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# UNIVERSITY OF CALIFORNIA RIVERSIDE 

Thermodynamic Signatures of Half-Quantum Vortices in $p+i p$ Josephson Junction Arrays

A Dissertation submitted in partial satisfaction of the requirements for the degree of

Doctor of Philosophy<br>in<br>Physics<br>by<br>Graham Joel Krahn

September 2012

Dissertation Committee:
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University of California, Riverside

## Acknowledgments

I am grateful to my advisor, without whose help, I would not have been here.

To my parents for all the support.

## ABSTRACT OF THE DISSERTATION

Thermodynamic Signatures of Half-Quantum Vortices in $p+i p$ Josephson Junction Arrays
by

Graham Joel Krahn

Doctor of Philosophy, Graduate Program in Physics
University of California, Riverside, September 2012
Dr. Kirill Shtengel, Chairperson

A very interesting type of excitation in a chiral p-wave superconductor is a half-quantum vortex. As the name suggests, a half-quantum vortex carries half of a superconducting flux quantum, and are only possible in superconductors with spin-triplet pairing. An astonishing feature of these excitations is the presence of topologically protected Majorana zero modes. Single half-quantum vortices were recently discovered (J. Jang et al, Science 331(6014): 186-188) in superconducting mesoscopic rings made of $\mathrm{Sr}_{2} \mathrm{RuO}_{4}$, yet to this date they have not been observed in macroscopic samples. We propose a method for detecting half-quantum vortices in Josephson junction arrays, which could host a large number of these vortices. Contrary to a 3D setting, we argue that half-quantum vortices can be energetically preferable in quasi-2D chiral spin-triplet superconductors. As a result, half-quantum vortices rather than full vortices could drive a Berezinskii-Kosterlitz-Thouless transition (which manifests itself as a resistive transition). We propose to look for their signatures by comparing transition temperatures in $p+i p$ Josephson junction arrays in a transverse magnetic field in both unfrustrated and frustrated cases.

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## Chapter 1

## Introduction

In this chapter we will review some of the concepts that are necessary to understand in order for us to present our results on phase transitions in $p+i p$ Josephson junction arrays. These concepts will include the superconducting order parameter, chiral p-wave $(p+i p)$ superconductors, Majorana fermions, and half-quantum vortices. The motivation behind this research is driven by the growing interest[1] in trying to experimentally detect Majorana fermions in a variety of condensed matter systems. This introductory chapter will lay the groundwork for the thesis, and in subsequent chapters we will show that half-quantum vortices and Majorana fermions are expected to leave various thermodynamic signatures accessible to the experimentalist.

### 1.1 Superconductivity

The phenomena of superconductivity was first discovered in 1911 by H. Kamerlingh Onnes[2]. He made the remarkable discovery that the electrical resistance of various metals, such as mercury, lead, and tin drops suddenly to zero at a critical temperature $T_{C}$. The transition temperature is a material-dependent property, and all supercon-
ducting materials discovered in the early years of superconductivity research had critical temperatures of just a few degrees kelvin. The transition from finite to zero electrical resistance at the critical temperature has been most sensitively probed by observing the magnetic field produced by persistent currents in superconducting rings[3]. Measurements have shown that there is zero change in magnetic field or current on the time scale of years, and thus perfect conductivity has traditionally been one of the first properties used to identify a superconductor.

An additional property unique to superconductors was discovered in 1933 by Meissner and Ochsenfeld[4], and is now simply called the Meissner effect. It was found that superconductors exhibit perfect diamagnetism, which means that no magnetic flux can penetrate into a superconductor. Magnetic field is zero inside the bulk of a superconductor, regardless of whether or not there is an applied external field. To understand the uniqueness of this property to superconducting materials, consider both a hypothetical perfect conductor and a superconductor at a very low temperature in zero magnetic field. If an external magnetic field is subsequently turned on, the perfect conductor would completely expel all magnetic flux by generating screening currents induced in accordance with Faraday's law. The exact same behavior is found in superconductors.

Now consider the case where both materials are normal, that is to say, in their resistive states (above the critical temperature). In the presence of a constant applied magnetic field, the screening currents in both materials will die out in some finite time due to resistivity, eventually allowing for full penetration of magnetic flux. If the temperature is now reduced below the critical value, the perfect conductor would remain fully penetrated by flux, because the magnetic field has remained a constant, and no screening currents can be generated via Faraday's law, which requires a time-dependent magnetic field. On the other hand, it is found experimentally that a superconductor will
completely expel magnetic flux as the temperature decreases below its critical temperature. At the critical temperature, a metal which is superconducting will suddenly switch from a state of flux penetration to complete flux expulsion. Therefore, superconductivity is fundamentally different than simply the classical limit of perfect conductivity, in that it characterizes a distinct thermodynamic state. It should be noted that in real materials, some flux will always leak into a superconductor. In real superconductors there is a finite surface layer into which magnetic fields can penetrate before they are screened out. Additionally, the magnetic field is typically not identically zero in the bulk of a real superconductor. Nevertheless the overall point remains: nearly all flux is expelled and continues to be expelled from a superconductor below $T_{\mathrm{C}}$.

### 1.2 Type I and Type II Superconductors

Superconductors can be divided into two general classes, type I and type II. Each type differs drastically in its response to an externally applied magnetic field. We have seen so far that two fundamental properties of superconductors are zero electrical resistivity and perfect diamagnetism. It has been experimentally demonstrated, however, that the Meissner effect (perfect diamagnetism) will not persist to arbitrarily high values of an applied magnetic fields. There is a temperature-dependant critical field, $H_{C}(T)$, at which pure superconductivity begins to break down. Above this critical field, magnetic flux begins to leak into the material. The manner in which magnetic field penetrates a superconductor allows us to distinguish between type I and type II superconducting materials.

The response of type I and type II superconductors in the presence of sufficiently small applied magnetic fields is the same: magnetic flux is screened out and does
not enter the bulk of the material. For sufficiently strong magnetic fields, superconductivity begins to break down. As the field strength increases in a type I superconductor, it enters a geometry-dependent intermediate state, where the sample contains alternating regions of normal metal and superconductor. Once the critical field $H_{C}(T)$ is reached, a type I superconductor will expel flux completely and enter a normal state of complete flux penetration. Examples of such materials that exhibit this property are mercury, aluminum, and tin, where superconductivity is completely destroyed at a well-defined critical field.

Type II superconductors, on the other hand, switch from a Meissner state to a state of discretized, microscopic partial flux penetration at a different critical field, $H_{C 1}(T)$. What is interesting in the type II case is that this mixed state does not completely destroy superconductivity, but allows superconductivity to persist around these microscopic regions of flux penetration. If the applied field is increased above $H_{C 1}(T)$ to an upper critical field $H_{C 2}$, the state of microscopic partial flux penetration crosses continuously over to full flux penetration, at which point superconductivity is completely destroyed. Typically, $H_{C 1}(T)$ is lower than the critical field $H_{C}(T)$ of a type I superconductor, while $H_{C 2}(T)$ is much higher[3]. One advantage of type II superconductors is that superconductivity persists even in the presence of very strong fields.

The partial flux penetration in the so-called mixed state of a type II superconductor is carried by quantized flux tubes, known as vortices, whose existence was first predicted by A. Abrikosov[5]. Each vortex is of microscopic size and suppresses superconductivity in its core region, which is surrounded by circulating supercurrents. A single vortex carries a quantum of magnetic flux

$$
\begin{equation*}
\Phi_{0}=\frac{h c}{2 e} \tag{1.1}
\end{equation*}
$$

A superconductor with many vortices, which is typical for any macroscopic sample, arranges itself into a regular array of flux lines referred to as a vortex lattice. The existence of a vortex array was first demonstrated experimentally in 1967[6]. As the applied field increases above $H_{C 1}(T)$, the concentration of these flux-carrying vortices increases until their separation distance is of the same order as their size. This occurs at the critical field $H_{C 2}(T)$, at which point superconductivity is destroyed throughout the entire sample.

The difference between type I and type II superconductors can be understood by considering the following two quantities: the bulk penetration depth, $\lambda$, and the microscopic coherence length, $\xi[3]$. In our discussion of the Meissner effect, we found that magnetic fields are completely screened and consequently cannot enter the bulk of a superconductor. However, this is not entirely accurate on the microscopic level, as magnetic field can in fact leak into a superconductor over short distances. The orientation of the field to the superconducting surface turns out to be important for field penetration. A magnetic field perpendicular to the surface of a superconductor is identically zero inside the material, a consequence of Maxwell's equations. However, parallel fields are permitted to enter a superconductor, but are exponentially screened over a characteristic distance called $\lambda$. The length scale $\lambda$ therefore measures the region over which there is non-zero field penetrating into the bulk of a superconductor. The experimental value for $\lambda$ in various pure superconducting samples is of the order of several hundred ångströms[7].

The coherence length, $\xi$, is a measure of the distance over which supercon-
ductivity is suppressed. Superconductivity is entirely destroyed in the core region of a vortex. The core radius of a vortex is of order $\xi$, which characterizes the region of the vortex which is in the normal state, with superconductivity being strongly suppressed. Therefore, for each vortex flux line, there is a circular boundary between the normal and superconducting state which is a distance of order $\xi$ from the center of the vortex core. Microscopically, we can now understand a vortex as consisting of a normal state or core region of radius $\xi$, surrounded by circulating supercurrents, which decay exponentially with a characteristic length $\lambda$. It is in fact these supercurrents which regulate the quantum flux $\Phi_{0}$ through each vortex line.

It is the relative size of the physical lengths $\lambda$ and $\xi$ which determines whether a superconductor is type I or type II. The Ginzburg-Landau parameter is defined as

$$
\begin{equation*}
\kappa=\frac{\lambda}{\xi} \tag{1.2}
\end{equation*}
$$

and the exact breakpoint between the regimes of type I and type II is $\kappa=1 / \sqrt{2}[5]$. A material with $\kappa<1 / \sqrt{2}$ is type I, while $\kappa>1 / \sqrt{2}$ corresponds to a type II superconductor. There is a surface energy associated with the creation of a domain wall between a normal and superconducting region[8], which can be either positive or negative. It can be shown[3] that the energy cost of a domain wall is of order $\sim(\xi-\lambda)$. Since the parameters of a type I superconductor roughly satisfy $\xi>\lambda$, there is a positive surface energy associated with the creation of a domain wall. Domains are energetically unfavorable and the sample prefers a minimum number of domain separations, and will stabilize into a state where the scale of subdivision is of the order of the sample size.

For type II superconductors, where $\lambda$ is typically greater than $\xi$, there is a negative surface energy associated with a domain wall, which causes them to be ener-
getically favorable. Therefore, the process of subdivision tends to maximize the surface area between the normal and superconducting state, limited by the microscopic length scale $\xi$. Type II superconductors favor the existence of vortices rather than the destruction of superconductivity in macroscopic regions. The generation of multiple vortices of microscopic radius $\xi$ allows the system to maximize the domain surface area, thereby minimizing the surface energy and hence preserving superconductivity in all regions surrounding these microscopic vortices.

### 1.3 The Superconducting Order Parameter

The superconducting phase and the normal phase of a superconductor are distinct thermodynamic states with differing thermodynamic properties. The properties of the phase transition at $T_{C}$ depends upon the strength of the external magnetic field, and the order of the transition depends on whether the superconductor is type I or type II.

In a type I material the transition is first order because in the presence of a magnetic field the breakdown of superconductivity is singular. There is a discontinuity in the free energy of the system at the transition temperature, and hence an associated latent heat. The condition necessary for a critical transition from the normal state into the superconducting state is that the superconducting state becomes the most energetically favorable state. To see when this happens, we must examine the energy difference between the normal and superconducting state. The free energy of a superconducting condensate is lower than the free energy of a normal state material by an amount referred to as the condensation energy. However, there is also a diamagnetic energy cost required to expel all magnetic flux from a superconductor in the superconducting
state. A phase transition will occur when the gain of the condensation energy of the condensate is exactly compensated by the diamagnetic energy cost of expelling all the magnetic energy. This is the criterion which determines the temperature dependence of the critical field $[9,10]$. At zero field, however, the transition is second order because there is no competing energy required to screen against magnetic pressure.

In a type II superconductor, the phase transition in the presence of an applied field is second order. This is due to the fact that superconductivity does not break down discontinuously at the transition as it does in the type I case. Rather, there is a continuous increase in flux penetration starting at $H_{C 1}(T)$ and persisting up until $H_{C 2}(T)$. This indicates that the transition from the superconducting state to the normal state is a continuous transition, and the free energy is continuous across the transition.

The presence of a phase transition, whether it is first order or second order, is indicative of the onset of an order parameter which characterizes the superconducting state. The order parameter is a quantity introduced within the general framework of Landau's phenomenological theory of phase transitions. The superconducting order parameter was first identified by Ginzburg and Landau in 1950[11], and their theory of superconductivity is now called the phenomenological Ginzburg-Landau theory. They denoted the superconducting order parameter as $\psi$, which is a complex pseudowavefunction describing the superconducting electrons. In the normal state there are no superconducting electrons, and so $\psi=0$. In the superconducting state, the local density of superconducting electrons (the superfluid density) is non-zero and is given by

$$
\begin{equation*}
n_{\mathrm{s}}=|\psi(r)|^{2} \tag{1.3}
\end{equation*}
$$

and is related to the bulk penetration depth $\lambda$ via the London equations[3],

$$
\begin{equation*}
n_{\mathrm{s}}=\frac{m^{*} c^{2}}{4 \pi e^{* 2} \lambda^{2}} \tag{1.4}
\end{equation*}
$$

where $m^{*}$ is the effective mass of a superconducting electron and $e^{*}$ is its effective charge. Ginzburg-Landau theory doesn't tell us the values of the effective mass and charge, only that there is some finite density $n_{\mathrm{S}}$ of superconducting particles in the superconducting regime.

The identification of $\psi$ as the order parameter of the superconducting state was a huge step forward in our understanding of the phenomenology of superconductors. Physically, $\psi$ can be thought of as the wavefunction which describes the motion of the condensate as a whole. It is in this sense that the superconducting condensate can be thought of as a single, coherent, macroscopic quantum object. In order to go beyond a phenomenological description of superconductivity and to understand the details of the interaction, we must go to a microscopic theory.

### 1.4 Conventional Superconductors

The first successful microscopic theory of superconductivity was produced by Bardeen, Cooper, and Schrieffer (BCS theory) in 1957[12]. They were able to identify the microscopic mechanism responsible for superconductivity. Conventional superconductors were the first class of superconductors to be discovered. They all have a very low transition temperature, of the order of a few Kelvin, and are understood in terms of conventional BCS theory. The basic mechanism behind superconductivity is the formation of Cooper pairs, which are pairs of electrons that bind together in the presence of a weak attractive interaction to form a bound state[13]. It turns out that in an ordinary Fermi-sea of electrons, there is a small attractive interaction between electrons causing
them to bind together into pairs, known as Cooper pairs, provided the temperature is sufficiently low. Normally the electron-electron interaction is repulsive, but an attractive interaction is possible when a background lattice of positive ions is present, which allows electrons to interact via the exchange of phonons.

