Title
Shifting Hecke Eigensystems in Positive Characteristic

Permalink
https://escholarship.org/uc/item/9wn926w3

Author
Reduzzi, Davide Alessandro

Publication Date
2012

Peer reviewed|Thesis/dissertation

Shifting Hecke Eigensystems
in Positive Characteristic

by

Davide Alessandro Reduzzi

2012

Shifting Hecke Eigensystems
in Positive Characteristic

by

Davide Alessandro Reduzzi

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2012

Professor Chandrashekhar Khare, Chair

We study congruences modulo p between modular forms arising from different contexts. In the first part of the dissertation we use geometric methods to show that the (mod p) PEL Hecke eigensystems associated to a reductive group G coincide with the (mod p) algebraic Hecke eigensystems associated to an inner form of G. In the second part of the dissertation we use cohomological methods to construct weight shiftings for (mod p) automorphic forms associated to a definite quaternion algebra over a totally real field in which p is unramified. In particular, we construct cohomological avatars of the partial Hasse invariants.

The dissertation of Davide Alessandro Reduzzi is approved.

Joseph A. Rudnick

Haruzo Hida

Richard S. Elman

Chandrashekhar Khare, Committee Chair

University of California, Los Angeles

2012

Alla mia famiglia

ACKNOWLEDGMENTS

First I would like to express my gratitude to my advisor, Chandrashekhar Khare, for the great opportunity he gave to me by accepting me as a student of his, for his guidance and encouragement during the years of my doctorate, and for his patience and generosity in answering my questions and giving me advice.

Conversations and correspondence with Don Blasius, Najmuddin Fakhruddin, Benedict Gross, Florian Herzig, Haruzo Hida, Marc-Hubert Nicole, Gordan Savin and Jacques Tilouine have contributed to the development of my research during the last three years: I owe thanks to them. In particular, I am grateful to Benedict Gross for suggesting the problem investigated in Part I of this dissertation. I am grateful to Massimo Bertolini, for being a constant reference in the Department of Mathematics of the Univeristà degli Studi di Milano, and to David Gieseker, for many interesting conversations we had about mathematics and other subjects.

I thank Patrick Allen, with whom I have gone through many stages of the Ph.D. student life. My life in UCLA has been moreover made easier by the work of the staff of the Department of Mathematics, which I also sincerely thank.

I could not be where I am today without the love of my family: I thank my parents Patrizia and Lino, my sisters Chiara and Haregeweyn, and my brothers Alberto and Stefano, for always sustaining me, and for being close even when I am so far away from home.

A special thanks goes to Kien, for his patience and support, and for introducing me to National Parks: places in which, like in mathematics, "when we try to pick out anything by itself, we find it hitched to everything else in the Universe." (J. Muir, *My First Summer in the Sierra*, 1911).

<div align="center">

Vita

</div>

2002–2005   B.Sc. (Mathematics), Università degli Studi di Milano, Italy

2005–2007   M.Sc. (Mathematics), Università degli Studi di Milano, Italy

2007–2012   Ph.D. Student and Teaching Assistant, Department of Mathematics, UCLA

<div align="center">

Publications

</div>

Reduction mod p of Cuspidal Representations of $GL_2(\mathbb{F}_{p^n})$ and Symmetric Powers, Journal of Algebra 324 (2010), pp. 3507–3531.

Introduction

In the present work we study various techniques for producing congruences modulo p between systems of Hecke eigenvalues arising from spaces of modular forms.

More precisely, in Part I we consider spaces of $(\bmod p)$ PEL modular forms associated to a reductive group G and we show that, under suitable assumptions, the Hecke eigensystems that they afford coincide with the Hecke eigensystems arising from algebraically defined modular forms associated to an inner form of G. Our results generalize constructions of Serre ([Ser96]) and Ghitza ([Ghi04a]) and can be interpreted as an instance of a global Langlands correspondence, conjecturally relating $(\bmod p)$ Galois representations arising from geometric objects to Hecke eigensystems occurring in adelic spaces. Our proofs make use of a uniformization result for basic isogeny classes of Shimura varieties due to Rapoport and Zink ([RZ96]).

In Part II, we work with automorphic forms associated to some definite quaternion algebras over a totally real field. In this context, we produce congruences modulo p between eigenforms of fixed level and varying weights. These weight shiftings are obtained via cohomological methods, generalizing constructions of Ash-Stevens ([AS86b]) and Edixhoven-Khare ([EK03]). In particular, we define cohomological avatars of the partial Hasse invariant operators, geometrically constructed by Goren ([Gor01]) and Andreatta-Goren ([AG05]) in the context of Hilbert-Blumenthal modular varieties. The starting point of our constructions can be found in a note of Serre on the existence of some differential operators intertwining modular representations of $GL_2(\mathbb{F}_q)$ (cf. [Ser01] and [Red10]).

Part I

Hecke Eigensystems of PEL and Algebraic Types

CHAPTER 1

Moduli of PEL type

1.1 Moduli of p-divisible groups

Let G be a p-divisible group over a scheme S (cf. [Dem72], [Fon77]) and denote by $\hat{G}$ its Serre dual. An S-polarization of G is an S-quasi-isogeny $\lambda : G \to \hat{G}$ that is anti-symmetric (i.e., $\hat{\lambda} = -\lambda$). A $\mathbb{Q}_p$-homogeneous S-polarization of G is the set $\overline{\lambda} = \mathbb{Q}_p^\times \lambda$ of $\mathbb{Q}_p^\times$-multiples of an S-polarization λ of G; a principal S-polarization is an S-polarization that is also an isomorphism. If $(\mathfrak{O};^*)$ is a $\mathbb{Z}_p$-algebra with involution, an action of $(\mathfrak{O};^*)$ on G is a homomorphism of $\mathbb{Z}_p$-algebras $i : \mathfrak{O} \to \mathrm{End}_S(G)$; if G is endowed with the action i, $\hat{G}$ is endowed with the dual action $\hat{i}$ given by setting $\hat{i}(a) := i(a^*)\hat{\,}$ for any $a \in \mathfrak{O}$.

Let k be a perfect field of characteristic $p > 0$ and denote by $W = W(k)$ the ring of Witt vectors of k; let $K_0 = W[\frac{1}{p}]$ and denote by σ the absolute Frobenius morphism of K_0. An isocrystal (D_0, F) over K_0 is a finite dimensional K_0-vector space D_0 endowed with a Frobenius K_0-semilinear automorphism $F : D_0 \to D_0$. For $n \in \mathbb{Z}$, define $\mathbf{1}(n)$ to be the isocrystal $(K_0, p^n\sigma)$. A polarization of D_0 is a K_0-bilinear non-degenerate alternating pairing of isocrystals $\langle , \rangle : D_0 \times D_0 \to \mathbf{1}(1)$, so that $\langle Fx, Fy \rangle = p \langle x, y \rangle^\sigma$ for all $x, y \in D_0$; a $\mathbb{Q}_p$-homogeneous polarization of D_0 is the equivalence class of $\mathbb{Q}_p^\times$-multiples of a polarization.

Let D be a Dieudonné W-module (i.e., a finitely generated left module over the Dieudonné ring $W[F, V]$) that is finite free over W; a polarization of D is a W-bilinear non-degenerate alternating form $\langle , \rangle : D \times D \to W$ such that $\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma$ for all $x, y \in D$; a $\mathbb{Z}_p$-homogeneous polarization of D is the equivalence class of $\mathbb{Z}_p^\times$-multiples of a polarization. A polarization is principal if it is a perfect pairing.

When k is algebraically closed, denote by M (resp. M_*) the contravariant (resp. covari-

3

ant) Dieudonné functor, giving an additive anti-equivalence (resp. equivalence) between the category of p-divisible groups over k and the category of Dieudonné $W(k)$-modules that are free and of finite rank over $W(k)$ (cf. [Fon77]). The functor M_* preserves the dictionary of (principal) polarizations between the categories of p-divisible groups and of Dieudonné modules.

1.1.1 Local PEL data

Let k, W, K_0 and σ be as above, with k algebraically closed. Let B be a finite dimensional semi-simple $\mathbb{Q}_p$-algebra endowed with an involution *, and let $V \neq 0$ be a finitely generated left B-module, endowed with a non-degenerate, alternating, $\mathbb{Q}_p$-bilinear form $\langle,\rangle : V \times V \to \mathbb{Q}_p$ which is skew-Hermitian with respect to *. These objects define a reductive group G over $\mathbb{Q}_p$ whose R-points, for any $\mathbb{Q}_p$-algebra R, are given by:

$$G(R) = \{(g,s) \in GL_{B \otimes_{\mathbb{Q}_p} R}(V \otimes_{\mathbb{Q}_p} R) \times R^\times : \langle gv, gw \rangle = s \langle v, w \rangle \quad \forall v, w \in V \otimes_{\mathbb{Q}_p} R\}.$$

The map $G(R) \to R^\times$ given by $(g,s) \mapsto s$ defines a homomorphism of $\mathbb{Q}_p$-groups $c : G \to \mathbb{G}_m$, called the similitude character of G.

A $\mathbb{Q}_p$-PEL datum for moduli of p-divisible groups over k is the datum

$$\mathcal{D}_p = (B, {}^*, V, \langle,\rangle, \mathcal{O}_B, \Lambda, b, \mu),$$

where: (1) $(B, {}^*, V, \langle,\rangle)$ is as above; (2) $\mathcal{O}_B$ is a maximal $\mathbb{Z}_p$-order in B stable under the involution *; (3) $\Lambda \subset V$ is an $\mathcal{O}_B$-stable $\mathbb{Z}_p$-lattice of V which is self-dual with respect to $\langle,\rangle$; (4) b is a fixed element of $G(K_0)$; (5) $\mu : \mathbb{G}_{m/K} \to G_K$ is a co-character of G defined over a finite field extension K of K_0. We require that the following four conditions are satisfied: (a) (b, μ) is an admissible pair in the sense of [RZ96]; (b) the isocrystal $(N, \mathbf{F}) := (V \otimes_{\mathbb{Q}_p} K_0, b\sigma)$ has slopes in the interval $[0,1]$; (c) the weight decomposition of $V \otimes_{\mathbb{Q}_p} K$ with respect to μ contains only weights 0 and 1: $V \otimes_{\mathbb{Q}_p} K = V_0 \oplus V_1$; (d) let $\mathbb{D}$ be the universal cover of $\mathbb{G}_m$ in the sense of quasi-algebraic groups ([Ser60], 7.3) and let $\nu : \mathbb{D}_{K_0} \to G_{K_0}$ be the slope morphism associated to b, as defined in [Kot85], 4 (for any algebraic finite-dimensional $\mathbb{Q}_p$-

4

representation $\rho : G \to GL(\mathcal{U})$ of G, let $\nu_\rho \in \mathrm{Hom}_{K_0}(\mathbb{D}_{K_0}, GL(\mathcal{U}_{K_0}))$ be the morphism for which the action of $\mathbb{D}$ on the isotypical component of the isocrystal $(\mathcal{U} \otimes_{\mathbb{Q}_p} K_0, \rho(b)(id_{\mathcal{U}} \otimes \sigma))$ of slope $\lambda \in \mathbb{Q}$ is given by the character $\lambda \in X^*(\mathbb{D})$; ν is the only morphism such that $\rho \circ \nu = \nu_\rho$ for any ρ as above); then we ask that $c \circ \nu : \mathbb{D}_{K_0} \to \mathbb{G}_{m/K_0}$ is the character of $\mathbb{D}_{K_0}$ corresponding to the rational number 1.

The reflex field E of $\mathcal{D}_p$ is the field of definition of the conjugacy class of the co-character μ.

1.1.2 The moduli functor for p-divisible groups

Let us keep the above notation, so that a local PEL datum $\mathcal{D}_p$ with reflex field E is fixed; write $c(b) = p \cdot u\sigma(u)^{-1}$ for some $u \in W^\times$. Define a map $\Psi : N \times N \to K_0$ by setting $\Psi(v, w) = u^{-1} \langle v, w \rangle$ for $v, w \in N$. As $\Psi(\mathbf{F}v, \mathbf{F}w) = p\Psi(v, w)^\sigma$, Ψ defines a polarization of the isocrystal N. Any other choice of u gives rise to a $\mathbb{Q}_p^\times$-multiple of Ψ, so that $\mathcal{D}_p$ defines a $\mathbb{Q}_p$-homogeneously polarized K_0-isocrystal endowed with an action of B: $(N, \mathbf{F}, \mathbb{Q}_p^\times \Psi)$.

Fix a p-divisible group $\mathbf{X}$ over k whose K_0-isocrystal constructed via M_* is isomorphic to $(N, \mathbf{F})$; by functoriality there is a $\mathbb{Q}_p$-algebra homomorphism $i_{\mathbf{X}} : B \to \mathrm{End}(\mathbf{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The class $\mathbb{Q}_p^\times \Psi$ induces a $\mathbb{Q}_p$-homogeneous polarization $\bar{\lambda}_{\mathbf{X}} : (\mathbf{X}, i_{\mathbf{X}}) \to (\hat{\mathbf{X}}, i_{\hat{\mathbf{X}}})$ that respects the action of B (this polarization need not to be principal). The triple $(\mathbf{X}, i_{\mathbf{X}}, \bar{\lambda}_{\mathbf{X}})$ is determined up to quasi-isogeny, and it is assumed fixed.

Let $\breve{E} = EK_0$ and denote by $\mathcal{O}_{\breve{E}}$ the ring of integers of $\breve{E}$; let $NILP_{\mathcal{O}_{\breve{E}}}$ be the category of locally noetherian $\mathrm{Spec}\, \mathcal{O}_{\breve{E}}$-schemes $(S, \mathcal{O}_S)$ such that the ideal sheaf $p\mathcal{O}_S$ is locally nilpotent; for any such S, denote by $\overline{S}$ the closed subscheme of S defined by $p\mathcal{O}_S$. Following [RZ96], 3.21, let $\breve{\mathcal{M}}$ be the contravariant functor $NILP_{\mathcal{O}_{\breve{E}}} \to SETS$ defined as follows: if S is a scheme in $NILP_{\mathcal{O}_{\breve{E}}}$, $\breve{\mathcal{M}}(S)$ consists of the equivalence classes of tuples $(X, i, \bar{\lambda}; \rho)$ where: (1) X is a p-divisible group over S; (2) $i : \mathcal{O}_B \to \mathrm{End}_S X$ is a $\mathbb{Z}_p$-algebra homomorphism such that $\det_{\mathcal{O}_S}(a, \mathrm{Lie}\, X) = \det_K(a, V_0)$ for all $a \in \mathcal{O}_B$ (cf. [RZ96], 3.23); (3) $\lambda : (X, i) \to (\hat{X}, \hat{\imath})$ is a principal polarization of (X, i); (4) $\rho : (\mathbf{X}, i_{\mathbf{X}})_{\overline{S}} \to (X, i)_{\overline{S}}$ is a quasi isogeny of p-divisible groups over $\overline{S}$ that respects the $\mathcal{O}_B$-structure and such that $\hat{\rho} \circ \lambda_{\overline{S}} \circ \rho \in \mathbb{Q}_p^\times (\lambda_{\mathbf{X}})_{\overline{S}}$. Two tuples $(X, i, \bar{\lambda}; \rho), (X', i', \overline{\lambda}'; \rho') \in \breve{\mathcal{M}}(S)$ are equivalent if the $\overline{S}$-quasi-isogeny $\rho' \circ \rho^{-1}$ lifts to

an isomorphism $f : (X, i) \to (X', i')$ of p-divisible groups over S with $\mathcal{O}_B$-action, such that $\hat{f} \circ \lambda' \circ f \in \mathbb{Z}_p^\times \lambda$.

By [RZ96], 3.25, the functor $\breve{\mathcal{M}}$ is prorepresentable by a formal scheme $\breve{\mathcal{M}}$ which is formally locally of finite type over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$.

1.1.3 A variant of the moduli functor

Let us keep the above assumptions and further require that the polarization $\lambda_{\mathbf{X}}$ is principal and that $i_{\mathbf{X}}$ comes from an action of $\mathcal{O}_B$ on $\mathbf{X}$ (this will always be true in our applications). Define the set $\mathcal{M}'(\overline{\mathbb{F}}_p) := \mathcal{M}'_{(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})}(\overline{\mathbb{F}}_p)$ to be the collection of equivalence classes of quasi-isogenies $\rho : (\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}}) \to (\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$ of $\mathbf{X}$ over $\overline{\mathbb{F}}_p$ that respect the $\mathcal{O}_B$-structure and such that $\hat{\rho} \circ \lambda_{\mathbf{X}} \circ \rho \in \mathbb{Q}_p^\times \lambda_{\mathbf{X}}$. Two quasi-isogenies ρ and ρ' are said to be equivalent if the $\overline{\mathbb{F}}_p$-quasi-isogeny $f := \rho' \circ \rho^{-1}$ is an isomorphism $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}}) \to (\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$ of p-divisible groups over $\overline{\mathbb{F}}_p$ with $\mathcal{O}_B$-action, such that $\hat{f} \circ \lambda_{\mathbf{X}} \circ f \in \mathbb{Z}_p^\times \lambda_{\mathbf{X}}$.

$\mathcal{M}'(\overline{\mathbb{F}}_p)$ is a non-empty closed subset of $\breve{\mathcal{M}}(\overline{\mathbb{F}}_p)$. Let $J(\mathbb{Q}_p)$ denote the group of quasi-isogenies $\rho : (\mathbf{X}, i_{\mathbf{X}}) \to (\mathbf{X}, i_{\mathbf{X}})$ over $\overline{\mathbb{F}}_p$ such that $\hat{\rho} \circ \lambda_{\mathbf{X}} \circ \rho \in \mathbb{Q}_p^\times \lambda_{\mathbf{X}}$, and let $J(\mathbb{Z}_p)$ be the subgroup of isomorphisms $(\mathbf{X}, i_{\mathbf{X}}) \to (\mathbf{X}, i_{\mathbf{X}})$ preserving the polarization form up to a factor in $\mathbb{Z}_p^\times$. While the space $\breve{\mathcal{M}}(\overline{\mathbb{F}}_p)$ is somehow mysterious, by sending $[\rho] \in \mathcal{M}'(\overline{\mathbb{F}}_p)$ to the coset $\rho^{-1} J(\mathbb{Z}_p) \subset J(\mathbb{Q}_p)$ we have:

Proposition 1.1.1. There is a natural bijection $\mathcal{M}'(\overline{\mathbb{F}}_p) \simeq J(\mathbb{Q}_p) / J(\mathbb{Z}_p)$.

1.2 Moduli of abelian schemes

Let S be a locally noetherian scheme. If A is an abelian scheme over S, we denote by $\hat{A}$ its dual and by $A(p)$ its p-divisible group. Let $(\mathfrak{O}; *)$ be a $\mathbb{Z}_{(p)}$-algebra with involution. The category of abelian $\mathfrak{O}$-schemes over S up to isogeny of order prime to p, denoted by $AbSCH^*_{\mathfrak{O},S}$ or, when no confusion arises, by $AbSCH^*$, is defined as follows: its objects are pairs (A, i) where A is an abelian scheme over S, and i is a homomorphism of $\mathbb{Z}_{(p)}$-algebras $i : \mathfrak{O} \to \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$; a morphism $f : (A_1, i_1) \to (A_2, i_2)$ in $AbSCH^*_{\mathfrak{O},S}$ is an element of

$\operatorname{Hom}_{\mathfrak{O}}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. An isogeny in $AbSCH^*_{\mathfrak{O},S}$ is a quasi-isogeny of abelian S-schemes which is a morphism of $AbSCH^*_{\mathfrak{O},S}$; its kernel is the kernel of the corresponding isogeny of p-divisible groups. A quasi-isogeny in $AbSCH^*_{\mathfrak{O},S}$ is a quasi-isogeny of abelian schemes that respects the action of $\mathfrak{O}$. If (A, i) is an object of $AbSCH^*_{\mathfrak{O},S}$, its dual object is $(\hat{A}, \hat{\imath})$, where $\hat{\imath} : \mathfrak{O} \to \operatorname{End} \hat{A} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is given by $\hat{\imath}(b) = i\,(b^*)\,\hat{}\ (b \in \mathfrak{O})$.

There is an obvious notion of polarization in $AbSCH^*_{\mathfrak{O},S}$; a $\mathbb{Q}$-homogeneous (resp. $\mathbb{Z}_{(p)}$-homogeneous) polarization $\overline{\lambda} : (A, i) \to (\hat{A}, \hat{\imath})$ is the set of (locally on S) $\mathbb{Q}^{\times}$-multiples (resp. $\mathbb{Z}_{(p)}^{\times}$-multiples) of a polarization λ of (A, i) in $AbSCH^*_{\mathfrak{O},S}$; $\overline{\lambda}$ is said to be principal if there is an element $\lambda' \in \overline{\lambda}$ that is a principal polarization in $AbSCH^*_{\mathfrak{O},S}$. (Cf. [Lan08]).

1.2.1 Global PEL data

Let B be a finite dimensional semi-simple $\mathbb{Q}$-algebra endowed with a positive involution *; let $V \neq 0$ be a finitely generated left B-module and $\langle,\rangle : V \times V \to \mathbb{Q}$ a non-degenerate, alternating $\mathbb{Q}$-bilinear form which is skew-hermitian with respect to *. These objects define a reductive group G over $\mathbb{Q}$ whose R-points, for a fixed $\mathbb{Q}$-algebra R, are given by $G(R) = \{(g, s) \in GL_{B \otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \times R^{\times} : \langle gv, gw \rangle = s \langle v, w \rangle\ \forall v, w \in V \otimes_{\mathbb{Q}} R\}$. The map $G(R) \to R^{\times}$ given by $g \mapsto c(g)$ defines a homomorphism of $\mathbb{Q}$-groups $c : G \to \mathbb{G}_m$, called the similitude character of G.

A $\mathbb{Q}$-PEL datum for moduli of abelian schemes (at p) is a tuple

$$\mathcal{D} = (B, {}^*, V, \langle,\rangle, \mathcal{O}_B, \Lambda, h, K^p, \nu)$$

where: (1) $(B, {}^*, V, \langle,\rangle)$ is as above; (2) $\mathcal{O}_B$ is a $\mathbb{Z}_{(p)}$-order of B stable under the involution * and such that $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order in $B_{\mathbb{Q}_p}$; (3) $\Lambda \subset V_{\mathbb{Q}_p}$ is an $\mathcal{O}_B$-stable $\mathbb{Z}_p$-lattice such that the restriction of $\langle,\rangle_{\mathbb{Q}_p}$ to $\Lambda \times \Lambda$ is a perfect pairing of $\mathbb{Z}_p$-modules; (4) K^p is an open compact subgroup of $G(\mathbb{A}_f^p)$; (5) $\nu : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ is an embedding of fields; (6) $h : \mathbb{C} \to \operatorname{End}_B V \otimes_{\mathbb{Q}} \mathbb{R}$ is an $\mathbb{R}$-algebra homomorphism such that: (a) $h\,(\overline{z}) = h\,(z)^*$ for all z in $\mathbb{C}$; (b) the symmetric $\mathbb{R}$-bilinear form $(,) : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ defined by $(v, w) = \langle v, h(\sqrt{-1})w \rangle$ is positive definite.

The PEL datum $\mathcal{D}$ is said to have *good reduction at p* if the algebra $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is unramified

7

and, in case $\text{End}_B V \otimes_{\mathbb{Q}} \mathbb{R}$ has a factor isomorphic to $M_n(\mathbb{H})$ for some $n > 0$, then p is odd (here $\mathbb{H}$ denotes the division algebra of real quaternions). (Cf. [Wed99], 1.4). When we are in the good reduction case, the algebraic group $G_{\mathbb{Q}_p}$ has a reductive model $\mathcal{G}$ over $\mathbb{Z}_p$ whose R-valued points for a commutative $\mathbb{Z}_p$-algebra R are given by:

$$\mathcal{G}(R) = \{(g, s) \in GL_{\mathcal{O}_B \otimes_{\mathbb{Z}_p} R}(\Lambda \otimes_{\mathbb{Z}_p} R) \times R^{\times} : \langle gv, gw \rangle = s \langle v, w \rangle \quad \forall v, w \in \Lambda \otimes_{\mathbb{Z}_p} R \}.$$

If $\mathcal{D}$ has good reduction, such a reductive model of $G_{\mathbb{Q}_p}$ will be considered fixed without further mention.

The map h endows $V_{\mathbb{R}}$ with a complex structure. Let $\mu : \mathbb{G}_{m/\mathbb{C}} \to G_{\mathbb{C}}$ be the co-character associated to h as in [RZ96], 6.1, and write $V_{\mathbb{C}} = V_{\mathbb{C},0} \oplus V_{\mathbb{C},1}$, where $V_{\mathbb{C},0}$ (resp. $V_{\mathbb{C},1}$) is the subspace of $V_{\mathbb{C}}$ on which $\mathbb{G}_{m/\mathbb{C}}$ acts - via μ - through the trivial (resp. identity) character. We obtain a semi-simple representation $\rho : B \to \text{End}_{\mathbb{C}} V_{\mathbb{C},0}$ of the $\mathbb{Q}$-algebras B. The reflex field of the PEL datum $\mathcal{D}$ is the field of definition E of the isomorphism class of ρ. The ν-adic completion of E, denoted E_{ν}, is called the ν-adic reflex field.

1.2.2 The moduli functor for abelian schemes

Let us assume a global PEL datum $\mathcal{D}$ for moduli of abelian schemes is fixed; let G be the associated algebraic group, and E the reflex field; assume that $\mathcal{D}$ has good reduction at p. Let S be a locally noetherian base scheme over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Let $(A, i; \lambda)$ be a principally polarized abelian scheme in $AbSCH^*$; assume that S is connected and let s be a geometric point of S. The Tate $\mathbb{A}_f^p$-module $H_1(A_s, \mathbb{A}_f^p)$ of A_s is endowed with a continuous action of $\pi_1(S, s)$. The action of $\mathcal{O}_B$ on A endows $H_1(A_s, \mathbb{A}_f^p)$ with a structure of B-module, and the principal polarization λ of (A, i) induces a skew-symmetric $\mathbb{A}_f^p$-linear pairing $H_1(A_s, \mathbb{A}_f^p) \times H_1(A_s, \mathbb{A}_f^p) \to \mathbb{A}_f^p(1)$, which is non-degenerate and skew-Hermitian with respect to *. On the other side, by definition of PEL datum, $V_{\mathbb{A}_f^p}$ is endowed with an action of B and a skew-Hermitian non-degenerate $\mathbb{A}_f^p$-linear pairing with values in $\mathbb{A}_f^p$. A level structure of type K^p based at s on $(A, i; \lambda)$ is the left K^p-orbit $\overline{\alpha}$ of an isomorphism $\alpha : H_1(A_s, \mathbb{A}_f^p) \to V_{\mathbb{A}_f^p}$ of skew-Hermitian B-modules such that $\overline{\alpha}$ is fixed by $\pi_1(S, s)$. Here by isomorphism of skew-Hermitian B-modules we mean an isomorphism of

8

B-modules carrying one alternating form into a $(\mathbb{A}_f^p)^\times$-multiple of the other. The choice of s is immaterial in practice (and it will not be mentioned later): if s' is another geometric point of S, there is a canonical bijection between level structures of type K^p based at s on $(A, i; \lambda)$ and level structures of type K^p based at s' on $(A, i; \lambda)$ ([Lan08], Cor. 1.3.7.13).

Assume that G has a reductive model $G_{/\mathbb{Z}}$ over $\mathbb{Z}$ such that for any commutative ring R one has:

$$G(R) = \{(g, s) \in GL_{\mathcal{O}_B \otimes_{\mathbb{Z}} R}(L \otimes_{\mathbb{Z}} R) \times R^\times : \langle gv, gw \rangle = s \langle v, w \rangle \quad \forall v, w \in L \otimes_{\mathbb{Z}} R \},$$

where L is an $\mathcal{O}_B$-stable $\mathbb{Z}$-lattice of V which is self-dual with respect to $\langle, \rangle$ and such that $L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \Lambda$. (In our later applications such a choice for G over $\mathbb{Z}$ will always be possible and it will be fixed without further mention). If $N \geq 1$ is an integer not divisible by p, a principal level-N structure on $(A, i; \lambda)$ is a level structure of type $U(N) := \mathrm{Ker}(G(\hat{\mathbb{Z}}^p) \to G(\hat{\mathbb{Z}}^p/N\hat{\mathbb{Z}}^p))$. If K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$ contained in $U(N)$ for some $N \geq 3$ not divisible by p, then K^p is neat (cf. [Lan08], 1.4.1.9-10).

Define the moduli problem $\mathbf{M} := \mathbf{M}(\mathcal{D})$ associated to the PEL datum $\mathcal{D}$ to be the contravariant functor from the category $SCH_{\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}}$ of locally noetherian schemes over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ to the category of sets as follows: if S is an object of $SCH_{\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p}$, then $\mathbf{M}(S)$ is the set of isomorphism classes of tuples $(A, i, \overline{\lambda}, \overline{\alpha})$ where: (1) (A, i) is an object in $AbSCH^*$ satisfying Kottwitz determinant condition: for any locally noetherian S-scheme S', $\det_{\mathcal{O}_{S'}}(a, \mathrm{Lie}\, A_{S'}) = \det_E(a, V_0)$ ($a \in \mathcal{O}_B \otimes \mathcal{O}_{S'}$; cf. [RZ96], 3.23 or [Kot92], 5, for the precise definition); (2) $\overline{\lambda} : (A, i) \to (\hat{A}, \hat{i})$ is a $\mathbb{Q}$-homogeneous principal polarization in $AbSCH^*$; (3) $\overline{\alpha}$ is a level structure of type K^p on $(A, i, \overline{\lambda})$. Two tuples $(A_1, i_1, \overline{\lambda}_1, \overline{\alpha}_1)$ and $(A_2, i_2, \overline{\lambda}_2, \overline{\alpha}_2)$ as above are isomorphic if there is an isomorphism $f : (A_1, i_1; \overline{\lambda}_1) \to (A_2, i_2; \overline{\lambda}_2)$ such that $\alpha_2 \circ H_1(f, \mathbb{A}_f^p) \circ \alpha_1^{-1} \in K^p$ and $c(\alpha_2) \cdot c(\alpha_1)^{-1} \in r \cdot c(K^p)$, where $r \in \mathbb{Z}_{(p)}^\times$ is such that $r \cdot \hat{f} \circ \lambda_2 \circ f = \lambda_1$.

We assume from now on that the subgroup K^p is neat. We have the following result (cf. [Kot92]; [Lan08], 2):

Theorem 1.2.1. *The functor $\mathbf{M}(\mathcal{D})$ is representable by a quasi-projective smooth scheme $\mathcal{S}_{\mathcal{D}, K^p}$ over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.*

Let k be an algebraic closure of the residue field of E_ν. Since the polarizations considered in our moduli problem are separable, we have the following (cf. [Lan08], 2.2.4.16, 2.3.2.1):

Proposition 1.2.2. The canonical map $\mathcal{S}_{\mathcal{D},K^p}(W(k)) \to \mathcal{S}_{\mathcal{D},K^p}(k)$ is surjective.

If $K_1^p \subseteq K_2^p$ are two neat open compact subgroups of $G(\mathbb{A}_f^p)$, the transition map $\mathcal{S}_{\mathcal{D},K_1^p} \to \mathcal{S}_{\mathcal{D},K_2^p}$ induced by $(A, i, \overline{\lambda}, K_1^p\alpha) \mapsto (A, i, \overline{\lambda}, K_2^p\alpha)$ is a finite étale covering which is Galois if K_1^p is normal in K_2^p. Denote by $\mathcal{S}_{\mathcal{D}}$ the projective system of the family of schemes $\{\mathcal{S}_{\mathcal{D},K^p}\}_{K^p}$ where the K^p's are small enough; we define the *Hecke action* of $G(\mathbb{A}_f^p)$ on $\mathcal{S}_{\mathcal{D}}$ as follows: if $g \in G(\mathbb{A}_f^p)$, then g acts on the right on $\mathcal{S}_{\mathcal{D}}$ via the isomorphism $g : \mathcal{S}_{\mathcal{D},K^p} \to \mathcal{S}_{\mathcal{D},g^{-1}K^pg}$ defined by $[(A, i, \overline{\lambda}, \overline{\alpha})] \cdot g := [(A, i, \overline{\lambda}, \overline{g^{-1} \circ \alpha})]$.

1.2.3 Modular forms of PEL type

For brevity, if we are given a group scheme X over Y with zero section $e : Y \to X$, we set $\mathfrak{t}_{X/Y}^* := e^*\Omega_{X/Y}^1$, where $\Omega_{X/Y}^1$ is the sheaf of relative invariant differentials of X over Y.

We keep assuming that $\mathcal{D}$ is a global PEL datum having good reduction at p, with associated group G having a model over $\mathbb{Z}$ as in 1.2.2. Let E be the reflex field of $\mathcal{D}$ and let $g = \dim_{\mathbb{C}} V_{\mathbb{C},0}$; fix an integer $N \geq 3$ not divisible by p and assume $K^p = U(N)$. Let $\mathcal{S} := \mathcal{S}_{\mathcal{D},U(N)}$ be the quasi-projective smooth scheme over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ representing $\mathbf{M}(\mathcal{D})$. Let $\pi : \mathcal{X} \to \mathcal{S}$ be the corresponding universal abelian scheme, with zero section 0. Set $\mathbb{E} := 0^*\Omega_{\mathcal{X}/\mathcal{S}}^1$.

Let $\rho : GL_g \to GL_m$ be a morphism of algebraic groups defined over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. We denote by $\mathbb{E}_\rho$ the locally free sheaf of rank m on $\mathcal{S}$ obtained by twisting $\mathbb{E}$ via ρ. More precisely, let $\{U_i\}_{i \in I}$ be an open cover of $\mathcal{S}$ trivializing $\mathbb{E}$ via isomorphisms $f_i : (\mathcal{O}_{\mathcal{S}|U_i})^g \to \mathbb{E}_{|U_i}$; let $g_{ij} = f_{j|U_i \cap U_j}^{-1} \circ f_{i|U_i \cap U_j} \in GL_g(\mathcal{O}_{\mathcal{S}|U_i \cap U_j})$ so that $g_{ij}g_{jk}g_{ki} = 1$ in $GL_g(\mathcal{O}_{\mathcal{S}|U_i \cap U_j \cap U_k})$ for all indices i,j and k. For any $i \in I$ define $(\mathbb{E}_\rho)_i := (\mathcal{O}_{\mathcal{S}|U_i})^m$; for indices $i, j \in I$ the element $\rho(g_{ij})$ defines an isomorphism $(\mathbb{E}_\rho)_{i|U_i \cap U_j} \to (\mathbb{E}_\rho)_{j|U_j \cap U_i}$. Since the $\rho(g_{ij})$'s satisfy the necessary cocycle identities, we can glue the $(\mathbb{E}_\rho)_i$'s to obtain a well defined sheaf $\mathbb{E}_\rho$ on $\mathcal{S}$.

For any $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-algebra $\mathfrak{R}$, the space of *PEL modular forms* over R of weight ρ relative

to the moduli problem $\mathbf{M}(\mathcal{D})$ is the $\mathfrak{R}$-module

$$M_\rho(\mathcal{D};\mathfrak{R}) := H^0(\mathcal{S}\otimes_{\mathcal{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}}\mathfrak{R};\mathbb{E}_\rho \otimes \mathfrak{R}).$$

We have:

Proposition 1.2.3. An element of $M_\rho(\mathcal{D};\overline{\overline{\mathbb{F}}}_p)$ is a rule f that assigns to any $\overline{\overline{\mathbb{F}}}_p$-rational tuple $(A,i,\overline{\lambda},\overline{\alpha},\eta)$ such that $[(A,i,\overline{\lambda},\overline{\alpha})]$ is an element of $\mathcal{S}(\overline{\mathbb{F}}_p)$ and η is an ordered basis for $\mathfrak{t}^*_{A/\overline{\mathbb{F}}_p}$ over $\overline{\overline{\mathbb{F}}}_p$, an element $f(A,i,\overline{\lambda},\overline{\alpha},\eta) \in \overline{\mathbb{F}}_p^m$ in such a way that:

(a) $f(A,i,\overline{\lambda},\overline{\alpha},\eta M) = \rho(M)^{-1} \cdot f(A,i,\overline{\lambda},\overline{\alpha},\eta)$ for all $M \in GL_g(\overline{\mathbb{F}}_p)$;

(b) if $(A,i,\overline{\lambda},\overline{\alpha},\eta) \simeq (A',i',\overline{\lambda}',\overline{\alpha}',\eta')$ then $f(A,i,\overline{\lambda},\overline{\alpha},\eta) = f(A',i',\overline{\lambda}',\overline{\alpha}',\eta')$.

It is natural to ask what is the relation between the modular forms defined above and those defined using the toroidal or minimal compactifications of the PEL Shimura varieties under consideration. In the case of modular curves, there are "more" modular forms defined using the open modular curves than those defined via the compactified modular curves. On the other hand, for Siegel modular varieties (and other PEL Shimura varieties) the so-called "Koecher's principle" can be applied. As the open PEL varieties have no less global sections than their compactifications, we content ourselves to work with non-compactified varieties.

CHAPTER 2

Uniformization of the superspecial locus of PEL Shimura varieties

2.1 A theorem of Rapoport and Zink

Let $\mathcal{D} = (B, {}^*, V, \langle, \rangle, \mathcal{O}_B, \Lambda, h, K^p, \nu)$ be a global PEL datum with good reduction at p, and neat level K^p. Let G be the associated group, and E the reflex field. In the rest of the dissertation, unless otherwise stated, we assume G is connected and satisfies the Hasse principle (the latter condition is required only for simplicity, as it is explained in the remark at the end of 2.1.2). The completion E_ν of E at ν coincide with the field of definition of the $G^0(\mathbb{Q}_p)$-conjugacy class of the co-character μ associated to $\mathcal{D}$; under the good reduction assumption E_ν is an unramified extension of $\mathbb{Q}_p$. Let $k = \overline{\mathbb{F}}_p$ be a fixed algebraic closure of the residue field of E_ν, and let $W = W(\overline{\mathbb{F}}_p)$, $K_0 = W[\frac{1}{p}]$ and σ be the Frobenius morphism of W. Fix a finite extension K of K_0 such that μ and the weight decomposition $V_K = V_{K,0} \oplus V_{K,1}$ are defined over K. Set $\breve{E}_\nu = E_\nu K_0 (= K_0)$, $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $\langle, \rangle_p = \langle, \rangle \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $G_p = G_{\mathbb{Q}_p}$, and $\mathcal{O}_{B_p} = \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

2.1.1 From global PEL data to local PEL data

Let $\mathcal{S}_{\mathcal{D},K^p}$ be the $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-scheme representing the functor $\mathbf{M} = \mathbf{M}(\mathcal{D})$ of Th. 1.2.1, and fix a point $[(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)] \in \mathcal{S}_{\mathcal{D},K^p}(\overline{\mathbb{F}}_p)$, where λ_0 is a principal polarization. Correspondingly we have a p-divisible group $\mathbf{X} = A_0(p)$ over $\overline{\mathbb{F}}_p$, endowed with the action $i_{\mathbf{X}} : \mathcal{O}_{B_p} \to \text{End}\,\mathbf{X}$ induced by i_0, and with the principal polarization $\lambda_{\mathbf{X}} : \mathbf{X} \to \hat{\mathbf{X}}$ induced by λ_0. The polarization $\lambda_{\mathbf{X}}$ respects the $\mathcal{O}_{B_p}$-action and it is well defined up to a constant in $\mathbb{Q}_p^\times$. The

triple $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$ is determined modulo isomorphisms by $[(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)]$. We associate to $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$ the isocrystal $(N := M_*(\mathbf{X})[\frac{1}{p}], \mathbf{F})$ over K_0 endowed with an action of B_p and with a non-degenerate bilinear form of isocrystals $\Psi : N \times N \to \mathbf{1}(1)$ (well defined up to an element of $\mathbb{Q}_p^\times$) that is skew-Hermitian with respect to *.

Fix an isomorphism of $B \otimes_{\mathbb{Q}} K_0$-modules $N \simeq V \otimes_{\mathbb{Q}} K_0$ that respects the skew-symmetric forms on both sides. Write the action of Frobenius on the right hand side as $\mathbf{F} = b \otimes \sigma$ for a unique element $b \in G_p(K_0)$; by construction $c(b) = p$. The isocrystal $V \otimes_{\mathbb{Q}} K_0$ has slopes in the interval $[0, 1]$, and in the decomposition of the K-vector space $V \otimes_{\mathbb{Q}} K$ under the co-character μ only the weights 0 and 1 appear. Since λ_0 is a separable polarization, Prop. 1.2.2 implies that the pair (b, μ) is admissible in the sense of [RZ96], 1. We conclude that $\mathcal{D}$ together with the fixed point $[(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)] \in \mathcal{S}_{\mathcal{D}, K^p}\left(\overline{\mathbb{F}}_p\right)$ determines a local $\mathbb{Q}_p$-PEL datum $\mathcal{D}_p = (B_p, {}^*, V_p, \langle, \rangle_p, \mathcal{O}_{B_p}, \Lambda, b, \mu)$, having reflex field E_ν. Denote by $\breve{\mathcal{M}}$ the formally smooth scheme over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}_\nu}$ that represents the functor associated in 1.1.2 to the pair $(\mathcal{D}_p; (\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}}))$.

Let $J(\mathbb{Q}_p)$ be the group of K_0-automorphisms of the isocrystal $(V \otimes_{\mathbb{Q}} K_0; b \otimes \sigma)$ equivariant for the action of B_p and preserving the polarization form induced by $\langle, \rangle$ up to a non-zero scalar in $\mathbb{Q}_p$. $J(\mathbb{Q}_p)$ is the group of $\mathbb{Q}_p$-rational points of an algebraic group J defined over $\mathbb{Q}_p$ and it is isomorphic to the group $J'(\mathbb{Q}_p)$ of quasi-isogenies $f : (\mathbf{X}, i_{\mathbf{X}}) \to (\mathbf{X}, i_{\mathbf{X}})$ of p-divisible groups over $\overline{\mathbb{F}}_p$ such that $\hat{f} \circ \lambda_{\mathbf{X}} \circ f \in \mathbb{Q}_p^\times \lambda_{\mathbf{X}}$. Since a choice of $b \in G_p(K_0)$ has been fixed, we identify $J(\mathbb{Q}_p)$ with $J'(\mathbb{Q}_p)$. The group $J(\mathbb{Q}_p)$ acts on $\breve{\mathcal{M}}$ from the left by the rule $g \cdot [(X, i, \overline{\lambda}; \rho)] := [(X, i, \overline{\lambda}; \rho \circ g^{-1})]$, where $[(X, i, \overline{\lambda}; \rho)] \in \breve{\mathcal{M}}(S)$ for some scheme S in $NILP_{\mathcal{O}_{\breve{E}_\nu}}$.

Denote by $I(\mathbb{Q})$ the group of quasi-isogenies of $\overline{\mathbb{F}}_p$-abelian variety $(A_0, i_0) \to (A_0, i_0)$ that send $\overline{\lambda}_0$ into itself; $I(\mathbb{Q})$ is the group of rational points of an algebraic group I defined over $\mathbb{Q}$. It acts by quasi-isogenies on the tuple $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$, hence on $((N, \mathbf{F}), i, \mathbb{Q}_p^\times \Psi)$, defining a morphism $\alpha_{0,p} : I(\mathbb{Q}) \to J(\mathbb{Q}_p)$ factoring through $I(\mathbb{Q}) \hookrightarrow I(\mathbb{Q}_p)$. Since $I(\mathbb{Q})$ acts by skew-Hermitian symplectic B-equivariant similitudes on $H_1(A_0, \mathbb{A}_f^p) \simeq V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$, there is a homomorphism $\alpha_0^p : I(\mathbb{Q}) \to G(\mathbb{A}_f^p)$ depending of the choice of a representative α_0 for the class $\overline{\alpha}_0$.

