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Essays on Decision Theory and Economic Networks

A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy
in

## Economics

by

Pedram Heydari

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University of California San Diego<br>2018

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# ABSTRACT OF THE DISSERTATION 

Essays on Decision Theory and Economic Networks

by

## Pedram Heydari

Doctor of Philosophy in Economics

University of California San Diego, 2018

Professor Christopher Chambers, Co-Chair
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This dissertation contains three chapters, the first two of which are on decision theory and the last of which is on social and economic networks. In the first chapter, I model a decision maker who is able to evaluate the available options in multiple subjective dimensions or attributes, but is not generally able to aggregate the values of these attributes through a single utility function to rank the available options. This inability arises whenever there are multiple attributes that lead to conflicting rankings over the options. The model that I propose is able to account for a number of violations of standard utility maximization models such as stochastic choice and context dependence, in particular the attraction effect and the compromise effect. I provide an
axiomatic characterization of my model. An important feature of this model is that attributes are identified through the observed choices.

The second chapter extends the random utility framework to the realm of choices over menus of alternatives. More specifically, I assume that as modelers we can only observe the aggregate choices of a heterogenous population of decision makers over menus (e.g. restaurants). Furthermore, I assume that each member of the population behaves according to the standard model of choice over menus. Specifically, each decision maker has a well-defined strict preference ordering over individual alternatives (e.g. meals) and chooses an available restaurant that offers the best meal among all the available restaurants. The potential heterogeneity of the population makes the aggregate choices over menus appear stochastic. In the main result of the paper, I provide a characterization of all stochastic choice data over menus that are consistent with a population of standard decision makers.

The third chapter is on social and economic networks and studies the integration of two communities that face a barrier to communicate with one another. We assume that nodes benefit from both direct and indirect connections, but direct connections come also with a cost. We capture the barrier of communication between the two communities by assuming that intercommunity links are more costly than intra-community links. In this context, we characterize all integrated structures that are efficient. Next, we determine which of these efficient structures can be also stable. A notable feature of our work is a new class of multi core and periphery structures that we introduce in this paper as parallel hyperstars.

## Chapter 1

## Luce Arbitrates: Stochastic Resolution of Inner Conflicts


#### Abstract

I propose and axiomatically characterize a multi-attribute stochastic choice model that simultaneously generalizes the standard model of deterministic choice and the Luce model. Attributes are cardinal, independent measures of desirability that are endogenously inferred from observed choices, and thus subjective. An item is chosen with a positive probability from a set of available items (menu) if and only if it is undominated attribute-wise in that menu. This randomness of choice reflects the complexity of decision making and a possible lack of decisiveness in the face of conflicting motives due to the multiplicity of attributes. The model leads to the context-dependent evaluation of items by assigning every menu a distinct reference point against which the menu items are re-evaluated. The reference point of every menu takes the minimum values of attributes in that menu. The reference point can be interpreted as a context-dependent version of the commonly used status quo. Moreover, I provide an intuitive characterization of this reference point when attributes are objective. With appropriate choice of parameters, the model can accommodate a range of empirical findings that are incompatible with the standard model of deterministic choice and the Luce model, such as the attraction effect, the compromise effect, and violations of stochastic transitivity. Finally, explicit modeling of attributes allows the model to avoid the duplicate problem of the Luce model.


### 1.1 Introduction

Imagine that you are about to choose between two items, one of which is clearly not worse than the other in any attribute you care about, yet strictly better in at least one of them. In other words, one dominates the other. As such, the decision is easy, and you would always choose the dominant item. The same is true when you are choosing among additional items; you would never want to choose an item that is clearly dominated by another available item. Without such a dominance relationship does not exist among items, making a decision is more difficult. For example, in purchasing a car one might care about both fuel efficiency and price. If car $A$ is better than $B$ in fuel efficiency but also more expensive, the trade-off that the two cars create between these attributes makes it difficult to choose between them. The decision is difficult because there may not exist an obvious way to reconcile the two attributes into a single score that measures the value of the two cars.

In this paper, I propose a descriptive model of individual choice. In this model, the decision maker is able to evaluate every option in multiple subjective attributes. However, he/she has difficulty in aggregating these evaluations to make the final choice. My model makes two key contributions. First, it makes it possible to infer the relevant attributes from the observed choices. Second, it accommodates some of the empirically documented patterns of choice in multi-attribute environments that violate standard models. These violating patterns fall into two general categories: stochasticity and context-dependent properties of choices.

Stochastic choice refers to the phenomenon that an individual's choices from a given set of items are not always the same. Often times, the documented stochastic choices are incompatible with standard deterministic theories and reasons such as preference for variety, measurement error, unobservable factors affecting choices, changes in preferences, intertemporal bundling of goods, etc. ${ }^{1}$ In this paper, I interpret stochastic choice behaviorally that results from bounded

[^0]rationality. In particular, I accredit stochastic choice to the presence of conflicting motives caused by the multiplicity of attributes and the inability of the decision maker to consistently resolve such conflicts. ${ }^{2}$

Context-dependence refers to the phenomenon that the tendency of people in choosing one item over another can be affected by context, in particular the menu of available items. ${ }^{3}$ One way of formalizing this idea is that the ratio of choice probabilities of a pair of items depends on the other items in the menu.

Two of the most robust manifestations of context-dependence are the attraction effect, also known as the asymmetric dominance effect and compromise effect, also known as extremeness aversion. To understand the above effects, suppose that cars are differentiated along fuel efficiency and price (affordability). Imagine two cars $A$ and $B$, as initial choice alternatives, neither of which dominates the other. (Figure 1) Imagine $C$ to be another item that is dominated by $A$, but not by $B$. In the attraction effect, although $C$ is never or hardly chosen, adding it to the menu $\{A, B\}$ increases the choice probability of $A$. Now imagine another alternative, $D$, such that $A$ is a compromise between $B$ and $D$ (i.e. the value of $A$ falls in between the values of $B$ and $D$ for every attribute). In the compromise effect, adding $D$ to the menu $\{A, B\}$ increases the ratio of the choice probability of $A$ to $B$. In a more extreme case, in which $D$ is chosen with a sufficiently low probability, the choice probability of $A$ increases as a result. ${ }^{4}$

In the model that I propose, context-dependence stems from a menu-dependent reference point, against which the available items are re-evaluated.

[^1]

## Figure 1.1. An illustration of four choice alternatives in which $A$ and $B$ do not dominate each other, $A$ dominates $C$, and $A$ is a compromise between $B$ and $D$.

One of the earliest and most popular models of stochastic choice is the Luce model (Luce (1959)), also known as the logit in the econometrics literature. Although stochastic, this model carries over the basic assumption of the standard model of deterministic choice (i.e., the utility maximization paradigm) that choices can be described by fixed values or psychological weights. In particular, it assumes that the choice probability of an item, whenever positive, is proportional to the weight of that item. The Luce model is characterized by a property called independence of irrelevant alternatives (IIA). IIA states that the ratio of pairwise choice probabilities of two items is equal to the ratio of their choice probabilities in any other menu, given that at least one of them is chosen with a positive probability in that menu. ${ }^{5}$

This makes the model inadequate in explaining the context-dependent properties of choices, such as the attraction and compromise effects, as both effects violate IIA. The attraction effect and sometimes the compromise effect also violate regularity, a property of all random utility models, including the Luce model. Regularity states that the choice probability of an item does not increase as we expand the set of available options.

Nevertheless, the Luce model is extremely useful due to its simplicity and computational tractability. The model that I propose largely maintains the computational simplicity of the

[^2]Luce model. At the same time, it incorporates the context-dependent properties of choices. In particular, it is compatible with the attraction and compromise effects.

In my model, every item represents a finite dimensional vector, every coordinate of which is the value of a relevant attribute of that item. Attributes are independent, cardinal measures of desirability. For example, an item may be a certain kind of product with different attributes, a bundle of different economic goods, a consumption stream over a finite number of periods, or a lottery over a finite set of prizes. I assume that the value of every item is determined entirely by its attribute values (i.e., the vector that it represents). Therefore, the essence of the decision is to choose one of the available vectors. As such, the decision can be described as if it were made over two stages. In the first stage, the decision maker chooses a vector of attributes from among those available. In the second stage, the decision maker chooses the available items that represent the chosen vector with equal probabilities.

In the first stage, an available vector is chosen with a positive probability, if and only if it is undominated. Therefore, choices are Pareto efficient, attribute-wise. In choosing among the undominated vectors, the decision maker is not completely decisive and can assign only ill-formed values to them. This leads to the stochasticity of choices in the first stage. These values play similar roles as those of psychological weights in the Luce model, with the difference being that that they depend on the context. In particular, the weight of every available vector is a continuous function of its improvement relative to the reference point of the menu. The function that measures such improvements is called the quasi-utility function.

The model, as described above, potentially may be operationalized with any reference point. In this paper, I focus on a specific kind: I assume that the the reference point is a vector of attributes whose value in every dimension is the minimum value of the corresponding attribute in the menu. In the axiomatic characterization of the model, I argue that this reference point has the intuitive interpretation of being the status-quo of the menu. The status-quo is one of the most commonly used forms of the reference point in the literature. Beyond this intuitive interpretation, in a different framework I show that a number of intuitive properties characterize this reference
point.
Excluding a few minor cases, ${ }^{6}$ the continuity of the quasi-utility function leads to a continuous change in the choice probability of an item with respect to changes in its attribute values. This can facilitate comparative statics, when we want to understand how changing an attribute of some item changes its choice probability in a certain context.

This model reduces to the standard model of deterministic choice when the items vary along only one attribute. With more than one attribute, if the quasi-utility function is log-linear, and every vector has only one representative item in the menu, the relative choice probabilities of items become independent of the context. Therefore, the model reduces to the Luce model. When there are multiple available items that represent a single vector of attribute values, my model treats them as if they were the same items, thereby avoiding the duplicate problem or the red bus/blue bus problem of the Luce model.

Although a log-linear quasi-utility function eliminates the effect of context on the relative choice probabilities, other functional forms allow us to address several behavioral effects related to the influence of context on choices. In section 4, I introduce a property called the generalized compromise effect, which is a generalization of the compromise effect to situations with larger menus and an arbitrary number of attributes. I show that this property is satisfied in my model if and only if the quasi-utility function is super-modular and log-concave in every proper subset of the arguments. I also demonstrate that when there are two relevant attributes, this type of quasiutility function is equivalent to the satisfaction of a property called the generalized attraction effect, which is a generalization of the attraction effect to arbitrary menus.

Therefore, when items vary along only two attributes, the two effects are unified in my model through a quasi-utility function that is log-concave in every argument and supermodular. The log-concavity of the quasi-utility function in every argument means that the marginal benefit to every attribute is decreasing. Super-modularity implies that there is a degree of complementarity between the two attributes.

[^3]In section 2 , I formally describe my model and characterize (axiomatize) the observable choice data that can be represented by it. In this characterization, I assume that the observable data describes the choice probabilities of different items in different menus. However, I do not assume that we directly observe the relevant attributes. Instead I infer them from the observable choice data. This is important since the relevance of attributes is subjective in many applications. For example, a professional photographer and an amateur photographer might care about different attributes of a camera. Moreover, some attributes are not hedonic, ${ }^{7}$, which makes the relative desirability of their different values subjective. For example, one person might prefer bigger cars, while another might have an arbitrary optimal size in mind.

To characterize the model, I define an item $x$ to be absolutely preferred to another item $y$ if $x$ is invariably chosen over $y$ in binary comparisons. Moreover, I define two items to be absolutely indifferent if they are chosen with equal probabilities in every menu containing the two. I say that $x$ is lightly preferred to $y$ if $x$ is absolutely preferred to $y$ or is absolutely indifferent to it.

There are four substantive axioms and two technical ones. The substantive axioms are transitivity, monotonicity, weak consistency of relative values, and independence of status-quo preserving alternatives.

Transitivity states that the light preference relation is transitive. Monotonicity has two parts. The first part states that in every menu an item that is absolutely dispreferred to another item in that menu is never chosen. Now imagine that $x$ is absolutely preferred to $y$, and $m$ is a menu that does not contain any item that is absolutely indifferent to $x$ or $y$. The second part of the monotonicity axiom states that $x$ is chosen with a higher probability than $y$ against $m$.

Weak consistency of relative values is a weaker version of a property satisfied by the Luce model. In the Luce model, if two triplets of items $\left\{x_{1}, x_{2}, x_{3}\right\}$ and $\left\{y_{1}, y_{2}, y_{3}\right\}$ are such that the pairwise choice probabilities of $x_{i}$ and $x_{i+1}$ are non-trivial ${ }^{8}$ and equal to the pairwise

[^4]choice probabilities of $y_{i}$ and $y_{i+1}$ for all $i \in\{1,2\}$, then the choice probability of $x_{j}$ for every $j \in\{1,2,3\}$ in every subset of $\left\{x_{1}, x_{2}, x_{3}\right\}$ is the same as the choice probability of $y_{j}$ in the corresponding subset of $\left\{y_{1}, y_{2}, y_{3}\right\}$. In a situation where attributes are objective, my model satisfies the described property when $y_{i} \mathrm{~S}$ are modifications of $x_{i} \mathrm{~S}$ in the same single attribute. This is translated by the weak consistency of relative values to a language that relies entirely on the observed choices.

Independence of status quo preserving alternatives weakens the IIA. It requires the ratio of choice probabilities of the set of absolutely indifferent items to $x$ and the set of absolutely indifferent items to $y$ to be equal in two menus with the same status quo. The status quo of a menu is defined as the most lightly preferred item that is lightly preferred to all items in the menu.

The two technical axioms are richness and poorness and continuity. Richness and poorness are two sets of conditions on the variety of items. These conditions help us to identify the parameters of the model. Continuity states that the choice probabilities change continuously in a specific sense that is defined formally.

My main result in section 2 shows that the above axioms characterize my model. I further show that the parameters of my model are essentially uniquely identified. The number of attributes is pinned down uniquely. Also, if the light preference relation is incomplete, the attributes are identified up to a linear transformation. Also, fixing the attributes, the quasi-utility function is identified up to a scaler multiplier.

In section 3, I talk about the empirically documented behavioral effects, such as the attraction and compromise effects and violations of stochastic transitivity. Stochastic transitivity is a property of stochastic choices analogous to the standard notion of transitivity in a deterministic setting. I show how these effects can be accommodated by my model.

In section 4, I talk about the discontinuities of my model and some issues regarding the reference point. In section 5, I take a novel approach in characterizing the reference point. I define a reference function to be one that takes a menu of $N$ dimensional vectors as its input
and delivers another $N$ dimensional vector as the output. I will then introduce a number of intuitive properties and prove that the only reference functions that satisfy these properties are those whose outputs either takes the minimum values of the $N$ dimensions in the menu, or the the maximum values of them. Although my model can alternatively be written with the latter reference point, it can not accommodate the attraction effect in that case. Therefore, I choose the former reference point.

In section 6, I review the related literature. Finally, I conclude the paper in section 7, with a summary and some ideas for future work.

### 1.2 Model

### 1.2.1 Preliminaries

Denote the universal set of alternatives by $X$. Generic items in $X$ are denoted by $x, y, z, s, t$ and so forth. A тепи is a finite, non-empty subset of $X$, representing the set of items that are available to the decision maker. Menus are denoted by $m, m^{\prime}, m^{\prime \prime}$, etc.. I assume that for every item $x \in X$ and menu $m$, we observe the probability of choosing $x$ when $m$ is available. This probability is given by $\rho: X \times 2^{X} \rightarrow[0,1]$, called the Random Choice Rule (RCR), such that for every $m, \sum_{x \in m} \rho(x \mid m)=1$. Also, for a set $m^{\prime} \subseteq m$, let $\rho\left(m^{\prime} \mid m\right)=\sum_{x \in m^{\prime}} \rho(x \mid m)$. Therefore, $\rho\left(m^{\prime} \mid m\right)$ denotes the probability that the chosen item from $m$ belongs to $m^{\prime}$.

### 1.2.2 Representation Formula

To see how the model works formally, let $N$ be the number of revealed attributes. $A: X \rightarrow$ $\mathbb{R}^{N}$ is called the attribute function. For every item $x, A(x)$ is its vector of attribute values, with $A_{n}(x)$ being the value of its $n^{\text {th }}$ attribute. An item $x$ dominates $y$, if the vector that $x$ represents dominates that of $y$, i.e., $A(x)>A(y)$, where $>$ is the canonical order in $\mathbb{R}^{N} .{ }^{9}$ Denote the reference point of menu $m$ by $r_{m}$. The value of $r_{m}$ in every attribute is the minimum value of that

[^5]

Figure 1.2. $r_{m}$ is the reference point of $m=\{x, y, z\}$.
attribute in $m$. Thus, $r_{m}=\left(\min A_{n}(x)\right)_{n=1}^{N}$. Since the value of every item is determined by the vector it represents, the decision essentially boils down to choosing a vector that is represented by some item in $m$. Let $V(m)=\left\{v \in \mathbb{R}^{N} \mid \exists x \in m, A(x)=v\right\}$. Therefore, $V(m)$ represents the set of all vectors that have a representative item in $m$. For every vector $v$, let $v_{m}=\{x \in m \mid A(x)=v\}$. Therefore, $v_{m}$ represents the set of all items in $m$ that represent vector $v$. Two items that represent the same vector are called duplicates.

If all the items in $m$ represent the same vector, i.e., $|V(m)|=1$, then they are all chosen with equal probabilities. Otherwise, the decision maker first chooses an undominated vector $v \in V\left(m^{u}\right)$, with a probability proportional to $U\left(v-r_{m}\right) . U: \mathbb{R}_{\geq 0}^{N} \rightarrow \mathbb{R}_{\geq 0}$ is called the quasiutility function, which is continuous and strictly increasing and satisfies $U(0)=0$. Intuitively, $U$ measures the value of the improvement of an undominated vector over the reference point. Once a vector is chosen, the items representing that vector are chosen with equal probabilities.

Let $S(m)=\sum_{v \in V\left(m^{u}\right)} U\left(v-r_{m}\right)$. The resulting choice probabilities are given by the following formula:

$$
\rho(x \mid m)=\left\{\begin{array}{lr}
1_{\left\{x \in m^{u}\right\}} \cdot \frac{U\left(A(x)-r_{m}\right)}{S(m)} \cdot \frac{1}{\left|A(x)_{m}\right|} & |V(m)|>1 \\
\frac{1}{|m|} & |V(m)|=1
\end{array}\right.
$$

One can alternatively interpret the model as one with multiple selves, each of whom is
associated with an attribute, plus an arbitrator that makes the final decision. Similar to the Nash bargaining model, for any vector $v \in V(m)$ we can interpret $v_{i}-r_{m_{i}}\left(r_{m_{i}}\right.$ is the value of $r_{m}$ in coordinate $i$ ) as the amount of surplus that self $i$ receives if $v$ is chosen. The arbitrator is not completely decisive. Instead, he can only aggregate the vector of surplus values $v-r_{m}$ into an ill-formed valuation $U\left(v-r_{m}\right)$. These valuations operate similarly as the weights in the Luce model, hence the representation formula.

Based on the above interpretation, I call the above representation formula the random arbitration rule. When the range of the attribute function is an open box in the sense of the box topology, I call the representation an open random arbitration rule.

### 1.2.3 Axiomatic Characterization

In this setion, I axiomatically characterize RCRs that can be represented by an open random arbitration rule with a finite number of endogenous attributes. Axioms are stated using three binary relations that are inferred from observed choices, namely the absolute preference relation, the absolute indifference relation and their join, the light preference relation, as defined below.

Absolute Preference Relation: An item $x$ is absolutely preferred to $y$, denoted by $x \succ y$, if $x \neq y$ and $\rho(x \mid\{x, y\})=1$.
$x \succ y$ means that $x$ is chosen invariably over $y$ in binary comparisons. In the random arbitration model, $\succ$ coincides with the dominance relation with respect to the revealed attributes. Next, I define two items to be absolutely indifferent if they are chosen with equal probabilities in every menu that contains the two. Absolutely indifferent items would correspond to items that are duplicates of one another with respect to the revealed attributes.

Absolute Indifference Relation: $x$ is absolutely indifferent to $y$, denoted by $x \sim y$, if for every menu $m, \rho(x \mid m \cup\{x, y\})=\rho(y \mid m \cup\{x, y\})$.

Note that according to this definition, every item is absolutely indifferent to itself: for every $x, x \sim x .{ }^{10}$ I call the join of the absolute preference and indifference relations the light preference relation. ${ }^{11}$

Light Preference Relation: $x$ is lightly preferred to $y$, denoted by $x \succeq y$, if and only if $x \succ y$ or $x \sim y$.

The light preference relation is generally not a complete relation. However, the axiom below requires it to be transitive.

1. Transitivity The light preference relation $\succeq$ is transitive.

For every menu $m$ and $x, x$ is reasonable (unreasonable) relative to $m$ if it belongs to $m$ and is not absolutely dispreferred (is absolutely dispreferred) to any (some) item in $m$. Let $m^{r}=\{x \in m \mid \nexists y \in m, y \succ x\}$. Therefore, $m^{r}$ represents the set of all items that are reasonable relative to $m$. Also, let $m_{x}=\{y \in \mid x \sim y\}$. Therefore, $m_{x}$ represents the set of items in $m$ that are absolutely indifferent to $x$.

The next axiom states that an item that is unreasonable relative to a menu is never chosen in that menu. Also, it states that if $x$ is absolutely preferred to $y$ and is reasonable relative to the menu $m \cup\{x\}$, then it is chosen with a higher probability than $y$ against $m$, provided that no item in $m$ is absolutely indifferent to $x$ or $y$.
2. Monotonicity: For all $m, x \succ y$ :

[^6](a) If $\{x, y\} \subseteq m$, then $\rho(y \mid m)=0$.
(b) If $x \in(m \cup\{x\})^{r}$ and $m_{x}=m_{y}=\emptyset$, then $\rho(x \mid m \cup\{x\})>\rho(y \mid m \cup\{y\})$.

Call $x$ and $y$ are comparable if $x \succeq y$ or $y \succeq x$. For every $x$ and $y$, let $(x, y)=\{z \in X \mid x \prec z \prec$ $y\}$, which represents the set of items in $X$ that are absolutely preferred to $y$, but absolutely dispreferred to $x$. A set $S \subseteq X$ is called totally ordered, if it is empty or all elements of $S$ are comparable, that is for every $x, y \in S, x \succeq y$ or $y \succeq x$.

When $\rho$ can be represented by an open random arbitration rule, the interval between a pair of items is totally ordered according to $\succeq$ if and only if the two items are different in at most one of the revealed attribute. Based on this observation, if the interval between any pair of items in a non-empty, totally ordered set is totally ordered, I call that set univariate. Now, if a univariate set is maximal, i.e. it is not a proper subset of another univariate set, then it represents a maximal set of items that differ in at most one attribute. I call such a set maximal univariate. Figure 3 shows examples of univariate and maximal univariate sets in an environment with 2 attributes.


Figure 1.3. $\{x, y\}$ is univariate. $L_{1}$ and $L_{2}$ correspond to two sets of vectors represented by two maximal univariates that are also homo-variate.

The next definition formalizes the ideas above.

Definition: A non-empty, totally ordered set $S \subseteq X$ is univariate, if for every $x, y \in S$, $(x, y)$ is totally ordered. A univariate set, $L$ is maximal univariate, if there is no other univariate set $S^{\prime}$, such that $L \subset S^{\prime}$.

Note that according to the above definition, every singleton set $\{x\}$ is univariate.
Now, if two maximal univariates $L_{1}, L_{2}$ are such that for any $x \in L_{1}$, we can find $y \in L_{2}$ that is absolutely preferred to $x$, then variations in $L_{1}$ and $L_{2}$ must be in the same attribute. This is because if $x \in L_{1}$ is large enough, ${ }^{12}$ it surpasses all items in $L_{2}$ in the attribute that varies in $L_{1}$. In that case, no item in $L_{2}$ would be absolutely preferred to $x$. This observation leads us to the following definition:

Definition: Maximal univariates $L_{1}, L_{2}$ are homo-variate, denoted by $L_{1} \sim^{H} L_{2}$, if for some $i \in\{1,2\}$ and $j \neq i$, we have:

$$
\forall x \in L_{i}, \exists y \in L_{j} \quad \text { s.t. } \quad y \succ x
$$

Given that $\rho$ is represented by a random arbitration rule, if we equally modify all items in a menu in a given revealed attribute, the reference point would also be modified by the same amount in that attribute. This implies that when two incomparable items $x$ and $y$ are both modified in the same attribute, their relative value does not change if and only if they are both equally modified. Therefore, if for $L \sim^{H} L^{\prime},\{x, y\} \subset L$, and $\left\{x^{\prime}, y^{\prime}\right\} \subset L^{\prime}$, we have $\rho\left(x \mid\left\{x, x^{\prime}\right\}\right)=\rho\left(y \mid\left\{y, y^{\prime}\right\}\right) \in(0,1)$, it is an indication that $y$ and $y^{\prime}$ are equal modifications of $x$ and $x^{\prime}$ in the attribute that varies in $L$ and $L^{\prime}$.

This leads us to the following axiom:
3. Weak Consistency Of Relative Values For $L_{1} \sim^{H} L_{2}, L_{1} \sim^{H} L_{3}$ and $\left\{x_{1}, y_{1}\right\} \subset L_{1}$,

[^7]$\left\{x_{2}, y_{2}\right\} \subset L_{2}$, and $\left\{x_{3}, y_{3}\right\} \subset L_{3}$, if
\[

$$
\begin{aligned}
& \rho\left(x_{1} \mid\left\{x_{1}, x_{2}\right\}\right)=\rho\left(y_{1} \mid\left\{y_{1}, y_{2}\right\}\right) \in(0,1) \\
& \rho\left(x_{2} \mid\left\{x_{2}, x_{3}\right\}\right)=\rho\left(y_{2} \mid\left\{y_{2}, y_{3}\right\}\right) \in(0,1)
\end{aligned}
$$
\]

## Then

$$
\begin{aligned}
\rho\left(x_{1} \mid\left\{x_{1}, x_{3}\right\}\right) & =\rho\left(y_{1} \mid\left\{y_{1}, y_{3}\right\}\right) \\
\rho\left(x_{1} \mid\left\{x_{1}, x_{2}, x_{3}\right\}\right) & =\rho\left(y_{1} \mid\left\{y_{1}, y_{2}, y_{3}\right\}\right)
\end{aligned}
$$

To better understand the above axiom, note that since $L_{2}$ and $L_{3}$ are both homo-variate with $L_{1}$, we can conclude that the variation of items in all three sets is in the same single attribute. Now, the two equations in the antecedent imply that $y_{1}, y_{2}$, and $y_{3}$ are equal modifications of $x_{1}, x_{2}$, and $x_{3}$ respectively in the attribute that varies in $L_{1}, L_{2}$ and $L_{3}$. Therefore, the relative values of $x_{1}, x_{2}, x_{3}$ must be the same as the relative values of $y_{1}, y_{2}, y_{3}$. This implies the two equations in the consequent.

The Luce model satisfies the above axiom for arbitrary triplets of $\left\{x_{1}, x_{2}, x_{3}\right\}$ and $\left\{y_{1}, y_{2}, y_{3}\right\}$.

Definition: An upper bound of a set $S$ is an element $y^{u} \in X$ such that $y^{u} \succeq x$ for every $x \in X$. A lower bound of a set $S$ is an element $y_{l}$ such that $x \succeq y_{l}$ for every $x \in S$. A set is unbounded if it does not have an upper bound or lower bound. $\inf S$ and $\sup S$, are the sets of greatest lower bounds and smallest upper bounds of $S$, respectively. ${ }^{13}$

For a menu $m$, call $\inf m$ the status quo of $m$. Intuitively, $\inf m$ is the worst thing that can happen when a choice is made from $m$. In other words, no matter what item is chosen from

[^8]$m$, the decision maker is lightly better off relative to choosing $\inf m$ had it been available. Moreover, $\inf m$ is the most lightly preferred item for which the latter statement is true. In the random arbitration model, the status-quo of $m$ coincides with its reference point. ${ }^{14}$ The next axiom treats absolutely indifferent items as if they were the same items. Moreover, it states that if the status quo of two menus are the same, the ratios of the choice probabilities of every pair of reasonable items in their intersection are equal.
4. Independence of Status Quo Preserving Alternatives: For all items $x, y$ and menus $m, m^{\prime}$ such that $\{x, y\} \subseteq m^{r} \cap m^{\prime r}$ and $\inf m=\inf m^{\prime}$, we have:
$$
\frac{\rho\left(m_{x} \mid m\right)}{\rho\left(m_{y} \mid m\right)}=\frac{\rho\left(m_{x}^{\prime} \mid m^{\prime}\right)}{\rho\left(m_{y}^{\prime} \mid m^{\prime}\right)}
$$

In addition to the above axioms, we need two technical axioms, richness and poorness and continuity to be able to fully identify the parameters of our representation. Richness is a set of conditions that guarantee enough variety in $X$. On the other hand, poorness requires that we do not have too much variety. To state these axioms, take $X^{\prime} \subseteq X$ to be a set that contains exactly one item from each equivalence class of the duplicate relation in $X$. In the statement of the next two axioms, I restrict my attention to $X^{\prime}$. In other words, I require that the two axioms be true when $\rho$ is restricted to any such $X^{\prime}$.

## 5. Richness and Poorness:

## Richness:

(a) For finite $S \subset X^{\prime}$, sup $S$ and $\inf S$ exist.
(b) Every maximal univariate is unbounded.
(c) For $m$, $x \prec y$, and $\delta \in(\rho(x \mid m \cup\{x\}), \rho(y \mid m \cup\{y\}))$, there exists $z \in(x, y)$ such that $\rho(z \mid m \cup\{z\})=\delta$.

[^9]
## Poorness:

(a) If for each $k \in\{1,2, \ldots\},\left\{x, x_{k}\right\}$ is univariate, then there exists $i$ and $j$ such that $x_{i} \succeq x_{j}$
(b) For $x, y, z$ there is at most one $t$ s.t. $\inf \{z, t\}=x$ and $\sup \{z, t\}=y$.

Richness (a) requires two things. First, for every item $x \in X^{\prime}$, there is an item that is absolutely preferred to it and there is an item that is absolutely dispreferred to $x$. Also, there is a smallest and biggest of such items, respectively. The latter can be interpreted as a form of compactness. Richness (b) implies that the range of every revealed attribute is open. Finally, Richness (c) requires that by moving an item in a none-empty interval of items, one can generate arbitrary choice probabilities against any menu $m$ that fall in between the choice probabilities of the boundary items of that interval against $m$. Poorness (a) guarantees that the number of revealed attributes is finite. Also, the second part of Poorness is a condition that ensures the lattice structure generated by the partial order $\succeq$ is a distributive one.

Lastly, I present the continuity axiom. Continuity axioms are usually stated in metric spaces. However, since we do not have an explicit metric space here, I state it in a more abstract form.
6. Continuity: For any increasing sequence of items $\left\{y_{n}\right\}_{n=1}^{\infty}$ s.t. $\sup \left\{y_{n}\right\}_{n=1}^{\infty}=y$, and every menu $m$ for which $m \cap\left[\left\{y_{i}\right\}_{i=1}^{\infty} \cup\{y\}\right]=\emptyset$ and any $x \in m$, we have:

$$
\lim _{n \rightarrow \infty} \rho\left(x \mid m \cup\left\{y_{n}\right\}\right)=\rho(x \mid m \cup\{y\})
$$

Continuity says that for every menu $m$, every item $x$ in $m$, and every increasing sequence of items $\left\{y_{n}\right\}_{n=1}^{\infty}$ with a smallest upper bound $y$, the choice probability of $x$ in the sequence of menus $m \cup\left\{y_{i}\right\} i=1,2, \ldots$ converges to the choice probability of $x$ in $m \cup\{y\}$.

Now, we are ready to state the main result of the section.

Theorem 1 A random choice rule $\rho$ satisfies axioms 1-6, if and only if it can be represented by an open random arbitration rule.

Proof: The detailed proof is in the appendix. However, a sketch of the proof is provided below.

Sketch of the proof: First, for every item $x$, I define $V(x)$ to be the set of all maximal univariates that contain $x$. I show that the cardinality of $V(x)$ is item-invariant and finite. Next, I define the extended homo-variability relation. Specifically, for two maximal univariates $L_{1}$ and $L_{2}$, I put $L_{1} \sim \sim^{H^{\prime}} L_{2}$ if there exists a maximal univariate $L$, such that $L_{1} \sim^{H} L$ and $L_{2} \sim^{H} L$. I show that $\sim^{H^{\prime}}$ is an equivalence relation. Then I prove that each equivalence classes of $\sim^{H^{\prime}}$ has exactly one element in $V(x)$ for every $x$. Therefore, $|V(x)|$ is equal to the cardinality of the set of equivalence classes of $\sim^{H^{\prime}}$, which is going to be the same as the number of attributes. Next, I show that $\succeq$ is the intersection of $N$ total orders $\succeq_{1}, \ldots, \succeq_{N}$, representing attributes 1 through $N$, respectively. Moreover, all items in a univariate set are equal according to all these orders, except one of them. In other words, all items in univariate sets vary in only one of the total orders. We will call a univariate set whose items vary in attribute $i$ (i.e., according to $\succeq_{i}$ ), $i$-variate. Up until this stage, transitivity, monotonicity, and richness and poorness have been used.

