using the Newton-Raphson method for given b, and finally we optimize b.

For example, we find for the dimensionless coefficient $va^2 = 1$ and $N = 20$ that the optimal b is $2.8a$. We define the energy density

$$\mathcal{U} \equiv \frac{1}{2}\,\mathrm{tr}\left[(D_r\widetilde{\Phi})^2 + (D_1\widetilde{\Phi})^2 + (D_2\widetilde{\Phi})^2\right] + \frac{1}{2}r^2\,\mathrm{tr}\left[F_{12}^2 + F_{1r}^2 + F_{2r}^2\right]\,, \tag{3.95}$$

for the exact solution we have

$$\mathcal{U} = \mathcal{U}_{\mathrm{BPS}} \equiv r\,\mathrm{tr}\left[F_{12}D_r\widetilde{\Phi} + F_{2r}D_1\widetilde{\Phi} + F_{r1}D_2\widetilde{\Phi}\right]\,. \tag{3.96}$$

The total energy is then

$$\mathcal{E}_{\mathrm{BPS}} = \frac{1}{4}\int \frac{1}{\mathbf{V}}\mathcal{U}_{\mathrm{BPS}}d\mathbf{V}d\mathbf{U}\,.$$

We present in Figure 3.4 our[1] numerical results for $\Theta \equiv \mathcal{U}/\mathbf{V}$ as well as for the gauge invariant absolute value of the scalar field

$$|\widetilde{\Phi}| \equiv (\widetilde{\Phi}^a\widetilde{\Phi}^a)^{1/2} = \sqrt{\mathbf{U}(\mathbf{f}^2 + \mathbf{h}^2) + \mathbf{g}^2}\,.$$

The results are for $va^2 = 1$, and it is interesting to note that for such a relatively small value of va^2, the core of the soliton (where $|\widetilde{\Phi}| = 0$) is at $r \approx 2.9$ ($\mathbf{V} = 8.3$ in the graph of Figure 3.4), which is far from $a = 1$.

[1]Graph drawn by `Mathematica`, Version 9.0, (Wolfram Research, Inc.).

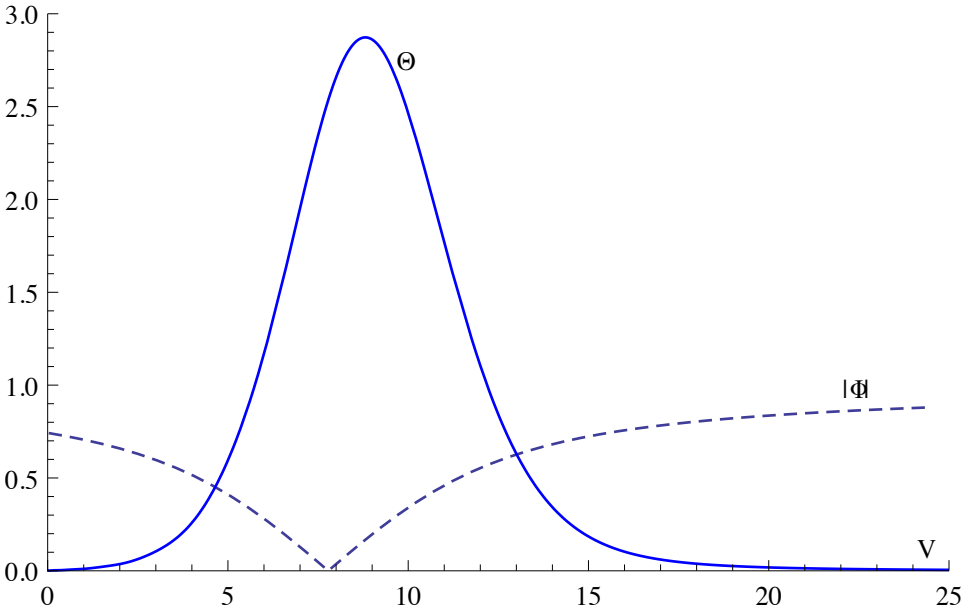


Figure 3.4: Results of a numerical simulation with parameters $b = 2.80$ and $N = 22$. The graphs show the energy density $\Theta \equiv \mathcal{U}/\mathbf{V}$ (solid line) and the gauge invariant absolute value of the scalar field $|\widetilde{\Phi}| \equiv (\widetilde{\Phi}^a \widetilde{\Phi}^a)^{1/2}$ (dashed line) for VEV $v = 1$ and soliton center at $a = 1$. The graphs are on the axis $\mathbf{U} = 0$ and the horizontal axis is $\mathbf{V}$. At $\mathbf{V} = 0$ the value of Θ is 2.0×10^{-3} and the value of $|\widetilde{\Phi}|$ is 0.76. The value of the excess energy $\mathcal{E}$ for this simulation is less than 2×10^{-5} of $\mathcal{E}_{\rm BPS}$.

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Appendix A

A proof of the determinant identity and the Smith normal form of the coupling constant matrix

Molinari gave an elegant proof [13] to a generalization of (2.2) using only polynomial analysis. Here we present an alternative basic linear-algebra proof for (2.2). At the same time we also demonstrate that the Smith normal form of the coupling constant matrix K defined in (2.5),

$$K = \begin{pmatrix} \mathbf{k}_1 & -1 & 0 & \ddots & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & \ddots & 0 & -1 & \mathbf{k}_n \end{pmatrix},$$

is identical to the Smith normal form of

$$H = W - \mathbf{I} = \begin{pmatrix} \mathbf{a} - 1 & \mathbf{b} \\ \mathbf{c} & \mathbf{d} - 1 \end{pmatrix},$$

where W was defined in (2.11).

We begin by moving the first row of K to the end, to get K_1'. We have

$$\det K = (-1)^n \det K_1'$$

but both K and K_1' have the same Smith normal form. For clarity, we will present explicit matrices for the $n = 5$ case. We get:

$$K_1' \equiv \begin{pmatrix} -1 & \mathbf{k}_2 & -1 & 0 & 0 \\ 0 & -1 & \mathbf{k}_3 & -1 & 0 \\ 0 & 0 & -1 & \mathbf{k}_4 & -1 \\ -1 & 0 & 0 & -1 & \mathbf{k}_5 \\ \mathbf{k}_1 & -1 & 0 & 0 & -1 \end{pmatrix},$$

We will now show how to successively define a series of matrices

$$K'_2, \ldots, K'_{n-1} = \begin{pmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & \mathbf{a}-1 & \mathbf{b} \\ & & & \mathbf{c} & \mathbf{d}-1 \end{pmatrix},$$

related to each other by row and column operations that preserve the Smith normal form. At each step, we need to keep track of a 2×2 block of K'_m formed from the elements on the $(n-1)^{th}$ and n^{th} rows and the m^{th} and $(m+1)^{st}$ columns.

$$H'_m \equiv \begin{pmatrix} [K'_m]_{(n-1)\,m} & [K'_m]_{(n-1)\,(m+1)} \\ [K'_m]_{n\,m} & [K'_m]_{n\,(m+1)} \end{pmatrix}$$

At the outset we have

$$H'_1 \equiv \begin{pmatrix} [K'_1]_{(n-1)\,1} & [K'_1]_{(n-1)\,2} \\ [K'_1]_{n\,1} & [K'_1]_{n\,2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ \mathbf{k}_1 & -1 \end{pmatrix}.$$

As will soon be clear from the construction, the matrix K'_m has the following block form:

$$K'_m = \begin{pmatrix} -\mathbf{I}_{m-1} & & & & & & \\ & -1 & \mathbf{k}_{m+1} & -1 & * & * & * \\ & & -1 & \mathbf{k}_{m+2} & * & * & * \\ & & & & X_{n-m-4} & * & * \\ & [H'_m]_{11} & [H'_m]_{12} & & & -1 & \mathbf{k}_n \\ & [H'_m]_{21} & [H'_m]_{22} & & & & -1 \end{pmatrix}, \tag{A.1}$$

where $\mathbf{I}_{m-1}$ is the $(m-1) \times (m-1)$ identity matrix, $*$ represents a block of possibly nonzero elements, X_{n-m-4} represents a nonzero $(n-m-4) \times (n-m-4)$ matrix and empty positions are zero. To get K'_{m+1} from K'_m we perform the following row and column operations on K'_m:

- Add $[H'_m]_{11}$ times the m^{th} row to the $(n-1)^{st}$ row;
- Add $[H'_m]_{21}$ times the m^{st} row to the n^{th} row;
- For $j = m+1, \ldots, n$, add $[K'_m]_{mj}$ times the m^{th} column to the j^{th} column.