Let $\mathbb{D}_{K_0}$ be the universal cover of $\mathbb{G}_{m/K_0}$ in the sense of quasi-algebraic groups and let $\nu : \mathbb{D}_{K_0} \to G_{K_0}$ be the slope morphism associated to b (cf. 1.1.1). Following [Kot85], we say that the p-divisible group $\mathbf{X}$ of $(A_0, i_0, \overline{\lambda}_0)$ is *basic* if ν factors through the center of G_{K_0}. This is equivalent to say that the algebraic group $J_{\mathbb{Q}_p}$ is an inner form of G_p (cf. [RZ96], 1.15).

2.1.2 Uniformization of basic isogeny classes

The formal scheme $\breve{\mathcal{M}}$ is not in general defined over $\mathcal{O}_{E_\nu}$. There is a suitable completion of $\breve{\mathcal{M}}$ that can be written as $\mathcal{M} \otimes_{\mathrm{Spf}\,\mathcal{O}_{E_\nu}} \mathrm{Spf}\,\mathcal{O}_{\breve{E}_\nu}$ for a pro-formal scheme $\mathcal{M}$ over $\mathrm{Spf}(\mathcal{O}_{E_\nu})$ (cf. [RZ96], 3.41). The action of $J(\mathbb{Q}_p)$ on $\breve{\mathcal{M}}$ descends to an action on $\mathcal{M}$.

We let

$$Z([(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)])(\overline{\mathbb{F}}_p) \subseteq \mathcal{S}_{\mathcal{D}, K^p}(\overline{\mathbb{F}}_p)$$

to be the set of points $[(A, i, \overline{\lambda}, \overline{\alpha})] \in \mathcal{S}_{\mathcal{D}, K^p}(\overline{\mathbb{F}}_p)$ such that there exists an isogeny $(A_0, i_0) \to (A, i)$ in $AbSCH^*$ sending $\overline{\lambda}_0$ into $\overline{\lambda}$. If the p-divisible group $\mathbf{X}$ of $(A_0, i_0, \overline{\lambda}_0)$ is basic in the sense of 2.1.1, then $Z([(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)])(\overline{\mathbb{F}}_p)$ is the set of $\overline{\mathbb{F}}_p$-valued points of a closed subset $Z := Z([(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)])$ of $\mathcal{S}_{\mathcal{D}, K^p} \otimes \overline{\mathbb{F}}_p$ (cf. [RR96]).

We have the following result, due to M. Rapoport and Th. Zink (cf. [RZ96], 6):

Theorem 2.1.1. *Let us fix* $[(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)] \in \mathcal{S}_{\mathcal{D}, K^p}(\overline{\mathbb{F}}_p)$ *such that the p-divisible group of* $(A_0, i_0, \overline{\lambda}_0)$ *is basic; denote by Z the closed subspace of $\mathcal{S}_{\mathcal{D}, K^p}$ define above. Let $\widehat{\mathcal{S}}_{\mathcal{D}, K^p/Z}$ be the formal completion of $\mathcal{S}_{\mathcal{D}, K^p}$ along Z. Then there is a canonical isomorphism of formal schemes over* $\mathrm{Spf}\,\mathcal{O}_{E_\nu}$:

$$\vartheta_{K^p} : I(\mathbb{Q}) \backslash \mathcal{M} \times G(\mathbb{A}_f^p) / K^p \to \widehat{\mathcal{S}}_{\mathcal{D}, K^p/Z},$$

where $I(\mathbb{Q})$ acts on $\mathcal{M}$ via $\alpha_{0,p}$, and on $G(\mathbb{A}_f^p)$ via α_0^p. The system of morphisms $\{\vartheta_{K^p}\}_{K^p}$ is equivariant with respect to the right Hecke $G(\mathbb{A}_f^p)$-action on the projective systems of both sides above.

The action of $G(\mathbb{A}_f^p)$ on $\{\widehat{\mathcal{S}}_{\mathcal{D}, K^p/Z}\}_{K^p}$ is the Hecke action defined in 1.2.2. If $K_1^p \subseteq K_2^p$ are open compact subgroups of $G(\mathbb{A}_f^p)$, there is a transition map $I(\mathbb{Q}) \backslash \mathcal{M} \times G(\mathbb{A}_f^p) / K_1^p \to$

$I(\mathbb{Q})\backslash\mathcal{M}\times G(\mathbb{A}_f^p)/K_2^p$. A projective system of formal schemes remains therefore defined and, if $g \in G(\mathbb{A}_f^p)$, the map $xK^p \mapsto g^{-1}xK^pg$ $(x \in G(\mathbb{A}_f^p))$ induces a morphism

$$g : I(\mathbb{Q})\backslash\mathcal{M} \times G(\mathbb{A}_f^p)/K^p \to I(\mathbb{Q})\backslash\mathcal{M}\times G(\mathbb{A}_f^p)/g^{-1}K^pg.$$

This is the Hecke $G(\mathbb{A}_f^p)$-action on $I(\mathbb{Q})\backslash\mathcal{M} \times G(\mathbb{A}_f^p)/K^p$.

One also sees that I is an inner form of G such that $I(\mathbb{R})$ is compact modulo the center, and there are canonical identifications $I(\mathbb{A}_f^p) = G(\mathbb{A}_f^p)$, $J(\mathbb{Q}_p) = I(\mathbb{Q}_p)$: this is a consequence of the basicity of $\mathbf{X}$ (cf. [RZ96], 6.30).

Assume from now on that $\mathbf{X}$ is basic. There is then a morphism of functors Θ_{K^p} : $I(\mathbb{Q})\backslash\breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \to \mathcal{S}_{\mathcal{D},K^p} \otimes_{\mathcal{O}_{E_\nu}} \mathcal{O}_{\breve{E}_\nu}$ over $NILP_{\mathcal{O}_{\breve{E}_\nu}}$, such that:

Proposition 2.1.2. The morphism Θ_{K^p} induces a canonical bijection of sets:

$$\Theta_{K^p}(\overline{\mathbb{F}}_p) : I(\mathbb{Q})\backslash\breve{\mathcal{M}}(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p)/K^p \to Z(\overline{\mathbb{F}}_p)$$

which is equivariant for the Hecke $G(\mathbb{A}_f^p)$-action.

Since it will be needed later, we recall the definition of Θ_{K^p}, that we shall call the *uniformization morphism* for the isogeny class Z. We have the well known:

Lemma 2.1.3. Let $S \in NILP_{\mathbb{Z}_p}$, and let A' be an object in $AbSCH^*$; set $X' = A'(p)$. For any quasi-isogeny $\xi : X' \to X''$ of p-divisible groups over S that respects the $\mathcal{O}_{B_p}$-action, there exists an element A'' of $AbSCH^*$ whose p-divisible group is X'' and a quasi-isogeny $\xi : A' \to A''$ of $AbSCH^*$ inducing $\xi : X' \to X''$. Furthermore the arrow $\xi : A' \to A''$ in $AbSCH^*$ is uniquely determined; we denote A'' by ξ_*A'. This construction is functorial, i.e. $(\xi_2\xi_1)_* A' = \xi_{2*}(\xi_{1*}A')$.

Under the hypothesis of the above lemma, if A' comes with a polarization λ, then ξ defines a polarization $\xi_*\lambda := (\xi^{-1})^\wedge \lambda\xi^{-1}$ on A''. If furthermore A' comes with a rigidification $\alpha : H_1(A', \mathbb{A}_f^p) \to V \otimes_\mathbb{Q} \mathbb{A}_f^p$ (i.e., a symplectic $\mathcal{O}_B$-equivariant isomorphism), then A'' comes with the rigidification $\xi_*\alpha := \alpha \circ H_1(\xi^{-1}, \mathbb{A}_f^p)$.

Let S be a fixed scheme in $NILP_{\mathcal{O}_{\breve{E}_\nu}}$. Denote by

$$(\tilde{A}_0, \tilde{\imath}_0, \overline{\tilde{\lambda}}_0, \overline{\tilde{\alpha}}_0)$$

a fixed lifting of $(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)$ over $\mathcal{O}_{\breve{E}_\nu}$ (Prop. 1.2.2) and let $(\tilde{\mathbf{X}}, \tilde{\imath}_{\mathbf{X}}, \overline{\tilde{\lambda}}_{\mathbf{X}})$ be the corresponding lifting of $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$ to $\mathcal{O}_{\breve{E}_\nu}$. Consider a p-divisible group with additional structure $[(X, i, \overline{\lambda}; \rho)] \in \check{\mathcal{M}}(S)$, so that $\rho : (\mathbf{X}, i_{\mathbf{X}})_{\overline{S}} \to (X, i)_{\overline{S}}$ is an $\overline{S}$-quasi-isogeny; by the rigidity property of quasi-isogenies of p-divisible groups, ρ lifts uniquely to an S-quasi-isogeny $\tilde{\rho} : (\tilde{\mathbf{X}}, \tilde{\imath}_{\mathbf{X}})_S \to (X, i)$. By the above lemma, we obtain therefore an abelian scheme $\tilde{\rho}_*(\tilde{A}_{0/S})$ over S endowed with an action $\tilde{\rho}_*(\tilde{\imath}_0)$ of $\mathcal{O}_B$, a polarization $\tilde{\rho}_*(\overline{\tilde{\lambda}}_0)$ and a level structure $\tilde{\rho}_*(\overline{\tilde{\alpha}}_0)$, such that

$$[(\tilde{\rho}_*(\tilde{A}_{0/S}), \tilde{\rho}_*(\tilde{\imath}_0), \tilde{\rho}_*(\overline{\tilde{\lambda}}_0), \tilde{\rho}_*(\overline{\tilde{\alpha}}_0))] \in \mathcal{S}_{\mathcal{D}, K^p}(S).$$

We define a morphism of functors Θ_{K^p} over $NILP_{\mathcal{O}_{\breve{E}_\nu}}$ by letting, for any $S \in \mathrm{Obj}(NILP_{\mathcal{O}_{\breve{E}_\nu}})$:

$$\Theta_{K^p}(S) \quad : \quad \check{\mathcal{M}}(S) \times G(\mathbb{A}_f^p)/K^p \to \mathcal{S}_{\mathcal{D}, K^p}(S),$$

$$[(X, i, \overline{\lambda}; \rho)] \times gK^p \quad \longmapsto \quad [(\tilde{\rho}_*(\tilde{A}_{0/S}), \tilde{\rho}_*(\tilde{\imath}_0), \tilde{\rho}_*(\overline{\tilde{\lambda}}_0), g^{-1} \cdot \tilde{\rho}_*(\overline{\tilde{\alpha}}_0))].$$

The family $\{\Theta_{K^p}\}_{K^p}$ is equivariant with respect to the right $G(\mathbb{A}_f^p)$-action on the projective systems of both sides above (this $G(\mathbb{A}_f^p)$-action is what we refer to as the Hecke $G(\mathbb{A}_f^p)$-action or, simply, as the Hecke action).

Fix $S \in \mathrm{Obj}(NILP_{\mathcal{O}_{\breve{E}_\nu}})$, $[(X, i, \overline{\lambda}; \rho)] \in \check{\mathcal{M}}(S)$, $g \in G(\mathbb{A}_f^p)$ and $\xi \in I(\mathbb{Q})$; we define

$$\xi \cdot \left([(X, i, \overline{\lambda}; \rho)] \times gK^p \right) := [(X, i, \overline{\lambda}; \rho \circ \alpha_{0,p}(\xi^{-1})_{\overline{S}})] \times \alpha_0^p(\xi)gK^p.$$

As Θ_{K^p} is invariant under this $I(\mathbb{Q})$-action, it induces a morphism of functors

$$\Theta_{K^p} : I(\mathbb{Q}) \backslash \check{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \to \mathcal{S}_{\mathcal{D}, K^p} \otimes_{\mathcal{O}_{E_\nu}} \mathcal{O}_{\breve{E}_\nu},$$

whose $\overline{\mathbb{F}}_p$-points give the Hecke isomorphism of Prop. 2.1.2.

Remark 2.1.4. If G is connected but does not satisfy the Hasse principle, slight variants of the above results can be proved: one replaces the closed set $Z(\overline{\mathbb{F}}_p) \subseteq \mathcal{S}_{\mathcal{D}, K^p}(\overline{\mathbb{F}}_p)$ defined above by the set of points $[(A, i, \overline{\lambda}, \overline{\alpha})] \in \mathcal{S}_{\mathcal{D}, K^p}(\overline{\mathbb{F}}_p)$ such that the homogenously principally

16

polarized B_p-isocrystal of $(A, i, \overline{\lambda})$ is isomorphic to $((N, \mathbf{F}), i, \mathbb{Q}_p^\times \Psi)$. This set, that we still denote by $Z(\overline{\mathbb{F}}_p)$, defines a closed subscheme Z of $\mathcal{S}_{\mathcal{D}, K^p} \otimes \overline{\mathbb{F}}_p$, if $\mathbf{X}$ is basic ([RZ96], 6.27). The uniformization morphism associated to the fixed element $[(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)]$ gives now an isomorphism of formal schemes over $\mathrm{Spf}\, \mathcal{O}_{E_\nu}$:

$$I(\mathbb{Q}) \backslash \mathcal{M} \times G(\mathbb{A}_f^p) / K^p \to \widehat{\mathcal{S}}_{\mathcal{D}, K^p / \mathcal{Z}},$$

for some $\mathcal{Z} \subseteq Z$ open and closed. If G satisfies the Hasse principle, $Z = \mathcal{Z}$ is an isogeny class on $\mathcal{S}_{\mathcal{D}, K^p} \otimes \overline{\mathbb{F}}_p$ and we recover the results stated above; in general $Z(\overline{\mathbb{F}}_p) = \coprod_{i=1}^q Z_i(\overline{\mathbb{F}}_p)$ is the disjoint union of finitely many isogeny classes on $\mathcal{S}_{\mathcal{D}, K^p}(\overline{\mathbb{F}}_p)$ ([RZ96], 6.30). If we denote by Θ_i the uniformization morphism associated to the ith isogeny class Z_i contained in Z, we have an isomorphism of formal schemes over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}_\nu}$:

$$\coprod_{i=1}^q \Theta_i : I(\mathbb{Q}) \backslash \left(\coprod_{i=1}^q \breve{\mathcal{M}} \right) \times G(\mathbb{A}_f^p) / K^p \to \widehat{\mathcal{S}}_{\mathcal{D}, K^p / Z} \otimes \mathcal{O}_{\breve{E}_\nu},$$

where $\coprod_{i=1}^q \breve{\mathcal{M}}$ is the sum of q copies of $\breve{\mathcal{M}}$ (notice in fact that $\breve{\mathcal{M}}$ only depends upon the quasi-isogeny class of the p-divisible group of $(A_0, i_0, \overline{\lambda}_0)$, and this quasi-isogeny class is constant along Z).

2.2   Restriction to the superspecial locus

2.2.1   Superspecial abelian varieties

Let $G_{1/2}$ be the p-divisible group over $\overline{\mathbb{F}}_p$ having slope $1/2$. An abelian variety A of dimension g over $\overline{\mathbb{F}}_p$ is said to be supersingular (resp. superspecial) if $A(p)$ is isogenous (resp. isomorphic) to $G_{1/2}^g$ over $\overline{\mathbb{F}}_p$. An elliptic curve over $\overline{\mathbb{F}}_p$ is supersingular if and only if it is superspecial (cf. [Dem72], page 92).

If $E/\overline{\mathbb{F}}_p$ is a supersingular elliptic curve, there exists a canonical $\mathbb{F}_{p^2}$-rational model E' of E such that the geometric Frobenius $E' \to E'^{(p^2)} = E'$ equals $[-p]$. The association $E \mapsto E'$ is functorial and $\mathrm{End}_{\mathbb{F}_{p^2}} E' = \mathrm{End}_{\overline{\mathbb{F}}_p} E$. Moreover, the cotangent space of E has a canonical $\mathbb{F}_{p^2}$-structure and $E'(p)$ is a canonical model of $E(p)$ whose covariant Dieudonné module

$A'_{1/2} := M_*(E'(p))$ over $W(\mathbb{F}_{p^2})$ is isomorphic to:

$$\left(W(\mathbb{F}_{p^2})^2; F = \begin{pmatrix} & 1 \\ -p & \end{pmatrix} \sigma', V = \begin{pmatrix} & -1 \\ p & \end{pmatrix} \sigma'^{,-1} \right),$$

where σ' is the Frobenius morphism of $W(\mathbb{F}_{p^2})$. One has $A_{1/2} := M_*(E(p)) \simeq \frac{W(\overline{\mathbb{F}}_p)[F,V]}{W(\overline{\mathbb{F}}_p)[F,V](F+V)}$, and the ring of Dieudonné module endomorphisms $\operatorname{End} A_{1/2} = \operatorname{End} A'_{1/2}$ is isomorphic to the maximal order in the quaternion division algebra over $\mathbb{Q}_p$ (cf. [Ghi04a], Cor. 7). Notice $F + V = 0$ in $A'_{1/2}$ and $A_{1/2}$. By a result of Deligne and Ogus ([Ogu79], Th. 6.2, and [Shi], Th. 3.5), if $g \geq 2$ and $E_1, ..., E_{2g}$ are supersingular elliptic curves over $\overline{\mathbb{F}}_p$, there is an isomorphism $E \times ... \times E_g \simeq E_{g+1} \times ... \times E_{2g}$ over $\overline{\mathbb{F}}_p$.

If A is an abelian variety over $\overline{\mathbb{F}}_p$ of dimension $g \geq 2$, then A is superspecial if and only if A is isomorphic over $\overline{\mathbb{F}}_p$ to E^g, for some supersingular elliptic curve $E/\overline{\mathbb{F}}_p$; this happens if and only if $M_*(A(p)) \simeq A_{1/2}^g$ as Dieduonné $W(\overline{\mathbb{F}}_p)$-modules (cf. [K Z98], 1.6.). It follows that a superspecial abelian variety $A/\overline{\mathbb{F}}_p$ of dimension $g \geq 2$ has a canonical model A' over $\mathbb{F}_{p^2}$, in which the geometric Frobenius equals $[-p]$. Furthermore the association $A \mapsto A'$ is functorial. If $A = E^g$ is a superspecial abelian variety over $\overline{\mathbb{F}}_p$ of dimension $g \geq 1$, then A comes with a canonical principal polarization induced from the canonical polarization of E. In the rest of the dissertation we will identify canonically E and $\hat{E}$, so that A will be endowed with the identity principal polarization.

2.2.2   Uniformization of the superspecial locus

Let us fix integers $g \geq 1$, $N \geq 3$ and a prime number p not dividing N; let us denote by $\mathcal{A}_{g,N}$ the quasi-projective smooth scheme over $\mathbb{Z}_{(p)}$ classifying prime-to-p isogeny classes of tuples $(A, \overline{\lambda}, \overline{\alpha})$, where A is an abelian projective scheme of relative dimension g over some $S \in SCH_{\mathbb{Z}_{(p)}}$, $\overline{\lambda}$ is a principal homogeneous polarization of A, and $\overline{\alpha}$ is a principal level N structure on $(A, \overline{\lambda})$ (this scheme is sometimes referred to as the Siegel moduli scheme of principal level N). If $g = 1$, $\mathcal{A}_{1,N}(\overline{\mathbb{F}}_p)$ contains a finite number of supersingular elliptic curves, which form an isogeny class; if $g > 1$, the supersingular abelian varieties in $\mathcal{A}_{g,N}(\overline{\mathbb{F}}_p)$ define a closed subset of positive dimension (cf. [K Z98], 4.9). This situation also occurs for other Shimura varieties of PEL type and motivates what follows.

Fix $\mathcal{D} = (B,^*, V, \langle, \rangle, \mathcal{O}_B, \Lambda, h, K^p, \nu)$ a global PEL datum with good reduction at p, and neat level K^p. Denote by G the associated algebraic group, assumed connected and satisfying the Hasse principle. The objects E, E_ν, $\breve{E}_\nu$, $\overline{\mathbb{F}}_p$, W, K_0, σ, K , μ, B_p, V_p, $\langle, \rangle_p$, G_p, $\mathcal{O}_{B_p}$ are defined as at the beginning of section 2.1.

Let $\mathcal{S}_{\mathcal{D},K^p}$ be the quasi-projective smooth scheme representing $\mathbf{M}(\mathcal{D})$ over $\operatorname{Spec} \mathcal{O}_{\breve{E}_\nu}$. Suppose the common dimension of the abelian schemes parametrized by $\mathcal{S}_{\mathcal{D},K^p}$ is $g :=$ $\dim_{\mathbb{C}} V_{\mathbb{C},0} \geq 2$; fix a supersingular elliptic curve E_0 over $\overline{\mathbb{F}}_p$, and denote its canonical model over $\mathbb{F}_{p^2}$ by E_0'. Let $A_0 = E_0^g$ be the corresponding superspecial abelian variety over $\overline{\mathbb{F}}_p$, endowed with the identity principal polarization $\lambda_0 = id_{E_0}^g$.

Assume that $\mathcal{S}_{\mathcal{D},K^p}$ contains a point of the form $[(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)] \in \mathcal{S}_{\mathcal{D},K^p}\left(\overline{\mathbb{F}}_p\right)$ that we fix (an obvious necessary condition for this to happen is the existence of a $\mathbb{Q}$-algebra homomorphism $B \to M_g(\mathfrak{B})$, where $\mathfrak{B}$ is the quaternion $\mathbb{Q}$-algebra ramified at p and ∞). The p-divisible group $\mathbf{X} = A_0(p)$ over $\overline{\mathbb{F}}_p$ is isomorphic to $G_{1/2}^g$ and it comes with an additional PE-structure $(i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$; by Dieudonné functoriality, we obtain an isocrystal $(N := M_*(\mathbf{X})[\frac{1}{p}], \mathbf{F})$ over K_0 endowed with an action of B_p and with a non-degenerate bilinear form of isocrystals $\Psi : N \times N \to \mathbf{1}(1)$. We define b as in 2.1.1, noticing that since N is isoclinic, the slope morphism associated to G and b over K_0 has image contained in the center of G, so that b is basic by [Kot85], 5.

As in 2.1.1, a local PEL datum $\mathcal{D}_p$ remains defined and we can consider the closed subscheme $\mathcal{M}'(\overline{\mathbb{F}}_p) := \mathcal{M}'_{(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})}(\overline{\mathbb{F}}_p)$ of $\breve{\mathcal{M}}(\overline{\mathbb{F}}_p)$ as defined in 1.1.3. We identify $\mathcal{M}'(\overline{\mathbb{F}}_p)$ with $J(\mathbb{Q}_p)/J(\mathbb{Z}_p)$.

Definition 2.2.1. We let $Z'(\overline{\mathbb{F}}_p) := Z'([(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)])(\overline{\mathbb{F}}_p) \subseteq \mathcal{S}_{\mathcal{D},K^p}(\overline{\mathbb{F}}_p)$ be the set of points $[(A, i, \overline{\lambda}, \overline{\alpha})] \in Z\left(\overline{\mathbb{F}}_p\right)$ such that the principally polarized p-divisible group $(A(p), i, \overline{\lambda})$ of $(A, i, \overline{\lambda})$ is isomorphic to $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$. We call $Z'(\overline{\mathbb{F}}_p)$ the superspecial locus associated to $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$.

The set $Z'(\overline{\mathbb{F}}_p)$ is a closed subset of $Z(\overline{\mathbb{F}}_p)$; furthermore if the class $[(A, i, \overline{\lambda}, \overline{\alpha})]$ belongs to $Z'\left(\overline{\mathbb{F}}_p\right)$, the p-divisible group of A is isomorphic to $G_{1/2}^g$, so that $A \simeq A_0$ is superspecial.

Remark 2.2.2. In [Eke87], it is shown that the isomorphism classes of principal polarizations on A_0 form a single genus class, so that if λ and λ' are two principal polarizations on A_0, the p-adic polarizations associated to λ and λ' respectively on the Dieudonné module of $A_0(p)$ are isomorphic. Hence $(A_0(p), \overline{\lambda}) \simeq (A_0(p), \overline{\lambda}')$ as principally polarized p-divisible groups over $\overline{\mathbb{F}}_p$. Let us denote by $\mathcal{A}_{g,N}$ the Siegel moduli scheme over $\mathbb{Z}_{(p)}$ introduced at the beginning of this paragraph. As a consequence of the cited above result of [Eke87], we have on $\mathcal{A}_{g,N}(\overline{\mathbb{F}}_p)$:

$$
\begin{aligned}
Z'(\overline{\mathbb{F}}_p) \;=\; & \{[(A_0, \overline{\lambda}, \overline{\alpha})] : \lambda \text{ a principal polarization on } A_0, \\
& \alpha \text{ a principal level } N \text{ structure on } A_0\}.
\end{aligned}
$$

Proposition 2.2.3. The uniformization morphism $\Theta_{K^p}(\overline{\mathbb{F}}_p)$ of Prop. 2.1.2 induces a canonical isomorphism:

$$
\Theta'_{K^p}(\overline{\mathbb{F}}_p) : I(\mathbb{Q}) \backslash \mathcal{M}'(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p)/K^p \to Z'(\overline{\mathbb{F}}_p),
$$

which is equivariant for the Hecke $G(\mathbb{A}_f^p)$-action. We call $\Theta'_{K^p}(\overline{\mathbb{F}}_p)$ the uniformization morphism for the superspecial locus.

Proof. Under our assumptions on G, and by the basicity of b, the map $\Theta_{K^p}(\overline{\mathbb{F}}_p)$ is a well-defined Hecke equivariant isomorphism. The action of $I(\mathbb{Q})$ on $\check{\mathcal{M}}(\overline{\mathbb{F}}_p)$ determines an action on $\mathcal{M}'(\overline{\mathbb{F}}_p) \subseteq \check{\mathcal{M}}(\overline{\mathbb{F}}_p)$, so that we obtain a natural injective Hecke equivariant map:

$$
I(\mathbb{Q}) \backslash \mathcal{M}'(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p)/K^p \hookrightarrow I(\mathbb{Q}) \backslash \check{\mathcal{M}}(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p)/K^p.
$$

Define $\Theta'_{K^p}(\overline{\mathbb{F}}_p)$ by precomposing this map with $\Theta_{K^p}(\overline{\mathbb{F}}_p)$. In order to determine the image of $\Theta'_{K^p}(\overline{\mathbb{F}}_p)$, we follow the construction of the uniformization morphism over the field $\overline{\mathbb{F}}_p$ (cf. 2.1.2).

Pick an element $[\rho] \in \mathcal{M}'(\overline{\mathbb{F}}_p)$; the quasi-isogeny $\rho : (\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}}) \to (\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$ determines, by Lemma 2.1.3, a principally polarized abelian variety $(\rho_* A_0, \rho_* i_0, \rho_* \overline{\lambda}_0)$ in $AbSCH^*$, whose p-divisible group is isomorphic to $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$, so that the image of $\overline{\Theta}'_{K^p}(\overline{\mathbb{F}}_p)$ is contained inside the superspecial locus.

On the other hand, let $[(A, i, \overline{\lambda}, \overline{\alpha})] \in Z'(\overline{\mathbb{F}}_p)$ and choose a quasi-isogeny of principally polarized abelian varieties $\rho : (A_0, i_0, \overline{\lambda}_0) \to (A, i, \overline{\lambda})$. Then ρ defines a quasi-isogeny of the corresponding p-divisible groups $\rho : (\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}}) \to (A(p), i, \overline{\lambda})$. Precomposing ρ with an isomorphism $\mu : (A(p), i, \overline{\lambda}) \to (\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$ we obtain an element $[\mu \circ \rho] \in \mathcal{M}'(\overline{\mathbb{F}}_p)$ such that $(\mu \circ \rho)_*(A_0, i_0, \overline{\lambda}_0) = \mu_*(A, i, \overline{\lambda}) \simeq (A, i, \overline{\lambda})$. Let now $g \in G(\mathbb{A}_f^p)$ be defined by $g := (\mu \circ \rho)_* \overline{\alpha}_0 \circ \overline{\alpha}^{-1}$; the pre-image of $[(A, i, \overline{\lambda}, \overline{\alpha}] \in Z'(\overline{\mathbb{F}}_p)$ under $\Theta'_{K^p}(\overline{\mathbb{F}}_p)$ is the $I(\mathbb{Q})$-class represented by $[\mu \circ \rho] \times gK^p$. ∎

We have seen that the group $I(\mathbb{Q}) := (\mathrm{End}_{\mathcal{O}_B}(A_0, \overline{\lambda}_0) \otimes_{\mathbb{Z}} \mathbb{Q})^\times$ acts on the left upon $\mathcal{M}'(\overline{\mathbb{F}}_p)$ through the map $\alpha_{0,p} : I(\mathbb{Q}) \to J(\mathbb{Q}_p)$. We have therefore the canonical identification:

$$I(\mathbb{Q}) \backslash \mathcal{M}'(\overline{\mathbb{F}}_p) \xrightarrow{\simeq} I(\mathbb{Q}) \backslash J(\mathbb{Q}_p) / J(\mathbb{Z}_p),$$

where the action of $I(\mathbb{Q})$ on $\mathcal{M}'(\overline{\mathbb{F}}_p)$ is the one described in 2.1.1, so that the action of $I(\mathbb{Q})$ on the coset space $J(\mathbb{Q}_p)/J(\mathbb{Z}_p)$ is given by $x \cdot gJ(\mathbb{Z}_p) = (M_*(x) \cdot g)J(\mathbb{Z}_p)$, for all $x \in I(\mathbb{Q})$ and all $g \in J(\mathbb{Q}_p)$. We will write $x \cdot gJ(\mathbb{Z}_p) = xgJ(\mathbb{Z}_p)$ to shorten notation.

Corollary 2.2.4. There is a canonical isomorphism equivariant for the Hecke $G(\mathbb{A}_f^p)$-action:

$$\Theta'_{K^p}(\overline{\mathbb{F}}_p) : I(\mathbb{Q}) \backslash \left(J(\mathbb{Q}_p)/J(\mathbb{Z}_p) \times G(\mathbb{A}_f^p)/K^p \right) \to Z'(\overline{\mathbb{F}}_p),$$

where the action of $I(\mathbb{Q})$ on $J(\mathbb{Q}_p)/J(\mathbb{Z}_p)$ is the one described above. Furthermore, $Z'(\overline{\mathbb{F}}_p)$ is a finite set.

Proof. We just need to show the finiteness of $Z'(\overline{\mathbb{F}}_p)$. By [RZ96], 6.29 and the basicity of $\mathbf{X}$, we have canonical identifications $I(\mathbb{A}_f^p) = G(\mathbb{A}_f^p)$ and $J(\mathbb{Q}_p) = I(\mathbb{Q}_p)$, so that we can rewrite the domain of the morphism $\Theta'_{K^p}(\overline{\mathbb{F}}_p)$ as:

$$I(\mathbb{Q}) \backslash \left(I(\mathbb{Q}_p)/I(\mathbb{Z}_p) \times I(\mathbb{A}_f^p)/C^p \right) = I(\mathbb{Q}) \backslash I(\mathbb{A}_f)/C,$$

where C^p is the image of K^p in $I(\mathbb{A}_f^p)$ and $C = I(\mathbb{Z}_p) \times C^p$. By the proof of 6.23 in [RZ96], $I(\mathbb{Q})$ is a discrete subgroup of $I(\mathbb{A}_f)$; by Prop. 1.4 of [Gro99], the quotient space $I(\mathbb{Q}) \backslash I(\mathbb{A}_f)$ is therefore compact, so that $I(\mathbb{Q}) \backslash I(\mathbb{A}_f)/C$ is finite. ∎

CHAPTER 3

Comparison of Hecke eigensystems

We apply the results of the last section to study systems of Hecke eigenvalues coming from unitary modular forms.

3.1 Superspecial points on unitary Shimura varieties

3.1.1 PEL data of type A

Let us fix from now on embeddings $\overline{\mathbb{Q}} \to \mathbb{C}$ and $\nu : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$; we assume the notation of 1.2.1. Fix a choice i of square root of -1 in $\mathbb{C}$. Let $B = k = \mathbb{Q}(\sqrt{\alpha})$ be a quadratic imaginary field ($\alpha \in \mathbb{Z}, \alpha < 0$). Fix an embedding $\tau : k \hookrightarrow \mathbb{C}$ such that $\tau(\sqrt{\alpha}) = i\sqrt{-\alpha}$ (where $\sqrt{-\alpha} > 0$) and identify $k \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathbb{C}$ via τ. Assume that $p \neq 2$ is a prime which is inert in $k/\mathbb{Q}$. (The fact that p is odd is used in the proof of Lemma 3.1.3. Moreover, in order to work with PEL data having good reduction at p, we require that p is unramified in the extension $k/\mathbb{Q}$. We exclude the case of p split in k in order to guarantee that k embeds in the quaternion $\mathbb{Q}$-algebra ramified at $\{p, \infty\}$: cf. the beginning of proof of Prop. 3.1.1).

Let $x \mapsto \overline{x}$ denote the non-trivial field automorphism of k. Set $V = k^g$ for a positive even integer $g = 2n$; fix two non-negative integers r and s whose sum is g and let $H = \begin{pmatrix} -\sqrt{\alpha}I_r & \\ & \sqrt{\alpha}I_s \end{pmatrix}$. Let us denote by $\langle , \rangle : V \times V \to \mathbb{Q}$ the map defined by setting $\langle v, w \rangle = Tr_{k/\mathbb{Q}}(\overline{v}^t H w)$ for every $v, w \in V$: $\langle , \rangle$ is a $\mathbb{Q}$-bilinear non-degenerate skew-Hermitian pairing. Let $h : \mathbb{C} \to \mathrm{End}_k V \otimes_{\mathbb{Q}} \mathbb{R} = M_g(k) \otimes_{\mathbb{Q}} \mathbb{R}$ be the $\mathbb{R}$-algebra homomorphism defined by

$$a + bi \longmapsto 1 \otimes a - H \otimes \frac{b}{\sqrt{-\alpha}} \qquad (a, b \in \mathbb{R}).$$

Set $\mathcal{O}_B = \mathcal{O}_{k,(p)} = \mathbb{Z}_{(p)}[\sqrt{\alpha}]$, and $\Lambda = \left(\mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \right)^g$.

The algebraic group G associated to the datum $(B, ^*, V, \langle, \rangle)$ as in 1.2.1 is $GU_g(k; r, s)$: for a $\mathbb{Q}$-algebra R, $GU_g(k \otimes_{\mathbb{Q}} R; r, s) = \{A \in GL_g(k \otimes_{\mathbb{Q}} R) : \overline{A}^t HA = c(A)H, c(A) \in R^\times\}$. Notice that G is connected and it has a reductive model over $\mathbb{Z}$ as defined in 1.2.2; such a model will be from now on considered fixed and denoted by the same letter G. Observe that $G(\mathbb{Z}_p)$ can then be canonically identified with the group of unitary similitudes on Λ, endowed with its $\mathbb{Z}_p$-valued Hermitian pairing induced by $\langle, \rangle$.

The group $\ker c = U_g(k; H)$ is an inner form of the quasi-split unitary group over $\mathbb{Q}$ associated to the extension $k/\mathbb{Q}$, hence it is a group of type A_{g-1} when viewed over $\overline{\mathbb{Q}}$; being g even, G satisfies the Hasse principle ([Kot92], 7).

If $\{e_a, f_b\}_{1 \leq a, b \leq g}$ denotes the standard ordered basis of $\mathbb{C}^{2g}$,

$$V_{\mathbb{C},0} = \langle ie_a \otimes 1 - e_a \otimes i, if_b \otimes 1 + f_b \otimes i \rangle_{\substack{1 \leq a \leq r \\ 1 \leq b \leq s}}$$

and the corresponding representation $\rho : B \to \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C},0}) \simeq M_g(\mathbb{C})$ is induced by the assignment $\sqrt{\alpha} \longmapsto H$. The field of definition of ρ is $E = \mathbb{Q}$ if $r = s$, and $E = k$ otherwise; furthermore the determinant polynomial is:

$$f(X_1, X_2) := \det(X_1 + \sqrt{\alpha}X_2; V_{\mathbb{C},0}) = (X_1 - \sqrt{\alpha}X_2)^r (X_1 + \sqrt{\alpha}X_2)^s \in \mathcal{O}_E[X_1, X_2].$$

Let $N \geq 3$ be an integer prime to p and let $K^p = U(N)$ (notice that, instead of working with $U(N)$, we could work more generally with a neat open compact subgroup of $G(\mathbb{A}_f^p)$). We have so far defined a $\mathbb{Q}$-PEL datum $\mathcal{D}_A$ with good reduction at p and reflex field E.

If $\rho : GL_g \to GL_m$ is a rational $\mathbb{Z}_{(p)}$-representation and if $\mathfrak{R}$ is any $\mathbb{Z}_{(p)}$-algebra, $M_\rho(\mathcal{D}_A; \mathfrak{R})$ is the space of unitary (or Picard) $\mathfrak{R}$-modular forms of signature (r, s) for the field k, having level N and weight ρ (cf. 1.2.3).

Let μ be the co-character associated to h. Let $\overline{\mathbb{F}}_p$ be a fixed algebraic closure of the residue field of $E_\nu \subset \overline{\mathbb{Q}}_p$; set $W = W(\overline{\mathbb{F}}_p)$, $K_0 = W[\frac{1}{p}] \subset \overline{\mathbb{Q}}_p$ and denote by σ the Frobenius morphism of W. Fix a finite extension $K \subset \overline{\mathbb{Q}}_p$ of K_0 such that μ is defined over K; set $\breve{E}_\nu = E_\nu K_0 (= K_0)$. Define $B_p, V_p, \langle, \rangle_p, G_p, \mathcal{O}_{B_p}$ as before.

3.1.2 Choice of a superspecial point

The embedding $\nu : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ identifies k_ν with the degree two unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$; if $\mathbb{F}_{p^2}$ denotes the residue field of k_ν, we obtain an embedding $\mathbb{F}_{p^2} \subset \overline{\mathbb{F}}_p$.

Proposition 3.1.1. There is a supersingular elliptic curve E_0 over $\overline{\mathbb{F}}_p$ whose endomorphism ring $\operatorname{End} E_0$ contains an element φ_α such that:

1. $\varphi_\alpha^2 = \alpha$;

2. the tangent map $\operatorname{Lie} \varphi_\alpha : \operatorname{Lie} E_0 \to \operatorname{Lie} E_0$ is multiplication by the scalar $\sqrt{\alpha}(\operatorname{mod} p) \in$ $\mathbb{F}_{p^2} \subset \overline{\mathbb{F}}_p$, where $\sqrt{\alpha} \in k$ is viewed as an element of k_ν via ν.

Proof. Let E be any fixed supersingular elliptic curve over $\overline{\mathbb{F}}_p$. Since p does not split in $k/\mathbb{Q}$, Th. 3.8 at page 78 of [Vig80] implies that there is an embedding of $\mathbb{Q}$-algebras $j : k \hookrightarrow \operatorname{End}^0 E$; there is a maximal order R of $\operatorname{End}^0 E$ containing $j(\sqrt{\alpha})$: in fact we can write $\operatorname{End}^0 E = j(k) \oplus j(k)u$ for some $u \in \operatorname{End}^0 E$ (cf. [Vig80], Cor. 2.2, page 6), and the left order of the ideal $\mathbb{Z} + \mathbb{Z}j(\sqrt{\alpha}) + \mathbb{Z}u + \mathbb{Z}j(\sqrt{\alpha})u$ of $\operatorname{End}^0 E$ clearly contains $j(\sqrt{\alpha})$. By work of Deuring, there is a an elliptic curve E_0 over $\overline{\mathbb{F}}_p$ and a quasi-isogeny $f : E_0 \to E$ such that $R = f \circ \operatorname{End} E_0 \circ f^{-1}$, so that $\operatorname{End} E_0$ contains an element φ'_α whose square equals α (cf. [Wed07], 2.15).

The tangent morphism $\operatorname{Lie} \varphi'_\alpha$ can be canonically identified with an element of $\overline{\mathbb{F}}_p$; since $(\operatorname{Lie} \varphi'_\alpha)^2 = \alpha \operatorname{mod} p$, we have $\operatorname{Lie} \varphi'_\alpha = \pm\sqrt{\alpha}(\operatorname{mod} p) \in \mathbb{F}_{p^2}$. We define $\varphi_\alpha := \pm\varphi'_\alpha$ depending on $\operatorname{Lie} \varphi'_\alpha$ being equal to $\pm\sqrt{\alpha}(\operatorname{mod} p)$ respectively. The pair (E_0, φ_α) we just constructed satisfies the requirement of the proposition. ∎

Fix a pair (E_0, φ_α) over $\overline{\mathbb{F}}_p$ as in the above proposition (the choice of isomorphism class of E_0 will not be relevant later on, since $g \geq 2$); let E'_0 denote the canonical model of E_0 over $\mathbb{F}_{p^2}$. Denote by $\mathfrak{R}$ the $\mathbb{Z}$-algebra $\operatorname{End} E_0 = \operatorname{End}_{\mathbb{F}_{p^2}} E'_0$: $\mathfrak{R}$ is a maximal order in $\mathfrak{B} := \operatorname{End}^0 E_0 = \operatorname{End} E_0 \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathfrak{B}$ is a quaternion algebra over $\mathbb{Q}$ whose ramification set is $\{p, \infty\}$. If η is a place of $\mathbb{Q}$, we denote by $\mathfrak{B}_\eta$ the $\mathbb{Q}_\eta$-algebra $\mathfrak{B} \otimes_{\mathbb{Q}} \mathbb{Q}_\eta$; we also denote by $\bar{\cdot}$ the canonical involution of $\mathfrak{B}$.

24

Set $A_0 = E_0^g$, $\lambda_0 = id_{E_0}^g$. We obtain canonical isomorphisms $\operatorname{End} A_0 = M_g(\mathfrak{R})$ and $\operatorname{End}^0 A_0 = M_g(\mathfrak{B})$. Under this identification and the canonical isomorphism $A_0 \xrightarrow{\sim} \hat{A}_0$, the principal polarization $\lambda_0 = id_{E_0}^g$ of A_0 coincides with the identity matrix $I_g \in M_g(\mathfrak{R})$, so that the auto-quasi-isogenies of the principally (homogeneously) polarized abelian variety $(A_0, \overline{\lambda}_0)$ are identified with the elements of the unitary quaternion similitude group:

$$GU_g(\mathfrak{B}; I_g) := \{X \in GL_g(\mathfrak{B}) : X^*X = c(X) \cdot I_g, \ c(X) \in \mathbb{Q}^\times\},$$

where $X^* := \overline{X}^t$. Similarly, the automorphisms of the pair $(A_0, \overline{\lambda}_0)$ are given by $GU_g(\mathfrak{R}; I_g)$, and the automorphisms of $(A_0, \overline{\lambda}_0)$ viewed as a polarized abelian variety up to prime-to-p isogeny are given by $GU_g(\mathfrak{R} \otimes_\mathbb{Z} \mathbb{Z}_{(p)}; I_g)$. Notice that $GU_g(\mathfrak{B}; I_g)$ defines a reductive algebraic group over $\mathbb{Q}$, and that $U_g(\mathfrak{B}; I_g)$ is compact at infinity, since $U_g(\mathfrak{B}_\infty; I_g) \subset O(4g)$.