Next, using the weak consistency of relative values and continuity, I come up with a way of determining if the cardinal distance of an $i$-variate pair of items $x, y$ is the same as that of another pair of $i$-variate items $x^{\prime}, y^{\prime}$. Specifically, this is the case if there exists another $i$-variate pair $x^{\prime \prime}, y^{\prime \prime}$ such that

$$
\rho\left(x, x^{\prime \prime}\right)=\rho\left(y, y^{\prime \prime}\right) \in(0,1) \quad \rho\left(x^{\prime}, x^{\prime \prime}\right)=\rho\left(y^{\prime}, y^{\prime \prime}\right) \in(0,1)
$$

Then, I use this to determine the distance ratio of two arbitrary pairs of items $x, y$ and $x^{\prime}, y^{\prime}$ along each attribute. Once these ratios are determined, I proceed in defining the revealed attribute
function $A: X \rightarrow \mathbb{R}^{N}$, which will respect the identified ratios of items values in different attributes.
In the next step, I define the revealed quasi-utility function $U$. Specifically, choose an arbitrary vector $b>0$ and put $U(b)=1$. Now, for every $a>0$ that is not comparable with $b^{15}$ define:

$$
U(a)=\frac{\rho(x \mid\{x, y, z\})}{\rho(y \mid\{x, y, z\})}
$$

where

1. $A(x)-A(z)=a$
2. $A(y)-A(z)=b$

Also, for every $c>0$ that is bigger than $b$, define:

$$
U(c)=\frac{\rho\left(y^{\prime} \mid\left\{x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)}{\rho\left(x^{\prime} \mid\left\{x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)} \times \frac{\rho(x \mid\{x, y, z\})}{\rho(y \mid\{x, y, z\})}
$$

Where

1. $A\left(x^{\prime}\right)-A\left(z^{\prime}\right)=A(x)-A(z)=a>0$ for some $a$ that is incomparable with $b$ and $c$.
2. $A\left(y^{\prime}\right)-A\left(z^{\prime}\right)=c$
3. $A(y)-A(z)=b$

A similar definition applies to $U(d)$ for any $d$ that is less than $b$. Using independence of status quo preserving alternatives, I show that $U$ as described above is well-defined and together with the attribute function $A$ represent $\rho$ according to the random arbitration model.

In addition to the above representation theorem, we have the following uniqueness result.

[^10]Proposition 1 If $\rho$ can be represented by an open random arbitration model, then $N$ is unique. If $\succeq$ is complete, then $N=1$ and $A$ is identified up to an ordinal transformation and $U$ is unidentified. If $\succeq$ is incomplete, then $N \geq 2$, the revealed attribute function $A$ is unique up to $a$ linear transformation and permutation of coordinates. Also, fixing the revealed attribute function, the revealed quasi-utility function $U$ is unique up to a scalar multiplier.

## Proof: Appendix.

Note that a revealed attribute in the above result does not necessarily correspond to a single observable property of choice objects. For example, the decision maker might be able to blend the values of two or more observable attributes into a single score (perhaps because for some reason they do not create much conflict). For instance, in purchasing a car, one might blend several different attributes of the car, such as fuel efficiency, comfort, reliability, etc into a single index of quality. This index would then be treated as a single attribute in the above representation. In the extreme case, the decision maker is able to blend all these attributes into a single score. This reduces the model to the standard deterministic choice via utility maximization. As such, the number of revealed attributes may also be interpreted as as a measure of "irrationality" or "indecisiveness" of the decision maker in a crude sense of the word.

### 1.2.4 Notable Special Cases

The random arbitration model generalizes the classical deterministic choice and the Luce model, as explained below.

Standard model of deterministic choice: When there is only one relevant attribute, $N=1$, the model reduces to the standard deterministic choice that satisfies the weak axiom of revealed preferences (WARP). In this case, $\succeq$ is a complete preference relation on $X$ and describes choices. In particular, let $C(m)=\{x \mid x \in m, \rho(x \mid m) \neq 0\}$, representing all items in $m$ that are chosen with
positive probabilities. When $N=1$, we have:

$$
C(m)=\{x \mid x \in m, \forall y \in m, x \succeq y\}
$$

According to this preference relation, duplicates are indifferent. Note that the standard deterministic choice does not normally specify the choice profitabilities of two indifferent items. The random arbitration model, on the other hand, requires them to be chosen with equal probabilities.

Luce model: Formally, the Luce model is characterized by the following two axioms: ${ }^{16}$

1. If $\rho(x \mid\{x, y\})=1$, then for any menu containing $x$ and $y$ and every $z \in m-\{x, y\}$

$$
\rho(z \mid m-\{y\})=\rho(z \mid m)
$$

Axiom 1 says that if $x$ is invariably chosen over $y$, then removing $y$ from any menu that contains both $x$ and $y$ does not affect the choice probabilities of other items. Remember that $m^{r}=\{x \mid x \in m, \forall y \in m-\{x\}, \rho(x \mid\{x, y\})>0\}$. The next axiom says that the ratio of choice probabilities of a pair of items in $m^{r}$ is independent of other items in the menu.
2. For every menu $m$ and $s, t \in m^{r}$ :

$$
\frac{\rho(s \mid m)}{\rho(t \mid m)}=\frac{\rho(s \mid\{s, t\})}{\rho(t \mid\{s, t\})}
$$

The two axioms above guarantee the existence of a weighting function $w: X \rightarrow \mathbb{R}_{+}$, so that the choice probability of every item $x$ is proportional to $w(x)$ when $x \in m^{r}$ and zero when $x \in m-m^{r}:$

[^11]$$
\rho(x \mid m)=1_{\left\{x \in m^{r}\right\}} \frac{w(x)}{\sum_{t \in m_{r}} w(t)}
$$

Therefore, in the Luce model, an unreasonable item relative to menu $m$ is never chosen in $m$ and does not affect the choice probabilities of other items in $m$. In the random arbitration model, an unreasonable item is dominated and never chosen either but may affect the choice probabilities of other items by affecting the reference point. More generally, in the random arbitration model, the ratio of choice probabilities of two items is a function of the number of their available duplicates and the reference point of the menu containing them.

However, when the quasi-utility function $U$ is log-linear, the reference point becomes irrelevant in determining the relative choice probabilities of the available vectors of attributes. To see this, let $u=\log U$. Therefore, $u$ is a linear function. Now, take two menus $m$ and $m^{\prime}$ and two items, $x, x^{\prime} \in m^{u} \cap m^{\prime u}$ and put $A(x)=v$ and $A\left(x^{\prime}\right)=v^{\prime}$. The linearity of $u$ means

$$
u\left(v-r_{m}\right)-u\left(v^{\prime}-r_{m}\right)=u\left(v-r_{m}^{\prime}\right)-u\left(v^{\prime}-r_{m}^{\prime}\right)
$$

which in turn implies that

$$
\begin{array}{r}
\frac{\rho\left(v_{m} \mid m\right)}{\rho\left(v_{m}^{\prime} \mid m\right)}=\frac{\rho\left(v_{m^{\prime}} \mid m^{\prime}\right)}{\rho\left(v_{m^{\prime}}^{\prime} \mid m^{\prime}\right)} \\
\Rightarrow \frac{\rho(x \mid m)}{\rho\left(x^{\prime} \mid m\right)} \cdot \frac{\left|v_{m}\right|}{\left|v_{m}^{\prime}\right|}=\frac{\rho\left(x \mid m^{\prime}\right)}{\rho\left(x^{\prime} \mid m^{\prime}\right)} \cdot \frac{\left|v_{m}^{\prime}\right|}{\left|v_{m^{\prime}}^{\prime}\right|} \tag{*}
\end{array}
$$

When the attribute function $A$ is one-to-one, then every vector is represented by only one item. Therefore, $\left|v_{m}\right|=\left|v_{m}^{\prime}\right|=\left|v_{m^{\prime}}\right|=\left|v_{m^{\prime}}^{\prime}\right|=1$. In this case, $(*)$ is equivalent:

$$
\frac{\rho(x \mid m)}{\rho\left(x^{\prime} \mid m\right)}=\frac{\rho\left(x \mid\left\{x, x^{\prime}\right\}\right)}{\rho\left(x^{\prime} \mid\left\{x, x^{\prime}\right\}\right)}
$$

Therefore, with a log-linear quasi-utility function and in the absence of non-trivial
duplicates, ${ }^{17}$ the random arbitration model reduces to the Luce model.

### 1.2.5 Resolution of the Duplicate Problem

When there are several duplicates of an item available, the random arbitration model avoids a well-known shortcoming of the Luce model called the duplicate problem, also known as the red bus/blue bus problem. This problem was first proposed by Debreu(1960)Debreu (1960) and it goes as the following. Imagine that a decision maker wants to choose a mode of transportation, and he/she does not care about the color of the mode. When asked to choose from a red bus (rb) or a car (c), he/she chooses them with equal probabilities, respectively. When given the option of choosing a red bus, blue bus (bb), or a car, he/she chooses them with $0.25,0.25,0.5$ probabilities, respectively. Also, since color is not an important attribute, bb and rb are duplicates and they are chosen with equal probabilities. Therefore

$$
\begin{equation*}
\rho(r b \mid\{r b, b b\})=0.5 \quad \rho(c \mid\{r b, c\})=0.5 \quad \rho(c \mid\{r b, b b, c\})=0.5 \tag{3}
\end{equation*}
$$

Note that this behavior, although plausible, is not compatible with the Luce model: Since the red bus and the car are chosen with equal probabilities, it must be that $w(r b)=w(c)$. On the other hand, the fact that the red bus and blue bus are duplicates implies $w(r b)=w(b b)$. Thus, the Luce model predicts that the red bus, blue bus, and the car are chosen with equal probabilities in $\{r b, b b, c\}$.

The take away from the above example is the observation that adding several duplicates of an item to a menu should not logically affect the choice probabilities of other available items. This is precisely what the random arbitration model does. What allows the model to address this observation is the fact that in every menu, weights are essentially assigned to the vectors of attribute values, rather than individual items. Of course, in a setting where the relevant attributes are exogenous, one can easily identify the duplicates and adjust the Luce model accordingly.

[^12]However, in situations where the relevant attributes are less obvious, duplicates must be derived from the observed choices.

In the characterization of the random arbitration model above, two items that are chosen with equal probabilities in every menu that contains both items are absolutely indifferent and are revealed to be duplicates. However, due to IIA, if a pair of items are chosen with equal probabilities when they are the only items available would be absolutely indifferent, and the Luce model is forced to deem them as duplicates. However, this is not necessarily true, as evident in the red bus/blue bus example above: although the red bus and the car are chosen with equal probabilities in the pairwise comparison, they are not duplicates of one another. Therefore, the Luce model is not able to resolve the duplicate problem, when the relevant attributes are unknown. On the other hand, independence of status quo preserving alternatives in the random arbitration model is conditioned on the sets of absolutely indifferent items in the menu, rather the individual items, which avoids the duplicate problem.

### 1.3 Behavioral Effects

### 1.3.1 Attraction and Compromise Effects

As far as the author knows, all experiments that have documented the attraction and compromise effects do so in a very special setting. In this setting, all items have only two attributes. Moreover, choices between two items $x$ and $y$ are only studied once when they are the only available items and once, when they are accompanied by a third item $z$ whose availability leads to the observation of the two effects, as explained above. Call the setting of these experiments the standard setting. It is unclear whether or not we get to observe the attraction and compromise effects exactly as they happen in the standard setting when there are more attributes and menus are larger. Could it not be the case that the standard notion of these two effects are manifestations of a more general phenomenon that manifests itself differently in other settings? Although the ultimate answer to this question can only be given empirically, my
model suggests that the answer might be affirmative. More modestly, I am suggesting that there might be many forces involved in producing such effects, one of which is identified in my model.

When the quasi-utility function is log-concave in every argument and super-modular, the random arbitration model can generate the two effects in the standard setting. However, the IRPA axiom does not allow for any change in the ratio of probabilities whenever the menu's reference point stays in tact. For instance, if the added item $z$ is dominated by $x \in m$, but at the same time it dominates the menu's reference point, $r_{m}$, we are not going to observe any changes in the choice probabilities of items in $m$. The same goes for the compromise effect. That is to say when adding $z$ does not change the menu's reference point, the relative choice probabilities of $x$ and $y$ will not change, even though $x$ might be a compromise between $y$ and $z$.

In general, with such quasi-utility function, whenever $z$ changes the reference point of $m$ only in attributes in which $x$ is worse than $y, x$ gets a stronger boost than $y$, which in turn increases the choice probability of $x$ relative to $y$. This phenomenon happens in the standard setting of attraction and compromise effect.

In this section, I introduce a new property called the generalized compromise effect that is an extension of the standard compromise effect to more diverse settings. In theorem 2 below, I will show that an open random arbitration model satisfies the modified compromise effect if and only if $U$ is log-concave in every proper subset of the arguments and super-modular.

For the sake of better exposition, I will assume that there are $N$ exogenously given attributes. In particular, the attributes are given by $A: X \rightarrow \mathbb{B}$, where $\mathbb{B} \subseteq \mathbb{R}^{N}$ is an open box. As in the standard definition, $x$ is a compromise between $y$ and $z$ if $x$ is not the only extreme item among $x, y, z$ in any of the attributes, i.e. for every attribute $i \in\{1, \ldots, N\}, A_{i}(x) \in$ $\left[\min \left\{A_{i}(y), A_{i}(z)\right\}, \max \left\{A_{i}(y), A_{i}(z)\right\}\right]$.

Generalized Compromise Effect If $x$ is a compromise between $y$ and $z$ and $\{x, y, z\} \subseteq m^{\prime u}$, where $m^{\prime}=m \cup\{z\}$, then the following is true:

$$
\frac{\rho(x \mid m)}{\rho(y \mid m)} \leq \frac{\rho(x \mid m \cup\{z\})}{\rho(y \mid m \cup\{z\})}
$$

To better understand the generalized compromise effect, note that if the reference point of the menus $m$ and $m^{\prime}=m \cup\{z\}$ coincide, then the ratio of the probability of choosing $x$ to the probability of choosing $y$ would not change after adding $z$. Therefore, to see any change in this ratio, the reference point of $m$ and $m^{\prime}$ must be different. The generalized compromise effect says that whenever this ratio changes as a result of adding $z$, this change is going to be in the positive direction.

Theorem 2: An open random arbitration model satisfies the generalized compromise effect, if and only if $U$ is log-concave in every proper subset of its arguments and weakly super-modular.

## Proof: Appendix.

In the proof of theorem 2, I show that the generalized compromise effect is equivalent to the property that if the added item $z$ decreases the reference point of the menu only in attributes for which $x$ is inferior to $y$, then the ratio of choice probability of $x$ to $y$ increases. I call this property the equivalence property. In the rest of the proof, I show that the equivalence property is equivalent to the fact that the utility function $U$ is log-concave in every proper subset of arguments and super-modular.

The next proposition shows that when the number of revealed attributes is only 2 , then the following version of the attraction effect is also equivalent to the conditions of theorem 2 on the quasi-utility function.

Generalized Attraction Effect: If $x \in m^{u}$ is the only item in $m$ that dominates $z$, then:

$$
\rho(x \mid m \cup\{z\}) \geq \rho(x \mid m)
$$

According to the generalized attraction effect, when $z$ is dominated by only $x$ in $m$, then adding it to the menu (weakly) increases the choice probability of $x$.

Proposition 2: When $N=2$, an open random arbitration model satisfies the generalized attraction effect if and only if $U$ is log-concave in every argument and super-modular.

## Proof: Appendix.

Therefore, when there are two attributes, the (generalized) attraction and compromise effects are unified in the random arbitration model through a quasi-utility function that is logconcave in every argument and super-modular. The log-concavity of the quasi-utility function in every argument means that the marginal benefit to every attribute is decreasing. Super-modularity means that there is a degree of complementarity between the two attributes.

Notice that to increase the choice probability of $x$, the new item $z$ should not necessarily be dominated by it. For example, as in the standard setting of the compromise effect, imagine there are only two attributes and initially two items to choose from, $x$ and $y$. Also, $x$ is a compromise between $y$ and $z$. With a quasi-utility function that is log-concave in every argument and super-modular, we know that adding $z$ to $m=\{x, y\}$ increases the relative choice probability of $x$ to $y$. Now, if we let the value of $z$ on the first attribute converge to that of $x$, while fixing its second attribute, at some point along the way we will have:

$$
\rho(x \mid m \cup\{z\}) \geq \rho(x \mid m)
$$

This is true, because all along the path the ratio of choice probabilities remains constant, as the reference point does not change. However, the choice probability of $z$ keeps decreasing along the path, which eventually results in an increase in the choice probability of $x$, relative to when $z$ is not available. Although a consequence of the compromise effect, this phenomenon was coined the relative inferiority effect by Huber and Puto (1983) Huber \& Puto (1983). This
is different than the attraction effect in the sense that $z$ itself is chosen with a strictly positive probability. Like the attraction effect, the relative inferiority effect is a violation of regularity.

As a final note, one can state the generalized compromise effect without an explicit reference to the attributes. In general, it is easy to show that $x$ is a compromise between $y$ and $z$ in the sense defined above, if and only if $\inf \{y, z\} \prec x \prec \sup \{y, z\}$. The latter notion has the following interpretation: $x$ is absolutely preferred to every item absolutely dispreferred to both $y$ and $z$, and it is absolutely dispreferred to every item that is absolutely preferred to both $x$ and $z$.

### 1.3.2 Violations of Stochastic Transitivity

Say $x$ is stochastically preferred to $y$, denoted by $x \succeq^{s} y$, if $\rho(x \mid\{x, y\}) \geq 0.5$, A simple property of the Luce model, called stochastic transitivity, states that $\succeq^{s}$ is transitive. However, there is good evidence that this property is sometimes violated. Tversky(1969) Tversky (1969) provides one template for generating such violations. In a two attribute environment, consider three items $x=\left(x_{1}, x_{2}\right), y=\left(y_{1}, y_{2}\right)$ and $z=\left(z_{1}, z_{2}\right)$, such that $y_{1}-x_{1}=z_{1}-y_{1}=d$ and $x_{2}-y_{2}=$ $y_{2}-z_{2}=2 d$. Tversky(1969)Tversky (1969) documents the following pattern for small values of $d$ :

$$
x \succeq^{s} y \quad y \succeq^{s} z \quad z \succ^{s} x
$$

hence a violation of stochastic transitivity. It is easy to see that in my model, the above pattern amounts to:

$$
U(0,2 d) \geq U(d, 0) \quad U(0,4 d)<U(2 d, 0)
$$

Now, put $f(a)=U(0, a)$ and $g(a)=U(a, 0)$. The above inequalities are equivalent to

$$
f(2 d) \geq g(d) \quad f(4 d)<g(2 d)
$$

These inequalities indicate that the marginal return to $f$ is less than the marginal return to $g$.

Assuming the functional form that generates the generalized attraction and compromise effect introduced in the previous section, the latter amounts to $f$ being more log-concave than $g$. Therefore, the pattern provided in Tversky(1969) can be accommodated in my model, whenever there is enough asymmetry across the two attributes. On the other hand, the next proposition shows that if there are only two relevant attributes and $U$ is symmetric, i.e. for all $a$ and $b$, $U(a, b)=U(b, a)$, then $\succeq^{s}$ is transitive.

Proposition 3: For $N=2$ and a symmetric $U$, stochastic preference is transitive.

## Proof: Appendix

### 1.4 Discussion

In this section, I address a few issues regarding the reference point and also talk about the discontinuities of my model.

### 1.4.1 Reference Point and the Effect of Dominated Items

In the random arbitration model, a dominated item is never chosen, but it may affect the choice probabilities of other items by affecting the reference point. Although this implication is to some extent in line with empirical evidence on the attraction effect (asymmetric dominance effect), it may sound implausible in situations of symmetric dominance, such as when one of the available items is dominated by all the other available items. In the latter scenario, it may be more plausible that the decision maker would immediately discard such obviously bad item and treat the menu of the remaining items as if the bad item had never been available. This point may be addressed in the following two ways.

First, experiments suggest that in making a decision, people attend back and forth to
different attributes while comparing the available options along the attended attribute. ${ }^{18}$ This procedure can be thought of as the one where the value of the reference point in each attribute is formed. Since no option has been discarded yet, the reference point can be influenced by all items. Even though the reference point is constructed without discarding any item, the decision maker does not still want to choose any dominated item. This story is in line with the random arbitration model. This is especially plausible since the extreme values of attributes are usually more salient and attract the attention of the decision maker. On the other hand, the fact that symmetrically dominated items can influence choice probabilities of other items is consistent with Masatlioglu \& Uler (2013). Although their experiments take reference points as exogenous, they suggest that even if the reference point is dominated by two of the available items, it can influence how people compare those items.

Second, if we still find it more plausible that symmetrically dominated items do not affect the choice probabilities of other items, it is possible to adjust our model accordingly by modifying the way the reference point is constructed. It seems that the alteration that is most aligned with the criticism and the motivation of our model is the following: the reference point is constructed by items that are not dominated by every undominated item in the menu. The interpretation is that the DM immediately discards every item that is dominated by all undominated items. The reason is that such items do not pose any trade offs between different attributes with any Pareto efficient item, i.e. undominated items. This change does not alter our representation drastically, as it only modifies the way that the reference point is constructed. Also, the only axiom that needs to be changed to incorporate this modification is independence of status quo preserving alternatives. Formally, the altered axiom would be:

$$
\forall m, m^{\prime} \text { s.t. } \inf \left\{x \in m \mid \exists y \in m^{u} \quad \text { s.t. } \quad y \nsucc x\right\}=\inf \left\{x \in m^{\prime} \mid \exists y \in m^{\prime u} \quad \text { s.t. } \quad y \nsucc x\right\} \text { and }
$$ $x, y \in m^{u} \cap m^{\prime u}$, we have:

$$
\frac{\rho\left(m_{x} \mid m\right)}{\rho\left(m_{y} \mid m\right)}=\frac{\rho\left(m_{x}^{\prime} \mid m^{\prime}\right)}{\rho\left(m_{y}^{\prime} \mid m^{\prime}\right)}
$$

[^13]
### 1.4.2 Similarity Effect and Discontinuities

A limitation of my model is also a well-known short-coming of the Luce model. As discussed in section 2, the Luce model suffers from the duplicate problem, which is an extreme version of a more general phenomenon that is called the similarity effect. Similarity effect states that making a new item available lowers the choice probabilities of similar items more than it lowers those of less-similar items. The common explanation of this effect, which has been confirmed empirically ${ }^{19}$ is that very similar items are treated as approximate substitutes, and therefore seen as almost identical. This approximate substitutability becomes perfect substitutability when the two items become exactly the same in all attributes that are relevant to the decision maker, i.e. when they are duplicates. The random arbitration model deals with the perfect substitutability scenario, as it treats duplicates as if they were the same items. However, it still suffers from a lack of ability to address the imperfect substitutability scenario.

For the Luce model, there are well known econometric tests that can tell us when such imperfect substitutability is causing serious problems and when it is not. ${ }^{20}$ The same tests can also be applied to the independence of status quo preserving alternatives in the random arbitration model.

This feature of the random arbitration model also gives rise to some discontinuities in the choice probabilities. Imagine that $x$ is a duplicate of $y$. Consider the menu $m=\{x, y, z\}$, where $z$ is neither a duplicate of $x$, nor dominated by it. Also, consider another item $x^{\prime}$ that is very close to $x$ (and therefore $y$ ) in all attributes, but not dominated by $x$ nor $z$. In this scenario, one expects that the choice probability of $z$ in $m^{\prime}=\left\{x^{\prime}, y, z\right\}$ should not jump discontinuously relative to its choice probability in $m$. However, my model predicts the opposite. In other words, the choice probability of $z$ moves discontinuously in $\{x, y, z\}$, as we change $x$ continuously.

Another related type of discontinuity in my model is when in the above example we

[^14]move $x$ marginally to $x^{\prime}$ that is dominated by $y$. In this case, the choice probability of $x$ out of $\{x, y, z\}$ falls discontinuously from $\rho(x \mid\{x, y, z\})>0$ to zero. However, such discontinuity seems less problematic than the previous one, given the rational assumption that a dominated item is never chosen.

### 1.5 Characterization of the Reference Point

One way of incorporating the effect of context on choices is by means of a menudependent reference point. In the previous sections, I showed that my specific choice of reference point helps us to accommodate the standard notions of the attraction and compromise effects. I also argued that such reference point can be interpreted as a context-dependent version of the status quo.

In this section, I take a different approach to motivate my choice of the reference point. I define a reference function $R: M \rightarrow \mathbb{R}^{N}$, where $M$ is the set of all finite and non-empty subsets of $\mathbb{R}^{N}$. Every set $m$ in $M$ is interpreted as a menu, and $R(m)$ is its reference point. I provide a number of reasonable properties on $R$ that imply that the reference point of a menu can only either take the minimum of all attributes or the maximum of them in the menu. However, as I will explain later, it is only the former that can generate the attraction effect.

The first property, scale invariance, states that transforming all items in a menu by an affine function, transforms the reference point by the same function.

1. Scale Invariance $a \in \mathbb{R}_{+}^{N}, b \in \mathbb{R}^{N}, m^{\prime}=\{a \cdot x+b \mid x \in m\}$ imply

$$
R\left(m^{\prime}\right)=a \cdot R(m)+b
$$

According to the next property, path independence, the reference point of the union of two menus is the reference point of the menu of their reference points. It is reminiscent of the path independence property of choice functions; $C\left(m \cup m^{\prime}\right)=C\left(\left\{C(m), C\left(m^{\prime}\right)\right\}\right)$.
2. Path Independence For all $m$ and $m^{\prime}$,

$$
R\left(m \cup m^{\prime}\right)=R\left(\left\{R(m), R\left(m^{\prime}\right)\right\}\right)
$$

The next property, symmetry, states that if we permute all vectors in $m$ by $\pi$, the reference point is also permeated by $\pi$. This implies that all attributes are treated similarly by the reference point function. In other words, when we exchange the value of two different attributes in all the menu items, the values of those attributes must be exchanged in the reference point as well.
3. Symmetry For all permutation $\pi$,

$$
R(\pi(m))=\pi(R(m))
$$

where $\pi(m)=\{\pi(x) \mid x \in m\}$
According to the next property, Range preservation, the value of the reference point of a menu for every attribute falls into the range of that attribute in the menu.

For every $x \in \mathbb{R}^{N}$ and $i \in\{1, . ., N\}$, let $x_{i}$ represent the $i^{t h}$ coordinate of $x$. Also, for every $m$ and $i \in\{1, \ldots, N\}, R_{i}(m)$ represents the $i^{t h}$ coordinate of the $R(m)$.
4. Range Preservation If for all $x \in m, x_{i} \in[l, h]$, then $R_{i}(m) \in[l, h]$

The last property, separability across attributes, requires that the value of the reference point not to change in an attribute, if we change only the value of other attributes in the menu items.
5. Separability Across Attributes For $m=\left\{x^{1}, x^{2} \ldots, x^{k}\right\}$ and $m^{\prime}=\left\{y^{1}, y^{2}, \ldots, y^{k}\right\}$,

$$
\left(x_{i}^{1}, \ldots, x_{i}^{k}\right)=\left(y_{i}^{1}, \ldots, y_{i}^{k}\right) \Rightarrow R_{i}(m)=R_{i}\left(m^{\prime}\right)
$$

Where $R_{i}(m)$ is the $i^{t h}$ component of $R(m)$.

Theorem $3 \quad R$ satisfies properties 1-5, iff

$$
\forall i \in\{1,2, \ldots, N\}, R_{i}(m)=\min _{x \in m} x_{i}
$$

or

$$
\forall i \in\{1,2, \ldots, N\}, R_{i}(m)=\max _{x \in m} x_{i}
$$

## Proof Appendix.

Note that a reference point that takes the maximum of all items for each attribute makes the random arbitration model incompatible with the attraction effect, regardless of the shape of the quasi-utility function. This is because adding a dominated item to a menu does not change the menu's reference point, thereby the weights of the already existing items. In principle, however, one can rewrite the random arbitration model with such reference point. This alternative model has the exact same axiomatization as the original one, with the only difference being that infimum must be exchanged with supremum in the ISPA axiom in section 2.

### 1.6 Related Literature

### 1.6.1 Reference-Dependent Deterministic Models

In economics, reference-dependent modeling came into prominence with the work of Markowitz (1952) and Kahneman \& Tversky (1979) in the context of choice under uncertainty, where final outcomes are evaluated based on their gain or losses relative to a reference point. Since then, numerous theoretical and experimental studies have appeared that document and
model the effect of different kinds of reference points on decisions. While most papers in the literature consider an exogenous reference point, a few of them endogenize the reference point.

Most notably, Simonson \& Tversky (1992) and more recently Tserenjigmid (2015) attribute context-dependence to a menu-dependent reference point that is constructed similar to the one in the random arbitration model. In an environment with two exogenously known attributes, these papers model choices as the outcome of maximizing a menu-dependent utility function, where the utility of an item $x=\left(x^{1}, x^{2}\right)$ in menu $m$, denoted by $U(x \mid m)$ is the following:

$$
U(x \mid m)=f\left(u\left(x^{1}\right)-u\left(r_{m}^{1}\right)\right)+g\left(w\left(x^{2}\right)-w\left(r_{m}^{2}\right)\right)
$$

Where $r_{m}=\left(\min _{x \in m} x^{1}, \min _{x \in m} x^{2}\right)$ and $u, w, f, g: \mathbb{R} \rightarrow \mathbb{R}$ are strictly increasing.
While Simonson \& Tversky (1992) only mention that this model can account for preference reversals such as the compromise effect when $f$ and $g$ are concave, Tserenjigmid (2015) also gives an axiomatization of the model. ${ }^{21}$

What the random arbitration model adds to these papers is that it captures the complexity of decision making in the presence of conflicting motives and the resulting stochasticity of decisions that stems from a lack of decisiveness in such situations. As a result, the random arbitration model provides a much smoother representation, where choice probabilities change continuously with respect to changes in the attributes values. Such smoothness facilitates comparative statics, when we want to understand how changing the value of an item's attributes changes its choice probability in a certain setting. Moreover, the random arbitration model allows for the inference of the relevant attributes from the observed stochastic choices. This inference is possible because the model generates both stochastic and deterministic choices. Specifically, attributes are inferred by means of the absolute preference relation that is derived from the deterministic part of observed choices. Since in Simonson \& Tversky (1992) and Tserenjigmid (2015) all choices are modeled deterministically, one can not infer their relevant

[^15]attributes from the observed choices. Lastly, I also provide a novel interpretation, as well as an intuitive characterization of the reference point.

A related work, Ok et al. (2014) also endogenizes the reference point. In particular, alternatives in every menu are compared to the menu's reference point, which is either one of the items in that menu or some exogenously given item. In contrast, the reference point of a menu in my model is not necessarily an element of the menu. Ok et al. (2014) axiomatically characterize such preferences in a deterministic environment and show how it is compatible with some documented anomalies such as the attraction effect.

Similar in spirit to the above works, De Clippel \& Eliaz (2012), propose a model in which every item has $N$ relevant attributes and decision is made according to the following procedure: For every menu $m$, item $x \in m$, and attribute $i$, the decision maker counts the number of items in $m$ that are ranked below $x$ in attribute $i$. This number serves as the score of $x$ in attribute $i$. The chosen items, $C(m)$, are those whose minimum attribute scores are the largest in the menu. This model generates the attraction effect by making the attractive item the only chosen one in the menu. A similar thing happens for the compromise item, which leads to the compromise effect. De Clippel \& Eliaz (2012) characterize the model for an arbitrary finite number of exogenously given attributes. Also, the model is characterized when there are only 2 endogenous attributes. Note that in contrast to my paper where attributes are cardinal measures of desirability, each attribute in De Clippel \& Eliaz (2012) is an ordinal one. Also, the model leads to deterministic choices, as opposed to our paper where choices are modeled stochastically.

### 1.6.2 Stochastic Choice

In general, there might be several reasons why choices appear to be stochastic, and each model of stochastic choice addresses a subset of these reasons. For example, the choices of an individual decision maker might look stochastic due to the deterministic variation of factors hidden to an outside observer, which are known to the decision maker himself/herself. Also, choice data might have come from a heterogeneous population with deterministic preferences.

This also makes the observed choices stochastic.
The random utility model, first characterized by Block et al. (1960), is a natural framework to model stochastic choice in the situations described above. Considering a finite set of alternatives $X$, this model assumes a probability distribution over the set of strict preference relations over $X$. As such, the probability of choosing $x$ in menu $m$ would be the probability of a preference relation that ranks $x$ above all other items in $m$. Due to its neutrality over the source of randomness of preferences, this model embodies the majority of stochastic choice models introduced in the literature. Note that under this interpretation of stochastic choice, there is nothing necessarily "irrational" or non-standard in the behavior of individual decision makers, rather it is a modeler's limited information that makes choice data appear stochastic.

On the other hand, stochastic choice can be interpreted as behavioral optimization. According to this interpretation, a decision maker's choices do not just look random to an outside observer with limited information, but they are truly random. People may choose randomly because of a variety of reasons such as uncertainty regarding their preferences, cognitive limitations, lack of enough information about available options, or other behavioral biases. The Luce model, as well the random arbitration model of this paper fall into this category.

In addition to the axioms provided in section 2, there are other notable characterizations of the Luce model. Gul et al. (2014) show that in a sufficiently rich environment with positive choice probabilities, the Luce model is the only one that admits a complete stochastic preference relation, as defined in section 2. This result reinforces the interpretation of the Luce model as a behavioral model of optimization.

Matejka \& McKay (2014) consider a setting where the values of items are contingent on some underlying uncertain state. The decision maker can choose an information channel that generates signals about the underlying state. More informative channels acquire more attention and are more costly. Matejka \& McKay (2014) show that when the decision maker optimally allocates her costly attention to acquiring information, then her choices will be in accordance with the Luce model.

Ahn et al. (2017) consider a setting where the universal set of items $X$ is a convex subset of $\mathbb{R}^{N}$ and introduce a property of average choices called partial path independence that resembles the path independence property of choice correspondences in deterministic settings. According to this property, the average choice of the union of two disjoint menus is a convex combination of the average of choice of the two menus. Ahn et al. (2017) show that partial path independence characterizes the Luce model.

Another behavioral strand of the literature explains stochastic choice as the outcome of deliberate randomization by the decision maker. This explanation first appeared in Machina (1985). Relying on the empirical evidence of violations of the expected utility hypothesis, in particular, the independence axiom in the realm of objective uncertainty, Machina (1985) argued that a decision maker may naturally want to randomize if she prefers a lottery over the available items to any of them individually. More recently, Fudenberg et al. (2015) and Cerreia-Vioglio et al. (2017) have proposed stochastic choice models that are special cases of the general model of Machina (1985).

Many of the models that interpret stochastic choice behaviorally are special cases of the random utility model, mathematically. Block et al. (1960) show that when all choice probabilities are positive, the Luce model can be represented by a random utility. Also, Holman and Marley (see Suppes \& Luce (1965)) and McFadden (1978) proved that the random part of utilities of items in the Luce model have i.i.d. Gumbel distributions. Although the random arbitration model is closely related to the Luce model, it can not be represented by a random utility. This is because the random arbitration model may violate regularity and also allows the absolutely dispreferred items to affect the choice probabilities of other items, contrary to the random utility models.