The ground state of a conventional superconductor is a coherent state of Cooper pairs. Each Cooper pair of electrons is paired in momentum space rather than real space. Pairs have a relatively large spatial extension (compared to the spacing of the electrons in the electron gas) and a vanishing total momentum. In the original BCS model, only electrons with opposite linear momentum can attract one another. The origin of this attractive potential is electron-phonon coupling, which causes electrons in a thin layer around the fermi surface to be attracted. This explanation for superconductivity (and the electron-lattice interaction) was first experimentally confirmed through observation of the isotope effect[14]. The physical idea is that as one electron moves it attracts a cloud of positive ions, and it is this disturbance of positive ions that then attracts the second electron. Therefore, the effective electron-electron attractive interaction is mediated through phonon exchange (lattice vibrations).

The only channel of attractive potential in conventional superconductors is the isotropic channel, that is, where each Cooper pair is in a state of relative s-wave symmetry, which means that each Cooper pair has zero net angular momentum. The wavefunction of a Cooper pair must be antisymmetric with respect to the exchange of the electrons, since electrons are fermions and must obey fermionic exchange statistics. The wavefunction of a Cooper pair can be written as

$$
\begin{equation*}
|\Psi\rangle=\chi\left(k_{1}, k_{2}\right)|S m\rangle \tag{1.5}
\end{equation*}
$$

where $\chi$ is the orbital wavefunction, $k_{i}$ is the momentum of the $i$ th electron, $S$ is the total spin of the electron pair, and $m$ is the total spin projection in the $z$-direction. In conventional superconductors, the electrons pair in a state of zero total spin ( $\mathrm{S}=0$ ). This is the spin singlet state and can be written[15]

$$
\begin{equation*}
|00\rangle=\frac{1}{\sqrt{2}}(|\uparrow \downarrow\rangle-|\downarrow \uparrow\rangle) \tag{1.6}
\end{equation*}
$$

where $|\uparrow \downarrow\rangle$ denotes electron 1 in a spin up state and electron 2 in a spin down state. Since the spin part of the wavefunction is antisymmetric under exchange, the orbital part must be symmetric (even parity) in order to preserve overall antisymmetry. This means that the orbital angular momentum $L$ is restricted to be an even integer: $L=0$, which is called s-wave, $L=2$, which is called d-wave, etc. Electron spin-pairing in conventional s-wave superconductors is always of spin-singlet symmetry[3].

The formation of Cooper pairs is responsible for the phenomena of superconductivity in conventional superconductors. Below the transition temperature, Cooper pairs become more energetically favorable than unpaired electrons, and it takes a finite amount of energy to break a pair. In BCS theory, one can calculate this finite excitation energy as a function of temperature, which physically creates a gap in the energy spectrum between the ground state and the first excited state. This energy is called the superconducting gap energy, and it has a maximum value at $T=0$ and vanishes at $T=T_{C}[3]$. The superconducting gap energy of BCS theory is related to the order parameter $\psi$ of the condensate. The order parameter (i.e. gap energy) of a conventional superconductor can be represented by a single complex gap function $\Delta(\mathbf{k})$, which is a function of the Cooper pair electron momentum $\mathbf{k}$. The function $\Delta(\mathbf{k})$ is very useful as it allows one to describe and understand the phase transition and all temperature
dependent properties of conventional superconductors.
A type II conventional superconductor allows for the presence of vortices. The internal structure of these vortices can be understood by examining the complex superconducting order parameter, which can be written in terms of its magnitude and phase,

$$
\begin{equation*}
\Delta(\mathbf{k})=|\Delta(\mathbf{k})| e^{i \theta} \tag{1.7}
\end{equation*}
$$

A vortex is a line defect in the superconducting phase. At every point in a superconductor, the order parameter (1.7) is well-defined, except at the core of a vortex. Imagine winding around the core of a vortex; the phase must wind by an integer multiple of $2 \pi$ in order to leave the order parameter single-valued. A non-trivial winding of the phase is what constitutes a vortex. The most energetically stable configuration of a vortex is one in which the phase winds by $2 \pi$ around some closed path that encloses its core. This leads to the notion of quantized topological charge, where $2 \pi$-winding of the phase corresponds to exactly one unit of topological charge. This topological charge is physically manifest by magnetic flux concentrated in the core region of a vortex. The core of a vortex is surrounded by circulating supercurrents, which generate a flux that is quantized in integer units of the magnetic flux quantum, $\Phi_{0}=h c / 2 e$.

In summary, we have seen that the order parameter of a superconductor can be identified with the macroscopic condensate wavefunction. A vortex excitation can be described in terms of a singularity in the order parameter, surrounded by a non-trivial winding of the superconducting phase. Vortices can only occur in type II superconducting materials. They are a very special kind of excitation in that a vortex is topologically stable. This means that a vortex cannot be removed by any continuous deformation of
the order parameter.

### 1.5 Unconventional Superconductors

For the first six decades of research on superconductivity, all known superconductors were of the conventional type. In recent years, a whole new class of superconductors has emerged with unconventional properties, which break symmetries that remain unbroken in conventional superconductors[16]. Strong electron correlations play an important role in the behavior of these materials. Examples of such materials with unconventional like properties are heavy fermion intermetallic alloys, organic materials, and cuprates (copper oxides). A major breakthrough in superconductivity occurred in 1986, with the discovery of high $T_{C}$ cuprates[17]. It was realized that an essential ingredient for high transition temperature materials were quasi-2D electronic states from planar copper oxides. Most high $T_{C}$ superconductors are unconventional, however there are some superconductors with a high $T_{C}$ which are conventional.

It could be that a superconductor is unconventional because the Cooper pair electrons have d-wave spin singlet symmetry, or possibly even p-wave spin-triplet symmetry. The vast majority of superconductors known to date, whether conventional or unconventional, are spin-singlet paired. The best known example of triplet-pairing occurs in the context of superfluidity. Liquid He-3 is known to form a condensate of spin-triplet atomic Cooper pairs[18]. The spin-triplet configuration is when two spin- $1 / 2$ fermions combine together to form a net spin of $S=1$. The three spin-triplet states $|\mathrm{Sm}\rangle$ are [15]

$$
\begin{aligned}
|11\rangle & =|\uparrow \uparrow\rangle \\
|10\rangle & =\frac{1}{\sqrt{2}}(|\uparrow \downarrow\rangle+|\downarrow \uparrow\rangle) \\
|1-1\rangle & =|\downarrow \downarrow\rangle
\end{aligned}
$$

In 1994, superconductivity was discovered in strontium ruthenate, $\mathrm{Sr}_{2} \mathrm{RuO}_{4}$, at a transition temperature of $T_{C}=1.5 K[19]$. Many researchers now suspect that $\mathrm{Sr}_{2} \mathrm{RuO}_{4}$ is a spin-triplet superconductor[20]. The most direct evidence for tripletpairing in $\mathrm{Sr}_{2} \mathrm{RuO}_{4}$ are the spin susceptibility measurements in the superconducting state performed by Ishida et al[21]. Triplet superconductivity may not be unique to $\mathrm{Sr}_{2} \mathrm{RuO}_{4}[22,23,24,25]$, but it is a prominent candidate for understanding spin-triplet superconductors because it has a relatively simple electronic structure[20].

As we have already seen, symmetry requires a Cooper pair wavefunction to be antisymmetric under exchange. A spin-triplet state is symmetric under exchange, and thus the orbital wave function of a triplet superconductor must be antisymmetric. This requires the spatial part of the Cooper pair wavefunction to have an odd integer value of angular momentum (odd parity). The simplest case is p-wave symmetry, which is $L=1 . \mathrm{Sr}_{2} \mathrm{RuO}_{4}$ is a possible example of a p-wave spin-triplet superconductor which breaks time-reversal symmetry[16]. A Cooper pair of finite angular momentum is a more complicated state, and it is not yet entirely understood why p-wave pairing may be preferred over conventional s-wave pairing in certain materials[20]. An intuitive picture of this effect is that certain unconventional materials have a strong on-site Coulomb repulsion, which in turn favors the formation of a Cooper pair wavefunctions with larger amplitudes at a finite distance from the origin. This reduces the on-site repulsion by energetically favoring Cooper pairs which have finite relative angular momentum.

P-wave superconductors are of great interest since triplet-pairing of Cooper pairs introduces a spin degree of freedom in addition to the orbital phase $\theta$ of the condensate wavefunction. This concept was originally discussed in the context of superconductors in the early $1960 \mathrm{~s}[26,27]$, but the first candidate superconducting material with p-wave symmetry was not discovered until 1994[19]. In the presence of spin degrees of freedom, in addition to the orbital phase $\theta$, the superconducting order parameter cannot be represented by a single complex quantity, but must have multiple components. In spin space, the superconducting order parameter takes the form of a $2 \times 2$ matrix $[16,20]$

$$
\hat{\Delta}(\mathbf{k})=\left(\begin{array}{cc}
\Delta_{\uparrow \uparrow}(\mathbf{k}) & \Delta_{\uparrow \downarrow}(\mathbf{k})  \tag{1.8}\\
\Delta_{\downarrow \uparrow}(\mathbf{k}) & \Delta_{\downarrow \downarrow}(\mathbf{k})
\end{array}\right)=\left(\begin{array}{cc}
-d_{x}+i d_{y} & d_{z} \\
d_{z} & d_{x}+i d_{y}
\end{array}\right)
$$

This multicomponent order parameter has been parameterized by the complex vector $\mathbf{d}(\mathbf{k})=\left(d_{x}(\mathbf{k}), d_{y}(\mathbf{k}), d_{z}(\mathbf{k})\right)$, which is a useful representation since $\mathbf{d}(\mathbf{k})$ transforms as a headless vector in spin space[8]. The director vector $\mathbf{d}(\mathbf{k})$ measures the amplitude of the three independent spin-triplet wavefunctions at a given value of $\mathbf{k}$.

Exotic vortices with a complex internal structure are permitted in p-wave superconductors due to the extra spin degree of freedom. The multicomponent order parameter depends now not only on the phase $\theta$, but also on the symmetry direction $\mathbf{d}(\mathbf{k})$ of Cooper pair spin. The director vector $\mathbf{d}(\mathbf{k})$ can be parameterized by the angle $\phi$ if $\mathbf{d}(\mathbf{k})$ is planar. An interesting type of vortex is the half-quantum vortex $\left(\frac{1}{2} Q V\right)$, which traps exactly one-half of the superconducting flux quantum $\Phi_{0}$. A $\frac{1}{2} \mathrm{QV}$ is an exotic topological defect with an internal structure generated by mixing both the spin and phase degrees of freedom. In a $\frac{1}{2} \mathrm{QV}$, simultaneous rotations of the complex vector $\mathbf{d}(\mathbf{k})$ by $\pi$ are combined with rotations of the phase $\theta$ by $\pi$ upon encircling the vortex core. The simultaneous winding of both degrees of freedom is permitted because it leaves
the multicomponent order parameter single-valued upon encircling the vortex core. In a spin-singlet SC , the existence of $\frac{1}{2} \mathrm{QVs}$ is topologically forbidden because the order parameter must remain single-valued. Therefore, $2 \pi$-windings in the superconducting phase are the only permissible vortex excitations in spin-singlet superconductors.

Relative windings between $\theta$ and $\phi$ tend to destabilize the superconducting state[16], and the smaller vorticity of a $\frac{1}{2} \mathrm{QV}$ may make them energetically preferable over the full QV. However, there are many other factors that must be taken into account when analyzing the stability of $\frac{1}{2} \mathrm{QVs}$ in quasi-two-dimensional $p+i p$ superconductors[28]. First, there is the issue of locking the $\hat{\mathbf{d}}$-vector into a planar orbit. In order for a $\frac{1}{2}$ QVs to exist, the director vector $\hat{\mathbf{d}}$ must be confined to rotations in a 2 D plane, and one possible way to do this is to apply an out-of-plane B-field[29]. Secondly, $\frac{1}{2}$ QVs carry an unscreened spin current that diverges logarithmically with size. In order to eliminate this divergence, $\frac{1}{2} \mathrm{QVs}$ must be paired with opposite spin windings. Thirdly, the stability of $\frac{1}{2} \mathrm{QVs}$ largely depends on the ratio $\rho_{\mathrm{sp}} / \rho_{\mathrm{s}}$, which is the ratio of the neutral spin superfluid density $\rho_{\mathrm{sp}}$ (spin stiffness) to the superfluid density $\rho_{\mathrm{s}}$ (superfluid stiffness). The physical meaning of spin stiffness and superfluid stiffness is discussed more in Section 1.7.3. It has been shown by S.B. Chung et al.[28] that pairs of $\frac{1}{2}$ QVs with opposite spin windings are energetically favorable over full QVs in $p+i p$ superconductors when the stiffness ratio $\rho_{\mathrm{sp}} / \rho_{\mathrm{s}}$ is small (softer in the spin sector than in the superfluid sector). This means that given the right conditions, full QVs can decay into a pair of $\frac{1}{2} \mathrm{QVs}$ with opposite spin windings (note that these two configurations are topologically equivalent). However, this process is limited by the core interaction energy of $\frac{1}{2} \mathrm{QVs}$, which can be substantial when core separation distances are of order of the coherence length $\xi$.

### 1.6 Majorana Fermions

Majorana fermions are spin- $1 / 2$ particles that are their own antiparticles, whose existence was first predicted by Ettore Majorana[30]. Examples of Majorana fermions in nature are thought to be very rare, but there are a number of interesting candidates in various fields of physics[1]. In particle physics, neutrinos are candidate particles that could possibly be Majorana fermions. In condensed matter physics, there are a variety of systems which may host exotic quasiparticle excitations that are thought to obey Majorana fermionic statistics(see [31] for a review of the various proposals). In fact, an experimental group led by Leo Kouwenhoven at the Delft University of Technology have recently detected signatures of Majorana fermions through electrical conductance measurements of an indium antimonide nanowire[32]. The emergence of Majorana modes at the end of a 1D nanowire was predicted theoretically[33, 34] in 2010, and the experimental discovery by Kouwenhoven's group in 2012 confirms these predictions and represents the strongest evidence to date of Majorana fermions.

Ordinary spin- $1 / 2$ fermions such as electrons were first described by Dirac in 1928[35] by his famous equation,

$$
\begin{equation*}
\left(i \hbar \gamma^{\mu} \partial_{\mu}-m c\right) \psi=0 \tag{1.9}
\end{equation*}
$$

The $\gamma^{\mu}$ matrices are four complex $4 \times 4$ matrices and the field $\psi$ is a complex 4 -component column matrix called a Dirac-spinor[36]. In the quantum field theoretical description of the electron, the field $\psi^{\dagger}$ creates an electron while $\psi$ destroys it, distinct operations since $\psi$ is complex. Majorana discovered a set of $\gamma^{\mu}$ matrices that were purely imaginary, which means that there exist solutions to the Dirac equation which are purely real, namely $\psi=\psi^{\dagger}$. Such solutions are known as Majorana fermions, spin- $1 / 2$ particles that
are their own anti-particles.

In the 2 D setting, Majorana fermions are predicted to emerge in the context of chiral p-wave superconductors $[37,38]$. Majoranas in solid state systems must arise as nontrivial emergent excitations, and superconductors provide just the right environment to look for Majorana modes. A superconducting condensate involves the pairing of fermions (electrons). Superconductors also screen electric and magnetic fields, thereby enabling charge conservation to be spontaneously violated (adding or subtracting a Cooper pair has little effect on the condensate). This is the first clue to finding Majoranas in superconductors, since Majoranas must be chargeless modes. Most superconductors are s-wave, where the condensate consists of Cooper pairs of electrons of opposite spin. Quasiparticles formed in an s-wave superconductor involve superpositions of electrons and holes, and therefore the quasiparticle operators will be physically distinct from each other (Majorana fermions must be identical with their antiparticle). What is needed to realize Majorana fermions are spinless superconductors (where Cooper pairing is between electrons of equal spin). Spinless superconductors are symmetric with respect to spin pairing, and therefore the orbital pairing must be p-wave.