Let $\iota : k \hookrightarrow \mathfrak{B}$ be the $\mathbb{Q}$-algebra homomorphism such that $\iota(\sqrt{\alpha}) = \varphi_\alpha$. Define a $\mathbb{Z}_{(p)}$-algebra monomorphism $i_0 : \mathcal{O}_{k,(p)} \hookrightarrow \operatorname{End}(A_0) \otimes_\mathbb{Z} \mathbb{Z}_{(p)} = M_g(\mathfrak{R} \otimes_\mathbb{Z} \mathbb{Z}_{(p)})$ by requiring that $\sqrt{\alpha} \mapsto \begin{pmatrix} -\varphi_\alpha I_r \\ & \varphi_\alpha I_s \end{pmatrix}$. On the dual variety $\hat{A}_0 = A_0$ we have the dual action $\hat{i}_0 : \mathcal{O}_{k,(p)} \hookrightarrow M_g(\mathfrak{R} \otimes_\mathbb{Z} \mathbb{Z}_{(p)})$ defined by $\hat{i}_0(b) = i_0(\overline{b})^*$ for any $b \in \mathcal{O}_{k,(p)}$. Since k is embedded into $\mathfrak{B}$ by ι, the conjugation on $\mathfrak{B}$ induces on $\iota(k)$ the only non-trivial automorphism; we therefore have:

$$i_0(\overline{\sqrt{\alpha}}) = -i_0(\sqrt{\alpha}) = i_0(\sqrt{\alpha})^*,$$

so that $\lambda_0 \circ i_0(b) = \hat{i}_0(b) \circ \lambda_0$ for any $b \in \mathcal{O}_{k,(p)}$, and λ_0 is a principal polarization for (A_0, i_0) (equivalently, $i_0(b^*) = i_0(b)^\dagger$ for any $b \in \mathcal{O}_{k,(p)}$, where $\dagger$ denotes the Rosati involution associated to λ_0).

Fix an ordered basis $\{t_1, ..., t_g\}$ for the $\overline{\mathbb{F}}_p$-vector space $\operatorname{Lie} A_0 = (\operatorname{Lie} E_0)^g$ such that $\{t_i\}$ is (the natural image of) a basis for the Lie algebra of the simple ith factor of $A_0 = E_0^g$ ($1 \leq i \leq g$). With respect to $\{t_1, ..., t_g\}$, $\operatorname{Lie} i_0(\sqrt{\alpha})$ acts on $\operatorname{Lie} A_0$ via the matrix $\begin{pmatrix} -\sqrt{\alpha}(\operatorname{mod} p) \cdot I_r \\ & \sqrt{\alpha}(\operatorname{mod} p) \cdot I_s \end{pmatrix} \in GL_g(\mathbb{F}_{p^2}) \subset GL_g(\overline{\mathbb{F}}_p)$. We conclude that the fixed pair (A_0, i_0)

satisfies Kottwitz' determinant condition, since we have the following equalities in $\mathbb{F}_{p^2}$:

$$
\begin{aligned}
\det(X_1 + \sqrt{\alpha}X_2; \operatorname{Lie} A_0) &= \det \begin{pmatrix} (X_1 - \sqrt{\alpha}X_2)I_r & \\ & (X_1 + \sqrt{\alpha}X_2)I_s \end{pmatrix} (\operatorname{mod} p) \\
&= (X_1 - \sqrt{\alpha}X_2)^r (X_1 + \sqrt{\alpha}X_2)^s (\operatorname{mod} p) \\
&= f(X_1, X_2) (\operatorname{mod} p).
\end{aligned}
$$

We fix a $U(N)$-orbit of an isomorphism $\alpha_0 : H_1(A_0, \mathbb{A}_f^p) \to V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ of skew-hermitian modules with k-action. Let $\mathcal{S}_{\mathcal{D}_A, U(N)}$ be the quasi-projective smooth scheme over $\mathcal{O}_{\breve{E}_\nu}$ defined in Th. 1.2.1; by definition of our moduli variety, we have determined a point:

$$
[(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)] \in \mathcal{S}_{\mathcal{D}_A, U(N)}(\overline{\mathbb{F}}_p)
$$

that we consider fixed for the remaining of this section. Correspondingly we have associated the closed subspace $Z'(\overline{\mathbb{F}}_p) = Z'([(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)])(\overline{\mathbb{F}}_p)$ of $\mathcal{S}_{\mathcal{D}_A, U(N)}(\overline{\mathbb{F}}_p)$ as in Def. 2.2.1.

The p-divisible group $\mathbf{X} := A_0(p)$ over $\overline{\mathbb{F}}_p$ is isomorphic to $G_{1/2}^g$ and is endowed with the action $i_{\mathbf{X}}$ of $\mathcal{O}_{B_p}$ induced by i_0 and the principal polarization $\lambda_{\mathbf{X}} : \mathbf{X} \to \hat{\mathbf{X}} \simeq \mathbf{X}$. We associate to $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$ the Dieudonné module $\mathbf{M} := M_*(\mathbf{X})$ over W, endowed with an action $i_{\mathbf{M}}$ of $\mathcal{O}_{B_p}$ and a principal polarization $e_{\mathbf{M}} : \mathbf{M} \times \mathbf{M} \to W$ of Dieudonné modules, which is skew-Hermitian with respect to $*$, and well defined only up to a scalar factor in $\mathbb{Z}_p^\times$. By inverting p, we obtain an isocrystal $(N := \mathbf{M}[\frac{1}{p}], \mathbf{F})$ over K_0 endowed with an action of B_p and with a non-degenerate bilinear form of isocrystals $\Psi : N \times N \to \mathbf{1}(1)$. We fix an isomorphism of $B \otimes_{\mathbb{Q}} K_0$-modules $N \simeq V \otimes_{\mathbb{Q}} K_0$ that respects the skew-symmetric forms on both sides and we then write the action of Frobenius on the right hand side as $\mathbf{F} = b \otimes \sigma$ for some $b \in G_p(K_0)$ (recall that since N is isoclinic, b is basic). We have a $\mathbb{Q}_p$-PEL datum for moduli of p-divisible groups $\mathcal{D}_p = (B_p, *, V_p, \langle, \rangle_p, \mathcal{O}_{B_p}, \Lambda, b, \mu)$ having reflex field E_ν. Associated to $\mathcal{D}_p$ and $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$ we have the moduli functor $\breve{\mathcal{M}}$ and hence the closed subspace $\mathcal{M}'(\overline{\mathbb{F}}_p) \subseteq \breve{\mathcal{M}}(\overline{\mathbb{F}}_p)$ (cf. 1.1.3).

Cor. 2.2.4 gives a canonical isomorphism (equivariant with respect to the Hecke $G(\mathbb{A}_f^p)$-action) associated to $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$:

$$
\Theta'_{U(N)}(\overline{\mathbb{F}}_p) : I(\mathbb{Q}) \backslash \left(J(\mathbb{Q}_p)/J(\mathbb{Z}_p) \times G(\mathbb{A}_f^p)/U(N) \right) \to Z'(\overline{\mathbb{F}}_p).
$$

Recall that both sides above are finite sets.

3.1.3 The groups I and J

We now describe the groups appearing in the domain of $\Theta'_{U(N)}(\overline{\mathbb{F}}_p)$. As we saw above, $G = GU_g(k; r, s)$ and

$$U(N) = \operatorname{Ker}(G(\hat{\mathbb{Z}}^p) \to G(\hat{\mathbb{Z}}^p/N\hat{\mathbb{Z}}^p))$$

is a compact open subgroup of $G(\mathbb{A}_f^p)$ (recall that in 3.1.1 we have fixed a specific integral model over $\mathbb{Z}$ for the unitary group $GU_g(k; r, s)$).

By definition, $I(\mathbb{Q})$ is the group of auto-$\mathcal{O}_{k,(p)}$-quasi-isogenies of the homogeneously principally polarized abelian variety $(A_0, \overline{\lambda}_0)$, so that if we let $\Phi := \Phi_\alpha = \begin{pmatrix} -\varphi_\alpha I_r & \\ & \varphi_\alpha I_s \end{pmatrix} \in M_g(\mathfrak{R})$, then $I(\mathbb{Q}) = \{X \in GU_g(\mathfrak{B}; I_g) : X\Phi = \Phi X\}$, and for any $\mathbb{Q}$-algebra S we have $I(S) = \{X \in GU_g(\mathfrak{B} \otimes_{\mathbb{Q}} S; I_g) : X\Phi = \Phi X\}$.

On the other side, $J(\mathbb{Q}_p)$ is the group of K_0-automorphisms of the homogeneously principally polarized isocrystal with k-action $((N, \mathbf{F}); i, \mathbb{Q}_p^\times \Psi)$; it has a compact subgroup $J(\mathbb{Z}_p)$ that is given by the W-automorphisms of the homogeneously principally polarized Dieudonné module $(\mathbf{M}, i_\mathbf{M}, \overline{e}_\mathbf{M})$. Since $\mathbf{M} = M_*(A_0(p)) \simeq A_{1/2}^{\oplus g}$ as principally polarized Dieudonné modules, where $A_{1/2}^{\oplus g}$ is endowed with the product polarization coming from the polarization $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $A_{1/2}$, one deduces ([Ghi04a], Cor. 10), keeping track of the action of $\mathcal{O}_{k,(p)}$:

$$\begin{aligned} J(\mathbb{Q}_p) &\simeq \{X \in GU_g(\mathfrak{B}_p; I_g) : X\Phi = \Phi X\} = I(\mathbb{Q}_p), \\ J(\mathbb{Z}_p) &\simeq \{X \in GU_g(\mathfrak{R}_p; I_g) : X\Phi = \Phi X\} =: I(\mathbb{Z}_p), \end{aligned}$$

where $\mathfrak{R}_p$ denotes the unique maximal order of the skew-field $\mathfrak{B}_p$. The above isomorphisms are canonical.

We deduce from Cor. 2.2.4:

Proposition 3.1.2. There is a canonical isomorphism equivariant with respect to the Hecke $G(\mathbb{A}_f^p)$-action:

$$\Theta'_{U(N)}(\overline{\mathbb{F}}_p) : I(\mathbb{Q}) \backslash I(\mathbb{A}_f)/U_N \to Z'(\overline{\mathbb{F}}_p),$$

where $U_N := J(\mathbb{Z}_p) \times U(N)$ is viewed as an open compact subgroup of $I(\mathbb{A}_f) = J(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$.

If $\mathbb{L}_q$ is a finite field of cardinality q, there is, up to isomorphism, a unique Hermitian space of dimension $m \geq 1$ associated to the quadratic extension $\mathbb{L}_{q^2}/\mathbb{L}_q$ (cf. [Lew82]). We denote the associated unitary group - defined over $\mathbb{L}_q$ - by U_m; in particular $GU_m(\mathbb{L}_{q^2}) = \{X \in GL_m(\mathbb{L}_{q^2}) : X^*X = c(X) \cdot I_m, c(X) \in \mathbb{L}_q^\times\}$. (Notice we do not denote this group by $GU_m(\mathbb{L}_q)$).

We will also need to consider the algebraic group $G(U_{m_1} \times U_{m_2}) \subset GU_{m_1} \times GU_{m_2}$ defined over $\mathbb{L}_q$, whose $\mathbb{L}_q$-points are:

$$G(U_{m_1} \times U_{m_2})(\mathbb{L}_{q^2}) = \left\{ g = \begin{pmatrix} X & 0_{m_1,m_2} \\ 0_{m_2,m_1} & Y \end{pmatrix} : g \in GU_{m_1+m_2}(\mathbb{L}_{q^2}) \right\} =$$

$$= \left\{ \begin{pmatrix} X & 0_{m_1,m_2} \\ 0_{m_2,m_1} & Y \end{pmatrix} : \begin{matrix} X^*X=cI_{m_1} \\ Y^*Y=cI_{m_2} \end{matrix}, \ c \in \mathbb{L}_q^\times \right\}.$$

Via the embedding $\iota : k \hookrightarrow \mathfrak{B}$ that we fixed, we obtain a natural epimorphism $\mathfrak{R}_p \twoheadrightarrow \mathbb{F}_{p^2}$, indeed we can write $\mathfrak{R}_p = \mathbb{Z}_p[\varphi_\alpha] \oplus \mathbb{Z}_p[\varphi_\alpha]\Pi$ for a choice of uniformizer Π of $\mathfrak{R}_p$ such that $\Pi^2 = p$, so that:

$$\frac{\mathfrak{R}_p}{\Pi\mathfrak{R}_p} \cong \frac{\mathbb{Z}_p[\varphi_\alpha]}{p\mathbb{Z}_p[\varphi_\alpha]} \xrightarrow{\simeq} \frac{\mathbb{Z}_p[\sqrt{\alpha}]}{p\mathbb{Z}_p[\sqrt{\alpha}]} = \mathbb{F}_{p^2}.$$

Lemma 3.1.3. Let $\bar{G} := G(U_r \times U_s)(\mathbb{F}_{p^2})$; there is a short exact sequence of groups (defining U_p):

$$1 \to U_p \to J(\mathbb{Z}_p) \xrightarrow{\pi} \bar{G} \to 1,$$

where the map $\pi : J(\mathbb{Z}_p) \to \bar{G}$ is induced by the canonical epimorphism $\mathfrak{R}_p \twoheadrightarrow \mathbb{F}_{p^2}$ arising from the fixed embedding $\iota : k \hookrightarrow \mathfrak{B}$.

Proof. By previous considerations, we have a natural identification $J(\mathbb{Z}_p) = \{X \in GU_g(\mathfrak{R}_p; I_g) : X\Phi = \Phi X\}$. Via the embedding ι, we identify $\varphi_\alpha \pmod{\Pi}$ with $\sqrt{\alpha} \pmod{p} \in \mathbb{F}_{p^2}$; if $X \in J(\mathbb{Z}_p)$, then $\pi(X) \in GU_g(\mathbb{F}_{p^2})$ and the equation $X\Phi = \Phi X$ for an (r,s)-block matrix $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GU_g(\mathfrak{R}_p; I_g)$ reduces to the equation in $M_g(\mathbb{F}_{p^2})$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} (\mathrm{mod}\,\Pi) \cdot \begin{pmatrix} -\sqrt{\alpha} \cdot I_r & \\ & \sqrt{\alpha} \cdot I_s \end{pmatrix} (\mathrm{mod}\,p) = \begin{pmatrix} -\sqrt{\alpha} \cdot I_r & \\ & \sqrt{\alpha} \cdot I_s \end{pmatrix} (\mathrm{mod}\,p) \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} (\mathrm{mod}\,\Pi).$$

28

We deduce that $B(\operatorname{mod}\Pi) = 0_{r,s}$ and $C(\operatorname{mod}\Pi) = 0_{s,r}$, since $p \neq 2$, so that $\pi(X) \in \bar{G}$.

On the other side, if $Y = \begin{pmatrix} T & 0_{r,s} \\ 0_{s,r} & S \end{pmatrix} \in \bar{G}$, then we can lift Y to a matrix in $G(U_r \times U_s)(\mathcal{O}_{k_\nu})$, and we can see $G(U_r \times U_s)(\mathcal{O}_{k_\nu}) \subset J(\mathbb{Z}_p)$ via ι, so that π is onto. $\blacksquare$

3.2 Unitary Dieudonné modules and invariant differentials

We have the following general fact:

Lemma 3.2.1. Let L be a subfield of $\bar{\mathbb{F}}_p$ and let σ denote the restriction of the Frobenius of W to $W(L)$. Let M be a Dieudonné module over $W(L)$ endowed with the $\mathbb{Z}_{(p)}$-linear action of a $\mathbb{Z}_{(p)}$-algebra with involution $(\mathfrak{O}; *)$; assume M is endowed with a principal polarization $e : M \times M \to W(L)$ that is skew-Hermitian with respect to the $\mathfrak{O}$-action. Then the assignment:

$$\langle,\rangle : \frac{M}{FM} \times \frac{M}{VM} \to L, \quad (\bar{x}, \bar{y}) \mapsto e(x, Fy)(\operatorname{mod} p)$$

is a well-defined perfect pairing which is L-linear in the first variable and L-semilinear (with respect to σ) in the second variable. Furthermore, $\langle,\rangle$ is skew-Hermitian with respect to the action of $\mathfrak{O}$.

If furthermore $F + V = 0$ on M, then $\langle,\rangle$ defines a σ^{-1}-alternating pairing $\frac{M}{VM} \times \frac{M}{VM} \to L$.

Proof. For $x, y, m, m' \in M$ we have:

$$
\begin{aligned}
e(x + Fm, F(y + Vm')) &= e(x, Fy) + e(Fm, Fy) + e(x + Fm, FVm') \\
&= e(x, Fy) + e(m, VFy)^\sigma + e(x + Fm, pm') \\
&\equiv e(x, Fy)(\operatorname{mod} p),
\end{aligned}
$$

so that $\langle,\rangle$ is well defined; it is clearly L-linear in the first variable, and σ-semilinear in the

second since F is σ-semilinear. If $b \in \mathfrak{O}$ and $x, y \in M$ we have:

$$
\begin{aligned}
\langle b\overline{x}, \overline{y} \rangle &= \langle \overline{bx}, y \rangle = e(bx, Fy) \pmod{p} \\
&= e(x, b^* Fy) \pmod{p} \\
&= e(x, Fb^* y) \pmod{p} \\
&= \langle \overline{x}, b^* \overline{y} \rangle .
\end{aligned}
$$

To show that $\langle , \rangle$ is non-degenerate we need to show that the L-linear map $\varepsilon : \frac{M}{FM} \to$ $\mathrm{Hom}_{\sigma\text{-semi-lin}}(\frac{M}{VM}, L)$ induced by $\langle , \rangle$ is an isomorphism of L-vector spaces. Let $x \in M$ such that $\langle \overline{x}, \overline{y} \rangle = 0$ for all $y \in M$. By assumption we have $e(y, Vx) \in pW(L)$ for all $y \in M$. If we denote by $\mu : M \to \mathrm{Hom}_W(M, W)$ the isomorphism defined by $\mu(m) := e(\cdot, m)$, we have that $\mu(Vx)$ has image contained inside $pW(L)$, so that there is $z \in M$ such that $\frac{1}{p}\mu(Vx) = \mu(z)$; we obtain $Vx = pz$, so that $px = FVx = pFz$, hence $x = Fz \in FM$. We deduce that ε is injective. Similarly one can show that $\langle , \rangle$ induces an injective σ-semilinear map of L-spaces $\frac{M}{VM} \to \mathrm{Hom}_L(\frac{M}{FM}, L)$. We conclude that $\dim_L \frac{M}{VM} = \dim_L \frac{M}{FM}$ and ε is forced to be an isomorphism.

Finally, if $F + V = 0$, we have $FM = VM$ and if $x, y \in M$ one computes:

$$
\begin{aligned}
\langle \overline{x}, \overline{y} \rangle &= e(x, Fy) \pmod{p} = -e(x, Vy) \pmod{p} \\
&= -e(Fx, y)^{\sigma^{-1}} \pmod{p} = e(y, Fx)^{\sigma^{-1}} \pmod{p} \\
&= \langle \overline{y}, \overline{x} \rangle^{\sigma^{-1}} .
\end{aligned}
$$

This says that $\langle , \rangle$ is σ^{-1}-alternating. ∎

Fix a point $[(A_0, i, \overline{\lambda}, \overline{\alpha})] \in Z'(\overline{\mathbb{F}}_p)$. By 2.2.1, $(A_0, i, \overline{\lambda})$ has a canonical $\mathbb{F}_{p^2}$-structure, that we denote by $(A'_0, i, \overline{\lambda}')$; by functoriality, the various object that we associated to $(A_0, i, \overline{\lambda})$ (as p-divisible groups, Dieudonné modules, polarizations, actions of algebras) are obtained as base changes to $\overline{\mathbb{F}}_p$ (resp. W) of analogous objects defined over $\mathbb{F}_{p^2}$ (resp. $W(\mathbb{F}_{p^2})$).

Let $\mathbf{X}' = A'_0(p)$ be the p-divisible group of A'_0: it is defined over $\mathbb{F}_{p^2}$; its covariant Dieudonné module is $\mathbf{M}' = (A'_{1/2})^g$ (cf. 2.2.1). Denote by $i_{\mathbf{M}'}$ the action of $\mathcal{O}_{B_p}$ on $\mathbf{M}'$ and by $e_{\mathbf{M}'}$:

$\mathbf{M}' \times \mathbf{M}' \to W(\mathbb{F}_{p^2})$ the principal polarization of Dieudonné modules which base change to $i_{\mathbf{M}}$ and $e_{\mathbf{M}}$ respectively over W.

By [P 82] 3.3.1, there is a positive integer m such that the canonical map of cotangent spaces (at the origin) $\mathfrak{t}^*_{A'_0[p^m]} \to \mathfrak{t}^*_{\mathbf{X}'}$ is an isomorphism. As a consequence, the closed immersion $A'_0[p^m] \hookrightarrow A'_0$ of $\mathbb{F}_{p^2}$-group schemes induces an epimorphism of $\mathbb{F}_{p^2}$-vector spaces $\mathfrak{t}^*_{A'_0} \to \mathfrak{t}^*_{\mathbf{X}'}$. Since A'_0 is superspecial, $\dim \mathbf{X}' = g = \dim A'_0$, so that we obtain canonical identifications $\mathfrak{t}^*_{A'_0} = \mathfrak{t}^*_{\mathbf{X}'}$ and $\mathrm{Lie}\, A'_0 = \mathrm{Lie}\, \mathbf{X}'$.

By covariant Dieudonné theory, we have a natural isomorphism of $\mathbb{F}_{p^2}$-vector spaces:

$$\mathrm{Lie}(\mathbf{X}') = \frac{\mathbf{M}'}{V\mathbf{M}'}.$$

All the above isomorphisms respect the actions of $\mathcal{O}_{k,(p)}$ on the modules considered, and also the polarizations induced by λ.

By the explicit description of $A'_{1/2}$ given in 2.2.1, we have $F + V = 0$ on $\mathbf{M}'$, so that by Prop. 3.2.1, the principal polarization $e_{\mathbf{M}'} : \mathbf{M}' \times \mathbf{M}' \to W(\mathbb{F}_{p^2})$ induces a non-degenerate pairing of $\mathbb{F}_{p^2}$-spaces:

$$\langle,\rangle : \frac{\mathbf{M}'}{V\mathbf{M}'} \times \frac{\mathbf{M}'}{V\mathbf{M}'} \to \mathbb{F}_{p^2}$$

which is linear in the first argument, σ-linear in the second argument, and σ-alternating (i.e., $\langle x, y \rangle = \langle y, x \rangle^\sigma$, as $\sigma^2 = 1$ on $\mathbb{F}_{p^2}$). Hence $(\frac{\mathbf{M}'}{V\mathbf{M}'}; i_{\mathbf{M}'}, \langle,\rangle)$ is a Hermitian space over $\mathbb{F}_{p^2}$ of dimension $g = 2n$, endowed with an action of $\mathcal{O}_{k,(p)}$ with respect to which the pairing $\langle,\rangle$ is skew-symmetric. Since $e_{\mathbf{M}'}$ is determined only up to a constant in $\mathbb{Z}_p^\times$, the pairing $\langle,\rangle$ is determined up to a constant in $\mathbb{F}_p^\times$.

Via the fixed embedding ν we obtain the decomposition:

$$\mathbf{M}' = \mathbf{M}'_- \oplus \mathbf{M}'_+,$$

where:

$$\mathbf{M}'_\pm := \{m \in \mathbf{M}' : i_{\mathbf{M}'}(\sqrt{\alpha})m = \pm\sqrt{\alpha}m\}; \quad \mathrm{rk}_{W(\mathbb{F}_{p^2})}\, \mathbf{M}'_\pm = g\ .$$

It is easily seen that $V\mathbf{M}'_\pm \subseteq \mathbf{M}'_\mp$, $F\mathbf{M}'_\pm \subseteq \mathbf{M}'_\mp$ and that $e_{\mathbf{M}'}(\mathbf{M}'_+, \mathbf{M}'_+) = 0, e_{\mathbf{M}'}(\mathbf{M}'_-, \mathbf{M}'_-) =$

0 (i.e., $\mathbf{M}'_-$ and $\mathbf{M}'_+$ are totally isotropic with respect to the principal polarization). We have:

$$\frac{\mathbf{M}'}{V\mathbf{M}'} = \frac{\mathbf{M}'_-}{V\mathbf{M}'_+} \oplus \frac{\mathbf{M}'_+}{V\mathbf{M}'_-}.$$

This is a decomposition as $\mathbb{F}_{p^2}$-vector spaces with action of $\mathcal{O}_{k,(p)}$, where $\sqrt{\alpha}$ acts as the element $-\sqrt{\alpha}(\mathrm{mod}\,p)$ of $\mathbb{F}_{p^2}$ on the first summand, and as $\sqrt{\alpha}(\mathrm{mod}\,p)$ on the second. Furthermore:

$$\dim_{\mathbb{F}_{p^2}}\frac{\mathbf{M}'_-}{V\mathbf{M}'_+} = r, \quad \dim_{\mathbb{F}_{p^2}}\frac{\mathbf{M}'_+}{V\mathbf{M}'_-} = s.$$

Proposition 3.2.2. Let $(A_0, i, \overline{\lambda})$ be a triple over $\overline{\overline{\mathbb{F}}}_p$ such that for some level structure $\overline{\alpha}$ we have $[(A_0, i, \overline{\lambda}, \overline{\alpha})] \in Z'(\overline{\mathbb{F}}_p)$; let $(A_0', i, \overline{\lambda}')$ be the canonical $\mathbb{F}_{p^2}$-structure of $(A_0, i, \overline{\lambda})$. The automorphism group of the $\mathbb{F}_{p^2}$-Hermitian space with $\mathcal{O}_{k,(p)}$-action $(\frac{\mathbf{M}'}{V\mathbf{M}'}; i_{\mathbf{M}'}, \langle,\rangle)$ is isomorphic to the finite group $\bar{G} = G(U_r \times U_s)(\mathbb{F}_{p^2})$.

Proof. Let $\mathfrak{L}_\pm := \frac{\mathbf{M}'_\pm}{V\mathbf{M}'_\mp}$. Let $\mathcal{B}^+$ (resp. $\mathcal{B}^-$) be a fixed ordered basis of $\mathfrak{L}_+$ (resp. $\mathfrak{L}_-$). If $\mathfrak{X}$ is an automorphism of $\frac{\mathbf{M}'}{V\mathbf{M}'}$ which commutes with the action of $\mathcal{O}_{k,(p)}$, we have $\mathfrak{X}\,\mathfrak{L}_\pm \subseteq \mathfrak{L}_\pm$, so that the matrix representing $\mathfrak{X}$ with respect to $\mathcal{B} := \mathcal{B}^- \cup \mathcal{B}^+$ is of the form:

$$X = \begin{pmatrix} X_- & 0_{r,s} \\ 0_{s,r} & X_+ \end{pmatrix} \in GL_g(\mathbb{F}_{p^2}).$$

Any such matrix represents - with respect to $\mathcal{B}$ - an automorphism of $\frac{\mathbf{M}'}{V\mathbf{M}'}$ commuting with the action of $\mathcal{O}_{k,(p)}$.

Since $F\mathbf{M}'_+ \subseteq \mathbf{M}'_-$ and $e_{\mathbf{M}'}(\mathbf{M}'_-, \mathbf{M}'_-) = 0$, we deduce that $\langle \mathfrak{L}_-, \mathfrak{L}_+ \rangle = 0$, by definition of the pairing $\langle,\rangle$. This implies that $\langle,\rangle$ is represented, with respect to $\mathcal{B}$, by a Hermitian diagonal matrix $\begin{pmatrix} U_- & 0_{r,s} \\ 0_{s,r} & U_+ \end{pmatrix} \in GL_g(\mathbb{F}_{p^2})$, so that if X is as above, it represents an automorphism of $(\frac{\mathbf{M}'}{V\mathbf{M}'}; i_{\mathbf{M}'}, \langle,\rangle)$ with respect to $\mathcal{B}$ if and only if:

$$X_\pm^* \cdot U_\pm \cdot X_\pm = cU_\pm,$$

where $c \in \mathbb{F}_p^\times$ is a scalar depending only on X.

We conclude that the automorphism group of $(\frac{\mathbf{M}'}{V\mathbf{M}'}; i_{\mathbf{M}'}, \langle, \rangle)$ is isomorphic to:

$$\mathcal{G} = \{(X_-, X_+) \in GU_r(\mathbb{F}_{p^2}; U_-) \times GU_s(\mathbb{F}_{p^2}; U_+) : c_-(X_-) = c_+(X_+)\},$$

where c_- and c_+ are the similitude factor homomorphisms of the two unitary groups above. The unitary spaces $(\mathbb{F}_{p^2}^r; U_-)$ and $(\mathbb{F}_{p^2}^r; I_r)$ are isomorphic, hence we can find an isomorphism $GU_r(\mathbb{F}_{p^2}; U_-) \simeq GU_r(\mathbb{F}_{p^2})$ preserving the similitude factor of corresponding matrices in each group; we can proceed similarly for $GU_s(\mathbb{F}_{p^2}; U_+)$. We obtain therefore an isomorphism $\mathcal{G} \simeq \bar{G}$. $\blacksquare$

3.2.1 Invariant differentials

We now work with cotangent spaces. As usual, $\mathfrak{t}_{A_0'}^*$ denotes the cotangent space (at the origin) of A_0'. As vector spaces over $\mathbb{F}_{p^2}$, we have a canonical identification:

$$\mathfrak{t}_{A_0'}^* = \mathrm{Hom}_{\mathbb{F}_{p^2}} \left(\frac{\mathbf{M}'}{V\mathbf{M}'}, \mathbb{F}_{p^2} \right).$$

Let $\mathfrak{L} := \frac{\mathbf{M}'}{V\mathbf{M}'} = \mathfrak{L}_- \oplus \mathfrak{L}_+$, where $\mathfrak{L}_\pm := \frac{\mathbf{M}'_\pm}{V\mathbf{M}'_\mp}$, so that $\mathfrak{t}_{A_0'}^* = \mathfrak{L}^*$. The action of $\mathcal{O}_{k,(p)}$ on $\mathfrak{L}$ induces by functoriality an algebra homomorphism:

$$i^\vee : \mathcal{O}_{k,(p)}^{opp} = \mathcal{O}_{k,(p)} \to \mathrm{End}_{\mathbb{F}_{p^2}}(\mathfrak{L}^*)$$

defined by $i^\vee(b)(\eta) := \eta \circ i(b)$ for all $b \in \mathcal{O}_{k,(p)}$ and $\eta \in \mathfrak{t}_{A_0'}^*$. Notice that $\sqrt{\alpha} \in \mathcal{O}_{k,(p)}$ acts on $\mathfrak{L}_\pm^*$ as $\pm\sqrt{\alpha}(\mathrm{mod}\, p)$, via $i^\vee$.

The non-degenerate Hermitian pairing $\langle, \rangle$ on $\mathfrak{L}$ induces a σ-semilinear isomorphism of $\mathbb{F}_{p^2}$-spaces $\varepsilon : \mathfrak{L} \to \mathfrak{L}^*$ by setting $\varepsilon_v : w \mapsto \langle w, v \rangle$ for all $v, w \in \mathfrak{L}$. This allows us to define a pairing $(,)$ on $\mathfrak{L}^*$ by setting $(\varepsilon_{v_1}, \varepsilon_{v_2}) := \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in \mathfrak{L}$. We have obtained a non-degenerate pairing $(,) : \mathfrak{L}^* \times \mathfrak{L}^* \to \mathbb{F}_{p^2}$, which is σ-semilinear in the first variable, linear in the second, and such that $(\eta_1, \eta_2) = (\eta_2, \eta_1)^\sigma$ for all $\eta_1, \eta_2 \in \mathfrak{L}^*$. Furthermore $(i^\vee(b)\eta_1, \eta_2) = (\eta_1, i^\vee(\bar{b})\eta_2)$ for all $b \in \mathcal{O}_{k,(p)}$, and $(\mathfrak{L}_-^*, \mathfrak{L}_+^*) = 0$. We have that

$$(\mathfrak{t}_{A_0'}^* = \mathfrak{L}^*; i^\vee, (,))$$

33

is a $\mathbb{F}_{p^2}$-Hermitian space of dimension g, endowed with an action $i^{\vee}$ of $\mathcal{O}_{k,(p)}$ with respect to which the pairing $(,)$ is skew-Hermitian.

Lemma 3.2.3. There is an isomorphism of groups:

$$\mathrm{Aut}_{\mathbb{F}_{p^2}}(\mathfrak{t}^*_{A'_0}; i^{\vee}, (,)) \simeq \bar{G}.$$

Proof. The result follows from Prop. 3.2.2, since the map $\mathfrak{X} \mapsto (\mathfrak{X}^*)^{-1}$ defines an isomorphism of groups $\mathrm{Aut}_{\mathbb{F}_{p^2}}(\mathfrak{t}^*_{A'_0}; i^{\vee}, (,)) \to \mathrm{Aut}_{\mathbb{F}_{p^2}}(\mathrm{Lie}\, A'_0; i, \langle, \rangle)$. ∎

We can now give the following:

Definition 3.2.4. Let $(A_0, i, \bar{\lambda})$ be a triple over $\overline{\mathbb{F}}_p$ such that for some level structure $\bar{\alpha}$ we have $[(A_0, i, \bar{\lambda}, \bar{\alpha})] \in Z'(\overline{\mathbb{F}}_p)$; let $(A'_0, i, \bar{\lambda}')$ be the canonical $\mathbb{F}_{p^2}$-structure of $(A_0, i, \bar{\lambda})$. A basis of invariant differentials of $(A'_0, i, \bar{\lambda}')$ (over $\mathbb{F}_{p^2}$) is a choice of an ordered (similitude) Hermitian basis $\eta = (\eta_-, \eta_+)$ of the Hermitian module $(\mathfrak{t}^*_{A'_0}; i^{\vee}, (,))$ such that $\eta_{\pm}$ is a basis for $(\mathfrak{t}^*_{A'_0})_{\pm} := \left(\frac{\mathbf{M}'_{\pm}}{V\mathbf{M}'_{\mp}} \right)^*$.

For $(A'_0, i, \bar{\lambda}')$ as above there is a basis $\mathcal{B}$ of $\mathfrak{t}^*_{A'_0}$ with respect to which the automorphisms of $(\mathfrak{t}^*_{A'_0}; i^{\vee}, (,))$ are represented by the matrices of $\bar{G} = G(U_r \times U_s)(\mathbb{F}_{p^2})$. Let $\eta \in \mathbb{F}^g_{p^2}$ be the coordinate column vector of a basis of invariant differentials for $(A'_0, i, \bar{\lambda}')$ with respect to $\mathcal{B}$. Then any other coordinate vector (with respect to $\mathcal{B}$) of a basis of invariant differentials for $(A'_0, i, \bar{\lambda}')$ is of the form $M\eta$ for a unique $M \in \bar{G}$.

3.3 Superspecial modular forms

We assume fixed from now on a basis η_0 of invariant differentials for $(A'_0, i_0, \bar{\lambda}'_0)$. We denote by $Z'_{diff}(\overline{\mathbb{F}}_p)$ the set of equivalence classes of tuples $(A, i, \bar{\lambda}, \bar{\alpha}, \eta)$, where $(A, i, \bar{\lambda}, \bar{\alpha})$ is a representative for an equivalence class $[(A, i, \bar{\lambda}, \bar{\alpha})] \in Z'(\overline{\mathbb{F}}_p)$, and η is a choice of basis of invariant differentials for the triple $(A', i, \bar{\lambda}')$ defined over $\mathbb{F}_{p^2}$. Two tuples $(A, i, \bar{\lambda}, \bar{\alpha}, \eta)$ and $(A_1, i_1, \bar{\lambda}_1, \bar{\alpha}_1, \eta_1)$ are equivalent if there is an isomorphism $f : (A, i, \bar{\lambda}, \bar{\alpha}) \to (A_1, i_1, \bar{\lambda}_1, \bar{\alpha}_1)$ such that $f^*(\eta_1) = \eta$, where $f^* : \mathfrak{t}^*_{A_1} \to \mathfrak{t}^*_A$ is the cotangent map induced by f.

We remark that if $g \in J(\mathbb{Z}_p) \subset GL_g(\mathfrak{R}_p)$ and $v \in \text{Lie } A_0' \simeq \mathbb{F}_{p^2}^g$, $\omega \in \mathfrak{t}_{A_0'}^* = (\text{Lie } A_0')^*$, then g acts on v and on ω as follows:

$$g \cdot v \; : \quad = g(\text{mod }\Pi)v$$

$$g \cdot \omega \; : \quad = \omega \circ g(\text{mod }\Pi),$$

where Π is a uniformizer for $\mathfrak{R}_p$.

Proposition 3.3.1. The uniformization map $\Theta'_{U(N)}(\overline{\mathbb{F}}_p)$ induces an isomorphism $\Theta'_{diff}(\overline{\mathbb{F}}_p)$ equivariant with respect to the Hecke $G(\mathbb{A}_f^p)$-action:

$$\Theta'_{diff}(\overline{\mathbb{F}}_p) : I(\mathbb{Q}) \backslash \left(J(\mathbb{Q}_p)/U_p \times G(\mathbb{A}_f^p)/U(N) \right) \to Z'_{diff}(\overline{\mathbb{F}}_p).$$

Proof. (In this proof, for $\xi \in I(\mathbb{Q})$ we will sometimes write ξ to denote $\alpha_{0,p}(\xi) \in J(\mathbb{Q}_p)$, if no ambiguity arises). Fix a left transversal $\mathcal{Y}$ (resp. $\mathcal{G}$) of $J(\mathbb{Z}_p)$ in $J(\mathbb{Q}_p)$ (resp. of U_p in $J(\mathbb{Z}_p)$). Let $I(\mathbb{Q}) \cdot (\rho^{-1}U_p \times xU(N))$ be a fixed element in the left hand side above; there are uniquely determined $y^{-1} \in \mathcal{Y}$ and $g^{-1} \in \mathcal{G}$ such that $\rho^{-1}U_p = y^{-1}g^{-1}U_p$. By the definition of $\Theta'_{U(N)}(\overline{\mathbb{F}}_p)$ (cf. 2.2.2), we obtain a well defined tuple $(y_* A_0, y_* i_0, y_* \overline{\lambda}_0, x^{-1} \cdot y_* \overline{\alpha}_0)$ representing a class in $Z'(\overline{\mathbb{F}}_p)$. Since the p-divisible group of $(y_* A_0, y_* i_0, y_* \overline{\lambda}_0)$ coincides with $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$, η_0 is a basis of invariant differentials for the $\mathbb{F}_{p^2}$-model of $(y_* A_0, y_* i_0, y_* \overline{\lambda}_0)$, via the canonical identification:

$$(\text{Lie } y_* A_0')^* = (\text{Lie } \mathbf{X}')^* \, ;$$

then $\eta_0 g$ is also a basis of invariant differentials for the model of $(y_* A_0, y_* i_0, y_* \overline{\lambda}_0)$ over $\mathbb{F}_{p^2}$. We set:

$$\Theta'_{diff}(\overline{\mathbb{F}}_p) : I(\mathbb{Q}) \cdot \left(y^{-1}g^{-1}U_p \times xU(N) \right) \longmapsto [(y_* A_0, y_* i_0, y_* \overline{\lambda}_0, x^{-1} \cdot y_* \overline{\alpha}_0, \eta_0 g)].$$

As we fixed the transversals $\mathcal{Y}$ and $\mathcal{G}$, to show that the above assignment is well defined we only need to check that the map constructed factors through $I(\mathbb{Q})$. Let $\xi \in I(\mathbb{Q})$, so that $\xi_* A_0 = A_0$. Then $\xi y^{-1} g^{-1} U_p = y_1^{-1} g_1^{-1} U_p$ for uniquely determined $y_1^{-1} \in \mathcal{Y}$ and $g_1^{-1} \in \mathcal{G}$. Write $y_1 = f \cdot y\xi^{-1}$ with $f = g_1^{-1} u g \in J(\mathbb{Z}_p)$, for some $u \in U_p$. The isomorphism f induces an isomorphism:

$$f_{ab} : (y_* A_0, y_* i_0, y_* \overline{\lambda}_0) \xrightarrow{\simeq} (y_{1,*} A_0, y_{1,*} i_0, y_{1,*} \overline{\lambda}_0).$$

35

By definition of the action of $I(\mathbb{Q})$ on $G(\mathbb{A}_f^p)/U(N)$ we have:

$$\xi \cdot xU(N) = \alpha_0^p(\xi)xU(N) = \alpha_0 \circ H_1(\xi) \circ \alpha_0^{-1} \circ xU(N).$$

The level structure on $(y_{1,*}A_0, y_{1,*}i_0, y_{1,*}\overline{\lambda}_0)$ associated to $y_1^{-1}g_1^{-1}U_p \times \alpha_0^p(\xi)xU(N)$ is therefore induced by:

$$
\begin{aligned}
x^{-1}\alpha_0^p(\xi^{-1})y_{1,*}(\alpha_0) &= x^{-1}\alpha_0 H_1(\xi^{-1})\alpha_0^{-1}\alpha_0 H_1(y_1^{-1}) \\
&= x^{-1}\alpha_0 H_1(y^{-1}f^{-1}) \\
&= x^{-1}y_*(\alpha_0) \cdot H_1(f^{-1}),
\end{aligned}
$$

so that f_{ab} is an isomorphism

$$f_{ab} : (y_*A_0, y_*i_0, y_*\overline{\lambda}_0, x^{-1} \cdot y_*\overline{\alpha}_0) \xrightarrow{\simeq} (y_{1,*}A_0, y_{1,*}i_0, y_{1,*}\overline{\lambda}_0, x^{-1}\alpha_0^p(\xi^{-1})y_{1,*}(\alpha_0)).$$

The cotangent map induced by f_{ab} gives:

$$f_{ab}^*(\eta_0 g_1) = \eta_0 g_1 f = \eta_0 u g = \eta_0 g,$$

so that f_{ab} preserves the choices of invariant differentials.

The map $\Theta'_{diff}(\overline{\mathbb{F}}_p)$ is surjective: let $[(A, i, \overline{\lambda}, \overline{\alpha}, \eta)] \in Z'_{diff}(\overline{\mathbb{F}}_p)$; we can find $y^{-1} \in \mathcal{Y}$, $\widetilde{g}^{-1} \in J(\mathbb{Z}_p)$ and $x \in G(\mathbb{A}_f^p)$ such that $(A, i, \overline{\lambda}, \overline{\alpha}, \eta)$ is isomorphic to a tuple of the form:

$$(y_*A_0, y_*i_0, y_*\overline{\lambda}_0, x^{-1} \cdot y_*\overline{\alpha}_0, \eta_0\widetilde{g}).$$

Let $g^{-1} \in \mathcal{G}$ such that $\widetilde{g}^{-1}U_p = g^{-1}U_p$; then

$$[(A, i, \overline{\lambda}, \overline{\alpha}, \eta)] = \Theta'_{diff}(\overline{\mathbb{F}}_p)(I(\mathbb{Q}) \cdot (y^{-1}g^{-1}U_p \times xU(N))).$$

The map $\Theta'_{diff}(\overline{\mathbb{F}}_p)$ is injective: let us fix $y^{-1}, y_1^{-1} \in \mathcal{Y}$, $g^{-1}, g_1^{-1} \in \mathcal{G}$ and $x, x_1 \in G(\mathbb{A}_f^p)$ such that there is an isomorphism:

$$f_{ab} : (y_*A_0, y_*i_0, y_*\overline{\lambda}_0, x^{-1} \cdot y_*\overline{\alpha}_0, \eta_0 g) \xrightarrow{\simeq} (y_{1,*}A_0, y_{1,*}i_0, y_{1,*}\overline{\lambda}_0, x_1^{-1} \cdot y_{1,*}\overline{\alpha}_0, \eta_0 g_1).$$

Denote by $f \in J(\mathbb{Z}_p)$ the automorphism induced by f_{ab} on $(\mathbf{X}, i_\mathbf{X}, \overline{\lambda}_\mathbf{X})$ and let

$$\xi = (y^{-1}f^{-1}y_1)_{ab}$$

36

be the auto-quasi-isogeny of $(A_0, i_0, \overline{\lambda}_0)$ inducing $y^{-1}f^{-1}y_1$ on $(\mathbf{X}, i_{\mathbf{X}}, \overline{\lambda}_{\mathbf{X}})$. Since f is an isomorphism, we have:

(i) $x_1^{-1}y_{1,*}(\overline{\alpha}_0)H_1(f) = x^{-1}y_*(\overline{\alpha}_0)$, so $x \equiv \alpha_0 H_1(y^{-1}f^{-1}y_1)\alpha_0^{-1}x_1 (\mathrm{mod}\, U(N))$;

(ii) $\eta_0 g_1 f = \eta_0 g$, hence $g_1 f \equiv g (\mathrm{mod}\, U_p)$.