## Multi-Attribute and Context-Dependent Stochastic Choice

In the general formulation of the random utility model introduced earlier, attributes are not explicitly modeled. In discrete choice estimation, however, the random utilities of items are
conditioned on the observable attributes of choice objects and decision makers. In addition to that, the most related multi-attribute stochastic choice models to my work are the elimination by aspect (EBA) introduced by Tversky (1972) and the attribute rule introduced by Gul et al. (2014). Both of these models interpret stochastic choice behaviorally. While extending the Luce model ${ }^{22}$, these models primarily address the duplicate problem (red bus/blue bus problem) of the Luce model.

According to EBA, a decision maker randomly chooses some aspect (or attribute) at each stage with a probability proportional to its importance and eliminates all items that do not have the chosen aspect. He/she keeps repeating this process until only one item remains. Notice that this model does not run into the red bus/blue bus problem. This is the case because once the aspect "car" is chosen, both the red bus and the blue bus would be eliminated. Therefore, car would be chosen with a probability equal to the probability of choosing the "car" aspect, just like when blue bus is not available.

In the attribute rule, each item is a bundle of attributes with different intensities that are either zero, in case the item does not have the associated attribute or some positive number, in case the object has that attribute. Similar to EBA, the decision maker first chooses an attribute with a probability proportional to its importance. Next, out of the items that have the chosen attribute, he/she chooses each one with a probability proportional to its intensity in the chosen attribute. Now, if a new item $z$ that shares an attribute with $x$, but not with $y$ is made available, the probability of choosing $x$ gets hurt more than that of $y$ in a way almost similar to the EBA model. This can account for the duplicate problem.

Although these papers both take attributes into account, they work in slightly different environments from my model and address different issues. In these papers, unlike my model, two items might have different sets of attributes. For example, the attribute of being a bus is unique to a bus and is not shared with a car. Also, there is no criterion to measure the quality of an attribute in the EBA. In the attribute rule, while an intensity function measures the quality of

[^16]an attribute, it is not a cardinal measure as in my model. Moreover, unlike my model, an item $x$ might have a lower intensity than $y$ in every attribute, but still be chosen with positive probability over $y$.

A number of recent papers aim at explaining the context-dependent nature of stochastic choices, but do so primarily without invoking the relevant attributes. Ravid (2015) introduces a stochastic model of choice, in which the decision maker first randomly chooses an available item $x$, called the focal option, and compares it with all other items available. In comparison with an item $y$, the focal option $x$ is preferred with a fixed probability $\pi(x, y)$. If in all such pairwise comparisons, the item turns out to be preferred, it will be chosen at the end. If not, the process starts again, by choosing another random focal option. Therefore, the probability of choosing $x$ is the probability that it is the focal item and is preferred in all pairwise comparisons.

This model interprets the attraction effect as the following: if $\pi\left(x^{-}, x\right)$ is sufficiently small and $\pi\left(x, x^{-}\right)>\pi\left(y, x^{-}\right)$, then the probability of choosing $x$ increases when $x^{-}$is added to the menu of $\{x, y\}$. On the other hand, if $\pi(x, z)>\pi(y, z)$, adding $z$ to the menu of $\{x, y\}$ increases the relative choice probability of $x$ to $y$. This is interpreted as a demonstration of compromise effect.

Natenzon (2010) introduces an informational stochastic model of choice, in which the decision maker gradually learns the true utility of the available items by receiving random signals in continuous time. The signals received from different items might be correlated, depending on their true utilities. At every point in time, the probability of choosing an item $x$ from a menu $m$ is the posterior belief that $x$ is the best item in $m$. It is shown that under appropriate assumptions about the signal structure, if made sufficiently early, the decision maker's choices generate the attraction and compromise effects. These effects diminish gradually as the decision maker becomes more informed. Once again, this model does not explicitly take attributes into account.

Manzini \& Mariotti (2014) propose a model in which the decision maker suffers from limited attention and attends to an item $x$ randomly when it is available, with a fixed probability
$\alpha_{x}$, called the attention parameter. Other than that the decision maker is a standard one with a well defined strict preference ordering over the set of items. Therefore, he/she chooses the most preferred item out of the attended subset of available options, i.e. the consideration set. Due to the random nature of attention, the observed choices are generally stochastic and the relative choice probabilities are generally context-dependent. Brady \& Rehbeck (2016), extend Manzini \& Mariotti (2014) and allow for a correlation structure in attention, where every subset of the universal set of items is endowed with an attention parameter, which is not necessarily the product of the attention parameters of its individual members. Therefore, the probability that an item is chosen is the total sum of probabilities that the item is the most preferred one in the random subset of items in the menu. Although Manzini \& Mariotti (2014) is a RUM, it turns out that Brady \& Rehbeck (2016) is not.

### 1.7 Conclusion

In this paper, I developed a multi-attribute, stochastic model of choice, the random arbitration model, that generalizes the Luce model and the standard model of deterministic choice. Moreover, it leads to context-dependent evaluation of items, which enables it to accommodate the famous attraction and compromise effects, and violations of stochastic transitivity. In this model, context dependence stems from a menu-dependent reference point against which the available items are re-evaluated. Due to its ability to violate regularity, as manifested in the attraction effect, the random arbitration model is not a random utility model.

A key contribution of this model is that it allows us to infer the relevant attributes from the observed stochastic choices. This is important in situations where we do not know a priori which attributes are really important to a decision maker, and how he/she ranks the different values of those attributes.

I showed that my model is characterized by axioms 1-6; transitivity, monotonicity, weak consistency of relative value, independence of status quo preserving alternatives, and two
technical axioms; richness and poorness, and continuity. Moreover, I showed that when there is only one relevant attribute, the model is essentially equivalent to the standard deterministic. Also, when there are no duplicates and improvements over the reference point are evaluated by a log-linear quasi-utility function, the random arbitration model reduces to the Luce model.

In section 4, I introduced the generalized attraction and generalized compromise effects that generalize the standard notions of the two effects to more diverse settings. I showed that when there are only two attributes involved, the two effects in the random arbitration model are equivalent to a quasi-utility function $U$ that is log-concave in every argument and super-modular. I also showed that for an arbitrary number of attributes, the generalized compromise effect is equivalent to the quasi-utility function $U$ being log concave in every proper subset of the arguments and super-modular.

In section 5, I provided a number of intuitive properties of the reference function. These properties imply that in every menu the value of the reference point for every attribute is either the maximum or minimum of that attribute among the menu items. Since the former is not sensitive to the dominated items, it can not accommodate the attraction effect. This partly justifies using the latter. However, an alternative model can be written with minimal changes, in which the reference point takes the maximum values of the attributes among the available items.

More generally, one can theoretically think about the random arbitration model with other types of reference point. Other choices of the reference point with some desirable properties can sometimes accommodate the attraction and compromise effects. For example, when the quasi-utility function is log-concave in every argument and super modular, and the reference point takes the average of every attribute among the menu items, we are still going to observe the compromise effect. However, this choice of reference point and quasi-utility function does not necessarily lead to the attraction effect. In future research, the validity of each of these reference points may be investigated empirically in different settings.

Another venue of exploration is generalizing how the decision maker measures the improvement of an item in an attribute relative to that of the reference point. Currently, this
measure is linear, but in general it does not have to be. For example, we can consider a general family of functions $I_{1}, \ldots, I_{N}: \mathbb{R}^{2 N} \rightarrow \mathbb{R}$ such that the weight of an item $x=\left(x^{1}, \ldots, x^{N}\right) \in m$ is the following:

$$
U\left(I_{1}\left(x^{1}, r_{m}^{1}\right), \ldots, I_{N}\left(x^{N}, r_{m}^{N}\right)\right)
$$

Where for every $j \in\{1, \ldots, N\}, I_{j}$ is strictly increasing in the first coordinate and strictly decreasing in the second coordinate. With this generalization, all our substantive axioms still hold, except weak consistency of relative values. This is because $I_{j}$ is not in general linear, that is to say $I_{j}(x+k, y) \neq I_{j}(x+k, y)+k$ and $I_{j}(x, y+k) \neq I_{j}(x, y)-k$. However, weak consistency of relative value should be relaxed.

Also, in practice, a decision maker might not pay attention to all the relevant attributes in all her decisions. Formally, one may address this phenomenon by allowing the decision maker to attend to the attributes randomly and make the final decision based on the attended subset of attributes. To every subset of attributes, a distinct quasi-utility function can be assigned that determines the choice probabilities of items according to the random arbitration model, when that subset of attributes is attended. The characterization of this model remains for future research.

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Chapter 1, in full, is currently being prepared for submission for publication of the material. Heydari, Pedram. The dissertation author was the sole author of this paper.

## Chapter 2

## Stochastic Choice over Menus


#### Abstract

Models of choice over menus aim at capturing the effect of some behavioral or nonstandard element of decision making on the behavior of a single decision maker. These models are usually compared with the standard model of choice over menus, in which the decision maker chooses a menu whose best item is better than that of all other available ones. However, in many empirical settings such as experimental studies, choice data comes from a population of decision makers with possibly heterogeneous attitudes and tastes. This heterogeneity can make the observed choices over menus stochastic. This fact calls for a stochastic characterization of models of choice over menus to be able to better compare and contrast different models empirically. In this paper, I do this task for the standard model, which would be an extension of the random utility model to the realm of choice over menus. In particular, I provide the necessary and sufficient conditions; i.e. axioms on (stochastic) choice data over menus for it to be consistent with a population of decision makers each of whom behaves according to the standard model. The axioms that characterize the model are the axiom of revealed stochastic preferences over singletons, and three rationality axioms. In the end, I also briefly discuss an extension of the model to a setting with potentially time-inconsistent decision makers.


### 2.1 Introduction

This paper is an attempt to extend the random utility model (RUM) to the realm of dynamic choice. The framework that I am going to use is the one introduced by Kreps (1979). In this framework, the decision is made in two stages. In the first stage, the decision maker (she) chooses a menu from a set of available menus, out of which she would choose an individual item in the second stage. For example, she might first choose a restaurant to then choose a meal from its menu once in the restaurant. In this framework, menus derive all their values from what they contain. In other words, we abstract away from situations in which menus have intrinsic values.

Following Kreps (1979), choices over menus have been employed by numerous papers to model various issues and behavioral biases that are unidentifiable within the classical framework when we only look at choices from menus, i.e. choices in the second stage. ${ }^{1}$ All of these models characterize the behavior of a single decision maker. These models are often compared with the standard model where the decision maker chooses a menu that offers the best item according to her preferences over individual items. ${ }^{2}$ However, in many empirical studies, such as experimental settings choice data comes in an aggregate form from a population of decision makers who might defer in their preferences and attitudes. In such scenario, choice data will appear stochastic in the sense that different menus might be observed to be chosen from the same set of available menus. This fact calls for modeling choices over menus in a stochastic fashion in order to better test, calibrate, compare and contrast different models of behavior.

This paper does precisely the above task when the underlying model of individual behavior is the standard model. To do so, I assume that choices are generated via random utility maximization. Specifically, I assume that there exists a probability distribution over the set of all possible preferences over individual items, which in turn leads to a probability distribution

[^17]over the set of preferences over menus of items, according to the standard model. For example, restaurant $A$ is better than $B$ if the best meal of $A$ is better than the best meal of $B$, according to the realized preference relation over meals. ${ }^{3}$ I provide the necessary and sufficient conditions on stochastic choices over menus for them to be compatible with the described model. In section 4, I will turn my attention to a more general framework, where decision makers are allowed to be time-inconsistent, i.e. to have different first and second stage preferences over individual items. I provide some implications of this time-inconsistent model and compare these implications with the properties of the standard model.

### 2.2 Related Literature

On the one hand, this paper is related to the literature on stochastic choice and in particular the random utility model. To understand what the random utility model entails, let $X$ be the finite universal set of items. Menus, denoted by $m, m^{\prime}, m_{1}$, etc. are non-empty subsets of $X$. Denote the probability of choosing an item $x$ in a menu $m$ by $\rho(x \mid m) . \rho: X \times 2^{X} \backslash \emptyset \rightarrow[0,1]$ is called the random choice rule and satisfies $\sum_{x \in m} \rho(x \mid m)=1$. This condition is a feasibility condition requiring the chosen option to be available, i.e. to belong to $m$. Also, let $O(X)$ be the set of all strict orderings of $X$ and denote a generic element of it by $u$. According to RUM, there exists $\mu \in \Delta(O(X))$, such that

$$
\rho(x \mid m)=\mu(u \in O(X) \mid \forall y \in m, u(x) \geq u(y))
$$

In words, $\rho(x \mid m)$ is equal to the likelihood of having an strict ordering that places $x$ above all other items in $m .{ }^{4}$

[^18]Not all random choices are consistent with RUM. McFadden \& Richter (1990) provided the condition that characterizes RUM. Block et al. (1960) also gave an alternative axiomatizations of the random utility model. The condition is called the axiom of revealed stochastic preferences (ARSP), which goes as the following:

Axiom of Revealed Stochastic Preferences: For any list $x_{1} \in m_{1}, \ldots, x_{n} \in m_{n}$, the following is true:

$$
\sum_{i=1}^{n} \rho\left(x_{i} \mid m_{i}\right) \leq \max _{u \in O(X)} \sum_{i=1}^{n} 1_{\left\{\forall x_{j} \in m_{i}, u\left(x_{i}\right) \geq u\left(x_{j}\right)\right\}}
$$

The intuition for ARSP is the following: For every $u \in O(X)$, let $T(u)$ be the number of items in the list that are ranked above all other items in their respective menus, according to $u$. Also, let the right-hand side of the inequality above be equal to $T^{*}$. Therefore, for any $u \in O(X), T(u) \leq T^{*}$. On the other hand, the left-hand side of the inequality above is some weighted average of $(T(u))_{u \in O(X)}$ and therefore is less than $T^{*}$, hence the inequality above.

On the other hand, this paper is closely related to the literature that studies choices over menus. Menus are meant to represent the first stage of a dynamic decision process that determine the available options in future stages. Apart from their relevance to everyday decisions, in the theoretical literature, menus have been used to model and identify some anticipated behavioral aspect or irregularity in decision making that can not be fully identified in the classical static environments.

In the first paper on the subject, Kreps (1979) first characterizes deterministic preferences over menus that are compatible with a single standard decision maker. It turns out a single axiom is all that is needed, which requires the union of any two menus to be indifferent to the more preferred one. Furthermore, Kreps (1979) studies the effect of subjective uncertainty regarding future mood (i.e. second stage preferences) on preferences over menus and contrasts it with preferences of a standard decision maker, who does not perceive such uncertainty. He shows that while a standard decision maker does not find the combination of two menus more valuable than
both, a decision maker who perceives uncertainty regarding her future preferences might do so. In other words, such uncertainty leads to a preference for flexibility.

Gul \& Pesendorfer (2001) models the effect of temptation and costly self-control in the second stage on preferences over menus. In contrast to Kreps (1979), the decision maker in Gul \& Pesendorfer (2001) might prefer commitment rather than flexibility. This means that the combination of any two menus will always be ranked in between the two.

Ortoleva (2013) studies the effect of costly contemplation of choice or thinking in the second stage on preferences over menus. In this model, such costly contemplation might also lead to a preference for commitment. Also, Kopylov (2012) studies the effect of perfectionism and guilt aversion on choices over menus. In all these model, in the first stage, the decision maker anticipates these irregularities and adjusts her choices over menus accordingly to maximize her own utility. In some cases, the preferences of the first and second stage selves are aligned, such as in Kreps (1979), Ortoleva (2013) and temptation with self control model of Gul \& Pesendorfer (2001). In some other cases, there might be a conflict of interest between the two, such as the temptation without self-control model of Gul \& Pesendorfer (2001).

Another close paper to mine is a more recent work by Fudenberg \& Strzalecki (2015). This paper also aims at importing stochastic properties of choice to a dynamic environment. In some aspects, Fudenberg \& Strzalecki (2015) is more general than my approach. In particular, this paper allows for environments with an arbitrary finite number of stages, rather than only two stages. Moreover, at each stage, they allow the menus to not only determine the future available options, but also to have intrinsic values of their own. In contrast to my approach, Fudenberg \& Strzalecki (2015) assumes that we observe choices in all stages. Moreover, it focuses on specific stochastic choice models, such as logit and does not characterize the general effect of having random utilities on choices over menus.

In the next section, I introduce the stochastic version of the standard model and provide a characterization of it.

### 2.3 Standard Model

### 2.3.1 Preliminaries

$X$ is the universal set of items and is finite. Menus are subsets of $X$ and are denoted by $m, m^{\prime}, m_{1}$ and so on. ${ }^{5} M \subseteq 2^{X} \backslash\{\{\emptyset\}\}$ denotes a generic collection of menus that contain more than just the empty menu. $S_{M}$ denotes the union of all menus in $M$. Call $m$ a singleton if $|m|=1$. The set of all strict orderings of $X$ is denoted by $O(X)$, and the set of weak orderings of $X$ is denoted by $W O(X)$. Without loss of generality, we can represent each ordering of $X$, weak or strict, by an ordinal utility function. For the sake of simplicity, I use orderings and utility functions interchangeably.

I assume that observable data consists of the probability of choosing an arbitrary menu $m$ in an arbitrary collection of menus $M$ that contains $m$. Such probabilities are represented via a function called the random menu choice rule, $(R M C R)$, denoted by $\rho: 2^{X} \times 2^{2^{X}} \backslash\{\{\emptyset\}\} \rightarrow[0,1]$, such that $\sum_{m \in M} \rho(m \mid M)=1$. Also, for all menus $m$ and collection of menus $M, \rho$ satisfies $\rho(m \mid M \cup\{\emptyset\})=\rho(m \mid M)$. The latter means that empty menus do not affect the choice probability of other available menus.

### 2.3.2 Representation

Before the first stage, $u \in O(X)$ is realized according to a probability distribution that is denoted by $\mu \in \Delta(O(X))$. In the first stage, one of the available menus in $M$ whose best item is better than that of all others according to $u$ is chosen. In this model, indifferences can arise whenever the intersection of $m$ with some other menu in $M$ is not empty. To be more precise, whenever the best item according to the realized utility function $u$ lies in the intersection of $m$ and some other menu in $M$, then the decision maker becomes indifferent between $m$ and those menus containing that best item. This model stays silent about how these ties are broken. In other words, conditional on a realized utility function, $u, \mathrm{I}$ only assume that the chosen menu is

[^19]one of the best available ones according to $V_{u} .{ }^{6}$ Formally, this translates to the following: Every $u \in O(X)$ induces a weak ordering $V_{u} \in W O\left(2^{X} \backslash\{\emptyset\}\right)$ of menus in the following way:
$$
V_{u}(m)=\max _{x \in m} u(x)
$$

As such, the standard model is equivalent to:

$$
\begin{equation*}
\mu\left[u \mid \forall m^{\prime} \in M, V_{u}(m)>V_{u}\left(m^{\prime}\right)\right] \leq \rho(m \mid\{m\} \cup M) \leq \mu\left[u \mid \forall m^{\prime} \in M, V_{u}(m) \geq V_{u}\left(m^{\prime}\right)\right] \tag{1}
\end{equation*}
$$

The inequality on the left requires that the probability of choosing $m$ against the menus in $M$ to be weakly greater than the probability of having a utility function $u$ such that $V_{u}$ ranks $m$ strictly above all menus in $M$. The inequality on the right requires the probability of choosing $m$ against $M$ not to be greater than the probability of having a utility function $u$ such that $V_{u}$ ranks $m$ weakly above all menus $M$.

### 2.3.3 Axioms

In this section, I provide the axioms that characterize the standard model. Notice that in this model, the restriction of observed choices to the collection of singletons are similar to choices that are generated by a random utility maximizer in the realm of static choice. Thus, the restriction of $\rho$ to the set of singletons must satisfy ARSP. This is stated formally in the following axiom.

1. ARSP Over Singletons: Let $n$ be a an arbitrary natural number. Also, for every $i \in$ $\{1,2, \ldots, N\}$, let $M_{i}$ be a collection of singletons and $\left\{x_{i}\right\} \in M_{i}$. Then, the following holds:

$$
\sum_{i=1}^{n} \rho\left(\left\{x_{i}\right\} \mid M_{i}\right) \leq \max _{u \in O(X)} \sum_{i=1}^{n} 1_{\left\{\forall\left\{x_{j}\right\} \in M_{i}, u\left(x_{i}\right) \geq u\left(x_{j}\right)\right\}}
$$

[^20]Now consider three available restaurants with mutually exclusive menus $m, m^{\prime}, m^{\prime \prime}$. Suppose that after a while, $m$ and $m^{\prime}$ merge into a single restaurant that serves the menu $m \cup m^{\prime}$. After this change, the probability of choosing the new restaurant, $m \cup m^{\prime}$ would be the same as the probability of choosing either $m$ or $m^{\prime}$, before the merging took place. This observation leads us to the next merging additivity axiom.
2. Merging Additivity: For all mutually exclusive menus $m, m^{\prime}$, and $m^{\prime \prime}$, we have:

$$
\rho\left(m \cup m^{\prime} \mid\left\{m \cup m^{\prime}, m^{\prime \prime}\right\}\right)=\rho\left(m \mid\left\{m, m^{\prime}, m^{\prime \prime}\right\}\right)+\rho\left(m^{\prime} \mid\left\{m, m^{\prime}, m^{\prime \prime}\right\}\right)
$$

To see how the next axiom works, consider a set of available restaurants $M$ whose menus do not overlap with restaurant $m$. The probability of choosing $m$ against the restaurants in $M$ is only sensitive to the union of the items offered by restaurants in $M$ and not to how these items are distributed across different restaurants. This property is formalized in the following reorganization invariance axiom.
3. Reorganization Invariance: For every menu $m$ and sets of menus $M$ and $M^{\prime}, S_{M}=S_{M^{\prime}}$ and $m \cap S_{M}=\emptyset$ imply:

$$
\rho(m \mid\{m\} \cup M)=\rho\left(m \mid\{m\} \cup M^{\prime}\right)
$$

The next axiom is monotonicity. To understand the monotonicity axiom, consider an arbitrary set of available restaurants. If one of the restaurants discards all items that are found in other restaurants, its choice probability would weakly decrease. Moreover, the level of decrease in its choice probability is not greater than its choice probability when it offers only the discarded items and those items are removed from all other available restaurants. The described story is formalized below.
4. Monotonicity: For every menu $m$ and set of menus $M$, we have:

$$
\left.0 \leq \rho(m \mid\{m\} \cup M)-\rho\left(m-S_{M} \mid\left\{m-S_{M}\right\} \cup M\right\}\right) \leq \rho\left(m \cap S_{M} \mid\left\{m \cap S_{M}\right\} \cup(M-m)\right)
$$

Where $M-m=\left\{m^{\prime}-m \mid m^{\prime} \in M\right\}$.

Now we are ready to state the theorem:

Theorem: An RMCR $\rho$ satisfies axioms 1-4 if and only if there is a probability distribution over $O(X)$ such that the $\rho$ satisfies (1), i.e. $\rho$ is rationalizable by the standard model.

Proof: The formal proof is in the appendix. Here, I provide an informal sketch for the "only if" part. First, by ARSP over singletons and the fact that ARSP characterizes RUM, it is guaranteed that a probability measure $\mu$ over preferences exists such that the restriction of $\rho$ to singletons can be represented by (1). Next, using merging additivity and reorganization invariance, for any such $\mu$ I prove the validity of (1) for bigger menus and when the menu in consideration is disjoint with other available menus. Finally, by using the monotonicity axiom I prove the validity of (1) for cases where the available menus might overlap.

Identification: Note that when we confine ourselves to menus of deterministic outcomes in $X$, the inferred probability measure $\mu \in O(X)$ above may not be unique. ${ }^{7}$ However, in an alternative framework we can allow menus to contain lotteries over $X$ and assume that the decision maker has a random cardinal utility function that evaluates lotteries in $\Delta(X)$. In this framework, if we require the decision maker to be an expected utility maximizer in the realm of lotteries, then we can replace the axioms of random expected utility model of Gul \& Pesendorfer (2006) with our ARSP axiom over singletons. Since the random expected utility model of Gul \& Pesendorfer

[^21](2006) uniquely identifies the probability distribution of the cardinal utility function, our model would do so as well. Other than that, the rest of our axioms remain unchanged.

One important feature of the model is that adding an item $x$ to a menu $m$ will always increase the chance of choosing that menu against any collection of menus $M$ that are disjoint with $m \cup\{x\}$. This is because all the preferences that rank $m$ above all menus in $M$ still do so for $m \cup\{x\}$. However, there are extra new preferences that rank $m \cup\{x\}$ above all menus in $M$, and those are the preferences that place $x$ above all items that found in $m$ and the menus in $M$. Therefore, all such preferences also choose $m \cup\{x\}$ over all menus in $M$. One can think of this property as a form of stochastic monotonicity.

### 2.4 Time-Inconsistence Model

In this section, I discuss some implications of the model that allows for a general probability distribution over pairs of first and second stage strict preferences $(u, v)$. For every decision maker in the population, the value of this pair is realized at the beginning of stage 1 , before the decision maker chooses from among the available menus. This setting is meant to capture situations where a decision makers might suffer from temptations that they can not control in the second period. Moreover, decision makers are sophisticated about their lack of self control and the nature of their temptations. Therefore, this model is the random version of the temptation without self-control model of Gul \& Pesendorfer (2001), also known as the Strotzian ${ }^{8}$ model.

Formally, for every realized pair $(u, v)$, the associated decision maker chooses an available menu that has the highest value according to $V_{(u, v)}$, defined as the following:

$$
V_{(u, v)}(m)=u(\underset{y \in m}{\operatorname{argmax}} \quad v(y))
$$

Now, let $v$ stand for the probability density of $(u, v)$ in the population. Therefore, for every

[^22]menu $m$ and a collection of menus $M$ that are disjoint with $m,{ }^{9}$ we can represent $\rho(m \mid\{m\} \cup M)$ as the following:
$$
\rho(m \mid\{m\} \cup M)=v\left[(u, v) \mid \forall m^{\prime} \in M, V_{(u, v)}(m)>V_{(u, v)}\left(m^{\prime}\right)\right]
$$

As implied by the above representation, the assumption is that the decision maker is certain that her second-period preferences are represented by $v$. Since we are allowing for a general distribution over $(u, v)$, this model generalizes both the standard and the simple timeinconsistent model. Therefore, it will generate weaker empirical predictions. Although I have not fully characterized this model, in what follows I present a number of its implications to compare them with the axioms of the standard model.

Implication 1: The restriction of $\rho$ to collection of singletons satisfies ARSP.

Now, consider the following two implications of the time-inconsistent model and compare them with merging additivity and reorganization invariance of the standard model.

Implication 2: For all mutually exclusive menus $m, m^{\prime}, m^{\prime \prime}$, we have:

$$
\rho\left(m \cup m^{\prime} \mid\left\{m \cup m^{\prime}, m^{\prime \prime}\right\}\right) \leq \rho\left(m \mid\left\{m, m^{\prime}, m^{\prime \prime}\right\}\right)+\rho\left(m^{\prime} \mid\left\{m, m^{\prime}, m^{\prime \prime}\right\}\right)
$$

Implication 3: Iffor all $i \in\{1, \ldots, n\}, m \cap m_{i}=\emptyset$, then:

$$
\rho\left(m \mid\{m\} \cup\left\{m_{1}, m_{2}, \ldots, m_{n}\right\}\right) \leq \rho\left(m \mid\{m\} \cup\left\{m_{1} \cup m_{2}, \ldots, m_{n}\right\}\right)
$$

The first implication is the same as before since any probability distribution over pairs of $(u, v)$ gives us a marginal probability distribution over $u$. The second implication is a weakening

[^23]of the merging additivity axiom. It stems from the fact that whenever a sophisticated decision maker, equipped with a certain pair of first and second stage preferences $(u, v)$, prefers $m \cup m^{\prime}$ to $m^{\prime \prime}$, then she either prefers $m$ to $m^{\prime}$ and $m^{\prime \prime}$, or $m^{\prime}$ to $m$ and $m^{\prime \prime}$. However, the opposite is not necessarily true. The third implication says that when we combine two competing menus, $m_{1}$ and $m_{2}$, then we weakly increase the choice probability of a third competing menu, $m$. This is because any pair $(u, v)$ who prefers $m$ to both $m_{1}$ and $m_{2}$ will also prefer $m$ to $m_{1} \cup m_{2}$. Therefore, this implication can be thought of as the stochastic version of the set betweenness axiom introduced in Gul \& Pesendorfer (2001). Set betweenness states that the union of two menus is ranked (weakly) in between the two menus. Implication 3 is a weaker version of the reorganization invariance axiom of the standard model.

### 2.5 Conclusion

This paper is a first attempt to characterize the implications of random preferences over a set of items, when the actual choice objects are menus of such items. I assumed that menus are not intrinsically valuable, and they draw all their values from what they contain. First, I studied choices over menus generated by a population of decision makers, each of whom behaves according to the standard model. I showed that such choices are characterized by four axioms, namely ARSP over singletons, merging additivity, reorganization invariance, and monotonicity. Second, I investigated a more general setting that allows for a population of decision makes with (possibly) time-inconsistent preferences. I provided some implications of the time-inconsistent model and compared them with those of the standard model. The exact characterization of the time-inconsistent model, however, remains for future research. Another natural future step is to incorporate the stochasticity of choice in other famous models of choice over menus, such as temptation with self-control model of Gul \& Pesendorfer (2001) and the model of preference for flexibility of Kreps (1979), among others.

### 2.6 Acknowledgements

Chapter 2, in full, is currently being prepared for submission for publication of the material. Heydari, Pedram. The dissertation author was the sole author of this paper.

## Chapter 3

## Integration in Multi Community Networks


#### Abstract

There are many social and economic situations where two or more communities need to be integrated in an efficient and stable way that facilitates overall resource access throughout the network. We study structures for efficient integration of multi-community networks where building bridges across communities incur an additional link cost compared to links within a community. Building on the connections models with direct link cost and direct and indirect benefits, we show that the efficient structure for homogeneous cost and benefit parameters, and for communities of arbitrary size, always has a diameter no greater than 3 . We further show that if the internal cost is not small enough to justify a full graph for each community, integration always follows one of these three structures: Single star, two hub-connected stars, and a new structure we introduce in this paper as parallel hyperstar, which is a special multi-core/periphery structure with parallel bridges that connect the core nodes of different communities and includes a wide range of efficiently integrated structures. Then we investigate stability conditions of these structures, using two different definitions: The standard pairwise stability, as well as a new stability notion we introduce in this paper as post transfer pairwise stability, which allows for bilateral utility transfers. We show that while the parallel hyperstar structure can never be both efficient and pairwise stable, once post transfer pairwise stability is used, efficiency guarantees stability. Furthermore, we show that all possible efficient structures can be simultaneously post


transfer pairwise stable. In the end, we discuss future extensions of this model to multiple communities.

### 3.1 Introduction

There are many social and economic situations where two or more communities of agents need to be integrated by establishing inter-community links that act as integration bridges. Examples of such cases include post-merger and acquisition integration of firms (Epstein (2004), Cloodt et al. (2006)), integration and assimilation policies for immigrants, especially following massive waves of immigrations (Carrera et al. (2009), Abramitzky et al. (2014)) or motivating cross-disciplinary research and development activities (Huutoniemi et al. (2010)). In general, inter-community bridges are more costly than links within the community because of additional transaction costs resulting from differences in culture, language, trust level or organizational processes and routines. In the presence of this additional cost component, the extent of integration success depends on both stability and efficiency of the integrated structures. Stability determines whether the integrated structure is aligned with the interests of individual agents. Efficiency of the structure drives not only the success of integration from the perspective of a central planner, but in many cases, the perception of individuals regarding the overall benefit of integration.

While integration is achieved through establishing bridges across communities, its success depends on how those bridges are laid out in conjunction with links within each community. In other words, successful integration is the result of modifying internal structure of communities, in addition to creating integration bridges.

Because of externalities related to higher order connections, in general, efficiency and stability of network structures are not aligned with each other. Thus, a central question in successful integration is that whether there are integrated structures that can simultaneously achieve efficiency and stability. on how integration bridges are laid out in conjunction with links within each community, on the other.

One way to tackle this issue, that we employ in this paper, is to break down the problem into two consecutive questions: First identify efficient integrated structures, then check stability conditions on the set of efficient structures. This two step process also provides an extra benefit for the types of network where a central planner can design the initial structure of the network according to efficiency criteria. Once the efficient integration is implemented, it will remain intact if efficiency and stability are aligned.

In this paper, we investigate conditions for efficient integration of two communities, followed by finding the stability conditions of the resulting structures in the presence of a finite inter-community interaction barrier.

We follow this two step process for integration of two communities in the presence of a finite inter-community interaction barrier. Following connection cost model introduced in Jackson \& Wolinsky (1996), we assume that links are costly, that agents receive both direct benefits from their neighbors and indirect benefits from nodes at larger distance. These benefits are assumed to be a non-increasing function of distance. The inter-community interaction barrier is modeled with an extra connection cost parameter for links that involve nodes from two different communities. Throughout this paper, we refer to such nodes as interface nodes (otherwise internal nodes) and to the links between them as bridges (internal links). We assume arbitrary sizes for each community, but focus on the case where communities are identical with respect to cost and benefit parameters.

In general, for efficient integration we expect the network topology of local connections (those that link nodes of the same community) to depend on the topology of bridges. However, for the special case, where the connection of local links is small enough to justify a complete graph for individual communities, the efficient integration problem becomes that of only determining efficient structure of bridges. Although this special case was previously presented by Jackson \& Rogers (2005), we deal with the general (and more realistic) case where local links and bridges are interdependent. For this general case, our results show that when integration makes sense at all (i.e., if there is any link between the two communities), then depending on cost and benefit
parameters only three classes for efficient structures are possible: $i$ ) Single star with a node from one of the communities that acts as the global hub, ii) Two hub-connected stars, and iii) A new structure we introduce in this paper as parallel hyperstar where each community includes a super-hub, i.e., a subset of nodes that act as local hubs and are then connected to all non-hub members of their own communities. The bridges between the two communities are parallel links between the two super hubs. ${ }^{1}$ The efficient connections within each super-hub is such that the combination of the two super-hubs creates a full graph, thus there is the possibility of a large number of equally efficient structures because of this super-hub flexibility. Figure 3.1 shows these two structures for the case of homogeneous community cost (where local costs of the two communities are the same). However, results presented in the paper also apply to the case of heterogeneous connection cost.