It is not hard to see that these operations produce a matrix that fits the general form (A.1) with $m \to m+1$. Tracking how the bottom two rows transform, we find that for $m < n-2$,

$$H'_{m+1} = \begin{pmatrix} [H'_{m+1}]_{11} & [H'_{m+1}]_{12} \\ [H'_{m+1}]_{21} & [H'_{m+1}]_{22} \end{pmatrix} = \begin{pmatrix} [H'_m]_{12} + \mathbf{k}_{m+1}[H'_m]_{11} & -[H'_m]_{11} \\ [H'_m]_{22} + \mathbf{k}_{m+1}[H'_m]_{21} & -[H'_m]_{21} \end{pmatrix} = H'_m \begin{pmatrix} \mathbf{k}_{m+1} & 1 \\ -1 & 0 \end{pmatrix}.$$

Since, by definition, $H'_1 = \begin{pmatrix} -1 & 0 \\ \mathbf{k}_1 & -1 \end{pmatrix}$, it follows that

$$H'_{n-2} = \begin{pmatrix} -1 & 0 \\ \mathbf{k}_1 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{k}_2 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \mathbf{k}_{n-2} & 1 \\ -1 & 0 \end{pmatrix} .$$

It can then be easily checked that the last two steps yield:

$$H'_n = H'_{n-2} \begin{pmatrix} \mathbf{k}_{n-1} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{k}_n & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Appendix B

Compatibility of the supersymmetric Janus configuration and the duality twist

In this section we describe the details of the supersymmetric Lagrangian. As explained in §2.2, the system is composed of two ingredients: (i) the supersymmetric Janus configuration; and (ii) an $SL(2,\mathbb{Z})$ duality twist. We will now review the details of both ingredients and demonstrate that their combination preserves supersymmetry.

B.1 Supersymmetric Janus

Extending the work of [17]-[19], Gaiotto and Witten [52] have constructed a supersymmetric deformation of $\mathcal{N}=4$ Super-Yang-Mills theory with a complex coupling constant τ that varies along one direction, which we denote by x_3. We will now review this construction, using the same notation as in [52]. First, the real and imaginary parts of the coupling constant are defined as

$$\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{\mathbf{e}^2}\,, \tag{B.1}$$

It is taken to vary along a semi-circle on the upper half τ-plane, centered on the real axis:

$$\tau = a + 4\pi D e^{2i\psi}\,, \tag{B.2}$$

where $\psi(x_3)$ is an arbitrary function.

The action is defined as

$$I = I_{\mathrm{N}=4} + I' + I'' + I'''$$

where $I_{\mathrm{N}=4}$ is the standard $\mathcal{N}=4$ action, modified only by making τ a function of x_3, and I', I'', and I''' are correction terms listed below. We will list the actions for a general gauge group, as derived by Gaiotto and Witten, although the application in this paper is for a $U(1)$ gauge group, and so several terms drop out. The bosonic fields are: a gauge field A_μ ($\mu = 0,1,2,3$), 3 adjoint-valued scalar fields X^a ($a = 1,2,3$) and 3 adjoint-valued scalar

fields Y^p ($p = 1, 2, 3$). In the $U(1)$ case, X^a and Y^p are real scalar fields. In the type-IIB realization on D3-branes, the D3-brane is in directions $0, 1, 2, 3$, X^a corresponds to fluctuations in directions $4, 5, 6$, and Y^p corresponds to directions $7, 8, 9$. The fermionic fields are encoded in a 16-dimensional Majorana-Weyl spinor Ψ on which even products of the 9+1D Dirac matrices $\Gamma_0, \ldots, \Gamma_9$ act. Products of pairs from the list $\Gamma_0, \ldots, \Gamma_3$ correspond to generators of the Lorentz group $SO(1, 3)$, while products of pairs from the list $\Gamma_4, \Gamma_5, \Gamma_6$ correspond to generators of the R-symmetry subgroup $SO(3)_X$ acting on X^1, X^2, X^3, and products of pairs from the list $\Gamma_7, \Gamma_8, \Gamma_9$ correspond to generators of the R-symmetry subgroup $SO(3)_Y$ acting on Y^1, Y^2, Y^3. We have the identity $\Gamma_{0123456789} = 1$.

The additional terms are

$$
\begin{aligned}
I' &= \frac{i}{\mathbf{e}^2} \int d^4x \, \mathrm{Tr}\, \overline{\Psi}(\alpha\Gamma_{012} + \beta\Gamma_{456} + \gamma\Gamma_{789})\Psi \,, \\
I'' &= \frac{1}{\mathbf{e}^2} \int d^4x \, \mathrm{Tr}\, \left(\mathbf{u}\epsilon^{\mu\nu\lambda}(A_\mu\partial_\nu A_\lambda + \tfrac{2}{3}A_\mu A_\nu A_\lambda) + \tfrac{\mathbf{v}}{3}\epsilon^{abc}X_a[X_b, X_c] + \tfrac{\mathbf{w}}{3}\epsilon^{pqr}Y_p[Y_q, Y_r]\right) \,, \\
I''' &= \frac{1}{2\mathbf{e}^2} \int d^4x \, \mathrm{Tr}\, \left(\mathbf{r}X_aX^a + \widetilde{\mathbf{r}}Y_pY^p\right) \,,
\end{aligned}
$$

where

$$
-\tfrac{1}{4}\mathbf{u} = \alpha = -\tfrac{1}{2}\psi' \,, \qquad -\tfrac{1}{4}\mathbf{v} = \beta = -\frac{\psi'}{2\cos\psi} \,, \qquad -\tfrac{1}{4}\mathbf{w} = \gamma = \frac{\psi'}{2\sin\psi} \,, \tag{B.3}
$$

$$
\mathbf{r} = 2(\psi' \tan\psi)' + 2(\psi')^2 \,, \qquad \widetilde{\mathbf{r}} = -2(\psi' \cot\psi)' + 2(\psi')^2 \,. \tag{B.4}
$$

As we are working with a $U(1)$ gauge group, we will not need the cubic terms in I''. They are nevertheless listed here for reference, and they will become relevant for extensions to a nonabelian gauge group.

To describe the preserved supersymmetry we follow Gaiotto-Witten and work in a spinor representation where

$$
\Gamma_{0123} = -\Gamma_{456789} = \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \,, \quad \Gamma_{3456} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \,, \quad \Gamma_{3789} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \,,
$$

where $\mathbf{I}$ is an 8×8 identity matrix. The surviving supersymmetries are those parameterized by a 16-component ε_{16} which takes the form

$$
\varepsilon_{16} = \begin{pmatrix} \cos(\frac{\psi}{2})\varepsilon_8 \\ \sin(\frac{\psi}{2})\varepsilon_8 \end{pmatrix} \,, \tag{B.5}
$$

where ε_8 is an arbitrary constant 8-component spinor.