Then we have:

$$I(\mathbb{Q}) \cdot \left(y_1^{-1}g_1^{-1}U_p \times x_1 U(N)\right) = I(\mathbb{Q}) \cdot \left(\xi y_1^{-1}g_1^{-1}U_p \times \alpha_0^p(\xi)x_1 U(N)\right)$$
$$\overset{(ii)}{=} I(\mathbb{Q}) \cdot \left(y^{-1}g^{-1}U_p \times \alpha_0 H_1(y^{-1}f^{-1}y_1)\alpha_0^{-1}x_1 U(N)\right)$$
$$\overset{(i)}{=} I(\mathbb{Q}) \cdot \left(y^{-1}g^{-1}U_p \times x U(N)\right).$$

The Hecke $G(\mathbb{A}_f^p)$-equivariance of $\{\Theta'_{diff,K^p}(\overline{\mathbb{F}}_p)\}_{K^p}$ (with respect to the projective systems of domain and codomain obtained by varying the prime-to-p level structures) is a consequence of the definition of the Hecke operators in this context: for $\gamma \in G(\mathbb{A}_f^p)$ let us denote by H_γ the corresponding Hecke operator acting on the domain or codomain of $\Theta'_{diff}(\overline{\mathbb{F}}_p)$, for a suitable level subgroup. For $y^{-1} \in \mathcal{Y}$, $g^{-1} \in \mathcal{G}$ and $x \in G(\mathbb{A}_f^p)$ we have:

$$I(\mathbb{Q}) \cdot \left(y^{-1}g^{-1}U_p \times x U(N)\right) \overset{H_\gamma}{\longmapsto} I(\mathbb{Q}) \cdot \left(y^{-1}g^{-1}U_p \times \gamma^{-1}x\gamma \cdot \gamma^{-1}U(N)\gamma\right)$$
$$\overset{\Theta'_{diff}(\overline{\mathbb{F}}_p)}{\longmapsto} [(y_*A_0, y_*i_0, y_*\overline{\lambda}_0, \gamma^{-1}x^{-1}\gamma \cdot y_*(\gamma^{-1}\overline{\alpha}_0), \eta_0 g);$$
$$I(\mathbb{Q}) \cdot \left(y^{-1}g^{-1}U_p \times x U(N)\right) \overset{\Theta'_{diff}(\overline{\mathbb{F}}_p)}{\longmapsto} [(y_*A_0, y_*i_0, y_*\overline{\lambda}_0, x^{-1} \cdot y_*(\overline{\alpha}_0), \eta_0 g)$$
$$\overset{H_\gamma}{\longmapsto} [(y_*A_0, y_*i_0, y_*\overline{\lambda}_0, \gamma^{-1}x^{-1} \cdot y_*(\overline{\alpha}_0), \eta_0 g),$$

and $\gamma^{-1}x^{-1}\gamma \cdot y_*(\gamma^{-1}\overline{\alpha}_0) = \gamma^{-1}x^{-1} \cdot y_*(\overline{\alpha}_0)$. $\blacksquare$

The isomorphism $\Theta'_{diff}(\overline{\mathbb{F}}_p)$ described above depends on the choices of transversals $\mathcal{G}$ and $\mathcal{Y}$. We assume from now on that such transversals have been fixed.

For an algebraic $\overline{\mathbb{F}}_p$-representation $\rho : GL_g \to GL_d$ of the group GL_g, denote by $M_\rho(N; \overline{\mathbb{F}}_p)$ the $\overline{\mathbb{F}}_p$-vector space $M_\rho(\mathcal{D}_A; \overline{\mathbb{F}}_p)$ of unitary modular forms (mod p) of signature (r, s) for the field k, having level N and weight ρ. Let $\iota : Z' \hookrightarrow \mathcal{S}_{\mathcal{D}_A, U(N)} \otimes \overline{\mathbb{F}}_p$ be the closed immersion of $\overline{\mathbb{F}}_p$-schemes giving on geometric points the inclusion $Z'(\overline{\mathbb{F}}_p) \subset \mathcal{S}_{\mathcal{D}_A, U(N)}(\overline{\mathbb{F}}_p)$. Let

$\tau : G(U_r \times U_s) \to GL_m$ be an algebraic $\mathbb{F}_p$-representation of the $\mathbb{F}_p$-group $G(U_r \times U_s)$. We consider restrictions of modular forms to the superspecial locus, as in [Ghi04a]:

Definition 3.3.2. The space $M_\tau^{ss}(N; \overline{\mathbb{F}}_p)$ of unitary superspecial modular forms $(\text{mod } p)$ of signature (r, s) for the field k, having weight τ and level N is the finite dimensional $\overline{\mathbb{F}}_p$-vector space whose elements f are rules that assign, to any tuple $(A, i, \overline{\lambda}, \overline{\alpha}, \eta)/\overline{\mathbb{F}}_p$ such that $[(A, i, \overline{\lambda}, \overline{\alpha})]$ is an element of $Z'(\overline{\mathbb{F}}_p)$ and η is an ordered basis of invariant differentials for $(A', i, \overline{\lambda}')$, an element $f(A, i, \overline{\lambda}, \overline{\alpha}, \eta) \in \overline{\mathbb{F}}_p^m$ in such a way that:

(a) $f(A, i, \overline{\lambda}, \overline{\alpha}, \eta M) = \tau(M)^{-1} f(A, i, \overline{\lambda}, \overline{\alpha}, \eta)$ for all $M \in \bar{G} \simeq \mathrm{Aut}_{\mathbb{F}_{p^2}}(\mathfrak{t}_{A'}^*; i^\vee, (,))$;

(b) if $(A, i, \overline{\lambda}, \overline{\alpha}, \eta) \simeq (A_1, i_1, \overline{\lambda}_1, \overline{\alpha}_1, \eta_1)$ then $f(A, i, \overline{\lambda}, \overline{\alpha}, \eta) = f(A_1, i_1, \overline{\lambda}_1, \overline{\alpha}_1, \eta_1)$.

One reason for which the restriction of modular forms to the superspecial locus $Z'(\overline{\mathbb{F}}_p)$ is of interest to us is the finiteness of the set $Z'(\overline{\mathbb{F}}_p)$ (Cor. 2.2.4). We have the following description of the space of superspecial modular forms:

Proposition 3.3.3. For any algebraic $\mathbb{F}_p$-representation $\rho : GL_g \to GL_d$, denote by $\mathrm{Res}\,\rho$ its restriction to $G(U_r \times U_s)$. Then:

$$M_{\mathrm{Res}\,\rho}^{ss}(N; \overline{\mathbb{F}}_p) = H^0(Z'(\overline{\mathbb{F}}_p), \iota^* \mathbb{E}_\rho).$$

Proof. If $f \in H^0(Z'(\overline{\mathbb{F}}_p), \iota^* \mathbb{E}_\rho)$, then f satisfies (a) and (b) of Def. 3.3.2 (cf. Prop. 1.2.3). On the other hand, let $f \in M_{\mathrm{Res}\,\rho}^{ss}(N; \overline{\mathbb{F}}_p)$ so that f is an assignment on tuples $(A, i, \overline{\lambda}, \overline{\alpha}, \eta)/\overline{\mathbb{F}}_p$ as in Def. 3.3.2; in particular η here is a basis of invariant differentials for $(A', i, \overline{\lambda}')$. If ω is any ordered basis of $\mathfrak{t}_A^*$, there is a unique $X_{\omega,\eta} \in GL_g(\overline{\mathbb{F}}_p)$ such that $\omega = \eta X_{\omega,\eta}$ and we define $f(A, i, \overline{\lambda}, \overline{\alpha}, \omega) := \rho(X_{\omega,\eta})^{-1} \cdot f(A, i, \overline{\lambda}, \overline{\alpha}, \eta)$. This assignment is well defined as ρ is a representation of GL_g, and allows us to view f as an element of $H^0(Z'(\overline{\mathbb{F}}_p), \iota^* \mathbb{E}_\rho)$. $\blacksquare$

The definition of $M_\tau^{ss}(N; \overline{\mathbb{F}}_p)$ depends upon $Z'(\overline{\mathbb{F}}_p)$, hence upon the choice of $(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)$ that we have fixed at the beginning. The Hecke operators act upon $M_\tau^{ss}(N; \overline{\mathbb{F}}_p)$.

3.3.1 Algebraic modular forms

Recall that we have identifications $I(\mathbb{Q}_p) = J(\mathbb{Q}_p)$ and $I(\mathbb{A}_f^p) = G(\mathbb{A}_f^p)$. Set $U := U_p \times U(N)$ and view it as an open compact subgroup of $I(\mathbb{A}_f)$. The group $I(\mathbb{Q})$ is discrete inside $I(\mathbb{A}_f)$, by [RZ96], 6.23, so that the double coset space $I(\mathbb{Q})\backslash I(\mathbb{A}_f)/U$ is finite, because $I(\mathbb{Q})\backslash I(\mathbb{A}_f)$ is compact (cf. [Gro99], Prop. 1.4). Assume fixed an algebraic $\mathbb{F}_p$-representation $\tau : G(U_r \times U_s) \to GL_m$. Following [Gro99], we define the space of *algebraic modular forms* $(\mathrm{mod}\, p)$ for the group I, having level U and weight τ to be the finite dimensional $\overline{\mathbb{F}}_p$-vector space:

$$
\begin{aligned}
M_\tau^{\mathrm{alg}}(N; \overline{\mathbb{F}}_p) \quad &: \quad = \{f : I(\mathbb{Q})\backslash I(\mathbb{A}_f)/U \to \overline{\mathbb{F}}_p^m : \\
f(xM) \quad &= \quad \tau(M)^{-1} f(x),\ M \in \bar{G},\ x \in I(\mathbb{Q})\backslash I(\mathbb{A}_f)/U\},
\end{aligned}
$$

where the right action of $\bar{G}$ on $I(\mathbb{Q})\backslash I(\mathbb{A}_f)/U$ is induced by the identification $\bar{G} = J(\mathbb{Z}_p)/U_p$.

The space $M_\tau^{\mathrm{alg}}(N; \overline{\mathbb{F}}_p)$ is endowed with a natural action of the Hecke algebra and our previous computations give:

Proposition 3.3.4. There is an isomorphism of finite dimensional $\overline{\mathbb{F}}_p$-vector spaces endowed with the Hecke $G(\mathbb{A}_f^p)$-action:

$$
M_\tau^{\mathrm{alg}}(N; \overline{\mathbb{F}}_p) \simeq M_\tau^{ss}(N; \overline{\mathbb{F}}_p).
$$

Proof. By Prop. 3.3.1, we have an isomorphism of $M_\tau^{\mathrm{alg}}(N; \overline{\mathbb{F}}_p)$ with the space of functions $Z'_{diff}(\overline{\mathbb{F}}_p) \to \overline{\mathbb{F}}_p^m$ satisfying condition (a) of Def. 3.3.2. (Note that if $g^{-1} \in \mathcal{G}$, $mU_p \in J(\mathbb{Z}_p)/U_p$ then by definition $\eta_0 g \cdot mU_p = \eta_0 m^{-1} g$). $\blacksquare$

3.4 The correspondence between Hecke eigensystems

Set for convenience of notation $\mathcal{S} := \mathcal{S}_{\mathcal{D}_A, U(N)} \otimes \overline{\mathbb{F}}_p$. Let $\mathcal{I}$ be the ideal sheaf associated to the closed immersion of $\overline{\mathbb{F}}_p$-schemes $\iota : Z' \hookrightarrow \mathcal{S}$ (recall that Z' is discrete), so that $\mathcal{I}$ is a coherent ideal sheaf on $\mathcal{S}$ and we have the following exact sequence:

$$
0 \to H^0(\mathcal{S}, \mathcal{I} \otimes \mathbb{E}_\rho) \to H^0(\mathcal{S}, \mathbb{E}_\rho) \to H^0(\mathcal{S}, \iota_* \mathcal{O}_{Z'} \otimes \mathbb{E}_\rho).
$$

The projection formula gives $\iota_*\iota^*\mathbb{E}_\rho = \iota_*\mathcal{O}_{Z'}\otimes\mathbb{E}_\rho$, so that $H^0(\mathcal{S}, \iota_*\mathcal{O}_{Z'}\otimes\mathbb{E}_\rho) = H^0(Z', \iota^*\mathbb{E}_\rho)$. The above exact sequence of vector spaces can therefore be written, by Prop. 3.3.3, as:

$$0 \to H^0(\mathcal{S}, \mathcal{I}\otimes\mathbb{E}_\rho) \to M_\rho(N; \overline{\mathbb{F}}_p) \xrightarrow{r} M^{ss}_{\mathrm{Res}\,\rho}(N; \overline{\mathbb{F}}_p),$$

where the map r need not to be surjective. Recall that $\mathrm{Res}\,\rho$ is the restriction of ρ to the algebraic group $G(U_r \times U_s)$. Furthermore, observe that r is equivariant with respect to the Hecke $G(\mathbb{A}_f^p)$-action.

Proposition 3.4.1. There exists a positive integer m_0 such that the map r induce a surjection $M_{\rho\otimes\det^m}(N; \overline{\mathbb{F}}_p) \to M^{ss}_{\mathrm{Res}(\rho\otimes\det^m)}(N; \overline{\mathbb{F}}_p)$ for any $m > m_0$.

Proof. The invertible sheaf of $\mathcal{O}_\mathcal{S}$-modules $\bigwedge^g \mathbb{E} = \mathbb{E}_{\det}$ is ample over $\mathcal{S}$ by [Lan08], Th. 7.2.4.1. The proposition then follows in the same way as [Ghi04a], Prop. 24. (The reader should be aware of some typos contained in the proof of Prop. 24 of loc.cit.: in line 3 of the proof of the proposition, "locally free" should be changed into "coherent"). ∎

We have:

Theorem 3.4.2. *Let p be an odd prime and $k/\mathbb{Q}$ be a quadratic imaginary field extension in which p is inert. Let n be a positive integer and let r, s be non-negative integers such that $r + s = g := 2n$. Let furthermore $N \geq 3$ be an integer not divisible by p. Denote by I the reductive $\mathbb{Q}$-group whose $\mathbb{Q}$-rational points are given by $I(\mathbb{Q}) = \{X \in GU_g(\mathfrak{B}; I_g) : X\Phi = \Phi X\}$, where $\mathfrak{B}$ is the quaternion algebra over $\mathbb{Q}$ ramified at p and ∞, and Φ is as in 3.1.3.*

The systems of Hecke eigenvalues arising from (r, s)-unitary PEL modular forms $(\mathrm{mod}\,p)$ for the quadratic imaginary field k, having fixed level N and any possible weight $\rho : GL_g \to GL_{m(\rho)}$ defined over $\mathbb{F}_p$, are the same as the systems of Hecke eigenvalues arising from $(\mathrm{mod}\,p)$ algebraic modular forms for the group I having level $U = U_p \times U(N) \subset I(\mathbb{A}_f)$ and any possible weight $\rho' : G(U_r \times U_s) \to GL_{m'(\rho')}$ defined over $\mathbb{F}_p$.

Proof. We first show that any system of Hecke eigenvalues occurring in the spaces $\{M_\rho(N; \overline{\mathbb{F}}_p)\}_\rho$ for variable weights $\rho : GL_g \to GL_{m(\rho)}$ defined over $\mathbb{F}_p$ also occurs in the

spaces $\{M_\tau^{ss}(N; \overline{\mathbb{F}}_p)\}_\tau$ for variable weights $\tau : G(U_r \times U_s) \to GL_{m'(\tau)}$ defined over $\mathbb{F}_p$. Then we follow the proof of Th. 28 in [Ghi04a] to show that the converse is also true, and finally we apply Prop. 3.3.4. Notice that the first part of this proof is different from the one given in [Ghi04a].

For any integer $i \geq 0$ we have an exact sequence of $\mathcal{O}_{\mathcal{S}}$-modules:

$$0 \to \mathcal{I}^{i+1} \to \mathcal{I}^i \to \mathcal{I}^i/\mathcal{I}^{i+1} \to 0$$

giving rise to the exact sequence in cohomology:

$$0 \to H^0(\mathcal{S}, \mathcal{I}^{i+1} \otimes \mathbb{E}_\rho) \to H^0(\mathcal{S}, \mathcal{I}^i \otimes \mathbb{E}_\rho) \xrightarrow{r_i} H^0(\mathcal{S}, \mathcal{I}^i/\mathcal{I}^{i+1} \otimes \mathbb{E}_\rho),$$

which defines the homomorphisms r_i for any $i \geq 0$ ($r_0 = r$ in the previous notation).

For any $j \geq 1$ we also have the exact sequence of sheaves of $\mathcal{O}_{\mathcal{S}}$-modules:

$$\mathcal{I} \otimes \mathcal{I}^j/\mathcal{I}^{j+1} \to \mathcal{O}_{\mathcal{S}} \otimes \mathcal{I}^j/\mathcal{I}^{j+1} \to \iota_* \mathcal{O}_{Z'} \otimes \mathcal{I}^j/\mathcal{I}^{j+1} \to 0.$$

The image of the first map is zero, so that we obtain isomorphisms of $\mathcal{O}_{\mathcal{S}}$-modules $\mathcal{I}^j/\mathcal{I}^{j+1} \simeq \iota_* \mathcal{O}_{Z'} \otimes \mathcal{I}^j/\mathcal{I}^{j+1}$ for any $j \geq 1$. In cohomology we have therefore isomorphisms ξ_j for any $j \geq 1$:

$$\xi_j : H^0(\mathcal{S}, \mathcal{I}^j/\mathcal{I}^{j+1} \otimes \mathbb{E}_\rho) \xrightarrow{\simeq} H^0(Z', \iota^*(\mathcal{I}^j/\mathcal{I}^{j+1} \otimes \mathbb{E}_\rho)).$$

Let $0 \neq f \in M_\rho(N; \overline{\mathbb{F}}_p) = H^0(\mathcal{S}, \mathbb{E}_\rho)$ be a Hecke eigenform of some fixed weight ρ : $GL_g \to GL_m$ defined over $\mathbb{F}_p$. If $r(f) \neq 0$ then the system of Hecke eigenvalues associated to f occurs in $M^{ss}_{\text{Res }\rho}(N; \overline{\mathbb{F}}_p)$, since r is Hecke equivariant. If $r(f) = 0$, then $f \in H^0(\mathcal{S}, \mathcal{I} \otimes \mathbb{E}_\rho)$. We claim that in this case there is a positive integer h such that $f \in H^0(\mathcal{S}, \mathcal{I}^h \otimes \mathbb{E}_\rho)$ and $r_h(f) \neq 0$. Assume not: then $r_1(f) = 0$ and $f \in H^0(\mathcal{S}, \mathcal{I}^2 \otimes \mathbb{E}_\rho)$, so that we can compute $r_2(f)$ and we have $r_2(f) = 0$; therefore $f \in H^0(\mathcal{S}, \mathcal{I}^3 \otimes \mathbb{E}_\rho)$, etc. Reiterating this procedure we deduce that $f \in H^0(\mathcal{S}, \mathcal{I}^i \otimes \mathbb{E}_\rho)$ for all integers $i > 0$, so that $f = 0$, contradicting our assumption $0 \neq f$. Then there exists a positive integer h such that $f \in H^0(\mathcal{S}, \mathcal{I}^h \otimes \mathbb{E}_\rho)$ and $r_h(f) \neq 0$. Let:

$$f^{ss} := \xi_h(r_h(f)) \in H^0(Z', \iota^*(\mathcal{I}^h/\mathcal{I}^{h+1} \otimes \mathbb{E}_\rho)).$$

41

Since ξ_h is injective, f^{ss} is non-zero. Observe that $\mathcal{I}^h/\mathcal{I}^{h+1} = \mathrm{Sym}^h(\mathcal{I}/\mathcal{I}^2)$ and that $\iota^*(\mathcal{I}/\mathcal{I}^2) = \iota^*(\Omega^1_{\mathcal{S}})$ (cf. [Har77] II, 8.17). Notice that the Hecke operators act on

$$H^0(Z', \iota^*(\mathcal{I}^h/\mathcal{I}^{h+1} \otimes \mathbb{E}_\rho))$$

by definition of the $G(\mathbb{A}_f^p)$-action on $\mathcal{S}$ (cf. also the identification made below of this space of sections with a space of superspecial modular forms).

Let $\mathcal{X}$ denote the universal abelian scheme over $\mathcal{S}$, endowed with the principal $\mathcal{S}$-polarization $\lambda_{univ} : \mathcal{X} \to \hat{\mathcal{X}}$ and the action i_{univ} of the ring $\mathcal{O}_{k,(p)}$. By Prop. 2.3.4.2. in [Lan08], the Kodaira-Spencer map induces an isomorphism of $\mathcal{O}_{\mathcal{S}}$-sheaves:

$$\underline{KS} : \frac{\mathbb{E} \otimes_{\mathcal{O}_{\mathcal{S}}} \widehat{\mathbb{E}}}{J'} \to \Omega^1_{\mathcal{S}},$$

where $\widehat{\mathbb{E}} = 0^*\Omega^1_{\hat{\mathcal{X}}/\mathcal{S}}$, and:

$$J' = \begin{pmatrix} \lambda^*_{univ}(y) \otimes z - \lambda^*_{univ}(z) \otimes y \\ i_{univ}(b)^*(x) \otimes y - x \otimes \hat{i}_{univ}(b)^*(y) \end{pmatrix} : x \in \mathbb{E}; \ y, z \in \widehat{\mathbb{E}}; \ b \in \mathcal{O}_{k,(p)}.$$

(Here λ^*_{univ} denotes the pull-back morphism $\lambda^*_{univ} : \widehat{\mathbb{E}} \to \mathbb{E}$; $i_{univ}(b)^*$, resp. $\hat{i}_{univ}(b)^*$, denotes the endomorphism of $\mathbb{E}$, resp. $\widehat{\mathbb{E}}$, induced by $i_{univ}(b)$, resp. $\hat{i}_{univ}(b)$). Precomposing $\underline{KS}$ with $id \otimes \lambda^*_{univ}$ we get the isomorphism of sheaves:

$$\frac{\mathrm{Sym}^2 \mathbb{E}}{J} \to \Omega^1_{\mathcal{S}},$$

where $J = \left(i_{univ}(b)^*(x) \otimes y - x \otimes i_{univ}(b)^*(y) : x, y \in \mathbb{E}, \ b \in \mathcal{O}_{k,(p)} \right)$. Write

$$\mathrm{Sym}^2_\flat \mathbb{E} := \frac{\mathrm{Sym}^2 \mathbb{E}}{J}$$

and notice that while J is not preserved by the group GL_g (if $rs > 0$), it has nevertheless an action of $GL_r \times GL_s$, because of the determinant condition imposed in the definition of the moduli problem.

We have:

$$H^0(Z', \iota^*(\mathcal{I}^h/\mathcal{I}^{h+1} \otimes \mathbb{E}_\rho)) = H^0(Z', \iota^*((\mathrm{Sym}^h \mathrm{Sym}^2_\flat \mathbb{E}) \otimes \mathbb{E}_\rho)).$$

The group $GL_r \times GL_s$ acts on the sheaf $\iota^*(\mathrm{Sym}^h(\mathrm{Sym}^2_\flat \mathbb{E}) \otimes \mathbb{E}_\rho)$; as the space of superspecial modular forms is defined for representations τ of $G(U_r \times U_s) \subset GL_r \times GL_s$, we conclude that:

$$f^{ss} \in H^0(Z', \iota^*(\mathrm{Sym}^h(\mathrm{Sym}^2_\flat \mathbb{E}) \otimes \mathbb{E}_\rho)) = M^{ss}_{\mathrm{Sym}^h \mathrm{Sym}^2_\flat (\mathrm{std}) \otimes \mathrm{Res}\, \rho}(N; \overline{\mathbb{F}}_p),$$

where we are viewing $\mathrm{Sym}^h(\mathrm{Sym}^2_\flat(\mathrm{std})) \otimes \mathrm{Res}\, \rho$ as a representation of $G(U_r \times U_s)$ by restriction ($\mathrm{std} : GL_g \to GL_g$ is the standard representation of GL_g).

The maps r_h and ξ_h are Hecke equivariant; as it is shown in [Fak09], the Kodaira-Spencer map is also Hecke equivariant, modulo a rescaling on the Hecke operators acting on $\mathrm{Sym}^2_\flat(\mathbb{E})$ (this rescaling on the Hecke operators can be interpreted as a "Tate twist"). We deduce that, after performing the mentioned rescaling, the non-zero form $f^{ss} \in M^{ss}_{\mathrm{Sym}^h \mathrm{Sym}^2_\flat (\mathrm{std}) \otimes \mathrm{Res}\, \rho}(N; \overline{\mathbb{F}}_p)$ is an Hecke eigenform with the same eigenvalues as our original f.

On the other hand, let us assume now that we are given a non-zero eigenform $f^{ss} \in M^{ss}_{\rho'}(N; \overline{\mathbb{F}}_p)$ for some weight $\rho' : G(U_r \times U_s) \to GL_{m'}$ defined over $\mathbb{F}_p$. There is a rational $\mathbb{F}_p$-representation $\tilde{\rho} : GL_g \to GL_m$ whose restriction to $G(U_r \times U_s)$ contains ρ'. Indeed the algebraically induced representation $\rho'' := \mathrm{Ind}^{GL_g}_{G(U_r \times U_s)} \rho'$ contains (non-canonically) a finite dimensional $G(U_r \times U_s)$-invariant subspace τ that is $G(U_r \times U_s)$-isomorphic to ρ'; by local finiteness there is a finite dimensional GL_g-submodule $\tilde{\rho}$ of ρ'' containing τ as a $G(U_r \times U_s)$-submodule.

By Prop. 3.4.1, there is an integer $c > 0$ divisible by $p^2 - 1$ such that the map:

$$r : M_{\tilde{\rho} \otimes \det^c}(N; \overline{\mathbb{F}}_p) \to M^{ss}_{\mathrm{Res}(\tilde{\rho} \otimes \det^c)}(N; \overline{\mathbb{F}}_p) = M^{ss}_{\mathrm{Res}\, \tilde{\rho}}(N; \overline{\mathbb{F}}_p)$$

is surjective; since $M^{ss}_{\rho'}(N; \overline{\mathbb{F}}_p) \subseteq M^{ss}_{\mathrm{Res}\, \tilde{\rho}}(N; \overline{\mathbb{F}}_p)$ and since r is Hecke equivariant, we see that a system of Hecke eigenvalues occurring in $M^{ss}_{\rho'}(N; \overline{\mathbb{F}}_p)$ also occurs in $M_{\tilde{\rho} \otimes \det^c}(N; \overline{\mathbb{F}}_p)$.

We conclude that the system of Hecke eigenvalues arising from the spaces of modular forms $M_\rho(N; \overline{\mathbb{F}}_p)$ for varying $\rho : GL_g \to GL_m$, coincide with the systems of Hecke eigenvalues arising from the spaces $M^{ss}_{\rho'}(N; \overline{\mathbb{F}}_p)$ for varying $\rho' : G(U_r \times U_s) \to GL_{m'}$. The theorem now follows from Prop. 3.3.4. ∎

We presented the construction of the Hecke correspondence for PEL Shimura varieties

associated to some unitary groups. One obtains the result of [Ghi04a] (for $g > 1$) by forgetting about the action of the algebra with involution that appears in our computations; observe that for Siegel modular forms, the superspecial locus has an easier shape, as explained in Rem. 2.2.2.

Let $\mathcal{D}$ be a PEL datum of rank g with associated reductive $\mathbb{Q}$-group G of type A or C. If $\mathcal{D}$ has good reduction at a prime $p > 2$ and the associated Shimura variety $\mathcal{S}_{\bar{\mathbb{F}}_p}$ has a non-empty superspecial locus, then one obtains a result analogous to Th. 3.4.2 by using similar techniques. More precisely, let I be the algebraic $\mathbb{Q}$-group of automorphisms of a fixed triple $(A_0, i_0, \bar{\lambda}_0)$ defining a point in the superspecial locus of $\mathcal{S}_{\bar{\mathbb{F}}_p}$. The basicity of the p-divisible group of A_0 implies that I is an inner form of G such that $I(\mathbb{R})$ is compact modulo center, and $I_{\mathbb{Q}_v} \simeq G_{\mathbb{Q}_v}$ for every place v of $\mathbb{Q}$ different from p and ∞ ([RZ96], Th. 6.30). The group of $W(\bar{\mathbb{F}}_p)$-linear automorphisms of the homogeneously principally polarized Dieudonné module with $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-action associated to $(A_0, i_0, \bar{\lambda}_0)$ defines an integral model $\Im$ of $I_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$. We can then set (cf. Lemma 3.1.3; notice that $I_{\mathbb{Q}_p} \simeq J_{\mathbb{Q}_p}$ in the notation of 3.1.3, by Cor. 6.29 of [RZ96]):

$$U_p := \ker(\Im(\mathbb{Z}_p) \to \Im(\mathbb{F}_p)), \quad \bar{G} := \Im_{\mathbb{F}_p}.$$

Fix an open compact neat subgroup K^p of $G(\mathbb{A}_f^p)$. Then the Hecke eigensystems arising from $(\mathrm{mod}\, p)$ modular forms of PEL type associated to the group G, having level K^p and varying weight $\rho : GL_g \to GL_{m(\rho)}$ defined over $\mathbb{F}_p$ coincide with the Hecke eigensystems arising from $(\mathrm{mod}\, p)$ algebraic modular forms associated to I, having level $K^p \times U_p$ and varying weight $\rho' : \bar{G} \to GL_{m'(\rho')}$ defined over $\mathbb{F}_p$. (Here we are identifying $G(\mathbb{A}_f^p)$ and $I(\mathbb{A}_f^p)$).

CHAPTER 4

On the number of unitary Hecke eigensystems

We keep the assumptions and the notation introduced in the previous section: in particular, we work with unitary $(\bmod p)$ PEL modular forms of signature (r, s), with $p \neq 2$. We give an estimate of the number of $(\bmod p)$ Hecke eigensystems occurring in the spaces $M_\rho(N; \overline{\mathbb{F}}_p)$ for N fixed and varying ρ, extending a result of [Ghi04b].

Denote by $\mathcal{N} := \mathcal{N}(p; k, r, s; N)$ the number of Hecke eigensystems occurring in the totality of spaces $M_\rho(N; \overline{\mathbb{F}}_p)$ for ρ varying over the set of $\mathbb{F}_p$-rational representations of GL_g; by Th. 3.4.2 and Prop. 3.3.4, $\mathcal{N}$ is the number of distinct Hecke eigensystems occurring in the totality of spaces $M_\rho^{ss}(N; \overline{\mathbb{F}}_p)$ where ρ now runs over the finite set $\mathrm{Irr}(\bar{G})$ of isomorphism classes of irreducible finite-dimensional representations of $\bar{G} := G(U_r \times U_s)(\mathbb{F}_{p^2})$ over $\overline{\overline{\mathbb{F}}}_p$. If $\rho : \bar{G} \to GL(W_\rho)$ is any fixed element representing a class in $\mathrm{Irr}(\bar{G})$, we have:

$$M_\rho^{ss}(N; \overline{\mathbb{F}}_p) = \{f : Z'_{diff}(\overline{\mathbb{F}}_p) \to W_\rho : f([(A, i, \overline{\lambda}, \overline{\alpha}, \eta M)]) =$$

$$\rho(M)^{-1} f([(A, i, \overline{\lambda}, \overline{\alpha}, \eta)]), \text{ all } M \in \bar{G}, [(A, i, \overline{\lambda}, \overline{\alpha}, \eta)] \in Z'_{diff}(\overline{\mathbb{F}}_p)\},$$

so that, by definition of $Z'(\overline{\mathbb{F}}_p)$, we have $\dim_{\overline{\mathbb{F}}_p} M_\rho^{ss}(N; \overline{\mathbb{F}}_p) \leq \#Z'(\overline{\mathbb{F}}_p) \cdot \dim_{\overline{\mathbb{F}}_p} W_\rho$, and:

$$\mathcal{N} \leq \#Z'(\overline{\mathbb{F}}_p) \cdot \sum\nolimits_{[\rho] \in \mathrm{Irr}(\bar{G})} \dim_{\overline{\mathbb{F}}_p} W_\rho. \tag{4.1}$$

4.1 Estimate of the cardinality of the superspecial locus

In order to compute $\#Z'(\overline{\mathbb{F}}_p)$, one would like to have an explicit mass formula for principally polarized superspecial varieties of the PEL type considered here; lacking such an explicit formula, we can instead using what is known for Siegel varieties. Let us denote by $\mathcal{A}$ the Siegel moduli scheme over $\mathcal{O}_{k,(p)}$ classifying prime-to-p isogeny classes of tuples $(A, \overline{\lambda}, \overline{\overline{\alpha}})$,

where A is an abelian projective scheme of relative dimension g over some $S \in SCH_{\mathcal{O}_{k,(p)}}$, $\overline{\lambda}$ is a principal homogeneous polarization of A, and $\overline{\overline{\alpha}}$ is a full level N structure on $(A, \overline{\lambda})$. There is a natural mapping j from the moduli $\mathcal{O}_{k,(p)}$-scheme $\mathcal{S}$ associated to the PEL datum $\mathcal{D}_A$ that we fixed, to $\mathcal{A}$. More precisely, by fixing an isomorphism of $\mathbb{Q}$-vector spaces $V = k^g \simeq \mathbb{Q}^{2g}$ we obtain a monomorphism of $\mathbb{Q}$-groups $GU_g(k; r, s) \hookrightarrow GSp_{2g}(J) \simeq GSp_{2g}$, where J is some symplectic form on $\mathbb{Q}^{2g}$; then by definition, if S is a locally noetherian $\mathcal{O}_{k,(p)}$-scheme, j sends the class $[(A, i, \overline{\lambda}, \overline{\alpha})] \in \mathcal{S}(S)$ to the class $[(A, \overline{\lambda}, \overline{\overline{\alpha}})] \in \mathcal{A}(S)$, where $\overline{\overline{\alpha}}$ is the $U'(N)$ orbit of the symplectic isomorphism $\alpha : H_1(A, \mathbb{A}_f^p) \to V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$, with $U'(N) := Ker(GSp_{2g}(\hat{\mathbb{Z}}^p; J) \to GSp_{2g}(\hat{\mathbb{Z}}^p/N\hat{\mathbb{Z}}^p; J))$ (notice that $U(N) = U'(N) \cap GU_g(\mathcal{O}_k \otimes \hat{\mathbb{Z}}^p; r, s)$).

Since j is a closed embedding, it sends injectively the superspecial locus $Z'(\overline{\mathbb{F}}_p)$ of the unitary PEL variety $\mathcal{S} \otimes \overline{\mathbb{F}}_p$ - relative to our choice of $(A_0, i_0, \overline{\lambda}_0, \overline{\alpha}_0)$ - into the superspecial locus of $\mathcal{A} \otimes \overline{\mathbb{F}}_p$. The explicit mass formula for superspecial principally polarized abelian varieties due to Ekedahl ([Eke87]) and based on work of Hashimoto-Ibukiyama ([HI80]) gives:

$$\#Z'(\overline{\mathbb{F}}_p) \le C_g \cdot \#GSp_{2g}(\mathbb{Z}/N\mathbb{Z}) \cdot \prod_{i=1}^g (p^i + (-1)^i), \tag{4.2}$$

where the constant C_g is:

$$C_g := \frac{(-1)^{g(g+1)/2}}{2^g} \cdot \prod_{i=1}^g \zeta(1 - 2i) = \frac{1}{2^{2g}g!} \cdot \prod_{i=1}^g B_{2i}.$$

(Here ζ is the Riemann zeta function, and B_{2i} denotes the $2i$th Bernoulli number).

4.2   Estimate of the size of the irreducible representations of $\bar{G}$

All the representations we consider in this paragraph are finite dimensional over the appropriate field. The number of pairwise non-isomorphic irreducible representations of the finite group $\bar{G}$ over $\overline{\mathbb{F}}_p$ coincides with the number $k^p(\bar{G})$ of p-regular conjugacy classes of $\bar{G}$; a matrix element X of $\bar{G}$ is p-regular if and only if its minimal polynomial has only simple roots over $\overline{\mathbb{F}}_p$, that is to say if and only if X is semi-simple (over $\overline{\mathbb{F}}_p$).

The group $\bar{G}$ is the set of $\mathbb{F}_p$-points of the connected reductive algebraic group $\mathbb{G} := G(U_r \times U_s)$ defined over $\mathbb{F}_p$; one can compute the center Z and the derived subgroup $\mathbb{G}'$ of

$\mathbb{G}$ and find:

$$Z = Z^0 \simeq \begin{cases} G(U_1 \times U_1) & \text{if } rs \neq 0, \\ GU_1 & \text{if } rs = 0, \end{cases}$$

$$\mathbb{G}' = SU_r \times SU_s.$$

Since $\mathbb{G}'$ is connected, simply-connected and semi-simple with rank:

$$rk(\mathbb{G}') = \begin{cases} g - 2 & \text{if } rs \neq 0, \\ g - 1 & \text{if } rs = 0, \end{cases}$$

by applying Th. 3.7.6 of [Car85], we have:

$$k^p(\bar{G}) = \#Z^0(\mathbb{F}_p) \cdot p^{rk(\mathbb{G}')} = \begin{cases} p^{g-2} \cdot (p-1)(p+1)^2 & \text{if } rs \neq 0, \\ p^{g-1} \cdot (p-1)(p+1) & \text{if } rs = 0. \end{cases} \tag{4.3}$$

If $t \geq 2$, then $SU_t(\mathbb{F}_{p^2})$ is the set of $\mathbb{F}_p$-points of a simply connected group of type $^2A_{t-1}(p)$, and its order is $\#SU_t(\mathbb{F}_{p^2}) = p^{\frac{t(t-1)}{2}} \cdot \prod_{i=2}^t (p^i - (-1)^i)$ (cf. [Car85] 2.9; we set $SU_0(\mathbb{F}_{p^2}) = SU_1(\mathbb{F}_{p^2}) := \{1\}$). Using the exactness of the sequence $1 \to SU_t(\mathbb{F}_{p^2}) \to U_t(\mathbb{F}_{p^2}) \xrightarrow{\det} U_1(\mathbb{F}_{p^2}) \to 1$ for $t > 0$, one deduces that $\#U_t(\mathbb{F}_{p^2}) = \#SU_t(\mathbb{F}_{p^2}) \cdot (p+1)$. We conclude that for any choice of non-negative integers r and s such that $r + s = g$ we have:

$$\begin{aligned} \#G(U_r \times U_s)(\mathbb{F}_{p^2}) &= \#U_r(\mathbb{F}_{p^2}) \cdot \#U_s(\mathbb{F}_{p^2}) \cdot (p-1) \\ &= p^{\frac{r(r-1)+s(s-1)}{2}} \cdot \prod_{i=1}^r (p^i - (-1)^i) \cdot \\ &\quad \prod_{i=1}^s (p^i - (-1)^i) \cdot (p-1). \end{aligned}$$

In particular, a p-Sylow subgroup of $G(U_r \times U_s)(\mathbb{F}_{p^2})$ has order $p^{\frac{r(r-1)+s(s-1)}{2}}$. Since $\bar{G}$ is a group with a split (B, N)-pair ([Car85] 1.18), we deduce that if $\rho : \bar{G} \to GL(W_\rho)$ is an irreducible representation of $\bar{G}$ over $\bar{\mathbb{F}}_p$, then:

$$\dim_{\bar{\mathbb{F}}_p} W_\rho \leq p^{\frac{r(r-1)+s(s-1)}{2}}. \tag{4.4}$$

(The proof of this fact is contained in [Cur70]; cf. esp. Cor. 3.5 and 5.11). Putting together formulae (4.3) and (4.4) we obtain:

$$\sum_{[\rho] \in \mathrm{Irr}(\bar{G})} \dim_{\bar{\mathbb{F}}_p} W_\rho \leq \begin{cases} p^{\frac{r(r-1)+s(s-1)}{2}} \cdot p^{g-2}(p-1)(p+1)^2 & \text{if } rs \neq 0, \\ p^{\frac{g(g-1)}{2}} \cdot p^{g-1}(p-1)(p+1) & \text{if } rs = 0. \end{cases} \tag{4.5}$$

47

4.3 Upper bound for the number of Hecke eigensystems

Putting formulae (4.2) and (4.5) together into formula (4.1), we obtain:

Theorem 4.3.1. *Let p, k, r, s, N be fixed as above (in particular $r, s \geq 0$, $r+s = g$ and $p \neq 2$) and set $C_g := 2^{-2g}(g!)^{-1} \prod_{i=1}^{g} B_{2i}$. The number $\mathcal{N} := \mathcal{N}(p; k, r, s; N)$ of distinct $(\mathrm{mod}\, p)$ Hecke eigensystems occurring in the totality of spaces $M_\rho(N; \overline{\mathbb{F}}_p)$ for varying ρ satisfies the following inequality:*

$$
\mathcal{N} \leq C_g \cdot \#GSp_{2g}(\mathbb{Z}/N\mathbb{Z}) \cdot \prod_{i=1}^{g}(p^i + (-1)^i) \cdot
$$
$$
\begin{cases}
p^{\frac{r(r-1)+s(s-1)}{2}} \cdot p^{g-2}(p-1)(p+1)^2 & \text{if } rs \neq 0, \\
p^{\frac{g(g-1)}{2}} \cdot p^{g-1}(p-1)(p+1) & \text{if } rs = 0.
\end{cases}
$$

In particular, if we keep k, r, s, N fixed and let $p > 2$ vary:

$$
\mathcal{N} \in O(p^{g^2+g+1-rs}), \quad \text{for } p \to \infty.
$$

For an estimate of $\mathcal{N}$ in the case of Siegel modular forms, cf. [Ghi04b]; for elliptic modular forms, a conjectural mass formula for the asymptotic with respect to p of two-dimensional odd and irreducible Galois representations of $\mathbb{Q}$ can be found in [Cen09].

Cohomological Weight Shiftings for Automorphic Forms on Definite Quaternion Algebras

CHAPTER 5

Weight shiftings for $GL_2(\mathbb{F}_q)$-modules

5.1 Untwisted $GL_2(\mathbb{F}_q)$-modules

Fix a rational prime p, a positive integer g, and set $q = p^g$. Denote by $\mathbb{F}_q$ a finite field with q elements and fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$; denote by $\sigma \in \mathrm{Gal}\,(\mathbb{F}_q/\mathbb{F}_p)$ the arithmetic Frobenius element. Let $G = GL_2\,(\mathbb{F}_q)$ and let M be a representation of G over $\mathbb{F}_q$; for any $n \in \mathbb{Z}$, the Frobenius element σ^n induces a map $G \to G$ obtained by applying σ^n to each entry of the matrices in G: composing this map with the action of G on M, we give to the latter a new structure of G-module, that is denoted $M^{[n]}$ and called the nth Frobenius twist of M. If $f : M \to N$ is a G-homomorphism and $n \in \mathbb{Z}$, denote by $f^{[n]} : M^{[n]} \to N^{[n]}$ the map defined by $f^{[n]}(x) = f(x)$ for all $x \in M^{[n]}$: $f^{[n]}$ is a G-homomorphism.

Let M_1 denote the standard representation of G on $\mathbb{F}_q^2$ and, for any positive integer k, define $M_k = \mathrm{Sym}^k M_1$ to be the kth symmetric power of M_1. We identify M_k with the $\mathbb{F}_q$-vector space of homogeneous polynomials over $\mathbb{F}_q$ in two variables and of degree k, endowed with the action of G induced by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY.$$

We set M_0 to be the trivial representation of G. Denote by $\det : G \to \mathbb{F}_q^\times$ the determinant character of G, so that $\det^{[n]} = \det^{p^n}$.