We then investigate the stability conditions of these efficient integrated structures, starting by applying the standard pairwise stability condition. We show that of the three integrated structures, the single star structure and the two hub-connected stars can simultaneously satisfy both efficiency and pairwise stability conditions, while the parallel hyper star that covers a wide range of efficient structures can never be simultaneously efficient and pairwise stable. The standard definition of pairwise stability however does not allow for any utility transfer between the pairs of agents, which can affect agents' decisions regarding their connections in the network. To incorporate this factor, we then introduce a more general notion of stability that we call post transfer pairwise stability, that allows for bilateral utility transfer between pairs of nodes. With this new definition we prove that for the parallel hyperstar structure, efficiency ensures post transfer pairwise stability. For the other two structures, simultaneous efficiency and post transfer pairwise stability is possible, depending on the cost and benefit parameters.

The rest of this paper is organized as follows. First in Section 3.2, we provide a brief description of related literature. In Section 3.3, we explain the model and definitions we use

[^24]

Figure 3.1. Three efficient structures for two community networks.
throughout the paper. Next, in Section 3.4 that constitutes the main part of the paper, we offer a number of propositions and their proofs about the efficient structure of integration. In Section 3.5, we discuss the stability conditions of these efficient integration structures using both pairwise stability, as well as the new stability notion we introduce, before presenting concluding remarks in Section 3.6. Details of the proofs of some of the lemmas used in the main proposition of Section 3.4 are provided at the end of the paper as an appendix.

### 3.2 Related Literature

Structure of interaction networks has been demonstrated to affect a wide range of social and economic activities, such as engaging in criminal activities (Calvó-Armengol \& Zenou (2004)), emergence of pro-social norms such as trust, cooperation and fairness (Nowak (2006), Mosleh \& Heydari (2017)), financial systemic risk Acemoglu et al. (2015), and immigration and immigrant integration (Bisin et al. (2008)). Our work also discusses the structure of interaction network and builds on the literature of network formation models, which has interdisciplinary roots. It also contributes to the literature related to homophily and segregation, which is of interest in several of social sciences disciplines, and the growing literature of core/periphery networks. Here, without intending a comprehensive literature review, we point to a number of key papers in these three strands of literature that are mostly related to this work.

Network Formation Models: Our work follows the literature of strategic network formation where it is assumed that the net benefit of a relationship depends -in general- on the larger interaction network, rather than simply the dyad in question (Jackson (2010)). Within the
strategic network formation models two general approaches exist: One assumes directed ties that can be formed unilaterally (Bala \& Goyal (2000)), as in citation or social media followership networks. The other one, which is the basis of our work, assumes non-directed ties and mutual consent between connecting agents in link formation. This line originated from a model that Jackson \& Wolinsky (1996) established and called the connections model, which captures the value of externalities in a network. Following this work, a number of models have been proposed to introduce heterogeneity into the costs and benefits of forming links across agents (Heydari et al. (2015), Galeotti et al. (2006), Carayol \& Roux (2009)). Our work also generalizes the model of multi-island network of Jackson \& Rogers (2005) in which a truncated connections model was used and it was assumed that the local connection cost is small enough that makes each efficient community a full graph. We lift both of these constraints, that is we assume a non-truncated benefit model, and we do not put any constraint on connection costs. Our work also substantially adds to the discussion of efficiency for the case of two-cost level model in Galeotti et al. (2006) where the link cost of within and across groups are assumed to be different. Our approach however is different from this work from two key perspectives: Whereas this work considers a non-decaying, constant benefit, based on existence of a path between two nodes, we discuss efficiency based on a general benefit that decays as a function of distance between nodes. Moreover, our stability condition is different from this work. In particular, while they study structures that arise in the Nash equilibria of the link announcement game where links can be formed unilaterally, we use the solution concept of pairwise stability, according to which formation of a link requires bilateral agreement.

Homophily and Segregation: The issue of integration and segregation have received substantial attention in the literature of network science, and often in conjunction with the notion of homophily. Homophily refers to a rather consistent observation that similarities in status and value characteristics (race, language, religion, age, beliefs, etc.) of two given nodes are highly correlated with the probability of those nodes being connected (McPherson et al. (2001)), which in turn can cause segregation in many networks. Segregation as a result of homophily has
been reported in several contexts of friendship (De Martí \& Zenou (2017)), school networks (Leszczensky \& Pink (2015)), and a variety of online social networks (Colleoni et al. (2014), Halberstam \& Knight (2016)). Homophily is attributed to either forces that affect the way people meet and interact, or to the way people decide on who to form ties to and interact with (Currarini et al. (2009)). On the decision-related aspect, similarities in values and status are argued to create an ease of communication and coordination (Marks (1994), McPherson et al. (2001)), or avoid strategic manipulation of information (Galeotti et al. (2013)), which motivates the use of an additional cost parameter in this paper for inter-community links to capture the extra effort needed in communication, coordination, and trust formation.

Core/Periphery Networks: Our results also contribute to the growing interdisciplinary literature of core/periphery structures. Existence of core/periphery structures has been demonstrated in a wide variety of contexts such as financial networks (Van Der Leij et al. (2016), Fricke \& Lux (2015)), software architecture (Amrit \& Van Hillegersberg (2010), MacCormack (2010)), and networks of research and development (König et al. (2012); Choi (2011)). A number of papers have been addressing the formation mechanism and criteria of core periphery networks (Hojman \& Szeidl (2008), Csermely et al. (2013)). The latter reference also provides a comprehensive review of the literature of core periphery structures.

Our paper expands this literature by offering a generalization of the concept to multi-core /periphery structures since the parallel hyperstar structure that we will introduce in this paper is an example of a multi core/periphery network, and to the best of our knowledge, the simplest description of efficiency of a multi core-periphery structure. Some empirical evidences of multicore periphery structures have been recently reported, for example in architecture of complex systems (Baldwin et al. (2014)), and in R\&D networks (Tomasello et al. (2014)).

### 3.3 Model and Background Definitions

Agents, Networks, and Communities

We focus our attention to networks consisting of two communities. For $n+n^{\prime} \geq 3$, let $\mathscr{N}=\left\{1, \ldots, n+n^{\prime}\right\}$ be a set of agents, each belonging to one of the two communities $I$ or $I^{\prime}$ (also called community 1 and 2 respectively), where $|I|=n$ and $\left|I^{\prime}\right|=n^{\prime}$. A network $\mathscr{G}$ is a set of pairs of agents $\{i, j\}$ that describes which agents are connected. We denote the link between $i$ and $j$ by $i j$, so $i j \in \mathscr{G}$ means that there is a link between $i$ and $j$ in network $\mathscr{G}$. Throughout the paper, we assume that the links are undirected and binary (i.e., we do not assume any weight on the links). For a given network $\mathscr{G}, N_{i}(\mathscr{G})$ indicates the neighborhood of $i$ under $\mathscr{G}$, that is the set of all agents linked to $i$ in the network $\mathscr{G}$.

The distance between two agents $i$ and $j$ in network $\mathscr{G}$, denoted by $d_{i j}(\mathscr{G})$, is the minimum path length between $i$ and $j$. The diameter of $\mathscr{G}$ is defined as the maximum distance between any two connected nodes, which is always less than $n+n^{\prime}-1$.

## Benefits, Costs and Utility

The benefit to two arbitrary nodes whose distance in the network is $d$ is equal to $b(d)$, where $b: \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$. The benefits can be interpreted in various ways and can be attributed to the favor, information, or transaction opportunities associated with social and economic relationships. There is also a possibility of receiving such benefits from indirect relationships, through intermediary agents. We do not make any assumption on these benefits other than that the benefit that two nodes receive from one another is a non-increasing function of their distance, i.e., $b(k) \geq b(k+1) \geq 0$ for any integer $k>0$. Also, two nodes that are not connected through a path do not receive any benefit from one another. We formalize this by the assumption that for any $k \geq n+n^{\prime}, b(k)=0$. Also we assume the cost of forming a direct link for two nodes is $c$ if they are from the same community and $c+\delta$ otherwise, for some $\delta \geq 0$.

Let $u_{i}(\mathscr{G})$ represent the net utility that agent $i$ receives under $\mathscr{G}$, considering all costs and benefits. Also let $U(\mathscr{G})$ denote the total utility attributed to network $\mathscr{G}$. We assume $U(\emptyset)=0$.

So, we can represent the utility of a given node and the total utility as follows:

$$
\begin{align*}
u_{i}(\mathscr{G}) & =\sum_{j \in \mathscr{N}-\{i\}} b\left(d_{i j}(\mathscr{G})\right)-\sum_{j \in N_{i}(\mathscr{G})} c_{i j}  \tag{3.1}\\
U(\mathscr{G}) & =\sum_{i=1}^{n} u_{i}(\mathscr{G})
\end{align*}
$$

where we assume $c_{i j}$ to be equal to $c$ when $i$ and $j$ are from the same community, and equal to $c+\delta$ when they belong to different communities.

Following a utilitarian measure, we assume that a network $\mathscr{G}$ is strongly efficient, or simply efficient if it maximizes $U(\mathscr{G})$. In an abuse of the term, we will also compare the total utility of two networks by referring to them as more or less efficient. Most of the definitions up to this point are standard notions in the literature. Besides those, we introduce two more definitions that we will use in developing our results.

Now, we turn into defining various useful structural features that allow us to state and prove our results. First, following the standard in the literature we call a network $\mathscr{N}$ a star if there exists $i \in \mathscr{N}$ such that for any two distinct nodes $j, k \in \mathscr{N}, j k \in \mathscr{G}$ if and only if $i \in\{j, k\}$. In addition to this standard structure, we introduce the following novel structural features.

Interface Component: An interface component is a maximal collection of interface nodes across the two communities, between each pair of which there exists a path that only uses bridges. We denote the interface-components by $C_{1}, C_{2}, \ldots, C_{M}$, when there are $M$ of them. Also, denote the number of (interface) nodes in $C_{i}$ that are in $I$ and $I^{\prime}$ by $h_{i}$ and $h_{i}^{\prime}$ respectively. That is:

$$
\begin{aligned}
& \left|C_{i} \cap I\right|=h_{i} \\
& \left|C_{i} \cap I^{\prime}\right|=h_{i}^{\prime}
\end{aligned}
$$

An interface-component that has only one interface node in each community is called singular
and otherwise, non-singular.
Small World Property: A network satisfies the small world property if the following conditions hold:

- Distance of non-interface nodes from different communities is 3 .
- Distance of two nodes from the same community is at most 2 .
- Distance of an interface node with any foreign node is at most 2 .

Parallel Hyperstar: A two-community network $\mathscr{N}$ is a parallel hyperstar if all its interface components are singular, and for every two interface nodes $i, j \in I$ that are connected to $i^{\prime}, j^{\prime} \in I^{\prime}$ respectively, $i j \in \mathscr{G}$ if and only if $i^{\prime} j^{\prime} \notin \mathscr{G}$.

One can interpret a parallel hyperstar as a structure in which each community includes a super-hub, i.e., a subset of nodes that act as local hubs and are then connected to all nonhub members of their own communities while connecting the two communities in parallel connections between local hub members. The connections within each super-hub are such that the combination of the two super-hubs creates a full graph, thus there is the possibility of a large number of equally efficient structures because of this super-hub flexibility. It is worth noting that the two hub-connected starts is a special case of this parallel hyperstar structure with super-hubs being single nodes in each community. An illustrative figure of the parallel hyperstar structure is depicted in Figure 3.2.

### 3.4 Efficient Integrated Structures

In this section, we present our results on the shape of the efficently integrated structures for the model offered in the section 3.3. The case where $c \leq b(1)-b(2)$ has been characterized by Jackson \& Rogers (2005) for the case of truncated benefit function and geometric benefit function. For this case, each community always is a full graph internally, thus the structure of bridges and


Figure 3.2. Parallel hyperstar structure
internal links are disentangled. It has been shown that, depending on the bridge cost, efficient network is either two separate full graphs, a single full graph, or two full graphs with a number of parallel links between them. Here, we base our analysis on the case where $c>b(1)-b(2)$, which makes internal links and bridges interdependent. Furthermore, we consider the general case with no truncation assumption for benefits and assume general decreasing benefits, rather than the geometric decaying function.

Our first result below shows that when integration does not make sense at all (i.e., it is not efficient to create any bridges between the two communities), then the efficient network is either empty, or it would consist of two separate stars, each residing in one of the two communities. On the other hand, when integration makes sense, then depending on the cost and benefit parameters only two classes for efficient structures are possible: Single star with a node from the larger community that acts as the global hub; and the new class of structures that we introduced in the previous section as parallel hyperstar.

Theorem 1: If $b(1)-c<b(2)$, the global efficient network is either an empty network, two separate stars, a parallel hyperstar, or a single star.

Proof: The detailed proof is provided in the appendix. In what follows, we present a sketch of the proof.

Proof sketch of Theorem 1: We start the characterization by claiming that any efficient network satisfies the small world property. In lemma 2, we demonstrate that the most efficient network is essentially the one that satisfies the small world property with the minimum number of links. In lemma 4, we show that no network with more bridges than the size of either community is efficient. Also, in the boundary case, where $l=\min \left\{n, n^{\prime}\right\}$, the most efficient network is a single star encompassing both communities. Thus, from there after, we will focus on cases where $l<\min \left\{n, n^{\prime}\right\}$. Next, assuming the small world property, by means of lemma 5 , we find lower bounds on the number of bridges and internal links as functions of the number of interface components, the number of internal and interface nodes. We then use these lower bounds to show that a specific transformation always makes our networks more efficient, unless they are one of the structures introduced above, namely a parallel hyperstar, a single star or a two-connected-stars, which is a special case of parallel hyperstar.

### 3.4.1 Efficient Structures for Slow Decay Benefits

In the previous section, we proved that the globally efficient network, if connected, is either a parallel hyperstar or a single star. Now, we show that when the indirect benefit decays relatively slowly with distance, or the cost of forming internal links is too high, then a parallel hyperstar with more than 1 bridge can not be efficient. In other words, the following theorem holds:

Theorem 2: When $b(1)-c<b(3)$, a parallel hyperstar with more than 1 bridge is not efficient.

Proof: Assume $b(1)-c<b(3)$ and the network takes the form of a parallel hyperstar with more than 1 bridge. Suppose that $n \geq n^{\prime}$, which implies that $k \geq k^{\prime}$. We will show that the following alteration improves the utility of the network. Take two of such bridges, $A A^{\prime}$ and $B B^{\prime}$ where $A$ and $B$ are from community 1 , and without loss of generality assume that $A$ and $B$ are connected, but $A^{\prime}$ and $B^{\prime}$ are not (Note that this follows from the parallel hyperstar structure). Now, remove $A A^{\prime}$ and all the links from $A$ to the internal nodes in community 1. Finally add $B A^{\prime}$. These alterations change the utility of our network by:

$$
2 k(b(2)-b(1)+c)+2 k^{\prime}(b(3)-b(2)) \geq 2 k(b(3)-b(1)+c)>0
$$

Thus, making the network more efficient. Therefore, when $b(1)-c<b(3)$, the efficient network can not be a parallel hyperstar with multiple bridges.

Theorem 2 above leads to the following corollary:

Corollary: When $b(1)-c<b(3)$, the efficient structure, if connected, is either a single star or a two-connected-stars.

### 3.4.2 Cost Thresholds and Boundary Conditions

Here, we discuss how to determine which of the possible efficient structures - namely the two-separate-stars, the parallel hyperstar, and the single star structure- is realized for a given set of cost and benefit parameters. We do this in a straightforward way: First, we determine the total utility of each class of structures as a function of network parameters. For simplicity, we focus on the symmetric communities, $n=n^{\prime}$. The total utility of the network, $U(\mathscr{G})$, for each possible structure is:

- Two separate stars:

$$
U(\mathscr{G})=4(n-1)(b(1)-c)+2(n-1)(n-2) b(2)
$$

- Parallel hyperstar with $l$ bridges:

$$
U(\mathscr{G})=l(4 n-3 l+1)(b(1)-c)+[2 n(n-1)+l(l-1)] b(2)+2(n-l)^{2} b(3)-2 l \delta
$$

- A single star:

$$
U(\mathscr{G})=2(2 n-1)(b(1)-c)+2(2 n-1)(n-1) b(2)-2 n \delta
$$

Using these total utility equations, one can use simple computational methods to find the structure of choice. This can be achieved by taking the following steps: First we need to find the optimal number of bridges for the parallel hyperstar, which can be done by taking the first order conditions for the fast decay scenario. Also, for the slow decay scenario we already know the optimal parallel hyperstar is the two-connected-stars. Afterwards, we need to algebraically compare the total utility of the optimal parallel hyperstar with that of a single star and two separate stars. This will give us the inequalities that determine the most efficient structure.

### 3.5 Stability of Efficient Structures

In this section, we investigate the stability conditions of the efficient structures that were introduced in the previous section. We first will use the notion of pairwise stability to test whether integration structures that simultaneously satisfy stability and efficiency are possible. We show that under the restrictive definition of pairwise stability, this is only possible for one of the three structures. Then in the second part of this section, we introduce a new intuitive notion of stability, i.e. pairwise stability with bilateral transfers and prove a more encouraging result that shows that all three possible efficient structures are also stable.

### 3.5.1 Pairwise Stability Conditions

A network $\mathscr{G}$ is pairwise stable, if for every two nodes $i, j$, the following two conditions hold:

- If $i j \in \mathscr{G}$, then $u_{i}(\mathscr{G}-i j) \leq u_{i}(\mathscr{G})$ and $u_{j}(\mathscr{G}-i j) \leq u_{j}(\mathscr{G})$.
- If $i j \notin \mathscr{G}$ and $u_{i}(\mathscr{G}+i j) \geq u_{i}(\mathscr{G})$, then $u_{j}(\mathscr{G}+i j)<u_{j}(\mathscr{G})$.

In simple terms, in a network that is pairwise stable, fixing all other links, a link exists between two nodes whenever it is mutually benefitial for both nodes.

Theorem 3: The single star structure is both efficient and stable when $\delta$ is small enough. $A$ parallel hyperstar can be both stable and efficient only when it has just one bridge; i.e., it is a two-connected-stars.

Proof: The single star structure is stable when $b(1)-c-\delta>0$ (which guarantees no link is severed) and $b(2)>b(1)-c$ (which guarantees that no new link is formed). The second condition is satisfied throughout the paper. Therefore, it boils down to $b(1)-c-\delta>0$. As a consequence, if $\delta$ is sufficiently small, then the single star is stable. On the other hand, naturally when $\delta$ is small enough, the single star would become efficient. Therefore, a single star can both be efficient and stable.

The parallel hyperstar structure with $l>1$ is never stable. This is because an interface node, $H$, that is connected to at least one local interface node, $H^{\prime}$, does not have the incentive to stay connected to a local internal node, $P$. This is due to the fact that $H$ only loses a direct benefit of $b(1)$ by losing its connection with $P$, because $H$ can still get to $P$ in two steps through $H^{\prime}$. So, the net benefit of disconnecting from $P$ is $c+b(2)-b(1)$ which is by assumption bigger than 0 . So, $H$ wants to disconnect from $P$. In this argument, we used the assumption that $H$ is connected to at least one local interface node. But this is certainly true for at least one interface node in the parallel hyper star structure, unless there is only one interface node in each
community, which makes the structure a two-connected-stars. Here, we will only show that there are ranges of parameters for which the two-connected-stars structure is both stable and efficient, without characterizing all such ranges. For the slow decay case, we know that if there are any bridges between the two communities in the efficient structure, then the structure must be either a two-connected-stars or a single star. In such a case, when $\delta$ is not too small, then the single star structure is not going to be efficient, which only leaves the two-connected-stars as the candidate for the efficient structure.

On the other hand, since we assume that $b(1)-c<b(2)$, for the two-connected-stars to be stable we only need to satisfy the following inequalities:

1. $b(1)-c-b(3)<\delta<b(1)-c+(n-1) b(2)$

The inequality on the left insures that no two internal nodes across the two community have the incentive to form a bridge. The inequality on the right hand side guarantees that the interface nodes do not have the incentive to disconnected from each other.
2. $b(1)-c>0$

This simultaneously guarantees that no internal node in either communities has the incentive to disconnect from its local interface node and vice versa.

Slow decay means that $b(1)-c-b(3)<0$. Therefore the left inequality in (1) above is automatically satisfied given that $\delta>0$. Therefore, we only need to show that in the slow decay case, the two-connected-stars can be efficient when $\delta<b(1)-c+(n-1) b(2)$ and $b(1)-c>0$. In other words, we need to show that under these conditions, the single star structure is not always efficient. On the other hand, it is easy to show that under the slow decay condition the single star is efficient when:

$$
\delta<(n-1)(b(2)-b(3))
$$

Therefore, for the two-connected-stars to be more efficient than the single-star, it suffices to show that there is a non-empty range of vector of parameters $(b(1), b(2), b(3), n, c)$ such that:

$$
(n-1)(b(2)-b(3))<b(1)-c+(n-1) b(2)
$$

$$
b(1)-c>-(n-1) b(3)
$$

So, whenever for example $b(1)-c>0$, the above as well as the second stability condition for the two-connected-stars are both satisfied. This shows that the two-connected-stars can be both stable and efficient.

The above theorem demonstrates that efficiency and stability are not always aligned. This is expected, as indirect benefits result in externalities when two nodes form/severe their connection. These externalities create a wedge between what is optimal for society and what is optimal for individuals, in terms of forming or severing a link. Similar results can be found in Jackson \& Wolinsky (1996) and Galeotti et al. (2006). An important implication of theorem 3 is that a structure with an intermediate number of bridges (i.e., any number between one and the size of the smaller community) can never be both stable and efficient simultaneously.

### 3.5.2 Pairwise Stability with Bilateral Transfers

In this section, we introduce a new notion of stability that allows for bilateral monetary exchanges between pairs of nodes. We consider a quasi-linear environment, where utility is transferable in the form of money. Specifically, denote the amount of monetary transfer from $j$ to $i$ by $t_{i j}$, which is positive when $j$ pays $i$ and negative otherwise. Then the utility of every node $i$ in the network $\mathscr{G}$ is:

$$
u_{i}(\mathscr{G})=\sum_{j \in \mathscr{N}-\{i\}} b\left(d_{i j}(\mathscr{G})\right)-\sum_{j \in N_{i}(\mathscr{G})} c_{i j}+\sum_{j \in \mathscr{N}-\{i\}} t_{i j}
$$

Notice that this way, given that $t_{i j}=-t_{j i}$, the total utility of the network $\mathscr{G}$ is the same as the one defined in previous sections. Therefore, introducing monetary transfers does not change
the efficient structures.
In the standard notion of pairwise stability discussed in the previous section, stability of $\mathscr{G}$ implies that for any link $i j \in \mathscr{G}$ the change in the utilities of $i$ and $j$ after removing $i j$, denoted by $\Delta_{i}^{-i j}(G)$ and $\Delta_{j}^{-i j}(G)$ respectively, are both negative. Therefore, if one of these changes is positive, then the link $i j$ would not be maintained. Now, imagine that for a network $\mathscr{G}$ one of these changes is positive and the other one negative, for example $\Delta_{i}^{-i j}(G) \geq 0$ and $\Delta_{j}^{-i j}(G) \leq 0$. Now, if $\Delta_{i}^{-i j}(G) \leq-\Delta_{j}^{-i j}(G)$, then $j$ would be wiling to pay $-\Delta_{i}^{-i j}(G)-\Delta_{j}^{-i j}(G)$ to node $i$ for maintaining the link $i j$ and $i$ would accept it. Therefore, if $\Delta_{i}^{-i j}(G)+\Delta_{j}^{-i j}(G) \leq 0$, then the link $i j$ would be maintained.

Similarly, the standard notion of pairwise stability implies that for any $i j \notin \mathscr{G}$, at least one of the changes in the utilities of $i$ and $j$ after adding the link $i j$ is negative. These levels of change in the utilities of $i$ and $j$ are equal to $\Delta_{i}^{+i j}(\mathscr{G})$ and $\Delta_{i}^{+i j}(\mathscr{G})$, respectively. Imagine that exactly one of them is negative, for example, assume $\Delta_{i}^{+i j}(\mathscr{G})>0$ and $\Delta_{j}^{+i j}(\mathscr{G})<0$. Therefore, $j$ incurs a loss from adding $i j$, while $i$ benefits from it. Now, if $\Delta_{i}^{+i j}(\mathscr{G}) \geq-\Delta_{j}^{+i j}(\mathscr{G})$, then $i$ would be wiling to pay $\Delta_{i}^{+i j}(\mathscr{G})+\Delta_{j}^{+i j}(\mathscr{G})$ to node $j$ for maintaining the link $i j$ and $j$ would accept it. Therefore, if $\Delta_{i}^{+i j}(\mathscr{G})+\Delta_{j}^{+i j}(\mathscr{G}) \geq 0$, then the link $i j$ would be created, and otherwise, it would not be created.

This motivates the following definition that puts the above two scenarios in a single formula:

Definition: Post Transfer Pairwise Stability: A network $\mathscr{G}$ is post transfer pairwise stable if for every $i, j \in \mathscr{N}$, we have:

1. $i j \in \mathscr{G} \Rightarrow \Delta_{i}^{-i j}(\mathscr{G})+\Delta_{i}^{-i j}(\mathscr{G}) \leq 0$
2. $i j \notin \mathscr{G} \Rightarrow \Delta_{i}^{+i j}(\mathscr{G})+\Delta_{i}^{+i j}(\mathscr{G})<0$

In the next theorem, we show that the parallel hyperstar structure with more than 2 bridges is post transfer pairwise stable whenever it is efficient. For proving this claim, the
following lemma comes handy. The lemma implies that when a network is efficient and when we allow for bilateral transfers as explained above, then no pair of nodes would have the incentive to connect, if they are not already connected.

Lemma: If $\mathscr{G}$ is efficient, then for any $i j \notin \mathscr{G}$, we have:

$$
\Delta_{i}^{+i j}(\mathscr{G})+\Delta_{j}^{+i j}(\mathscr{G})<0
$$

Proof: First, note that the cost of link $i j$ is only borne by $i$ and $j$. On the other hand, the link $i j$, if added, potentially shortens the path between any pair of nodes in the network and therefore benefits all nodes other than $i$ and $j$. Therefore, for every $k \in \mathscr{N} \backslash\{i, j\}, \Delta_{k}^{+i j}(\mathscr{G}) \geq 0$. At the same time, efficiency of $\mathscr{G}$ requires that $\sum_{k \in \mathscr{N}} \Delta_{k}^{+i j}(\mathscr{G}) \leq 0$. The last two claims imply:

$$
\Delta_{i}^{+i j}(\mathscr{G})+\Delta_{j}^{+i j}(\mathscr{G})<0
$$

Theorem 4: An efficient parallel hyperstar structure with more than 2 bridges is post transfer pairwise stable.

Proof: First, based on lemma (?), we only need to check the first condition of the post transfer pairwise stability. Therefore, we only need to check that no pair of nodes have the incentive to sever an already existing link. By the proof of theorem 3, we know that pairwise instability of a parallel hyperstar with more than 2 bridges stems from the tendency of an interface node to sever its link with a local internal node. Therefore, we only need to check that this link would be preserved when the structure is efficient and we allow for bilateral transfers.

Imagine that this link is in the first community. It is easy to see that the change in the utility of the internal node after severing this link is $b(1)-c+h^{\prime}[b(1)-b(3)]+k^{\prime}[b(3)-b(4)]$
and the change in the utility of the interface node is $[b(2)-b(1)+c]$. Therefore, for this link to be maintained, we need to have:

$$
b(1)-c+h^{\prime}[b(1)-b(3)]+k^{\prime}[b(3)-b(4)] \geq b(2)-b(1)+c
$$

or equivalently

$$
h^{\prime}[b(1)-b(3)]+k^{\prime}[b(3)-b(4)] \geq 2[b(2)-b(1)+c] \quad(*)
$$

At the same time, according to theorem 2, efficiency of a parallel hyperstar with more than 2 bridges implies that $b(1)-c \geq b(3)$ or $b(1)-b(3) \geq c$. Now, plug in $b(1)-b(3)$ in place of $c$ in $(*)$. After simplification, the condition turns into:

$$
\left(h^{\prime}-2\right)(b(2)-b(3))+k^{\prime}(b(3)-b(4) \geq 0
$$

But the above is satisfied for the parallel hyperstar with more than two bridges, since $h^{\prime}-2 \geq 0, b(2)-b(3) \geq 0, k^{\prime} \geq 0, b(3)-b(4) \geq 0$. Therefore, when a parallel hyperstar with more than two bridges is efficient, it would be post transfer pairwise stable as well.

### 3.6 Concluding Remarks

We have studied structures for efficient integration of multi-community networks where building bridges across communities incur an additional link cost compared to links within a community. We built our analysis based on the connection models, a framework for strategic network formation, where we assumed direct link cost and direct and indirect benefits. We showed that the efficient structure for homogeneous cost and benefit parameters, and for communities of arbitrary size always has a diameter no greater than 3 . We further showed that if the internal cost is not small enough to justify a full graph for each community, integration always follows one of these two structures: Either a single star, or a new structure we introduced in this paper, called
parallel hyperstar, which is a special multi-core/periphery structure with parallel links among core nodes of different communities. We offered cost and benefit conditions where each structure is efficient and discuss the stability conditions of those structures. An important implication of our result on stability is that a structure with a medium number of bridges (i.e. between one and the size of the smaller community) is never stable and efficient simultaneously.

We would like to point out some of the limitations of our analysis. While we have introduced a degree of heterogeneity to the connection model by assuming multiple communities, the model still assumes all cost and benefit parameters are identical for all communities. Identifying efficient and stable structures for general heterogeneous parameters can be intractable, however, certain classes of heterogeneity, such as those with separable connection costs might be appropriate next steps of this analysis. Moreover, we confined our analysis to integration of two communities. While we expect much of the building blocks of our analysis to apply to the general number of communities, our work leaves investigating the general case (i.e., multiple island with identical internal and bridge costs) an open question for future exploration. Finally, in many cases, the notion of pairwise stability might not be the most intuitive stability solution, since in some cases, decisions of building a link across communities depend not only on the connecting agent, but on some measures of her community agreement. Formulating this notion of stability makes a priority for future research.

### 3.7 Acknowledgements

Chapter 3, in full, is currently being prepared for submission for publication of the material. Heydari, Pedram., Heydari, Babak. and Mosleh, Mohsen. were the primary authors of this paper.

## Appendix A

## Luce Arbitrates: Stochastic Resolution of Inner Conflicts

## A. 1 Proof of Theorem 1

From now on, WCRV and ISPA stand for weak consistency of relative values and independence of status quo preserving items, respectively.

It is easy to prove the "if" part of the theorem. So, I just prove the "only if" part. I break down the proof to two separate cases:
$1 . \succeq$ is complete.
2. $\succeq$ is not complete.

## Case 1:

If $\succeq$ is complete, then by the transitivity and monotonicity axioms, it will be a complete preference relation. In this case, any attribute function $A: X \rightarrow \mathbb{R}$ satisfying $A(x) \geq A(y)$ if and only if $x \succeq y$ and any increasing quasi-utility function $U: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$represent $\rho$ according to the random arbitration model. In fact, in this case, the quasi-utility function would not play any role, as the deterministic choices are all made on the basis of dominance and any nondeterministic choice is due to the presence of duplicates. Therefore, both the deterministic and
non-deterministic choices are purely based on A. Q.E.D.

## Case 2:

From now on, assume that $\succeq$ is not complete. Throughout the proof, I sometimes denote $\inf S$ by $\wedge S$ and if elements are specified; $S=\left\{x_{1}, \ldots, x_{n}\right\}$ we denote it by $x_{1} \wedge x_{2} \wedge \ldots \wedge x_{n}$. Also, I do the same things for $\sup S$ by $\vee$.

Fact 1: The light preference relation $\succeq$ is a partial order, i.e. it is:

- Reflexive: $\forall x, x \succeq x$

This part follows from the fact that $x \sim x$.

- Anti-symmetric: $\forall x, y$, s.t. $x \succeq y$ and $y \succeq x, \quad x \sim y$

This is usually stated with the equality sign replacing $\sim$.

- Transitive: $\forall x, y, z$ s.t. $x \succeq y, \quad$ and $\quad y \succeq z, \quad x \succeq y$

This follows from the transitivity axiom.

Based on the above definition of partial order, for a set $S, \inf S$ and $\sup S$ are in general sets of items. Specifically, for a partially ordered set $\left(X^{*}, \succeq^{*}\right), x \in \inf S$ if the following conditions hold:

1. $\forall t \in S, t \succeq^{*} x$
2. $\forall t \in S, t \succeq^{*} x$ implies $x \succeq^{*} y$.

Definition: A partially ordered set $\left(X^{*}, \succeq^{*}\right)$ is called a lattice if for every $x, y \in X^{*}, \inf \{x, y\}$ and $\sup \{x, y\}$ are non-empty.

Let $\wedge S$ and $\bigvee S$ denote arbitrary elements of $\inf S$ and $\sup S$, respectively. Also, $x_{1} \wedge x_{2} \wedge$ $\ldots \wedge x_{n}$ and $x_{1} \vee x_{2} \vee \ldots \vee x_{n}$ stand for the latter two things, respectively when $S=\left\{x_{1}, \ldots, x_{n}\right\}$.

Some simple properties: Given that $\left(X^{*}, \succeq^{*}\right)$ is a lattice, the following are true:

1. $(x \wedge y) \wedge z \sim^{*} x \wedge(y \wedge z)$
2. $x \wedge y \sim^{*} y$ if and only if $x \succeq^{*} y$
3. $x \vee y \sim^{*} y$ if and only if $y \succeq^{*} x$
4. Absorption Law: $x \vee(x \wedge y) \sim^{*} x \wedge(x \vee y) \sim^{*} x$

Definition: Consider a lattice $\left(X^{*}, \succeq^{*}\right)$. For an interval $I=[c, d] \subseteq X$, we say $b \in I$ is a complement of $a \in X$ relative to $I$ if $a \wedge b \sim^{*} c$ and $a \vee b \sim^{*} d$.