Chiral p-wave superconductors are special because they consist of topological phases that support exotic excitations at their boundary and at topological defects such as vortices[39]. In the $2 \mathrm{D} p_{x}+i p_{y}$ case $(p+i p$ for short), superconducting vortices bind zero-energy (with respect to the chemical potential) Majorana fermions to their cores[40]. Thus, 2D $p+i p$ superconductors are the most natural environment to look for Majorana fermions since vortices will bind Majoranas. As pointed out by [41], both full quantum vortices for spinless (or spin-polarized) fermions and half-quantum vortices for spinful fermions in 2D $p+i p$ superconductors contain zero-energy Majorana fermions. Typically half-quantum vortices are more energetically costly than full quantum vortices; however,
there are ways of stabilizing half-quantum vortices such that they are the energetically preferable excitations at low temperatures[29].
$\mathrm{Sr}_{2} \mathrm{RuO}_{4}$ is a possible physical realization of a chiral p-wave superconductor. In fact, half-height magnetization steps were recently discovered in annular-shaped mesoscopic samples of superconducting $\mathrm{Sr}_{2} \mathrm{RuO}_{4}$ through torque magnetometery techniques[42]. This is strong evidence that $\mathrm{Sr}_{2} \mathrm{RuO}_{4}$ behaves as a 2D $p+i p$ superconductor and that $\frac{1}{2} \mathrm{QVs}$ are possible topological excitations. Majorana fermions on vortex cores are very interesting for at least two reasons; first, Majoranas entangle non-locally, and second, Majoranas are characterized by non-Abelian exchange statistics[41, 39, 43]. With regards to non-local entanglement, (1.9) shows that Majorana operators are actually fractionalized fermionic modes. Each Majorana fermion constitutes half of a fermion, such that two Majoranas can combine via (1.9) to create a single fermionic mode. This means that Majoranas entangle non-locally to create zero-energy degenerate fermions, a topologically stable mode which can be singly occupied or empty, a necessary ingredient for topological quantum computation.

Secondly, the statistics of Majoranas in 2D is inherently non-Abelian. It turns out that for topological reasons, 2D systems allow for the emergence of particles called anyons, which are neither bosons nor fermions. Upon exchange of two identical Abelian anyons, the wavefunction acquires a statistical phase $\mathrm{e}^{i \theta}$. Non-abelian anyons, such as Majorana fermions on vortex cores in 2D $p+i p$ superconductors, are even more interesting. They can change to a fundamentally different quantum state upon exchange of identical particles. Such exchanges do not in general commute since any exchange may result in a unitary transformation in the degenerate space of ground states. Thus, vortices and phase transitions in 2D $p+i p$ superconductors are interesting because Majorana modes will be bound to their cores. Majoranas are not only of interest from
a fundamental physics point of view, but also because they could potentially be used to encode topological qubits and built a quantum computer[40].

We have seen that Majorana fermions will bind to vortices in $p+i p$ superconductors. The operator which creates a Majorana fermion on the core of vortex $i$ can be denoted by $\gamma_{i}$, and it satisfies the properties $\gamma_{i}=\gamma_{i}^{\dagger}$ and $\gamma_{i}^{2}=1$. The Majorana mode operators satisfy the algebra $\left\{\gamma_{i}, \gamma_{j}\right\}=2 \delta\left(\mathbf{r}_{i}-\mathbf{r}_{j}\right)$, which is different then that of conventional fermions[44]. Consider taking two Majorana operators $\gamma_{1}$ and $\gamma_{2}$ and combining them into a single complex operator[41]

$$
\begin{equation*}
\gamma=\frac{1}{2}\left(\gamma_{1}+i \gamma_{2}\right) \tag{1.10}
\end{equation*}
$$

This operator corresponds to a usual fermion since one can easily verify that

$$
\begin{equation*}
\left\{\gamma, \gamma^{\dagger}\right\}=1 \tag{1.11}
\end{equation*}
$$

We see that the combination of two Majorana fermions creates a a normal fermionic mode, which is either singularly occupied or empty. Therefore, a single Majorana fermion can be thought of as half of a normal fermionic state. In order to generate a legitimate fermionic mode of zero energy in a $p+i p$ superconductor, there must be a linear combination of two zero energy (Majorana) modes bound to two well-separated and distinct $\frac{1}{2} \mathrm{QVs}[41]$.

### 1.7 Vortex Energetics

The energy of a vortex depends on the type of vortex and the geometry within which it is confined. It is important to know whether half-quantum vortices or full quantum vortices have lower excitation energies, since the existence of both are permitted
in $p+i p$ superconductors. We have already seen that vortices are only possible in type II superconductors. An isolated full vortex costs an energy that diverges logaritmically [see equation (2.5)] with system size, and it follows that single vortices do not arise in the thermodynamic limit in a superconductor. Pairs of vortices, on the other hand, arise as thermal fluctuations at low temperatures, because the energy cost is finite as long as the net vorticity of the pair is zero. We will now consider the energetics of vortices in two different confining geometries: vortices in a bulk superconductor (3D), known as Abrikosov vortices, and vortices in thin superconducting films (2D), known as Pearl vortices. The interaction energies below are the unrenormalized expressions. Vortex screening of the vortex-vortex interaction will be dealt with in subsequent chapters.

### 1.7.1 Abrikosov Vortices

In a 3 D superconductor the lines of flux quanta that penetrate in small microscopic regions of radius $\xi$ are known as Abrikosov vortices. There is energy associated with the circulation of supercurrents around the core of a vortex. The contributions to the energy come from both the magnetic field energy and the kinetic energy of the moving particles. In the London limit of a type II superconductor $(\lambda \gg \xi)$, the energy per unit length (line energy) of an Abrikosov vortex (AV) is[8]

$$
\begin{equation*}
E_{\mathrm{AV}}=\frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda^{2}} \ln \frac{\lambda}{\xi} \tag{1.12}
\end{equation*}
$$

where $\xi$ is the core radius, $\Phi_{0}$ is the flux quantum, and $\lambda$ is the bulk penetration depth. The interaction energy (per unit length) between two like Abrikosov vortices, that is, vortices with flux orientated in the same direction a distance $r$ apart is

$$
\begin{equation*}
U_{\mathrm{AV}}^{+}=\frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda^{2}} 2 K_{0}\left(\frac{r}{\lambda}\right) \tag{1.13}
\end{equation*}
$$

where $K_{0}(x)$ is a zeroth-order Hankel function of imaginary argument[3]. The interaction is repulsive for like vortices, and attractive for oppositely charged vortices. Physically, the source of interaction is magnetic, where the charged Cooper pairs constituting the supercurrent of the first vortex experience a Lorentz force due to the magnetic field of the second vortex. The energy per unit length of a vortex-antivortex pair follows from (1.12) and (1.13),

$$
\begin{equation*}
E_{\mathrm{AV}}^{ \pm}=2 \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda^{2}}\left(\ln \frac{\lambda}{\xi}-K_{0}\left(\frac{r}{\lambda}\right)\right) \tag{1.14}
\end{equation*}
$$

The second term in (1.14) is the attractive interaction between two vortices, and has the following limiting forms,

$$
U_{\mathrm{AV}}^{ \pm}= \begin{cases}\frac{\Phi_{0}^{2}}{8 \pi^{2} \lambda^{2}} \ln \frac{r}{\lambda} & \text { if } \xi \ll r \ll \lambda  \tag{1.15}\\ -\frac{\Phi_{0}^{2}}{8 \pi^{2} \lambda^{2}}\left(\frac{\pi \lambda}{2 r}\right)^{1 / 2} e^{-r / \lambda} & \text { if } r \gg \lambda\end{cases}
$$

Abrikosov vortices interact logarithmically at small distances, but the interaction decreases exponentially at large distances. The reason for this is that the magnetic field of a vortex is exponentially screened with distance from entering the bulk, and so the field seen at large distances is very small, making the interaction very weak.

### 1.7.2 Pearl Vortices

The interaction of vortices in 2D superconductors was first obtained in closed analytic form by J. Pearl[45]. The interaction is long-ranged, as opposed to the shortranged interaction of Abrikosov vortices. The difference stems from the fact that in thin
superconducting films, the currents which circulate around vortex cores live largely on the surface of the film and thus persist without screening to much larger distances. This allows Pearl vortices to interact logarithmically over large distance scales. By applying a magnetic field perpendicular to a thin film, screening currents will be automatically generated to ensure that field only penetrates the film in small localized regions of quantized flux, which are referred to as Pearl vortices.

The total amount of flux penetrating through a Pearl vortex is $\Phi_{0}$, and it is distributed over the characteristic length $\lambda_{\perp}$. The flux is concentrated in a core region of radius $\xi$, but small amounts of flux will leak into the superconductor up to the characteristic length $\lambda_{\perp}$. It was shown by Pearl[46] that this length, called the effective transverse penetration depth is given by

$$
\begin{equation*}
\lambda_{\perp}=\frac{\lambda^{2}}{d} \tag{1.16}
\end{equation*}
$$

where $d$ is the film thickness and $\lambda$ is the bulk penetration depth. For very thin films, this penetration depth can be substantial. The energy of a Pearl vortex (PV) is $[7]$

$$
\begin{equation*}
E_{\mathrm{PV}}=\frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}} \ln \frac{\lambda_{\perp}}{\xi} \tag{1.17}
\end{equation*}
$$

This is exactly the same form as (1.12), which is to be expected. The interaction energy between two like Pearl vortices is of the form[47]

$$
\begin{equation*}
U_{\mathrm{PV}}^{+}=-\pi \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}}\left\{N_{0}\left(\frac{r}{2 \lambda_{\perp}}\right)-H_{0}\left(\frac{r}{2 \lambda_{\perp}}\right)\right\} \tag{1.18}
\end{equation*}
$$

where $N_{0}(r)$ and $H_{0}(r)$ are the Neumann and Struve functions respectively. The interaction energy for a Pearl vortex-antivortex pair now follows from (1.17) and (1.18),

$$
\begin{equation*}
E_{\mathrm{PV}}^{ \pm}=\frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}}\left[2 \ln \frac{\lambda_{\perp}}{\xi}+\pi\left\{N_{0}\left(\frac{r}{2 \lambda_{\perp}}\right)-H_{0}\left(\frac{r}{2 \lambda_{\perp}}\right)\right\}\right] \tag{1.19}
\end{equation*}
$$

The second term in (1.19) is the interaction energy, which has the limiting forms

$$
U_{\mathrm{PV}}^{ \pm}= \begin{cases}\frac{\Phi_{0}^{2}}{8 \pi^{2} \lambda_{\perp}^{2}} \ln \frac{r}{\lambda_{\perp}} & \text { if } \xi \ll r \ll \lambda_{\perp}  \tag{1.2}\\ -\frac{\Phi_{0}^{2}}{4 \pi^{2} r} & \text { if } r \gg \lambda_{\perp}\end{cases}
$$

The interaction is logarithmic at short distances, and exhibits a power law decay at long distances.

The long range behavior of the interaction is due to the fact there is far less screening in the 2D case than in the 3D case. In the 2D case, vortices can interact via the distorted magnetic field above and below the thin film superconductor, for which there is no screening effects. In a thin superconducting film $\lambda_{\perp}$ is typically of the order of centimeters, in which case one can fabricate samples where the distance between vortices is always much less than the effective penetration depth[48]. Therefore, the interaction energy of Pearl vortices can always be taken to be purely logarithmic, since any finite sample corresponds to the regime where $r \ll \lambda_{\perp}$. We can now write the energy (1.19) of a pair of Pearl vortices in the simple form

$$
\begin{equation*}
E_{\mathrm{PV}}^{ \pm}=2 \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}} \ln \frac{r}{\xi} \tag{1.21}
\end{equation*}
$$

We see that the energy of a pair of Pearl vortices is logarithmic in the regime $\xi \ll r \ll \lambda_{\perp}$. In order for there to be a legitimate Kosterlitz-Thouless phase transition, the logarithmic interaction must persist to all length scales in a thermodynamically large system. Since the interaction decays as a power law at large length scales, there will be no singular behavior at the transition, but instead a crossover. However, since
we are dealing with finite-sized 2D films where the transverse penetration depth $\lambda_{\perp}$ can greatly exceed its lateral dimensions, it becomes impossible to experimentally distinguish between a true phase transition and a crossover since all divergences and non-analytic points will be smoothed over.

### 1.7.3 Half-Quantum Pearl Vortices

Up until this point we have only been discussing the energetics of full-quantum vortices. Let us now consider the energetics of half-quantum Pearl vortices in 2D superconductors. Hereafter we will simply refer to these as $\frac{1}{2}$ QVs. Taking the result (1.19), we first simply make the substitution $\Phi_{0} \rightarrow \Phi_{0} / 2$ to account for half quantum vortices. However, there is an additional correction due to the presence of spin current in half-quantum vortices. The spin current energy of a $\frac{1}{2} \mathrm{QV}$ diverges logarithmically as [16, 28]

$$
\begin{equation*}
E_{\mathrm{spin}}=\frac{\pi}{4} \rho_{\mathrm{sp}} d\left(\frac{\hbar^{2}}{2 m}\right) \ln \frac{R}{\xi} \tag{1.22}
\end{equation*}
$$

where $R$ is the lateral size of the film and $d$ is its thickness. The quantity $\rho_{\mathrm{sp}}$ is the spin stiffness or spin fluid density. It is a measure of the energy cost for twisting the director vector $\hat{d}$. If the spin stiffness is lower than the superfluid stiffness, it is easier to twist the $\hat{\mathbf{d}}$ than it is to twist the superconducting phase $\theta$.

It is the absence of screening of the spin current which causes the divergence in the spin current energy. However, a pair of $\frac{1}{2} \mathrm{QVs}$ with opposite spin winding is free of such a divergent energy cost. The spin current energy of a pair of $\frac{1}{2} \mathrm{QVs}$ with opposite spin windings a distance $r$ apart is

$$
\begin{equation*}
E_{\mathrm{spin}}=\frac{1}{2} \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}} \frac{\rho_{\mathrm{sp}}}{\rho_{\mathrm{s}}} \ln \frac{r}{\xi} \tag{1.23}
\end{equation*}
$$

where we have used (1.4) and (1.16) to rewrite this expression in terms of $\rho_{\mathrm{sp}}$ and $\lambda_{\perp}$. In this expression we see the ratio of the spin stiffness to the superfluid stiffness. The stiffness is a measure of the change in ground state energy as a rotor twists. It measures the energy cost of applying a twist. In the limit where the twists vary slowly, the stiffness can be thought of as the rigidity of two well-separated rotors to relative twistings. In the case of superconductors, the rotor is $\theta$, the superfluid phase of the quantum mechanical wavefunction of the condensate. The superfluid stiffness $\rho_{\mathrm{s}}$ measures the phase rigidity for applied twists. In chiral p-wave superconductors, in addition to the superconducting phase $\theta$, there is also a director vector $\hat{\mathbf{d}}$ which can wind. The spin stiffness $\rho_{\text {sp }}$ is similarly a measure of the energy cost for applying a twist or gradient in $\hat{d}$. Thermal excitations will renormalize each of these stiffnesses, but their ratio should remain constant, with the spin stiffness $\rho_{\mathrm{sp}}$ less than the superfluid stiffness $\rho_{\mathrm{s}}[49]$.