Recall (cf. [Ste63], [Ste68] §13) that the irreducible representations of G over $\mathbb{F}_q$ are all

and only of the form:

$$\det{}^m \otimes_{\mathbb{F}_q} \bigotimes_{i=0}^{g-1} M_{k_i}^{[i]},$$

where $k_0, ..., k_{g-1}$ and m are integers such that $0 \leq k_i \leq p-1$ for $i = 0, ..., g-1$, $0 \leq m < q-1$, and all the tensor products are over $\mathbb{F}_q$. The above representations are pairwise non-isomorphic.

We denote by $K_0(G)$ the Grothendieck group of finitely generated $\mathbb{F}_q[G]$-modules: it can be identified with the free abelian group generated by the isomorphism classes of irreducible representations of G over $\mathbb{F}_q$ ([Ser77]). If M is an $\mathbb{F}_q[G]$-module, we denote by $[M]$ its class in $K_0(G)$ and set $e = [\det]$; if no confusion arises we also write M to denote $[M]$. Tensor product over $\mathbb{F}_q$ induces on $K_0(G)$ a structure of commutative ring with identity; we denote the product in $K_0(G)$ by $\cdot$ or by juxtaposition.

5.1.1 Identities in $K_0(G)$ (I)

We present some identities between virtual representations in $K_0(G)$ that we will need later.

Negative weights We extend the definition of $M_k \in K_0(G)$ for $k < 0$ in a way that is coherent with Brauer character computations, as suggested by Serre in [Ser01].

Let $\mathbf{G} = GL_2$ as an algebraic group over $\mathbb{F}_q$, and let $\mathbf{T} \subset \mathbf{G}$ be the maximal split torus of diagonal matrices. Identify the character group $X(\mathbf{T})$ of $\mathbf{T}$ with $\mathbb{Z}^2$ in the usual way, so that the roots associated to $(\mathbf{G}, \mathbf{T})$ are $(1, -1)$ and $(-1, 1)$; fix a choice of positive root $\alpha = (1, -1)$. The corresponding Borel subgroup $\mathbf{B}$ is the group of upper triangular matrices in $\mathbf{G}$; we denote by $\mathbf{B}^-$ the opposite Borel subgroup. For a fixed $\lambda \in X(\mathbf{T})$, let $\mathbf{M}_\lambda$ be the one dimensional left $\mathbf{B}^-$-module on which $\mathbf{B}^-$ acts (through $\mathbf{T}$) via the character λ. Denote by $\operatorname{ind}_{\mathbf{B}^-}^{\mathbf{G}} \mathbf{M}_\lambda$ the left $\mathbf{G}$-module given by algebraic induction from $\mathbf{B}^-$ to $\mathbf{G}$ of $\mathbf{M}_\lambda$. Define the following generalization of the dual Weyl module for λ (cf. [Jan03], II.5):

$$W(\lambda) = \sum_{i \geq 0} (-1)^i \cdot R^i \operatorname{ind}_{\mathbf{B}^-}^{\mathbf{G}} (\mathbf{M}_\lambda),$$

where $R^i \operatorname{ind}_{\mathbf{B}^-}^{\mathbf{G}} (\cdot)$ denote the ith right derived functor of $\operatorname{ind}_{\mathbf{B}^-}^{\mathbf{G}} (\cdot)$. $W(\lambda)$ is an element

of the Grothendieck group $K_0(\mathbf{G})$ of $\mathbf{G}$, because each $R^i \operatorname{ind}_{\mathbf{B}^-}^{\mathbf{G}}(\mathbf{M}_\lambda)$ is a finite dimensional $\mathbf{G}$-module, and $R^i \operatorname{ind}_{\mathbf{B}^-}^{\mathbf{G}}(\mathbf{M}_\lambda)$ is zero for $i > 1$ ([Jan03], II.4.2). For $\lambda_k = (k, 0) \in X(\mathbf{T})$ with k any integer we have:

$$R^i \operatorname{ind}_{\mathbf{B}^-}^{\mathbf{G}}(\mathbf{M}_{\lambda_k}) \simeq H^i(\mathbb{P}^1_{\mathbb{F}_q}, \mathcal{O}(k)).$$

If $k \geq 0$, $H^1(\mathbb{P}^1_{\mathbb{F}_q}, \mathcal{O}(k)) = 0$ so that $W(\lambda_k) = H^0(\mathbb{P}^1_{\mathbb{F}_q}, \mathcal{O}(k)) = \operatorname{Sym}^k \mathbb{F}_q^2$; if $k < 0$ we have $H^0(\mathbb{P}^1_{\mathbb{F}_q}, \mathcal{O}(k)) = 0$ and $W(\lambda_k) = -H^1(\mathbb{P}^1_{\mathbb{F}_q}, \mathcal{O}(k))$; the canonical perfect pairing of $\mathbf{G}$-modules:

$$H^0(\mathbb{P}^1_{\mathbb{F}_q}, \mathcal{O}(-k-2)) \times H^1(\mathbb{P}^1_{\mathbb{F}_q}, \mathcal{O}(k)) \to H^1(\mathbb{P}^1_{\mathbb{F}_q}, \mathcal{O}(-2)) \simeq \det{}^{-1} \otimes \mathbb{G}_{a, \mathbb{F}_q},$$

brings naturally to the following:

Definition 5.1.1. Let $k < 0$ be an integer. Define the element M_k of the Grothendieck group $K_0(G)$ of G over $\mathbb{F}_q$ by:

$$M_k = \begin{cases} 0 & \text{if } k = -1 \\ -e^{1+k} \cdot M_{-k-2} & \text{if } k \leq -2 \end{cases}.$$

Lemma 5.1.2. For any $k \in \mathbb{Z}$ we have in $K_0(G)$ the identity:

$$M_k + e^{1+k} \cdot M_{-k-2} = 0. \tag{$\Delta_{g,k}$}$$

An identity of Serre   Let us fix an embedding $\iota : \mathbb{F}_{q^2} \to M_2(\mathbb{F}_q)$ corresponding to a choice of $\mathbb{F}_q$-basis for the degree 2 extension of $\mathbb{F}_q$ inside $\overline{\mathbb{F}}_q$. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of the p-adic field $\mathbb{Q}_p$ and let us fix an isomorphism between $\overline{\mathbb{F}}_q$ and the residue field of the ring of integers $\overline{\mathbb{Z}}_p$ of $\overline{\mathbb{Q}}_p$; denoting by $\chi : \overline{\mathbb{F}}_q^\times \to \overline{\mathbb{Z}}_p^\times$ the corresponding Teichmüller character, the Brauer character $G_{reg} \to \overline{\mathbb{Q}}_p$ of the representations M_k $(k \geq 1)$ is given as follows:

$$\begin{pmatrix} a & \\ & a \end{pmatrix} \mapsto (k+1)\chi(a)^k, \qquad\qquad a \in \mathbb{F}_q^\times$$

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \mapsto \frac{\chi(a)^{k+1} - \chi(b)^{k+1}}{\chi(a) - \chi(b)}, \qquad a, b \in \mathbb{F}_q^\times, a \neq b$$

$$\iota(c) \mapsto \frac{\chi(c)^{q(k+1)} - \chi(c)^{k+1}}{\chi(c)^q - \chi(c)}, \qquad c \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times.$$

Using the above formulae, the following is proved in [Ser01]:

Lemma 5.1.3. For any $k \in \mathbb{Z}$ we have in $K_0(G)$ the identity:

$$M_k - e \cdot M_{k-(q+1)} = M_{k-(q-1)} - e \cdot M_{k-2q}. \qquad (\Sigma_{g,k})$$

Product formula It is a result of Glover that for any positive integers n, m there exists a short exact sequence of $\overline{\mathbb{F}}_q[SL_2(\overline{\mathbb{F}}_q)]$-modules of the form:

$$0 \to M_{n-1} \otimes_{\overline{\mathbb{F}}_q} M_{m-1} \xrightarrow{j} M_n \otimes_{\overline{\mathbb{F}}_q} M_m \xrightarrow{\pi} M_{n+m} \to 0,$$

where j is induced by the assignment $u \otimes v \mapsto uX \otimes vY - uY \otimes vX$ and π is induced by multiplication inside the algebra $\overline{\mathbb{F}}_q[X, Y]$. The following is an easy extension to GL_2 of Glover's result:

Lemma 5.1.4. For any $n, m \in \mathbb{Z}$ we have in $K_0(G)$ the identity:

$$M_n M_m = M_{n+m} + e M_{n-1} M_{m-1}. \qquad (\Pi_{g,n,m})$$

Proof. Let τ be the Brauer character of the virtual representation $M_n M_m - M_{n+m} - e M_{n-1} M_{m-1}$. Let $a, b \in \mathbb{F}_q^\times$ such that $a \neq b$; denote by $\tilde{x}$ the Teichmüller lift of $x \in \overline{\mathbb{F}}_q^\times$ taken via χ. We have:

$$\tau \begin{pmatrix} a & \\ & a \end{pmatrix} = (n+1)(m+1)\tilde{a}^{n+m} - (n+m+1)\tilde{a}^{n+m} +$$
$$-\tilde{a}^2 \cdot nm\tilde{a}^{(n-1)+(m-1)};$$

$$\tau \begin{pmatrix} a & \\ & b \end{pmatrix} = \frac{\tilde{a}^{n+1} - \tilde{b}^{n+1}}{\tilde{a} - \tilde{b}} \frac{\tilde{a}^{m+1} - \tilde{b}^{m+1}}{\tilde{a} - \tilde{b}} - \frac{\tilde{a}^{n+m+1} - \tilde{b}^{n+m+1}}{\tilde{a} - \tilde{b}} +$$
$$-\tilde{a}\tilde{b}\frac{(\tilde{a}^n - \tilde{b}^n)(\tilde{a}^m - \tilde{b}^m)}{(\tilde{a} - \tilde{b})^2}.$$

Both these expressions are trivially zero. If $c \in \mathbb{F}_{q^2}^\times \backslash \mathbb{F}_q^\times$ and $\iota : \mathbb{F}_{q^2} \to M_2(\mathbb{F}_q)$ is as above, then $\det \iota(c) = c^{1+q}$, so that:

$$\tau(\iota(c)) = \frac{\tilde{c}^{q(n+1)} - \tilde{c}^{n+1}}{\tilde{c}^q - \tilde{c}} \frac{\tilde{c}^{q(m+1)} - \tilde{c}^{m+1}}{\tilde{c}^q - \tilde{c}} +$$
$$-\frac{\tilde{c}^{q(n+m+1)} - \tilde{c}^{n+m+1}}{\tilde{c}^q - \tilde{c}} - \tilde{c}^{1+q} \frac{(\tilde{c}^{qn} - \tilde{c}^n)(\tilde{c}^{qm} - \tilde{c}^m)}{(\tilde{c}^q - \tilde{c})^2},$$

and this is also zero. As τ is identically zero on G^{reg}, $M_n M_m - M_{n+m} - eM_{n-1}M_{m-1}$ is the zero element of $K_0(G)$. ∎

We summarize the three identities obtained so far:

Proposition 5.1.5. Let $q = p^g$ $(g \geq 1)$ and let $k, n, m \in \mathbb{Z}$. The following identities hold in $K_0(G)$:

$$M_k = -e^{1+k} \cdot M_{-k-2} \qquad (\Delta_{g,k})$$

$$M_k - e \cdot M_{k-(q+1)} = M_{k-(q-1)} - e \cdot M_{k-2q} \qquad (\Sigma_{g,k})$$

$$M_n M_m = M_{n+m} + eM_{n-1}M_{m-1}. \qquad (\Pi_{g,k})$$

5.1.2 Intertwining operators for the periods $q+1$ and $q-1$

Recall that the irreducible complex representations of G (of dimension larger than one) that are not twists of the Steinberg representation are of two types: the principal series representations, having dimension $q+1$ and obtained by inducing to G characters of a Borel subgroup of G, and the cuspidal representations, having dimension $q-1$ and characterized by the property that they do not occur as a factor of a principal series.

The two periods $q+1$ and $q-1$ appear in the identity $(\Sigma_{g,k})$ and suggest the existence of intertwining operators that shift weights by $q+1$ and $q-1$ respectively; furthermore one expects these operators to give a bridge between the modular representations of G and the above mentioned characteristic zero representations of G.

The period $q+1$ Let $k > q$ be an integer and let $\Theta_q = XY^q - X^q Y \in \mathbb{F}_q[X,Y]$. Dickson proved that this polynomial is one of the two generators of the ring of $SL_2(\mathbb{F}_q)$-invariants

in the symmetric algebra $\mathrm{Sym}^* \mathbb{F}_q^2$, so we will call it the Dickson invariant. Let us denote by Θ_q also the G-equivariant map $\det \otimes M_{k-(q+1)} \to M_k$ given by multiplication by Θ_q.

Proposition 5.1.6. For $k > q$, there is an exact sequence of G-modules:

$$0 \to \det \otimes M_{k-(q+1)} \overset{\Theta_q}{\to} M_k \to \mathrm{Ind}_B^G \left(\eta^k \right) \to 0,$$

where B is the subgroup of G consisting of upper triangular matrices, and η is the character of B defined extending the character $\mathrm{diag}(a, b) \mapsto a$ of the standard maximal torus of G. Furthermore, for any integer $\lambda \geq 0$ there are isomorphisms of G-modules:

$$\frac{M_k}{\det \otimes M_{k-(q+1)}} \simeq \frac{M_{k+\lambda(q-1)}}{\det \otimes M_{k+\lambda(q-1)-(q+1)}},$$

where the inclusion $\det \otimes M_{k+\lambda(q-1)-(q+1)} \hookrightarrow M_{k+\lambda(q-1)}$ is induced by the multiplication by Θ_q.

Proof. The above result is standard; cf. [Red10], Prop. 2.7. ∎

The period $q - 1$ For the period $q - 1$, the starting point is the G-equivariant derivation $D : \mathbb{F}_q[X, Y] \to \mathbb{F}_q[X, Y]$ defined by Serre as:

$$D : \quad f(X, Y) \longmapsto X^q \frac{\partial f}{\partial X}(X, Y) + Y^q \frac{\partial f}{\partial Y}(X, Y).$$

This map defines by restriction an intertwining operator $M_k \to M_{k+(q-1)}$ for any $k \geq 0$, giving rise to a weight shifting by $q - 1$. The kernel of D is large, as shown by G. Savin:

Proposition 5.1.7. The kernel of the map $D : \mathbb{F}_q[X, Y] \to \mathbb{F}_q[X, Y]$ is given by $\ker D = \mathbb{F}_q[X^p, Y^p, \theta_q]$.

Proof. Let $A = \mathbb{F}_q[X^p, Y^p, \theta_q]$ and $B = \ker D$; notice that we have the inclusions of rings $\mathbb{F}_q[X^p, Y^p] \subseteq A \subseteq B \subseteq \mathbb{F}_q[X, Y]$. The polynomial $t^p - (X^{pq}Y^p - X^pY^{pq}) \in \mathbb{F}_q(X^p, Y^p)[t]$ is irreducible in $\mathbb{F}_q(X^p, Y^p)[t]$ since $X^{pq}Y^p - X^pY^{pq}$ does not have a pth-root in $\mathbb{F}_q(X^p, Y^p)$, so that we have $[Q(A) : \mathbb{F}_q(X^p, Y^p)] = p$, where we denote by $Q(R)$ the field of fractions of an integral domain R inside some extension of R.

Now observe that $Q(B)$ is properly contained inside $\mathbb{F}_q(X, Y)$: if not, we could write $X = \frac{f}{g}$ with $f, g \in B$, $g \neq 0$ and $1 = af + bg$ for some $a, b \in B$; this would imply $X = f \cdot (aX + b)$ so that $f \in B$ would be an associate of X in $\mathbb{F}_q[X, Y]$. Since $[\mathbb{F}_q(X, Y) : \mathbb{F}(X^p, Y^p)] = p^2$ we have therefore $Q(A) = Q(B)$. Notice that $\mathbb{F}_q[X, Y]/A$ is an integral extension, so that B/A is too.

The domain A is normal, since the corresponding variety has equation $X_1^q X_2 - X_1 X_2^q - X_3^p = 0$, and then it defines an hypersurface of $\mathbb{A}^3_{/\mathbb{F}_q}$ that is non-singular in codimension one. We conclude $A = B$. $\blacksquare$

We assume for the rest of this paragraph that p is an odd prime. If we restricted ourselves to weights $2 \leq k \leq p - 1$ we have the following exact sequence:

$$0 \to \det \otimes M_{k-2} \xrightarrow{\overline{\Theta}_q} \frac{M_{k+(q-1)}}{D(M_k)} \to \operatorname{coker} \overline{\Theta}_q \to 0, \tag{5.1}$$

where $\overline{\Theta}_q = \Theta_q(\operatorname{mod} D(M_k))$ is induced by the Dickson invariant.

The main result we can prove is the following:

Theorem 5.1.8. *Let $q \neq 2$, $2 \leq k \leq p - 1$ with $k \neq \frac{q+1}{2}$ and let us denote by $\Xi\left(\chi^k\right)$ the cuspidal $\overline{\mathbb{Q}}_p$-representation of G associated to the kth-power of the Teichmüller character χ. Let $\mathcal{C}$ be the Deligne-Lusztig variety of $SL_{2/\mathbb{F}_q}$. There exists a canonical $W(\mathbb{F}_q)$-integral model*

$$\widetilde{\Xi}\left(\chi^k\right) := H^1_{\operatorname{cris}}(\mathcal{C}_{/\mathbb{F}_q})_{-k}$$

of $\Xi\left(\chi^k\right)$, arising from the $(-k)$-eigenspace of the first crystalline cohomology group of $\mathcal{C}_{/\mathbb{F}_q}$, such that there is an isomorphism of $\mathbb{F}_q[G]$-modules:

$$\frac{M_{k+(q-1)}}{D(M_k)} \simeq \widetilde{\Xi}\left(\chi^k\right) \otimes_{W(\mathbb{F}_q)} \mathbb{F}_q.$$

(The $(-k)$-eigenspace of $H^1_{\operatorname{cris}}(\mathcal{C}_{/\mathbb{F}_q})$ is computed with respect to the natural action of $\mu := \ker(\operatorname{Nm}_{\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times})$ on $H^1_{\operatorname{cris}}(\mathcal{C}_{/\mathbb{F}_q})$.)

Proof. Let

$$U_2\left(\mathbb{F}_{q^2}\right) = \left\{g \in GL_2\left(\mathbb{F}_{q^2}\right) : gA\overline{g}^t = A\right\},$$

56

where $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We consider $\mathbb{F}_{q^2}^\times$ embedded in $U_2(\mathbb{F}_{q^2})$ via $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-q} \end{smallmatrix}\right)$, so that $U_2(\mathbb{F}_{q^2}) = \mathbb{F}_{q^2}^\times \cdot SL_2(\mathbb{F}_q)$. The group $U_2(\mathbb{F}_{q^2})$ acts on $\mathcal{C}_{/\mathbb{F}_{q^2}}$ via the embedding:

$$U_2(\mathbb{F}_{q^2}) \hookrightarrow GL_3(\mathbb{F}_{q^2}) : g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Fix a rational prime $l \neq p$ and field isomorphisms $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$ and $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$; fix embeddings $W(\mathbb{F}_q) \subset W(\mathbb{F}_{q^2}) \hookrightarrow \overline{\mathbb{Q}}_p \simeq \mathbb{C}$. By [HJ90], §2.10, we have isomorphisms of $U_2(\mathbb{F}_{q^2})$-modules:

$$H^1_{\mathrm{cris}}(\mathcal{C}_{/\mathbb{F}_{q^2}}) \otimes_{W(\mathbb{F}_{q^2})} \mathbb{C} = H^1_{\mathrm{dR}}(\mathcal{C}_{/W(\mathbb{F}_{q^2})}) \otimes_{W(\mathbb{F}_{q^2})} \mathbb{C} \simeq H^1(\mathcal{C}, \overline{\mathbb{Q}}_l).$$

If $1 \leq k \leq q$ we deduce $H^1_{\mathrm{dR}}(\mathcal{C}_{/W(\mathbb{F}_{q^2})})_{-k} \otimes_{W(\mathbb{F}_{q^2})} \mathbb{C} \simeq H^1(\mathcal{C}, \overline{\mathbb{Q}}_l)_{-k}$, hence also an isomorphism of $SL_2(\mathbb{F}_q)$-modules:

$$H^1_{\mathrm{dR}}(\mathcal{C}_{/W(\mathbb{F}_q)})_{-k} \otimes_{W(\mathbb{F}_q)} \mathbb{C} \simeq H^1(\mathcal{C}, \overline{\mathbb{Q}}_l)_{-k}.$$

Here the $(-k)$-eigenspace of $H^1_{\mathrm{dR}}(\mathcal{C}_{/W(\mathbb{F}_q)})$ is computed with respect to the action of $\mu \subset \mathbb{F}_{q^2}^\times \subset U_2(\mathbb{F}_{q^2})$ on $H^1_{\mathrm{dR}}(\mathcal{C}_{/W(\mathbb{F}_q)}) = H^1_{\mathrm{cris}}(\mathcal{C}_{/\mathbb{F}_q})$.

Since $H^1(\mathcal{C}, \overline{\mathbb{Q}}_l)_{-k}$ is the subspace of $H^1(\mathcal{C}, \overline{\mathbb{Q}}_l)$ on which μ acts via the character $\vartheta_{-k} : \mu \to \overline{\mathbb{Q}}_l^\times : t \mapsto t^{-k}$, by [Lus78], Example 2.20, the $\overline{\mathbb{Q}}_l$-representation of $SL_2(\mathbb{F}_q)$ afforded by $H^1(\mathcal{C}, \overline{\mathbb{Q}}_l)_{-k}$ is of the following type: if ϑ_{-k} is in general position (i.e. $\vartheta_{-k}^2 \neq 1$ or, equivalently, $k \neq (q+1)/2$), $H^1(\mathcal{C}, \overline{\mathbb{Q}}_l)_{-k}$ is an irreducible cuspidal representation of $SL_2(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_l$. If $k = (q+1)/2$, then $H^1(\mathcal{C}, \overline{\mathbb{Q}}_l)_{-k} = V \oplus V^*$, with V cuspidal.

If ϑ_{-k} is in general position, there is an indecomposable character $\varsigma : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times \simeq \overline{\mathbb{Q}}_l^\times$ for which there is an isomorphism of $SL_2(\mathbb{F}_q)$-modules:

$$H^1(\mathcal{C}, \overline{\mathbb{Q}}_l)_{-k} \simeq \mathrm{Res}^{GL_2(\mathbb{F}_q)}_{SL_2(\mathbb{F}_q)}(\Xi(\varsigma)).$$

Notice that ς is not unique and can be changed into ς^{-1} or into any other indecomposable character that equals ς on μ. If now we take any p-adic integral model of each side of the above isomorphism (e.g. we can take $H^1_{\mathrm{dR}}(\mathcal{C}_{/W(\mathbb{F}_q)})_{-k}$ for the étale cohomology group) and we reduce mod p, we find the $SL_2(\mathbb{F}_q)$-module isomorphism

$$H^1_{\mathrm{dR}}(\mathcal{C}_{/\mathbb{F}_q})^{ss}_{-k} \simeq \left(\widehat{\Xi}(\varsigma)\right)^{ss}.$$

57

If $2 \leq k \leq p-1$ the left-hand side is isomorphic to $\left(\frac{M_{k+(q-1)}}{D(M_k)}\right)^{ss}$ by Prop. 4.1 of [Red10], and hence to $\left(\overline{\Xi'\left(\chi^k\right)}\right)^{ss}$ for any choice of integral model $\Xi'\left(\chi^k\right)$ of $\Xi\left(\chi^k\right)$ (using a computation of Brauer characters). Therefore the reductions of the Brauer characters of $\overline{\hat{\Xi}\left(\varsigma\right)}$ and $\overline{\Xi'\left(\chi^k\right)}$ need to coincide.

Assume the embedding $\iota : \mathbb{F}_{q^2} \to M_2\left(\mathbb{F}_q\right)$ is given by setting

$$c = x + y\sqrt{\varepsilon} \mapsto \begin{pmatrix} x & y\varepsilon \\ y & x \end{pmatrix},$$

where $x, y \in \mathbb{F}_q$ and ε is a generator of $\mathbb{F}_q^{\times}$. If $\iota(c) \in SL_2\left(\mathbb{F}_q\right)$ we have $c \in \mu$; the formulae giving the Brauer characters of the cuspidal representations of $GL_2\left(\mathbb{F}_q\right)$ imply that, if $\varsigma_{|\mu} = \chi_{|\mu}^h$ $(0 \leq h \leq q)$, we have

$$\chi(c)^k + \chi(c)^{-k} = \chi(c)^h + \chi(c)^{-h}$$

for any $c \in \mu$, so that $\left(\chi(c)^{k+h} - 1\right)\left(\chi(c)^k - \chi(c)^h\right) = 0$. We conclude that

$$k \equiv \pm h (\operatorname{mod} q + 1)$$

and $\varsigma_{|\mu} = \chi_{|\mu}^{\pm k}$. We can assume without loss of generality $\varsigma = \chi^k$; this implies that the $SL_2\left(\mathbb{F}_q\right)$-action on $H^1\left(\mathcal{C}, \overline{\mathbb{Q}}_l\right)_{-k}$ extends to a $GL_2(\mathbb{F}_q)$-action giving an isomorphism

$$H^1\left(\mathcal{C}, \overline{\mathbb{Q}}_l\right)_{-k} \simeq \Xi\left(\chi^k\right).$$

If $\tilde{\Xi}\left(\chi^k\right)$ is the $W\left(\mathbb{F}_q\right)$-model of $\Xi\left(\chi^k\right)$ corresponding to $H^1_{\mathrm{dR}}(\mathcal{C}_{/W(\mathbb{F}_q)})_{-k}$ in the above isomorphism, we have $\overline{\tilde{\Xi}\left(\chi^k\right)} \simeq H^1_{\mathrm{dR}}(\mathcal{C}_{/\mathbb{F}_q})_{-k}$. In [Red10], §3.4 and §4.1, a canonical isomorphism

$$H^1_{\mathrm{dR}}(\mathcal{C}_{/\mathbb{F}_q})_{-k} \simeq \frac{M_{k+(q-1)}}{D\left(M_k\right)}$$

is constructed by identifying the exact sequence (5.1) with:

$$0 \quad \to \quad H^0(\mathcal{C}_{/\mathbb{F}_q}, \Omega^1_{\mathcal{C}})_{-k} \quad \to \quad H^1_{\mathrm{dR}}(\mathcal{C}_{/\mathbb{F}_q})_{-k} \quad \to \quad H^1(\mathcal{C}_{/\mathbb{F}_q}, \mathcal{O}_{\mathcal{C}})_{-k} \quad \to \quad 0.$$

This concludes the proof of the theorem. ∎

5.1.3 Determination of Jordan-Hölder constituents: the case $g = 1$

Assume g is any positive integer. For convenience, we give the following non standard definition:

Definition 5.1.9. Let $M \in K_0(G)$ be of the form $M = e^m \prod_{i=0}^{g-1} M_{k_i}^{[i]}$ where $k_0, ..., k_{g-1}$ and m are integers. We say that the Jordan-Hölder factors of M can be computed in the standard form (using $(\Delta_g), (\Sigma_g)$ and (Π_g)) if, by applying finitely many times the identities of Prop. 5.1.5, together with the identities $e^{q-1} = 1$ and $\sigma^g = 1$, we can write M as:

$$M = \sum_{j \in J} n_j \left(e^{m_j} \prod_{i=0}^{g-1} M_{k_i^{(j)}}^{[i]} \right),$$

where J is a finite set and for any $j \in J$ we have $n_j, m_j, k_0^{(j)}, ..., k_{g-1}^{(j)} \in \mathbb{Z}$ such that $n_j \neq 0$, $0 \leq m_j < q - 1$, $0 \leq k_0^{(j)}, ..., k_{g-1}^{(j)} \leq p - 1$ and, if $j, j' \in J$ with $j \neq j'$ then $(m_j, k_0^{(j)}, ..., k_{g-1}^{(j)}) \neq (m_{j'}, k_0^{(j')}, ..., k_{g-1}^{(j')})$. (Notice that the integers $n_j, m_j, k_0^{(j)}, ..., k_{g-1}^{(j)}$ are uniquely determined by M).

Similarly one defines the notion of computability in standard form for an element of $K_0(G)$ that is given as an algebraic sum of products of elements of the form $M = e^m \prod_{i=0}^{g-1} M_{k_i}^{[i]}$. Also, in an obvious way, one defines computability in standard form using any subset or superset of the identities $(\Delta), (\Sigma)$ and (Π) (together with the identities $e^{q-1} = 1$ and $\sigma^g = 1$).

Lemma 5.1.10. Let g be any positive integer and let $n, m \in \mathbb{Z}$ such that $n, m \geq 0$. By applying (Π_g) we obtain the following identity in $K_0(G)$:

$$M_n M_m = \sum_{i=0}^{\min\{n,m\}} e^i M_{n+m-2i}.$$

Proof. We induct on n. For $n = 0$ the statement is true; for $n \geq 0$ we have, assuming $m > 0$:

$$
\begin{aligned}
M_{n+1} M_m & = M_{n+m+1} + e M_n M_{m-1} = \\
& = M_{n+m+1} + \sum_{i=0}^{\min\{n,m-1\}} e^{i+1} M_{n+m-1-2i} = \\
& = M_{n+m+1} + \sum_{i=1}^{\min\{n+1,m\}} e^i M_{(n+1)+m-2i} = \\
& = \sum_{i=0}^{\min\{n+1,m\}} e^i M_{(n+1)+m-2i}. \quad \blacksquare
\end{aligned}
$$

59

Corollary 5.1.11. For any positive integer t and any integers $n_1, ..., n_t \geq 0$ we have:

$$\prod_{i=1}^{t} M_{n_i} = \sum_{\alpha \in A} e^{s_\alpha} M_{r_\alpha},$$

where A is a finite set and $s_\alpha, r_\alpha \geq 0$ for any $\alpha \in A$.

Proof. It follows from applying (Π_g) and inducting on t. ∎

The following proposition guarantees that, if $g = 1$, (Δ_1) and (Σ_1) are enough to compute explicitly the Jordan-Hölder factors of any of the modules M_k for $k \in \mathbb{Z}$.

Proposition 5.1.12. Let $g = 1$. For any $m, k \in \mathbb{Z}$, we can compute the Jordan-Hölder factors of $e^m M_k$ in the standard form, using (Δ_1) and (Σ_1). Furthermore, by using also (Π_1), we can compute the Jordan-Hölder factors in the standard form for any algebraic sum of products of $e^m M_k$'s.

Proof. The second assertion in the statement of the proposition follows from the first one, together with Lemma 5.1.10. To prove the first assertion, we can assume $m = 0$ and, using (Δ_1), we also suppose $k \geq 0$. Write $k = np + r$ where n is a non-negative integer and r is an integer such that $0 \leq r \leq p - 1$. We induct on n.

If $n = 0$, there is nothing to prove. Assume $n \geq 1$ is fixed and that we can compute the Jordan-Hölder factors of M_k in the standard form, using $(\Delta_1), (\Sigma_1)$ and (Π_1), for any k of the form $k = n'p + r'$ where $0 \leq n' \leq n-1$ and $0 \leq r' \leq p-1$. If $0 \leq r \leq p-1$ we have, applying (Σ_1), that $M_{np+r} = M_{(n-1)p+(r+1)} + e(M_{(n-1)p+(r-1)} - M_{(n-2)p+r})$. If $r \neq 0, p-1$ we are done by induction assumption. If $r = 0$, then $M_{np} = M_{(n-1)p+1} + e(M_{(n-2)p+(p-1)} - M_{(n-2)p})$ and we are done. If $r = p-1$, just notice that $M_{(n-1)p+p} = M_{np} = M_{(n-1)p+1} + e(M_{(n-2)p+(p-1)} - M_{(n-2)p})$. (When $n = 1$ one sometimes needs to apply (Δ_1) to canonically compute the constituents of the virtual representations appearing in these identities). ∎

5.1.4 Application to elliptic modular forms

In this section we present some weight shifting results for elliptic modular forms modulo p in terms of cohomology of groups. We assume $p > 3$; by a modular form mod p we mean the

reduction modulo p of a form in characteristic zero - as defined by Serre and Swinnerton-Dyer, unless otherwise specified. In this paragraph we assume $g = 1$.

Let $N \geq 5$ be a positive integer not divisible by p and denote by $M_k(N, \overline{\mathbb{F}}_p)$ the $\overline{\mathbb{F}}_p$-vector space of mod p modular forms for the group $\Gamma_1(N)$ having weight $k \geq 2$ and with coefficients in $\overline{\mathbb{F}}_p$; the Hecke algebra $\mathcal{H}_N$, generated over $\overline{\mathbb{F}}_p$ by the operators T_l for $l \neq p$, acts on this space. The q-expansion homomorphism is an injective map $M_k(N, \overline{\mathbb{F}}_p) \hookrightarrow \overline{\mathbb{F}}_p[[q]]$.

The theta operator $\Theta : M_k(N, \overline{\mathbb{F}}_p) \to M_{k+(p+1)}(N, \overline{\mathbb{F}}_p)$ is defined on q-expansion by the formula $\Theta(\sum_n a_n q^n) = \sum_n n a_n q^n$; it satisfies $\Theta T_l = l T_l \Theta$ for any prime $l \neq p$. Denote by E_{p-1} the normalized form of the classical characteristic zero Eisenstein series whose q-expansion is given by:

$$E_{p-1} = 1 - 2(p-1)/B_{p-1} \sum_n \sigma_{p-2}(n) q^n;$$

then $E_{p-1} \in M_{p-1}(1, \mathbb{Z}_{(p)})$ and $E_{p-1} \equiv 1 (\mathrm{mod}\, p\mathbb{Z}_{(p)}[[q]])$, as $2\zeta(2-p)^{-1} \equiv 0 (\mathrm{mod}\, p)$ by the Clausen-von Staudt theorem. Multiplication by the reduction mod p of E_{p-1} gives rise to a Hecke-equivariant map $M_k(N, \overline{\mathbb{F}}_p) \to M_{k+(p-1)}(N, \overline{\mathbb{F}}_p)$, that we refer to as the Hasse invariant.

In view of the Eichler-Shimura isomorphism, the study of Hecke eigensystems of mod p modular forms of weight $k \geq 2$ and level N leads to the study of the eigenvalues of the Hecke algebra $\mathcal{H}_N$ acting on the cohomology group $H^1(\Gamma_1(N), M_{k-2})$, where $\Gamma_1(N)$ acts on M_{k-2} via its reduction mod p, and the action of $\mathcal{H}_N$ comes from the G-action on M_{k-2} and it is defined as in [AS86b]. The weight shiftings realized on the spaces of modular forms by the theta operator and the Hasse invariant have cohomological counterparts. In [AS86b], Ash and Stevens identifies a group-theoretical analogue of the Θ-operator in the Hecke-equivariant map induced in cohomology by the Dickson invariant (cf. 5.1.2):

$$\Theta_{p,*} : H^1(\Gamma_1(N), \det \otimes M_{k-2}) \longrightarrow H^1(\Gamma_1(N), M_{k+p-1}).$$

Here the twisting by det on the left hand side is a manifestation of the fact that the Θ operator on spaces of modular forms is twist-Hecke-equivariant.

Edixhoven and Khare identifies in [EK03] a cohomological analogue of the Hasse invariant

in the case $k = 2$ by studying the degeneracy map

$$H^1\left(\Gamma_1(N), M_0\right)^{\oplus 2} \to H^1\left(\Gamma_1(N) \cap \Gamma_0(p), M_{p-1}\right).$$

The D-derivation defined in 5.1.2 can be used to produce weight shifting by $p - 1$ for $3 \leq k \leq p + 1$:

Theorem 5.1.13. *Let $\mathfrak{M}$ be a non-Eisenstein maximal ideal of the Hecke algebra $\mathcal{H}_N$.*

(1) If $k \geq 0$ and $H^1(\Gamma_1(N), M_k)_{\mathfrak{M}} \neq 0$, then also $H^1(\Gamma_1(N), M_{k+(p-1)})_{\mathfrak{M}} \neq 0$.

(2) If $0 \leq k \leq p - 1$, there is a Hecke-equivariant embedding

$$H^1(\Gamma_1(N), M_k)_{\mathfrak{M}} \hookrightarrow H^1(\Gamma_1(N), M_{k+(p-1)})_{\mathfrak{M}}.$$

   Proof. If $k \geq 0$ and $k \not\equiv 0 \pmod{p + 1}$, then $M_{k+(p-1)} - M_k$ is positive in $K_0\left(G\right)$, giving the first assertion. If $1 \leq k \leq p - 1$ we have the exact sequence of G-modules:

$$0 \to M_k \xrightarrow{D} M_{k+(p-1)} \to \operatorname{coker} D \to 0.$$

By passing to the long exact sequence in cohomology and localizing with respect to the non-Eisenstein maximal ideal $\mathfrak{M}$ we get the second statement for $1 \leq k \leq p - 1$ (cf. [Kha01]). If $k = 0$, the existence of a monic map $\alpha : H^1\left(\Gamma_1(N), M_0\right)_{\mathfrak{M}} \hookrightarrow H^1(\Gamma_1(N), M_{p-1})_{\mathfrak{M}}$ is the cited above result of Edixhoven and Khare ([EK03]).

   The existence of α also implies the first statement for $k \equiv 0 \pmod{p + 1}$: if $k = s(p + 1)$ for some $s \geq 0$, formula $(\Sigma_{g,k})$ gives the following identity in $K_0\left(G\right)$:

$$M_{s(p+1)+p-1} = e^s \cdot M_{p-1} + \left(M_{s(p+1)} - e^s \cdot M_0\right).$$

Notice that $M_{s(p+1)} - \det{}^s \cdot M_0 > 0$ because of the existence of the monic map $\Theta_p : \det \otimes M_0 \hookrightarrow M_{p+1}$. If $H^1(\Gamma_1(N), M_{s(p+1)})_{\mathfrak{M}} \neq 0$ then

$$H^1(\Gamma_1(N), \det{}^s \otimes M_0)_{\mathfrak{M}} \neq 0 \text{ or } H^1(\Gamma_1(N), M_{s(p+1)}/\det{}^s \otimes M_0)_{\mathfrak{M}} \neq 0.$$

In the first case, by applying α we deduce $H^1(\Gamma_1(N), \det{}^s \otimes M_{p-1})_{\mathfrak{M}} \neq 0$ and hence

$$H^1(\Gamma_1(N), M_{s(p+1)+p-1})_{\mathfrak{M}} \neq 0.$$

62

If it is $H^1(\Gamma_1(N), M_{s(p+1)}/\det^s \otimes M_0)_{\mathfrak{m}} \neq 0$, the same conclusion holds. $\blacksquare$

Notice that the above theorem cannot be deduced only by the existence of the map D, as for $k = 0$ the virtual representation $M_{p-1} - M_0$ is not positive in $K_0(G)$. A similar situation will occur later on when we will consider the more general case of Hilbert modular forms (cf. section 6.2).

5.2 Twisted $GL_2(\mathbb{F}_q)$-modules and intertwining operators for $g > 1$

We keep the notation of the previous section, so that p is a prime number, g a positive integer, and $q = p^g$; we denote by $\mathbb{F}_q$ a finite field with q elements and we fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$; we let $\sigma \in \mathrm{Gal}\,(\mathbb{F}_q/\mathbb{F}_p)$ be the arithmetic Frobenius element and $G = GL_2\,(\mathbb{F}_q)$. If $k \in \mathbb{Z}$, $M_k^{[i]}$ is its ith Frobenius twist of the virtual representation M_k, for any integer i.

5.2.1 Identities in $K_0(G)$ (II)

None of the identities in $K_0(G)$ appearing in Prop. 5.1.5 contains a Frobenius twist; this implies that, while $(\Delta_g), (\Sigma_g), (\Pi_g)$ are all we need to compute the Jordan-Hölder factors of products of virtual representations of the form $M_k\ (k \in \mathbb{Z})$ when $g = 1$ (Prop. 5.1.12), these same three families of identities are not enough to work out such a computation when $g > 1$. For example, when $g > 1$, the Jordan-Hölder factors of M_p are $\{M_1^{[1]}, eM_{p-2}\}$ and they cannot be found using (Σ_g).

Proposition 5.2.1. Let $g \geq 1$. For any $k \in \mathbb{Z}$ we have in $K_0(G)$ the identity:

$$M_k = M_{k-p}M_1^{[1]} - e^p M_{k-2p}. \qquad (\Phi_{g,k})$$

Proof. Fix an embedding $\iota : \mathbb{F}_{q^2} \to M_2\,(\mathbb{F}_q)$ and denote by $\tilde{x} \in \overline{\mathbb{Z}}_p$ the Teichmüller lift of $x \in \overline{\mathbb{F}}_q^{\times}$ taken via the Teichmüller character we previously fixed. Let τ be the Brauer character of the virtual representation $M_k - M_{k-p}M_1^{[1]} + e^p M_{k-2p}$. Let $a, b \in \mathbb{F}_q^{\times}$ such that $a \neq b$. We have:

$$\tau \begin{pmatrix} a & \\ & a \end{pmatrix} = (k+1)\tilde{a}^k - (k-p+1)\tilde{a}^{k-p} \cdot 2\tilde{a}^p +$$

$$+ \tilde{a}^{2p} \cdot (k - 2p + 1)\tilde{a}^{k-2p};$$

$$\tau \begin{pmatrix} a & \\ & b \end{pmatrix} = \frac{\tilde{a}^{k+1} - \tilde{b}^{k+1}}{\tilde{a} - \tilde{b}} - \frac{\tilde{a}^{k-p+1} - \tilde{b}^{k-p+1}}{\tilde{a} - \tilde{b}} \frac{\tilde{a}^{2p} - \tilde{b}^{2p}}{\tilde{a}^p - \tilde{b}^p} +$$

$$+ \tilde{a}^p \tilde{b}^p \frac{\tilde{a}^{k-2p+1} - \tilde{b}^{k-2p+1}}{\tilde{a} - \tilde{b}}.$$

64

Both these expressions are zero. If $c \in \mathbb{F}_{q^2}^{\times} \backslash \mathbb{F}_q^{\times}$ then $\det \iota(c) = c^{1+q}$; also notice that $\operatorname{tr}(\iota(c); M_1^{[1]}) = \operatorname{tr}(\iota(c)^{\sigma}; M_1) = \operatorname{tr} \iota(c)^{\sigma} = c^p + c^{pq}$, so that:

$$
\tau(\iota(c)) = \frac{\tilde{c}^{q(k+1)} - \tilde{c}^{k+1}}{\tilde{c}^q - \tilde{c}} - \frac{\tilde{c}^{q(k-p+1)} - \tilde{c}^{k-p+1}}{\tilde{c}^q - \tilde{c}}(c^p + c^{pq}) +
$$
$$
+ \tilde{c}^{(1+q)p} \frac{\tilde{c}^{q(k-2p+1)} - \tilde{c}^{k-2p+1}}{\tilde{c}^q - \tilde{c}},
$$

and this is also zero. As τ is identically zero on G^{reg}, $M_k - M_{k-p} M_1^{[1]} + e^p M_{k-2p}$ is the zero element of $K_0(G)$. ■

Corollary 5.2.2. Let $g \geq 1$; for any $k, h \in \mathbb{Z}$ the following identity holds in $K_0(G)$:

$$
M_k M_h^{[1]} - e^p M_{k-p} M_{h-1}^{[1]} = M_{k-p} M_{h+1}^{[1]} - e^p M_{k-2p} M_h^{[1]}. \tag{$\Phi'_{g,k,h}$}
$$

Proof. Multiplying $(\Phi_{g,k})$ by $M_h^{[1]}$ we obtain the identity

$$
M_k M_h^{[1]} = M_{k-p} (M_1 M_h)^{[1]} - e^p M_{k-2p} M_h^{[1]}.
$$

Applying $(\Pi_{g,1,h})$ and distributing the Frobenius action we deduce that the left hand side of this equation equals $M_{k-p} \left(M_{h+1}^{[1]} + e^p M_{h-1}^{[1]} \right) - e^p M_{k-2p} M_h^{[1]}$. ■

Corollary 5.2.3. Let $g \geq 1$. For any $k, h, i \in \mathbb{Z}$ we have in $K_0(G)$ the identity:

$$
M_k^{[i]} M_h^{[i+1]} - e^{p^{i+1}} M_{k-p}^{[i]} M_{h-1}^{[i+1]} = M_{k-p}^{[i]} M_{h+1}^{[i+1]} - e^{p^{i+1}} M_{k-2p}^{[i]} M_h^{[i+1]}.
$$

Proof. Just apply ith Frobenius twist to $(\Phi'_{g,k,h})$. ■

Remark 5.2.4. Notice the following:

1. By applying the product formula, one sees that (Φ_1) and (Σ_1) are equivalent.

2. Equation (Φ'_g) $(g > 1)$ has a structure similar to equation (Σ_1): the weight shiftings appearing in the latter are by $p + 1$ and $p - 1$ (corresponding respectively to the degree of the Dickson invariant and of Serre's derivation map); in equation (Φ'_g), the

65

weight shiftings occurring are by $(p, 1, 0, ..., 0)$ and $(p, -1, 0, ..., 0)$ - the commas separate the shifting constants for tensor factors corresponding to Frobenius twistings by $\sigma^0, \sigma^1, ..., \sigma^{g-1}$. In this sense we can think of (Φ'_g) as a generalization of (Σ_1) for $g > 1$.