Definition: A lattice $\left(X^{*}, \succeq^{*}\right)$ is distributive if for all $x, y, z \in X^{*}$ we have: ${ }^{1}$

$$
\begin{aligned}
& x \wedge(y \vee z) \sim^{*}(x \wedge y) \vee(x \wedge z) \\
& x \vee(y \wedge z) \sim^{*}(x \vee y) \wedge(x \vee z)
\end{aligned}
$$

Fact 2: $\quad(X, \succeq)$ is a lattice.

Proof: According to fact $1, \succeq$ is partial order. This fact, together with richness (1) implies that $(X, \succeq)$ is a lattice. Q.E.D.

Theorem (Gratzer (2011)Grätzer (2011) A lattice is distributive if and only if every element in it has at most one relative complement in every interval.

[^25]Some Definitions: For every $x, y \in X$, let $[x, y]=\{z \in X \mid x \preceq z \preceq y\}$. For two sets of items $S, S^{\prime} \subseteq X$, let $S \wedge S^{\prime}=\bigcup_{x \in S, x^{\prime} \in S^{\prime}} \inf \left\{x, x^{\prime}\right\}$ and $S \vee S^{\prime}=\bigcup_{x \in S, x^{\prime} \in S^{\prime}} \sup \left\{x, x^{\prime}\right\}$. Also, for every item $x \in X$ and set $S$, let $S \wedge x=S \wedge\{x\}$ and $S \vee x=S \vee\{x\}$.

Lemma 1. $(X, \succeq)$ is a distributive lattice.

Proof: Poorness (2) implies that every element of $X$ has at most one relative complement in every interval, modulo the duplicates. Therefore, by $\operatorname{Gratzer}(2011)$ Grätzer (2011) $(X, \succeq)$ is a distributive lattice.

Lemma 2. If $\{x, y\}$ is univariate, then $[x, y]$ is univariate.

Proof: To prove that $[x, y]$ is univariate, we need to show that for all $z, t \in[x, y],[z, t]$ is totally ordered according to $\succeq$. Since $\{x, y\}$ is univariate, $[x, y]$ is totally ordered according to $\succeq$. Also, $z, t \in[x, y]$, implies that $[z, t] \subseteq[x, y]$, which means that $[z, t]$ is totally ordered as well. Therefore, $[x, y]$ is univariate.

Lemma 3. For incomparable $x, y$, and $t \in(x \wedge y, x), s \in(x \wedge y, y)$, $t$ and $s$ are incomparable.

Proof: By means of contradiction, assume the contrary, and without loss of generality, suppose $t \preceq s$. This implies that $t \wedge s \sim t \quad(*)$

By assumption, $t \prec x$ and $s \prec y$. Therefore, $t \wedge s \preceq x \wedge y \quad(* *)$ On the other hand, $x \wedge y \prec t$ and $x \wedge y \prec s$. Therefore, $x \wedge y \preceq t \wedge s \quad(* * *)$.
$(*),(* *)$, and $(* * *)$ imply that $t \wedge s \sim x \wedge y \sim t$, which contradicts $t \succ x \wedge y . Q . E . D$.

Lemma 4. If $[x \wedge y, x]$ and $[x \wedge y, y]$ are both totally ordered, then $[x, x \vee y]$ and $[y, x \vee y]$ are both totally ordered.

Proof: $\forall t \in[y, x \vee y], y \wedge x \preceq t \wedge x \preceq x$, therefore, $t^{*}=t \wedge x \in[x \wedge y, x]$.
Now:

$$
y \vee t^{*} \sim y \vee(t \wedge x) \sim(y \vee t) \wedge(y \vee x) \sim t \wedge(x \vee y) \sim t
$$

Therefore, $[x, x \vee y]=y \vee[x \wedge y, x]$. Therefore, since $[x \wedge y, x]$ is totally ordered by assumption, so is $[x, x \vee y] . Q . E . D$.

Lemma 5. If $\{x, y\}$ is univariate, then for any $z,\{x \wedge z, y \wedge z\}$ and $\{x \vee z, y \vee z\}$ are univariate.

Proof: We prove that the function $f:(x, y) \rightarrow(x \wedge z, y \wedge z)$ where $f(t)=t \wedge z$ is monotonic and is a bijection. The fact that it is monotonic is easy to check, so the range of $f$ is a subset of $(x \wedge z, y \wedge z)$.

Now, we show that it is onto. Consider $s \in(x \wedge z, y \wedge z)$, then:

$$
((s \vee x) \wedge y) \wedge z \sim(s \vee x) \wedge(y \wedge z) \sim^{(*)}(s \wedge y \wedge z) \vee(y \wedge x \wedge z) \sim^{(* *)} s \vee(y \wedge x \wedge z)
$$

Note that $(*)$ follows from the distributive property and $(* *)$ follows from the absorption law, given that $s \preceq y \wedge z$.

However, since $s \succeq x \wedge z$ and $x \wedge z \succeq y \wedge x \wedge z$, we have $s \succeq y \wedge x \wedge t$ by transitivity of $\succeq$. Therefore, using the absorption law once again, we can see that $s \vee(y \wedge x \wedge z) \sim s$, which means that

$$
((s \vee x) \wedge y) \wedge z \sim s \quad(a)
$$

On the other hand, $s \vee x \succeq x$ and $y \succ x$. This implies $y \succeq(s \vee x) \wedge y \succeq x$. Therefore,

$$
\begin{equation*}
(s \vee x) \wedge y \in[x, y] \tag{b}
\end{equation*}
$$

Therefore, $(a)$ and (b) together imply that $f$ is onto. Therefore, since $f$ is monotonically increasing, $[x \wedge z, y \wedge z]$ is a chain, and therefore univariate. The proof for $[x \vee z, y \vee z]$ is similar.
Q.E.D.

Lemma 6. Imagine that $[s, t],[t, x]$, and $[x, y]$ are all totally ordered, and $\{s, x, y\}$ is not univariate. Then, there exists $z$ such that $\{z, x, y\}$ is univariate and $z \succ s$.

Proof: By richness 2, we know that there exists $z$ such that $\{z, x, y\}$ is univariate. Now, if $z \succ s$, then we are done. If not, then $z$ must be incomparable with $s$, since otherwise $\{s, x, y\}$ would be univariate, which contradicts our assumption. Now, take the set $z \vee(s, t)=$ $(z \vee s, z \vee t)=(z \vee s, x)$ is non-empty and totally ordered, since $(s, t)$ is totally ordered. Now, take any $z^{\prime} \in(z \vee s, x)$, it is clear that $z^{\prime} \succ s$ and $\left\{z^{\prime}, x, y\right\}$ is univariate. Q.E.D.

Lemma 7. Imagine that $\{z \wedge t, t\}$ and $\{z \wedge t, z\}$ and $\{x, z \wedge t, t\}$ is univariate. Then, for any $w_{2} \prec w_{1} \prec z$ such that $\left\{w_{1}, w_{2}, z, z \vee t\right\}$ is univariate and $z \wedge t \prec h_{2} \prec h_{1} \prec z, w_{1} \wedge h_{2}$ and $w_{2} \wedge h_{1}$ are incomparable.

Proof: First, we show that $w_{i}$ and $h_{j}$ are incomparable and $w_{i} \vee h_{j} \sim z$ for all $(i, j) \in$ $\{1,2\}^{2}$, which implies that $\left(w_{i}, w_{i} \vee h_{j}\right)$ and $\left(h_{j}, w_{i} \vee h_{j}\right)$ are both totally ordered. First, note that by lemma $3,[z \wedge t, z]=z \wedge[t, z \vee t]$. This means that we can find $h_{i}^{\prime} \in[t, z \vee t]$ such that $h_{i} \sim h_{i}^{\prime} \wedge z$ for $i \in\{1,2\}$. Therefore, $w_{i} \vee h_{j} \sim w_{i} \vee\left(z \wedge h_{j}^{\prime}\right) \sim\left(w_{i} \vee z\right) \wedge\left(w_{i} \vee h_{j}^{\prime}\right) \sim z \wedge\left(w_{i} \vee h_{j}^{\prime}\right)$. Now, we want to show that $w_{i} \vee h_{j}^{\prime} \sim z \vee t$.

It is easy to show that this is equivalent to showing $w_{i} \vee t \sim z \vee t$ : Note that since $w_{i} \prec z$, we have $w_{i} \vee t \preceq z \vee t$, which implies that $w_{i} \preceq w_{i} \vee t \preceq z \vee t$. Now, since [ $w_{i}, z \vee t$ ] is totally ordered, one of the following must be true:

1. $w_{i} \vee t \in\left[w_{i}, z\right)$
2. $w_{i} \vee t \in[z, z \vee t]$
(1) implies that $w_{i} \vee t \prec z$. This means that $z \succ t$, which is a contradiction, since by assumption $z$ and $t$ are incomparable. Therefore, only (2) can be true. But (2) implies that
$w_{i} \vee t \succeq z$ and therefore, $w_{i} \vee t \succeq z \vee t$. But since we also know that $w_{i} \vee t \preceq z \vee t$, it follows that $w_{i} \vee t \sim z \vee t$. Now, as mentioned before, this fact implies that $w_{i} \vee h_{j}^{\prime} \sim z \vee h_{j}^{\prime}$, which in turn leads us to: $w_{i} \vee h_{j} \sim z \wedge\left(z \vee h_{j}^{\prime}\right) \sim z$, which is what we were after. Now, we proceed to proving the claim of the lemma. First, we show that $w_{i}$ and $h_{j}$ are incomparable. Notice that by what we just proved $w_{i} \vee h_{j} \sim z$. Also, $z \succ w_{i}$ and $z \succ h_{j}$. Therefore, $w_{i} \vee h_{j} \nsim w_{i}$ and $w_{i} \vee h_{j} \nsim h_{j}$, which implies that $w_{i}$ and $h_{j}$ are incomparable. Now, as an intermediate step, we show that $\left(w_{2} \wedge h_{1}\right) \vee\left(w_{1} \wedge h_{2}\right) \sim w_{1} \wedge h_{1}$. So, let's expand the left hand side:

$$
\begin{aligned}
& \left(w_{2} \wedge h_{1}\right) \vee\left(w_{1} \wedge h_{2}\right) \sim\left[\left(w_{2} \wedge h_{1}\right) \vee w_{1}\right] \wedge\left[\left(w_{2} \wedge h_{1}\right) \vee h_{2}\right] \sim \\
& \sim w_{1} \wedge\left[\left(w_{2} \vee h_{2}\right) \wedge\left(h_{1} \vee h_{2}\right)\right] \sim w_{1} \wedge\left(z \wedge h_{1}\right) \sim w_{1} \wedge h_{1}
\end{aligned}
$$

Now, note that by lemma 5, $w_{1} \wedge\left[h_{2}, h_{1}\right]=\left[w_{1} \wedge h_{2}, w_{1} \wedge h_{1}\right]$ and $h_{1} \vee\left[w_{1} \wedge h_{2}, w_{1} \wedge h_{1}\right]=$ [ $h_{2}, h_{1}$ ]. Therefore, since $h_{1} \succ h_{2},\left(h_{2}, h_{1}\right)$ is non-empty. Therefore, $\left(w_{1} \wedge h_{2}, w_{1} \wedge h_{1}\right)$ is also nonempty, which implies that $w_{1} \wedge h_{2} \succ w_{1} \wedge h_{1}$. The same way, we can show that $w_{2} \wedge h_{1} \succ w_{1} \wedge h_{1}$. Therefore, $\left(w_{2} \wedge h_{1}\right) \vee\left(w_{1} \wedge h_{2}\right) \sim w_{1} \wedge h_{1}$. Also, $w_{1} \wedge h_{1} \nsim w_{1} \wedge h_{2}$ and $w_{1} \wedge h_{1} \nsim w_{2} \wedge h_{1}$. Therefore, $w_{1} \wedge h_{2}$ and $w_{2} \wedge h_{1}$ are incomparable. Q.E.D.

Corollary If $\{z \wedge t, t\},\{z \wedge t, t\}$ and $\{x, z \wedge t, t\}$ are univariate, then $\{x, z\}$ is not univariate.

Proof: Since $w_{i} \vee h_{j} \in[x, z]$ for all $(i, j) \in\{1,2\}^{2}$, the above lemma means that we have found two items in $[x, z]$, namely $w_{1} \wedge h_{2}$ and $w_{2} \wedge h_{1}$ that are incomparable, which implies that $[x, z]$ is not totally ordered. Q.E.D.

Lemma 8. If $[z, x]$ and $[z, y]$ are both totally ordered and $x$ and $y$ are incomparable, then $z \sim x \wedge y$.

Proof: It is obvious that $z \preceq x \wedge y$. By means of contradiction, suppose that $z \prec x \wedge y$. This means that $x \wedge y \in(z, x) \cap(z, y)$. Therefore, $[x \wedge y, x]$ and $[x \wedge y, y]$ are both totally ordered, and $\{z, x \wedge y, x\}$ is univariate. But by the corollary of lemma 7, this means that $\{z, y\}$ is not
univariate. Therefore, $[z, y]$ is not be totally ordered as well, which implies $z \sim x \wedge y . Q . E . D$. Using similar techniques, we can prove the following lemma.

Lemma 9. For incomparable $x$ and $y$ if $[x, z]$ and $[y, z]$ are both totally ordered, then $x \vee y \sim z$.

Proof: Proof is omitted.

Lemma 10. If $x \nsim y$ and $\{x, y, z\}$ and $\{x, y, t\}$ are both univariate, then $\{x, y, z, t\}$ is univariate.

Proof: Since both sets $\{x, y, z\}$ and $\{x, y, t\}$ are univariate, they are both totally ordered. Without loss of generality, assume $x \preceq y$. Now, we can have one of the following cases:

1. $t \preceq x \prec y$ and $x \prec y \preceq z$
2. $z \preceq x \prec y$ and $x \prec y \preceq t$
3. $x \prec y \preceq z$ and $x \prec y \preceq t$
4. $z \preceq x \prec y$ and $t \preceq x \prec y$

The first two case, by the transitivity of $\succeq$ imply that $z$ and $t$ are comparable. Case 1 implies that $t \preceq x \preceq y \preceq z$. Now, we will show that $[z, t]$ is totally ordered. Note that since $\{x, y, z\}$ and $\{x, y, t\}$ are univariate, $[z, y]$ and $[x, t]$ are totally ordered. Therefore, to show that $[z, t]$ is totally ordered, it is enough to demonstrate that for every $w \in[z, x]$ and $s \in[y, t], w$ and $s$ are comparable. But $w \preceq x$ and $s \succeq y$, and $x \preceq y$, which by transitivity of $\succeq$ imply that $w \preceq s$. Therefore, $[z, t]$ is totally ordered, which implies that $\{x, y, z, t\}$ is univariate. Similarly, we can prove the lemma for case 2.

Now, consider case 3. By means of contradiction, assume that $z$ and $t$ are incomparable. Then, since $[y, z]$ and $[y, t]$ are both totally ordered, lemma 10 implies that $t \wedge z \sim y$. On the other hand, $[x, z]$ and $[x, t]$ are also totally ordered. Therefore, by lemma $10, t \wedge z \sim x$. Therefore, $x \sim y$, which contradicts the lemma's assumption that $x \nsim y$. Therefore, $z$ and $t$ must be comparable.

Without loss of generality, assume $t \succeq z$. Now, since $\{x, y, t\}$ is univariate, $[x, t]$ is totally ordered, and $\{z, y\} \subseteq[x, t]$. Therefore, $\{x, y, t, z\}$ is univariate.

For case 4, if $z$ and $t$ are incomparable, then lemma 9 implies that $z \vee t \sim x$ and $z \vee t \sim y$. This implies that $x \sim y$, which again is a contradiction. Now, once again, without loss of generality, assume that $t \succeq z$. Since, $\{x, y, z\}$ is univariate, $[z, y]$ is totally ordered and $\{x, t\} \subseteq[z, y]$. Therefore, $\{x, y, z, t\}$ is univariate. Q.E.D.

Lemma 11. For every $x \succ y$, there exists at most one maximal univariate set that contains $x$ and $y$.

Proof: As a consequence of lemma 10, if two univariates $S$ and $S^{\prime}$ contain $x, y$, then $S \cup S^{\prime}$ is also univariate and contains both $S$ and $S^{\prime}$. Now, take two maximal univariates $L$ and $L^{\prime}$ that contain $x$ and $y$. Note that $L \subseteq L \cup L^{\prime}, L^{\prime} \subseteq L \cup L^{\prime}$, and $L \cup L^{\prime}$ is univariate. But since $L$ and $L$ are maximal univariates, we must have $L=L \cup L^{\prime}$ and $L^{\prime}=L \cup L^{\prime}$. Therefore, $L=L^{\prime} . Q . E . D$.

Some definitions: For every set $S \subseteq X$, let $\max S=\{x \in S \mid \forall y \in S, x \succeq y\}$ and $\min S=\{x \in$ $S \mid \forall y \in S, y \succeq x\}$. Therefore, $\max S$ and $\min S$ represent the set of most lightly preferred items and least lightly preferred items in $S$. Note that these sets might be empty for some $S$. When $\max S(\min S)$ is non-empty, we say that $S$ has a maximum (minimum). Also, for every set $S$ and $x \in X$, let $S^{\succeq x}=\{y \in S \mid y \succeq x\}$ and $S^{\preceq x}=\{y \in S \mid y \preceq x\}$.

Lemma 12. For all $y \in X$ and maximal univariate $L, L^{\preceq y}$ has a maximum and $L^{\succeq y}$ has a minimum.

Proof By the second part of richness, $L$ is unbounded. Therefore, there exists $t \in L$ such that $t \nprec y$ and $\left\{x, r_{1}, t\right\}$ is univariate. Obviously, $t \succeq L^{\preceq y}$, i.e., $t \succeq z$ for all $z \in L^{\preceq y}$. Now, take $z^{*}=t \wedge y$. Since, $t \succeq L^{\preceq y}$ and $y \succeq L^{\preceq y}$, we will also have $t \wedge y \succeq L^{\preceq y . ~(*) ~}$

On the other hand, we know that $t \wedge y \preceq t$, which follows by our definition of $\wedge$. Now, pick an item on $x \in L^{\preceq y}$. We will have:

$$
x \preceq t \wedge y \preceq t
$$

But, $\{x, t\} \in L$. Therefore, $t \wedge y \in[x, t] \subset L$. This, together with $(*)$, implies that:

$$
t \wedge y \in \max L^{\preceq y}
$$

Similarly, we can prove that $L^{\succeq y}$ has a minimum. Q.E.D.

Definition: For every item $x$, denote by $V(x)$ the set of all maximal univariates that contain $x$ :

$$
V(x)=\{L \mid L=\text { maximal univariate }, x \in L\}
$$

we will show that $|V(x)|<\infty$ and the cardinality of $V(x)$ is independent of $x$.
Lemma 13. $\forall x, x^{\prime} \in X,|V(x)|=\left|V\left(x^{\prime}\right)\right|<\infty$.

Proof: To show that for every $x,|V(x)|<\infty$. For all $L \in V(x)$, pick an arbitrary item $z \in L$, such that $z \succ x$, and put them in a set $S$. Note that this is possible by the richness axiom part (b). Therefore, $|S|=|V(x)|$. Now, if $V(x)$ is not finite, then $S$ has infinite number of elements, out of which we can pick $\left\{x_{i}\right\}_{i=1}^{n}$ that are pairwise incomparable and $\left\{x, x_{i}\right\}$ is univariate for all $i \in \mathbb{N}$. However, this is in contradiction with the first part of the poorness axiom. Therefore, $V(x)<\infty$.

Now, we will show that for every $x$ and $x^{\prime},|V(x)|=\left|V\left(x^{\prime}\right)\right|$. Put $V(x)=\left\{L_{i}\right\}_{i=1}^{N}$, and $V\left(x^{\prime}\right)=\left\{L_{i}^{\prime}\right\}_{i=1}^{M}$. We prove the lemma in two cases:

1. $x$ and $x^{\prime}$ are comparable
2. $x$ and $x^{\prime}$ are incomparable.

Case 1: Without loss of generality assume that $x^{\prime} \succeq x$. In each $L_{i}$, choose an $t_{i}$ that is incomparable with $x^{\prime}$, and in each $L_{i}^{\prime}$ choose $t_{i}^{\prime}$ so that it is incomparable with $x$. Note that by the
richness axiom part b this is always possible. Because of lemma 3 and 4, elements of $\left\{y \vee t_{i}\right\}_{i=1}^{N}$ are pairwise incomparable and for all $1 \leq i \leq N,\left(y, y \vee t_{i}\right)$ is totally ordered, and therefore, it can be embedded in a maximal univariate. Moreover, by lemma 11, this embedding maximal univariate is unique. Therefore, $N \leq M$. Also, elements of $\left\{x \wedge t_{i}^{\prime}\right\}_{i=1}^{M}$ are pairwise incomparable and for all $1 \leq j \leq M,\left(x \wedge t_{i}^{\prime}, x\right)$ is totally ordered and therefore, it can be embedded in a maximal univariate. Therefore, $M \leq N$. The two inequalities imply that $M=N$.

Case 2: If $x$ and $x^{\prime}$ are incomparable, then $x \wedge x^{\prime} \prec x$ and $x \wedge x^{\prime} \prec x^{\prime}$. By case 1, $|V(x)|=\left|V\left(x \wedge x^{\prime}\right)\right|$ and $\left|V\left(x^{\prime}\right)\right|=\left|V\left(x \wedge x^{\prime}\right)\right|$. The latter two equations imply that $|V(x)|=\left|V\left(x^{\prime}\right)\right|$. Q.E.D.

From now on, we put $|V(x)|=N$.

Definition: $\quad L_{1}$ supersedes $L_{2}$, if the following holds:

$$
\forall x \in L_{2}, \exists y \in L_{1} \quad \text { s.t. } \quad y \succ x
$$

In other words, for $L_{1} \sim^{H} L_{2}$, either $L_{1}$ supersedes $L_{2}$ or vice-versa.
Now, we extend the homo-variability relation, in the following way:

Extended Homo-Variability Relation: The binary relation $\sim \sim^{H^{\prime}}$ extends the notion of homovariate sets in the following way: For two maximal univariates $L_{1}$ and $L_{2}, L_{1} \sim^{H^{\prime}} L_{2}$ if one of the following happens:

1. $L_{1} \sim^{H} L_{2}$
2. There exists another univariate $L$, such that $L_{1} \sim^{H} L$ and $L_{2} \sim^{H} L$.

We know that if $L_{1} \sim^{H} L_{2}$ and $L_{2}$ supersedes $L_{1}$, then for every $s_{1} \in L_{1}$ there exists $s_{2} \in L_{2}$ such that

$$
L_{2}^{\succeq s_{2}}=L_{1}^{\succeq s_{1}} \vee s_{2}
$$

$$
L_{2}^{\preceq s_{2}} \wedge s_{1}=L_{1}^{\preceq s_{1}}
$$

Also, it is easy to see that $\forall z \in L_{2}^{\preceq s_{2}}, z$ is incomparable with $s_{1}$.

Fact 3: It is not hard to check that according to the definition of extended homo-variability relation, two maximal univariate $L_{1}$ and $L_{2}$ are homo-variate, i.e $L_{1} \sim H^{H^{\prime}} L_{2}$, if and only if one of the following is true:

1. $L_{1} \wedge L_{2}=L_{1}$ or $L_{1} \wedge L_{2}=L_{2}$
2. There exists another maximal univariate set $L$ such that $L, L_{1}$ and $L, L_{2}$ satisfy 1 .

In other words, $L_{1} \sim^{H^{\prime}} L_{2}$, if they both supersede some maximal univariate $L$.

Lemma 14. For all maximal univariates $L_{2}, L_{3}, L_{2} \sim^{H^{\prime}} L_{3}$ if and only if $L_{2} \wedge L_{3}$ is a maximal univariate.

Proof: By fact 2, $L_{2} \sim^{H^{\prime}} L_{3}$ implies the existence of another maximal univariate $L_{1}$ such that $L_{1} \wedge L_{2}=L_{1}$ and $L_{1} \wedge L_{3}=L_{1}$. Now, since they both supersede $L_{1}$ we can find $s_{1}, s_{2}, s_{3}$ such that

$$
\begin{array}{ll}
L_{2}^{\succeq s_{2}}=L_{1}^{\succeq s_{1}} \vee s_{2} & L_{1}^{\preceq s_{1}}=L_{2}^{\preceq s_{2}} \wedge s_{1} \\
L_{3}^{\succeq s_{3}}=L_{1}^{\succeq s_{1}} \vee s_{3} & L_{1}^{\preceq s_{1}}=L_{3}^{\preceq s_{3}} \wedge s_{1}
\end{array}
$$

Now, for all items $y_{2} \in L_{2}^{\succeq s_{2}}$ and $y_{3} \in L_{3}^{\succeq s_{3}}$, there exists $x_{2}, x_{3} \in L_{1}^{\succeq s_{1}}$, such that:

$$
y_{2}=s_{2} \vee x_{2} \quad y_{3}=s_{3} \vee x_{3}
$$

Also, assume without loss of generality $x_{2} \succeq x_{3}$, then:

$$
y_{2} \wedge y_{3}=\left(s_{2} \vee x_{2}\right) \wedge\left(s_{3} \vee x_{3}\right)
$$

But:
$\left(s_{2} \wedge s_{3}\right) \vee x_{3} \sim\left(s_{2} \vee x_{3}\right) \wedge\left(s_{3} \vee x_{3}\right) \preceq\left(s_{2} \vee x_{2}\right) \wedge\left(s_{3} \vee x_{3}\right) \preceq\left(s_{2} \vee x_{2}\right) \wedge\left(s_{3} \vee x_{2}\right) \sim\left(s_{2} \wedge s_{3}\right) \vee x_{2}$

However, since $\left[x_{3}, x_{2}\right]$ is totally ordered, then $\left[\left(s_{2} \wedge s_{3}\right) \vee x_{3},\left(s_{2} \wedge s_{3}\right) \vee x_{2}\right]$ is totally ordered, according to lemma 2 . Therefore, since $y_{2} \wedge y_{3} \in\left[\left(s_{2} \wedge s_{3}\right) \vee x_{3},\left(s_{2} \wedge s_{3}\right) \vee x_{2}\right]$. This means that

$$
L_{2}^{\succeq s_{2}} \wedge L_{3}^{\succeq s_{3}}=\left(s_{2} \wedge s_{3}\right) \vee L_{1}^{\succeq s_{1}}
$$

And the above set is univariate according to lemma 5. Now,as we make $s_{1}$ smaller and smaller, $s_{2}$ and $s_{3}$ also get smaller, so the set $L_{2}^{\succeq s_{2}} \wedge L_{3}^{\succeq s_{3}}$ converges to $L_{2} \wedge L_{3}$. This implies that $L_{2} \wedge L_{3}$ is a maximal univariate. Q.E.D.

Now, we will prove that the extended homo-variability relation over pairs of maximal univariates is an equivalence relation.

Lemma 15. $\sim^{H^{\prime}}$ is an equivalence relation.

Proof: Obviously, each maximal univariate $L \sim^{H} L$ since it supersedes itself, due to the richness assumption.

Also, if $L_{1} \sim^{H^{\prime}} L_{2}$ and $L_{2} \sim^{H^{\prime}} L_{3}$ then, according to the lemma above, $L_{1} \wedge L_{2}$ and $L_{1} \wedge L_{3}$ are both maximal univariates.

Now, we show that $\left(L_{1} \wedge L_{2}\right) \wedge\left(L_{1} \wedge L_{2}\right)=L_{1} \wedge\left(L_{1} \wedge L_{2}\right)$ is also a maximal univariate. Once again choose $\left(s_{1}, s_{1,2}, s_{1,3}\right) \in L_{1} \times\left(L_{1} \wedge L_{2}\right) \times\left(L_{1} \wedge L_{3}\right)$, such that

$$
\begin{aligned}
& \left(L_{1} \wedge L_{2}\right)^{\preceq s_{1,2}}=L_{1}^{\preceq s_{1}} \wedge s_{1,2} \\
& \left(L_{1} \wedge L_{3}\right)^{\preceq s_{1,3}}=L_{1}^{\preceq s_{1}} \wedge s_{1,3}
\end{aligned}
$$

Therefore, we have

$$
\left(L_{1} \wedge L_{2}\right)^{\preceq s_{1,2}} \wedge\left(L_{1} \wedge L_{3}\right)^{\preceq s_{1,3}}=L_{1}^{\preceq s_{1}} \wedge\left(s_{1,2} \wedge s_{1,3}\right)
$$

But according to lemma 5, $\left(L_{1} \wedge L_{2}\right)^{\preceq s_{1,2}} \wedge\left(L_{1} \wedge L_{3}\right)^{\preceq s_{1,3}}=L_{1}^{\preceq s_{1}} \wedge\left(s_{1,2} \wedge s_{1,3}\right)$ is univariate, since $L_{1}^{\preceq s_{1}}$ is univariate. Now, as $s_{1} \rightarrow \infty$, we will have:

$$
\left(L_{1} \wedge L_{2}\right)^{\preceq s_{1,2}} \uparrow L_{1} \wedge L_{2}
$$

and

$$
\left(L_{1} \wedge L_{3}\right)^{\underline{〔} s_{1,3}} \uparrow L_{1} \wedge L_{3}
$$

Therefore:

$$
\left(L_{1} \wedge L_{2}\right)^{\preceq s_{1,2}} \wedge\left(L_{1} \wedge L_{3}\right)^{\preceq s_{1,3}} \uparrow\left(L_{1} \wedge L_{2}\right) \wedge\left(L_{1} \wedge L_{3}\right)
$$

Which implies that $\left(L_{1} \wedge L_{2}\right) \wedge\left(L_{1} \wedge L_{3}\right)=L_{1} \wedge L_{2} \wedge L_{3}$ is also maximal univariate. But by the absorption law, $L_{2} \wedge\left(L_{1} \wedge L_{2} \wedge L_{3}\right)=L_{3} \wedge\left(L_{1} \wedge L_{2} \wedge L_{3}\right)=L_{1} \wedge L_{2} \wedge L_{3}$. Therefore, by definition $L_{2} \sim^{H^{\prime}} L_{3}$. Q.E.D.

Next, we will define the total orders implied by the attributes. We already know that for every item $x \in X$ there exists a set, $V(x)$, contains $N$ maximal univariates each containing $x$. Also, we can partition the set of maximal univariates into $N$ equivalence classes based on the homo-variability relation. We enumerate these equivalence classes by $i \in\{1,2, \ldots, N\}$, and denote the maximal univariate that contains $x$ and falls into the $i^{\text {th }}$ equivalence class by $L_{(x, i)}$.

## A.1.1 Ordinal Structure of Attributes

Definition: For each $i \in\{1,2, \ldots, N\}$, define the binary relation $\succeq_{i}$ as the following:

$$
x \succeq_{i} y \Leftrightarrow y \in \min L_{(y, i)}^{\succeq x \wedge y}
$$

Lemma 16. If $L_{1} \sim^{H} L_{2}$ and both supersede $L$, then $L_{1} \wedge L_{2}$ also supersedes $L$.

Proof: $\quad$ Since $L_{1}$ and $L_{2}$ both supersede $L$, by definition we have:

$$
L_{1} \wedge L=L_{2} \wedge L=L
$$

Therefore

$$
\left(L_{1} \wedge L_{2}\right) \wedge L=L_{1} \wedge\left(L_{2} \wedge L\right)=L_{1} \wedge L=L
$$

Which implies that $L_{1} \wedge L_{2}$ supersedes L. Q.E.D.

Notation: It is easy to see that for every maximal univariate $L$ and $x \in X$, $\max L^{\preceq x}=\min L^{\succeq x}$. Denote an arbitrary element of these sets by $s_{L}^{x}$.

Lemma 17. For any $x, y \in X, i \in\{1, \ldots, N\}$, and $L_{0}=L_{(x \wedge y, i)}$, we have:

$$
s_{L_{0}}^{y} \sim x \wedge y \quad \text { or } \quad s_{L_{0}}^{x} \sim x \wedge y
$$

Proof First of all, it is easy to see that we have:

$$
\begin{equation*}
x \in \min L_{(x, i)}^{\succeq s_{L_{0}}^{x_{0}}} \quad \text { and } \quad y \in \min L_{(y, i)}^{\succeq s_{L_{0}}^{y}} \tag{*}
\end{equation*}
$$

Since $s_{L_{0}}^{x}, s_{L_{0}}^{y} \in L_{0}$, we can conclude that $s_{L_{0}}^{x}, s_{L_{0}}^{y}$ are comparable. Without loss of generality, assume $s_{L_{0}}^{x} \preceq s_{L_{0}}^{y}$. Now, since $x \wedge y \in L_{0}$, and $x \wedge y \preceq x,(*)$ implies that $s_{L_{0}}^{x} \succeq x \wedge y$. However, we also know that $s_{L_{0}}^{x} \preceq x$ and $s_{L_{0}}^{x} \preceq s_{L_{0}}^{y} \preceq y$. This implies that $s_{L_{0}}^{x} \preceq x \wedge y$. Therefore, $s_{L_{0}}^{x} \sim x \wedge y$. Q.E.D.

Lemma 18. The binary relation $\succeq_{i}$ is complete for any $i \in\{1, \ldots, N\}$.

Proof: By lemma 17, we know that we have:

$$
s_{L_{0}}^{y} \sim x \wedge y \quad \text { or } \quad s_{L_{0}}^{x} \sim x \wedge y
$$

Suppose, without loss of generality, that $s_{L_{0}}^{y} \sim x \wedge y$. It is easy to see that $y \sim \min L_{(y, i)}^{\succeq x \wedge y}$ and thus $x \succeq_{i} y$. Therefore, we can conclude that $x \succeq_{i} y$ if and only if $s_{L_{0}}^{x} \succeq s_{L_{0}}^{y}$. Therefore, $\succeq_{i}$ is complete. Q.E.D.