By combining the spin current energy (1.23) with (1.19) modified with half a quantum flux, we find that the total energy of a pair of oppositely charged $\frac{1}{2} \mathrm{QVs}$ is

$$
\begin{equation*}
E_{\mathrm{HPV}}^{ \pm}=\frac{1}{2} \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}}\left(1+\frac{\rho_{\mathrm{sp}}}{\rho_{\mathrm{s}}}\right) \ln \frac{r}{\xi} \tag{1.24}
\end{equation*}
$$

The energy of half-quantum vortices in 2D p-wave superconductors is purely logarithmic in the asymptotic limit $\xi \ll r \ll \lambda_{\perp}$. This limit can generally be applied to Pearl vortices since $\lambda_{\perp}$ typically exceeds the lateral dimensions of a 2 D film.

### 1.8 Conclusion

In this chapter we have introduced the various classes of superconductors and described their properties. We will be particularly interested in the thermodynamic properties of vortices in the subsequent chapters. The most interesting type of vortex is the $\frac{1}{2} \mathrm{QV}$. The $\frac{1}{2} \mathrm{QV}$ has exotic exchange statistics due to the presence of a bound Majorana fermion, and this type is excitation is only permitted in $p+i p$ superconducting materials. However, a $p+i p$ superconductor permits the existence of both half-quantum vortices and full-quantum vortices. At low temperatures, vortices will be produced as thermal fluctuations of neutral bound pairs of zero net vorticity. Vortices with the lowest energy are the relevant excitations The low energy physics of various $p+i p$ superconducting systems can be modeled in terms of the relevant excitations, which will be vortices with the lowest energy. It is easy to verify using (1.19) and (1.24) that $\frac{1}{2} \mathrm{QVs}$ are energetically favorable over full QVs as long as the condition $\rho_{\mathrm{sp}} / \rho_{\mathrm{s}}<3$ is satisfied. However, A.J. Leggett has shown[49] under very general conditions that $\rho_{s p} / \rho_{s}<1$ (the spin sector is always softer than the superfluid sector). Therefore, pairs of $\frac{1}{2}$ QVs with opposite windings are expected to be energetically favorable in thin superconducting films.

## Chapter 2

## The XY Model

### 2.1 Introduction

The two-dimensional classical xy model is a system of lattice spins which interact with nearest neighbors only. Each spin degree of freedom is continuous and the Hamiltonian for the system is

$$
\begin{equation*}
H=-J \sum_{\langle i j\rangle} \mathbf{S}_{i} \cdot \mathbf{S}_{j}=-J \sum_{\langle i j\rangle} \cos \left(\theta_{i}-\theta_{j}\right) \tag{2.1}
\end{equation*}
$$

where $\mathbf{S}_{i, j}$ denote classical spin variables of unit modulus, confined to rotations in the plane of the lattice. The sum over $\langle i j\rangle$ is restricted to nearest neighbors and for simplicity we consider a square lattice of $N$ spins. Since we are taking $N$ to be very large, we assume periodic boundary conditions which ensures there are exactly $2 N$ individual terms (interaction energies) that contribute to the sum in (2.1).

In addition to describing a lattice of nearest neighbor interacting spins, the xy model can also be used to describe a classical 2D Coulomb gas, the roughening transition of a fluctuating surface, and the physics of superfluid/superconducting thin films[50]. The partition function for a system of $N$ lattice spins (xy model) can be written as

$$
\begin{equation*}
Z=\int d \theta_{1} \ldots \int d \theta_{N} \exp \left[\beta J \sum_{\langle i j\rangle} \cos \left(\theta_{i}-\theta_{j}\right)\right] \tag{2.2}
\end{equation*}
$$

where $\beta=1 / k_{B} T$ and $0<\theta_{i}<2 \pi$ corresponds to the angle that the spin at site $i$ makes with some arbitrary axis. To simplify the following calculations, we assume that this 2D lattice is periodic, which physically means that the lattice is shaped like a torus.

Our purposes for studying the xy model is that this model can be used to describe a Josephson junction array (see Chapter 3). Strictly speaking, the xy model only applies to a Josephson junction array in the semi-classical limit, where the individual superconducting grains used to form the lattice are large enough that quantum fluctuations can be ignored. The classical xy model also does not account for any external magnetic field. When an external field is applied perpendicular to the plane of the array, the simple xy model is modified and becomes the frustrated xy model.

A frustrated model is condensed matter physics is one in which competing forces forbid the simultaneous minimization of the interaction energies at any given lattice site. In the case of the xy model, the application of a perpendicular magnetic field makes it impossible for all lattice bonds to be simultaneously minimized, even at zero temperature. This in turn leads to nonzero entropy and degenerate ground states at zero temperature, which physically means that there is no unique state into which the system freezes at low temperatures. These additional complications will be studied further in Chapter 3. The details of the low energy physics and phase transitions in the xy model lay the ground work for understanding and presenting the central results of this thesis. The major focus of this thesis is the study of phase transitions and low energy physics in p-wave Josephson junction arrays, and the major results of this work can be found in Chapters 4-6.

In this chapter we will review in detail the Kosterlitz-Thouless (KT) phase transition of the xy model, as originally proposed by [51]. A complete renormalization group treatment of the KT transition will also be reviewed (accounts of this procedure for the xy model can also be found in $[50,52,53])$. The details of this calculation will be important when we apply a similar procedure to include Majorana fermions and half-quantum vortices (see Section 2.9 and Chapter 6 respectively). The major result of this chapter is found in Section 2.9, where we allow Majorana modes to bind to vortex cores in 2D spin-polarized chiral p-wave superconductors. We find that the additional entropy generated via vortex Majorana modes has a small effect (around $2 \%$ ) on the KT transition of full-quantum vortices.

### 2.2 Kosterlitz-Thouless Transition

There is a remarkable phase transition that occurs in the xy model which is not associated with the usual macroscopic, spontaneous magnetization. In fact, the Mermin-Wagner-Hohenberg theorem guarantees that a two dimensional system with continuous symmetry cannot have an ordered phase[54, 55]. This theorem tells us that there cannot be any broken continuous symmetry at finite temperature in dimensions less than or equal to two, because low energy spin wave excitations will destroy long range order. Therefore, the phase transition in the xy model is not a conventional phase transition, it turns out to be a topological phase transition known as the KosterlitzThouless (KT) transition.

To understand the KT transition, we need to consider the correlation function in the xy model. In Appendices A and B we calculate the correlation function in the high and low temperature regimes, and find that its behavior differs drastically:

$$
G\left(\mathbf{r}_{1}-\mathbf{r}_{2}\right)= \begin{cases}\left|\frac{r_{0}}{\mathbf{r}_{1}-\mathbf{r}_{2}}\right|^{1 / 2 \pi \beta J} & \text { if } k_{B} T \ll J  \tag{2.3}\\ \exp \left[-\frac{\left|\mathbf{r}_{1}-\mathbf{r}_{2}\right|}{\xi}\right] & \text { if } k_{B} T \gg J\end{cases}
$$

At high temperatures, the correlations decay exponentially, as one would expect for the usual paramagnetic phase (see Appendix B). At low temperatures, the correlation function decays as a power law, and the system is said to have quasi-long range order (see Appendix A). Since it is known that the correlation function is dominated by exponential decay for sufficiently high temperatures, there must be a phase transition at some point where a crossover from power law to exponential decay occurs.

This transition will not involve normal ordering, because the Mermin-WagnerHohenberg theorem forbids it. Rather, this phase transition will involve a novel type of ordering known as topological order. Each $\theta$ in the xy model is a periodic variable which should be identified modulo $2 \pi$. This turns out to be of significance because one can imagine low energy configurations where all neighboring spins $\theta$ point nearly along the same direction, but where $\theta$ changes by a non-zero multiple of $2 \pi$ if we trace its behavior along some large closed contour. These types of configurations, see Figure 2.1 for an example, are topological excitations known as vortices. Notice that these types of excitations are topologically stable, they cannot by undone by any smooth change of the order parameter. Vortices in which the order parameter winds by $2 \pi$ upon circulating once around the core are unit vortices, and carry one unit of topological charge. A vortex which involves rotating $\theta$ by some higher integer multiple of $2 \pi$ are well-defined excitations, but are typically much more costly in terms of energy.

The energy of a single vortex is found by expanding the hamiltonian for the xy model around the ground state. After expanding the cosine in (2.1), we can write[50]


Figure 2.1: The angle $\theta$ in each of these vortices winds by $2 \pi$ when we travel around any closed contour that includes the center of the vortex. These vortices each carry a single unit of topological charge. This entails that they are topologically equivalent, which means each vortex can be continuously deformed until it looks identical to the second. However, neither vortex can be undone by any smooth and continuous deformations, making them stable topological excitations.

$$
\begin{equation*}
E_{\mathrm{V}}=H-E_{0}=\frac{J}{2} \sum_{\langle i j\rangle}\left(\theta_{i}-\theta_{j}\right)^{2} \tag{2.4}
\end{equation*}
$$

where $E_{0}$ is simply the ground state energy, and $E_{\mathrm{V}}$ is the excitation energy of a vortex.
For a large system, the lattice constant, denoted as $a$, between spins is very small and it is convenient to write (2.4) in continuous notation,

$$
\begin{align*}
E_{\mathrm{V}} & \simeq \frac{J}{2} \int d^{2} r\left(\frac{\theta_{i}-\theta_{j}}{a}\right)^{2} \\
& \simeq \frac{J}{2} \int d^{2} r\left(\nabla \theta_{r}\right)^{2} \\
& =\frac{J}{2} \int d^{2} r\left(\frac{2 \pi n}{2 \pi r}\right)^{2} \\
& =\frac{J n^{2}}{2} \int_{0}^{2 \pi} d \phi \int_{a}^{R} r d r\left(\frac{1}{r^{2}}\right) \\
& =\pi J n^{2} \ln \frac{R}{a} \tag{2.5}
\end{align*}
$$

In the third line, we have considered the case of a circular contour, in which case the
rate of change of $\theta$ is given by the total change of $\theta=2 \pi n$ in going around the vortex core, divided by the length of the contour. The quantity $n$ is known as the vorticity or winding number of a vortex. The distance $R$ is the radius of the sample, and we see that the energy of a single vortex diverges with the system size.

Single vortices cost an infinite amount of energy in the thermodynamic limit and should not appear as excitations. However, it can be shown that a system of two vortices has finite energy as long as the total vorticity vanishes. Two vortices, for example, one with positive vorticity $(n=+1)$ and one with negative vorticity ( $n=-1$ ) would only cost a finite energy, even if the system was infinite in extent. Therefore, we do expect to see vortices being generated in pairs at low temperatures, even though single vortices are infinitely costly in terms of energy.

There is a simple energy-entropy argument known as the Kosterlitz-Thouless criterion[51] which determines the temperature at which vortex generation becomes important. A single vortex can live in roughly $R^{2} / a^{2}$ different locations, and so its entropy is

$$
\begin{equation*}
S_{\mathrm{V}}=2 k_{B} \ln \frac{R}{a} \tag{2.6}
\end{equation*}
$$

Note that $R$ is the size of the sample and $a$ is the lattice constant. In thermal equilibrium, the Helmholtz free energy will tend towards the global minima by minimizing the system's energy and maximizing it's entropy. The free energy for a single vortex is

$$
\begin{equation*}
F=E_{\mathrm{V}}-T S_{\mathrm{V}}=\left(\pi J n^{2}-2 k_{B} T\right) \ln \frac{R}{a} \tag{2.7}
\end{equation*}
$$

From this expression we see that when $T<\pi J / 2 k_{B}$ the free energy is positive and the existence of free unit vortices $(n= \pm 1)$ is highly improbable. In this temperature regime
it is energetically very costly for single vortices to occur as thermal fluctuations, and they are therefore suppressed. Although free vortices are strongly suppressed in this low temperature regime, tightly bound vortices can form dipole pairs of zero net vorticity, since the energy cost is finite. When $T>\pi J / 2 k_{B}$, the free energy is negative and single vortices with $n= \pm 1$ begin to appear in the system. The temperature at which vortices first begin to proliferate is

$$
\begin{equation*}
T_{\mathrm{KT}}=\frac{\pi J}{2 k_{B}} \tag{2.8}
\end{equation*}
$$

This is known as the Kosterlitz-Thouless transition temperature[51]. It is important to note that (2.8) is not exact, but is a first-order estimate of the vortex proliferation temperature. A more detailed analysis, which includes the screening effect of vortices, must be introduced in order to improve upon the first estimate.

The physical picture of the Kosterlitz-Thouless transition is that at low temperatures vortices emerge as neutral bound pairs. These dipoles carry zero net vorticity and single vortices (free vortex charges) are prohibited. This can be thought of as similar to the insulating phase of a metal. Above the transition temperature, the proliferation of vortices begins and hence the number of effective free vortex charges increases very rapidly. This phase can be thought of as the metallic phase of a metal; as positive and negative vortices become free to move about, they are able to screen each other. The above argument has ignored the screening interaction of vortices, therefore the result in (2.8) above gives only an upper bound on the critical temperature. A more accurate result requires a renormalization group treatment. Nevertheless, we have argued that there is a phase transition of topological nature, involving the unbinding of topological defects. In fact, it turns out that (2.8) gives a very good first estimate for the

Kosterlitz-Thouless transition temperature.

### 2.3 Vortex Excitations in the XY Model

In Appendix $A$ and $B$ we show the closed analytic expressions for the correlation function in both the low temperature and the high temperature limit [see (A.25) and (B.8) respectively]. At low temperature, the alignment of spins is the energetically favored configuration, which allowed to us to replace the exact partition function (2.2) by a pure Gaussian model (A.2), which is so-called the spin-wave approximation. However, in this treatment, we have neglected the $2 \pi$-periodic character of the angles $\theta_{i}$. By identifying the spins modulo $2 \pi$, one finds that topological defects begin to appear in the system. These topological defects are vortex excitations, which interact with each other and undergo a Kosterlitz-Thouless vortex unbinding transition when the temperature crosses over from a low temperature regime to a high temperature regime.