3. The reason for which only three (possibly) non-zero terms appear in (Φ_g) instead of four - as one could have expected by looking at (Σ_1), is that by applying weight-shifting of $(p, 1, 0, ..., 0)$ to M_k we obtain $e^p M_{k-p} M^{[1]}_{-1}$ that is the zero module: this phenomenon cannot happen when $g = 1$.

4. The reason for which, when $g > 1$, we were expecting an identity in $K_0(G)$ involving weight shiftings by $(p, \pm 1, 0, ..., 0)$ (and cyclic permutations of this) resides in the existence of the partial Hasse invariants and theta operators acting on spaces of mod p Hilbert modular forms of genus g. Also, for good reasons we do not have weight shiftings by $(\pm 1, p, 0, ..., 0)$, as long as $g > 2$: cf. 6.1.1.

5.2.2 Determination of Jordan-Hölder constituents: the case $g > 1$

We know show that equations $(\Delta_g), (\Phi_g), (\Pi_g)$ are enough to compute the Jordan-Hölder constituents of products of virtual representations of the form $e^m \prod_{i=0}^{g-1} M^{[i]}_{k_i}$ $(m, k_0, ..., k_{g-1} \in \mathbb{Z})$.

Lemma 5.2.5. Let $g \geq 1$; for any $k \in \mathbb{Z}$, we can compute the Jordan-Hölder factors of M_k in the standard form, using $(\Delta_g), (\Phi_g), (\Pi_g)$.

Proof. By applying (Δ_g) if necessary we can assume $k \geq 0$. If $g = 1$, the lemma follows from the last remark and Prop. 5.1.5. For $g \geq 2$, write $k = np + r$ where $n, r \in \mathbb{Z}$ are such that $n \geq 0$ and $0 \leq r \leq p - 1$. We induct on n.

If $n = 0$, there is nothing to prove. Assume $n \geq 1$ is fixed and that we can compute the Jordan-Hölder factors of M_k in the standard form, using $(\Delta_g), (\Phi_g)$ and (Π_g), for any k of the form $k = n'p + r'$ where $0 \leq n' \leq n - 1$ and $0 \leq r' \leq p - 1$. We have $M_{np+r} = M_{(n-1)p+r} M^{[1]}_1 - e^p M_{(n-2)p+r}$ by (Φ_g); the Jordan-Hölder factors of $e^p M_{(n-2)p+r}$ can be computed in the standard form by induction (if $n = 1$ then $M_{(n-2)p+r} = -e^{1-p+r} M_{p-r-2}$ by

66

(Δ_g)). Also, by induction we have an algorithm that allows us to write $M_{(n-1)p+r} = \sum_{i \in I} J_i$ where I is a finite set and each J_i is of the form $e^m \prod_{i=0}^{g-1} M_{k_i}^{[i]}$ for some integers $m, k_0, ..., k_{g-1}$ such that $0 \leq m < q-1$, $0 \leq k_0, ..., k_{g-1} \leq p-1$. It is therefore enough to show that we can compute the factors of $\left(\prod_{i=0}^{g-1} M_{k_i}^{[i]} \right) M_1^{[1]}$ in standard form, where $0 \leq m < q-1$, $0 \leq k_0, ..., k_{g-1} \leq p-1$. The product formula gives:

$$\left(\prod_{i=0}^{g-1} M_{k_i}^{[i]} \right) M_1^{[1]} = \left(\prod_{\substack{i=0 \\ i \neq 1}}^{g-1} M_{k_i}^{[i]} \right) M_{k_1+1}^{[1]} + e^p \left(\prod_{\substack{i=0 \\ i \neq 1}}^{g-1} M_{k_i}^{[i]} \right) M_{k_1-1}^{[1]}.$$

If $k_1 \neq p-1$, each of the two summands is either a Jordan-Hölder factor in standard form, or it is zero. Otherwise we are left with the determination of the constituents of the first summand. If $g = 2$ the latter equals $M_{k_0} M_p^{[1]} = M_{k_0} M_1 + e^p M_{k_0} M_{p-2}^{[1]} = M_{k_0+1} + e M_{k_0-1} + e^p M_{k_0} M_{p-2}^{[1]}$ and this is not in standard form if and only if $k_0 = p-1$, in which case we can compute the constituents of $M_{k_0+1} = M_p$ in standard form by using (Φ_g): $M_p = M_1^{[1]} + e M_{p-2}$.

Assume now $g > 2$ and $k_1 = p-1$. We have, applying (Φ_g):

$$\left(\prod_{\substack{i=0 \\ i \neq 1}}^{g-1} M_{k_i}^{[i]} \right) M_p^{[1]} = \left(\prod_{\substack{i=0 \\ i \neq 1,2}}^{g-1} M_{k_i}^{[i]} \right) (M_{k_2} M_1)^{[2]} + e^p \left(\prod_{\substack{i=0 \\ i \neq 1}}^{g-1} M_{k_i}^{[i]} \right) M_{p-2}^{[1]}.$$

The second summand is already in standard form; for the first summand we have:

$$\left(\prod_{\substack{i=0 \\ i \neq 1,2}}^{g-1} M_{k_i}^{[i]} \right) (M_{k_2} M_1)^{[2]} = \left(\prod_{\substack{i=0 \\ i \neq 1,2}}^{g-1} M_{k_i}^{[i]} \right) M_{k_2+1}^{[2]} + e^{p^2} \left(\prod_{\substack{i=0 \\ i \neq 1,2}}^{g-1} M_{k_i}^{[i]} \right) M_{k_2-1}^{[2]}.$$

If $k_2 \neq p-1$, each of the two summands is either a Jordan-Hölder factor in standard form, or it is zero. Otherwise we are left with the determination of the constituents of the first summand. We proceed as before, distinguishing the cases $g = 3$ and $g > 3$. It is easily seen by induction that the algorithm produces the Jordan-Hölder factors of the virtual representations appearing in each step as long as $k_i \neq p-1$ for some $1 \leq i \leq g-1$. If $k_1 = ... = k_{g-1} = p-1$, we are left with the determination the Jordan-Hölder factors of $M_{k_0} M_p^{[g-1]} = M_{k_0}(M_1 + e^{p^{g-1}} M_{p-2}^{[g-1]})$. By the product formula, we just need to find the constituents of M_{k_0+1}: if $k_0 \neq p-1$ this is an irreducible representation; otherwise $M_p = M_1^{[1]} + e M_{p-2}$ and we are done. ∎

Corollary 5.2.6. Let $g \geq 1$. Then $(\Delta_g), (\Phi_g), (\Pi_g)$ imply (Σ_g).

Proof. By the previous lemma, we can compute the Jordan-Hölder factors of each summand appearing in (Σ_g) (in standard form). Since we know a priori that the Jordan-Hölder factors appearing in the right and left hand sides of (Σ_g) have to appear with the same multiplicities, (Σ_g) is a consequence of $(\Delta_g), (\Phi_g), (\Pi_g)$. ∎

We can finally prove:

Theorem 5.2.7. *Let $g \geq 1$. Using $(\Delta_g), (\Phi_g), (\Pi_g)$ we can compute the Jordan-Hölder factors in the standard form for any algebraic sum of products of virtual representations of the form $e^m \prod_{i=0}^{g-1} M_{k_i}^{[i]}$ $(m, k_0, ..., k_{g-1} \in \mathbb{Z})$.*

Proof. If $g = 1$, this is just Prop. 5.1.5. Assume $g \geq 2$; by applications of (Δ_g) and of Lemma 5.1.11, it is enough to prove that we can compute the Jordan-Hölder factors in the standard form for the representation $M = \bigotimes_{i=0}^{g-1} M_{k_i}^{[i]}$ $(k_0, ..., k_{g-1} \geq 0)$. We induct on $\dim_{\mathbb{F}_q} M$. If $\dim_{\mathbb{F}_q} M = 1$, we are done, otherwise we distinguish two cases.

<u>*Case 1*</u>: There is some i, $0 \leq i \leq g - 1$, such that M_{k_i} is reducible.

By applying an appropriate Frobenius twist, we can assume without loss of generality that M_{k_0} is reducible. By the previous lemma, we can compute the Jordan-Hölder factors of M_{k_0} in the standard form, say $M_{k_0} = \sum_{h \in I} J_h$ in $K_0(G)$, where I is a finite set with at least two elements and each J_h is a non-zero composition factor of M_{k_0}, written in standard form. It is then enough to compute in standard form the constituents of $J_h \prod_{i=1}^{g-1} M_{k_i}^{[i]}$ for each $h \in I$. Fix an element $h \in I$; up to twisting by a power of e we can assume $J_h = \prod_{i=0}^{g-1} M_{r_i}^{[i]}$ where $0 \leq r_0, ..., r_{g-1} \leq p - 1$, so that an application of Lemma 5.1.10 gives:

$$
\begin{aligned}
J_h \prod_{i=1}^{g-1} M_{k_i}^{[i]} &= M_{r_0} \prod_{i=1}^{g-1} (M_{r_i} M_{k_i})^{[i]} = \\
&= M_{r_0} \prod_{i=1}^{g-1} \left(\sum_{j=0}^{\min\{r_i, k_i\}} e^{jp^i} M_{r_i+k_i-2j}^{[i]} \right) = \\
&= \sum_{\substack{j_i = 0 \\ (1 \leq i \leq g-1)}}^{\min\{r_i, k_i\}} e^{s(j_1, ..., j_{g-1})} M_{r_0} M_{r_1+k_1-2j_1}^{[1]} ... M_{r_{g-1}+k_{g-1}-2j_{g-1}}^{[g-1]},
\end{aligned}
$$

where $s(j_1, ..., j_{g-1}) \in \mathbb{Z}$ and the last summation is over the $g - 1$ indices $j_1, ..., j_{g-1}$. Since

$$
\dim_{\mathbb{F}_q} \left(M_{r_0} \otimes M_{r_1+k_1-2j_1}^{[1]} \otimes ... \otimes M_{r_{g-1}+k_{g-1}-2j_{g-1}}^{[g-1]} \right) < \dim_{\mathbb{F}_q} M
$$

68

for any value of $j_1, ..., j_{g-1}$, by induction assumption we can compute the Jordan-Hölder constituents of $J_h \prod_{i=1}^{g-1} M_{k_i}^{[i]}$ in the standard form.

Case 2: For any i, $0 \le i \le g-1$, the representation M_{k_i} is irreducible.

By the previous lemma, we can assume - up to twistings by powers of det - that we have written $M_{k_i}^{[i]} = \prod_{j=0}^{g-1} M_{r_j^{(i)}}^{[j]}$ for any $0 \le i \le g-1$, where $0 \le r_0^{(i)}, ..., r_{g-1}^{(i)} \le p-1$. Then:

$$M = \prod_{i=0}^{g-1} M_{k_i}^{[i]} = \prod_{j=0}^{g-1} \left(\prod_{i=0}^{g-1} M_{r_j^{(i)}} \right)^{[j]}. \tag{5.2}$$

Applying the product formula (cf. Cor. 5.1.11), we can write:

$$\left(\prod_{i=0}^{g-1} M_{r_j^{(i)}} \right)^{[j]} = \sum_{\alpha_j \in A_j} e^{s_{\alpha_j}} M_{r_{\alpha_j}}^{[j]}, \tag{5.3}$$

where, for any $0 \le j \le g-1$, A_j is a non-empty finite set and $s_{\alpha_j}, r_{\alpha_j} \ge 0$ for $\alpha_j \in A_j$. Combining (5.2) and (5.3) we obtain:

$$M = \sum_{\substack{\alpha_j \in A_j \\ (0 \le j \le g-1)}} e^{s(\alpha_0, ..., \alpha_{g-1})} M_{r_{\alpha_0}} M_{r_{\alpha_1}}^{[1]} ... M_{r_{\alpha_{g-1}}}^{[g-1]}, \tag{5.4}$$

where $s(\alpha_0, ..., \alpha_{g-1}) \in \mathbb{Z}$ and the summation is over the g-tuples $(\alpha_0, ..., \alpha_{g-1}) \in A_0 \times ... \times A_{g-1}$. If each of the sets $A_0, ..., A_{g-1}$ contains exactly one element, then for any $0 \le j \le g-1$, at most one element in $\{r_j^{(0)}, ..., r_j^{(g-1)}\}$ is positive. Indeed, if this were not the case, there would be some j such that $r_j^{(a)}, r_j^{(b)} > 0$ for some a, b with $0 \le a < b \le g-1$; then by only applying the product formula we would obtain:

$$\left(\prod_{i=0}^{g-1} M_{r_j^{(i)}} \right)^{[j]} = \left[\left(\prod_{\substack{i=0 \\ i \ne a,b}}^{g-1} M_{r_j^{(i)}} \right) \left(M_{r_j^{(a)} + r_j^{(b)}} + e M_{r_j^{(a)} - 1} M_{r_j^{(b)} - 1} \right) \right]^{[j]}.$$

Since $r_j^{(a)} - 1, r_j^{(b)} - 1 \ge 0$, the left hand side above contains at least two non-zero summand, contradicting the fact that by only applying the product formula we could also write $\left(\prod_{i=0}^{g-1} M_{r_j^{(i)}} \right)^{[j]} = e^{s_j} M_{r_j}^{[j]}$ for some integers s_j, r_j. We conclude that if each of the sets $A_0, ..., A_{g-1}$ contains exactly one element, then M is irreducible and (5.2) is the standard Jordan-Hölder form of M.

If there is $0 \le j \le g-1$ such that A_j has at least two elements, then in (5.4) at least tow non-zero terms appear, so that each of the summand of (5.4) has dimension strictly less than $\dim_{\mathbb{F}_q} M$, and by induction we are done. $\blacksquare$

5.2.3 Families of intertwining operators for $g > 1$

For $g = 1$, one has available two intertwining operators acting on $\mathbb{F}_p[G]$-modules and shifting weights by $p \pm 1$, namely the Dickson invariant Θ_p and the derivation map D (cf. 5.1.2). For $g > 1$, equation (Φ_g) and the existence of partial Hasse invariants and theta operators acting on spaces of mod p Hilbert modular forms (cf. [AG05]) suggest that there should be other intertwining operators between modular representations of G, generalizing Θ_q and D. In this section we will construct such operators.

Unless otherwise specified, we will always assume $g > 1$, and we will consider all the tensor product over $\mathbb{F}_q$ $(q = p^g)$.

Generalized Dickson invariants

Definition 5.2.8. For any integer β such that $1 \le \beta \le g - 1$, the (non-twisted) generalized βth Dickson operator is the element

$$\Theta_\beta = X \otimes Y^{p^{g-\beta}} - Y \otimes X^{p^{g-\beta}}$$

of the G-module $M_1 \otimes M_{p^{g-\beta}}^{[\beta]}$.

For integers α, β such that $0 \le \alpha \le g - 1$ and $1 \le \beta \le g - 1$, the α-twisted generalized βth Dickson operator is the element

$$\Theta_\beta^{[\alpha]} = X \otimes Y^{p^{g-\beta}} - Y \otimes X^{p^{g-\beta}}$$

of the G-module $M_1^{[\alpha]} \otimes M_{p^{g-\beta}}^{[\alpha+\beta]}$.

Lemma 5.2.9. Let k, h be two non-negative integers and let α, β be two integers such that $0 \le \alpha \le g - 1$ and $1 \le \beta \le g - 1$. Multiplication by $\Theta_\beta^{[\alpha]}$ in the $\mathbb{F}_q[G]$-algebra $\mathbb{F}_q[X, Y]^{[\alpha]} \otimes \mathbb{F}_q[X, Y]^{[\alpha+\beta]}$ induces an injective G-homomorphism:

$$\Theta_\beta^{[\alpha]} : \det^{p^\alpha} \otimes M_k^{[\alpha]} \otimes M_h^{[\alpha+\beta]} \rightarrow M_{k+1}^{[\alpha]} \otimes M_{h+p^{g-\beta}}^{[\alpha+\beta]}.$$

Proof. We can assume $\alpha = 0$. To prove G-equivariance of the map Θ_β, it is enough to

70

show that $\gamma\Theta_\beta = \det\gamma \cdot \Theta_\beta$ for all $\gamma \in G$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$; then $\gamma\Theta_\beta$ equals:

$$(aX + cY) \otimes (b^{p^\beta}X + d^{p^\beta}Y)^{p^{g-\beta}} - (bX + dY) \otimes (a^{p^\beta}X + c^{p^\beta}Y)^{p^{g-\beta}}$$

$$= (aX + cY) \otimes (b^q X^{p^{g-\beta}} + d^q Y^{p^{g-\beta}}) - (bX + dY) \otimes (a^q X^{p^{g-\beta}} + c^q Y^{p^{g-\beta}})$$

$$= (aX + cY) \otimes (bX^{p^{g-\beta}} + dY^{p^{g-\beta}}) - (bX + dY) \otimes (aX^{p^{g-\beta}} + cY^{p^{g-\beta}})$$

$$= adX \otimes Y^{p^{g-\beta}} + bcY \otimes X^{p^{g-\beta}} - bcX \otimes Y^{p^{g-\beta}} - adY \otimes X^{p^{g-\beta}}$$

$$= \det\gamma \cdot \left(X \otimes Y^{p^{g-\beta}} - Y \otimes X^{p^{g-\beta}} \right)$$

$$= \det\gamma \cdot \Theta_\beta.$$

To show injectivity of Θ_β, notice that there is an isomorphism of $\mathbb{F}_q[G]$-algebras $\mathbb{F}_q[X,Y] \otimes \mathbb{F}_q[X,Y]^{[\beta]} \simeq \mathbb{F}_q[Z,W,T^{p^\beta},U^{p^\beta}]$ obtained by sending the ordered tuple $(X \otimes 1, Y \otimes 1, 1 \otimes X, 1 \otimes Y)$ into the ordered tuple $(Z,W,T^{p^\beta},U^{p^\beta})$, were we are letting G acts on Z,W,T,U as follows: for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$,

$$\gamma Z = aZ + cW,$$

$$\gamma W = bZ + dW,$$

$$\gamma T = aT + cU,$$

$$\gamma U = bT + dU.$$

Under the above identification, the map Θ_β corresponds to multiplication by $ZU^q - WT^q$ on $\mathbb{F}_q[Z,W,T^{p^\beta},U^{p^\beta}]$, and it is therefore injective. $\blacksquare$

In addition to the above operators, the classical Dickson invariant also gives rise to an intertwining map:

Proposition 5.2.10. Let k be a non-negative integer and let α be an integer such that $0 \leq \alpha \leq g - 1$. Let $\Theta^{[\alpha]} = XY^q - YX^q$ be the classical Dickson invariant, viewed as an element of $M_{q+1}^{[\alpha]}$. Multiplication by $\Theta^{[\alpha]}$ in the $\mathbb{F}_q[G]$-algebra $\mathbb{F}_q[X,Y]^{[\alpha]}$ induces an injective G-homomorphism:

$$\Theta^{[\alpha]} : \det{}^{p^\alpha} \otimes M_k^{[\alpha]} \longrightarrow M_{k+(q+1)}^{[\alpha]}.$$

Proof. This follows from section 5.1.2. ∎

Notice that the operators $\Theta_\beta^{[\alpha]}$ and $\Theta^{[\alpha]}$ ($0 \leq \alpha \leq g-1$, $1 \leq \beta \leq g-1$) pairwise commute, as it follows by seeing them as multiplication by polynomials in some polynomial algebra over $\mathbb{F}_q$ (cf. the end of the proof of Lemma 5.2.9).

Remark 5.2.11. Let us fix a convention that will make the notation easier in the sequel. For non-negative integers $k_0, ..., k_{g-1}$, the G-module $M_{k_0} \otimes M_{k_1}^{[1]} \otimes ... \otimes M_{k_{g-1}}^{[g-1]}$ will be identified with the G-module obtained by permuting in any possible way the tensor factors. Also, for integers α, β and any G-module M, the notation $M^{[\alpha+\beta]}$ will denote the γth Frobenius twist of M, where γ is the smallest non-negative integer such that $\gamma \equiv \alpha + \beta \pmod{g}$.

We can summarize the above results as follows:

Theorem 5.2.12. *Let us fix non-negative integers $k_0, ..., k_{g-1}$. For any integers α, β subject to the constraints $0 \leq \alpha \leq g-1$ and $1 \leq \beta \leq g-1$, there are pairwise commuting injective G-intertwining operators as follows:*

$$\Theta_\beta^{[\alpha]} : \det^{p^\alpha} \otimes \bigotimes_i M_{k_i}^{[i]} \longrightarrow \left(\bigotimes_{i \neq \alpha, \alpha+\beta} M_{k_i}^{[i]} \right) \otimes M_{k_\alpha+1}^{[\alpha]} \otimes M_{k_{\alpha+\beta}+p^{g-\beta}}^{[\alpha+\beta]};$$

$$\Theta^{[\alpha]} : \det^{p^\alpha} \otimes \bigotimes_i M_{k_i}^{[i]} \longrightarrow \left(\bigotimes_{i \neq \alpha} M_{k_i}^{[i]} \right) \otimes M_{k_\alpha+(q+1)}^{[\alpha]},$$

where the tensor products indices run over the integers i such that $0 \leq i \leq g-1$, unless otherwise specified.

Remark 5.2.13. The operators $\Theta_{g-1}^{[\alpha]}$ for $0 \leq \alpha \leq g-1$ give, under suitable assumptions, cohomological analogues of the theta operators defined in [AG05] in the context of Hilbert modular forms. We do not know of any geometric interpretation of the other generalized Dickson operators.

We can picture the weight shiftings allowed by the $g(g-1) + g = g^2$ generalized Dickson operators with the following self-explanatory tables:

Θ_1	$(1, p^{g-1}, 0, 0, ..., 0, 0)$
$\Theta_1^{[1]}$	$(0, 1, p^{g-1}, 0, ..., 0, 0)$
$\Theta_1^{[2]}$	$(0, 0, 1, p^{g-1}, ..., 0, 0)$
...	...
$\Theta_1^{[g-2]}$	$(0, 0, 0, 0, ..., 1, p^{g-1})$
$\Theta_1^{[g-1]}$	$(p^{g-1}, 0, 0, 0, ..., 0, 1)$

Θ_2	$(1, 0, p^{g-2}, 0, ..., 0, 0)$
$\Theta_2^{[1]}$	$(0, 1, 0, p^{g-2}, ..., 0, 0)$
$\Theta_2^{[2]}$	$(0, 0, 1, 0, ..., 0, 0)$
...	...
$\Theta_2^{[g-2]}$	$(p^{g-2}, 0, 0, 0, ..., 1, 0)$
$\Theta_2^{[g-1]}$	$(0, p^{g-2}, 0, 0, ..., 0, 1)$

...

Θ_{g-1}	$(1, 0, 0, 0, ..., 0, p)$
$\Theta_{g-1}^{[1]}$	$(p, 1, 0, 0, ..., 0, 0)$
$\Theta_{g-1}^{[2]}$	$(0, p, 1, 0, ..., 0, 0)$
...	...
$\Theta_{g-1}^{[g-2]}$	$(0, 0, 0, 0, ..., 1, 0)$
$\Theta_{g-1}^{[g-1]}$	$(0, 0, 0, 0, ..., p, 1)$

Θ	$(q + 1, 0, 0, ..., 0, 0)$
$\Theta^{[1]}$	$(0, q + 1, 0, ..., 0, 0)$
$\Theta^{[2]}$	$(0, 0, q + 1, ..., 0, 0)$
...	...
$\Theta^{[g-2]}$	$(0, 0, 0, ..., q + 1, 0)$
$\Theta^{[g-1]}$	$(0, 0, 0, ..., 0, q + 1)$

For example, if $g = 2$ the generalized Dickson operators give all and only the weight shiftings of the form:

$$a_1(1, p) + a_2(p, 1) + a_3(0, p^2 + 1) + a_4(p^2 + 1, 0),$$

for any non-negative integers a_1, a_2, a_3, a_4. For $g > 2$ a new phenomenon occurs, as the operators do not allow weight shiftings of the form:

$$(1, p, 0, ..., 0, 0), (0, 1, p, ..., 0, 0), ..., (0, 0, 0, ..., 1, p), (p, 0, 0, ..., 0, 1).$$

This happens not because of limitations intrinsic to our intertwining maps, but because of the structure of G-modules:

Proposition 5.2.14. Assume $g > 2$ and let k, h be integers such that $0 \leq k, h \leq p - 1$. For any integer α such that $0 \leq \alpha \leq g - 1$ and any integer m, there are no G-module morphisms $\det^m \otimes M_k^{[\alpha]} \otimes M_h^{[\alpha+1]} \rightarrow M_{k+1}^{[\alpha]} \otimes M_{h+p}^{[\alpha+1]}$.

73

Proof. It is enough to prove the non existence of morphisms for $\alpha = 0$. Using (Φ_g) and (Δ_g) we have, in $K_0(G)$:

$$M_{h+p}^{[1]} = M_h^{[1]} M_1^{[2]} + e^{p(h+1)} M_{p-h-2}^{[1]}.$$

If $k \neq p - 1$, as $g > 2$, we deduce that the Jordan-Hölder factors of $M_{k+1} \otimes M_{h+p}^{[1]}$ are $M_{k+1} \otimes M_h^{[1]} \otimes M_1^{[2]}$ and $\det^{p(h+1)} \otimes M_{k+1} \otimes M_{p-h-2}^{[1]}$, unless $h = p - 1$, in which case only the first factor occurs. None of these factors coincides with $\det^m \otimes M_k \otimes M_h^{[1]}$.

If $k = p - 1$, write $M_p = M_1^{[1]} + eM_{p-2}$ in $K_0(G)$. Applying (Π_g) we obtain:

$$
\begin{aligned}
M_p M_{h+p}^{[1]} &= \left(M_1^{[1]} + eM_{p-2} \right) \left(M_h^{[1]} M_1^{[2]} + e^{p(h+1)} M_{p-h-2}^{[1]} \right) \\
&= M_{h+1}^{[1]} M_1^{[2]} + e^p M_{h-1}^{[1]} M_1^{[2]} + e^{p(h+1)} M_{p-h-1}^{[1]} + e^{p(h+2)} M_{p-h-3}^{[1]} \\
&\quad + eM_{p-2} M_h^{[1]} M_1^{[2]} + e^{p(h+1)+1} M_{p-2} M_{p-h-2}^{[1]}.
\end{aligned}
$$

If $h \neq p - 1$, the above formula shows that none of the Jordan-Hölder factors of $M_p \otimes M_{h+p}^{[1]}$ equals $\det^m \otimes M_{p-1} \otimes M_h^{[1]}$. If $h = p - 1$, we have:

$$
\begin{aligned}
M_p M_{2p-1}^{[1]} &= M_1^{[2]} M_1^{[2]} + e^p M_{p-2}^{[1]} M_1^{[2]} + e^p M_{p-2}^{[1]} M_1^{[2]} + eM_{p-2} M_{p-1}^{[1]} M_1^{[2]} \\
&= M_2^{[2]} + e^{p^2} + 2e^p M_{p-2}^{[1]} M_1^{[2]} + eM_{p-2} M_{p-1}^{[1]} M_1^{[2]},
\end{aligned}
$$

and $\det^m \otimes M_{p-1} \otimes M_{p-1}^{[1]}$ is not a constituent of $M_p \otimes M_{2p-1}^{[1]}$ if $p \neq 2$. If $p = 2$, decomposing $M_2^{[2]}$ we get to the same conclusion. $\blacksquare$

We conclude this section by noticing the following consequence of Prop. 5.1.6:

Proposition 5.2.15. Let us fix non-negative integers $k_0, ..., k_{g-1}$. For any integer α such that $0 \leq \alpha \leq g - 1$ consider the G-map $\Theta^{[\alpha]} : \det^{p^\alpha} \otimes \bigotimes_i M_{k_i}^{[i]} \to \left(\bigotimes_{i \neq \alpha} M_{k_i}^{[i]} \right) \otimes M_{k_\alpha + (q+1)}^{[\alpha]}$. We have:

$$\operatorname{coker} \Theta^{[\alpha]} \simeq \left(\bigotimes_{i \neq \alpha} M_{k_i}^{[i]} \right) \otimes \left[\operatorname{Ind}_B^G \left(\eta^{k_\alpha + 2} \right) \right]^{[\alpha]},$$

where B is the subgroup of G consisting of upper triangular matrices, and η is the character of B defined extending the character $\operatorname{diag}(a, b) \mapsto a$ of the standard maximal torus of G.

74

Remark 5.2.16. The Jordan-Hölder constituents of coker $\Theta_\beta^{[\alpha]}$ can be explicitly computed using the results we proved earlier, but we do not know of any interesting description of the cokernel of the operators $\Theta_\beta^{[\alpha]}$.

Generalized D-operators

Let us denote by ∂_X (resp. ∂_Y) the operator of partial derivation with respect to X (resp. Y) acting on the polynomial algebra $\mathbb{F}_q[X, Y]$; if $f \in \mathbb{F}_q[X, Y]$, denote by the same symbol the $\mathbb{F}_q$-vector space endomorphism of $\mathbb{F}_q[X, Y]$ induced by multiplication by f. The operators $\partial_X \otimes f, \partial_Y \otimes f, f \otimes \partial_X$ and $f \otimes \partial_Y$ are therefore derivation of the $\mathbb{F}_q$-algebra $\mathbb{F}_q[X, Y] \otimes \mathbb{F}_q[X, Y]$.

Definition 5.2.17. Let k, h be two non-negative integers. For any integer β such that $1 \leq \beta \leq g - 1$, the (non-twisted) generalized βth D-operator is the $\mathbb{F}_q$-vector space homomorphism:

$$D_\beta = \partial_X \otimes X^{p^{g-\beta}} + \partial_Y \otimes Y^{p^{g-\beta}} : M_k \otimes M_h^{[\beta]} \longrightarrow M_{k-1} \otimes M_{h+p^{g-\beta}}^{[\beta]}.$$

For any integers α, β such that $0 \leq \alpha \leq g-1$ and $1 \leq \beta \leq g-1$, the α-twisted generalized βth D-operator is the $\mathbb{F}_q$-vector space homomorphism:

$$D_\beta^{[\alpha]} = \partial_X \otimes X^{p^{g-\beta}} + \partial_Y \otimes Y^{p^{g-\beta}} : M_k^{[\alpha]} \otimes M_h^{[\alpha+\beta]} \longrightarrow M_{k-1}^{[\alpha]} \otimes M_{h+p^{g-\beta}}^{[\alpha+\beta]}.$$

Lemma 5.2.18. Let k, h be two non-negative integers and let α, β be integers such that $0 \leq \alpha \leq g - 1$ and $1 \leq \beta \leq g - 1$. The operator $D_\beta^{[\alpha]} : M_k^{[\alpha]} \otimes M_h^{[\alpha+\beta]} \to M_{k-1}^{[\alpha]} \otimes M_{h+p^{g-\beta}}^{[\alpha+\beta]}$ is a G-homomorphism; it is injective if $0 < k \leq p - 1$ and $0 \leq h \leq p - 1$.

Proof. By twisting, we can assume that $\alpha = 0$. Fix $f_1 \in M_k$, $f_2 \in M_h^{[\beta]}$ and let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$; denote by γ^{σ^β} the matrix $\left(\begin{smallmatrix} a^{\sigma^\beta} & b^{\sigma^\beta} \\ c^{\sigma^\beta} & d^{\sigma^\beta} \end{smallmatrix}\right)$, where σ denotes the arithmetic Frobenius

element of $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. $D_\beta\left(\gamma(f_1 \otimes f_2)\right)$ equals:

$$
\begin{aligned}
&[a \cdot (\partial_X f_1)\,(\gamma X, \gamma Y) + b \cdot (\partial_Y f_1)\,(\gamma X, \gamma Y)] \otimes X^{p^{g-\beta}} f_2(\gamma^{\sigma^\beta} X, \gamma^{\sigma^\beta} Y) \\
&+ [c \cdot (\partial_X f_1)\,(\gamma X, \gamma Y) + d \cdot (\partial_Y f_1)\,(\gamma X, \gamma Y)] \otimes Y^{p^{g-\beta}} f_2(\gamma^{\sigma^\beta} X, \gamma^{\sigma^\beta} Y) \\
={}& (\partial_X f_1)\,(\gamma X, \gamma Y) \otimes (aX^{p^{g-\beta}} + cY^{p^{g-\beta}}) f_2(\gamma^{\sigma^\beta} X, \gamma^{\sigma^\beta} Y) \\
&+ (\partial_Y f_1)\,(\gamma X, \gamma Y) \otimes (bX^{p^{g-\beta}} + dY^{p^{g-\beta}}) f_2(\gamma^{\sigma^\beta} X, \gamma^{\sigma^\beta} Y) \\
={}& \gamma D_\beta\,(f_1 \otimes f_2)\,.
\end{aligned}
$$

For the injectivity statement, notice that if $0 < k \leq p-1$ and $0 \leq h \leq p-1$, then $M_k \otimes M_h^{[\beta]}$ is an irreducible G-module, so it is enough to show that D_β is non-zero on $M_k \otimes M_h^{[\beta]}$. We have $D_\beta(X^k \otimes X^h) = kX^{k-1} \otimes X^{h+p^{g-\beta}}$, and this is non-zero as k is prime with p. $\blacksquare$

In addition to the above operators, the D-map defined by Serre also gives an intertwining map:

Proposition 5.2.19. Let k be a non-negative integer and let α be an integer such that $0 \leq \alpha \leq g-1$. Then the Frobenius twists of Serre's operator $D^{[\alpha]} = X^q \partial_X + Y^q \partial_Y$ define G-homomorphisms:

$$
D^{[\alpha]} : M_k^{[\alpha]} \longrightarrow M_{k+(q-1)}^{[\alpha]}
$$

which are injective if $1 \leq k \leq p-1$.

Proof. After twisting, we can assume $\alpha = 0$. The result then follows from section 5.1.2 and the irreducibility of $M_k^{[\alpha]}$ in the range $1 \leq k \leq p-1$. $\blacksquare$

We can summarize the above results as follows:

Theorem 5.2.20. *Let us fix non-negative integers $k_0, ..., k_{g-1}$. For any integers α, β subject to the constraints $0 \leq \alpha \leq g-1$ and $1 \leq \beta \leq g-1$, there are G-intertwining operators as follows:*

$$
D_\beta^{[\alpha]} : \bigotimes_i M_{k_i}^{[i]} \longrightarrow \left(\bigotimes_{i \neq \alpha, \alpha+\beta} M_{k_i}^{[i]}\right) \otimes M_{k_\alpha - 1}^{[\alpha]} \otimes M_{k_{\alpha+\beta}+p^{g-\beta}}^{[\alpha+\beta]};
$$

$$
D^{[\alpha]} : \bigotimes_i M_{k_i}^{[i]} \longrightarrow \left(\bigotimes_{i \neq \alpha} M_{k_i}^{[i]}\right) \otimes M_{k_\alpha+(q-1)}^{[\alpha]},
$$

where the tensor product indices run over the integers i such that $0 \le i \le g-1$, unless otherwise specified. If $0 < k_\alpha \le p-1$, then $D^{[\alpha]}$ is injective; if in addition $0 \le k_{\alpha+\beta} \le p-1$, then $D^{[\alpha]}_\beta$ is injective.

Remark 5.2.21. The operators $D^{[\alpha]}_{g-1}$ for $0 \le \alpha \le g-1$ give, under suitable assumptions, cohomological analogues of the partial Hasse invariants defined in [AG05] in the context of mod p Hilbert modular forms. We do not know of any geometric interpretation of the other D-maps introduced above.

We can picture the weight shiftings allowed by the $g(g-1)+g = g^2$ generalized D-maps as follows:

$$
\begin{array}{ll}
D_1 & (-1, p^{g-1}, 0, 0, ..., 0, 0) \\
D_1^{[1]} & (0, -1, p^{g-1}, 0, ..., 0, 0) \\
D_1^{[2]} & (0, 0, -1, p^{g-1}, ..., 0, 0) \\
... & \qquad ... \\
D_1^{[g-2]} & (0, 0, 0, 0, ..., -1, p^{g-1}) \\
D_1^{[g-1]} & (p^{g-1}, 0, 0, 0, ..., 0, -1)
\end{array}
\qquad
\begin{array}{ll}
D_2 & (-1, 0, p^{g-2}, 0, ..., 0, 0) \\
D_2^{[1]} & (0, -1, 0, p^{g-2}, ..., 0, 0) \\
D_2^{[2]} & (0, 0, -1, 0, ..., 0, 0) \\
... & \qquad ... \\
D_2^{[g-2]} & (p^{g-2}, 0, 0, 0, ..., -1, 0) \\
D_2^{[g-1]} & (0, p^{g-2}, 0, 0, ..., 0, -1)
\end{array}
$$

$$...$$

$$
\begin{array}{ll}
D_{g-1} & (-1, 0, 0, 0, ..., 0, p) \\
D_{g-1}^{[1]} & (p, -1, 0, 0, ..., 0, 0) \\
D_{g-1}^{[2]} & (0, p, -1, 0, ..., 0, 0) \\
... & \qquad ... \\
D_{g-1}^{[g-2]} & (0, 0, 0, 0, ..., -1, 0) \\
D_{g-1}^{[g-1]} & (0, 0, 0, 0, ..., p, -1)
\end{array}
\qquad
\begin{array}{ll}
D & (q-1, 0, 0, ..., 0, 0) \\
D^{[1]} & (0, q-1, 0, ..., 0, 0) \\
D^{[2]} & (0, 0, q-1, ..., 0, 0) \\
... & \qquad ... \\
D^{[g-2]} & (0, 0, 0, ..., q-1, 0) \\
D^{[g-1]} & (0, 0, 0, ..., 0, q-1)
\end{array}
$$

Similarly to what happened for the generalized Dickson operators, the non existence of shiftings of the form

$$(-1, p, 0, ..., 0, 0), (0, -1, p, ..., 0, 0), ..., (0, 0, 0, ..., -1, p), (p, 0, 0, ..., 0, -1)$$

when $g > 2$ is a consequence of the structure of the irreducible G-modules:

Proposition 5.2.22. Assume $g > 2$ and let k, h be integers such that $0 \leq k, h \leq p - 1$. For any integer α such that $0 \leq \alpha \leq g - 1$ and any integer m, there are no G-module morphisms $\det^m \otimes M_k^{[\alpha]} \otimes M_h^{[\alpha+1]} \to M_{k-1}^{[\alpha]} \otimes M_{h+p}^{[\alpha+1]}$.

Proof. It is enough to consider the case $\alpha = 0$; we can also assume that $k \neq 0$. Using formulae (Φ_g) and (Δ_g) we have, in $K_0(G)$:

$$M_{h+p}^{[1]} = M_h^{[1]} M_1^{[2]} + e^{p(h+1)} M_{p-h-2}^{[1]}.$$

As $g > 2$, the Jordan-Hölder factors of $M_{k-1} \otimes M_{h+p}^{[1]}$ are

$$M_{k-1} \otimes M_h^{[1]} \otimes M_1^{[2]}, \det^{p(h+1)} \otimes M_{k-1} \otimes M_{p-h-2}^{[1]},$$

unless $h = p - 1$, in which case only the first factor occurs: none of these factors coincides with $\det^m \otimes M_k \otimes M_h^{[1]}$. ∎

We conclude by noticing the following consequence of Th. 5.1.8:

Proposition 5.2.23. Let us fix non-negative integers $k_0, ..., k_{g-1}$; let α be an integer such that $0 \leq \alpha \leq g - 1$ and assume $2 \leq k_\alpha \leq p - 1$, $k_\alpha \neq \frac{q+1}{2}$. Consider the injective G-map $D^{[\alpha]} : D^{[\alpha]} : \bigotimes_i M_{k_i}^{[i]} \to \left(\bigotimes_{i \neq \alpha} M_{k_i}^{[i]} \right) \otimes M_{k_\alpha+(q-1)}^{[\alpha]}$. We have:

$$\operatorname{coker} D^{[\alpha]} \simeq \left(\bigotimes_{i \neq \alpha} M_{k_i}^{[i]} \right) \otimes \left[\bar{\Xi} \left(\chi^{k_\alpha} \right) \right]^{[\alpha]},$$

where: $\bar{\Xi} \left(\chi^{k_\alpha} \right) = H^1_{\mathrm{cris}}(\mathcal{C}_{/\mathbb{F}_q})_{-k_\alpha} \otimes_{W(\mathbb{F}_q)} \mathbb{F}_q$, $\mathcal{C}$ is the Deligne-Lusztig variety of $SL_{2/\mathbb{F}_q}$ and the $(-k_\alpha)$-eigenspace of $H^1_{\mathrm{cris}}(\mathcal{C}_{/\mathbb{F}_q})$ is computed with respect to the natural action of $\ker(\mathrm{Nm}_{\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times})$ on $H^1_{\mathrm{cris}}(\mathcal{C}_{/\mathbb{F}_q})$.

Remark 5.2.24. We do not know of any interesting description of the cokernel of the operators $D_\beta^{[\alpha]}$. The Jordan-Hölder constituents of $\operatorname{coker} D_\beta^{[\alpha]}$ can be explicitly computed using the results described in this dissertation.

CHAPTER 6

Weight shiftings for automorphic forms

We apply the results of the previous sections to obtain weight shiftings for automorphic forms on definite quaternion algebras whose center is a totally real field F unramified at the prime $p > 2$. First we treat the case in which the tensor factors - corresponding to the prime decomposition of p in F - of the weight that we want to shift are all of dimension greater than one: this is what we call a weight not containing a $(2, ..., 2)$-block. In section 6.2 we consider shiftings for irreducible weights that contain a $(2, ..., 2)$-block.

6.1 Shiftings for weights not containing $(2, ..., 2)$-blocks

Let us fix some notation that will be used throughout this section and the next one. Let F be a totally real number field of degree g over $\mathbb{Q}$, and let $p > 2$ be a rational prime which is unramified in $F/\mathbb{Q}$. Denote by $\mathcal{O}_F$ the ring of integers of F and write $p\mathcal{O}_F = \prod_{j=1}^{r} \mathfrak{P}_j$, where the $\mathfrak{P}_j$'s are distinct maximal ideals of $\mathcal{O}_F$.