Lemma 19. $x \succeq_{i} y$ if and only if for every maximal univariate $L$ that is superseded by both $L_{(x, i)}$ and $L_{(y, i)}\left(\right.$ i.e., $\left.L_{(x, i)} \wedge L=L_{(y, i)} \wedge L=L\right)$, we have $s_{L}^{x} \succeq s_{L}^{y}$.

Proof Put $L_{0}=L_{(x \wedge y, i)}$. First, note that $L_{(x \wedge y)_{i}}=L_{(x, i)} \wedge L_{(y, i)}$ and by lemma 16, $L_{(x, i)} \wedge L_{(y, i)}$ supersedes $L$. Therefore, by lemma $12, L^{\preceq s_{L_{0}}^{y}}$ and $L^{\preceq s_{L_{0}}^{x}}$ have maximums.

Now, we will show that for any $L$ superseded by $L_{x_{i}}$ and $L_{y_{i}}$, we have the following:

$$
s_{L}^{y} \in \max L^{\preceq s_{L_{0}}^{y}} \quad s_{L}^{x} \in \max L^{\preceq s_{L_{0}}^{x}}
$$

Note that $s_{L}^{y} \in \max L^{\preceq s_{L_{0}}^{y}}$ is equivalent to the following:

1. If $z \in L$ and $z \preceq s_{L_{0}}^{y}$, then $z \preceq s_{L}^{y}$
2. $s_{L}^{y} \preceq s_{L_{0}}^{y}$

First, we show (1). Note that by definition $s_{L_{0}}^{y} \preceq y$. Therefore, $z \preceq s_{L_{0}}^{y}$ implies that $z \preceq y$. Therefore, $z \in L$ and $z \preceq s_{L_{0}}^{y}$ imply that $z \in L^{\preceq y}$. Also, by definition, $s_{L}^{y} \in \max L^{\preceq y}$. Therefore, $z \preceq s_{L}^{y}$.

We show (2) by means of contradiction. Let $t \in \min L_{0}^{\succeq s_{L}^{y}}$ and imagine that $t \succ s_{L_{0}}^{y}$. Since $L_{(y, i)}$ supersedes $L_{0}$, we have $y \wedge t \sim s_{L_{0}}^{y}$. On the other hand, since $t \succ s_{L}^{y}$ and $y \succ s_{L}^{y}$, we also
have $t \wedge y \succeq s_{L}^{y}$. This implies that $s_{L_{0}}^{y} \succeq s_{L}^{y}$, which is a contradiction.

Therefore, the following statement is easily implied:

$$
s_{L}^{y} \preceq s_{L}^{x} \quad \Leftrightarrow \quad s_{L_{0}}^{y} \preceq s_{L_{0}}^{x}
$$

The above by the proof of lemma 17, implies that for all $L$ that is superseded by $L_{(x, i)}$ and $L_{(y, i)}$, we have the following:

$$
x \succeq_{i} y \quad \Leftrightarrow \quad s_{L}^{x} \succeq s_{L}^{y}
$$

Q.E.D.

Lemma 20. For every $i, \succeq_{i}$, as defined above, is transitive.

Proof: Imagine that $x \succeq_{i} y$ and $y \succeq_{i} z$. We want to show that $x \succeq_{i} z$. Put $L=L_{(x \wedge y \wedge z)_{i}}$. $L$ is dominated by $L_{(x, i)}, L_{(y, i)}, L_{(z, i)}$. Therefore, since $x \succeq_{i} y$ and $y \succeq_{i} z$, by lemma 19, we can conclude:

$$
\begin{aligned}
& s_{L}^{x} \succeq s_{L}^{y} \\
& s_{L}^{y} \succeq s_{L}^{z}
\end{aligned}
$$

Therefore, by transitivity of $\succeq$, we can conclude that $s_{L}^{x} \succeq s_{L}^{z}$. Therefore, since $L$ is superseded by $L_{(x, i)}$ and $L_{(z, i)}$, by lemma 19 , we conclude that $x \succeq_{i} z . Q . E . D$.

Lemma 21. If $x \succ y$, then there exists $t$ such that $\{y, t\}$ is univariate and $y \prec t \prec x$.

Proof: If $\{x, y\}$ is univariate, then any $t \in(y, x) \neq \emptyset$ suffices, since by definition, $\{y, t\}$ is univariate. If $\{x, y\}$ is not univariate, then by lemma 12 , the set $L_{(y, i)}^{\preceq x}$ for any $i \in\{1, \ldots, N\}$ has
a maximum, denoted by $s_{L_{(y, i)}}^{x}$. Note that since $\{x, y\}$ is not univariate $s_{L_{(y, i)}}^{x} \nsim x$, and therefore, $s_{L_{(y, i)}}^{x} \prec x$. So, we can put $t=s_{L_{(y, i)}}^{x}$ for an arbitrary i. Q.E.D.

Lemma 22. $\forall x, y$ and $i \in\{1,2, \ldots N\}$, if $x \succeq_{i} y$, then $x \wedge y \sim_{i} y$ and $x \vee y \sim_{i} x$.

Proof: We already know that $y \succeq_{i} x \wedge y$, since $y \succeq x \wedge y$ in general. So, we need to show that $x \wedge y \succeq_{i} y$. Note that $(x \wedge y) \wedge y \sim x \wedge y$. Therefore,

$$
y \in \min L_{(y, i)}^{\succeq(x \wedge y) \wedge y}=\min L_{(y, i)}^{\succeq x \wedge y}
$$

Therefore, $x \wedge y \sim_{i} y$. Similarly, we can prove that $x \vee y \sim_{i} x$. Q.E.D.

Lemma 23. $\succeq$ is the intersection of $\succeq_{1}, \succeq_{2}, \ldots, \succeq_{N}$.

Proof: First, suppose that $x \succeq y$. This means that $x \wedge y \sim y$. Therefore, we have the following:

$$
y \in \min L_{(y, i)}^{\succeq x \wedge y}=\min L_{(y, i)}^{\succeq y}
$$

which implies that $x \succeq_{i} y$. Now, suppose that $x \succeq_{i} y$ for all $i \in\{1, \ldots, N\}$. we want to show that $x \succeq y$. Note that by lemma 22, we know that $x \wedge y \sim_{i} y$ for all $i \in\{1, \ldots, N\}$. Now, if $x \wedge y \prec y$, then by lemma 21, we can find some $j \in\{1, \ldots, N\}$ and $t \in L_{(x \wedge y, j)}$ such that $y \succ t \succ x \wedge y$, which contradicts $x \wedge y \sim_{j} y$. Therefore, $x \wedge y \sim y$ and this implies that $y \preceq x . Q . E . D$.

Lemma 24. For $x \nsim y,(x, y) \neq \emptyset$ is totally ordered, if and only if there exists $j \in\{1,2 \ldots, N\}$ such that all elements in $(x, y)$ are equal according to all $\succeq_{i}, i \in\{1,2, \ldots, N\} /\{j\}$ and $y$ is bigger than $x$ according to $\succeq_{j}$. That is:

$$
\exists j \in\{1, \ldots, N\} \text { s.t. } \forall i \in\{1,2 \ldots, n\} /\{j\}, y \sim_{i} x \text { and } y \succ_{j} x
$$

Proof: First of all, by lemma 23, since $y \succ x$, we must have $y \succeq_{i} x$ for all $i \in\{1, \ldots, N\}$ and $y \succ_{j} x$ for at least one $j \in\{1, \ldots, N\}$. Now, by means of contradiction, assume that $y \succ_{j} x$ and $y \succ_{k} x$ for $j \neq k \in\{1, \ldots, N\}$. Consider $t^{j} \prec s^{j}$ both in $\left(x, s_{L_{(x, j)}}^{y}\right)$ and $t^{k} \prec s^{k}$ both in $\left(x, s_{L_{(x, k)}}^{y}\right)$. Then $t^{j} \vee s^{k}, s^{j} \vee t^{k} \in(x, y)$. Now we will prove that $t^{j} \vee s^{k}, s^{j} \vee t^{k}$ are incomparable. First, note that $\left(t^{j} \vee s^{k}\right) \wedge\left(t^{k} \vee s^{j}\right) \sim t^{k} \vee t^{j}$. Also,

$$
\begin{equation*}
\left(t^{j} \vee t^{k}\right) \vee s^{j} \sim t^{k} \vee s^{j} \succ s^{j} \tag{1}
\end{equation*}
$$

And

$$
\begin{equation*}
\left(t^{k} \vee t^{j}\right) \wedge s^{j} \sim\left(t^{k} \wedge s^{j}\right) \vee\left(t^{j} \wedge s^{j}\right) \sim\left(t^{k} \wedge s^{j}\right) \vee t^{j} \sim x \vee t^{j} \sim t^{j} \tag{2}
\end{equation*}
$$

Now, putting (1) and (2) together, we can conclude that $t^{k} \vee t^{j}$ and $s^{j}$ are incomparable.
This means that

$$
\left(t^{j} \vee t^{k}, t^{k} \vee s^{j}\right)=\left(t^{j} \vee t^{k},\left(t^{k} \vee t^{j}\right) \vee s^{j}\right) \neq \emptyset \quad(a)
$$

and similarly,

$$
\begin{equation*}
\left(t^{j} \vee t^{k}, t^{j} \vee s^{k}\right)=\left(t^{j} \vee t^{k},\left(t^{j} \vee t^{k}\right) \vee s^{k}\right) \neq \emptyset \tag{b}
\end{equation*}
$$

Now, by means of contradiction assume that $t^{j} \vee s^{k}$ and $s^{j} \vee t^{k}$ are comparable, and without loss of generality, suppose $t^{j} \vee s^{k} \succeq s^{j} \vee t^{k}$. This implies that $\left(t^{j} \vee s^{k}\right) \wedge\left(t^{k} \vee s^{j}\right) \sim t^{j} \vee s^{k} \sim t^{k} \vee t^{j}$, which means that $\left(t^{j} \vee t^{k}, t^{k} \vee s^{j}\right)=\emptyset$. However, the latter contradicts (a). Therefore, $t^{j} \vee s^{k}$ and $s^{j} \vee t^{k}$ are incomparable. Therefore, assuming that $y \succ_{j} x$ and $y \succ_{k} x$ for $j \neq k$, allows us to find two incomparable items in $(x, y)$, which contradicts the assumption that $(x, y)$ is totally ordered. This implies that there is only one $j \in\{1, \ldots, N\}$ for which we have $y \succ_{j} x$ and for all $i \in\{1, \ldots, N\} /\{j\}$, we have $x \sim_{i} y$. Q.E.D.

Corollary Elements in a maximal univariate vary in only a single attribute. Also, two maximal univariates are homo-variate if and only if the variation in them is in the same attribute.

Definition: Call a univariate set of items $m, i$-variate for some $i \in\{1, \ldots, N\}$, if all items in $m$ have the same value in all attributes $j \neq i$.

## A.1.2 Cardinal Distance Relation

Take any two pairs of items $\{x, y\}$ and $\left\{x^{\prime}, y^{\prime}\right\}$, for which W.L.O.G $y \succeq_{i} x$ and $y^{\prime} \succeq_{i} x^{\prime}$. Also, take $\left(s, s^{\prime}\right) \in L_{(x, i)} \times L_{\left(x^{\prime}, i\right)}$ such that $s \sim_{i} y$ and $s^{\prime} \sim_{i} y^{\prime}$. Now, we will have the following definition.

Definition: Put $\{x, y\}={ }_{i}\left\{x^{\prime}, y^{\prime}\right\}$, if there exists $i$-variate $\left\{x^{\prime \prime}, y^{\prime \prime}\right\}$ such that both of the following are true:

$$
\begin{gathered}
0<\rho\left(x \mid\left\{x, x^{\prime \prime}\right\}\right)=\rho\left(s \mid\left\{s, y^{\prime \prime}\right\}\right)<1 \\
0<\rho\left(x^{\prime} \mid\left\{x^{\prime}, x^{\prime \prime}\right\}\right)=\rho\left(s^{\prime} \mid\left\{s^{\prime}, y^{\prime \prime}\right\}\right)<1
\end{gathered}
$$

Remark Note that we are also allowing $x^{\prime \prime}, y^{\prime \prime}$ to be the same as our original pairs $x, y$ or $x^{\prime}, y^{\prime}$, in the above definition.

Now, our goal is to show that this binary relation is an equivalence relation. However, before tackling that problem, we need to prove two auxiliary lemmas.

Lemma 25. For any menu of items $m$, we can find an item $z$ that is incomparable with all items in $m$.

Proof: Since we are assuming that $\succeq$ is incomplete, $N \geq 2$. Therefore, pick two different arbitrary attributes $i$ and $j$. It is not hard to see that we can find some item $z$ such that for all $x \in m, z \prec_{i} x$ and $z \succ_{j} x$. Such $z$ is incomparable with all $x \in m$. Q.E.D.

Lemma 26. Extension of WCRV: Pick an arbitrary $n \in N, i \in\{1, \ldots, N\}$ and $n i$-variates $\left\{\left\{x_{j}, y_{j}\right\}\right\}_{j=1}^{n}$. If for all $j \in\{1, \ldots, n-1\}$,

$$
\rho\left(x_{j} \mid\left\{x_{j}, x_{j+1}\right\}\right)=\rho\left(y_{j} \mid\left\{y_{j}, y_{j+1}\right\}\right) \in(0,1)
$$

Then we will have:

$$
\rho\left(x_{1} \mid\left\{x_{1}, x_{n}\right\}\right)=\rho\left(y_{1} \mid\left\{y_{1}, y_{n}\right\}\right)
$$

Proof: By lemma 25 and the richness axiom, we are able to find an $i$-variate $\left\{x_{0}, y_{0}\right\}$ such that $x_{0}$ is incomparable with each $x_{i} \in\left\{x_{1}, \ldots, x_{n}\right\}$ and

$$
\rho\left(x_{1} \mid x_{1}, x_{0}\right)=\rho\left(y_{1} \mid y_{1}, y_{0}\right) \in(0,1)
$$

Now, by applying WCRV to $\{0,1,2\}$ we conclude that

$$
\rho\left(x_{2} \mid\left\{x_{2}, x_{0}\right\}\right)=\rho\left(y_{2} \mid\left\{y_{2}, y_{0}\right\}\right) \in(0,1)
$$

. Note that the probabilities are in $(0,1)$ because we have assumed that $x_{0}$ is incomparable with all $x_{i} \in\left\{x_{1}, \ldots, x_{n}\right\}$, which includes $x_{2}$. Now, we can apply WCRV to $\{1,2,3\}$, etc. By repeating this process, we will be able to show that $\rho\left(x_{n} \mid\left\{x_{n}, x_{0}\right\}\right)=\rho\left(y_{n} \mid\left\{y_{n}, y_{0}\right\}\right) \in(0,1)$. Finally, by applying WCRV to $\{0,1, n\}$, we can conclude:

$$
\rho\left(x_{1} \mid\left\{x_{1}, x_{n}\right\}\right)=\rho\left(y_{1} \mid\left\{y_{1}, y_{n}\right\}\right)
$$

Q.E.D.

Lemma 27. If there exists $\left(z_{0}, t_{0}\right)$ such that

$$
0<\rho\left(x \mid\left\{x, z_{0}\right\}\right)=\rho\left(s \mid s, t_{0}\right)<1 \quad 0<\rho\left(x^{\prime} \mid\left\{x^{\prime}, z_{0}\right\}\right)=\rho\left(s^{\prime} \mid\left\{s^{\prime}, t_{0}\right\}\right)<1
$$

Then for all $(z, t)$ such that $0<\rho(x \mid\{x, z\})=\rho(s \mid\{s, t\})<1$, and $z$ incomparable with $x$ and $x^{\prime}$, we have:

$$
\rho\left(x^{\prime} \mid\left\{x^{\prime}, z\right\}\right)=\rho\left(s^{\prime} \mid\left\{s^{\prime}, t\right\}\right)
$$

Proof: If for some $i$-variate set $\left\{z^{0}, t^{0}\right\}$ we have

$$
0<\rho\left(x \mid\left\{x, z_{0}\right\}\right)=\rho\left(s \mid s, t_{0}\right)<1 \quad 0<\rho\left(x^{\prime} \mid\left\{x^{\prime}, z_{0}\right\}\right)=\rho\left(s^{\prime} \mid\left\{s^{\prime}, t_{0}\right\}\right)<1
$$

Then if $0<\rho(x \mid\{x, z\})=\rho(s \mid\{s, t\})<1$, relabel items as:

$$
(z, t)=\left(x_{1}, y_{1}\right),(x, s)=\left(x_{2}, y_{2}\right),\left(z^{0}, t^{0}\right)=\left(x_{3}, y_{3}\right),\left(x^{\prime}, s^{\prime}\right)=\left(x_{4}, y_{4}\right)
$$

Then, by what we have assumed, we have:

$$
\forall i \in\{1,2,3\}, \quad 0<\rho\left(x_{i} \mid\left\{x_{i}, x_{i+1}\right\}\right)=\rho\left(y_{i} \mid\left\{y_{i}, y_{i+1}\right\}\right)<1
$$

Therefore, by lemma 26, we can conclude that:

$$
\rho\left(x_{1} \mid\left\{x_{1}, x_{4}\right\}\right)=\rho\left(y_{1} \mid\left\{y_{1}, y_{4}\right\}\right)
$$

which getting back to our original notation, means that :

$$
\rho\left(z \mid\left\{z, x^{\prime}\right\}\right)=\rho\left(t \mid\left\{t, s^{\prime}\right\}\right)
$$

or equivalently:

$$
\rho\left(x^{\prime} \mid\left\{z, x^{\prime}\right\}\right)=\rho\left(s^{\prime} \mid\left\{t, s^{\prime}\right\}\right)
$$

Q.E.D.

Lemma 28. For every $i \in\{1, \ldots, N\},={ }_{i}$ is an equivalence relation.

Proof: Take $\{z, t\}$ to be an $i$-variate, such that $z$ is incomparable with $\left\{x, x^{\prime}, x^{\prime \prime}\right\}$ and

$$
\rho(x \mid\{x, z\})=\rho(s \mid\{s, t\}) \in(0,1)
$$

Also, because $\{x, s\}={ }_{i}\left\{x^{\prime}, s^{\prime}\right\}$, by lemma 27, we must have:

$$
\rho\left(x^{\prime} \mid\left\{x^{\prime}, z\right\}\right)=\rho\left(s^{\prime} \mid\left\{s^{\prime}, t\right\}\right) \in(0,1)
$$

Note that the probabilities fall in $(0,1)$ because we are assuming that $z$ is incomparable with $x^{\prime}$.
Now, because $\left\{x^{\prime}, s^{\prime}\right\}={ }_{i}\left\{x^{\prime \prime}, s^{\prime \prime}\right\}$, once again according to lemma 27, we must have:

$$
\rho\left(x^{\prime \prime} \mid\left\{x^{\prime \prime}, z\right\}\right)=\rho\left(s^{\prime \prime} \mid\left\{s^{\prime \prime}, t\right\}\right) \in(0,1)
$$

Therefore, we have the following two equations:

$$
\begin{gathered}
\rho(x \mid\{x, z\})=\rho(s \mid\{t, s\}) \in(0,1) \\
\rho\left(x^{\prime \prime} \mid\left\{x^{\prime}, z\right\}\right)=\rho\left(s^{\prime \prime} \mid\left\{t, s^{\prime \prime}\right\}\right) \in(0,1)
\end{gathered}
$$

Therefore, by definition of $=_{i}$ we have $\left\{x^{\prime \prime}, s^{\prime \prime}\right\}={ }_{i}\{x, s\} . \quad$ Q.E.D.
We can easily generalize the idea of the above corollary to the following definition.

Definition Two menus $m=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ and $m^{\prime}=\left\{x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{n}^{\prime}\right\}$ are congruent if for all $i \in\{1,2 \ldots, n\}, \rho\left(x_{i} \mid m\right)=\rho\left(x_{i}^{\prime} \mid m^{\prime}\right)$. When this happens, we put $m \cong m^{\prime}$.

Lemma 29. Imagine that $m=\{x, y, r\}$ and $m^{\prime}=\left\{x^{\prime}, y^{\prime}, r^{\prime}\right\}$, are such that $r \preceq x \wedge y$ and $r^{\prime} \preceq x^{\prime} \wedge y^{\prime}$. Moreover, $\forall i \in\{1,2 \ldots, N\},\left\{r, r^{\prime}\right\}={ }_{i}\left\{x, x^{\prime}\right\}={ }_{i}\left\{y, y^{\prime}\right\}$. Then $\rho(x \mid m)=\rho\left(x^{\prime} \mid m^{\prime}\right)$.

Proof: To simplify matters, imagine that $\left\{x, x^{\prime}\right\},\left\{r, r^{\prime}\right\},\left\{y, y^{\prime}\right\}$ are all $i$-variates sets for some $i \in\{1, \ldots, N\}$. Other cases follow once we prove this case.

Now, take two monotonically increasing sequences $\left\{z_{j}\right\}_{j=1}^{\infty}$ and $\left\{z_{j}^{\prime}\right\}_{j=1}^{\infty}$ such that
$\bigvee\left\{z_{j} \wedge r\right\}_{j=1}^{\infty} \sim r, \bigvee\left\{z_{j}^{\prime} \wedge r^{\prime}\right\}_{j=1}^{\infty} \sim r^{\prime},\left\{z_{j}\right\}_{j=1}^{\infty} \cup\left\{z_{j}^{\prime}\right\}_{j=1}^{\infty} \subset L_{i}^{z}$, and $\forall j,\left\{z_{j}, z_{j}^{\prime}\right\}={ }_{i}\left\{x, x^{\prime}\right\}$.
Note that by richness, it is always possible to find such sequences. By the continuity and ISPA
axiom

$$
\begin{equation*}
\frac{\rho\left(x \mid\left\{x, y, r, z_{i}\right\}\right)}{\rho\left(y \mid\left\{x, y, r, z_{i}\right\}\right)} \rightarrow \frac{\rho(x \mid\{x, y, r\}}{\rho(y \mid\{x, y, r\})} \tag{*}
\end{equation*}
$$

On the other hand, by WCRV, we have:

$$
\begin{aligned}
& \left\{r, z_{i}, x\right\} \cong\left\{r^{\prime}, z_{i}^{\prime}, x^{\prime}\right\} \\
& \left\{r, z_{i}, y\right\} \cong\left\{r^{\prime}, z_{i}^{\prime}, y^{\prime}\right\}
\end{aligned}
$$

But, since $\bigwedge\left\{r, z_{i}, x\right\} \sim \bigwedge\left\{r, z_{i}, y\right\} \sim \bigwedge\left\{x, y, r, z_{i}\right\} \sim r \wedge z_{i}$, and $\bigwedge\left\{r^{\prime}, z_{i}^{\prime}, x^{\prime}\right\} \sim \bigwedge\left\{r^{\prime}, z_{i}^{\prime}, y^{\prime}\right\} \sim$ $\bigwedge\left\{x^{\prime}, y^{\prime}, r^{\prime}, z_{i}^{\prime}\right\} \sim r^{\prime} \wedge z_{i}^{\prime}$, by ISPA we have:

$$
\left\{r, z_{i}, x, y\right\} \cong\left\{r^{\prime}, z_{i}^{\prime}, x^{\prime}, y^{\prime}\right\}
$$

But, this together with $(*)$ means that:

$$
\{r, x, y\} \cong\left\{r^{\prime}, x^{\prime}, y^{\prime}\right\}
$$

which is what we are after.
Now, inductively we can prove the above for when $\left\{x, x^{\prime}\right\},\left\{r, r^{\prime}\right\},\left\{y, y^{\prime}\right\}$ are not all univariate. Q.E.D.

Definition Imagine $i \in\{1, \ldots, N\}$ and $x_{1} \preceq_{i} y_{1}$ and $x_{2} \preceq_{i} y_{2}$. Define $\left\{x_{1}, y_{1}\right\}>_{i}\left\{x_{2}, y_{2}\right\}$ if there exists $y \succ_{i} y_{2}$ such that $\left\{x_{2}, y\right\}={ }_{i}\left\{x_{1}, y_{1}\right\}$.

Ratio of cardinal distances: For any $k \in R_{+}$and two pairs $\left\{x_{1}, y_{1}\right\}$ and $\left\{x_{2}, y_{2}\right\}$, we will define what $k\left\{x_{1}, y_{1}\right\}={ }_{i}\left\{x_{2}, y_{2}\right\}$ means in the following way:

1. For $k \in \mathbb{N}, k\left\{x_{1}, y_{1}\right\}={ }_{i}\left\{x_{2}, y_{2}\right\}$ if and only if there exist strictly increasing $\left\{z_{i}\right\}_{i=0}^{k}$ such that

$$
z_{0}=x_{2} \quad, \quad z_{k}=y_{2} \quad, \quad\left\{z_{j}, z_{j+1}\right\}={ }_{i}\left\{x_{1}, y_{1}\right\}
$$

2. For $k=\frac{a}{b}$ where $a, b \in \mathbb{N}$, define $k\left\{x_{1}, y_{1}\right\}={ }_{i}\left\{x_{2}, y_{2}\right\}$ if and only if there exists $x_{3}, y_{3}$ such that $\left\{x_{1}, y_{1}\right\}={ }_{i} b\left\{x_{3}, y_{3}\right\}$ and $\left\{x_{2}, y_{2}\right\}={ }_{i} a\left\{x_{3}, y_{3}\right\}$ according to 1 .
3. For $k \in \mathbb{R}_{+}-\mathbb{Q}$, where $k$ is the Dedekind cut (1872) Dedekind (1965) created by ( $K_{l}, K_{h}$ ) where $K_{l}, K_{h}$ are sets of rational numbers. Now, define $k\left\{x_{1}, y_{1}\right\}={ }_{i}\left\{x_{2}, y_{2}\right\}$ if and only if $k_{l}\left\{x_{1}, y_{1}\right\}<_{i}\left\{x_{2}, y_{2}\right\}$ for all $k_{l} \in K_{l}$ and $k_{h}\left\{x_{1}, y_{1}\right\}>_{i}\left\{x_{2}, y_{2}\right\}$ for all $k_{h} \in K_{h}$.

Lemma 30. Now, for two arbitrary $i$-variate items $x, y, z$, such that $x \prec y \prec z$ we can find a real number $k<1$ such that $\{x, y\}={ }_{i} k\{x, z\}$.

Proof: For any natural $k$, there exist $y_{l} \in(x, z)$ such that $k\left\{x, y_{l}\right\}>_{i}\{x, z\}$ and $y_{h} \in(x, z)$ such that $k\left\{x, y_{h}\right\}<_{i}\{x, z\}$. Now, for any $y \in\left(y_{l}, y_{h}\right)$ such that $k\{x, y\}<_{i}\{x, z\}$, we are going to be able to find $y^{\prime} \in(x, z)$ such that $k\{x, y\}={ }_{i}\left\{x, y^{\prime}\right\}$. Now, as we move $y$ towards $y_{h}, y^{\prime}$ moves increases and moves continuously (i.e. it does not leave anything out of its path) towards $z$. Since we are going to eventually surpass $z$, for some $y_{k} \in\left(y_{l}, y_{h}\right), y_{k}$ has to hit $z$, i.e. $y_{k} \sim z$. For such $y_{k}$, by definition we have $k\left\{x, y_{k}\right\}={ }_{i}\{x, z\}$.

Now, for $a>b \in \mathbb{N}$, we will show that there exists $y \in(x, z)$ such that $\frac{a}{b}\{x, y\}={ }_{i}\{x, z\}$. To do this, note that by what we stated above, there exists $t \in[x, z]$ s.t. $\{x, z\}={ }_{i} a\{x, t\}$. Now, we want to show that there exists $y \in[t, z]$ s.t. $\{x, y\}={ }_{i} b\{x, t\}$. Since, $\{x, z\}={ }_{i} a\{x, t\}$, we can find an increasing sequence $\left\{s_{j}\right\}_{j=0}^{a} \subset[x, z]$ s.t. $s_{0}=x, s_{a}=z$ and for all $i,\left\{s_{j}, s_{j+1}\right\}={ }_{i}\{x, t\}$. Now, given $b<a$, we can put $y=s_{b}$, which implies that $\{x, y\}={ }_{i} b\{x, t\}$.

Finally, by using Dedekind cuts, for any irrational $k \in(1, \infty)$, we can find $y \in[x, z]$ such that $k\{x, y\}={ }_{i}\{x, z\} . Q . E . D$.

Definition: Attribute Function Lemma 30 implies that for any $x, y$ and $x^{\prime}, y^{\prime}$ we can find a real number $k$ such that $\{x, y\}={ }_{i} k\left\{x^{\prime}, y^{\prime}\right\}$. Now, choose an arbitrary pair of items $x_{0}, y_{0}$ such that $x_{0} \prec_{i} y_{0}$ for all $i$. For each $i \in\{1, \ldots, N\}$, define a function $A_{i}: X \rightarrow R$, in the following way:

- $A_{i}\left(x_{0}\right)=0 \quad A_{i}\left(y_{0}\right)=1$
- $\forall y \succeq_{i} x$ and $\left\{x_{0}, y\right\}={ }_{i} k\left\{x_{0}, y_{0}\right\}$, put $A_{i}(y)=k$.
- $\forall y \preceq_{i} x$ and $\left\{y, x_{0}\right\}={ }_{i} k\left\{x_{0}, y_{0}\right\}$, put $A_{i}(y)=-k$.

Finally for every $x \in X$, put $A(x)=\left(A_{1}(x), \ldots, A_{N}(x)\right)$.

Some Facts and Notation: By construction, it is easy to see that $x \succ y$ if and only if $x$ dominates $y$ with respect to $A$, i.e, $A(x)>A(y)$. Also, $x \sim y$ if and only if $x$ is a duplicate of $y$, with respect to $A$. Therefore, the set of reasonable items relative to $m$ coincides with the set of undominated items in $m$ with respect to the revealed attribute function $A$, i.e., $m^{r}=m^{u}=$ $\{x \in m \mid \nexists y \in m, A(x)<A(y)\}$. Also, the revealed reference point of every menu $m$ coincides with the vector represented by $\Lambda m$, that is $r_{m}=A(\bigwedge m)=\left(\min _{x \in m} A_{n}(x)\right)_{n=1}^{N}$. For notational simplicity, for any menu $m$, I sometimes denote $r_{m}$ by $R(m)$ and $R_{j}(m)$ represents the $j^{\text {th }}$ coordinate of $r_{m}$. Also, after this point, whenever $m^{u}, r_{m}$, or duplicates are used, they are meant to be with respect to the revealed attribute function $A$. Also, $v_{m}=\{x \in m \mid v=A(x)\}$ and $V(m)=\left\{v \in \mathbb{R}^{N} \mid \exists x \in m, A(x)=v\right\}$.

Lemma 31. For $\{x, y\} \subseteq m$ and $\left\{x^{\prime}, y^{\prime}\right\} \subseteq m^{\prime}$ such that $A(x)-A\left(x^{\prime}\right)=A(y)-A\left(y^{\prime}\right)=r_{m}-r_{m^{\prime}}$ we have:

$$
\frac{\rho(x \mid m)}{\rho(y \mid m)}=\frac{\rho\left(x^{\prime} \mid m^{\prime}\right)}{\rho\left(y^{\prime} \mid m^{\prime}\right)}
$$

Proof: Now, based on the definition of $A, A(x)-A\left(x^{\prime}\right)=A(y)-A\left(y^{\prime}\right)$ is equivalent to $\left\{x, x^{\prime}\right\}={ }_{i}\left\{y, y^{\prime}\right\}$ for all $i \in\{1, \ldots, N\}$. Therefore, it is enough to prove the lemma for the following case:

$$
\begin{gathered}
\exists j \in\{1, \ldots, N\} \quad \text { s.t. } \quad A_{j}(x)-A_{j}(y)=A_{j}\left(x^{\prime}\right)-A_{j}\left(y^{\prime}\right)=R_{j}(m)-R_{j}\left(m^{\prime}\right)>0 \\
\forall i \neq j, \quad A_{i}(x)-A_{i}(y)=A_{i}\left(x^{\prime}\right)-A_{i}\left(y^{\prime}\right)=R_{i}(m)-R_{i}\left(m^{\prime}\right)=0
\end{gathered}
$$

But, according to Lemma 29, we have (*):

$$
\begin{aligned}
& \rho(x \mid\{x, y, \bigwedge m\})=\rho\left(x^{\prime} \mid\left\{x^{\prime}, y^{\prime}, \bigwedge m^{\prime}\right\}\right) \\
& \rho(y \mid\{x, y, \bigwedge m\})=\rho\left(y^{\prime} \mid\left\{x^{\prime}, y^{\prime}, \bigwedge m^{\prime}\right\}\right)
\end{aligned}
$$

Moreover, since $\bigwedge\{x, y, \bigwedge m\} \sim \bigwedge m$ and $\bigwedge\left\{x^{\prime}, y^{\prime}, \bigwedge m^{\prime}\right\} \sim \bigwedge m^{\prime}$, by the ISPA axiom we have (**):

$$
\begin{gathered}
\frac{\rho(x \mid\{x, y, \wedge m\})}{\rho(y \mid\{x, y, \wedge m\})}=\frac{\rho(x \mid m)}{\rho(y \mid m)} \\
\frac{\rho\left(x^{\prime} \mid\left\{x^{\prime}, y^{\prime}, \wedge m^{\prime}\right\}\right)}{\rho\left(y^{\prime} \mid\left\{x^{\prime}, y^{\prime}, \wedge m^{\prime}\right\}\right)}=\frac{\rho\left(x^{\prime} \mid m^{\prime}\right)}{\rho\left(y^{\prime} \mid m^{\prime}\right)}
\end{gathered}
$$

$(*)$ and $(* *)$ imply that:

$$
\frac{\rho(x \mid m)}{\rho(y \mid m)}=\frac{\rho\left(x^{\prime} \mid m^{\prime}\right)}{\rho\left(y^{\prime} \mid m^{\prime}\right)}
$$

Q.E.D.