In order to allow for the existence of vortices, we must expand the partition function (2.2) in a manner that allows the angles $\theta_{i}$ to be periodic. To start this procedure, it will be useful to perform a duality transformation[50], which in effect maps the problem with variables defined on each lattice site to dual variables which live in between the lattice sites. These dual variables are defined on each plaquette and now characterize the degrees of freedom for low energy excitations in the system. To achieve the so-called duality mapping, we first consider the low temperature limit. In the low temperature limit, the spins are correlated and we can write $\theta_{i}-\theta_{j}=2 \pi q_{i j}+\epsilon_{i j}$, where $\epsilon_{i j} \ll 1$ and $q_{i j}$ is an integer characterizing the nearest neighbor phase difference for sites $i$ and $j$. We can now expand the cosine and write

$$
\begin{align*}
\cos \left(\theta_{i}-\theta_{j}\right) & =1-\frac{\epsilon_{i j}^{2}}{2} \\
& =1-\frac{\left(\theta_{i}-\theta_{j}-2 \pi q_{i j}\right)^{2}}{2} \tag{2.9}
\end{align*}
$$

The cosine is periodic in $2 \pi$, and at low temperatures neighboring spins will be nearly aligned, meaning that $\theta_{i}-\theta_{j}$ will be very close to zero, or in principal some integer multiple of $2 \pi$. Therefore, in order to retain important phase information we should allow for all possible integer values of $q_{i j}$ in the partition function by summing over all possible values[56, 52, 53]. This allows one to write the partition function (2.2) in the form

$$
\begin{equation*}
Z=\sum_{\left\{q_{i j}\right\}} e^{2 N \beta J} \int d \theta_{1} \cdots \int d \theta_{N} \exp \left[-\frac{\beta J}{2} \sum_{\langle i j\rangle}\left(\theta_{i}-\theta_{j}-2 \pi q_{i j}\right)^{2}\right] \tag{2.10}
\end{equation*}
$$

which is essentially a Gaussian approximation valid at low temperatures. The sum over $\left\{q_{i j}\right\}$ means that for each and every nearest neighbor link $q_{i j}$, we are summing over all possible integers. Explicitly, this means that each $q_{i j}$ link variable is summed from $q_{i j}=(-\infty \cdots-1,0,1 \cdots \infty)$. This transformation to a sum over integers is known as the Villain model[57], an approximation which picks out every periodic minima in the xy model centered about $2 \pi q_{i j}$.

A useful mathematical identity which allows us to rewrite (2.10) is Poisson's equation[53]

$$
\begin{equation*}
\sum_{q=-\infty}^{\infty} h(q)=\sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} d \tau h(\tau) e^{2 \pi i \ell \tau} \tag{2.11}
\end{equation*}
$$

which immediately casts (2.10) into the form

$$
\begin{equation*}
Z=\sum_{\left\{\ell_{i j}\right\}} e^{2 N \beta J} \int d \theta_{1} \cdots \int d \theta_{N} \int_{-\infty}^{\infty} \prod_{\langle i j\rangle} d \tau_{i j} \exp \left[-\frac{\beta J}{2} \sum_{\langle i j\rangle}\left(\theta_{i}-\theta_{j}-2 \pi \tau_{i j}\right)^{2}+2 \pi i \ell_{i j} \tau_{i j}\right] \tag{2.12}
\end{equation*}
$$

where each $\ell_{i j}$ is a new link variable, each being summed over all integers. To simplify this expression we can immediately factorize the integrand into a product of $2 N$ exponentials each containing only a single $\tau_{i j}$. Each $\tau_{i j}$-integration can easily be performed,

$$
\begin{equation*}
\int_{-\infty}^{\infty} d \tau_{i j} \exp \left[-\frac{\beta J}{2}\left(\theta_{i}-\theta_{j}-2 \pi \tau_{i j}\right)^{2}+2 \pi i \ell_{i j} \tau_{i j}\right]=\sqrt{\frac{1}{2 \pi \beta J}} \exp \left[-\left(\frac{\ell_{i j}^{2}}{2 \beta J}-i \ell_{i j}\left(\theta_{i}-\theta_{j}\right)\right)\right] \tag{2.13}
\end{equation*}
$$

There are a total of $2 N$ of these Gaussian $\tau_{i j}$-integrals, and hence we can write the partition function as

$$
\begin{equation*}
Z=\left(\frac{e^{2 \beta J}}{2 \pi \beta J}\right)^{N} \int d \theta_{1} \cdots \int d \theta_{N} \sum_{\left\{\ell_{i j}\right\}} \exp \left[-\sum_{\langle i j\rangle}\left(\frac{\ell_{i j}^{2}}{2 \beta J}-i \ell_{i j}\left(\theta_{i}-\theta_{j}\right)\right)\right] \tag{2.14}
\end{equation*}
$$

It is useful to write the variable $\ell_{i j}$, which is defined on every link, as a vector field $\ell_{\mu}(r)$, where $r$ denotes the lattice site and the index $\mu=x, y$ directs the bond from the lattice site $r$ to either the nearest neighbor on the right (x-direction) or the top (y-direction). By this prescription, each lattice site $r$ has two lattice bonds $\ell_{\mu}(r)$ associated with it, which avoids double counting of bonds. This allows us to rewrite the exponent in (2.14) as

$$
\begin{equation*}
-\sum_{r, \mu}\left[\frac{\ell_{\mu}^{2}(r)}{2 \beta J}-i \ell_{\mu}(r)\left(\theta_{r}-\theta_{r+\mu}\right)\right]=-\sum_{r, \mu}\left[\frac{\ell_{\mu}^{2}(r)}{2 \beta J}-i\left(\ell_{\mu}(r)-\ell_{\mu}(r-\mu)\right) \theta_{r}\right] \tag{2.15}
\end{equation*}
$$

where it is important to recognize that $r$ and $\mu$ are both vectors. The sum in the second term on the right hand side of (2.15) has been regrouped into terms that contain only
one lattice site. By plugging (2.15) into (2.14) we can factorize the exponent and then perform the $\theta_{r}$-integration $N$ times which leads simply to the result

$$
\begin{equation*}
Z=\left(\frac{e^{2 \beta J}}{\beta J}\right)^{N} \sum_{\left\{\ell_{\mu}(r)\right\}} \exp \left[\sum_{r, \mu}-\frac{\ell_{\mu}^{2}(r)}{2 \beta J}\right] \prod_{r} \delta_{\left\{\sum_{\mu} \ell_{\mu}(r)-\ell_{\mu}(r-\mu)\right\}, 0} \tag{2.16}
\end{equation*}
$$

where

$$
\delta_{i j}= \begin{cases}1 & \text { if } i=j  \tag{2.17}\\ 0 & \text { otherwise }\end{cases}
$$

is the Kronecker delta function.
It would be useful if one could remove the product of delta functions in the partition function by automatically satisfying the condition

$$
\begin{equation*}
\sum_{\mu} \ell_{\mu}(r)-\ell_{\mu}(r-\mu)=0 \tag{2.18}
\end{equation*}
$$

present in the delta function of (2.16). This can in fact be done by rewriting the partition function (2.16) in terms of dual variables. Notice that (2.18) is really a discretized version of a condition for zero-divergence in the vector field $\ell_{\mu}(r)$. By defining the vector $\vec{\ell}_{r} \equiv\left(\ell_{x}(r), \ell_{y}(r)\right)$, the condition (2.18) can be written in the continuous form

$$
\begin{equation*}
\vec{\nabla} \cdot \vec{\ell}_{r}=0 \tag{2.19}
\end{equation*}
$$

By the general principles of vector calculus, a divergentless vector field can be written as the curl of some new vector field,

$$
\begin{equation*}
\vec{\ell}_{r}=\vec{\nabla} \times \vec{\eta}_{r} \tag{2.20}
\end{equation*}
$$

where $\vec{\eta}_{r}=\left(0,0, \eta_{r}\right)$ is a one-component field. We can spell out the connection between the components of $\ell_{\mu}(r)$ and the scalar $\eta_{r}$ defined in (2.20) explicitly as

$$
\begin{align*}
\ell_{x}(r) & =\eta_{r}-\eta_{r-y} \\
\ell_{y}(r) & =-\eta_{r}+\eta_{r-x} \\
\ell_{z}(r) & =0 \tag{2.21}
\end{align*}
$$

By plugging these relations directly into (2.18), we see that $\eta_{r}$ automatically satisfies the condition for zero-divergence. This allows us to write the partition function (2.16) in a much simpler form,

$$
\begin{equation*}
Z=\left(\frac{e^{2 \beta J}}{\beta J}\right)^{N} Z^{\prime} \tag{2.22}
\end{equation*}
$$

where

$$
\begin{equation*}
Z^{\prime}=\sum_{\left\{\eta_{r}\right\}} \exp \left[-\sum_{r, \mu} \frac{1}{2 \beta J}\left(\eta_{r}-\eta_{r-\mu}\right)^{2}\right] \tag{2.23}
\end{equation*}
$$

An important step has been made here in that the partition function is now expressed in terms of integer-valued variables $\eta_{r}$ defined at site $r$ on the dual lattice. The dual lattice is the lattice with a site $r$ in each elementary plaquette of the original lattice (see Figure 2.2). It is a simple matter to show that the number of degrees of freedom has been preserved in the duality transformation of this problem. Initially, we started out with the xy model with N degrees of freedom, corresponding to each lattice spin, and then the problem was mapped onto $2 N$ different $\ell_{\mu}(r)$ lattice bond variables with $N$ constraints given by (2.18), leading to $2 N-N=N$ degrees of freedom. Finally, a duality transformation to $N$ independent variables $\eta_{r}$ was made, and hence the number


Figure 2.2: The original lattice is defined by the continuous variable $\theta_{r}$ which is defined at every point. The dual lattice is defined by the discrete variables $\eta_{r}$ which live in the plaquette of the original lattice, labeled here with a cross. Each cross can be thought of as a small piece of a much larger fluctuating surface (see surrounding text for details).
of degrees of freedom has been preserved in the problem.

The xy model originally referred to spins living on the square lattice, but the duality transformation has mapped the spins to dual variables, $\eta_{r}$, which live on the dual lattice (in plaquettes). A physical interpretation of the dual model is the so-called solid-on-solid model[50]. Each variable $\eta_{r}$ can be thought of as the height in the z-direction of a small piece of a large surface. Fluctuations in the height of the surface from square to square costs the system surface energy, proportional to $\left(\eta_{r}-\eta_{r-\mu}\right)^{2}$. This model then describes the roughening transition of a surface. At low temperatures, the Gaussian approximation in the xy model is good, and small fluctuations $(\delta \theta)$ of neighboring spins dominate the partition sum. Conversely, since $\beta$ appears in the denominator of (2.23), large fluctations $(\delta \eta)$ in $\eta$ dominate the partition sum, which physically corresponds to a rough surface state. Thus the dual model describes a rough surface at low temperature and a smooth surface at high temperature, indicating the presence of a phase transition in the intermediate regime. to obtain

$$
\begin{equation*}
Z^{\prime}=\int_{-\infty}^{\infty} \prod_{r} d \tau_{r} \sum_{\left\{m_{r}\right\}} \exp \left[-\frac{1}{2 \beta J} \sum_{r, \mu}\left(\triangle_{\mu} \tau_{r}\right)^{2}+2 \pi i \sum_{r} m_{r} \tau_{r}\right] \tag{2.24}
\end{equation*}
$$

where $\triangle_{\mu} \tau_{r}=\tau_{r}-\tau_{r-\mu}$ is a lattice derivative. It is possible to solve this N -dimensional integral by Fourier transform. By substituting (A.5) into the first term and using (A.6), it is a simple matter to show that

$$
\begin{equation*}
\sum_{r, \mu}\left(\triangle_{\mu} \tau_{r}\right)^{2}=\frac{1}{(2 \pi)^{2}} \int d^{2} k\left(4-2 \cos k_{x}-2 \cos k_{y}\right) \tau_{k} \tau_{-k} \tag{2.25}
\end{equation*}
$$

where $k$ has been rescaled into a dimensionless variable $(k a \rightarrow k)$ and the limits of integration extend from $-\pi<k_{x}, k_{y}<\pi$. Similarly, the second term can be shown to yield

$$
\begin{equation*}
\sum_{r} m_{r} \tau_{r}=\frac{1}{(2 \pi)^{2}} \int d^{2} k m_{k} \tau_{-k} \tag{2.26}
\end{equation*}
$$

where $m_{k}$ is the Fourier transform of $m_{r}$, defined in exactly the same manner as (A.4). We now combine (2.25) and (2.26) and rewrite the total exponent in (2.24) as

$$
\begin{equation*}
-\frac{1}{(2 \pi)^{2}} \int d^{2} k\left\{\frac{1}{2 \beta J}\left(4-2 \cos k_{x}-2 \cos k_{y}\right) \tau_{k} \tau_{-k}-2 \pi i m_{k} \tau_{-k}\right\} \tag{2.27}
\end{equation*}
$$

By making the substitution

$$
\begin{equation*}
\psi_{k}=\tau_{k}-\frac{4 \pi i \beta J}{4-2 \cos k_{x}-2 \cos k_{y}} m_{k} \tag{2.28}
\end{equation*}
$$

and completing the square in (2.27), the total exponent can be factorized and written, after some algebra, as

$$
\begin{align*}
& \exp \left[-\frac{1}{8 \pi^{2} \beta J} \int d^{2} k\left(4-2 \cos k_{x}-2 \cos k_{y}\right) \psi_{k} \psi_{-k}\right] \times \\
& \exp \left[-\frac{1}{(2 \pi)^{2}} \int d^{2} k \frac{2 \pi^{2} \beta J}{4-2 \cos k_{x}-2 \cos k_{y}} m_{k} m_{-k}\right] \tag{2.2}
\end{align*}
$$

We now proceed to Fourier transform $m_{k}$ back to $m_{r}$ via (A.5), the second term in (2.29) becomes

$$
\begin{equation*}
\exp \left[-2 \pi^{2} \beta J \sum_{r, r^{\prime}} m_{r} G_{r-r^{\prime}} m_{r^{\prime}}\right] \tag{2.30}
\end{equation*}
$$

where

$$
\begin{equation*}
G_{r-r^{\prime}}=\int_{-\pi}^{\pi} \frac{d k_{x}}{2 \pi} \int_{-\pi}^{\pi} \frac{d k_{y}}{2 \pi}\left(\frac{e^{i \mathbf{k} \cdot\left(\mathbf{r}-\mathbf{r}^{\prime}\right) / a}}{4-2 \cos k_{x}-2 \cos k_{y}}\right) \tag{2.31}
\end{equation*}
$$

If we define

$$
\begin{equation*}
Z_{V}=\sum_{\{m(r)\}} \exp \left[-2 \pi^{2} \beta J \sum_{r, r^{\prime}} m_{r} G_{r-r^{\prime}} m_{r^{\prime}}\right] \tag{2.32}
\end{equation*}
$$

then combining our results thus far, (2.24) can be written in the form

$$
\begin{equation*}
Z^{\prime}=Z_{V}\left(\int_{-\infty}^{\infty} \prod_{r} d \varphi_{r} \exp \left[-\frac{a^{2}}{8 \pi^{2} \beta J} \int d^{2} k\left(4-2 \cos k_{x} a-2 \cos k_{y} a\right) \psi_{k} \psi_{-k}\right]\right) \tag{2.33}
\end{equation*}
$$

Our goal now is to simplify the bracketed term in (2.33). First we employ the prescription $\int d^{2} k \rightarrow \sum_{k} \frac{(2 \pi)^{2}}{L^{2}}=\sum_{k} \frac{(2 \pi)^{2}}{N}$, where $L$ is the linear dimension of the system, and $N$ is the number of degrees of freedom. Note that $L^{2}=N$ since we have scaled distance into the units of lattice spacing $a$. Let us now write the differential in terms of its real and imaginary Fourier components,

$$
\begin{equation*}
\prod_{r} d \varphi_{r} \rightarrow \frac{1}{2} \prod_{k} d\left(\Re \varphi_{k}\right) d\left(\Im \varphi_{k}\right) \rightarrow \frac{1}{2} \prod_{k} d\left(\Re \psi_{k}\right) d\left(\Im \psi_{k}\right) \tag{2.34}
\end{equation*}
$$