B.2 Introducing an SL(2, $\mathbb{Z}$)-twist

Here ψ is a function of x_3 such that $\tau(x_3)$ traces a geodesic on τ-plane with metric $|d\tau|^2/\tau_2^2$. We pick the parameters a and D so that the semi-circle (B.2) will be invariant

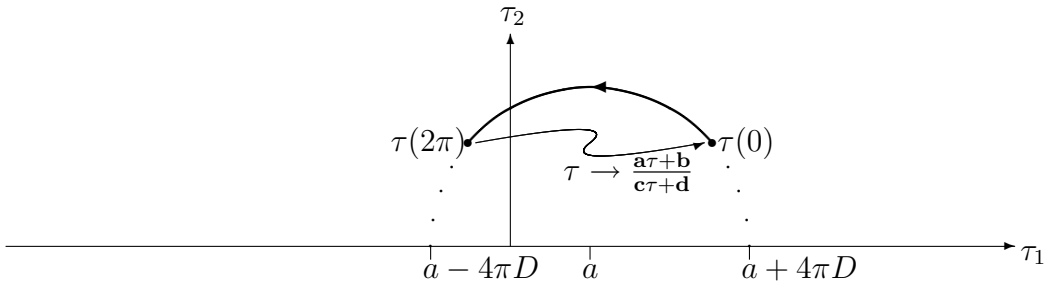

Figure B.1: In the Janus configuration the coupling constant τ traces a portion of a semi-circle of radius $4\pi D$ in the upper-half plane, whose center a is on the real axis. We augment it with an SL(2, $\mathbb{Z}$) duality twist that glues $x_3 = 2\pi$ to $x_3 = 0$.

under

$$\tau \to \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}\,.$$

This amounts to solving the two equations

$$(a - 4\pi D) = \frac{\mathbf{a}(a - 4\pi D) + \mathbf{b}}{\mathbf{c}(a - 4\pi D) + \mathbf{d}}\,, \qquad (a + 4\pi D) = \frac{\mathbf{a}(a + 4\pi D) + \mathbf{b}}{\mathbf{c}(a + 4\pi D) + \mathbf{d}}\,.$$

The solution is:

$$a = \frac{\mathbf{a} - \mathbf{d}}{2\mathbf{c}}\,, \qquad 4\pi D = \frac{\sqrt{(\mathbf{a} + \mathbf{d})^2 - 4}}{2|\mathbf{c}|}\,,$$

and is real for a hyperbolic element of SL(2, $\mathbb{Z}$) (with $|\mathbf{a} + \mathbf{d}| > 2$). Note that it is important to have both $(a \pm 4\pi D)$ as fixed-points of the SL(2, $\mathbb{Z}$) transformation, so as not to reverse the orientation of the $\tau(x_3)$ curve, and not create a discontinuity in $\tau'(x_3)$. So, given $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, our configuration is constructed by first calculating a and D, and then picking an arbitrary $\psi(2\pi)$ with a corresponding $\tau(2\pi) = a + 4\pi D e^{2i\psi(2\pi)}$. Next, we calculate the SL(2, $\mathbb{Z}$) dual $\tau(0) = (\mathbf{a}\tau(2\pi) + \mathbf{b})/(\mathbf{c}\tau(2\pi) + \mathbf{d})$ and match it to a point on the semicircle according to $\tau(0) = a + 4\pi D e^{2i\psi(0)}$. The function $\psi(x_3)$ can then be chosen arbitrarily, as long as it connects $\psi(0)$ to $\psi(2\pi)$. It can then be checked that $\mathbf{r}$ and $\widetilde{\mathbf{r}}$ are continuous at $x_3 = 2\pi$.

At low-energy, the mass parameters $\mathbf{r}$ and $\widetilde{\mathbf{r}}$ in I''' make the scalar fields (X^a and Y^p) massive. Note that in principle, the parameters can be locally negative [although this can be averted by choosing $\psi(x_3)$ so that $\psi'' = 0$], but the effective 2+1D masses, [obtained by solving for the spectrum of the operators $-d^2/dx_3^2 + \mathbf{r}(x_3)$, and $-d^2/dx_3^2 + \widetilde{\mathbf{r}}(x_3)$] have to be positive, since the configuration is supersymmetric and the BPS bound prevents us from having a profile of $X^a(x_3)$ or $Y^p(x_3)$ with negative energy. Similar statements hold for the fermionic masses in I'.

B.3 The supersymmetry parameter

As explained in [21], the $\mathrm{SL}(2,\mathbb{Z})$ duality transformation acts nontrivially on the SUSY generators. Define the phase φ by

$$e^{i\varphi} = \frac{|\mathbf{c}\tau + \mathbf{d}|}{\mathbf{c}\tau + \mathbf{d}} \,.$$

Then, the SUSY transformations act on the supersymmetry parameter as

$$\varepsilon \to e^{\frac{1}{2}\varphi\Gamma_{0123}}\varepsilon \,.$$

(See equation (2.25) of [21].)

We can now check that

$$\frac{|\mathbf{c}\tau + \mathbf{d}|}{\mathbf{c}\tau + \mathbf{d}} = e^{i(\tilde{\psi}-\psi)} \,, \tag{B.6}$$

where $\tilde{\psi}$ is defined by

$$\tilde{\tau} \equiv \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}} \equiv a + 4\pi D e^{2i\tilde{\psi}} \,.$$

It follows from (B.6) that the Gaiotto-Witten phase that is picked up by the supersymmetry parameter as it traverses the Janus configuration from $\eta = 0$ (corresponding to angular variable ψ) to $\eta = 2\pi$ (corresponding to $\tilde{\psi}$) is precisely canceled by the Kapustin-Witten phase of the $\mathrm{SL}(2,\mathbb{Z})$-duality twist. The entire "Janus plus twist" configuration is therefore supersymmetric.

B.4 Extending to a type-IIA supersymmetric background

In section §2.4 we assumed that there is a lift of the gauge theory construction to type-IIB string theory and, following a series of dualities, we obtained a type-IIA background with NSNS fields turned on. Here we would like to outline how such a lift might be constructed. We start with the well-known $AdS_3 \times S^3 \times T^4$ type-IIB background, and perform S-duality (if necessary) to get the 3-form flux to be NSNS. Then, take AdS_3 to be of Euclidean signature and replace T^4 with $\mathbb{R}^4$, which we then Wick rotate to $\mathbb{R}^{1,3}$. We take the AdS_3 metric in the form

$$ds^2 = \frac{r^2}{r_1 r_5}(-dt^2 + dx_5^2) + \frac{r_1}{r_5}\sum_{i=6}^{9} dx_i^2 + \frac{r_1 r_5}{r^2}dr^2 + r_1 r_5 d\Omega_3^2$$

$$H^{(RR)} = \frac{2r_5^2}{g}(\epsilon_3 + {}^*_6\epsilon_3) \,, \qquad e^{\phi} = \frac{g r_1}{r_5}$$

where ϵ_3 is the volume form on the unit sphere, and ${}^*_6$ is the Hodge dual in the six dimensions $x_0, \ldots, x_5$ (of $AdS_3 \times S^3$), and where r_1, r_5 are constants. (We follow the notation of [23].)

We need to change variables $r \to x_3$, $t \to ix_1$ and $x_9 \to ix_0$, and perform S-duality (where the RHS of arrows are the variables of §2.5). We then compactify directions x_1 and

x_2 so that $0 \leq x_i < 2\pi L_i$ $(i = 1, 2)$. As a function of x_3, we define the Kähler modulus of the $x_1 - x_2$ torus to be

$$\rho = i\frac{4\pi^2 r_1^2 L_1 L_2}{x_3^2}$$

Finally, we perform T-duality on direction x_5 to replace ρ with the complex structure τ of the resulting T^2. In an appropriate limit, this gives a solution where τ goes along a straight perpendicular line in the upper half plane. We can convert it to a semi-circle with an $\mathrm{SL}(2, \mathbb{R})$ transformation.