## A.1.3 Construction of the Quasi-Utility Function $U$

Based on the cardinal structure of attributes,i.e. $A(x)$, we can define the utility function as the following:

Definition: Quasi-utility function Choose an arbitrary vector $b>0$, and put $U(b)=1$, then for every $a>0$, not comparable with $b$, define:

$$
U(a)=\frac{\rho(x \mid\{x, y, z\})}{\rho(y \mid\{x, y, z\})}
$$

where $(*)$

1. $A(x)-A(z)=a$
2. $A(y)-A(z)=b$
which implies $z \sim \bigwedge\{x, y, z\}$. Note that due to Lemma 27 the ratio of probabilities above is constant for all $x, y, z$ that satisfy $(*)$. Now, for every $c>0$ such that $c>b$, define:

$$
U(c)=\frac{\rho\left(y^{\prime} \mid\left\{x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)}{\rho\left(x^{\prime} \mid\left\{x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)} \times \frac{\rho(x \mid\{x, y, z\})}{\rho(y \mid\{x, y, z\})}
$$

where ( $* *$ )

1. $A\left(x^{\prime}\right)-A\left(z^{\prime}\right)=A(x)-A(z)=a>0$ for some $a$ that is incomparable with $b$ and $c$.
2. $A\left(y^{\prime}\right)-A\left(z^{\prime}\right)=c$
3. $A(y)-A(z)=b$

Now, we need to show that $U(c)$ is well-defined, or in other words, the quantity on the right hand side of equation is constant for all $x, y, z, x^{\prime}, y^{\prime}, z^{\prime}$ satisfying (**).In particular, fixing $y, y^{\prime}, z, z^{\prime}$ we show that

$$
\frac{\rho\left(y^{\prime} \mid\left\{x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)}{\rho\left(x^{\prime} \mid\left\{x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)} \times \frac{\rho(x \mid\{x, y, z\})}{\rho(y \mid\{x, y, z\})}
$$

is independent of $a$. Take $w$ such that $A(w)-A\left(z^{\prime}\right)=A(y)-A(z)$. Then by the WCRV and ISPA:

$$
\begin{gathered}
\frac{\rho\left(y^{\prime} \mid\left\{x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)}{\rho\left(x^{\prime} \mid\left\{x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)} \times \frac{\rho(x \mid\{x, y, z\})}{\rho(y \mid\{x, y, z\})}= \\
\frac{\rho\left(y^{\prime} \mid\left\{w, y^{\prime}, x^{\prime}, z^{\prime}\right\}\right)}{\rho\left(x^{\prime} \mid\left\{w, y^{\prime}, x^{\prime}, z^{\prime}\right\}\right)} \times \frac{\rho\left(x^{\prime} \mid\left\{w, x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)}{\rho\left(w \mid\left\{w, x^{\prime}, y^{\prime}, z^{\prime}\right\}\right)}= \\
\frac{\rho\left(y^{\prime} \mid\left\{w, y^{\prime}, x^{\prime}, z^{\prime}\right\}\right)}{\rho\left(w \mid\left\{w, y^{\prime}, x^{\prime}, z^{\prime}\right\}\right)}
\end{gathered}
$$

which by ISPA is equal to:

$$
\frac{\rho\left(y^{\prime} \mid\left\{w, y^{\prime}, z^{\prime}\right\}\right)}{\rho\left(w \mid\left\{w, y^{\prime}, z^{\prime}\right\}\right)}
$$

which is independent of $x$ and $x^{\prime}$.
The value of $U(d)$ for $d<b$ is defined similarly.
Now, we need to prove the following proposition:

Lemma 32. For any menu $m$ that does not contain any pair of duplicates and $x, y \in m^{u}$, we have:

$$
\frac{\rho(x \mid m)}{\rho(y \mid m)}=\frac{U\left(A(x)-r_{m}\right)}{U\left(A(y)-r_{m}\right)}
$$

Proof: First of all, note that by the ISPA, we have:

$$
\frac{\rho(x \mid m)}{\rho(y \mid m)}=\frac{\rho(x \mid\{x, y, \inf m\})}{\rho(y \mid\{x, y, \inf m\})}
$$

So, equivalently we want to show that:

$$
\frac{U\left(A(x)-r_{m}\right)}{U\left(A(y)-r_{m}\right)}=\frac{\rho(x \mid\{x, y, \bigwedge m\})}{\rho(y \mid\{x, y, \bigwedge m\})}
$$

We break down the proof into three cases, which might also overlap:

1. $A(x)-r_{m}$ and $A(y)-r_{m}$ are both incomparable with $b$ as defined above.
2. Both $A(x)-r_{m}$ and $A(y)-r_{m}$ dominate $b$.
3. Both $A(x)-r_{m}$ and $A(y)-r_{m}$ are dominated by b.

For Case (1), by the definition of $U$ and the ISPA axiom, we have:

$$
U\left(A(x)-r_{m}\right)=\frac{\rho(x \mid\{x, t, \bigwedge m\})}{\rho(t \mid\{x, t, \bigwedge m\})}=\frac{\rho(x \mid\{x, t, y, \bigwedge m\})}{\rho(t \mid\{x, t, y, \bigwedge m\})}
$$

$$
U\left(A(y)-r_{m}\right)=\frac{\rho(y \mid\{y, t, \bigwedge m\})}{\rho(t \mid\{y, t, \bigwedge m\})}=\frac{\rho(x \mid\{x, t, y, \bigwedge m\})}{\rho(t \mid\{x, t, y, \bigwedge m\})}
$$

Where $A(t)-A\left(r_{m}\right)=b$. Note that the first equal signs in both lines follow from the definition of $U$ and the second equal signs follow from the ISPA, since $\wedge\{x, t, \bigwedge m\}=\bigwedge\{y, t, \wedge m\}=$ $\bigwedge\{x, t, y, \bigwedge m\}=\bigwedge m$.

Now, dividing the sides by each other, we get:

$$
\frac{U\left(A(x)-r_{m}\right)}{U\left(A(y)-r_{m}\right)}=\frac{\rho(x \mid\{x, t, y, \wedge m\})}{\rho(y \mid\{x, t, y, \wedge m\})}
$$

which once again by ISPA, is equal to:

$$
\frac{\rho(x \mid\{x, y, \bigwedge m\})}{\rho(y \mid\{x, y, \bigwedge m\})}
$$

For Case (2), note that by the definition of $U$ and the ISPA, we have:

$$
\begin{aligned}
& U\left(A(x)-r_{m}\right)=\frac{\rho(x \mid\{x, t, \bigwedge m\})}{\rho(t \mid\{x, t, \bigwedge m\})} \times \frac{\rho(t \mid\{t, z, \bigwedge m\})}{\rho(z \mid\{t, z, \bigwedge m\})}=\frac{\rho(x \mid\{x, t, y, \bigwedge m\})}{\rho(t \mid\{x, t, y, \bigwedge m\})} \times \frac{\rho(t \mid\{t, z, \bigwedge m\})}{\rho(z \mid\{t, z, \bigwedge m\})} \\
& U\left(A(y)-r_{m}\right)=\frac{\rho(y \mid\{x, t, \bigwedge m\})}{\rho(t \mid\{x, t, \bigwedge m\})} \times \frac{\rho(t \mid\{t, z, \bigwedge m\})}{\rho(z \mid\{t, z, \bigwedge m\})}=\frac{\rho(y \mid\{x, t, y, \bigwedge m\})}{\rho(t \mid\{x, t, y, \bigwedge m\})} \times \frac{\rho(t \mid\{t, z, \bigwedge m\})}{\rho(z \mid\{t, z, \bigwedge m\})}
\end{aligned}
$$

Where $A(z)-r_{m}=b$ and $A(t)-A(t)$ is incomparable with $A(x)-r_{m}$ and $A(y)-r_{m}$. Note that like case (1), the first equal signs follow from definition of $U$ and the second equal signs are due to the ISPA.

Once again, dividing the two sides and using the ISPA leads us to:

$$
\frac{U\left(A(x)-r_{m}\right)}{U\left(A(y)-r_{m}\right)}=\frac{\rho(x \mid\{x, y, \wedge m\})}{\rho(y \mid\{x, y, \bigwedge m\})}
$$

Case (3) can be proved similarly. Q.E.D.

Lemma 33. For every menu $m$ such that $|V(m)|>1$ and $v, v^{\prime} \in V\left(m^{u}\right)$, we have:

$$
\frac{\rho\left(v_{m} \mid m\right)}{\rho\left(v_{m}^{\prime} \mid m\right)}=\frac{U\left(v-r_{m}\right)}{U\left(v^{\prime}-r_{m}\right)}
$$

Proof: Let $x, y \in m$ be such that $x^{\prime} \in v_{m}$ and $A\left(x^{\prime}\right) \in v_{m}^{\prime}$. Therefore, $v_{m}=m_{x}$ and $v_{m}^{\prime}=m_{x}^{\prime}$. Now, given that the reference points of $m$ and $\left\{x, x^{\prime}, \bigwedge m\right\}$ are the same, by the IRPA axiom we have:

$$
\frac{\rho\left(m_{x} \mid m\right)}{\rho\left(m_{x^{\prime}} \mid m\right)}=\frac{\rho\left(x \mid\left\{x, x^{\prime}, \wedge m\right\}\right)}{\rho\left(y \mid\left\{x, x^{\prime}, \bigwedge m\right\}\right)}
$$

Also, by lemma 32

$$
\frac{\rho(x \mid\{x, y, \bigwedge m\})}{\rho(y \mid\{x, y, \bigwedge m\})}=\frac{U\left(A(x)-r_{m}\right)}{U\left(A(y)-r_{m}\right)}
$$

The proof follows from the above two equations. Q.E.D.

Lemma 34. For every $m$ such that $|V(m)|>1$ and $x \in m$, we have:

$$
\rho(x \mid m)=\sum_{v \in V\left(m^{u}\right)} \frac{U\left(v-r_{m}\right)}{S(m)} \cdot \frac{1_{\{A(x)=v\}}}{\left|v_{m}\right|}
$$

Where $S(m)=\sum_{v \in V\left(m^{u}\right)} U\left(v-r_{m}\right)$.
Proof: If $x \in m-m^{u}$, then $x$ is dominated by some item in $m^{u}$, which by the monotonicity axiom implies that $\rho(x \mid m)=0$, which is confirmed by the representation of the lemma.

Therefore, let $x \in m^{u}$ and $v=A(x)$ By lemma 32, for $v^{*} \in V\left(m^{u}\right)$, we have:

$$
\begin{equation*}
\frac{\rho\left(v_{m} \mid m\right)}{\rho\left(v_{m}^{*} \mid m\right)}=\frac{U\left(v-r_{m}\right)}{U\left(v^{*}-r_{m}\right)} \tag{*}
\end{equation*}
$$

Given that all items in $v_{m}$ are chosen with equal probabilities and $x \in v_{m}, \rho\left(v_{m} \mid m\right)=$ $\left|v_{m}\right| \rho(x \mid m)$. This together with $(*)$ implies

$$
\frac{\rho(x \mid m)}{\rho\left(v_{m}^{*} \mid m\right)}=\frac{U\left(v-r_{m}\right)}{U\left(v^{*}-r_{m}\right)} \cdot \frac{1}{\left|v_{m}\right|} \quad(* *)
$$

Also, by (*) we have

$$
\frac{\sum_{v^{\prime} \in m^{u}} \rho\left(v_{m}^{\prime} \mid m\right)}{\rho\left(v_{m}^{*} \mid m\right)}=\frac{\sum_{v^{\prime} \in V\left(m^{u}\right)} U\left(v^{\prime}-r_{m}\right)}{U\left(v^{*}-r_{m}\right)}=\frac{S(m)}{U\left(v^{*}-r_{m}\right)}
$$

Given that $\sum_{v^{\prime} \in V\left(m^{u}\right)} \rho\left(v_{m}^{\prime} \mid m\right)=1$, the above implies:

$$
\rho\left(v_{m}^{*} \mid m\right)=\frac{U\left(v^{*}-r_{m}\right)}{S(m)}
$$

Which by (**) implies that

$$
\rho(x \mid m)=\frac{U\left(A(x)-r_{m}\right)}{S(m)} \cdot \frac{1}{\left|A(x)_{m}\right|}
$$

Theorem 1 For any menu $m$ and $x \in m$, we have the following:

$$
\rho(x \mid m)=\left\{\begin{array}{lr}
1_{\left\{x \in m^{u}\right\}} \cdot \frac{U\left(A(x)-r_{m}\right)}{S(m)} \cdot \frac{1}{\left|A(x)_{m}\right|} & |V(m)|>1 \\
\frac{1}{|m|} & |V(m)|=1
\end{array}\right.
$$

Proof: The second case, $|V(m)|=1$, follows from the fact that $m$ contains just duplicates, and they are chosen with equal probabilities. The first case, $|V(m)|>1$, follows from lemma 34 . Q.E.D.

## A. 2 Proof Of Proposition 1

The number of attributes, $N$ is unique, since it is equal to the number of maximal univariates that contain an arbitrary item, which is pinned down by the RCR, $\rho$. Without loss of generality, we can fix the order of these attributes. Now, imagine that there are two pairs of utility and attribute functions $(U, A)$ and $\left(U^{\prime}, A^{\prime}\right)$ that rationalize $\rho$, according to the random arbitration model. The ratio of distances between $x$ and $y$ and $t$ and $z$ along attribute $i$ is determined uniquely by $\rho$. Therefore, for all $x, y, t$, and $z$ and every $i \in\{1, \ldots, N\}$, we have:

$$
\frac{A_{i}(x)-A_{i}(y)}{A_{i}(t)-A_{i}(z)}=\frac{A_{i}^{\prime}(x)-A_{i}^{\prime}(y)}{A_{i}^{\prime}(t)-A_{i}^{\prime}(z)}
$$

which implies that for all $x$ and $y$ the ratio $\frac{A_{i}^{\prime}(x)-A_{i}^{\prime}(y)}{A_{i}(x)-A_{i}(y)}$ is equal to a constant $c \geq 0$. Now, pick $y *$ such that $A(y)=0$. Then for all $x \in X$, we have:

$$
A_{i}^{\prime}(x)=A_{i}(y *)+c A_{i}(x)
$$

Therefore, $A^{\prime}$ is a linear transformation of $A$ along each dimension.

Uniqueness of $U$ : Fix an attribute function $A$, and suppose that $U$ and $U^{\prime}$ both rationalize $\rho$. First of all, for incomparable $a \in R_{+}^{N}$ and $b \in R_{+}^{N}$, we have:

$$
\frac{U(a)}{U(b)}=\frac{U^{\prime}(a)}{U^{\prime}(b)}=\frac{\rho(x \mid\{x, y, r\})}{\rho(y \mid\{x, y, r\})} \quad(*)
$$

where $A(x)-A(r)=a$ and $A(y)-A(r)=b$.
Also, if $a \geq b$, we can find $c$ incomparable with both $a$ and $b$. Then by ( $*$ ), we have:

$$
\begin{aligned}
& \frac{U(a)}{U(c)}=\frac{U^{\prime}(a)}{U^{\prime}(c)} \\
& \frac{U(c)}{U(b)}=\frac{U^{\prime}(c)}{U^{\prime}(b)}
\end{aligned}
$$

Therefore:

$$
\frac{U(a)}{U(b)}=\frac{U^{\prime}(a)}{U^{\prime}(b)}
$$

Therefore, for all $a$ and $b$ :

$$
\frac{U(a)}{U(b)}=\frac{U^{\prime}(a)}{U^{\prime}(b)}
$$

Which implies that $U$ is unique up to a scalar multiplier.

## A. 3 Proof of Theorem 2

To prove the proposition, we will show that the generalized compromise effect is equivalent to the following equivalence property:

Equivalence Property: If adding z to the menu m changes the reference point, but only in attributes in which $x$ is inferior to $y$, then: ${ }^{2}$

$$
\frac{\rho(x \mid m)}{\rho(y \mid m)} \leq \frac{\rho(x \mid m \cup\{z\})}{\rho(y \mid m \cup\{z\})}
$$

First, we prove that the generalized compromise effect implies the equivalence property.

[^26]Take $z$ to be an item such that for every $i \in\{1, \ldots, N\}$, we have:

$$
\begin{equation*}
\min \left\{A_{i}(y), A_{i}(z)\right\} \leq A_{i}(x) \leq \max \left\{A_{i}(y), A_{i}(z)\right\} \tag{*}
\end{equation*}
$$

For better notation, in this proof I use the reference function $R(m)$ for $r_{m}$ and $R_{i}(m)$ denotes the $i^{t} h$ coordinate of $R(m)$. To start with, note that for there to be any change in the ratio of probabilities, the reference point has to change. Therefore, assume that $R(m) \neq R\left(m^{\prime}\right)$, where $m^{\prime}=m \cup\{z\}$. Since $R_{m^{\prime}}=\left(\min \left\{R_{i}(m), A_{i}(z)\right\}\right)_{i=1}^{N}$, the following is true:

$$
\forall j \in\{1,2 \ldots, N\}, R_{j}\left(m^{\prime}\right) \neq R_{j}(m) \Rightarrow R_{j}\left(m^{\prime}\right)=A_{j}(z) \quad(* *)
$$

Now, suppose that for some $j, R_{j}\left(m^{\prime}\right) \neq R_{j}(m)$. According to $(* *), A_{j}(z)=R_{j}\left(m^{\prime}\right)$. Now, since $R_{j}\left(m^{\prime}\right) \leq A_{j}(x), A_{j}(y)$, we would have $A_{j}(z) \leq A_{j}(x)$ and $A_{j}(z) \leq A_{j}(y)$. Therefore, $\max \left\{A_{j}(y), A_{j}(z)\right\}=A_{j}(y)$ and $\min \left\{A_{j}(y), A_{j}(z)\right\}=A_{j}(z)$. This, together with (**) imply

$$
A_{j}(z) \leq A_{j}(x) \leq A_{j}(y)
$$

which is what we are after.

Now, I will show that if $z$ is an item that decreases the value $R(m)$ only in the attributes in which $x$ is inferior to $y$, then the effect of such $z$ on the relative choice probabilities of $x$ and $y$ can be replicated by some $z^{\prime}$ that makes $x$ a compromise between $y$ and $z^{\prime}$.

Let $I=\left\{i \mid A_{i}(x) \leq A_{i}(y)\right\}$, denoting the set of attributes in which $x$ is (weakly) inferior to $y$. It is easy to see that if $R\left(m^{\prime}\right)$ is different from $R(m)$ only in attributes in which $x$ is inferior to $y$, i.e. those in $I$, then for any such attribute $j \in I, A_{j}(z) \leq A_{j}(x) \leq A_{j}(y)$.

Now, if for some attribute $i R_{i}\left(m^{\prime}\right)=R_{i}(m)$ and $A_{i}(x)>\max \left\{A_{i}(y), A_{i}(z)\right\}$, then by virtue of the assumption that the random arbitration model is open, we can increase the value of $z$ in attribute $i$ (weakly) above $x$, without changing the value of the reference point in attribute $i$.

Similarly, if for some attribute $i, R_{i}\left(m^{\prime}\right)=R_{i}(m)$ and $A_{i}(x)<\min \left\{A_{i}(y), A_{i}(z)\right\}$, we can decrease the value of $A_{i}(z)$ (weakly) below $A_{i}(x)$, without changing the value of the reference point in attribute $i$.

Therefore, the generalized compromise effect is indeed equivalent to the equivalence property. Now, I show that the equivalence property is equivalent to $U$ being log-concave in every proper subset of its arguments and super-modular. Let, $u=\log U$. We need to show that $u$ is concave in every proper subset of the arguments and super-modular.

For simplicity, I assume that $N=2$, as the proof can easily be extended to $N>2$. Now, take two arbitrary items $x$ and $y$, such that $A_{1}(x)>A_{1}(y)$ and $A_{2}(x)<A_{2}(y)$. Also, consider a menu $m$ such that $x, y \in m^{u}$. Let $A(x)-r_{m}=\left(a_{1}, a_{2}\right)$ and $A(y)-r_{m}=\left(b_{1}, b_{2}\right)$. Put $m^{\prime}=m \cup\{z\}$ and assume that $z$ decreases the reference point of $m$ only in the attribute in which $x$ is inferior to $y$. Therefore, $A(x)-r_{m^{\prime}}=\left(a_{1}, a_{2}+s\right)$ and $A(y)-r_{m^{\prime}}=\left(b_{1}, b_{2}+s\right)$ for $s \geq 0$. Note that $b_{1}<a_{1}$, $b_{2}>a_{2}$ and $s>0$ are arbitrary. Therefore, by the equivalence property we have:

$$
\begin{equation*}
\forall a_{1}>b_{1}, b_{2}>a_{2}, s>0, u\left(a_{1}, a_{2}+s\right)-u\left(a_{1}, a_{2}\right) \geq u\left(b_{1}, b_{2}+s\right)-u\left(b_{1}, b_{2}\right) \tag{*}
\end{equation*}
$$

Now, by continuity of $u$ we have:

$$
\lim _{a_{1} \downarrow b_{1}} u\left(a_{1}, a_{2}+s\right)-u\left(a_{1}, a_{2}\right)=u\left(b_{1}, a_{2}+s\right)-u\left(b_{1}, a_{2}\right)
$$

Now, by (*), we have:

$$
\forall a_{1}>b_{1}, b_{2}>a_{2}, s \geq 0, u\left(b_{1}, a_{2}+s\right)-u\left(b_{1}, a_{2}\right) \geq u\left(b_{1}, b_{2}+s\right)-u\left(b_{1}, b_{2}\right)
$$

The above implies that the marginal return to the second argument is decreasing. Therefore, $u$ is concave in the second argument. A similar reasoning proves that it is concave in the
first argument.
Now, I will show that $u$ is super-modular. Continuity of $u$ implies:

$$
\lim _{a_{2} \uparrow b_{2}} u\left(a_{1}, a_{2}+s\right)-u\left(a_{1}, a_{2}+s\right)=u\left(a_{1}, b_{2}+s\right)-u\left(b_{1}, b_{2}\right)
$$

Now, the above and (*) imply:

$$
\forall a_{1}>b_{1}, b_{2}>a_{2}, s \geq 0, u\left(a_{1}, b_{2}+s\right)-u\left(a_{1}, b_{2}\right) \geq u\left(b_{1}, b_{2}+s\right)-u\left(b_{1}, b_{2}\right)
$$

The above implies that the marginal return to the second argument is increasing in the first attribute. Similarly, we can show this for the first attribute. Therefore, $u$ and subsequently $U$ is super-modular.

## A. 4 Proof of Proposition 2

It is easy to check the if part of the theorem. For the only if part, given the validity of theorem 2, it is enough to show that when $N=2$ the generalized attraction effect is equivalent to the equivalence property. Consider two menus $m, m^{\prime}$ and $x, y \in m^{u} \cap m^{\prime u}$. Let $r$ and $r^{\prime}$ be two items for which $A(r)=r_{m}$ and $A\left(r^{\prime}\right)=r_{m^{\prime}}$ and assume that $r^{\prime}$ is lower than $r$ only in the attribute in which $x$ is inferior to $y$, but equal to $r$ in all other attributes. In the random arbitration model, we have:

$$
\frac{\rho(x \mid m)}{\rho(y \mid m)}=\frac{\rho(x \mid\{x, y, r\})}{\rho(y \mid\{x, y, r\})} \quad \frac{\rho\left(x \mid m^{\prime}\right)}{\rho\left(y \mid m^{\prime}\right)}=\frac{\rho\left(x \mid\left\{x, y, r^{\prime}\right\}\right)}{\rho\left(y \mid\left\{x, y, r^{\prime}\right\}\right)}
$$

It is easy to see that we can find an item $z$ dominated by $x$ but not by $y$ such that $R(\{x, y, r, z\})=r_{m^{\prime}}$. Therefore:

$$
\frac{\rho(x \mid\{x, y, r, z\})}{\rho(y \mid\{x, y, r, z\})}=\frac{\rho\left(x \mid\left\{x, y, r^{\prime}\right\}\right)}{\rho\left(y \mid\left\{x, y, r^{\prime}\right\}\right)}=\frac{\rho\left(x \mid m^{\prime}\right)}{\rho\left(y \mid m^{\prime}\right)}
$$

According to the generalized attraction effect we have:

$$
\rho(x \mid\{x, y, r, z\}) \geq \rho(x \mid\{x, y, r\})
$$

Since $z$ and $r$ are never chosen and $R(\{x, y, r, z\})=r_{m^{\prime}}$, the above implies that:

$$
\frac{\rho(x \mid\{x, y, r\})}{\rho(y \mid\{x, y, r\})}=\frac{\rho(x \mid\{x, y, r, z\})}{\rho(y \mid\{x, y, r, z\})} \geq \frac{\rho(x \mid\{x, y, r\})}{\rho(y \mid\{x, y, r\})}
$$

Therefore

$$
\frac{\rho\left(x \mid m^{\prime}\right)}{\rho\left(y \mid m^{\prime}\right)} \geq \frac{\rho(x \mid m)}{\rho(y \mid m)}
$$

Which implies that the equivalence property is satisfied. Q.E.D.

## A. 5 Proof of Proposition 3

Let $N=2$ and by means of contradiction, imagine that $x \succeq^{s} y$ and $y \succeq^{s} z$ and $z \succ^{s} x$. As the first case, assume that $A_{1}(x) \geq A_{1}(y) \geq A_{1}(z)$ and $A_{2}(x) \leq A_{2}(y) \leq A_{2}(z)$. Then $x \succeq^{s} y$ is equivalent to $U\left(A_{1}(x)-A_{1}(y), 0\right) \geq U\left(0, A_{2}(y)-A_{2}(x)\right)$. Since $U$ is symmetric, the latter is equivalent to $A_{1}(x)-A_{1}(y) \geq A_{2}(y)-A_{2}(x)$. Similarly, by $y \succeq^{s} z$ and $z \succeq^{s} x$, we can conclude that $A_{1}(y)-A_{1}(z) \geq A_{2}(z)-A_{2}(y)$ and $A_{1}(x)-A_{1}(z)<A_{2}(z)-A_{2}(x)$. But $A_{1}(x)-A_{1}(y) \geq$ $A_{2}(y)-A_{2}(x), A_{1}(y)-A_{1}(z) \geq A_{2}(z)-A_{2}(y)$, and $A_{1}(x)-A_{1}(z)<A_{2}(z)-A_{2}(x)$ clearly lead to a contradiction. For other cases for which $x, y, z$ are positioned differently in terms of their attribute values, one can use similar arguments to show that stochastic transitivity is satisfied. Q.E.D.

## A. 6 Proof of Theorem 3

For notational simplicity, we work in two dimensions, but the proof works for any number of dimensions. To start, let $R(\{(x, y),(x+d, y)\})=(x(d), y)$ for some $d \geq 0$. By the range preservation property, we have $x(d) \in[x, x+d]$. Therefore, there is an $\alpha_{d} \in[0,1]$ such that

$$
x(d)=\alpha_{d} x+\left(1-\alpha_{d}\right)(x+d)=x+\left(1-\alpha_{d}\right) d
$$

Now, we want to show that $\alpha_{d}$ is $d$-invariant, i.e.

$$
\exists \alpha^{*} \quad \text { s.t. } \quad \forall d, \quad x(d)=x+\left(1-\alpha^{*}\right) d
$$

Or equivalently, $\frac{x(d)-x}{d}=\alpha^{*}$. To show above, note that we have:

$$
R(\{(x-x(d), 0),(x-x(d)+d, 0))\}=(0,0)
$$

Therefore, by the scale invariance property:

$$
\begin{aligned}
& R\left(\left\{\left(\frac{x-x(d)}{d}, 0\right),\left(\frac{x-x(d)}{d}+1,0\right)\right\}\right)=(0,0) \\
& \Rightarrow R(\{(0,0),(1,0)\})+\left(\frac{x-x(d)}{d}, 0\right)=(0,0)
\end{aligned}
$$

Therefore:

$$
\frac{x(d)-x}{d}=R_{1}(\{(1,0),(0,0)\})=\alpha^{*}
$$

Now, by the separability property, for any $y$ and $y^{\prime}$, we have:

$$
\frac{x(d)-x}{d}=R_{1}\left(\left\{(1, y),\left(0, y^{\prime}\right)\right\}\right)=\alpha^{*}
$$

Now, want to show that $\alpha^{*}$ can only be 0 or 1 . To show that consider a larger menu, consisting of three items: $m=\left\{\left(x_{1}, y\right),\left(x_{2}, y\right),\left(x_{3}, y\right)\right\}$, where $x_{1} \leq x_{2} \leq x_{3}$ Now, the path independence property implies:

$$
R(m)=R\left(\left\{R\left(\left\{\left(x_{1}, y\right),\left(x_{2}, y\right)\right\}\right),\left(x_{3}, y\right)\right\}\right)
$$

but by what we proved above:

$$
R\left(\left\{\left(x_{1}, y\right),\left(x_{2}, y\right)\right\}\right)=\left(\alpha^{*} x_{1}+\left[1-\alpha^{*}\right] x_{2}, y\right)
$$

Therefore, by invoking our result above again, we will have:

$$
\begin{equation*}
R(m)=\left(\alpha^{*}\left[\alpha^{*} x_{1}+\left[1-\alpha^{*}\right] x_{2}\right]+\left[1-\alpha^{*}\right] x_{3}, y\right) \tag{a}
\end{equation*}
$$

However, we can calculate $R(m)$ alternatively, by first finding $R\left(\left\{\left(x_{2}, y\right),\left(x_{3}, y\right)\right\}\right.$ and then combining that with $\left(x_{1}, y\right)$, which will result in:

$$
\begin{equation*}
R(m)=\left(\alpha^{*} x_{1}+\left[1-\alpha^{*}\right]\left[\alpha^{*} x_{2}+\left[1-\alpha^{*}\right] x_{3}\right], y\right) \tag{b}
\end{equation*}
$$

By setting the coefficients of $x_{1}$ equal, (a) and (b) together imply that:

$$
\left(\alpha^{*}\right)^{2}=\alpha^{*}
$$

Therefore, $\alpha^{*} \in\{0,1\}$.
Note that by the way we defined $\alpha^{*}$, the fact that it is either 0 or 1 is equivalent to the fact that $x(d) \in\{x, x+d\}$. Therefore, so far we have shown that when $|m| \leq 3$, the value of $R(m)$ in every component is either the maximum or the minimum of that component among the vectors in $m$. Now, by invoking the path independence property, we inductively prove the claim for larger menus of items. To do this, let $R_{i}(m)$ be the value of the $i^{\text {th }}$ component of $R(m)$, which

By the separability property $R_{i}(m)$ is only a function of the set of $i^{t h}$ components of the
items in $m$. When $m=\left\{\left(x_{1}, y_{1}\right), \ldots,\left(x_{n}, y_{n}\right)\right\}$, let $m^{1}=\left\{x_{1}, \ldots, x_{n}\right\}$ and $m^{2}=\left\{x_{1}, \ldots, x_{n}\right\}$. Now, we will show that if $\alpha^{*}=0$ then for every $i \in\{1,2\}, R_{i}(m)=\max \left\{m^{i}\right\}$.

Since the proof is similar for $i=1,2$, I only provide the proof for $i=1$. We have already seen that the claim is true for $n=2,3$. Now, assuming that the claim holds for $m=$ $\left\{\left(x_{1}, y_{1}\right), \ldots,\left(x_{n}, y_{n}\right)\right\}$, for some $n \geq 2$, I will show that it is also holds for $n+1$, that is for $m \cup\left\{\left(x_{n+1}, y_{n+1}\right)\right\}$. Therefore, by assumption we have:

$$
R_{1}(m)=\max \left\{x_{1}, \ldots, x_{n}\right\}
$$

. The latter together with path independence property and the fact the statement is true for $n=2$, leads to:

$$
\begin{gathered}
R_{1}\left(m \cup\left\{\left(x_{n+1}, y_{n+1}\right)\right\}\right)=R_{1}\left(\left\{R(m),\left(x_{n+1}, y_{n+1}\right)\right\}\right)=R_{1}\left(\left\{\max \left\{x_{1}, \ldots, x_{n}\right\}, x_{n+1}\right\}\right)= \\
=\max \left\{\max \left\{x_{1}, \ldots, x_{n}\right\}, x_{n+1}\right\}=\max \left\{x_{1}, \ldots, x_{n}, x_{n+1}\right\}
\end{gathered}
$$

Therefore, by induction, the claim is true for every $n \geq 2$. Similarly, we can prove that if $\alpha^{*}=1$, then $R_{1}(m)=\min m^{1}$. Now, by symmetry, if $\alpha^{*}=0$, then both $R_{1}$ and $R_{2}$ are max operators and if $\alpha^{*}=1$, they are both min operators. This proves the theorem. Q.E.D.

## Appendix B

## Stochastic Choice over Menus

## B. 1 Proof of Theorem 1

In order to prove theorem 1, the following lemma is useful.

Lemma 1: For all disjoint $m$ and $m^{\prime}$ the following is true:

$$
\mu\left[u \mid V_{u}\left(m-m^{\prime}\right)>V_{u}\left(m^{\prime}\right)\right]+\mu\left[u \mid V_{u}\left(m \cap m^{\prime}\right)>V_{u}\left(m^{\prime}-m\right)\right]=\mu\left[u \mid V_{u}(m) \geq V_{u}\left(m^{\prime}\right)\right]
$$

Proof: Consider a utility function $u$ for which $V_{u}(m) \geq V_{u}\left(m^{\prime}\right)$ and let $x_{u}^{*}=\underset{x \in m \cup m^{\prime}}{\operatorname{argmax}} u(x)$. Therefore

$$
x_{u}^{*} \in m
$$

This implies that $x_{u}^{*} \in m-m^{\prime}$ or $x_{u}^{*} \in m \cap m^{\prime}$. If the former is true, then $V_{u}\left(m-m^{\prime}\right)>V_{u}\left(m^{\prime}\right)$, and if the latter is true, then $V_{u}\left(m \cap m^{\prime}\right)>V_{u}\left(m^{\prime}-m\right)$. Therefore

$$
\begin{equation*}
\mu\left[u \mid V_{u}\left(m-m^{\prime}\right)>V_{u}\left(m^{\prime}\right)\right]+\mu\left[u \mid V_{u}\left(m \cap m^{\prime}\right)>V_{u}\left(m^{\prime}-m\right)\right] \geq \mu\left[u \mid V_{u}(m) \geq V_{u}\left(m^{\prime}\right)\right] \tag{*}
\end{equation*}
$$

Similarly, we can show that if for a utility function $u$ if $V_{u}\left(m-m^{\prime}\right)>V_{u}\left(m^{\prime}\right)$ or $V_{u}(m \cap$
$\left.m^{\prime}\right)>V_{u}\left(m^{\prime}-m\right)$, then $V_{u}(m) \geq V_{u}\left(m^{\prime}\right)$. This implies that

$$
\begin{equation*}
\mu\left[u \mid V_{u}\left(m-m^{\prime}\right)>V_{u}\left(m^{\prime}\right)\right]+\mu\left[u \mid V_{u}\left(m \cap m^{\prime}\right)>V_{u}\left(m^{\prime}-m\right)\right] \geq \mu\left[u \mid V_{u}(m) \leq V_{u}\left(m^{\prime}\right)\right] \tag{**}
\end{equation*}
$$

$(*)$ and $(* *)$ together will prove the lemma.
Now, we are ready to state the proof of theorem 1.