Fix an integer j with $1 \leq j \leq r$. Let f_j be the residual degree of $\mathfrak{P}_j$ over $p\mathbb{Z}$, so that $\mathbb{F}_{\mathfrak{P}_j} := \mathcal{O}_F/\mathfrak{P}_j$ is an extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of degree f_j. Let $F_{\mathfrak{P}_j}$ be the completion of F at $\mathfrak{P}_j$, and denote by $\mathcal{O}_{F_{\mathfrak{P}_j}}$ its ring of integers. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$; let n be the positive least common multiple of the integers $f_1, ..., f_r$ and let E be the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ having degree n over $\mathbb{Q}_p$, so that $\mathrm{Hom}(F, \overline{\mathbb{Q}}_p) = \mathrm{Hom}(F, E)$. Denote by $\mathcal{O}$ the ring of integers of E and let $\mathbb{F}$ be its residue field. Let σ be the arithmetic Frobenius of the extension $E/\mathbb{Q}_p$. Set:

$$\mathrm{Hom}(F_{\mathfrak{P}_j}, E) = \left\{ \sigma_i^{(j)} : 0 \leq i \leq f_j - 1 \right\},$$

where the labeling is chosen so that, for any i, we have:

$$\sigma \circ \sigma_i^{(j)} = \sigma_{i+1}^{(j)}.$$

Here the subscripts are taken modulo f_j and in the range $0 \le i \le f_j - 1$.

Denote by a bar the analogous morphisms for the residue fields, so that $\overline{\sigma}$ is the arithmetic Frobenius of the extension $\mathbb{F}/\mathbb{F}_p$, and:

$$\mathrm{Hom}(\mathbb{F}_{\mathfrak{P}_j}, \mathbb{F}) = \{\overline{\sigma}_i^{(j)} : 0 \le i \le f_j - 1\}$$

are labeled so that:

$$\overline{\sigma} \circ \overline{\sigma}_i^{(j)} = \overline{\sigma}_{i+1}^{(j)},$$

where the subscripts are taken modulo f_j and in the range $0 \le i \le f_j - 1$.

We let $\mathbb{A}_F$ be the topological ring of adèles of F, and we denote by $\mathbb{A}_F^\infty$ the subring of finite adèles. We let $\mathfrak{M}_{F,f}$ (resp. $\mathfrak{M}_{F,\infty}$) be the set of finite (resp. infinite) places of F and we identify $\mathfrak{M}_{F,f}$ with the set of maximal ideals of $\mathcal{O}_F$.

6.1.1 Some motivations: geometric Hilbert modular forms

Denote by d_F the discriminant of $F/\mathbb{Q}$ and fix a fractional ideal $\mathfrak{a}$ of F with its natural positive cone $\mathfrak{a}^+$, so that $(\mathfrak{a}, \mathfrak{a}^+)$ represents an element in the strict class group of F. Let $N \ge 4$ be an integer and recall that, by previous assumptions, p does not divide d_F. Let S be a scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{d_F}])$.

There is an S-scheme $\mathcal{M}$ parametrizing isomorphism classes $[(A, \lambda, \iota, \varepsilon)/T/S]$ of $(\mathfrak{a}, \mathfrak{a}^+)$-polarized Hilbert-Blumenthal abelian T-schemes (A, λ) of relative dimension g (T is an S-scheme), endowed with real multiplication ι by $\mathcal{O}_F$, μ_N-level structure ε, and satisfying the Deligne-Pappas condition (or, equivalently since d_F is invertible in S, satisfying the Rapoport condition). $\mathcal{M}$ has relative dimension g over S and is geometrically irreducible; see [DP94] and [AG05] for more details.

Let $\mathbb{G} = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}(\mathbb{G}_{m,\mathcal{O}_F})$ be the Weil restriction to $\mathbb{Z}$ of the algebraic $\mathcal{O}_F$-group $\mathbb{G}_{m,\mathcal{O}_F}$. For any scheme T, denote by $\mathbb{X}_T = \mathrm{Hom}(\mathbb{G}_T, \mathbb{G}_{m,T})$ the group of characters of the base

change $\mathbb{G}_T$ of $\mathbb{G}$ to T. If S is the scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{d_F}])$ fixed above, a geometric $(\mathfrak{a}, \mathfrak{a}^+)$-polarized Hilbert modular form f over S having weight $\chi \in \mathbb{X}_S$ and level μ_N is a rule that assigns to any affine scheme $\mathrm{Spec}(R) \to S$, any R-point $[(A, \lambda, \iota, \varepsilon)/R/S]$ of $\mathcal{M}$, and any generator ω of the $R \otimes_\mathbb{Z} \mathcal{O}_F$-module $\Omega^1_{A/R}$, an element $f(A, \lambda, \iota, \varepsilon, \omega) \in R$ such that:

$$f(A, \lambda, \iota, \varepsilon, \alpha^{-1}\omega) = \chi(\alpha) \cdot f(A, \lambda, \iota, \varepsilon, \omega)$$

for $\alpha \in \mathbb{G}(R)$, and such that some compatibility conditions are satisfied (cf. [AG05], 5). We denote by $M_\chi(\mu_N, S)$ the $\Gamma(S, \mathcal{O}_S)$-module of such functions.

We remark that the formation of spaces of geometric Hilbert modular forms does not commute with base change: for example, if $g > 1$ and $1 \leq j \leq r$, $0 \leq i \leq f_j - 1$, the (j, i)th partial Hasse invariant that we will consider below is a non-zero, non-cuspidal modular forms over $\mathrm{Spec}\,\mathbb{F}_{\mathfrak{P}_j}$ that cannot be lifted to a modular forms over $\mathrm{Spec}\,\mathcal{O}_F$: the natural reduction morphism $M_\chi(\mu_N, \mathcal{O}_F) \to M_\chi(\mu_N, \mathbb{F}_{\mathfrak{P}_j})$ is in general not surjective.

Assume $g > 1$ for the rest of this paragraph. We consider modular forms over $S = \mathrm{Spec}(\mathbb{F})$. The labeling of the embeddings $\overline{\sigma}_i^{(j)}$ for $1 \leq j \leq r$ and $0 \leq i \leq f_j - 1$ induces a canonical splitting:

$$\begin{aligned}
\mathbb{G}_\mathbb{F} &= \bigoplus_{j=1}^r \left(\mathrm{Res}_{\mathbb{F}_{\mathfrak{P}_j}/\mathbb{F}_p}(\mathbb{G}_{m,\mathbb{F}_{\mathfrak{P}_j}}) \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\mathbb{F} \right) \\
&= \bigoplus_{j=1}^r \bigoplus_{\overline{\sigma}_i^{(j)}:\mathbb{F}_{\mathfrak{P}_j} \hookrightarrow \mathbb{F}} \mathbb{G}_{m,\mathbb{F}},
\end{aligned}$$

such that the projection $\chi_{(j,i)}$ of $\mathbb{G}_\mathbb{F}$ onto the (j,i)th factor is induced by $\overline{\sigma}_i^{(j)}$. The character group $\mathbb{X}_\mathbb{F}$ of $\mathbb{G}_\mathbb{F}$ is the free $\mathbb{Z}$-module or rank g generated by these projections. A geometric Hilbert modular form over $\mathrm{Spec}(\mathbb{F})$ whose weight is $\prod_{j=1}^r \prod_{i=0}^{f_j-1} \chi_{(j,i)}^{a_i^{(j)}}$ for some $a_i^{(j)} \in \mathbb{Z}$ is also said to have weight vector $\vec{a} = (\vec{a}^{(1)}, ..., \vec{a}^{(r)})$ where $\vec{a}^{(j)} = (a_0^{(j)}, ..., a_{f_j-1}^{(j)})$ for $1 \leq j \leq r$.

Th. 2.1 of [Gor01] shows that, for any $1 \leq j \leq r$ and $0 \leq i \leq f_j - 1$, there is an $(\mathfrak{a}, \mathfrak{a}^+)$-polarized Hilbert modular form $h_{(j,i)}$ over $\mathrm{Spec}(\mathbb{F})$ having weight $\chi_{(j,i-1)}^p \chi_{(j,i)}^{-1}$ and level 1, whose q-expansion at every $(\mathfrak{a}, \mathfrak{a}^+)$-polarized unramified $\mathbb{F}_p$-rational cusp is one. $h_{(j,i)}$ is called the (j,i)th *partial Hasse invariant*. As mentioned earlier, the forms $h_{(j,i)}$ are not liftable to characteristic zero; even the total Hasse invariant, i.e., the form $h = \prod_{(j,i)} h_{(j,i)}$,

81

having parallel weight $(p-1, p-1, ..., p-1)$, is not always liftable to characteristic zero (cf. Prop. 3.1 in [Gor01]).

As a consequence of the existence of the partial Hasse invariants, one can produce (geometric) weight shiftings. More precisely, fix an integer j such that $1 \leq j \leq r$ and assume $\chi \in \mathbb{X}_{\mathbb{F}}$ is such that $M_\chi(\mu_N, \mathbb{F}) \neq 0$; denote the weight vector associated to χ by $\vec{a} = (\vec{a}^{(1)}, ..., \vec{a}^{(r)})$. Multiplication by $h_{(j,i)}$ for an integer i such that $0 \leq i \leq f_j - 1$ induces a Hecke injection of $M_\chi(\mu_N, \mathbb{F})$ into $M_{\chi'}(\mu_N, \mathbb{F})$, where the weight vector associated to χ' is $\vec{a} + \vec{t}$ and $\vec{t} = (\vec{t}^{(1)}, ..., \vec{t}^{(r)})$ is such that $\vec{t}^{(r)} = \vec{0}$ if $r \neq j$, while $\vec{t}^{(j)}$ is one of the following f_j-tuples:

$$
\begin{aligned}
(-1, 0, 0, ..., 0, p) & \quad \text{if } i = 0, \\
(p, -1, 0, ..., 0, 0) & \quad \text{if } i = 1, \\
(0, p, -1, ..., 0, 0) & \quad \text{if } i = 2, \\
... & \\
(0, 0, 0, ..., p, -1) & \quad \text{if } i = f_j - 1.
\end{aligned}
$$

In [Kat78] 2.5. and [AG05] 12, generalized theta operators acting on spaces of geometric Hilbert modular forms over $\mathrm{Spec}(\mathbb{F})$ are defined, allowing additional weight shiftings. For example, if p is inert in $F/\mathbb{Q}$, these operators induce shiftings by the vectors:

$$
\begin{aligned}
(1, 0, 0, ..., 0, p), \\
(p, 1, 0, ..., 0, 0), \\
(0, p, 1, ..., 0, 0), \\
... \\
(0, 0, 0, ..., p, 1).
\end{aligned}
$$

The reader will notice that the two sets of weight shifting vectors described above are contained in the sets of weight shifting vectors produced in 5.2.3 and 5.2.3 for $\bar{\mathbb{F}}_p$-representation of $GL_2(\mathbb{F})$. Exploiting the adelic definition of Hilbert modular forms, we will see that all the geometric weight shiftings can be obtained as cohomological weight shiftings via the

operators considered in Section 3. The purely cohomological picture will be reacher, as more shiftings will be allowed. The formation of spaces of adelic automorphic forms on definite quaternion algebra will have the advantage of being compatible with base changes, under suitable assumptions (Prop. 6.1.2). Finally, our cohomological weight shiftings translate into weight shiftings for $(\mathrm{mod}\, p)$ Galois representations arising from automorphic forms on $GL_2(\mathbb{A}_F)$.

6.1.2 Automorphic forms on definite quaternion algebras

We recall the definition and some properties of automorphic forms on definite quaternion algebras over totally real number fields. The exposition follows [Tay06] and [Kis09b]; cf. also [Tay89].

Fix a finite set $\Sigma \subset \mathfrak{M}_{F,f}$ disjoint from the set of places of F lying above p and such that $\#\Sigma + [F : \mathbb{Q}] \equiv 0 (\mathrm{mod}\, 2)$. Let D be a quaternion algebra over F whose ramification set is $\mathfrak{M}_{F,\infty} \cup \Sigma$. Let $\mathcal{O}_D$ be a fixed maximal order of D and for any $v \in \mathfrak{M}_{F,f} - \Sigma$ fix ring isomorphisms $(\mathcal{O}_D)_v \simeq M_2(\mathcal{O}_{F_v})$.

Let U be a compact open subgroup of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ such that:

1. $U = \prod_{v \in \mathfrak{M}_{F,f}} U_v$, where U_v is a subgroup of $(\mathcal{O}_D)_v^\times$;

2. $U_v = (\mathcal{O}_D)_v^\times$ if $v \in \Sigma$;

3. if $v | p$, then $U_v = GL_2(\mathcal{O}_{F_v})$.

Let A be a topological $\mathbb{Z}_p$-algebra. Let v be a place of F above p, say $v = v_j := \mathfrak{P}_j$ for some integer j such that $1 \leq j \leq r$; let W_{τ_j} be a free A-module of finite rank and fix a continuous homomorphism

$$\tau_j : U_{v_j} = GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}}) \longrightarrow \mathrm{Aut}(W_{\tau_j}),$$

where $\mathrm{Aut}(W_{\tau_j})$ is the group of continuous A-linear automorphisms of W_{τ_j}. Let $W_\tau = \bigotimes_{j=1}^r W_{\tau_j}$, where the tensor products are over A, and denote by τ the corresponding group

83

homomorphism $\tau : \prod_{j=1}^{r} U_{v_j} \to \mathrm{Aut}(W_\tau)$. If no confusion arises, we also denote by τ the action of U on W_τ induced by precomposing the latter morphism with the natural projection $U \to \prod_{j=1}^{r} U_{v_j}$.

For A as above, let $\psi : (\mathbb{A}_F^\infty)^\times / F^\times \to A^\times$ be a continuous character such that, for any $v \in \mathfrak{M}_{F,f}$:

$$\tau_{|U_v \cap \mathcal{O}_{F_v}^\times}(u) = \psi^{-1}(u) \cdot Id_{W_\tau}, \quad \text{for all } u \in U_v \cap \mathcal{O}_{F_v}^\times.$$

We say that such a Hecke character ψ is compatible with τ.

Definition 6.1.1. For D, U, A, τ, W_τ and ψ as above, the space $S_{\tau,\psi}(U, A)$ of automorphic forms on D having level U, weight τ, character ψ and coefficients in A is the A-module consisting of all the functions:

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \longrightarrow W_\tau$$

satisfying:

(a) $f(gu) = \tau(u)^{-1} f(g)$ for all $g \in (D \otimes_F \mathbb{A}_F^\infty)^\times$ and all $u \in U$;

(b) $f(gz) = \psi(z) f(g)$ for all $g \in (D \otimes_F \mathbb{A}_F^\infty)^\times$ and all $z \in (\mathbb{A}_F^\infty)^\times$.

As in [Kis09b], we will assume, unless otherwise stated, that for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, the finite group $(U \cdot (\mathbb{A}_F^\infty)^\times \cap t^{-1} D^\times t)/F^\times$ has order prime to p. This assumption is automatically satisfied if U is sufficiently small, as Lemma 1.1. of [Tay06] implies that in this case $(U \cdot (\mathbb{A}_F^\infty)^\times \cap t^{-1} D^\times t)/F^\times$ is a 2-group. We obtain as a consequence (cf. [Tay06], Cor. 1.2):

Proposition 6.1.2. Let B a topological A-algebra. The natural morphism

$$S_{\tau,\psi}(U, A) \otimes_A B \to S_{\tau \otimes_A B, \psi \otimes_A B}(U, B)$$

is an isomorphism of B-modules.

Define a left action of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ on the set of functions $D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \to W_\tau$ by setting $(gf)(x) := f(xg)$ for all $g, x \in (D \otimes_F \mathbb{A}_F^\infty)^\times$. Let S be a set of primes of F

containing the ramification set of D, the primes above p and the primes v for which U_v is not a maximal compact subgroup of $D_v^\times$. Let $\mathbb{T}_{S,A}^{univ} = A[T_v, S_v : v \notin S]$ be the commutative polynomial A-algebra in the indicated indeterminates. For each finite place $v \notin S$, let ϖ_v be a fixed uniformizer for F_v. $S_{\tau,\psi}(U, A)$ has a natural action of $\mathbb{T}_{S,A}^{univ}$, with S_v acting via the double coset $U \left(\begin{smallmatrix} \varpi_v & \\ & \varpi_v \end{smallmatrix} \right) U$ and T_v via $U \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix} \right) U$ (cf. [Tay06], 1); this action does not depend upon the choices of uniformizers that we made. The image of $\mathbb{T}_{S,A}^{univ}$ in the ring of A-module endomorphisms of $S_{\tau,\psi}(U, A)$ is the Hecke algebra $\mathbb{T}_{S,A}$ acting on $S_{\tau,\psi}(U, A)$. The isomorphism of Prop. 6.1.2 is Hecke equivariant.

6.1.3 Behavior of Hecke eigensystems under reduction modulo $\mathfrak{M}_R$

For a discrete valuation ring R, we will denote by $\mathfrak{M}_R$ its maximal ideal. If the residual characteristic of R is $p > 0$ and no confusion arises, we will also improperly refer to reduction modulo $\mathfrak{M}_R$ as reduction modulo p. If $\mathcal{T}$ is a commutative algebra, a system of eigenvalues of $\mathcal{T}$ with values in R is a set theoretic map $\Omega : \mathcal{T} \to R$; the reduction of Ω modulo p, denoted $\bar{\Omega}$, is the function obtained by composing Ω with the reduction morphism $R \to \frac{R}{\mathfrak{M}_R}$. Let $R\mathcal{T} = R \otimes_{\mathbb{Z}} \mathcal{T}$; if M is an $R\mathcal{T}$-module, we say that a system of eigenvalues $\Omega : \mathcal{T} \to R$ occurs in M if there is a non-zero element $m \in M$ such that $Tm = \Omega(T)m$ for all $T \in \mathcal{T}$. Such a non-zero m is called an Ω-eigenvector.

Fixing R and $\mathcal{T}$ as above. We have:

Lemma 6.1.3. Let M be an $R\mathcal{T}$-module which is finitely generated over R. If $\Omega : \mathcal{T} \to R$ is a system of eigenvalues of $\mathcal{T}$ occurring in M, then $\bar{\Omega} : \mathcal{T} \to \frac{R}{\mathfrak{M}_R}$ is a system of eigenvalues of $\mathcal{T}$ occurring in $\bar{M} := M \otimes_R \frac{R}{\mathfrak{M}_R}$.

Proof. Cf. [AS86a], Prop. 1.2.3. ∎

Lemma 6.1.4. Let M be an $R\mathcal{T}$-module which is finite and free over R. Let $\bar{\Omega} : \mathcal{T} \to \frac{R}{\mathfrak{M}_R}$ be a system of eigenvalues of $\mathcal{T}$ occurring in $\bar{M} = M \otimes_R \frac{R}{\mathfrak{M}_R}$. There exists a finite extension of discrete valuation rings R'/R such that $\mathfrak{M}_{R'} \cap R = \mathfrak{M}_R$ and a system of eigenvalues $\Omega' : \mathcal{T} \to R'$ of $\mathcal{T}$ occurring in $M \otimes_R R'$ such that, for all $T \in \mathcal{T}$, $\Omega'(T)(\mathrm{mod}\,\mathfrak{M}_{R'}) = \bar{\Omega}(T)$

in $\frac{R'}{\mathfrak{M}_{R'}}$. (Here we view $\frac{R}{\mathfrak{M}_R} \subseteq \frac{R'}{\mathfrak{M}_{R'}}$ by the given embedding $R \subseteq R'$).

Proof. Cf. [DS74], Lemme 6.11. A generalization of the result is given in [AS86a], Prop. 1.2.2. ∎

Let D, U, τ, W_τ and ψ be as in 6.1.2, and set $A = \mathcal{O}$. In particular, we assume that ψ is compatible with (τ, W_τ), U is small enough and p is odd. Denote by a bar the operation of tensoring over $\mathcal{O}$ with $\mathbb{F}$. From now on, unless otherwise stated, we assume fixed a set S of primes of F containing the ramification set of D, the primes above p and the primes v for which U_v is not a maximal compact subgroup of $D_v^\times$. The Hecke eigensystems considered below will always be with respect to the Hecke algebra $\mathbb{T}_{S,A'}^{univ}$ for some topological $\mathbb{Z}_p$-algebra A'.

Proposition 6.1.5. Fix an $\mathcal{O}$-valued weight $(\tau', W_{\tau'})$ together with a compatible Hecke character $\psi' : (\mathbb{A}_F^\infty)^\times / F^\times \to \mathcal{O}^\times$ such that $\bar{\psi}' = \bar{\psi}$. Let $\varphi : (\bar{\tau}, W_{\bar{\tau}}) \to (\bar{\tau}', W_{\bar{\tau}'})$ be a non-zero intertwining operator for $\mathbb{F}$-representations of U. φ induces a Hecke equivariant map $\varphi_* : S_{\bar{\tau}, \bar{\psi}}(U, \mathbb{F}) \to S_{\bar{\tau}', \bar{\psi}}(U, \mathbb{F})$.

Assume φ is injective: then if Ω is a Hecke eigensystem occurring in $S_{\tau, \psi}(U, \mathcal{O})$, there is a finite extension of E, with ring of integer $\mathcal{O}'$ such that $\mathfrak{M}_{\mathcal{O}'} \cap \mathcal{O} = \mathfrak{M}_{\mathcal{O}}$, and there is a Hecke eigensystem Ω' occurring in $S_{\tau', \psi'}(U, \mathcal{O}')$ such that:

$$\Omega'(\mathrm{mod}\, \mathfrak{M}_{\mathcal{O}'}) = \Omega(\mathrm{mod}\, \mathfrak{M}_{\mathcal{O}}) \quad \text{in } \frac{\mathcal{O}'}{\mathfrak{M}_{\mathcal{O}'}}.$$

Proof. For $f \in S_{\bar{\tau}, \bar{\psi}}(U, \mathbb{F})$ set $\varphi_*(f) := \varphi \circ f$. If $g \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, $u \in U$ and $z \in (\mathbb{A}_F^\infty)^\times$ we have:

$$\begin{aligned}
\varphi_*(f)(gu) &= \varphi(f(gu)) = \varphi\left(\bar{\tau}(u^{-1})f(g)\right) = \bar{\tau}'(u)^{-1}\varphi\left(f(g)\right), \\
\varphi_*(f)(gz) &= \varphi(f(gz)) = \varphi\left(\bar{\psi}(z)f(g)\right) = \bar{\psi}(z)\varphi\left(f(g)\right).
\end{aligned}$$

Since $\bar{\psi}' = \bar{\psi}$, we have that $\bar{\tau}'$ and $\bar{\psi}$ are compatible and we conclude that $\varphi_*(f) \in S_{\bar{\tau}', \bar{\psi}}(U, \mathbb{F})$.

If $g, x \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, we have:

$$
\begin{aligned}
(g \cdot \varphi_*(f))(x) &= (\varphi \circ f)(xg) \\
&= (\varphi \circ (g \cdot f))(x) \\
&= (\varphi_*(g \cdot f))(x),
\end{aligned}
$$

so that φ is Hecke-equivariant. Assume now that φ is injective and notice that this implies the injectivity of φ_*. Let Ω be a Hecke eigensystem occurring in the finite $\mathcal{O}$-module with Hecke action $S_{\tau, \psi}(U, \mathcal{O})$; by Prop. 6.1.2, reduction modulo p induces a Hecke equivariant surjection $\pi : S_{\tau, \psi}(U, \mathcal{O}) \longrightarrow S_{\bar{\tau}, \bar{\psi}}(U, \mathbb{F})$.

By Lemma 6.1.3, the Hecke eigensystem $\bar{\Omega} := \Omega(\mathrm{mod}\, \mathfrak{M}_{\mathcal{O}})$ occurs in $S_{\bar{\tau}, \bar{\psi}}(U, \mathbb{F})$, and hence in $S_{\bar{\tau}', \bar{\psi}}(U, \mathbb{F})$ as φ_* is Hecke equivariant and injective. Now, applying Lemma 6.1.4 to the Hecke equivariant surjection $S_{\tau', \psi'}(U, \mathcal{O}) \to S_{\bar{\tau}', \bar{\psi}}(U, \mathbb{F})$, we deduce the existence of a finite extension of discrete valuation rings $\mathcal{O}'/\mathcal{O}$ such that $\mathfrak{M}_{\mathcal{O}'} \cap \mathcal{O} = \mathfrak{M}_{\mathcal{O}}$, and of a Hecke eigensystem $\Omega' : \mathbb{T}_{S, \mathcal{O}'} \to \mathcal{O}'$ occurring in $S_{\tau', \psi'}(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}'$ whose reduction modulo $\mathfrak{M}_{\mathcal{O}'}$ has value in $\mathbb{F} \subset \frac{\mathcal{O}'}{\mathfrak{M}_{\mathcal{O}'}}$ and coincide with $\bar{\Omega}$. By Prop. 6.1.2, $S_{\tau', \psi'}(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}' \simeq S_{\tau', \psi'}(U, \mathcal{O}')$ as Hecke modules, and we are done. ∎

6.1.4 Holomorphic weights

For any integer j such that $1 \leq j \leq r$ let us fix two tuples $\vec{k}^{(j)} = (k_0^{(j)}, ..., k_{f_j-1}^{(j)}) \in \mathbb{Z}_{\geq 2}^{f_j}$ and $\vec{w}^{(j)} = (w_0^{(j)}, ..., w_{f_j-1}^{(j)}) \in \mathbb{Z}^{f_j}$. Define the finite free $\mathcal{O}$-module with $GL_2(\mathcal{O})$-action:

$$
W_{(\vec{k}^{(j)}, \vec{w}^{(j)})} := \bigotimes_{i=0}^{f_j-1} \mathrm{Sym}^{k_i^{(j)}-2} \mathcal{O}^2 \otimes \det^{w_i^{(j)}}
$$

where the tensor products are over $\mathcal{O}$.

If we let the group $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ act on the tensor factor $\mathrm{Sym}^{k_i^{(j)}-2} \mathcal{O}^2 \otimes \det^{w_i^{(j)}}$ (here $0 \leq i \leq f_j - 1$) via the embedding $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}}) \to GL_2(\mathcal{O})$ induced by $\sigma_i^{(j)} = \sigma^i \circ \sigma_0^{(j)}$, $W_{(\vec{k}^{(j)}, \vec{w}^{(j)})}$ can be seen as a representation of $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$. We view $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ as a subgroup of $GL_2(\mathcal{O})$ via the embedding $\sigma_0^{(j)}$, and we write the $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$-representation $W_{(\vec{k}^{(j)}, \vec{w}^{(j)})}$ as:

$$
W_{(\vec{k}^{(j)}, \vec{w}^{(j)})} = \bigotimes_{i=0}^{f_j-1} \left(\mathrm{Sym}^{k_i^{(j)}-2} \mathcal{O}^2 \otimes \det^{w_i^{(j)}} \right)^{[i]},
$$

87

where the superscript $[i]$ indicates twisting by the ith power of the Frobenius element σ. In the sequel, unless otherwise stated, we always view $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}}) \subseteq GL_2(\mathcal{O})$ via $\sigma_0^{(j)}$.

Denote by $\tau_{(\vec{k}^{(j)}, \vec{w}^{(j)})}$ the continuous action of $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ on $W_{(\vec{k}^{(j)}, \vec{w}^{(j)})}$ and let $\tau_{(\vec{k}, \vec{w})} = \bigotimes_{j=1}^{r} \tau_{(\vec{k}^{(j)}, \vec{w}^{(j)})}$, where the tensor products are over $\mathcal{O}$ and $\vec{k} = (\vec{k}^{(1)}, ..., \vec{k}^{(r)})$. We have:

$$\tau_{(\vec{k}, \vec{w})} : \prod_{j=1}^{r} GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}}) \longrightarrow \operatorname{Aut} W_{(\vec{k}, \vec{w})},$$

with $W_{(\vec{k}, \vec{w})} = \bigotimes_{j=1}^{r} W_{(\vec{k}^{(j)}, \vec{w}^{(j)})}$ (tensor product over $\mathcal{O}$).

If there is some integer j such that $\vec{k}^{(j)} = (2, ..., 2)$, we say that the weight $\tau_{(\vec{k}, \vec{w})}$ contains a $(2, ..., 2)$-*block* relative to the prime $\mathfrak{P}_j$. This terminology is not standard but it is used throughout the dissertation.

We say that $\tau_{(\vec{k}, \vec{w})}$ is a *holomorphic* weight if there exists an integer w such that:

$$k_i^{(j)} + 2w_i^{(j)} - 1 = w \tag{*}$$

for all $1 \leq j \leq r$ and all $0 \leq i \leq f_j - 1$ (cf. [Hid88]).

The pair $(\vec{k}, \vec{w}) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}^g$ is called the parameter pair for $\tau_{(\vec{k}, \vec{w})}$. If $\tau_{(\vec{k}, \vec{w})}$ is a holomorphic weight, it is also determined by the parameter pair $(\vec{k}, w) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}$, with w as in (*).

Some results on holomorphic weight shiftings

Lemma 6.1.6. Let us view the holomorphic weight $\tau_{(\vec{k}, w)}$ as an $\mathcal{O}$-representation of the fixed level $U \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$. A Hecke character $\psi : (\mathbb{A}_F^\infty)^\times / F^\times \to \mathcal{O}^\times$ is compatible with $\tau_{(\vec{k}, w)}$ if and only if the following two conditions are satisfied:

(a) $\psi(u) = 1$ for all $u \in U_v \cap \mathcal{O}_{F_v}^\times$, where $v \in \mathfrak{M}_{F,f}$ and $v \nmid p$;

(b) $\psi(u) = \left(\operatorname{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u) \right)^{1-w}$ for all $u \in \mathcal{O}_{F_{\mathfrak{P}_j}}^\times$, where $1 \leq j \leq r$.

Proof. The reason for condition (a) is clear, as the representation $\tau_{(\vec{k}, w)}$ factors through $\prod_{j=1}^{r} GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$. Let j be such that $1 \leq j \leq r$ and fix $u \in \mathcal{O}_{F_{\mathfrak{P}_j}}^\times$; recall that we embed $\mathcal{O}_{F_{\mathfrak{P}_j}}$

in $\mathcal{O}$ via $\sigma_0^{(j)}$. The matrix $\begin{pmatrix} u & \\ & u \end{pmatrix} \in GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ acts on $W_{(\vec{k}^{(j)},w)}$ as the automorphism:

$$
\begin{aligned}
\bigotimes_{i=0}^{f_j-1} \left(\sigma^i(u)^{k_i^{(j)}-2+2w_i^{(j)}} \cdot Id_i \right) &= \bigotimes_{i=0}^{f_j-1} \sigma^i(u)^{w-1} \cdot Id_i \\
&= \left(\mathrm{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u) \right)^{w-1} \cdot \bigotimes_{i=0}^{f_j-1} Id_i \\
&= \left(\mathrm{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u) \right)^{w-1} \cdot Id_{W_{(\vec{k}^{(j)},w)}},
\end{aligned}
$$

where Id_i denotes the identity map of the $\mathcal{O}$-vector space:

$$
\left(\mathrm{Sym}^{k_i^{(j)}-2} \mathcal{O}^2 \otimes \det^{w_i^{(j)}} \right)^{[i]},
$$

and we used the assumption that the local extension $F_{\mathfrak{P}_j}/\mathbb{Q}_p$ is unramified with Galois group generated by the restriction of σ to $F_{\mathfrak{P}_j}$. The result now follows, as we need to have $\tau_{|\mathcal{O}_{F_{\mathfrak{P}_j}}^\times}(u) = \psi^{-1}(u) \cdot Id_{W_{(\vec{k},w)}}$. $\blacksquare$

Lemma 6.1.7. Let w be an even integer. Then there exists a continuous character ψ : $(\mathbb{A}_F^\infty)^\times / F^\times \to \mathbb{Z}_p^\times$ such that:

(a) $\psi(u) = 1$ for all $u \in \mathcal{O}_{F_v}^\times$, where $v \in \mathfrak{M}_{F,f}$ and $v \nmid p$;

(b) $\psi(u) = \left(\mathrm{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u) \right)^w$ for all $u \in \mathcal{O}_{F_{\mathfrak{P}_j}}^\times$, where $1 \leq j \leq r$.

Proof. The adèles norm map $(\mathbb{A}_F^\infty)^\times \to (\mathbb{A}_\mathbb{Q}^\infty)^\times$ induces a continuous homomorphism $\mathrm{Nm} : (\mathbb{A}_F^\infty)^\times / F^\times \to (\mathbb{A}_\mathbb{Q}^\infty)^\times / \mathbb{Q}^\times$. The group-theoretic decomposition $(\mathbb{A}_\mathbb{Q}^\infty)^\times = \mathbb{Q}^\times \cdot \hat{\mathbb{Z}}^\times$ induces a continuous isomorphism $\beta : (\mathbb{A}_\mathbb{Q}^\infty)^\times / \mathbb{Q}^\times \to \hat{\mathbb{Z}}^\times / \langle -1 \rangle$. Finally, the map $\prod_l \mathbb{Z}_l^\times \to \mathbb{Z}_p^\times$ defined by sending the tuple $(a_l)_l \in \prod_l \mathbb{Z}_l^\times$ into $a_p^w \in \mathbb{Z}_p^\times$ defines a continuous homomorphism $\alpha : \hat{\mathbb{Z}}^\times / \langle -1 \rangle \to \mathbb{Z}_p^\times$ since w is even. We check that the composition $\psi := \alpha \circ \beta \circ \mathrm{Nm}$ is a Hecke character with the desired properties.

Assume $v = \mathfrak{P}_j | p$ and view a fixed $u \in \mathcal{O}_{F_{\mathfrak{P}_j}}^\times$ as an element of $(\mathbb{A}_F^\infty)^\times$ whose v-component is u and whose v'-component is 1 for all finite places $v' \neq v$ of F. Then $\mathrm{Nm}(u \cdot F^\times) = \mathrm{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u) \cdot \mathbb{Q}^\times$, where we identify $\mathrm{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u)$ with the adèle of $\mathbb{Q}$ whose p-component is the p-adic unit $\mathrm{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u) \in \mathbb{Z}_p^\times$ and whose other components are equal to 1. Then $(\alpha \circ \beta) \left(\mathrm{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u) \cdot \mathbb{Q}^\times \right) = \left(\mathrm{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u) \right)^w \in \mathbb{Z}_p^\times$.

Assume v is a finite place of F lying above some rational prime $l \neq p$ and let $u \in \mathcal{O}_{F_v}^\times$ viewed as an element of $(\mathbb{A}_F^\infty)^\times$ in the usual way. Write $\mathrm{Nm}(u \cdot F^\times) = \mathrm{Nm}_{F_v/\mathbb{Q}_l}(u) \cdot \mathbb{Q}^\times$; since the p-component of $\mathrm{Nm}_{F_v/\mathbb{Q}_l}(u) \in \hat{\mathbb{Z}}^\times$ is trivial, $\psi(u) = 1$. $\blacksquare$

Set $A = \mathcal{O}$ and let D, U, (τ, W_τ) and ψ be as in 6.1.2.

Proposition 6.1.8. Assume $\tau = \tau_{(\vec{k},w)}$ and $\tau' = \tau_{(\vec{k}',w')}$ are *holomorphic $\mathcal{O}$-linear weights* for automorphic forms on D, with $w \equiv w' (\mathrm{mod}\, p - 1)$ and w odd. Assume that $\tau_{(\vec{k},w)}$ and ψ are compatible and that $\bar{\tau}_{(\vec{k},w)}$ is isomorphic to an $\mathbb{F}$-linear U-subrepresentation of $\bar{\tau}_{(\vec{k}',w')}$. Then:

(a) There is a Hecke character $\psi' : (\mathbb{A}_F^\infty)^\times / F^\times \to \mathcal{O}^\times$ which is compatible with $\tau_{(\vec{k}',w')}$ and such that $\bar{\psi}' = \bar{\psi}$;

(b) For any Hecke eigensystem Ω occurring in $S_{\tau,\psi}(U, \mathcal{O})$ there is a finite extension of discrete valuation rings $\mathcal{O}'/\mathcal{O}$ with $\mathfrak{M}_{\mathcal{O}'} \cap \mathcal{O} = \mathfrak{M}_{\mathcal{O}}$ and a Hecke eigensystem Ω' occurring in $S_{\tau',\psi'}(U, \mathcal{O}')$ such that $\Omega'(\mathrm{mod}\, \mathfrak{M}_{\mathcal{O}'}) = \Omega(\mathrm{mod}\, \mathfrak{M}_{\mathcal{O}})$.

Proof. Since $p > 2$, the integer $1 - w'$ is even. By Lemma 6.1.7, there exists a Hecke character $\psi'' : (\mathbb{A}_F^\infty)^\times / F^\times \to \mathbb{Z}_p^\times \subset \mathcal{O}^\times$ such that $\psi''(u) = 1$ for all $v \in \mathfrak{M}_{F,f}$ not lying above p and all $u \in \mathcal{O}_{F_v}^\times$, and $\psi''(u) = \left(\mathrm{Nm}_{F_{\mathfrak{P}_j}/\mathbb{Q}_p}(u) \right)^{1-w'}$ for $u \in \mathcal{O}_{F_{\mathfrak{P}_j}}^\times$ $(1 \leq j \leq r)$. By Lemma 6.1.6, ψ'' is compatible with $\tau_{(\vec{k}',w')}$.

Let α denote the reduction modulo $\mathfrak{M}_{\mathcal{O}}$ of the Hecke character $\psi^{-1}\psi''$. Since $w \equiv w' (\mathrm{mod}\, p - 1)$, by the compatibility of ψ with $\tau_{(\vec{k},w)}$ and by the construction of ψ'', the continuous character α is trivial on the open subgroup

$$\textstyle\prod_{v \nmid p} \left(U_v \cap \mathcal{O}_{F_v}^\times \right) \times \prod_{j=1}^r \mathcal{O}_{F_{\mathfrak{P}_j}}^\times$$

of $\left(\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \right)^\times$. Therefore α factors through a finite discrete quotient of $(\mathbb{A}_F^\infty)^\times$. In particular, the Teichmüller lift $\tilde{\alpha}$ of α is a continuous character $(\mathbb{A}_F^\infty)^\times / F^\times \to \mathcal{O}^\times$. The $\mathcal{O}^\times$-valued Hecke character $\psi' := \psi'' \tilde{\alpha}^{-1}$ is compatible with $\tau_{(\vec{k}',w')}$ and satisfies $\bar{\psi}' = \bar{\psi}$, so that (a) is proved.

Part (b) follows by applying Prop. 6.1.5 with ψ' chosen as in (a). $\blacksquare$

90

Link with classical automorphic forms on $D^\times$

To conclude this paragraph, we make explicit the link between adelic automorphic forms for a definite quaternion algebra D having holomorphic weights, and classical automorphic forms for the algebraic $\mathbb{Q}$-group $\mathbb{D}$ associated to $D^\times$.

Set $A = E$ and let $\tau : \prod_{j=1}^r GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}}) \to \mathrm{Aut}(W_\tau)$ be a weight for adelic automorphic forms on D as considered in 6.1.2; suppose $W_\tau = W_{\tau^{\mathrm{alg}}} \otimes_E W_{\tau^{\mathrm{sm}}}$, where $W_{\tau^{\mathrm{sm}}}$ is a smooth irreducible E-representation of $\prod_{j=1}^r GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$, and

$$W_{\tau^{\mathrm{alg}}} = \bigotimes_{j=1}^r \bigotimes_{i=0}^{f_j-1} \left(\mathrm{Sym}^{k_i^{(j)}-2} E^2 \otimes \det^{w_i^{(j)}} \right)^{[i]}$$

is an irreducible algebraic representation of $\mathbb{D}(\mathbb{Q}_p) = (D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times = \prod_{j=1}^r GL_2(F_{\mathfrak{P}_j})$. We assume that $k_i^{(j)} + 2w_i^{(j)} - 1$ equals some fixed integer w for all $1 \leq j \leq r$ and all $0 \leq i \leq f_j - 1$. Recall that, as usual, we see $F_{\mathfrak{P}_j}$ embedded in E via $\sigma_0^{(j)}$ for $1 \leq j \leq r$; we can also write $W_{\tau^{\mathrm{alg}}} = \bigotimes_{\sigma:F\hookrightarrow E} \left(\mathrm{Sym}^{k_\sigma-2} E^2 \otimes \det^{w_\sigma} \right)$. Let $\psi : (\mathbb{A}_F^\infty)^\times / F^\times \to E^\times$ be a Hecke character compatible with τ.

Fix an isomorphism $\bar{\mathbb{Q}}_p \simeq \mathbb{C}$, inducing an embedding $E \hookrightarrow \mathbb{C}$. View $W_{\tau_{\mathbb{C}}^{\mathrm{alg}}} := W_{\tau^{\mathrm{alg}}} \otimes_E \mathbb{C}$ (resp. $W_{\tau_{\mathbb{C}}^{\mathrm{sm}}} := W_{\tau^{\mathrm{sm}}} \otimes_E \mathbb{C}$) as a complex representation of $\mathbb{D}(\mathbb{R}) := (D \otimes_{\mathbb{Q}} \mathbb{R})^\times \subset \mathbb{D}(\mathbb{C}) \simeq \mathbb{D}(\bar{\mathbb{Q}}_p)$ (resp. of $\prod_{j=1}^r GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$). Let $W_{\tau_{\mathbb{C}}} := W_\tau \otimes_E \mathbb{C}$ be the corresponding complex representation of $\prod_{j=1}^r GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}}) \times \prod_{v|\infty}(\mathcal{O}_D)_v^\times$.

Let U' be a compact open subgroup of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ such that $U' = \prod_{v \in \mathfrak{M}_{F,f}} U'_v$, where $U'_v = U_v$ if $v \nmid p$ and, for $v_j|p$, $U'_{v_j} \subseteq GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ acts trivially on $W_{\tau^{\mathrm{sm}}}$. Denote by

$$\mathcal{C}^\infty(D^\times \backslash (D \otimes_F \mathbb{A}_F)^\times / U')$$

the complex vector space of smooth functions $f : D^\times \backslash (D \otimes_F \mathbb{A}_F)^\times \to \mathbb{C}$ which are invariant by the action of U'. Let $W_{\tau_{\mathbb{C}}}^*$ be the $\mathbb{C}$-linear dual of $W_{\tau_{\mathbb{C}}}$.

Define a map:

$$\alpha : S_{\tau,\psi}(U, E) \longrightarrow \mathrm{Hom}_{(D\otimes_{\mathbb{Q}}\mathbb{R})^\times} \left(W_{\tau_{\mathbb{C}}}^*, \mathcal{C}^\infty(D^\times \backslash (D \otimes_F \mathbb{A}_F)^\times / U') \right)$$

by sending $f \in S_{\tau,\psi}(U, E)$ to the assignment:

$$w^* \longmapsto (g \longmapsto w^*(\tau_{\mathbb{C}}^{\mathrm{alg}}(g_\infty^{-1}) \tau^{\mathrm{alg}}(g_p) f(g^\infty))),$$

where $w^* \in W^*_{\tau_{\mathbb{C}}}$ and $g \in (D \otimes_F \mathbb{A}_F)^\times$. We have the following (cf. [Kis09b], 3.1.14):

Proposition 6.1.9. The map α identifies $S_{\tau,\psi}(U, E) \otimes_E \mathbb{C}$ with a space of automorphic forms for the group $D^\times$ having central character $\psi_{\mathbb{C}}$ given by

$$\psi_{\mathbb{C}}(g) = \mathrm{Nm}_{F/\mathbb{Q}}(g_\infty)^{1-w} \mathrm{Nm}_{F/\mathbb{Q}}(g_p)^{w-1} \psi(g^\infty)$$

for $g \in (D \otimes_F \mathbb{A}_F)^\times$.

If $\pi = \bigotimes_v \pi_v$ is an irreducible automorphic representation for the group $D^\times$, then π is generated by an element in $\alpha(f)(W^*_{\tau_{\mathbb{C}}})$ for some $f \in S_{\tau,\psi}(U, E')$, some U small enough and some $E' \supseteq E$ big enough, if and only if $\pi_\infty \simeq W^*_{\tau^{\mathrm{alg}}_{\mathbb{C}}}$ and $\bigotimes_{v|p} \pi_v$ contains $W^*_{\tau^{\mathrm{sm}}_{\mathbb{C}}}$ as a representation of $\prod_{j=1}^r GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$.