Appendix C

Recasting BPS equations in terms of a single potential

The action (3.64) is invariant under dilatations that act as

$$f(r,\rho) \to f(\lambda r, \lambda\rho)\,, \qquad \chi(r,\rho) \to \chi(\lambda r, \lambda\rho)\,.$$

The components of the associated Noether current are given by

$$J^r = \frac{\rho f_r^2}{2f^2} + \frac{\rho^2 f_r f_\rho}{r f^2} - \frac{\rho f_\rho^2}{2f^2} + \frac{\rho\chi_r^2}{2f^2} + \frac{\rho^2\chi_r\chi_\rho}{r f^2} - \frac{\rho\chi_\rho^2}{2f^2}\,,$$
$$J^\rho = \frac{\rho^2 f_\rho^2}{2r f^2} + \frac{\rho f_r f_\rho}{f^2} - \frac{\rho^2 f_r^2}{2r f^2} + \frac{\rho^2\chi_\rho^2}{2r f^2} + \frac{\rho\chi_r\chi_\rho}{f^2} - \frac{\rho^2\chi_r^2}{2r f^2}\,.$$

The equations of motion (3.63)-(3.62) imply the conservation equation[1]

$$(J^r)_r + (J^\rho)_\rho = 0,$$

which implies that there exists a potential function Φ such that

$$J^\rho = \Phi_r\,, \qquad J^r = -\Phi_\rho\,. \tag{C.1}$$

To proceed, we think of the functions f and χ as defining a change of coordinates from (f,χ) to (r,ρ) [similar to (3.66), except with the ϕ coordinate absent]. In (r,ρ) coordinates, the AdS_2 metric (3.65) becomes:

$$ds^2 = \mathfrak{G}_{rr}dr^2 + 2\mathfrak{G}_{r\rho}dr d\rho + \mathfrak{G}_{\rho\rho}d\rho^2\,, \tag{C.2}$$

where the metric $\mathfrak{G}$ can be expressed, using (C.1), as:

$$\mathfrak{G}_{rr} = -\tfrac{r^2}{r^2+\rho^2}(\Phi_{\rho\rho} + \Phi_{rr} + \tfrac{1}{r}\Phi_r + \tfrac{1}{\rho}\Phi_\rho)\,,$$
$$\mathfrak{G}_{\rho\rho} = -\tfrac{r^2}{r^2+\rho^2}(\Phi_{\rho\rho} + \Phi_{rr} - \tfrac{1}{r}\Phi_r - \tfrac{1}{\rho}\Phi_\rho)\,,$$
$$\mathfrak{G}_{r\rho} = \tfrac{r}{\rho(r^2+\rho^2)}(r\Phi_r - \rho\Phi_\rho)\,.$$

[1] We stress that this is not directly related to the stress-energy tensor in the original fields A_i and Φ. These generally vanish in BPS configurations [61].

Φ then satisfies a nonlinear differential equation that states that the Ricci scalar of (C.2) is $R = -2$. In order to incorporate the Dirac string for $r < a$, the function Φ must diverge like $\log\rho$ as $\rho \to 0$ and $r < a$. For large a, the solution to f and χ is given by adapting the Prasad-Sommerfield solution as given by [41]:

$$f = \frac{\rho \sinh \mathbf{R}}{\mathbf{R} + \mathbf{R}\cosh\mathbf{R}\cosh\mathbf{Z} - \mathbf{Z}\sinh\mathbf{Z}\sinh\mathbf{R}}\,, \qquad \chi = \frac{\mathbf{Z}\cosh\mathbf{Z}\sinh\mathbf{R} - \mathbf{R}\sinh\mathbf{Z}\cosh\mathbf{R}}{\mathbf{R} + \mathbf{R}\cosh\mathbf{R}\cosh\mathbf{Z} - \mathbf{Z}\sinh\mathbf{Z}\sinh\mathbf{R}} \tag{C.3}$$

where $\mathbf{Z}$ and $\mathbf{R}$ are given in (3.81), we have set the VEV $v = 1$, and we have used $\mathbf{R}$ as a substitute for the distance from the core of the monopole. From this we find,

$$\Phi \to -\tfrac{1}{4}\rho^2 + \tfrac{1}{2}\log\rho - \log\mathbf{R} + \log\sinh\mathbf{R}\,. \tag{C.4}$$

We also note that the abelian solution

$$f = \left(\frac{\mathbf{R}-\mathbf{Z}}{2a}\right) e^{-\frac{1}{2}vr^2}\,, \qquad \chi = 0\,,$$

can be derived from the potential

$$\Phi = \tfrac{1}{4}v^2r^2\rho^2 + \tfrac{1}{2}v(2a\mathbf{R} + r^2 - \rho^2) + \log\left[\frac{2a\mathbf{R}}{(\mathbf{R}-\mathbf{Z})(a+\mathbf{R}+\mathbf{Z})}\right].$$

Finally, we note that a change of variables,

$$r = ae^{\tau}\cos\sigma\,, \qquad \rho = ae^{\tau}\sin\sigma\,,$$

converts the metric to the more compact form

$$ds^2 = -\cos^2\sigma(\Phi_{\sigma\sigma} + \Phi_{\tau\tau})(d\sigma^2 + d\tau^2) + \cot\sigma\left[\Phi_\sigma(d\sigma^2 - d\tau^2) + 2\Phi_\tau d\sigma d\tau\right].$$

Appendix D

Large VEV expansion

In this section we will discuss the behavior of the solution to (3.40) for large VEV v. Since the dimensionless combination is va^2, we can just as well discuss fixed v and large a, which means that the core of the monopole solution is far from the tip. Let us set[1] $x_3 \equiv r - a$ and rescaling $\phi = a\Phi/\mathbf{k}R$, so that equations (3.40) can be rewritten as

$$(1 + \tfrac{x_3}{a})D_i\phi = \tfrac{1}{2}\epsilon_{ijk}F_{jk} \tag{D.1}$$

where in this section $i, j, k = 1, 2, 3$ refer to x_1, x_2, x_3 with Euclidean metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \,.$$

In the limit $a \to \infty$, (D.1) reduce to Bogomolnyi's equations and the one-monopole solution is [36]:

$$A_i^{a\,(0)} = \epsilon_{iaj}x_j K(R)\,, \qquad \phi^{a\,(0)} = x_a H(R)\,, \tag{D.2}$$

where

$$H \equiv \tfrac{1}{R}\coth R - \tfrac{1}{R^2}\,, \qquad K \equiv \tfrac{1}{R\sinh R} - \tfrac{1}{R^2}\,, \tag{D.3}$$

and here

$$R^2 = \sum_{i=1}^{3} x_i^2 \,.$$

We set

$$b \equiv \frac{1}{a}\,, \qquad \vec{\ell} \equiv (0, 0, b)\,,$$

so that $\frac{x_3}{a} = \vec{\ell}\cdot\vec{x}$, and (D.1) can be written as:

$$(1 + \vec{\ell}\cdot\vec{x})D_i\phi = \tfrac{1}{2}\epsilon_{ijk}F_{jk}\,. \tag{D.4}$$

[1] We hope the no confusion will arise with the coordinate x_3 that was used in §3.2.1. That coordinate plays no role here, and the only coordinates relevant for this section are x_1, x_2 and $r = a + x_3$.

We can now expand around the Prasad-Sommerfield solution:

$$A_i = A_i^{(0)} + bA_i^{(1)} + b^2 A_i^{(2)} + \cdots , \qquad \phi = \phi^{(0)} + b\phi^{(1)} + b^2\phi^{(2)} + \cdots ,$$

where we set the 0^{th} order terms to the Prasad-sommerfield solution (D.2).

At order $O(b)$ we write all possible terms that are allowed by spherical symmetry and we keep only the terms that are also invariant under the parity symmetry

$$\phi^a(\vec{x},\vec{\ell}) \to -\phi^a(-\vec{x},-\vec{\ell}) , \qquad A_i^a(\vec{x},\vec{\ell}) \to -A_i^a(-\vec{x},-\vec{\ell}) . \tag{D.5}$$

The general expression is then

$$b\phi^{a(1)} = \ell_a \mathbf{f}_{1,1}(R) + x_a(\ell_k x_k)\mathbf{f}_{1,2}(R) , \tag{D.6}$$

$$bA_i^{a(1)} = x_a\epsilon_{ijk}x_j\ell_k\mathbf{f}_{1,3}(R) + x_i\epsilon_{ajk}x_j\ell_k\mathbf{f}_{1,4}(R) + \epsilon_{aij}\ell_j\mathbf{f}_{1,5}(R) , \tag{D.7}$$

and we note the identity

$$x_{[i}\epsilon_{a]jk}x_j\ell_k = \tfrac{1}{2}\epsilon_{aij}\ell_j R^2 - \tfrac{1}{2}(\ell_k x_k)\epsilon_{aij}x_j , \tag{D.8}$$

which is the reason why we did not include a term of the form $(\ell_k x_k)\epsilon_{aij}x_j\mathbf{f}_{1,6}$. The coefficients $\mathbf{f}_{1,1},\ldots,\mathbf{f}_{1,5}$ are unknown functions of R.