## Proof of Theorem 1:

If part: It is easy to check that if $\rho$ satisfies (1), that is:

$$
\begin{equation*}
\mu\left[u \mid \forall m^{\prime} \in M, V_{u}(m)>V_{u}\left(m^{\prime}\right)\right] \leq \rho(m \mid\{m\} \cup M) \leq \mu\left[u \mid \forall m^{\prime} \in M, V_{u}(m) \geq V_{u}\left(m^{\prime}\right)\right] \tag{1}
\end{equation*}
$$

then it satisfies ARSP over singletons, merging additivity, and reorganization invariance. It is also easy to check that if $\rho$ satisfies (1) and $m \cap S_{M}=\emptyset$, then $\rho$ satisfies monotonicity. Therefore, we only need to show monotonicity follows by assuming (1) and $m \cap S_{M} \neq \emptyset$. Since $\left(m-S_{M}\right) \cap S_{M}=\left(S_{M}-m\right) \cap\left(m \cap S_{M}\right)=\emptyset$, we have the following:

$$
\begin{gather*}
\left.\rho\left(m-S_{M} \mid\left\{m-S_{M}\right\} \cup M\right\}\right)=\rho\left(m-S_{M} \mid\left\{m-S_{M}, S_{M}\right\}\right)=\mu\left[u \mid V_{u}\left(m-S_{M}\right)>V_{u}\left(S_{M}\right)\right]  \tag{2}\\
\rho\left(m \cap S_{M} \mid\left\{m \cap S_{M}\right\} \cup(M-m)\right)=\rho\left(m \cap S_{M} \mid\left\{m \cap S_{M}, S_{M}-m\right\}\right)=\mu\left[u \mid V_{u}\left(m \cap S_{M}\right)>V_{u}\left(S_{M}-m\right)\right] \tag{3}
\end{gather*}
$$

The first equalities in both lines above follow from merging additivity, which is a consequence of (1). The second equalities follow directly from (1). Now, (2) and (3) imply the following:

$$
\begin{gathered}
\rho\left(m-S_{M} \mid\left\{m-S_{M}, S_{M}\right\}\right)+\rho\left(m \cap S_{M} \mid\left\{m \cap S_{M}, S_{M}-m\right\}\right)= \\
\mu\left[u \mid V_{u}\left(m-S_{M}\right)>V_{u}\left(S_{M}\right)\right]+\mu\left[u \mid V_{u}\left(m \cap S_{M}\right)>V_{u}\left(S_{M}-m\right)\right]=
\end{gathered}
$$

$$
\begin{equation*}
\mu\left[u \mid V_{u}(m) \geq V_{u}\left(S_{M}\right)\right] \tag{4}
\end{equation*}
$$

Where the last equality follows from lemma 1 . On the other hand, it is easy to see that

$$
\mu\left[u \mid V_{u}(m) \geq V_{u}\left(S_{M}\right)\right]=\mu\left[u \mid \forall m^{\prime} \in M, V_{u}(m) \geq V_{u}\left(m^{\prime}\right)\right]
$$

The latter, together with (1) and (4) imply

$$
\rho(m \mid\{m\} \cup M) \leq \rho\left(m-S_{M} \mid\left\{m-S_{M}\right\} \cup M\right)+\rho\left(m \cap S_{M} \mid\left\{m \cap S_{M}\right\} \cup(M-m)\right)
$$

Also, (1) and (2) imply:

$$
\rho\left(m-S_{M} \mid\left\{m-S_{M}\right\} \cup M\right) \leq \rho(m \mid\{m\} \cup M)
$$

The last two inequalities establish Monotonicity.

Only if part: Assume that $\rho$ satisfies axioms ARSP, merging additivity, reorganization invariance, and monotonicity. ARSP over singletons guarantees that there exists a probability measure, $\mu$ over the set of strict orderings of singletons or equivalently $X$, such that for any set of singletons $\left\{x_{1}\right\},\left\{x_{2}\right\}, \ldots,\left\{x_{n}\right\}$, we have:

$$
\begin{equation*}
\rho\left(\left\{x_{1}\right\} \mid\left\{\left\{x_{1}\right\},\left\{x_{2}\right\}, \ldots,\left\{x_{n}\right\}\right\}\right)=\mu\left[u \mid \forall i \neq 1, u\left(x_{1}\right)>u\left(x_{i}\right)\right] \tag{5}
\end{equation*}
$$

Now, we show that $\rho$ together with $\mu$ satisfies (1). We break down the proof into two separate, exhausting cases:

1. $\rho(m \mid\{m\} \cup M)$ where $m$ is disjoint with all menus in $M$.
2. $\rho(m \mid\{m\} \cup M)$ where $m$ is allowed to overlap with menus in $M$.

## - Case 1

For any menu $m$, denote the set of all its subsets of cardinality one, i.e. the finest partition of it by $F P_{m}$. Since $m$ is disjoint with all menus in $M$, and it is the union of all singletons created by its elements, merging additivity, through induction implies:

$$
\rho(m \mid\{m\} \cup M)=\sum_{x \in m} \rho\left(\{x\} \mid F P_{m} \cup M\right)
$$

By reorganization invariance we have

$$
\rho\left(\{x\} \mid F P_{m} \cup M\right)=\rho\left(\{x\} \mid F P_{m} \cup F P_{S_{M}}\right)=\rho\left(\{x\} \mid F P_{S_{M} \cup m}\right)
$$

Now, since all menus in $F P_{S_{M}}$ are singletons, (5) implies:

$$
\rho\left(\{x\} \mid F P_{S_{M} \cup m}\right)=\mu\left[u \mid \forall x^{\prime} \in F P_{S_{M} \cup m}-\{x\}, u(x)>u\left(x^{\prime}\right)\right]
$$

Therefore

$$
\sum_{x \in m} \rho\left(x \mid F P_{m} \cup M\right)=\sum_{x \in m} \mu\left[u \mid \forall x^{\prime} \in F P_{S_{M} \cup m}-\{x\}, u(x)>u\left(x^{\prime}\right)\right]
$$

And finally

$$
\sum_{x \in m} \mu\left[u \mid \forall x^{\prime} \in F P_{S_{M} \cup m}-\{x\}, u(x)>u\left(x^{\prime}\right)\right]=\mu\left[u \mid \forall m^{\prime} \in M, V_{u}(m)>V_{u}\left(m^{\prime}\right)\right]
$$

Therefore

$$
\rho(m \mid\{m\} \cup M)=\mu\left[u \mid \forall m^{\prime} \in M, V_{u}(m)>V_{u}\left(m^{\prime}\right)\right]
$$

Which is equivalent to (1), when $m$ does not intersect with any of the menus in $M$.

## - Case 2

In this case $m$ is allowed to have a non-empty intersection with $S_{M}$. According to monotonicity and reorganization invariance, we have:

$$
0 \leq \rho(m \mid\{m\} \cup M)-\rho\left(m-S_{M} \mid\left\{m-S_{M}, S_{M}\right\}\right) \leq \rho\left(m \cap S_{M} \mid\left\{m \cap S_{M}, S_{M}-m\right\}\right)
$$

Now, since $\left(m-S_{M}\right) \cap S_{M}=\left(m \cap S_{M}\right) \cap\left(S_{M}-m\right)=\emptyset$, according to case 1:

$$
\begin{gathered}
\rho\left(m-S_{M} \mid\left\{m-S_{M}, S_{M}\right\}\right)=\mu\left[u \mid V_{u}\left(m-S_{M}\right)>V_{u}\left(S_{M}\right)\right] \\
\rho\left(m \cap S_{M} \mid\left\{m \cap S_{M}, S_{M}-m\right\}\right)=\mu\left[u \mid V_{u}\left(m \cap S_{M}\right)>V_{u}\left(S_{M}-m\right)\right]
\end{gathered}
$$

Therefore

$$
\begin{gathered}
\rho\left(m-S_{M} \mid\left\{m-S_{M}, S_{M}\right\}\right)+\rho\left(m \cap S_{M} \mid\left\{m \cap S_{M}, S_{M}-m\right\}\right)= \\
\mu\left[u \mid V_{u}\left(m-S_{M}\right)>V_{u}\left(S_{M}\right)\right]+\mu\left[u \mid V_{u}\left(m \cap S_{M}\right)>V_{u}\left(S_{M}-m\right)\right]= \\
\mu\left[u \mid \forall m^{\prime} \in M, V_{u}(m) \geq V_{u}\left(m^{\prime}\right)\right]
\end{gathered}
$$

where the last equality follows from lemma 1 . On the other hand, the following is true:

$$
\rho\left(m-S_{M} \mid\left\{m-S_{M}, S_{M}\right\}\right)=\mu\left[u \mid \forall m^{\prime} \in M, V_{u}(m)>V_{u}\left(m^{\prime}\right)\right]
$$

The last two equations, together with monotonicity imply

$$
\mu\left[u \mid V_{u}(m)>V_{u}\left(m^{\prime}\right) \quad \forall m^{\prime} \in M\right] \leq \rho(m \mid\{m\} \cup M) \leq \mu\left[u \mid V_{u}(m) \geq V_{u}\left(m^{\prime}\right) \quad \forall m^{\prime} \in M\right]
$$

## Appendix C

## Integration in Multi-Community Networks

## C. 1 Proof of Theorem 1

Notation: For every network $\mathscr{G}, h$ and $k$ denote the number of interface nodes and internal nodes in community $I$. Similarly, $h^{\prime}$ and $k^{\prime}$ denote the number of interface nodes and internal nodes in community $I^{\prime}$.

Lemma 1 below demonstrates that the small world property introduced above is an essential feature of any efficient network.

Lemma 1: Fixing $h, h^{\prime}, k$, and $k^{\prime}$, the most efficient scenario is when the small world property holds.

Proof: The small world property keeps the distance between any two categories of nodes at its minimum. The distance between:

- Two non-interface nodes is 3 .
- Two non-neighboring nodes from the same island is 2 .
- An interface node with a non-neighbor foreign node is 2 .

Since all such distances are minimized, efficiency is achieved under the conditions of the small world property.

Lemma 2: Fixing the number of bridges (internal links), $h, h^{\prime}, k$, and $k^{\prime}$, the most efficient network is the one with the least number of internal links (bridges) that satisfies the small world property.

Proof: Note that the distance of two internal nodes from different communities is 3 , and the distance of any other pair of nodes in the network is 1 or 2 . Therefore, fixing $h, h^{\prime}, k$, and $k^{\prime}$, if a network $\mathscr{G}$ has the same number of bridges (internal links) and fewer internal links (bridges) than $\mathscr{G}^{\prime}$, then it means that we have traded a number of $(b(1)-c)$ or $(b(1)-c-\delta)$ with $b(2)$ to get from $U(\mathscr{G})$ to $U\left(\mathscr{G}^{\prime}\right)$. Now, since $b(1)-c<b(2)$, such trade off improves the utility of network. Therefore, $U(\mathscr{G})>U\left(\mathscr{G}^{\prime}\right)$.

Lemma 3: For any $i$, the interface component $C_{i}$ has at least $h_{i}+h_{i}^{\prime}-1$ bridges.

Proof: Every interface component $C_{i}$ is connected using bridges only. Therefore, the number of bridges in $C_{i}$ must be at least $\left|C_{i}\right|-1=h_{i}+h_{i}^{\prime}-1$.

Corollary: Assuming there are $M$ interface-components in the network, there are at least $h+h^{\prime}-M$ bridges.

In lemma 4, we will show that it is inefficient to have more bridges than the size of either community. In particular, any such structure is dominated by a single star that encompasses all the network with its core in the larger community, in terms of efficiency.

Lemma 4: No network with $l>\min \left\{n, n^{\prime}\right\}$ is efficient. Also, the most efficient network with $l=\min \left\{n, n^{\prime}\right\}$ is a single star encompassing both communities, with its core in the larger community.

Proof: If $l>n$, and the network is connected, then in total we must have $n+n^{\prime}-1$ links, out of of which at least $n$ are bridges. However, due to the fact that $b(1)-c<b(2)$ and bridges are more costly than internal links, a single star encompassing both communities with its core in community 2 is clearly more efficient than the original network. The same reasoning can be applied to show that the most efficient network when $l=\min \left\{n, n^{\prime}\right\}$ is a single star encompassing both communities, with its core in the larger community.

Therefore, from now on, we will assume that $l \leq \min \left\{n, n^{\prime}\right\}$.

Lemma 5: This lemma is presented in three parts below. Lemma 5 sets a lower bound on the number of internal links as a function of the number of interface and internal nodes and interface components for each community (lemma 5 b does this conditional on $k, k^{\prime}>1$ ).

Lemma 5a Every network $\mathscr{G}=\left(h, h^{\prime}, k, k^{\prime}, M\right)$ that satisfies the small world property has at least:

$$
M\left(k+k^{\prime}\right)+M(M-1) / 2+h+h^{\prime}-2 M
$$

## internal links.

Proof: We prove the lemma by induction on the vector of number of hubs in the two communities; $\left(h, h^{\prime}\right)$, while keeping $M, k, k^{\prime}$ fixed. Since we want to find a lower bound on the number of internal links while satisfying the small world property, we can assume that in each interface component every interface node is connected to all the interface nodes in the other community. So, assume throughout that this is the case.

As the base case of the induction is when $\left(h, h^{\prime}\right)=(M, M)$.
To prove the statement for $\left(h, h^{\prime}\right)=(M, M)$, we need to show that there are at least a total of $M\left(k+k^{\prime}\right)+M(M-1) / 2$ internal links. But this is clear, since the small world property requires that each internal node is connected to at least one interface node in each interface
component. This makes up a total of $M\left(k+k^{\prime}\right)$ internal links in the two communities. On the other hand, since we want to get from every node in $C_{i} \cap I_{1}$ to every node in $C_{j} \cap I_{2}$ and vice versa in at most two steps, we would need to have at least one internal link between $C_{i}$ and $C_{j}$ for every $i \neq j$, either in community 1 or 2 . This makes up at least a total of $M(M-1) / 2$ extra internal links. So, we are done.

Next, suppose that the statement is true for all $\left(h, h^{\prime}\right) \leq\left(h_{0}, h_{0}^{\prime}\right)$ for $\left(h_{0}, h_{0}^{\prime}\right) \geq(M, M)$. We will show that it is also true for $\left(h, h^{\prime}\right) \leq\left(h_{0}, h_{0}^{\prime}+1\right)$. So, consider a network $N$ with the vector of parameters $\left(h, h_{0}^{\prime}+1\right)$ where $M \leq h \leq h_{0}$ that satisfies the small world property. Now, pick an interface node $H \in C_{i}^{\prime} \cap I^{\prime}$ where $\left|C_{i}^{\prime} \cap I^{\prime}\right|>1$. Next, consider another network $\mathscr{G}^{\prime}=\left(h_{0}, h^{\prime}, k, k^{\prime}, M\right)$ whose interface nodes have the same distribution as $\mathscr{G}$ across the interface components in both communities, except that $C_{i}^{\prime} \cap I_{1}^{\prime}$ has one less interface node than in $N$, namely node $H$. Now, note that adding $H$ to $\mathscr{G}^{\prime}$ does not shorten the path of any of the interface nodes in $C_{i}^{\prime} \cap I_{1}^{\prime}$ to community 2 . Therefore, the presence of $H$ does not make it easier for us to satisfy part (b) of the small world property. In other words, $H$ does not allow us to satisfy the small world property with fewer internal links. On the other hand, adding $H$ to $\mathscr{G}^{\prime}$ leaves us with even more pairs in community 1 that need to be at a distance of at most 2 through a path that only uses internal links. Therefore, our original network $\mathscr{G}$ requires at least one more internal link than $\mathscr{G}^{\prime}$ to satisfy the small world property. By our induction hypothesis, network $\mathscr{G}^{\prime}$ needs at least $M\left(k+k^{\prime}\right)+M(M-1) / 2+h+h_{0}^{\prime}-2 M$ internal links to satisfy the small world property. Therefore, $\mathscr{G}$ needs at least $M\left(k+k^{\prime}\right)+M(M-1) / 2+h+h_{0}^{\prime}+1-2 M$ internal links, which is what we want. Similarly, we can prove the statement for $\left(h_{0}+1, h^{\prime}\right)$ for any $h^{\prime} \geq h_{0}^{\prime}$. And finally, we can show it for $\left(h_{0}+1, h_{0}^{\prime}+1\right)$. Therefore, by strong induction the statement will be proved for all networks with arbitrary vector of parameters $\left(h, h^{\prime}\right) \geq(M, M)$.

Lemma 5b Assume that $h$ and $h^{\prime}$ are both strictly bigger than $M$, and both $k$ and $k^{\prime}$ are bigger than 1. Then every network $\mathscr{G}=\left(h, h^{\prime}, k, k^{\prime}, M\right)$ that satisfies the small world property has at
least:

$$
M\left(k+k^{\prime}\right)+M(M-1) / 2+h+h^{\prime}-2 M+1
$$

internal links, if one of the following happens:

- $M \geq 4$
- At least two interface components are non-singular.

Proof: The proof is by induction using similar techniques as the proof of lemma 5a. It is easy to see that the base cases of induction satisfy the given lower bound. If there are at least two nonsingular interface components, the base case is the one depicted in Figure C.1 3. It is not hard to check that we need at least 3 internal links in addition to the old $M(M-1) / 2+M\left(k+K^{\prime}\right)$ internal links to keep the small world property satisfied. Therefore, the total number of internal links required is at least $M(M-1) / 2+M\left(k+k^{\prime}\right)+3=M(M-1) / 2+M\left(k+k^{\prime}\right)+h+h^{\prime}-2 M+1$.

Also, when $M \geq 4$ and there is only one non-singular interface component, in the base case, we must have one interface component that has two interface nodes in each community and at least 3 singular interface components. Now, to satisfy the small world property, we need at least 2 links from $C_{1}$ to each of the other singular interface components. Since there are at least 3 singular interface components, we need at least 3 internal links between $C_{1}$ and other interface components in addition to the old $M(M-1) / 2+M\left(k+k^{\prime}\right)$. This will give us the desired lower bound for this lemma. The argument of the inductive step is also very similar to lemma 5a, which is therefore omitted.

Lemma 5c If $M \in\{2,3\}$, and the network has only one non-singular interface component and $l=h+h^{\prime}-M$, then the number of internal links is at least $M\left(k+k^{\prime}\right)+M(M-1) / 2+h+h^{\prime}-$ $2 M+1$.

Proof Once again, the proof is by induction. The inductive step is very similar to the proof of lemma 5a. The base case of the induction is when the non-singular interface component


Figure C.1. Lemma 5


Figure C.2. Lemma 5c
has 2 interface nodes in each community. Since we set $l$ at its lower bound, there are only 3 bridges in the non-singular interface component. Topologically, there is only one such possibility (Figure C.2). It is easy to check that we need $M(M-1) / 2+M\left(k+k^{\prime}\right)+3$ internal links to satisfy the small world property.

Next, we focus on the inductive step. Imagine that the statement is true for $\left(h, h^{\prime}\right) \geq$ $(M+1, M+1)$, we prove that it is true for $\left(h+1, h^{\prime}\right)$ as well. The proof is almost similar to the proof of the Lemma 5 a. Consider a network $\mathscr{G}$ that satisfies the small world property and has only one non-singular interface component who has the minimum number of bridges. Now, since the number of bridges in the non-singular interface component is at its minimum, there must be at least one interface node in it whose foreign degree is only 1 . Call this interface node $H$ (Figure C.1c). Now, without changing the arrangement of other interface nodes and their
foreign connections, and without changing the number of internal nodes, remove $H$ and all its links. Call the resulting network $\mathscr{G}^{\prime} . \mathscr{G}^{\prime}$ also has only one non-singular interface component with the minimum number of bridges. However, it has $h$ and $h^{\prime}$ interface nodes; one less than $\mathscr{G}$ in one of the communities. Since the interface components links in $\mathscr{G}^{\prime}$ have the exact same arrangement as in $\mathscr{G}$ modulo $H, \mathscr{G}$ would need a strictly higher number of internal links than $\mathscr{G}^{\prime}$ to satisfy the small world property. By the induction hypothesis we know that $\mathscr{G}^{\prime}$ needs at least $M\left(k+k^{\prime}\right)+M(M-1) / 2+h+h^{\prime}-2 M+1$ internal links to satisfy the small world property, so $N$ needs at least $M\left(k+k^{\prime}\right)+M(M-1) / 2+h+h^{\prime}-2 M+1+1$. So we are done.

In the next lemma, for any set of network parameters, $\left(h, h^{\prime}, k, k^{\prime}, M\right)$, we will show that the efficient network must be in one of the to be specified classes.

Lemma 6: Any network $\mathscr{G}=\left(h, h^{\prime}, k, k^{\prime}, M\right)$ can be transformed to another (weakly) more efficient network $\mathscr{G}^{t}=\left(h, h^{\prime}, k, k^{\prime}, M\right)$ with the following characteristics:

1. $M \geq 2$ :

- There are $M-2$ interface components that have only two nodes, one in each community.
- There is one interface component, with $h-M+1$ interface nodes in community 1 and one interface node in community 2, called B. Finally there is another interface component with $h^{\prime}-M+1$ interface nodes in community 2 and one interface node in community 1, called $A$.

The resulting structure is one of those depicted in Figure C.3. If both $h$ and $h^{\prime}$ are bigger than $M$, then the transformed structure is the one on the left that has $M(M-1) / 2+M(k+$ $\left.k^{\prime}\right)+h+h^{\prime}-2 M+1$ internal links. Otherwise, the transformed structure is the one on the right that has $M(M-1) / 2+M\left(k+k^{\prime}\right)+h+h^{\prime}-2 M$ internal links.
2. $M=1$ :


Figure C.3. Lemma 6

- There are interface nodes $A$ and $B$ in community 1 and 2 that are connected to all interface nodes in the other community.
- A and B are connected to all their local nodes.

Proof Case 1: $M \geq 2$

It is easy to see that the transformed version has $h+h-M$ bridges, which by the corollary to lemma 3 , is the minimum number of bridges that a network with parameters $\left(h, h^{\prime}, k, k^{\prime}, M\right)$ can have. Now, if one of $h$ or $h^{\prime}$ is equal to $M$, then the transformed structure requires $M(M-1) / 2+M\left(k+k^{\prime}\right)+h+h^{\prime}-2 M$ internal links, which is the minimum number of them required according to lemma 5a. Therefore, in this case the transformed version is weakly more efficient than the original network.

Now assume that both $h$ and $h^{\prime}$ are strictly bigger than $M$. There are two (possibly overlapping) sub-cases:

1. $M>4$
2. There are at least two non-singular interface components.

In both cases, the transformed structure is the one on the left. Since we have assumed $l \leq$ $\min \left\{n, n^{\prime}\right\}$, and $h, h>M$ and also $l \geq h+h^{\prime}-M$, it follows that $k, k^{\prime}>1$. Therefore, the transformed structure has the minimum number of internal links required according to lemma 4 b . So in all these cases, the transformation to $N^{t}$ makes the network more efficient.

The only case that remains is when $h, h^{\prime}>M, M \in\{2,3\}$, and there is only one nonsingular interface component with more than two interface nodes in each community. By lemma 5c, we know that $\mathscr{G}$ either has more than $h+h-M$ bridges or it has at least $M\left(k+k^{\prime}\right)+M(M-$ 1) $/ 2+h+h^{\prime}-2 M+1$ internal links. If the latter is true, then we are done. If the former is true, it means that $\mathscr{G}$ has more than $h+h^{\prime}-M$ bridges; at least one more than $\mathscr{G}^{t}$, and by lemma 5 a, at least $M\left(k+k^{\prime}\right)+M(M-1) / 2+h+h_{2}^{\prime} M$ internal links; which is at most one less than $\mathscr{G}^{t}$. Since bridges are more costly than internal links, and both networks satisfy the small world property, we can conclude that $\mathscr{G}^{t}$ is more efficient than $\mathscr{G}$.

Case 2: $M=1$

Proof follows from Lemma 5a, since this network achieves the lower bound on the number of bridges and internal links, while satisfying the small world property.

Theorem 1: If $b(1)-c<b(2)$, the global efficient network is either an empty network, two separate stars, a parallel hyperstar, or a single star.

Proof: First, it follows from Jackson \& Wolinsky (1996) that in the class of non-empty networks with no bridges, the efficient network is two separate stars. Therefore, from now on suppose that the network is non-empty and there is at least one bridge between the two communities. We prove the theorem by means of the following two auxiliary claims:

1. A network with $1<M<\max \left\{h, h^{\prime}\right\}$ can not be efficient.
2. The most efficient network when $M=1$ is the two-connected-stars.

For the first claim, assume that $1<M<\max \left\{h, h^{\prime}\right\}$. Since $M>1$, we know that in this case, the efficient network looks like one that is described in lemma 6. Also, since $M<\max \left\{h, h^{\prime}\right\}$ either $h-M+1>1$ or $h^{\prime}-M+1>1$. Without loss of generality, assume that $h^{\prime}-M+1>1$. This means that interface node $A$ is connected to more than two interface nodes in community 2 . Now, we find the conditions that guarantee optimality of turning an internal node in community 2 to an interface node that is connected to A , while keeping the structure of the network one that is described in lemma 6 . Take an internal node, $x$ in community 2 . Now, if we connect it to A and turn the network into one that is consistent with lemma 6, the total utility of the network changes by (factored out by 2 ):

$$
k(b(2)-b(3))+(M-1)[b(2)-b(1)+c]+b(1)-c-\delta-b(2)
$$

It is clear that this condition is independent of $k^{\prime}$ and $h^{\prime}$. Therefore, if it is more efficient to have 2 interface nodes connected to $A$ rather than 1 , it would also be more efficient to have 3 , $4, .$. , etc, until we run out of internal nodes in community 2 . Therefore, if $h^{\prime}-M+1>1$, then $k^{\prime}=0$. But this means that there are at least $n$ bridges. However, we know that the most efficient structure with $l=n$ is a single star encompassing the two communities. Therefore, a network with $1<M<\max \left\{h, h^{\prime}\right\}$ can not be efficient.

For the second claim, note that the most efficient network with $M=1$ has a structure similar to the one described in lemma 6. Now, similar to the proof of the first claim, we can show that if there are more than 1 interface node in community $i$ in the structure described in lemma 6 , then it is efficient to have no internal node at all in that community. This means that the most efficient network with $M=1$ is either a single star or a two-connected-stars, which is a special case of parallel hyperstar.

From the first claim, it follows that the globally most efficient network either has $M=1$ or is a parallel hyperstar. On the other hand, the second claim shows that when $M=1$, the
efficient network is either a single star or a two-connected-stars, which is a special case of parallel hyperstar. This proves the theorem.

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[^0]:    ${ }^{1}$ See Tversky (1969), Camerer (1989), Hey \& Orme (1994), Sippel (1997), Hey Hey (2001), Manzini et al. (2010), Agranov \& Ortoleva (2017), for a number of experiments that document stochastic choice at the individual level in repeated choice tasks.

[^1]:    ${ }^{2}$ This goes against the the standard deterministic approach in economics that describes choices by a stable preference relation whose maximization determines the final choice. This results in the same choices in multiple trials of a given choice task.
    ${ }^{3}$ In the psychology and marketing literature, where the term context-dependence has originated, context can go beyond the menu of available items. For example, it may refer to items that were available in the past, or the items that the decision maker sees now, but are unfeasible. In this paper, however, context only refers to the menu of available items.
    ${ }^{4}$ The attraction effect was first discovered by Huber, et.al. (1982) Huber et al. (1982) and the compromise effect was first discovered by Simonson (1989). Both effects have been replicated by other researchers in different settings, such as choices over political candidates, choices over lotteries, choices over policy issues, etc. For example, see Simonson \& Tversky (1992), Ariely \& Wallsten (1995), Herne (1998), Doyle et al. (1999), Sharpe et al. (2008),Herne (1999), Herne (1997), OCurry \& Pitts (1995).

[^2]:    ${ }^{5}$ In the more common treatment of the model in the literature, it is assumed that choice probabilities are all positive. In that case, IIA simply says that the ratio of the choice probabilities of two items is independent of other available items. For a more formal discussion of the Luce model see section 2.

[^3]:    ${ }^{6}$ These cases are discussed in section 4.

[^4]:    ${ }^{7}$ Hedonic attributes are those that have a natural order of desirability to them. For example, price (affordability) is a hedonic attribute, less (more) of which is always better, everything else equal.
    ${ }^{8}$ That is to say, they are strictly between zero and one.

[^5]:    ${ }^{9}$ For every $v, v^{\prime} \in \mathbb{R}^{N}, v>v^{\prime}$ if and only if for every $i \in\{1, \ldots, N\}, v_{i} \geq v_{i}^{\prime}$ and for at least one $j \in\{1, \ldots, N\}$, $v_{j}>v_{j}^{\prime}$.

[^6]:    ${ }^{10}$ Gul et al. (2014), define duplicate items as two items that can be replaced in every menu without affecting the choice probabilities of other items. My definition of absolute indifference, together with the independence of status quo preserving alternatives below implies the notion of duplicates in Gul et al. (2014).
    ${ }^{11}$ This is analogues to the weak preference relation in standard deterministic choice, which is defined as the join of the strict preference and the indifference relations.

[^7]:    ${ }^{12}$ The richness axiom below requires maximal unvariates to be unbounded.

[^8]:    ${ }^{13}$ Because of the possible existence of duplicates, $\inf S$ and $\sup S$ are generally non-singleton sets. In particular, if $x \in \inf S$ and $y \sim x$, then $y \in \inf S$. A similar thing holds for $\sup S$.

[^9]:    ${ }^{14}$ The richness axiom below ensures that for every $m, \inf m$ is non-empty.

[^10]:    ${ }^{15}$ All comparisons between vectors in $\mathbb{R}^{N}$ are with respect to the canonical order in $\mathbb{R}^{N}$.

[^11]:    ${ }^{16}$ This is how the model is introduced in Luce (1959). However, in subsequent treatment of the model in the literature, for instance in the logit formulation, it is usually assumed that all the choice probabilities are strictly positive. In that case, the model is simply characterized by the IIA. The IIA says that the ratio of choice probabilities of two items is independent of the menu containing them.

[^12]:    ${ }^{17} x$ and $y$ are non-trivial duplicates, if $x$ is a duplicate of $y$ and $x \neq y$.

[^13]:    ${ }^{18}$ See Noguchi \& Stewart (2014).

[^14]:    ${ }^{19}$ See Tversky \& Russo (1969) for a more detailed explanation and related experiments.
    ${ }^{20}$ For example, Hausman \& McFadden (1984) and Small \& Hsiao (1985) propose methods of testing IIA in the context of the multinomial logit models.

[^15]:    ${ }^{21}$ Tserenjigmid (2015) also extends the model to an environment with $N$ exogenously known attributes, for an $\operatorname{arbitrary} N \in \mathbb{N}$.

[^16]:    ${ }^{22}$ They do this under the assumption that all observed choice probabilities are strictly positive.

[^17]:    ${ }^{1}$ Some examples include Gul \& Pesendorfer (2001), Ortoleva (2013), Kopylov (2012). These models are discussed in more detail in section 2.
    ${ }^{2}$ This setting implies that the decision maker does not perceive any uncertainty regarding future mood, nor any temptation, cost of thinking, or any other behavioral biases in the second stage and her preferences are stable over time.

[^18]:    ${ }^{3}$ Like the standard random utility model, one can attribute the stochasticity of choice alternatively to a single decision maker or a population of decision makers who have random preferences in the first stage.
    ${ }^{4}$ RUM only focuses on strict orderings of $X$, because when we allow for arbitrary indifferences among items, the model loses its empirical content, and any random choice rule will be rationalizable by the model. This can be done if we put all the probability mass on the ordering that is indifferent between all items in $X$ and break the ties in a way that matches our random choice rule.

[^19]:    ${ }^{5}$ This treatment is a bit different from the standard treatment of menus in the literature such as the one introduced in the previous section. Here, for the ease of exposition, I allow menus to be empty as well.

[^20]:    ${ }^{6}$ This is because it is not clear how such ties should should be broken. Moreover, any tie breaking rule will only make the model more complicated, without adding much to its content.

[^21]:    ${ }^{7}$ It is a well known property of RUM that the inferred probability measure is not always uniquely identified. For an example of the multiplicity of such measure see McFadden (1978).

[^22]:    ${ }^{8}$ Strotz (1955) introduced a model of decision making where preferences are time-inconsistent, and the decision maker might have different short term and long term goals.

[^23]:    ${ }^{9}$ When $m$ is not disjoint with all menus in $M$, the representation would involve some inequalities of the type in the standard model. Here, for simplicity I only focus on the disjoint case.

[^24]:    ${ }^{1}$ The second structure, two hub-connected structure, can be considered a special case of parallel hyperstar with only one bridge. However for the sake of presentation and differentiating some results we use it as a separate category.

[^25]:    ${ }^{1}$ This treatment is a bit different from the usual definition of these concepts in the sense that the duplicate relation $\sim$ has replaced the equality sign. However, this does not change anything, given the fact that $\sim$ is an equivalence relation.

[^26]:    ${ }^{2}$ It is easy to see that the property can be more generally stated as: If $x, y \in m^{u} \cap m^{\prime u}$ and $r_{m^{\prime}}$ is lower than $r_{m}$ in attributes for which $x$ is inferior to $y$, but equal to $r_{m}$ in all other attributes, then:

    $$
    \frac{\rho(x \mid m)}{\rho(y \mid m)} \leq \frac{\rho\left(x \mid m^{\prime}\right)}{\rho\left(y \mid m^{\prime}\right)}
    $$