Now if we break $\psi_{k}$ into purely real and imaginary parts, i.e. $\psi_{k}=\Re \psi_{k}+i \Im \psi_{k}$, the integral in (2.33) factorizes into a product of $2 N$ identical Gaussian integrals, and a simple calculation yields

$$
\begin{equation*}
\frac{1}{2}(2 \pi N \beta J)^{N} \prod_{k} \frac{1}{4-2 \cos k_{x}-2 \cos k_{y}} \tag{2.35}
\end{equation*}
$$

By using the identity $\exp (\ln x)=x$, one can show that (2.35) can be written, after some simplifications, as

$$
\begin{equation*}
\frac{1}{2}\left\{(2 \pi N \beta J) \cdot \exp \left[\int_{-\pi}^{\pi} \frac{d k_{x}}{2 \pi} \int_{-\pi}^{\pi} \frac{d k_{y}}{2 \pi} \ln \left(4-2 \cos k_{x}-2 \cos k_{y}\right)\right]\right\}^{N} \tag{2.36}
\end{equation*}
$$

Putting everything together, we combine results (2.32), (2.33), and (2.36) with (2.22) which allows us to write the total partition function as

$$
\begin{equation*}
Z=Z_{\mathrm{V}} Z_{\mathrm{SW}} \tag{2.37}
\end{equation*}
$$

where

$$
\begin{equation*}
Z_{\mathrm{SW}}=\frac{1}{2}\left\{\left(2 \pi N e^{2 \beta J}\right) \cdot \exp \left[\int_{-\pi}^{\pi} \frac{d k_{x}}{2 \pi} \int_{-\pi}^{\pi} \frac{d k_{y}}{2 \pi} \ln \left(4-2 \cos k_{x}-2 \cos k_{y}\right)\right]\right\}^{N} \tag{2.38}
\end{equation*}
$$

is the contribution due to spin-wave excitations, and

$$
\begin{equation*}
Z_{\mathrm{V}}=\sum_{\left\{m_{r}\right\}} \exp \left[-2 \pi^{2} \beta J \sum_{r, r^{\prime}} m_{r} G_{r-r^{\prime}} m_{r^{\prime}}\right] \tag{2.39}
\end{equation*}
$$

is the contribution due to interacting vortices.
Our interest now is primarily in the vortex partition function $Z_{\mathrm{V}}$. In order to make use of (2.39), it is necessary to cast it into a more useful form, since $G_{r-r^{\prime}}$ is not well-behaved over its entire domain. The function $G_{r-r^{\prime}}$ defined in (2.31) is divergent
in the infrared (small wavevector $k$ ) regime. In order to peel of this divergent behavior, we can write

$$
\begin{equation*}
-2 \pi G_{r}=\Gamma_{r}-2 \pi G_{o} \tag{2.40}
\end{equation*}
$$

where

$$
\begin{equation*}
G_{o}=\int_{-\pi}^{\pi} \frac{d k_{x}}{2 \pi} \int_{-\pi}^{\pi} \frac{d k_{y}}{2 \pi}\left(\frac{1}{4-2 \cos k_{x}-2 \cos k_{y}}\right) \tag{2.41}
\end{equation*}
$$

captures the divergent behavior, and $\Gamma_{r}$ is a well-defined quantity, which can be seen to be

$$
\begin{equation*}
\Gamma_{r}=\int_{-\pi}^{\pi} \frac{d^{2} k}{2 \pi}\left(\frac{1-e^{i \mathbf{k} \cdot \mathbf{r} / a}}{4-2 \cos k_{x}-2 \cos k_{y}}\right) \tag{2.42}
\end{equation*}
$$

Note that this is the exact same function which was previously defined in (A.20), and hence also can be approximated by (A.22). Notice that the divergent piece $G_{o}$ is independent of $\mathbf{r}$ and so comes outside the sum in the exponent of (2.39). Therefore, the vortex partition function will be dominated by terms for which the condition

$$
\begin{equation*}
\sum_{r} m_{r}=0 \tag{2.43}
\end{equation*}
$$

is satisfied, which automatically ensures that this large divergent piece is suppressed. This is known as the neutrality condition, which physically means that the lowest energy excitations of the system are ones with zero net vorticity. When vortices arise as thermal fluctuations, they will always emerge as $+/-$ pairs, such that the total vorticity of the system is zero.

The vortex partition function (2.39) can now be written as

$$
\begin{equation*}
Z_{\mathrm{V}}=\sum_{\left\{m_{r}\right\}^{\prime}} \exp \left[\pi \beta J \sum_{r, r^{\prime}} m_{r} \Gamma_{r-r^{\prime}} m_{r^{\prime}}\right] \tag{2.44}
\end{equation*}
$$

The prime indicates that the sum over $\left\{m_{r}\right\}^{\prime}$ is restricted to the sets of $m_{r}$ that satisfy the neutrality condition (2.43). We can use the approximate form of $\Gamma_{r}$ given by (A.22), along with the neutrality condition (2.43), allowing us to write the exponent in (2.39) and (2.44) as

$$
\begin{aligned}
-2 \pi^{2} \beta J \sum_{r, r^{\prime}} m_{r} G_{r-r^{\prime}} m_{r^{\prime}} & =\pi \beta J \sum_{r, r^{\prime}} m_{r} \Gamma_{r-r^{\prime}} m_{r^{\prime}} \\
& =\pi \beta J \sum_{r \neq r^{\prime}} m_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} m_{r^{\prime}}+\pi \beta J \ln \frac{a}{r_{0}} \sum_{r \neq r^{\prime}} m_{r} m_{r^{\prime}}
\end{aligned}
$$

Now the first term can be written as

$$
\begin{equation*}
2 \pi \beta J \sum_{r \neq r^{\prime}}^{\prime} m_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} m_{r^{\prime}} \tag{2.45}
\end{equation*}
$$

where the prime indicates that we are restricting our sum to unique pairings of $r$ and $r^{\prime}$. The second term in (2.45) can be written as

$$
\begin{align*}
\pi \beta J \ln \frac{a}{r_{0}} \sum_{r \neq r^{\prime}} m_{r} m_{r^{\prime}} & =\pi \beta J \ln \frac{a}{r_{0}}\left[\sum_{r, r^{\prime}} m_{r} m_{r^{\prime}}-\sum_{r=r^{\prime}} m_{r} m_{r^{\prime}}\right] \\
& =\pi \beta J \ln \frac{a}{r_{0}}\left[\left(\sum_{r} m_{r}\right)^{2^{\prime}}-\sum_{r} m_{r}^{2}\right] \\
& =-\pi \beta J \ln \frac{a}{r_{0}} \sum_{r} m_{r}^{2} \tag{2.46}
\end{align*}
$$

where we have used the neutrality condition to drop the first term in second line above. Using (2.46) and (2.45) we can write the vortex partition function (2.39) in the very useful approximate form

$$
\begin{equation*}
Z_{\mathrm{V}}=\sum_{\left\{m_{r}\right\}^{\prime}} \exp \left[2 \pi \beta J \sum_{r \neq r^{\prime}}^{\prime} m_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} m_{r^{\prime}}-\pi \beta J \ln \frac{a}{r_{0}} \sum_{r} m_{r}^{2}\right] \tag{2.47}
\end{equation*}
$$

From (2.47) we are lead to the following physical picture: degrees of freedom $m_{r}$ represent vortices that live in elementary plaquettes and interact logarithmically with one another. The strength of a vortex $m_{r}$ can have any integer value, referred to as the vortex charge or vorticity. The system as a whole is subject to the constraint that the net vorticity is zero at low temperatures. Vortices with vorticity of $\pm 1$ are the most energetically favorable configuration, and a vortex will be attracted to an antivortex. This picture of logarithmically interacting vortices is analogous to a 2D Coulomb gas of interacting charged particles[58]. Therefore, both systems will be governed by the same physics and hence undergo the same type of phase transition.

Since the vortices interact logarithmically, we expect to see a Kosterlitz-Thouless phase transition. The energy of a pair of vortices a distance $r$ apart with opposite unit vorticity is

$$
\begin{equation*}
E_{\mathrm{V}}^{\text {pair }}=2 \pi J \ln \frac{r}{a}+2 \pi J \ln \frac{a}{r_{0}} \tag{2.48}
\end{equation*}
$$

which can be seen from (2.47). The interaction between oppositely charged vortices is attractive, however it is repulsive for like vortices. The interaction is indeed logarithmic and vortices will begin to proliferate at the transition temperature (2.8). A vortexantivortex pair is the most stable configuration below the Kosterlitz-Thouless transition, but above the transition pairs begin to unbind and free vortices begin to appear. The second term in (2.48) is a constant, and can be identified with the chemical potential of a pair of vortices. It follows that the chemical potential of a single vortex of unit vorticity is

$$
\begin{align*}
\mu & =\pi J \ln \frac{a}{r_{0}} \\
& =\pi J\left(\frac{3}{2} \ln 2+\gamma\right) \tag{2.49}
\end{align*}
$$

This number is positive and so it costs energy to add vortices to the system. The most important result of this section is the fact that the xy model can be mapped onto a system of logarithmically interacting vortices, described by the partition function (2.47).

### 2.4 Screening Effect of Vortices on Correlations

To include the effects of vortices in the calculation of the correlation function (A.13), we must go beyond the spin-wave approximation and use the vortex partition function (2.47) derived in the previous section. We will consider the more general correlation function,

$$
\begin{equation*}
g_{p}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\Re\left\langle e^{i p\left(\theta_{o}-\theta_{n}\right)}\right\rangle \tag{2.50}
\end{equation*}
$$

where $p$ is an arbitrary integer, which can later be set to unity. Throughout this section, we will generally follow the approach presented in the paper by L.P. Kadanoff et al.[56] and outlined in the book by C. Itzykson[52].

The calculation of the correlation function is most conveniently performed if we start with the paritition function in the form

$$
\begin{equation*}
Z=\left(\frac{e^{2 \beta J}}{2 \pi \beta J}\right)^{N} \sum_{\left\{\ell_{\mu}(r)\right\}} \int \prod_{r} d \theta_{r} \exp \left\{-\sum_{r, \mu}\left[\frac{\ell_{\mu}^{2}(r)}{2 \beta J}-i\left(\ell_{\mu}(r)-\ell_{\mu}(r-\mu)\right) \theta_{r}\right]\right\} \tag{2.51}
\end{equation*}
$$

In order to calculate the correlation function (2.50) with respect to the partition function (2.51), we first choose a specified path between sites $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$ which does not cross itself.

This allows us to write the exponent in (2.50) as

$$
\begin{equation*}
i p\left(\theta_{o}-\theta_{n}\right)=i p \sum_{j=1}^{n}\left(\theta_{j-1}-\theta_{j}\right) \tag{2.52}
\end{equation*}
$$

since all terms in the sum will cancel except the end points. We can now combine (2.52) and (2.51) and immediately evaluate (2.50) by performing all of the $\theta_{r}$-integrations. Just as in (2.16), the result gives a series of delta functions with various constraints similar to (2.18).

Previously, we found that these constraints could be automatically satisfied by introducing dual lattice variables defined in (2.21). The problem at hand has slightly more complex constraints, and one can show by performing the $\theta_{r}$-integration that by defining

$$
\begin{align*}
\ell_{x}(r) & =\eta_{r}-\eta_{r-y}+p \nu_{r, r-y} \\
\ell_{y}(r) & =-\eta_{r}+\eta_{r-x}+p \nu_{r-x, r} \\
\ell_{z}(r) & =0 \tag{2.53}
\end{align*}
$$

the constraints are automatically satisfied. In (2.53), the variable $\eta_{r}$ is again a dual lattice variable defined in each elementary plaquette, and $p$ is the integer introduced in (2.50). The other variable $\nu_{r r^{\prime}}$, is an antisymmetric dual lattice bond that points from site $\mathbf{r}$ to $\mathbf{r}^{\prime}$ on the dual lattice, and is non-vanishing only when $\mathbf{r}$ and $\mathbf{r}^{\prime}$ are nearest neighbors (see Figure 2.3). It can take on three discrete values, and is defined precisely as follows:


Figure 2.3: This is a picture of a particular dual lattice bond $\nu_{i j}$ that cuts through the path bond (for simplicity we haven't labeled the direction of the path bond). If the path bond is oriented from $\mathbf{r}_{o}$ to $\mathbf{r}_{n}$, then by the prescription laid out in the text, we conclude that $\nu_{i j}=1$.
$0 \quad$ if $\mathbf{r}, \mathbf{r}^{\prime}$ are not nearest neighbors OR if the dual bond
does not cross the specified path between sites $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$.
$\nu_{r r^{\prime}}=\left\{\begin{array}{l}1 \quad \text { if the bond does cross the path, orientated such that rotating the path } \\ \text { bond counterclockwise reproduces the directionality of the dual bond. }\end{array}\right.$
-1 if the bond does cross the path, orientated such that rotating the path
bond clockwise reproduces the directionality of the dual bond.
This duality mapping allows us to write (2.50) very simply as

$$
\begin{equation*}
Z g_{p}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\sum_{\left\{\eta_{r}\right\}} \exp \left[-\frac{1}{2 \beta J} \sum_{r, \mu}\left(\eta_{r}-\eta_{r-\mu}+p \nu_{r, r-\mu}\right)^{2}\right] \tag{2.54}
\end{equation*}
$$

This expression is now in a form similar to (2.23), and hence the procedure for rewriting this expression is analogous to that used in going from equation (2.23) to (2.39). We
start by using Poisson's equation (2.11) to rewrite (2.54) as

$$
\begin{equation*}
Z g_{p}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\int_{-\infty}^{\infty} \prod_{r} d \varphi_{r} \sum_{\left\{m_{r}\right\}} \exp \left[-\frac{1}{2 \beta J} \sum_{r, \mu}\left(\triangle_{\mu} \varphi_{r}+p \nu_{r, r-\mu}\right)^{2}+2 \pi i \sum_{r} m_{r} \varphi_{r}\right] \tag{2.55}
\end{equation*}
$$

Using (A.4) and (A.5) allows us to Fourier transform this equation, and after some tedious algebra, we find

$$
\begin{equation*}
Z g_{p}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=Z_{\mathrm{V}}^{\prime} Z_{\mathrm{SW}} \tag{2.56}
\end{equation*}
$$

where $Z_{\mathrm{SW}}$ is given by (2.38), and

$$
\begin{equation*}
Z_{\mathrm{V}}^{\prime}=\exp \left[-\frac{p^{2}}{2 \beta J} \sum_{\left\langle r, r^{\prime}\right\rangle} \nu_{r, r^{\prime}}^{2}\right] \sum_{\left\{m_{r}\right\}} \exp \left[\sum_{r, r^{\prime}}\left\{\frac{p^{2}}{2 \beta J} \chi_{r} \chi_{r^{\prime}}-2 \pi i p \chi_{r} m_{r^{\prime}}-2 \pi^{2} \beta J m_{r} m_{r^{\prime}}\right\} G_{r-r^{\prime}}\right] \tag{2.57}
\end{equation*}
$$