We also have the freedom to apply an infinitesimal $O(b)$ gauge transformation which takes the form

$$\delta\phi^a = \epsilon_{abc}\lambda^b\phi^c , \qquad \delta A_i^a = \partial_i\lambda^a - \epsilon_{abc}A_i^b\lambda^c$$

with

$$\lambda^a = \epsilon_{abc}x_b\ell_c\mathbf{g}_{1,1}(R) .$$

This gives

$$\delta\phi^a = \epsilon_{abc}\epsilon_{bde}x_d\ell_e\mathbf{g}_{1,1}x^c H = -x_a\ell_k x_k\mathbf{g}_{1,1}H + \ell_a R^2\mathbf{g}_{1,1}H , \tag{D.9}$$

$$\delta A_i^a = -\epsilon_{iab}\ell_b\mathbf{g}_{1,1} + \tfrac{1}{R}x_i\epsilon_{abc}x_b\ell_c\mathbf{g}'_{1,1} + x_a\epsilon_{ibc}x_b\ell_c\mathbf{g}_{1,1}K . \tag{D.10}$$

Using this gauge transformation we can set one of the parameters in (D.6)-(D.7) to zero. We choose to set

$$\mathbf{f}_{1,5} = 0. \tag{D.11}$$

We end up with the general form of the $O(b)$ correction:

$$b\phi^{a(1)} = \ell_a \mathbf{f}_{1,1}(R) + x_a(\ell_k x_k)\mathbf{f}_{1,2}(R) , \tag{D.12}$$

$$bA_i^{a(1)} = x_a\epsilon_{ijk}x_j\ell_k\mathbf{f}_{1,3}(R) + x_i\epsilon_{ajk}x_j\ell_k\mathbf{f}_{1,4}(R) . \tag{D.13}$$

Plugging (D.2) and (D.12)-(D.13) into (D.4) and comparing terms of order $O(b)$ we get:

$$HK - \tfrac{1}{R}H' = \tfrac{1}{R}(\mathbf{f}'_{1,2} + \mathbf{f}'_{1,3}) - K\mathbf{f}_{1,2} - K\mathbf{f}_{1,3} + (K-H)\mathbf{f}_{1,4} , \tag{D.14}$$

$$0 = \tfrac{1}{R}\mathbf{f}'_{1,1} + (1+R^2K)\mathbf{f}_{1,3} + R^2H\mathbf{f}_{1,4} , \tag{D.15}$$

$$0 = R\mathbf{f}'_{1,3} + K\mathbf{f}_{1,1} - \mathbf{f}_{1,2} + 3\mathbf{f}_{1,3} + (1+R^2K)\mathbf{f}_{1,4} \tag{D.16}$$

$$-H(1+R^2K) = K\mathbf{f}_{1,1} + (1+R^2K)\mathbf{f}_{1,2} + \mathbf{f}_{1,4} , \tag{D.17}$$

These are ordinary inhomogeneous linear differential equations in $\mathbf{f}_{1,1},\ldots,\mathbf{f}_{1,4}$. Note that $\mathbf{f}_{1,4}$ can be eliminated from (D.17), so the general solution is given be an arbitrary solution of the full equations (D.14)-(D.17) plus a linear combination of three linearly independent solutions of the homogeneous equations:

$$0 = \tfrac{1}{R}(\mathbf{f}'_{1,2}+\mathbf{f}'_{1,3}) - K\mathbf{f}_{1,2} - K\mathbf{f}_{1,3} + (K-H)\mathbf{f}_{1,4}\,, \tag{D.18}$$
$$0 = \tfrac{1}{R}\mathbf{f}'_{1,1} + (1+R^2K)\mathbf{f}_{1,3} + R^2H\mathbf{f}_{1,4}\,, \tag{D.19}$$
$$0 = R\mathbf{f}'_{1,3} + K\mathbf{f}_{1,1} - \mathbf{f}_{1,2} + 3\mathbf{f}_{1,3} + (1+R^2K)\mathbf{f}_{1,4} \tag{D.20}$$
$$0 = K\mathbf{f}_{1,1} + (1+R^2K)\mathbf{f}_{1,2} + \mathbf{f}_{1,4}\,, \tag{D.21}$$

The general solution to (D.14)-(D.17) that is nonsingular at $R=0$ is

$$\mathbf{f}_{1,1} = -\tfrac{R}{2\sinh R} + c_1(\tfrac{R\cosh^2 R}{\sinh^3 R} - \tfrac{\cosh R}{\sinh^2 R}) + c_2(\tfrac{3R}{\sinh R} - \tfrac{3R^2\cosh R}{\sinh^2 R} + \tfrac{R^3\cosh^2 R}{\sinh^3 R})\,, \tag{D.22}$$
$$\mathbf{f}_{1,2} = \tfrac{1}{2R^2} + \tfrac{1-2\cosh R}{2R\sinh R} + c_1(\tfrac{1}{R^4} - \tfrac{\cosh^2 R}{R\sinh^3 R} + \tfrac{\cosh R-1}{R^2\sinh^2 R}) + c_2(-\tfrac{R\cosh^2 R}{\sinh^3 R} + \tfrac{(2\cosh R-3)}{R\sinh R} + \tfrac{(3\cosh R-1)}{\sinh^2 R})\,, \tag{D.23}$$
$$\mathbf{f}_{1,3} = \tfrac{1}{2R^2} - \tfrac{1}{2R}\coth R + c_1(-\tfrac{1}{R^4} + \tfrac{\cosh R}{R\sinh^3 R}) + c_2(\tfrac{R\cosh R}{\sinh^3 R} + \tfrac{\cosh R}{R\sinh R} - \tfrac{2}{\sinh R})\,, \tag{D.24}$$
$$\mathbf{f}_{1,4} = c_1(-\tfrac{1}{R^3\sinh R} - \tfrac{\cosh R}{R^2\sinh^2 R} + \tfrac{1+\cosh^2 R}{R\sinh^3 R}) + c_2(\tfrac{R(1+\cosh R^2)}{\sinh^3 R} + \tfrac{3}{R\sinh R} - \tfrac{5\cosh R}{\sinh^2 R})\,, \tag{D.25}$$

where c_1, c_2 are undtermined constants. Note that the functions (D.22)-(D.25) have a regular power series expansion at $R=0$ with nonnegative even powers of R only. We note that there is another homogeneous solution that we discarded because it is singular at $R=0$:

$$\begin{aligned}
\mathbf{f}_{1,1} &= c_3(\tfrac{\cosh^2 R}{\sinh^3 R})\,, & \mathbf{f}_{1,2} &= -c_3(\tfrac{\coth R}{R^4} + \tfrac{1}{R^3\sinh^2 R} + \tfrac{\cosh^2 R}{R^2\sinh^3 R})\,,\\
\mathbf{f}_{1,3} &= c_3(\tfrac{\coth R}{R^4} + \tfrac{1}{R^3\sinh^2 R} + \tfrac{\cosh R}{R^2\sinh^3 R})\,, & \mathbf{f}_{1,4} &= c_3(\tfrac{\cosh R}{R^3\sinh^2 R} + \tfrac{1+\cosh^2 R}{R^2\sinh^3 R})\,.
\end{aligned} \tag{D.26}$$

We are left with two unknown parameters c_1, c_2 but one can be a adjusted to zero by shifting the center of the zeroth order solution: $\vec{x} \to \vec{x} + c_0\vec{\ell}$, followed by a suitable gauge transformation to fix the $\mathbf{f}_{1,6}=0$ gauge, to set $c_1=0$. The parameter c_2 is undetermined at this point, since it depends on the proper boundary conditions at $R=\infty$ and at $R=-1/b$.