Assume furthermore that $F/\mathbb{Q}$ has even degree and that we choose Σ to be the empty set. Let τ be a holomorphic weight with parameters $(\vec{k}, w) \in \mathbb{Z}^g_{\geq 2} \times \mathbb{Z}$ and let $\psi : \mathbb{A}^\times_F / F^\times \to \bar{\mathbb{Q}}^\times_p$ be a continuous character such that $\psi(a) = (\mathrm{Nm}\, a)^{1-w}$ for all a contained inside an open subgroup of $(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$. Fix an isomorphism $\bar{\mathbb{Q}}_p \simeq \mathbb{C}$ as before.

As a consequence of the classical Jacquet-Langlands theorem, we can identify the complexification of the space $S_{\tau,\psi}(U, \bar{\mathbb{Q}}_p)$ ($\vec{k} \neq \vec{2}$) with a space of regular algebraic cuspidal automorphic representations π of $GL_2(\mathbb{A}_F)$ such that π_∞ has weight $(\vec{k}, w)$ and π has central character ψ_∞. If $\vec{k} = \vec{2}$ the identification works if we consider, instead of $S_{\tau,\psi}(U, \bar{\mathbb{Q}}_p)$, the quotient of $S_{\tau,\psi}(U, \bar{\mathbb{Q}}_p)$ by the subspace of functions factoring through the reduced norm. For a detailed formulation of these last facts, cf. Th. 2.1 of [Hid88] and Lemma 1.3 of [Tay06].

6.1.5 Holomorphic weight shiftings via generalized Dickson invariants and D-operators

Let q be a power of p. The intertwining operators between $\mathbb{F}_q$-representations of $GL_2(\mathbb{F}_q)$ studied in Section 5.2 allow us to produce weight shiftings between spaces of automorphic forms having holomorphic weights.

Main theorem

Let us set $A = \mathcal{O}$ and let D, U, (τ, W_τ) and ψ be as in 6.1.2. Recall in particular that U is small enough and that ψ is compatible with τ. For simplicity, if τ is a holomorphic weight with parameters $(\vec{k}, w) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}$ and $f \in S_{\tau, \psi}(U, \mathcal{O})$, we also say that f has weight $(\vec{k}, w)$ or that f has weight $\vec{k}$. Recall that we write $\vec{k} = (\vec{k}^{(1)}, ..., \vec{k}^{(r)})$ with $\vec{k}^{(j)} = (k_0^{(j)}, ..., k_{f_j-1}^{(j)}) \in \mathbb{Z}_{\geq 2}^{f_j}$ for $1 \leq j \leq r$, and that we define the vector $\vec{w}^{(j)} = (w_0^{(j)}, ..., w_{f_j-1}^{(j)}) \in \mathbb{Z}^{f_j}$ by the relations $k_i^{(j)} + 2w_i^{(j)} - 1 = w$, for all $0 \leq i \leq f_j - 1$.

Theorem 6.1.10. *Assume τ is a holomorphic $\mathcal{O}$-linear weight with parameters $(\vec{k}, w) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}$ with w odd. Let $f = \min\{f_1, ..., f_r\}$ and fix an integer β such that $1 \leq \beta \leq f$. For any integers i, j with $1 \leq j \leq r$ and $0 \leq i \leq f_{j-1}$ choose:*

$$a_i^{(j)} \in \{p^\beta - 1, p^\beta + 1\}.$$

Set $\vec{a} = (\vec{a}^{(1)}, ..., \vec{a}^{(r)})$ with $\vec{a}^{(j)} = (a_0^{(j)}, ..., a_{f_j-1}^{(j)})$, and let $w' = w + (p^\beta - 1)$. Assume at least one of the following conditions is satisfied:

(*) Let j be any integer such that $1 \leq j \leq r$ and $\beta < f_j$. Then for any i with $0 \leq i \leq f_j - 1$ and $a_i^{(j)} = p^\beta - 1$, we have that $2 < k_i^{(j)} \leq p + 1$, $2 \leq k_{i+f_j-\beta}^{(j)} \leq p + 1$ and if $i' \neq i$ is another integer such that $0 \leq i' \leq f_j - 1$ and $a_{i'}^{(j)} = p^\beta - 1$, we also have $i \not\equiv i' - \beta \pmod{f_j}$.

Let j be any integer such that $1 \leq j \leq r$ and $\beta = f_j$. Then for any i with $0 \leq i \leq f_j - 1$ and $a_i^{(j)} = p^\beta - 1$, we have that $2 < k_i^{(j)} \leq p + 1$.

(**) The weight $(\vec{k}, w)$ is p-small and generic, i.e., $2 < k_i^{(j)} \leq p + 1$ for all i, j.

Let $\psi : (\mathbb{A}_F^\infty)^\times / F^\times \to \mathcal{O}^\times$ be a Hecke character compatible with τ. Then, if Ω is a Hecke eigensystem occurring in the space $S_{\tau, \psi}(U, \mathcal{O})$, there is a finite local extension of discrete valuation rings $\mathcal{O}'/\mathcal{O}$ and an $\mathcal{O}'$-valued Hecke eigensystem Ω' occurring in holomorphic weight $(\vec{k} + \vec{a}, w')$ and with associated Hecke character ψ' such that:

$$\Omega'(\mathrm{mod}\,\mathfrak{M}_{\mathcal{O}'}) = \Omega(\mathrm{mod}\,\mathfrak{M}_{\mathcal{O}}).$$

93

The character ψ' is compatible with the weight $(\vec{k} + \vec{a}, w')$ and it can be chosen so that $\bar{\psi}' = \bar{\psi}$.

Proof. Recall that τ is the $\mathcal{O}$-linear representation

$$\tau : \prod_{j=1}^{r} GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}}) \to \operatorname{Aut} W,$$

where $W = \bigotimes_{j=1}^{r} W_j$, $W_j = \bigotimes_{i=0}^{f_j-1} \left(\operatorname{Sym}^{k_i^{(j)}-2} \mathcal{O}^2 \otimes \det^{w_i^{(j)}} \right)^{[i]}$, $k_i^{(j)} + 2w_i^{(j)} - 1 = w$. The group $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ acts on W via the action on W_j induced by the embedding $\sigma_0^{(j)} : GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}}) \hookrightarrow GL_2(\mathcal{O})$. The superscript $[i]$ indicates twisting by the ith power of the arithmetic Frobenius element of $\operatorname{Gal}(E/\mathbb{Q}_p)$.

The $\mathbb{F}$-linear representation $\bar{W}_j := W_j \otimes_{\mathcal{O}} \mathbb{F}$ of $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ factors through the reduction map $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}}) \to GL_2(\mathbb{F}_{\mathfrak{P}_j})$; using the notation introduced in 5.1 we can identify $\bar{W}_j$ with the $\mathbb{F}[GL_2(\mathbb{F}_{\mathfrak{P}_j})]$-module

$$\bar{W}_j = \bigotimes_{i=0}^{f_j-1} \left(M_{k_i^{(j)}-2} \otimes \det^{w_i^{(j)}} \right)^{[i]},$$

where we see $GL_2(\mathbb{F}_{\mathfrak{P}_j}) \hookrightarrow GL_2(\mathbb{F})$ via $\bar{\sigma}_0^{(j)}$, and the superscript $[i]$ indicates twisting by the ith power of the arithmetic Frobenius element of $\operatorname{Gal}(\mathbb{F}/\mathbb{F}_p)$.

For any fixed integer j, $1 \le j \le r$, let $\mathcal{I}_j = \{i : a_i^{(j)} = p^{\beta}+1\}$ and $\mathcal{D}_j = \{i : a_i^{(j)} = p^{\beta}-1\}$. For $i \in \mathcal{I}_j$ set $\vartheta_i^{(j)} := \Theta_{f_j-\beta}^{[i]}$ if $\beta < f_j$ and $\vartheta_i^{(j)} := \Theta^{[i]}$ if $\beta = f_j$, where $\Theta_{f_j-\beta}^{[i]}$ and $\Theta^{[i]}$ are the generalized Dickson invariants for the group $GL_2(\mathbb{F}_{\mathfrak{P}_j}) \simeq GL_2(\mathbb{F}_{p^{f_j}})$ as defined in 5.2.3. For $i \in \mathcal{D}_j$ set $\delta_i^{(j)} := D_{f_j-\beta}^{[i]}$ if $\beta < f_j$ and $\delta_i^{(j)} := D^{[i]}$ if $\beta = f_j$, where $D_{f_j-\beta}^{[i]}$ and $D^{[i]}$ are the generalized D-operators for $GL_2(\mathbb{F}_{\mathfrak{P}_j})$ defined in 5.2.3. Set:

$$\Lambda_j = \left(\bigodot_{i \in \mathcal{I}_j} \vartheta_i^{(j)} \right) \circ \left(\bigodot_{i \in \mathcal{D}_j} \delta_i^{(j)} \right),$$

where the symbol $\bigodot$ denotes composition of functions, and each of the two composition factors above is computed by ordering $\mathcal{I}_j$ and $\mathcal{D}_j$ in the natural way. As seen in section 5.2, the operators $\vartheta_i^{(j)}$ and $\delta_i^{(j)}$ give rise to morphisms of $\mathbb{F}_{\mathfrak{P}_j}[GL_2(\mathbb{F}_{\mathfrak{P}_j})]$-modules, and hence to morphisms of $\mathbb{F}[GL_2(\mathbb{F}_{\mathfrak{P}_j})]$-modules via the scalar extension $\bar{\sigma}_0^{(j)} : \mathbb{F}_{\mathfrak{P}_j} \hookrightarrow \mathbb{F}$. We deduce that Λ_j induces a $GL_2(\mathbb{F}_{\mathfrak{P}_j})$-equivariant and $\mathbb{F}$-linear morphism:

$$\Lambda_j : \bar{W}_j \to \bar{W}_j',$$

94

where $\bar{W}_j'$ is the $\mathbb{F}[GL_2(\mathbb{F}_{\mathfrak{P}_j})]$-module:

$$\bar{W}_j' \ := \ \bigotimes_{i \in \mathcal{T}_j} \left(M_{k_i^{(j)}+(p^\beta+1)-2} \otimes \det{}^{w_i^{(j)}-1} \right)^{[i]}$$
$$\otimes \bigotimes_{i \in \mathcal{D}_j} \left(M_{k_i^{(j)}+(p^\beta-1)-2} \otimes \det{}^{w_i^{(j)}} \right)^{[i]}.$$

Indeed, by Th. 5.2.12, $\Theta_{f_j-\beta}^{[i]}$ increases $k_i^{(j)}$ by 1, $k_{i+f_j-\beta}^{(j)}$ by p^β, $w_i^{(j)}$ by -1, and does not change $k_s^{(j)}$ for $s \neq i, i+f_j-\beta$ or $w_s^{(j)}$ for $s \neq i$; $\Theta^{[i]}$ increases $k_i^{(j)}$ by $p^{f_j}+1$, $w_i^{(j)}$ by -1, and does not change $k_s^{(j)}$ or $w_s^{(j)}$ for $s \neq i$. On the other hand, by Th. 5.2.20, the operator $D_{f_j-\beta}^{[i]}$ increases $k_i^{(j)}$ by -1, $k_{i+f_j-\beta}^{(j)}$ by p^β, and does not change $k_s^{(j)}$ for $s \neq i, i+f_j-\beta$ or $w_s^{(j)}$ for any s; $D^{[i]}$ increases $k_i^{(j)}$ by $p^{f_j}-1$, and does not change $k_s^{(j)}$ for $s \neq i$ or $w_s^{(j)}$ for any s.

By Th. 5.2.12, $\bigodot_{i \in \mathcal{T}_j} \vartheta_i^{(j)}$ is injective. If $(*)$ is satisfied, the injectivity statement of Th. 5.2.20 implies that $\bigodot_{i \in \mathcal{D}_j} \delta_i^{(j)}$ is injective on $\bar{W}_j$. The image of

$$\bigotimes_{i=0}^{f_j-1} \left(X^{k_i^{(j)}-2} \otimes 1 \right)^{[i]} \in \bar{W}_j$$

under $\bigodot_{i \in \mathcal{D}_j} \delta_i^{(j)}$ is easily seen to be of the form $\prod_{i \in \mathcal{D}_j}(k_i^{(j)}-2) \cdot u$ for some non-zero $u \in \bar{W}_j$. If $(**)$ holds, $\prod_{i \in \mathcal{D}_j}(k_i^{(j)}-2)$ is non-zero in $\mathbb{F}$ and, being $\bar{W}_j$ an irreducible representation of $GL_2(\mathbb{F}_{\mathfrak{P}_j})$, we deduce that $\bigodot_{i \in \mathcal{D}_j} \delta_i^{(j)}$ is injective on $\bar{W}_j$. We conclude that under assumptions $(*)$ or $(**)$, all the maps Λ_j for $1 \leq j \leq r$ are injective.

Let $b_i^{(j)} = -1$ if $i \in \mathcal{T}_j$ and $b_i^{(j)} = 0$ if $i \in \mathcal{D}_j$. Define the $\mathcal{O}[GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})]$-module:

$$W_j' = \bigotimes_{i=0}^{f_j-1} \left(\mathrm{Sym}^{k_i^{(j)}+a_i^{(j)}-2} \mathcal{O}^2 \otimes \det{}^{w_i^{(j)}+b_i^{(j)}} \right)^{[i]},$$

so that $W_j' \otimes_{\mathcal{O}} \mathbb{F} = \bar{W}_j'$ as $\mathbb{F}$-representations of $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ or, equivalently, of $GL_2(\mathbb{F}_{\mathfrak{P}_j})$. Set $W' = \bigotimes_{j=1}^{r} W_j'$ and denote by τ' the action of U on W' induced by the projection $U \to \prod_{j=1}^{r} GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$. Let $w' = w + (p^\beta - 1)$; for all the values of i and j for which the following integers are defined, we have $k_i^{(j)} + a_i^{(j)} \geq k_i^{(j)} \geq 2$ and:

$$\left(k_i^{(j)} + a_i^{(j)} \right) + 2 \left(w_i^{(j)} + b_i^{(j)} \right) - 1$$
$$= \left(k_i^{(j)} + 2w_i^{(j)} - 1 \right) + p^\beta - 1$$
$$= w + (p^\beta - 1).$$

Therefore τ' is a holomorphic weight for automorphic forms on D with parameters $(\vec{k} + \vec{a}, w') \in \mathbb{Z}^g_{\geq 2} \times \mathbb{Z}$.

The injections Λ_j $(1 \leq j \leq r)$ constructed above allow us to see $\bar{W} = \bigotimes^r_{j=1} \bar{W}_j$ as an $\mathbb{F}$-linear U-subrepresentation of $\bar{W}' = \bigotimes^r_{j=1} \bar{W}'_j$. Since w is odd and $w \equiv w' (\mathrm{mod}\, p - 1)$, we can apply Prop. 6.1.8. We conclude that there exists a Hecke character $\psi' : (\mathbb{A}^\infty_F)^\times / F^\times \to \mathcal{O}^\times$ compatible with τ' and such that $\bar{\psi}' = \bar{\psi}$; furthermore, for any Hecke eigensystem Ω occurring in $S_{\tau,\psi}(U, \mathcal{O})$ there is a finite extension of discrete valuation rings $\mathcal{O}'/\mathcal{O}$ with $\mathfrak{M}_{\mathcal{O}'} \cap \mathcal{O} = \mathfrak{M}_\mathcal{O}$ and a Hecke eigensystem Ω' occurring in $S_{\tau',\psi'}(U, \mathcal{O}')$ such that $\Omega'(\mathrm{mod}\, \mathfrak{M}_{\mathcal{O}'}) = \Omega(\mathrm{mod}\, \mathfrak{M}_\mathcal{O})$. ∎

Corollary 6.1.11. Under the same notation and assumptions of Th. 6.1.10, any $\bar{\mathbb{F}}_p$-linear continuous Galois representation arising from a Hecke eigenform in $S_{\tau,\psi}(U, \mathcal{O})$, where τ is a holomorphic weight of parameter $\vec{k}$, also arises from an eigenform in $S_{\tau',\psi'}(U, \bar{\mathbb{Z}}_p)$, where τ' is a holomorphic weight of parameters $\vec{k} + \vec{a}$ and ψ' is some $\mathcal{O}^\times$-valued Hecke character compatible with τ' and such that $\bar{\psi}' = \bar{\psi}$.

Remark 6.1.12. We remark what follows:

1. Condition $(*)$ of Th. 6.1.10 is true if, for example, for any j with $1 \leq j \leq r$, there is at most one i, $0 \leq i \leq f_j - 1$, such that $a^{(j)}_i = p^\beta - 1$, and for these values of i and j we have $2 < k^{(j)}_i \leq p + 1$ and $2 \leq k^{(j)}_{i+f_j-\beta} \leq p + 1$.

2. The reason for which in the above result we limit $a^{(j)}_i$ to be in the set $\{p^\beta - 1, p^\beta + 1\}$ for all i, j is that we want to preserve the holomorphicity of the weights of the automorphic forms involved. More weight shiftings are possible using the generalized Dickson and D-operators if we do not impose the holomorphicity condition.

3. As a consequence of Rem. 5.2.13 and Rem. 5.2.21, the above result gives rise to more holomorphic weight shiftings than the ones obtained by the theory of generalized theta operators and Hasse invariants for geometric $(\mathrm{mod}\, p)$ Hilbert modular forms (cf. 6.1.1).

6.2 Shiftings for weights containing $(2, ..., 2)$-blocks

While the generalized Dickson invariants induce injective maps on the trivial $\mathbb{F}$-representation of $GL_2(\mathbb{F}_{\mathfrak{P}_j})$, the D-operators are identically zero on this module. Starting with automorphic forms whose weight contains a $(2, ..., 2)$-block (cf. definition in 6.1.4), we can then produce weight shiftings through the operators $\Theta_\alpha^{[\beta]}$ but we cannot always successfully use the operators $D_\alpha^{[\beta]}$. On the other side, the study of weight shiftings "by $\overrightarrow{p-1}$" for automorphic forms whose weight contains a $(2, ..., 2)$-block is motivated by the weight part of Serre's modularity conjecture for totally real fields (cf. Rem. 6.2.4 below).

In this section we slightly generalize a result of Edixhoven and Khare (cf. [EK03]) to produce weight shiftings "by $\overrightarrow{p-1}$" starting from forms whose weight is not necessarily parallel but contains $(2, ..., 2)$-blocks relative to some primes of F above p. We always assume that $p > 2$ is unramified in the totally real number field F.

We keep the notation introduced in 6.1, and we furthermore assume that F has even degree over $\mathbb{Q}$ and that the quaternion F-algebra D is ramified at all and only the infinite places of F, i.e., $\Sigma = \varnothing$. We fix an isomorphism $(D \otimes_F \mathbb{A}_F^\infty)^\times \simeq GL_2(\mathbb{A}_F^\infty)$.

The symbols $\mathbb{F}$, U, (τ, W_τ), ψ, S and $\mathbb{T}_{S,\mathbb{F}}^{univ}$ will have the same meaning as in 6.1.2. We assume that τ is a (non necessarily holomorphic) $\mathbb{F}$-linear weight with parameters $(\vec{k}, \vec{w}) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}^g$, where $\vec{k} = (\vec{k}^{(1)}, ..., \vec{k}^{(r)})$ and $\vec{k}^{(j)} = (k_0^{(j)}, ..., k_{f_j-1}^{(j)}) \in \mathbb{Z}_{\geq 2}^{f_j}$; $\vec{w} = (\vec{w}^{(1)}, ..., \vec{w}^{(r)})$ and $\vec{w}^{(j)} = (w_0^{(j)}, ..., w_{f_j-1}^{(j)}) \in \mathbb{Z}^{f_j}$, for $1 \leq j \leq r$.

We write $W_\tau = \bigotimes_{j=1}^r W_{\tau_j}$ where W_{τ_j} is the $\mathbb{F}$-representation of $GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ defined by:

$$W_{\tau_j} = \bigotimes_{i=0}^{f_j-1} \left(\text{Sym}^{k_i^{(j)}-2} \mathbb{F}^2 \otimes \det^{w_i^{(j)}} \right)^{[i]}.$$

If the weight τ is holomorphic, it is also determined by the pair $(\vec{k}, w) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}$ where $k_i^{(j)} + 2w_i^{(j)} - 1 = w$, for all i and j.

Choose a prime $\mathfrak{P}$ of F above p and let ϖ be a fixed choice of uniformizer for the ring of integers of the completion of F at $\mathfrak{P}$. We can assume, up to relabeling, that $\mathfrak{P} = \mathfrak{P}_1$. Define

the matrix of $GL_2(F_{\mathfrak{P}_1})$:

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix},$$

and view it as an element of $GL_2(\mathbb{A}_F^\infty)$ whose components away from $\mathfrak{P}_1$ are trivial.

If g is an element of $GL_2(\mathbb{A}_F^\infty)$ and Q is a finite set of finite places of F, we denote by g^Q the element of $GL_2(\mathbb{A}_F^\infty)$ whose components at each place of Q are trivial, and whose components away from Q coincide with those of g. We let $g_Q = g/g^Q$. A similar convention is used for subgroups of $GL_2(\mathbb{A}_F^\infty)$ which are products of subgroups of $GL_2(F_v)$ for v varying over the finite places of F. In particular, by assumption we have $U_p = GL_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$.

We denote the action by right translation of $GL_2(\mathbb{A}_F^\infty)$ on $S_{\tau,\psi}(U, \mathbb{F})$ by a dot.

Set:

$$U_0 = \left\{ u \in U : u_{\mathfrak{P}_1} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod}\,\varpi) \right\}.$$

By restricting τ to U_0, we define $S_{\tau,\psi}(U_0, \mathbb{F})$ as in Def. 6.1.1; notice that the level of the automorphic forms belonging to this space is not prime-to-p.

We have the following result, which is a not-prime-to-p version of Lemma 3.1 of [Tay06]:

Lemma 6.2.1. Assume that τ is an irreducible (non necessarily holomorphic) $\mathbb{F}$-linear weight with parameters $(\vec{k}, \vec{w}) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}^g$ such that $\vec{k}^{(1)} = \vec{2}$. Then the map:

$$\alpha : S_{\tau,\psi}(U, \mathbb{F}) \oplus S_{\tau,\psi}(U, \mathbb{F}) \longrightarrow S_{\tau,\psi}(U_0, \mathbb{F})$$

defined by:

$$(f_1, f_2) \longmapsto f_1 + \Pi \cdot f_2$$

is a Hecke-equivariant $\mathbb{F}$-morphism whose kernel is Eisenstein, i.e., the localization $(\ker \alpha)_{\mathfrak{M}}$ vanishes for all maximal ideals $\mathfrak{M}$ of $\mathbb{T}_{S,\mathbb{F}}^{univ}$ which are non-Eisenstein.

Proof. It is straightforward to check that α is well defined, using the fact that $GL_2(\mathcal{O}_{F_{\mathfrak{P}_1}})$ acts on W_{τ_1} via an integral power of the $(\mathrm{mod}\,\varpi)$ determinant character. Also, α is equivariant for the action of the algebra $\mathbb{T}_{S,\mathbb{F}}^{univ}$.

Write $\Pi U \Pi^{-1} = U^{\mathfrak{P}_1} \times \Pi GL_2(\mathcal{O}_{F_{\mathfrak{P}_1}})\Pi^{-1}$. Define an $\mathbb{F}$-linear action of the subgroup $\Pi U \Pi^{-1}$ of $GL_2(\mathbb{A}_F^\infty)$ on W_τ by letting $U^{\mathfrak{P}_1}$ act on $\bigotimes_{j=2}^r W_{\tau_j}$ via the restriction of τ to $U^{\mathfrak{P}_1}$, and by letting $\Pi GL_2(\mathcal{O}_{F_{\mathfrak{P}_1}})\Pi^{-1}$ act on W_{τ_1} via the reduction modulo ϖ of the determinant character raised to the power of $\sum_{i=0}^{f_1-1} w_i^{(1)} p^i$. Observe that this action is compatible with the given action τ of U on W_τ.

If $(f_1, f_2) \in \ker \alpha$, we see that $f_1(gu) = u^{-1}f_1(g)$ for all u in U and all u in $\Pi U \Pi^{-1}$, so that $f_1(gu) = u^{-1}f_1(g)$ for every u in $SL_2(F_{\mathfrak{P}_1})U \subset GL_2(\mathbb{A}_F^\infty)$. Here $SL_2(F_{\mathfrak{P}_1})$ acts on W_{τ_1} trivially.

Assume that $W_\tau^U \neq \{0\}$, i.e., that $W_\tau = \mathbb{F}$ is the trivial representation of U. If $(f_1, f_2) \in \ker \alpha$, then f_1 is invariant under right translations by elements of $D^\times U$; strong approximation for SL_2 then implies that f_1 is invariant under right translations by any element of $SL_2(\mathbb{A}_F^\infty)$, and hence it factors through the reduced norm map $D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \to F^\times \backslash (\mathbb{A}_F^\infty)^\times$. Since any maximal ideal of $\mathbb{T}_{S,\mathbb{F}}^{univ}$ in the support of the space of functions $D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \to W_\tau$ factoring through the reduced norm is Eisenstein, we obtain the desired result.

Assume now that $W_\tau^U = \{0\}$ and let $(f_1, f_2) \in \ker \alpha$. Using strong approximation, we see that for any $g \in GL_2(\mathbb{A}_F^\infty)$ and $u \in \prod_{j=1}^r GL_2(\mathcal{O}_{F_{\mathfrak{P}_j}})$ we can find an element $\delta \in D^\times \cap gSL_2(F_{\mathfrak{P}_1})Ug^{-1}$ such that for all $j = 1, ..., r$:

$$g_{\mathfrak{P}_j}^{-1} \delta g_{\mathfrak{P}_j} \in u_{\mathfrak{P}_j} + M_2(\mathfrak{P}_j).$$

In particular, we obtain:

$$f_1(g) = f_1(\delta^{-1}g) = f_1(g(g^{-1}\delta^{-1}g))$$

and, since $g^{-1}\delta^{-1}g \in SL_2(F_{\mathfrak{P}_1})U$:

$$f_1(g) = \left(g^{-1}\delta g\right) f_1(g) = uf_1(g).$$

Since u is arbitrary, we conclude that $f_1(g) \in W_\tau^U$ for any $g \in GL_2(\mathbb{A}_F^\infty)$, so that $f_1 = 0$, $f_2 = 0$ and α is injective. $\blacksquare$

Let $\mathcal{F}_\tau$ denote the space consisting of all the functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \to W_\tau,$$

99

and define a left $\mathbb{F}$-linear action of U on $\mathcal{F}_\tau$ by:

$$(uf)(g) = \tau(u)f(gu)$$

for all $u \in U$, $g \in (D \otimes_F \mathbb{A}_F^\infty)^\times$ and $f \in \mathcal{F}_\tau$. Set:

$$S_\tau(U, \mathbb{F}) = H^0(U, \mathcal{F}_\tau).$$

In what follows, we work for simplicity with the spaces $S_\tau(U, \mathbb{F})$, forgetting about the action of the center of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ on $\mathcal{F}_\tau$. Following the proof of Prop. 1 at page 48 of [EK03], and using Lemma 6.2.1, we obtain the following result:

Theorem 6.2.2. *Assume that τ is an irreducible (non necessarily holomorphic) $\mathbb{F}$-linear weight with parameters $(\vec{k}, \vec{w}) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}^g$ such that $\vec{k}^{(j)} = \vec{2}$ for some $1 \leq j \leq r$. Let τ' be the $\mathbb{F}$-linear weight associated to the parameters $\vec{k}' = (\vec{k}^{(1)}, ..., \overrightarrow{\vec{k}^{(j)}+p-1}, ..., \vec{k}^{(r)})$ and $\vec{w}' = \vec{w}$. For any non-Eisenstein maximal ideal $\mathfrak{M}$ of $\mathbb{T}_{S,\mathbb{F}}^{univ}$, there is an injective Hecke-equivariant $\mathbb{F}$-morphism:*

$$S_\tau(U, \mathbb{F})_\mathfrak{M} \hookrightarrow S_{\tau'}(U, \mathbb{F})_\mathfrak{M}.$$

Proof. Assume without loss of generality that $j = 1$. Via the surjection $U \to GL_2(\mathbb{F}_{\mathfrak{P}_1})$, the group U acts on the $\mathbb{F}_{\mathfrak{P}_1}$-points $\mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})$ of the projective $\mathbb{F}_p$-line, and we can identify the coset space U/U_0 with $\mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})$. Recall that we are viewing $\mathbb{F}_{\mathfrak{P}_1}$ as a subfield of $\mathbb{F}$ via the fixed embedding $\bar\sigma_0^{(1)}$.

By Shapiro's lemma applied to the pair (U, U_0) and the left $\mathbb{F}[U]$-module $\mathcal{F}_\tau$, we obtain an isomorphism:

$$H^0(U_0, \mathcal{F}_\tau) \xrightarrow{\sim} H^0(U, \mathcal{F}_\tau \otimes_\mathbb{F} \mathbb{F}[\mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})]). \tag{1}$$

Here U acts on $\mathbb{F}[\mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})] = \{\varphi : \mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1}) \to \mathbb{F}\}$ via its quotient $GL_2(\mathbb{F}_{\mathfrak{P}_1})$ and by the rule $(u\varphi)(P) = \varphi(u^{-1}P)$ for $u \in GL_2(\mathbb{F}_{\mathfrak{P}_1})$ and $P \in \mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})$. Furthermore U acts diagonally on $\mathcal{F}_\tau \otimes \mathbb{F}[\mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})]$. By Lemma 1.1.4 of [AS86a], the isomorphism (1) preserves the Hecke action on both sides.

By Lemma 2.6 of [Red10], there is an isomorphism of $\mathbb{F}[GL_2(\mathbb{F}_{\mathfrak{P}_1})]$-modules:

$$\mathbb{F}[\mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})] \simeq \mathbb{F} \oplus \mathrm{Sym}^{p^{f_1}-1}(\mathbb{F}^2) = M_0 \oplus M_{p^{f_1}-1},$$

inducing a surjection:

$$H^0(U, \mathcal{F}_\tau \otimes \mathbb{F}[\mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})]) \longrightarrow H^0(U, \mathcal{F}_\tau \otimes M_{p^{f_1}-1}). \tag{2}$$

Observe that the composition of the restriction map $H^0(U, \mathcal{F}_\tau) \to H^0(U_0, \mathcal{F}_\tau)$ with the surjection:

$$H^0(U_0, \mathcal{F}_\tau) \simeq H^0(U, \mathcal{F}_\tau \otimes \mathbb{F}[\mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})]) \to H^0(U, \mathcal{F}_\tau)$$

is given by $f \mapsto \frac{1}{[U:U_0]} \sum_{u \in U/U_0} 1 \otimes uf = 1 \otimes f$. This implies that the first summand of $H^0(U, \mathcal{F}_\tau)^{\oplus 2}$ is identified via the map α of Lemma 6.2.1 and the Shapiro isomorphism with the direct summand $H^0(U, \mathcal{F}_\tau)$ of $H^0(U, \mathcal{F}_\tau \otimes \mathbb{F}[\mathbb{P}^1(\mathbb{F}_{\mathfrak{P}_1})])$.

Using the map α, the Shapiro isomorphism, the projection (2), and the isomorphism of $\mathbb{F}[GL_2(\mathbb{F}_{p^{f_1}})]$-modules $M_{p^{f_1}-1} \simeq \bigotimes_{i=0}^{f_1-1} M_{p-1}^{[i]}$, we obtain a Hecke equivariant morphism:

$$\beta : H^0(U, \mathcal{F}_\tau)^{\oplus 2} \longrightarrow H^0\left(U, \mathcal{F}_\tau \otimes \bigotimes_{i=0}^{f_1-1} M_{p-1}^{[i]}\right).$$

By Lemma 6.2.1, precomposing β with the injection $H^0(U, \mathcal{F}_\tau) \hookrightarrow H^0(U, \mathcal{F}_\tau)^{\oplus 2}$ given by $f \mapsto (0, f)$ we obtain a Hecke equivariant injective morphism:

$$H^0(U, \mathcal{F}_\tau)_\mathfrak{M} \hookrightarrow H^0\left(U, \mathcal{F}_\tau \otimes \bigotimes_{i=0}^{f_1-1} M_{p-1}^{[i]}\right)_\mathfrak{M} \tag{3}$$

for any non-Eisenstein maximal ideal $\mathfrak{M}$ of $\mathbb{T}_{S,\mathbb{F}}^{univ}$.

Let $\vec{k}' = (\overrightarrow{p+1}, \vec{k}^{(2)}, ..., \vec{k}^{(r)})$ and set $\vec{w}' = \vec{w}$. Observe that if τ' is the representation of U associated to the parameters $(\vec{k}', \vec{w}')$ then $W_{\tau'} \simeq W_\tau \otimes_\mathbb{F} \bigotimes_{i=0}^{f_1-1} M_{p-1}^{[i]}$. The U-equivariant map $\mathcal{F}_\tau \otimes \bigotimes_{i=0}^{f_1-1} M_{p-1}^{[i]} \to \mathcal{F}_{\tau'}$ induced by the assignment:

$$f \otimes m \longmapsto [g \mapsto f(g) \otimes m]$$

for $g \in D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times$ is injective. We deduce that for any non-Eisenstein maximal ideal $\mathfrak{M}$ of $\mathbb{T}_{S,\mathbb{F}}^{univ}$, there is a Hecke equivariant monomorphism:

$$H^0\left(U, \mathcal{F}_\tau \otimes \bigotimes_{i=0}^{f_1-1} M_{p-1}^{[i]}\right)_\mathfrak{M} \hookrightarrow H^0(U, \mathcal{F}_{\tau'})_\mathfrak{M}.$$

Combining this with (3), we are done. ∎

Remark 6.2.3. Under the assumptions of the above theorem, τ' is an irreducible representation of U. This implies that, if the number of indices j such that $\vec{k}^{(j)} = \vec{2}$ is larger than one, Th. 6.2.2 can be further applied to obtain weight shiftings "in blocks" by $\overrightarrow{p-1}$.

Remark 6.2.4. The content of Th. 6.2.2 generalizes Lemma 4.6.8 of [Gee11], which is proved in loc. cit. via Lemma 1.5.5 of [Kis09a].

The weight shifting produced by Th. 6.2.2 is not in general of holomorphic type: for example, if $r > 1$ and τ is holomorphic, then τ' is never holomorphic. Nevertheless we have:

Corollary 6.2.5. Assume that τ is the irreducible holomorphic $\mathbb{F}$-linear weight with parameters $(\vec{2}, w) \in \mathbb{Z}_{\geq 2}^g \times (2\mathbb{Z} + 1)$. Let τ' be the holomorphic weight associated to the parameters $(\overrightarrow{p+1}, w + (p-1)) \in \mathbb{Z}_{\geq 2}^g \times (2\mathbb{Z} + 1)$. For any non-Eisenstein maximal ideal $\mathfrak{M}$ of $\mathbb{T}_{S,\mathbb{F}}^{univ}$, there is an injective Hecke-equivariant $\mathbb{F}$-morphism:

$$S_\tau(U, \mathbb{F})_{\mathfrak{M}} \hookrightarrow S_{\tau'}(U, \mathbb{F})_{\mathfrak{M}}.$$

Proof. Fix a non-Eisenstein maximal ideal $\mathfrak{M}$ of $\mathbb{T}_{S,\mathbb{F}}^{univ}$. Applying Th. 6.2.2 r times we obtain a Hecke equivariant injection $S_\tau(U, \mathbb{F})_{\mathfrak{M}} \hookrightarrow S_{\tau'}(U, \mathbb{F})_{\mathfrak{M}}$, where τ' is the irreducible $\mathbb{F}$-linear weight with parameters $(\overrightarrow{p+1}, \vec{v}) \in \mathbb{Z}_{\geq 2}^g \times \mathbb{Z}^g$ and each component of $\vec{v}$ equals the integer $\frac{w-1}{2}$. This weight is holomorphic with parameters $(\overrightarrow{p+1}, w + (p-1))$. ∎

The Jacquet-Langlands correspondence and Cor. 6.2.5 imply (cf. [EK03]):

Corollary 6.2.6. An irreducible continuous representation $\rho : \mathrm{Gal}(\bar{F}/F) \to GL_2(\bar{\mathbb{F}}_p)$ arising from a holomorphic Hilbert modular form of level $U \subset GL_2(\mathbb{A}_F^\infty)$ and parallel weight $\vec{2}$ also arises from a holomorphic Hilbert modular form of level U and parallel weight $\overrightarrow{p+1}$.

References

[AG05]   F. Andreatta and E. Z. Goren. *Hilbert modular forms: mod p and p-adic aspects*. Memoirs of the American Mathematical Society 819, 2005.

[AS86a] A. Ash and G. Stevens. "Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues." *J. Reine Angew. Math.*, **365**:192–220, 1986.

[AS86b] A. Ash and G. Stevens. "Modular forms in characteristic l and special values of their L-functions." *Duke Math. J.*, **53**:849–868, 1986.

[Car85] R. W. Carter. *Finite groups of Lie type. Conjugacy classes and complex characters*. John Wiley and Sons, 1985.

[Cen09] T. G. Centeleghe. *A Conjectural Mass Formula for (mod p) Galois representations*. PhD thesis, University of Utah, 2009.

[Cur70] C. W. Curtis. *Modular Representations of Finite Groups with split (B, N)-pairs, in "Seminar on Algebraic Groups and Related Finite Groups"*. Lecture Notes in Mathematics, vol. 131. Springer, Berlin, 1970.

[Dem72] M. Demazure. *Lectures on p-divisible Groups*. Lectures Notes in Mathematics, vol. 302. Springer, Berlin, 1972.

[DP94]   P. Deligne and G. Pappas. "Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant." *Compositio Math.*, **90**:59–74, 1994.

[DS74]   P. Deligne and J-P. Serre. "Formes modulaires de poids 1." *Ann. Scient. Ec. Norm. Sup.*, **7**:507–530, 1974.

[EK03]   B. Edixhoven and C. Khare. "Hasse Invariant and Group Cohomology." *Doc. Math.*, **8**:43–50, 2003.

[Eke87] T. Ekedahl. "On Supersingular Curves and Abelian Varieties." *Math. Scand.*, **60**:151–178, 1987.

[Fak09] N. Fakhruddin. *Correspondence with C. Khare*, 2009.

[Fon77] J-M. Fontaine. *Groupes p-divisibles sur les Corps Locaux*. Astérisque 47 - Société Mathématique de France, 1977.

[Gee11] T. Gee. "Automorphic lifts of prescribed types." *Math. Ann.*, **350**:107–144, 2011.

[Ghi04a] A. Ghitza. "Hecke Eigenvalues of Siegel Modular Forms (mod p) and of Algebraic Modular Forms." *J. Number Th.*, **106**:345–384, 2004.

[Ghi04b] A. Ghitza. "Upper Bound on the Number of Systems of Hecke Eigenvalues for Siegel Modular Forms (mod p)." *Int. Math. Research Notices*, **55**, 2004.

[Gor01]   E. Z. Goren. "Hasse invariants for Hilbert modular varieties." *Israel J. Math.*, **122**:157–174, 2001.

[Gro99]   B. H. Gross. "Algebraic Modular Forms." *Israel J. Math.*, **113**:61–93, 1999.

[Har77]   R. Hartshorne. *Algebraic Geometry.* Springer, New York, 1977.

[HI80]   K. Hashimoto and T. Ibukiyama. "On Class Number of Positive Definite Binary Quaternion Hermitian Forms." *J. Fac. Sci. Univ. Tokyo*, **27**:549–601, 1980.

[Hid88]   H. Hida. "On p-adic Hecke algebras for GL(2) over totally real fields." *Ann. Math.*, **128**:295–384, 1988.

[HJ90]   B. Haastert and J. C. Jantzen. "Filtrations of the discrete series of $SL_2(q)$ via crystalline cohomology." *J. Algebra*, **132**:77–103, 1990.

[Jan03]   J. C. Jantzen. *Representations of Algebraic Groups, Second Edition.* Mathematical Surveys and Monographs 107, AMS, 2003.

[K Z98]   F. Oort K-Z. Li. *Moduli of Supersingular Abelian Varieties.* Lectures Notes in Mathematics, vol. 1680. Springer, Berlin, 1998.

[Kat78]   N. M. Katz. "p-adic L-functions for CM field." *Invent. Math.*, **49**:199–297, 1978.

[Kha01]   C. Khare. "A local analysis of congruences in the (p, p) case: Part II." *Invent. Math.*, **143**:129–155, 2001.

[Kis09a]   M. Kisin. "The Fontaine-Mazur conjecture for GL_2." *J. Amer. Math. Soc.*, **22**:641–690, 2009.

[Kis09b]   M. Kisin. "Moduli of finite flat group schemes, and modularity." *Ann. of Math.*, **170**:1085–1180, 2009.

[Kot85]   R. E. Kottwitz. "Isocrystals with additional structure." *Comp. Math.*, **2**:201–220, 1985.

[Kot92]   R. E. Kottwitz. "Points on some Shimura Varieties over Finite Fields." *J. Amer. Math. Soc.*, **5**(2):373–444, 1992.

[Lan08]   K.-W. Lan. *Arithmetic compactification of PEL type Shimura varieties.* PhD thesis, Harvard University, 2008.

[Lew82]   D. W. Lewis. "The isometry classification of Hermitian forms over division algebras." *Linear Algebra Appl.*, **43**:245–272, 1982.

[Lus78]   G. Lusztig. "Representations of Finite Chevalley Groups." *CBMS Reg. Conf. Ser. Math.*, **39**, 1978.

[Ogu79]   A. Ogus. "Supersingular K3 crystals. Journées Géom. Algébr. Rennes, Vol. II, 1978." *Astérisque*, **64**:3–86, 1979.

[P 82]   W. Messing P. Berthelot, L. Breen. *Théorie de Dieudonné Cristalline II.* Lecture Notes in Mathematics, vol. 930. Springer, Berlin, 1982.

[Red10]   D. A. Reduzzi. "Reduction mod p of cuspidal representations of $GL_2(\mathbf{F}_{p^n})$ and symmetric powers." *J. Algebra*, **324**:3507–3531, 2010.

[RR96]   M. Rapoport and M. Richartz. "On the classification and specialization of F-isocrystals with additional structure." *Comp. Math.*, **103**:153–181, 1996.

[RZ96]   M. Rapoport and Th. Zink. *Period Spaces for p-divisible Groups.* Annals of Mathematics Studies, Princeton University Press, 1996.

[Ser60]   J-P. Serre. "Groupes proalgébriques." *Publications Mathematiques de l'I.H.E.S.*, **7**:5–67, 1960.

[Ser77]   J-P. Serre. *Linear Representations Of Finite Groups.* Springer, New York, 1977.

[Ser96]   J-P. Serre. "Two letters on quaternions and modular forms (mod p)." *Israel J. Math.*, **95**:281–299, 1996.

[Ser01]   J-P. Serre. "Lettre à Mme Hamer.", 2 Juillet 2001.

[Shi]   T. Shioda. "Supersingular K3 surfaces." *Algebraic Geometry, Copenhagen (1978). LNM 132, Springer-Verlag (1979), 564-591.*

[Ste63]   R. Steinberg. "Representations of algebraic groups." *Nagoya Math. J.*, **22**:33–56, 1963.

[Ste68]   R. Steinberg. "Lectures on Chevalley Groups." *Yale Univ. Math. Dept.*, 1968.

[Tay89]   R. Taylor. "On Galois representations associated to Hilbert modular forms." *Invent. Math.*, **98**:265–280, 1989.

[Tay06]   R. Taylor. "On the meromorphic continuation of degree two L-functions." *Doc. Math.*, pp. 729–779, 2006.

[Vig80]   M-F. Vignéras. *Arithmétique des Algèbres de Quaternions.* Lecture Notes in Mathematics, vol. 800. Springer, Berlin, 1980.

[Wed99]   T. Wedhorn. "Ordinariness in Good Reduction of Shimura Varieties of PEL-Type." *Ann. Scient. Ec. Norm. Sup.*, **32**:575–618, 1999.

[Wed07]   T. Wedhorn. "The Genus of The Endomorphisms of a Supersingular Elliptic Curve." *Astérisque*, **312**:25–47, 2007.