Notice that the last term in the exponent of (2.57) is exactly the same as the vortex partition function (2.39). The extra terms that appear are due to the presence of the term $p \nu_{r r^{\prime}}$ in (2.54). The variable $\chi_{r}$ has been defined here, and it is the inverse Fourier transform of $\chi_{k}$ which has been calculated to be

$$
\begin{equation*}
\chi_{k}=\sum_{r, \mu} \nu_{r, r-\mu} e^{i \mathbf{k} \cdot \mathbf{r}}\left(1-e^{-i \mathbf{k} \cdot \mu}\right) \tag{2.58}
\end{equation*}
$$

By taking the inverse Fourier transform (A.5) of the previous expression, one can show that $\chi_{r}$ is given by

$$
\begin{equation*}
\chi_{r}=\sum_{\left\langle r^{\prime}\right\rangle} \nu_{r r^{\prime}} \tag{2.59}
\end{equation*}
$$

where $\left\langle r^{\prime}\right\rangle$ denotes nearest neighbors of site $\mathbf{r}$. The exponent in (2.57) is a perfect square, and can be expressed as

$$
\begin{equation*}
2 \pi \beta J \sum_{r, r^{\prime}}^{\prime}\left(m_{r}+\hat{m}_{r}\right)\left(-2 \pi G_{r-r^{\prime}}\right)\left(m_{r^{\prime}}+\hat{m}_{r^{\prime}}\right) \tag{2.60}
\end{equation*}
$$

where the prime on the sum indicates unique pairings of $\mathbf{r}$ and $\mathbf{r}^{\prime}$, and

$$
\begin{equation*}
\hat{m}_{r}=\frac{i p}{2 \pi \beta J} \chi_{r} \tag{2.61}
\end{equation*}
$$

We can now write (2.57) as

$$
\begin{equation*}
Z_{\mathrm{V}}^{\prime}=\exp \left[-\frac{p^{2}}{2 \beta J}\right] \sum_{\left\{m_{r}\right\}} \exp \left[2 \pi \beta J \sum_{r, r^{\prime}}^{\prime}\left(m_{r}+\hat{m}_{r}\right)\left(-2 \pi G_{r-r^{\prime}}\right)\left(m_{r^{\prime}}+\hat{m}_{r^{\prime}}\right)\right] \tag{2.62}
\end{equation*}
$$

Strictly speaking, there are terms in the exponent which are very large and negative because of the function $G_{r-r^{\prime}}$, which highly suppresses a number of terms in the partition function. As before, we rewrite $G_{r-r^{\prime}}$ in terms of the well-defined piece $\Gamma_{r-r^{\prime}}$, and the divergent piece $G_{o}$, defined in (2.41). Using the neutrality condition (2.43), and the result

$$
\begin{equation*}
\sum_{r} \hat{m}_{r} \propto \sum_{\left\langle r, r^{\prime}\right\rangle} \nu_{r r^{\prime}}=0 \tag{2.63}
\end{equation*}
$$

which follows from (2.61) and (2.59), we find that once again the divergent piece $G_{o}$ drops out of the exponent in (2.62). These modifications allow $Z_{\mathrm{V}}^{\prime}$ to be written as

$$
\begin{equation*}
Z_{V}^{\prime}=\exp \left[-\frac{p^{2}}{2 \beta J}\right] \sum_{\{m(r)\}} \exp \left[2 \pi \beta J \sum_{r, r^{\prime}}^{\prime}\left(m_{r}+\hat{m}_{r}\right) \Gamma_{r-r^{\prime}}\left(m_{r^{\prime}}+\hat{m}_{r^{\prime}}\right)\right] \tag{2.64}
\end{equation*}
$$

Using this result, we can now write (2.56) as

$$
\begin{align*}
g_{p}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) & =\left\langle e^{i p\left(\theta_{o}-\theta_{n}\right)}\right\rangle=Z_{\mathrm{V}}^{\prime} / Z_{\mathrm{V}} \\
& =G_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) G_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \tag{2.65}
\end{align*}
$$

where we have defined the spin-wave correlation function as

$$
\begin{equation*}
G_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\exp \left[-\frac{p^{2}}{2 \beta J} \sum_{r, r^{\prime}} \nu_{r, r^{\prime}}^{2}-\frac{p^{2}}{2 \pi \beta J} \sum_{r, r^{\prime}}^{\prime} \nu_{r} \Gamma_{r-r^{\prime}} \nu_{r^{\prime}}\right] \tag{2.66}
\end{equation*}
$$

and the vortex correlation function as

$$
\begin{equation*}
G_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\left\langle\exp \left[i p \sum_{r, r^{\prime}} m_{r} \Gamma_{r-r^{\prime}} \nu_{r^{\prime}}\right]\right\rangle_{Z_{\mathrm{V}}} \tag{2.67}
\end{equation*}
$$

In (2.67) the thermal average is with respect to the usual vortex partition function (2.44) and in the previous expressions we have introduced the variable

$$
\begin{equation*}
\nu_{r}=\sum_{\left\langle r^{\prime}\right\rangle} \nu_{r r^{\prime}} \tag{2.68}
\end{equation*}
$$

The correlation function (2.65) has been factorized into the product of two functions. In order to simplify (2.66), first note that the prime in the second term in the exponent can be removed from the sum if we introduce a factor of $1 / 2$. Also, $\Gamma_{r-r^{\prime}}$ can be replaced by $-2 \pi G_{r-r^{\prime}}$, because the divergent piece $G_{o}$ gives no contribution since $\sum_{r} \nu_{r}=0$, which is essentially equivalent to (2.63).

We now define the function

$$
\begin{equation*}
\gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\sum_{\left\langle r, r^{\prime}\right\rangle} \nu_{r, r^{\prime}}^{2}-\sum_{r, r^{\prime}} \nu_{r} G_{r-r^{\prime}} \nu_{r^{\prime}} \tag{2.69}
\end{equation*}
$$

such that (2.66) can be written as

$$
\begin{equation*}
G_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\exp \left[-\frac{p^{2}}{2 \beta J} \gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)\right] \tag{2.70}
\end{equation*}
$$

Let us measure distance in units of the lattice constant $a$, such that $|\mathbf{r}| \rightarrow a|\mathbf{r}|$. We can now calculate $\gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)$ by considering the simple case where $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$ both lie on the
principle x -axis of the lattice (see Figure 2.4). The path connecting them is a straight line of length $\left|\mathbf{r}_{o}-\mathbf{r}_{n}\right|=r$, where $r$ is a dimensionless integer-valued distance measured in the units of the lattice constant. It now follows that (2.69) can be written as

$$
\begin{equation*}
\gamma(\mathbf{r})=r-\sum_{x, x^{\prime}=0}^{r-1}\left\{2 G_{x-x^{\prime}, 0}-G_{x-x^{\prime}, 1}-G_{x-x^{\prime},-1}\right\} \tag{2.71}
\end{equation*}
$$

where we have written $G_{r-r^{\prime}}=G_{x-x^{\prime}, y-y^{\prime}}$ explicitly in terms of its vector components. Using the definition of $G_{r-r^{\prime}}$ in (2.31), it is a simple matter to show that this sum collapses to the very convenient form

$$
\begin{equation*}
\gamma(\mathbf{r})=\frac{\Gamma(\mathbf{r})}{\pi} \tag{2.72}
\end{equation*}
$$

where $\Gamma(\mathbf{r})$ is the same function that has been defined in (2.42). This enables us to write (2.66) as

$$
\begin{equation*}
G_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\exp \left[-\frac{p^{2}}{2 \pi \beta J} \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)\right] \tag{2.73}
\end{equation*}
$$

If we take $p=1$ in this expression and compare with (A.21), it becomes clear why we called this the spin-wave correlation function.

$$
\mathrm{r}_{\mathrm{o}} \stackrel{-}{\bullet} \stackrel{-}{\bullet} \stackrel{-}{+} \stackrel{-}{+} \stackrel{-}{+} \stackrel{-}{+}+\stackrel{-}{+} \stackrel{-}{+} \stackrel{-}{+} \stackrel{-}{+} \mathrm{r}_{\mathrm{n}}
$$

Figure 2.4: This is the path connecting $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$. The variable $\nu_{r}$ plays the role of a discretized dipole potential. It takes the value +1 in the plaquette directly below the curve and the value -1 directly above it.

Our next project is to calculate $G_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)$. In order to proceed we use the cumulant expansion

$$
\begin{equation*}
\left\langle e^{V}\right\rangle_{0}=\exp \left[\langle V\rangle_{0}+\frac{\left\langle V^{2}\right\rangle_{0}-\langle V\rangle_{0}^{2}}{2}+\ldots\right] \tag{2.74}
\end{equation*}
$$

For the case of $G_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)$ in (2.67), we identify

$$
\begin{equation*}
V \equiv i p \sum_{a, a^{\prime}} m_{a} \Gamma\left(\mathbf{r}_{a}-\mathbf{r}_{a^{\prime}}\right) \nu_{a^{\prime}} \tag{2.75}
\end{equation*}
$$

where $a$ and $a^{\prime}$ simply label the spatial location of each plaquette. The thermal average of this quantity is

$$
\begin{equation*}
\langle V\rangle_{Z_{\mathrm{V}}}=i p \sum_{a, a^{\prime}} \nu_{a^{\prime}} \Gamma\left(\mathbf{r}_{a}-\mathbf{r}_{a^{\prime}}\right)\left\langle m_{a}\right\rangle_{Z_{\mathrm{V}}} \simeq 0 \tag{2.76}
\end{equation*}
$$

to first order, because each plaquette is equally likely to have a vorticity of 0 or $\pm 1$, which follows from the condition for neutrality (2.43). Therefore, the leading order term in the cumulant expansion is

$$
\begin{align*}
\left\langle V^{2}\right\rangle_{Z_{\mathrm{V}}} & =-p^{2}\left\langle\left(\sum_{a, a^{\prime}} m_{a} \Gamma\left(\mathbf{r}_{a}-\mathbf{r}_{a^{\prime}}\right) \nu_{a^{\prime}}\right)\left(\sum_{b, b^{\prime}} m_{b} \Gamma\left(\mathbf{r}_{b}-\mathbf{r}_{b^{\prime}}\right) \nu_{b^{\prime}}\right)\right\rangle_{Z_{\mathrm{V}}}  \tag{2.77}\\
& =-p^{2} \sum_{a, b}\left\langle m_{a} m_{b}\right\rangle_{Z_{\mathrm{V}}} \sigma_{a} \sigma_{b} \tag{2.78}
\end{align*}
$$

where

$$
\begin{equation*}
\sigma_{a}=\sum_{a^{\prime}} \Gamma\left(\mathbf{r}_{a}-\mathbf{r}_{a^{\prime}}\right) \nu_{a^{\prime}} \tag{2.79}
\end{equation*}
$$

Therefore, we find

$$
\begin{equation*}
G_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \simeq \exp \left[-\frac{p^{2}}{2} \sum_{a, b}\left\langle m_{a} m_{b}\right\rangle_{Z_{\mathrm{V}}} \sigma_{a} \sigma_{b}\right] \tag{2.80}
\end{equation*}
$$

The sums over $a^{\prime}$ and $b^{\prime}$ in $\sigma_{a, b}$ can be performed in the limit when $\mathbf{r}_{a, b}$ are far from curve $C$, which is the curve that connects $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$. In this case we can use the estimate for $\Gamma\left(\mathbf{r}_{a}-\mathbf{r}_{a^{\prime}}\right)$ given by (A.22) as

$$
\begin{align*}
\sigma_{a} & =\sum_{a^{\prime}} \Gamma\left(\mathbf{r}_{a}-\mathbf{r}_{a^{\prime}}\right) \nu_{a^{\prime}} \\
& =-\sum_{\text {Curve } C}\left\{\Gamma\left(x_{a}-x_{a^{\prime}}, y_{a}-y_{a^{\prime}}\right)-\Gamma\left(x_{a}-x_{a^{\prime}}, y_{a}-\left(y_{a^{\prime}}-1\right)\right)\right\} \\
& \simeq-\int_{z_{o}}^{z_{n}} d x \frac{\partial}{\partial y} \Gamma\left(\left|z_{a}-z\right|\right) \\
& \simeq-\int_{z_{o}}^{z_{n}} d x \frac{\partial}{\partial y} \ln \left|z_{a}-z\right| \\
& =-\int_{z_{o}}^{z_{n}} d x \frac{\partial}{\partial y} \Re\left[\ln \left(z_{a}-z\right)\right] \\
& =\int_{z_{o}}^{z_{n}} d x \frac{\partial}{\partial x} \Im\left[\ln \left(z_{a}-z\right)\right] \\
& =\Im\left[\ln \left(\frac{z_{a}-z_{n}}{z_{a}-z_{o}}\right)\right] \tag{2.81}
\end{align*}
$$

In the second line above we have used the fact that $\nu_{a^{\prime}}$ is a discretized dipole potential that has the value +1 in the plaquette directly below curve $C$ and the value -1 directly above it (see Figure 2.4). In the third line we denote 2D planar positions by complex variables, $z_{i}$, measured in units of the lattice constant, and convert the discrete sum and difference into a continuous version, valid when $z_{a, b}$ are far from curve $C$, i.e. $\left|z_{a, b}-z\right| \gg 1$. Next, we have used the approximate form of $\Gamma(\mathbf{r})$ given by (A.22), and made use of the Cauchy-Riemann equations for analytic functions to finally obtain (2.81).

Now let the coordinate $z_{R}$ be the center of the pair of vortices $m_{a, b}$ (see Figure 2.5). At low temperatures, vortices will form tightly bound dipole pairs whose correlation decreases rapidly with relative distance. Therefore, we can Taylor expand $\sigma_{a, b}$ as


Figure 2.5: Here is a schematic representation of the coordinates used in the calculation of the correlation function. We assume the coordinates $Z_{a, b}$ are far away from $Z_{o, n}$

$$
\begin{equation*}
\sigma_{a, b}=\sigma_{R} \pm \frac{1}{2}\left(\mathbf{r}_{a}-\mathbf{r}_{b}\right) \cdot \vec{\nabla} \sigma_{R} \tag{2.82}
\end{equation*}
$$

If we denote the relative separation as $\mathbf{r}=\mathbf{r}_{a}-\mathbf{r}_{b}$, we can write (2.80) as

$$
\begin{equation*}
G_{\mathrm{V}}\left(\mathbf{r}_{0}-\mathbf{r}_{N}\right) \simeq \exp \left[-\frac{p^{2}}{2} \sum_{r, R}\left\langle m_{R-\frac{r}{2}} m_{R+\frac{r}{2}}\right\rangle_{Z_{\mathrm{V}}}\left\{\sigma_{R}^{2}-\frac{1}{4}\left(\mathbf{r} \cdot \vec{\nabla} \sigma_{R}\right)^{2}\right\}\right] \tag{2.83}
\end{equation*}
$$

This result can be simplified by noting that the first term in the exponent vanishes,

$$
\begin{align*}
\sum_{r, R}\left\langle m_{R-\frac{r}{2}} m_{R+\frac{r}{2}}\right\rangle_{Z_{\mathrm{V}}} \sigma_{R}^{2} & =\sum_{r, R}\left\langle m_{0} m_{r}\right\rangle_{Z_{\mathrm{V}}} \sigma_{R}^{2} \\
& =\sum_{r, R}\left\langle m_{0} m_{r}\right\rangle_{Z_{\mathrm{V}}} \sigma_{R}^{2} \\
& =\left\langle m_{0} \sum_{r} m_{r}\right\rangle_{Z_{\mathrm{V}}} \sum_{R} \sigma_{R}^{2} \\
& \simeq 0 \tag{2.84}
\end{align*}
$$

because of the neutrality condition (2.43). In the first line of the above equation we have explicitly made use of translation invariance; the correlation of vortices depends only on the relative distance between them and not on the absolute position.