Now we move on to order $O(b^2)$. The general ansatz at this order is:

$$\begin{aligned}
b^2\phi^{a(2)} &= \ell^2 x_a \mathbf{f}_{2,1}(R)\\
&\quad + [\ell_a(\ell_k x_k) - \tfrac{1}{3}\ell^2 x_a]\mathbf{f}_{2,3}(R) + x_a[(\ell_k x_k)(\ell_m x_m) - \tfrac{1}{3}\ell^2 R^2]\mathbf{f}_{2,4}(R)\,,
\end{aligned} \tag{D.27}$$
$$\begin{aligned}
bA_i^{a(2)} &= \ell^2\epsilon_{iak}x_k\mathbf{f}_{2,2}(R) + x_a\epsilon_{ijk}x_j\ell_k(\ell_m x_m)\mathbf{f}_{2,5}(R) + x_i\epsilon_{ajk}x_j\ell_k(\ell_m x_m)\mathbf{f}_{2,6}(R)\\
&\quad + \epsilon_{aij}[\ell_j(\ell_m x_m) - \tfrac{1}{3}\ell^2 x_j]\mathbf{f}_{2,7}(R) + (\ell_i\epsilon_{ajk}x_j\ell_k - \tfrac{1}{3}\ell^2\epsilon_{aji}x_j)\mathbf{f}_{2,8}(R)\,,
\end{aligned} \tag{D.28}$$

where we have separated the different terms according to whether they can be expressed in terms of the spin-0 combination $\ell^2 \equiv \ell_k\ell_k$ or the spin-2 combination $\ell_k\ell_m - \frac{1}{3}\ell^2\delta_{km}$. We again used the identity (D.8) to eliminate the term $\epsilon_{aij}x_j(\vec{\ell}\cdot\vec{x})^2$, and we also note the

identity $\ell_{[i}\epsilon_{a]jk}x_j\ell_k = \frac{1}{2}\epsilon_{aij}\ell_j(\ell_k x_k) - \frac{1}{2}\ell^2\epsilon_{aij}x_j$, which we used to eliminate a term of the form $\ell_a\epsilon_{ijk}x_j\ell_k\mathbf{f}_{2,9}$. At order $O(b^2)$ the possible gauge parameters are of the form:

$$\lambda^a = \epsilon_{abc}x_b\ell_c(\ell_k x_k)\mathbf{g}_{2,1}(R)\,,$$

and we use the corresponding gauge transformation to gauge fix $\mathbf{f}_{2,8} = 0$. Our parameters $\mathbf{f}_{2,1}, \mathbf{f}_{2,2}$ have spin-0, while $\mathbf{f}_{2,3}, \ldots, \mathbf{f}_{2,7}$ have spin-2. The spin-2 equations are:

$$\begin{aligned}
0 &= \tfrac{1}{R}\mathbf{f}'_{2,4} - \tfrac{1}{R}\mathbf{f}'_{2,5} - K\mathbf{f}_{2,4} + K\mathbf{f}_{2,5} + (H-K)\mathbf{f}_{2,6} \\
&\qquad + \tfrac{1}{R}\mathbf{f}'_{1,2} - K\mathbf{f}_{1,2} - H\mathbf{f}_{1,4} - \mathbf{f}_{1,2}\mathbf{f}_{1,4} - \mathbf{f}_{1,3}\mathbf{f}_{1,4}\,, && \text{(D.29)}\\
0 &= \tfrac{1}{R}\mathbf{f}'_{2,7} + H\mathbf{f}_{2,7} + K\mathbf{f}_{2,3} + (1+R^2K)\mathbf{f}_{2,4} - 2\mathbf{f}_{2,6} \\
&\qquad + K\mathbf{f}_{1,1} + \mathbf{f}_{1,2} + R^2K\mathbf{f}_{1,2} - \mathbf{f}_{1,1}\mathbf{f}_{1,3}\,, && \text{(D.30)}\\
0 &= \tfrac{1}{R}\mathbf{f}'_{2,3} - (1+R^2K)\mathbf{f}_{2,5} + (1-R^2H)\mathbf{f}_{2,6} + (K-H)\mathbf{f}_{2,7} \\
&\qquad + \tfrac{1}{R}\mathbf{f}'_{1,1} + R^2H\mathbf{f}_{1,4} + \mathbf{f}_{1,1}\mathbf{f}_{1,3} + \mathbf{f}_{1,1}\mathbf{f}_{1,4} + R^2\mathbf{f}_{1,2}\mathbf{f}_{1,4} + R^2\mathbf{f}_{1,3}\mathbf{f}_{1,4}\,, && \text{(D.31)}\\
0 &= R\mathbf{f}'_{2,5} - \tfrac{1}{R}\mathbf{f}'_{2,7} - K\mathbf{f}_{2,3} + 2\mathbf{f}_{2,4} + 4\mathbf{f}_{2,5} + (2+R^2K)\mathbf{f}_{2,6} + K\mathbf{f}_{2,7} \\
&\qquad - K\mathbf{f}_{1,1} + \mathbf{f}_{1,2} + \mathbf{f}_{1,1}\mathbf{f}_{1,3} + R^2\mathbf{f}_{1,3}\mathbf{f}_{1,4}\,, && \text{(D.32)}\\
0 &= \mathbf{f}_{2,3} - R^2\mathbf{f}_{2,6} - \mathbf{f}_{2,7} - R^2\mathbf{f}_{1,1}\mathbf{f}_{1,3} - R^4\mathbf{f}_{1,3}\mathbf{f}_{1,4}\,. && \text{(D.33)}
\end{aligned}$$

The spin-0 equations are:

$$\begin{aligned}
0 &= \mathbf{f}'_{2,1} + \tfrac{1}{R}\mathbf{f}_{2,1} + \tfrac{2}{R}(1+R^2K)\mathbf{f}_{2,2} + \tfrac{1}{3}\mathbf{f}'_{1,1} + \tfrac{1}{3}R^2\mathbf{f}'_{1,2} + \tfrac{2}{3}R\mathbf{f}_{1,2} - \tfrac{2}{3}R\mathbf{f}_{1,1}\mathbf{f}_{1,4}\,, && \text{(D.34)}\\
0 &= \mathbf{f}'_{2,2} + (\tfrac{2}{R} + RH)\mathbf{f}_{2,2} + \tfrac{1}{R}(1+R^2K)\mathbf{f}_{2,1} \\
&\quad + \tfrac{1}{3}RK\mathbf{f}_{1,1} + \tfrac{1}{3}R(1+R^2K)\mathbf{f}_{1,2} + \tfrac{1}{2}R\mathbf{f}_{1,1}\mathbf{f}_{1,3} - \tfrac{1}{3}R^3\mathbf{f}_{1,3}\mathbf{f}_{1,4}\,. && \text{(D.35)}
\end{aligned}$$

We first solve the spin-0 equations. The general solution is given by:

$$\begin{aligned}
\mathbf{f}_{2,1} &= \tfrac{R^2}{36\sinh^2 R} + \tfrac{1}{6}R\coth R + c_5\big(\tfrac{1}{\sinh^2 R} - \tfrac{1}{R}\coth R\big) + c_6\big(\tfrac{1}{R\sinh^2 R}\big)\,, && \text{(D.36)}\\
\mathbf{f}_{2,2} &= \tfrac{R^2\cosh R}{36\sinh^2 R} - \tfrac{R}{8\sinh R} + c_5\big(\tfrac{\cosh R}{\sinh^2 R} - \tfrac{1}{R\sinh R}\big) + c_6\big(\tfrac{\cosh R}{R\sinh^2 R}\big)\,. && \text{(D.37)}
\end{aligned}$$

Since c_6 multiplies an R-odd and singular solution, we set $c_6 = 0$. The unknown c_5 needs to be determined by the boundary conditions at $R = \infty$ and $R = -a$.

Now, we move on to the spin-2 equations. First we look for a solution of the homogeneous spin-2 part:

$$\begin{aligned}
0 &= \tfrac{1}{R}\mathbf{f}'_{2,4} - \tfrac{1}{R}\mathbf{f}'_{2,5} - K\mathbf{f}_{2,4} + K\mathbf{f}_{2,5} + (H-K)\mathbf{f}_{2,6}\,, && \text{(D.38)}\\
0 &= \tfrac{1}{R}\mathbf{f}'_{2,7} + H\mathbf{f}_{2,7} + K\mathbf{f}_{2,3} + (1+R^2K)\mathbf{f}_{2,4} - 2\mathbf{f}_{2,6}\,, && \text{(D.39)}\\
0 &= \tfrac{1}{R}\mathbf{f}'_{2,3} - (1+R^2K)\mathbf{f}_{2,5} + (1-R^2H)\mathbf{f}_{2,6} + (K-H)\mathbf{f}_{2,7}\,, && \text{(D.40)}\\
0 &= R\mathbf{f}'_{2,5} - \tfrac{1}{R}\mathbf{f}'_{2,7} - K\mathbf{f}_{2,3} + 2\mathbf{f}_{2,4} + 4\mathbf{f}_{2,5} + (2+R^2K)\mathbf{f}_{2,6} + K\mathbf{f}_{2,7}\,, && \text{(D.41)}\\
0 &= \mathbf{f}_{2,3} - R^2\mathbf{f}_{2,6} - \mathbf{f}_{2,7}\,. && \text{(D.42)}
\end{aligned}$$

The general solution that is well behaved as $R \to \infty$ is:

$$\mathbf{f}_{2,3}^{(homog)} = c_7\{\tfrac{4R}{\sinh R}\} + c_8\{\tfrac{4}{R^4 \sinh R}\}\,, \tag{D.43}$$

$$\mathbf{f}_{2,4}^{(homog)} = c_7\{\tfrac{6\cosh R-4}{R\sinh R} - \tfrac{2}{\sinh^2 R}\} + c_8\{-\tfrac{4(\cosh R+1)}{R^6 \sinh R} - \tfrac{2}{R^5 \sinh^2 R}\}\,, \tag{D.44}$$

$$\mathbf{f}_{2,5}^{(homog)} = c_7\{-\tfrac{2\cosh R}{\sinh^2 R} - \tfrac{4\cosh R-6}{R\sinh R}\} + c_8\{-\tfrac{4(\cosh R+1)}{R^6 \sinh R} - \tfrac{2\cosh R}{R^5 \sinh^2 R}\}\,, \tag{D.45}$$

$$\mathbf{f}_{2,6}^{(homog)} = c_7\{-\tfrac{2}{R\sinh R} + \tfrac{2\cosh R}{\sinh^2 R}\} + c_8\{\tfrac{8}{R^6 \sinh R} + \tfrac{2\cosh R}{R^5 \sinh^2 R}\}\,, \tag{D.46}$$

$$\mathbf{f}_{2,7}^{(homog)} = c_7\{\tfrac{6R}{\sinh R} - \tfrac{2R^2\cosh R}{\sinh^2 R}\} + c_8\{-\tfrac{4}{R^4 \sinh R} - \tfrac{2\cosh R}{R^3 \sinh^2 R}\}\,. \tag{D.47}$$

Additionally, there are two more linearly independent solutions that grow exponentially as $R \to \infty$. They are given by:

$$\begin{aligned}\mathbf{f}_{2,3}^{(homog)} &= c_9\{-\tfrac{2\cosh^2 R}{R^2 \sinh R} + \tfrac{6\cosh R}{R^3} - \tfrac{6\sinh R}{R^4}\}\\ &\quad + c_{10}\{-\tfrac{6\cosh R}{R^4} - \tfrac{2\cosh R}{R^2} + \tfrac{6\cosh^2 R}{R^3 \sinh R}\}\,,\end{aligned} \tag{D.48}$$

$$\begin{aligned}\mathbf{f}_{2,4}^{(homog)} &= c_9\{\tfrac{6\sinh R}{R^6} - \tfrac{3(1+2\cosh R)}{R^5} + \tfrac{2\cosh^2 R}{R^4 \sinh R} + \tfrac{1}{R^3 \sinh^2 R}\}\\ &\quad + c_{10}\{\tfrac{6(1+\cosh R)}{R^6} - \tfrac{3(\cosh R+2\cosh^2 R)}{R^3 \sinh R} + \tfrac{2\cosh R}{R^4} - \tfrac{3}{R^4 \sinh^2 R}\}\,,\end{aligned} \tag{D.49}$$

$$\begin{aligned}\mathbf{f}_{2,5}^{(homog)} &= c_9\{\tfrac{6\sinh R}{R^6} - \tfrac{3(2+\cosh R)}{R^5} + \tfrac{2\coth R}{R^4} + \tfrac{\cosh R}{R^3 \sinh^2 R}\}\\ &\quad + c_{10}\{\tfrac{6\cosh R}{R^6} + \tfrac{6}{R^6} - \tfrac{3\cosh^2 R}{R^5 \sinh R} - \tfrac{6\coth R}{R^5} + \tfrac{2}{R^4} - \tfrac{3\cosh R}{R^4 \sinh^2 R}\}\,,\end{aligned} \tag{D.50}$$

$$\begin{aligned}\mathbf{f}_{2,6}^{(homog)} &= c_9\{-\tfrac{12\sinh R}{R^6} + \tfrac{9\cosh R}{R^5} - \tfrac{2\cosh^2 R}{R^4 \sinh R} - \tfrac{\cosh R}{R^3 \sinh^2 R}\}\\ &\quad + c_{10}\{-\tfrac{12\cosh R}{R^6} + \tfrac{9\cosh^2 R}{R^5 \sinh R} + \tfrac{3\cosh R}{R^4 \sinh^2 R} - \tfrac{2\cosh R}{R^4}\}\,,\end{aligned} \tag{D.51}$$

$$\begin{aligned}\mathbf{f}_{2,7}^{(homog)} &= c_9\{\tfrac{6\sinh R}{R^4} - \tfrac{3\cosh R}{R^3} + \tfrac{\cosh R}{R\sinh^2 R}\}\\ &\quad + c_{10}\{\tfrac{6\cosh R}{R^4} - \tfrac{3\cosh^2 R}{R^3 \sinh R} - \tfrac{3\cosh R}{R^2 \sinh^2 R}\}\,.\end{aligned} \tag{D.52}$$

Once we have a complete linearly independent set of solutions to the homogeneous problem, we can find the solution to the inhomogeneous problem by integration. When we perform the integration we obtain complicated expressions that contain polylogarithms

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \tfrac{z^k}{k^n}\,.$$

For example, if we set $c_2 = 0$ in (D.22)-(D.25), we get:

$$\begin{aligned}
\mathbf{f}^{(inhomog)}_{2,3} &= -\tfrac{9}{2R^4\sinh R}\,\mathrm{Li}_4(e^{-2R}) + \big[\tfrac{3}{R^3}(\sinh R - \tfrac{2}{\sinh R}) - (\tfrac{3}{R^4}+\tfrac{1}{R^2})\cosh R\big]\,\mathrm{Li}_3(e^{-2R}) \\
&\quad -\big[\tfrac{3}{R^2\sinh R} + (\tfrac{6}{R^3}+\tfrac{2}{R})\cosh R - \tfrac{6\sinh R}{R^2}\big]\,\mathrm{Li}_2(e^{-2R}) \\
&\quad +\big[(\tfrac{6}{R^2}+2)\cosh R - \tfrac{6}{R}\sinh R\big]\log(1-e^{-2R}) - \tfrac{1}{\sinh R}\big(\tfrac{1}{2}+\tfrac{45}{2R^3}+\tfrac{2}{R}+\tfrac{59}{120}R\big) \\
&\quad +\big(\tfrac{R^2}{8\sinh^2 R}+\tfrac{45}{2R^4}+\tfrac{15}{2R^2}+\tfrac{2}{R}+2+\tfrac{2}{3}R\big)\cosh R - \big(\tfrac{45}{2R^3}+\tfrac{2}{R}+2+\tfrac{2}{3}R\big)\sinh R\,,
\end{aligned} \tag{D.53}$$

$$\begin{aligned}
\mathbf{f}^{(inhomog)}_{2,4} &= (\tfrac{9}{2R^6\sinh R}+\tfrac{9}{2R^6}\coth R + \tfrac{9}{4R^5\sinh^2 R})\,\mathrm{Li}_4(e^{-2R}) \\
&\quad +\big[\tfrac{3}{R^6}+(\tfrac{3}{R^6}+\tfrac{1}{R^4})\cosh R + \tfrac{15}{2R^5}\coth R + \tfrac{6}{R^5\sinh R} + \tfrac{3}{R^4\sinh^2 R} - \tfrac{3}{R^5}\sinh R\big]\,\mathrm{Li}_3(e^{-2R}) \\
&\quad +\big[\tfrac{6}{R^5}+\tfrac{3}{2R^3\sinh^2 R}+(\tfrac{6}{R^5}+\tfrac{2}{R^3})\cosh R + \tfrac{6}{R^4}\coth R + \tfrac{3}{R^4\sinh R} - \tfrac{6}{R^4}\sinh R\big]\,\mathrm{Li}_2(E^{-2R}) \\
&\quad +\big[\tfrac{6}{R^3}\sinh R - \tfrac{3}{R^3}\coth R - (\tfrac{6}{R^4}+\tfrac{2}{R^2})\cosh R - \tfrac{6}{R^4}\big]\log(1-e^{-2R}) \\
&\quad +(\tfrac{45}{2R^5}+\tfrac{2}{R^3}+\tfrac{2}{R^2}+\tfrac{2}{3R})\sinh R + (\tfrac{45}{4R^4}+\tfrac{1}{R^2}+\tfrac{1}{4R}+\tfrac{37}{120})\tfrac{1}{\sinh^2 R} - \tfrac{\cosh R}{8\sinh^2 R} \\
&\quad +(\tfrac{45}{2R^5}+\tfrac{2}{R^3}+\tfrac{1}{2R^2}+\tfrac{59}{120R})\tfrac{1}{\sinh R} + (\tfrac{45}{4R^5}+\tfrac{2}{R^3}-\tfrac{1}{2R^2}+\tfrac{1}{5R})\coth R \\
&\quad -(\tfrac{45}{2R^6}+\tfrac{15}{2R^4}+\tfrac{2}{R^3}+\tfrac{2}{R^2}+\tfrac{2}{3R})\cosh R - \tfrac{45}{2R^6} - \tfrac{2}{R^3} + \tfrac{1}{8R^2}\,,
\end{aligned} \tag{D.54}$$

$$\begin{aligned}
\mathbf{f}^{(inhomog)}_{2,5} &= (\tfrac{9}{2R^6\sinh R}+\tfrac{9}{2R^6}\coth R + \tfrac{9\cosh R}{4R^5\sinh^2 R})\,\mathrm{Li}_4(e^{-2R}) \\
&\quad +(\tfrac{3}{R^6}+\tfrac{1}{R^4}+\tfrac{3}{R^6}\cosh R - \tfrac{3}{2R^5}\sinh R + \tfrac{6}{R^5}\coth R + \tfrac{15}{2R^5\sinh R} + \tfrac{3\cosh R}{R^4\sinh^2 R})\,\mathrm{Li}_3(e^{-2R}) \\
&\quad +(\tfrac{6}{R^5}+\tfrac{2}{R^3}+\tfrac{6}{R^5}\cosh R - \tfrac{3}{R^4}\sinh R + \tfrac{3}{R^4}\coth R + \tfrac{6}{R^4\sinh R} + \tfrac{3\cosh R}{2R^3\sinh^2 R})\,\mathrm{Li}_2(e^{-2R}) \\
&\quad -\big[\tfrac{6}{R^4}+\tfrac{2}{R^2}+\tfrac{6}{R^4}\cosh R - \tfrac{3}{R^3}\sinh R + \tfrac{3}{R^3\sinh R}\big]\log(1-e^{-2R}) \\
&\quad -\tfrac{45}{2R^6} - \tfrac{15}{2R^4} - \tfrac{2}{R^3} - \tfrac{3}{8R^2} - \tfrac{2}{3R} - \big(\tfrac{45}{2R^6}+\tfrac{2}{R^3}+\tfrac{1}{R^2}\big)\cosh R \\
&\quad +\big(\tfrac{45}{4R^5}+\tfrac{2}{R^3}+\tfrac{1}{R^2}\big)\sinh R + \big(\tfrac{45}{2R^5}+\tfrac{2}{R^3}+\tfrac{1}{2R^2}+\tfrac{13}{15R}\big)\coth R \\
&\quad +\big(\tfrac{45}{4R^5}+\tfrac{2}{R^3}-\tfrac{1}{2R^2}-\tfrac{27}{40R}\big)\tfrac{1}{\sinh R} + \big(\tfrac{45}{4R^4}+\tfrac{1}{R^2}+\tfrac{1}{4R}+\tfrac{37}{120}\big)\tfrac{\cosh R}{\sinh^2 R} - \tfrac{1}{8\sinh^2 R}\,,
\end{aligned} \tag{D.55}$$

$$\begin{aligned}
\mathbf{f}^{(inhomog)}_{2,6} &= -(\tfrac{9}{R^6\sinh R}+\tfrac{9\cosh R}{4R^5\sinh^2 R})\,\mathrm{Li}_4(e^{-2R}) \\
&\quad -\big[(\tfrac{6}{R^6}+\tfrac{1}{R^4})\cosh R - \tfrac{9}{2R^5}\sinh R + \tfrac{27}{2R^5\sinh R} + \tfrac{3\cosh R}{R^4\sinh^2 R}\big]\,\mathrm{Li}_3(e^{-2R}) \\
&\quad -\big[(\tfrac{12}{R^5}+\tfrac{2}{R^3})\cosh R - \tfrac{9}{R^4}\sinh R + \tfrac{9}{R^4\sinh R} + \tfrac{3\cosh R}{2R^3\sinh^2 R}\big]\,\mathrm{Li}_2(e^{-2R}) \\
&\quad +\big[(\tfrac{12}{R^4}+\tfrac{2}{R^2})\cosh R - \tfrac{9}{R^3}\sinh R + \tfrac{3}{R^3\sinh R}\big]\log(1-e^{-2R}) \\
&\quad -(\tfrac{45}{4R^4}+\tfrac{1}{R^2}+\tfrac{1}{4R}+\tfrac{13}{30})\tfrac{\cosh R}{\sinh^2 R} + (\tfrac{45}{R^6}+\tfrac{15}{2R^4}+\tfrac{4}{R^3}+\tfrac{3}{R^2}+\tfrac{2}{3R})\cosh R \\
&\quad -(\tfrac{135}{4R^5}+\tfrac{4}{R^3}+\tfrac{3}{R^2}+\tfrac{2}{3R})\sinh R - (\tfrac{135}{4R^5}+\tfrac{4}{R^3}-\tfrac{13}{30R})\tfrac{1}{\sinh R}\,,
\end{aligned} \tag{D.56}$$

$$\mathbf{f}^{(inhomog)}_{2,7} = \mathbf{f}^{(inhomog)}_{2,3} - R^2\mathbf{f}^{(inhomog)}_{2,6} - R^2\mathbf{f}_{1,1}\mathbf{f}_{1,3} - R^4\mathbf{f}_{1,3}\mathbf{f}_{1,4}\,. \tag{D.57}$$

We note that the combinations of polylogarithms that appear here are the results of the integrals

$$\int R^3\coth R\,dR = -\tfrac{3}{4}\,\mathrm{Li}_4(e^{-2R}) - \tfrac{3}{2}R\,\mathrm{Li}_3(e^{-2R}) - \tfrac{3}{2}R^2\,\mathrm{Li}_2(e^{-2R}) + R^3\log(1-e^{-2R}) + \tfrac{1}{4}R^4\,,$$

and

$$\int R^2 \coth R \, dR = -\tfrac{1}{2} \operatorname{Li}_3(e^{-2R}) - R \operatorname{Li}_2(e^{-2R}) + R^2 \log(1 - e^{-2R}) + \tfrac{1}{3} R^3 \, .$$

When we turn on $c_2 \neq 0$ we get additional terms, but they can be expressed as rational functions of e^R and R and do not cancel the polylogarithms. In any case, this demonstrates that a simple solution to the BPS equations (3.40), involving only basic functions, does not exist.