To deal with the second term in the exponent of (2.83), note that it can be written as

$$
\begin{equation*}
\frac{p^{2}}{8} \sum_{r}\left\langle m_{0} m_{r}\right\rangle_{Z_{\mathrm{V}}} r^{2} \cos ^{2} \theta \sum_{R}\left(\vec{\nabla} \sigma_{R}\right)^{2} \tag{2.85}
\end{equation*}
$$

If we take the form of $\sigma_{R}$ given by (2.81) and the form of $\Gamma(\mathbf{r})$ given by (A.22), it is a straight forward matter to establish the mathematical identity

$$
\begin{equation*}
\left(\vec{\nabla} \sigma_{R}\right)^{2}=\left[\vec{\nabla} \Gamma\left(\mathbf{R}-\mathbf{r}_{o}\right)-\vec{\nabla} \Gamma\left(\mathbf{R}-\mathbf{r}_{n}\right)\right]^{2} \tag{2.86}
\end{equation*}
$$

where all derivatives are taken with respect to $\mathbf{R}$. This allows us to write

$$
\begin{align*}
\sum_{R}\left(\vec{\nabla} \sigma_{R}\right)^{2} & \simeq \int d^{2} R\left[\vec{\nabla} \Gamma\left(\mathbf{R}-\mathbf{r}_{o}\right)-\vec{\nabla} \Gamma\left(\mathbf{R}-\mathbf{r}_{n}\right)\right]^{2} \\
& =-\int d^{2} R\left[\Gamma\left(\mathbf{R}-\mathbf{r}_{o}\right)-\Gamma\left(\mathbf{R}-\mathbf{r}_{n}\right)\right]^{2} \cdot \nabla^{2}\left[\Gamma\left(\mathbf{R}-\mathbf{r}_{o}\right)-\Gamma\left(\mathbf{R}-\mathbf{r}_{n}\right)\right] \tag{2.87}
\end{align*}
$$

where we have integrated by parts and dropped the boundary term.
The function $\Gamma(\mathbf{r})$ depends logarithmically on distance and describes the interaction between vortices, therefore, it is directly related to the 2 D electrostatic potential of a point charge. The function $\Gamma(\mathbf{r})$ satisfies the 2 D version of the electrostatic Poisson equation

$$
\begin{equation*}
\nabla^{2} \Gamma(\mathbf{r})=2 \pi \delta^{2}(\mathbf{r}) \tag{2.88}
\end{equation*}
$$

We can now write (2.87) simply as

$$
\begin{equation*}
\sum_{R}\left(\vec{\nabla} \sigma_{R}\right)^{2}=4 \pi \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \tag{2.89}
\end{equation*}
$$

and combining the previous results we obtain

$$
\begin{equation*}
G_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \simeq \exp \left[\frac{\pi p^{2}}{2} \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \sum_{r} r^{2} \cos ^{2} \theta\left\langle m_{0} m_{r}\right\rangle_{Z_{\mathrm{V}}}\right] \tag{2.90}
\end{equation*}
$$

The last thing left to do is to calculate $\left\langle m_{0} m_{r}\right\rangle_{Z_{\mathrm{v}}}$, which is done most conveniently with respect to the vortex partition function in the form (2.47). The first non-trivial configuration, which is constrained by the neutrality condition (2.43), is a pair of vortices of intensity $\pm 1$ a distance $r$ (in the units of $a$ ) apart. The corresponding correlation function, which follows with the help of (2.47), is given approximately by

$$
\begin{align*}
\left\langle m_{0} m_{r}\right\rangle_{Z_{\mathrm{V}}} & \simeq-\exp \left[-2 \pi \beta J \ln r-2 \pi \beta J \ln \frac{a}{r_{0}}\right] \\
& =-y^{2}\left(\frac{1}{r}\right)^{2 \pi \beta J} \tag{2.91}
\end{align*}
$$

where

$$
\begin{equation*}
y \equiv \exp [-\beta \mu]=\exp \left[-\pi \beta J \ln \frac{a}{r_{0}}\right] \tag{2.92}
\end{equation*}
$$

is the vortex fugacity. Since we are in the low temperature regime, $y$ is a very small quantity. We can now simplify (2.90) and write

$$
\begin{align*}
G_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) & \simeq \exp \left[-\frac{\pi p^{2} y^{2}}{2} \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \sum_{r} r^{2}\left(\frac{1}{r}\right)^{2 \pi \beta J} \cos ^{2} \theta\right] \\
& \simeq \exp \left[-\frac{\pi p^{2} y^{2}}{2} \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \int r d r d \theta r^{2}\left(\frac{1}{r}\right)^{2 \pi \beta J} \cos ^{2} \theta\right] \\
& =\exp \left[-\frac{y^{2} p^{2} \pi^{2}}{2}\left(\int_{1}^{\infty} d r\left(\frac{1}{r}\right)^{2 \pi \beta J-3}\right) \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)\right] \tag{2.93}
\end{align*}
$$

Finally, we can collect (2.73) and (2.93) and substitute these results into (2.65) to obtain the final form for spin correlation function,

$$
\begin{equation*}
g_{p}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\Re\left\langle e^{i p\left(\theta_{o}-\theta_{n}\right)}\right\rangle \simeq \exp \left[-\frac{p^{2}}{2 \pi \beta_{\mathrm{eff}} J} \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)\right] \tag{2.94}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{1}{\beta_{\mathrm{eff}} J}=\frac{1}{\beta J}+\pi^{3} y^{2} \int_{a}^{\infty} d r\left(\frac{a}{r}\right)^{2 \pi \beta J-3} \tag{2.95}
\end{equation*}
$$

plays the role of an effective temperature, and we have rescaled distances into the normal dimensions of length. If we use the approximation for $\Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)$ in (A.22), we can write the correlation as

$$
\begin{equation*}
g_{p}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\left(\frac{r_{0}}{\left|\mathbf{r}_{o}-\mathbf{r}_{n}\right|}\right)^{p^{2} / 2 \pi \beta_{\mathrm{eff}} J} \tag{2.96}
\end{equation*}
$$

Just as in the spin wave approximation, we see that the correlation decays as a power law and exhibits quasi-long range order. However, the inclusion of vortices has increased the effective temperature, causing the correlation to decrease more rapidly.

### 2.5 Renormalization Group Analysis

In the previous section we have calculated the effect vortices have on the correlation function, and found that an effective temperature naturally emerges, whose increased value causes correlations to decay more rapidly than in the spin-wave approximation. The parameters which control the behavior of the system are the vortex fugacity $y$ [see (2.92)] and the temperature $\beta$, both of which are dimensionless and are defined at a microscopic length scale. We now want to perform a real space renormalization by integrating out the short distance degrees of freedom. By scaling the cutoff length from $a \rightarrow \lambda a$ where $\lambda>1$, we can track how the parameters $y_{\lambda}$ and $\beta_{\lambda}$ scale as a function
of $\lambda$ such that the correlation functions retain the same form throughout the re-scaling procedure.

Let us first consider breaking the integral in (2.95) into two parts, from $a$ to $\lambda a$, and from $\lambda a$ to $\infty$. By defining the dimensionless distance $x=r / a$, the integral becomes

$$
\begin{align*}
\int_{1}^{\infty} d x\left(\frac{1}{x}\right)^{2 \pi \beta J-3} & =\int_{1}^{\lambda} d x x^{3-2 \pi \beta J}+\int_{\lambda}^{\infty} d x x^{3-2 \pi \beta J} \\
& =\frac{\lambda^{4-2 \pi \beta J}-1}{4-2 \pi \beta J}+\lambda^{4-2 \pi \beta J} \int_{1}^{\infty} d x x^{3-2 \pi \beta J} \tag{2.97}
\end{align*}
$$

In the limit of infinitesimal scaling, $\lambda \rightarrow 1$, we can approximate $\ln \lambda \simeq \lambda-1$ and (2.97) can be written as

$$
\begin{equation*}
\ln \lambda+(1+(4-2 \pi \beta J) \ln \lambda) \int_{1}^{\infty} d x x^{3-2 \pi \beta J} \tag{2.98}
\end{equation*}
$$

such that (2.95) now becomes

$$
\begin{equation*}
\frac{1}{\beta_{\mathrm{eff}} J}=\frac{1}{\beta J}+\pi^{3} y^{2} \ln \lambda+\pi^{3} y^{2}(1+(4-2 \pi \beta J) \ln \lambda) \int_{1}^{\infty} d x x^{3-2 \pi \beta J} \tag{2.99}
\end{equation*}
$$

In order to retain the original form of the correlation function in the rescaled dimensions, we make the following prescriptions;

$$
\begin{equation*}
\frac{1}{\beta_{\lambda} J}=\frac{1}{\beta J}+\pi^{3} y_{\lambda}^{2} \ln \lambda \tag{2.100}
\end{equation*}
$$

for the renormalized temperature at the distance scale $\lambda a$, and

$$
\begin{equation*}
y_{\lambda}^{2}=y^{2}\left(1+\left(4-2 \pi \beta_{\lambda} J\right) \ln \lambda\right) \tag{2.101}
\end{equation*}
$$

for the renormalized vortex fugacity. Since $y$ is a very small quantity at low temperatures, we may approximate (2.101) as

$$
\begin{equation*}
y_{\lambda} \simeq y+\left(2-\pi \beta_{\lambda} J\right) y \ln \lambda \tag{2.102}
\end{equation*}
$$

In the limit $\lambda \rightarrow 1$ we can approximate $\ln \lambda \simeq \Delta \lambda / \lambda$, and the expressions (2.100) and (2.102) can expressed in differential form,

$$
\begin{align*}
\frac{d}{d \ln \lambda} y_{\lambda} & =\left(2-\pi \beta_{\lambda} J\right) y_{\lambda}  \tag{2.103}\\
\frac{d}{d \ln \lambda} \frac{1}{\beta_{\lambda} J} & =\pi^{3} y_{\lambda}^{2} \tag{2.104}
\end{align*}
$$

These mathematical equations describe the renormalization group (RG) flow in the parameter space of $y$ and $\beta$, originally discussed by J.M. Kosterlitz[59], and are plotted in Figure 2.6.

We see that there are three regimes in Figure 2.6, separated by a linear flow called the separatrix. Region 1 is the low temperature regime, below the separatrix. The vortex fugacity $y$ loses importance with scaling, which corresponds to an increasing vortex chemical potential [see (2.92)]. Therefore, in the low temperature regime, the vortex chemical potential (2.49) is very high and so vortex excitations are suppressed. Region 3 corresponds to the high temperature regime, in which $y$ becomes increasingly more important with scaling. This corresponds to a chemical potential which decreases with scaling and hence begins to favor vortex excitations. In region 2 near $\beta J=2 / \pi$ is the Kosterlitz-Thouless transition, where vortices begin to proliferate upon moving from the low temperature regime to the high temperature regime.


Figure 2.6: Scaling behavior of vortex fugacity $y$ and coupling $J$. There are three distinct regimes, separated by a linear flow called the separatrix.

### 2.6 Universal Jump in Superfluid Stiffness

From the RG flow equations in (2.103) and (2.104), we find that there is a universal jump in the superfluid stiffness, which is a characteristic property of the Kosterlitz-Thouless transition. To see this, let us first write the RG flow equations as

$$
\begin{align*}
\frac{d y}{d \ell} & =(2-\pi K) y  \tag{2.105}\\
\frac{d K^{-1}}{d \ell} & =\pi^{3} y^{2} \tag{2.106}
\end{align*}
$$

where $\ell=\ln \lambda, y=\exp [-\beta \mu]$ is the vortex fugacity, and $K=\beta J$ is the dimensionless bare coupling. From the behavior of these equations as shown in Figure 2.6 (which is valid for $y \ll 1$ ), we see that the critical transition can identified as the largest temperature at which the vortex fugacity $(y)$ renormalizes to zero as we integrate out small length scales. Therefore, for $T \leq T_{\mathrm{KT}}$,

$$
\begin{equation*}
\lim _{\ell \rightarrow \infty} y(\ell)=0 \tag{2.107}
\end{equation*}
$$

which implies that the renormalized coupling, as the critical temperature is approached from below becomes

$$
\begin{equation*}
K_{\mathrm{R}}^{-1}\left(T_{\mathrm{KT}}\right)=\lim _{T \rightarrow T_{\mathrm{KT}}^{-}} K_{\mathrm{R}}^{-1}(T)=\frac{\pi}{2} \tag{2.108}
\end{equation*}
$$

which can be written as

$$
\begin{equation*}
\frac{k_{B} T_{\mathrm{KT}}}{J\left(T_{\mathrm{KT}}\right)}=\frac{\pi}{2} \tag{2.109}
\end{equation*}
$$

This fact follows the RG flow equations which are shown in Figure 2.6, when $K^{-1}>2 / \pi$ and we approach the critical line from below, we will flow to the point $\left(K^{-1}, y\right)=$ (2/pi,0).

When $T>T_{\mathrm{KT}}$, the renormalized fugacity begins to grow indefinitely as small length scales are integrated out and

$$
\begin{equation*}
K_{\mathrm{R}}\left(T_{\mathrm{KT}}\right)=\lim _{T \rightarrow T_{\mathrm{KT}}^{+}} K(T)=0 \tag{2.110}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
J\left(T \geq T_{\mathrm{KT}}\right)=0 \tag{2.111}
\end{equation*}
$$

Above the KT transition the Josephson coupling is zero, and below it satisfies equation (2.109). Therefore, the universal jump in the coupling energy can be written as

$$
\begin{equation*}
\frac{J\left(T_{\mathrm{KT}}\right)}{T_{\mathrm{KT}}}=\frac{2 k_{B}}{\pi} \tag{2.112}
\end{equation*}
$$

We can relate the Josephson coupling to the superfluid stiffness $\rho_{\mathrm{s}}$ via[58]

$$
\begin{equation*}
J=\rho_{\mathrm{s}}\left(\frac{\hbar}{m^{*}}\right)^{2} \tag{2.113}
\end{equation*}
$$

where $m^{*}$ is the mass of a Cooper pair, twice the mass of an electron $\left(m^{*}=2 m\right)$. Now (2.112) becomes

$$
\begin{equation*}
\frac{\rho\left(T_{\mathrm{KT}}\right)}{T_{\mathrm{KT}}}=\frac{8 k_{B}}{\pi}\left(\frac{m}{\hbar}\right)^{2} \tag{2.114}
\end{equation*}
$$

This relation is known as the universal jump in superfluid stiffness[60]. This is a signature of the Kosterlitz-Thouless transition, where the jump to zero in the superfluid stiffness $\rho_{\mathrm{s}}$ at $T_{\mathrm{KT}}$ is universal.

### 2.7 Linearized RG Flow

The line of fixed points for the RG flow equations (2.103) and (2.104) occur when $y=0$. There is also a unique fixed point when $\beta J>2 / \pi$. The line $y=0$, $\beta J>2 / \pi$ is an attractive or stable fixed line, while $y=0, \beta J<2 / \pi$ is a repulsive or unstable fixed line. We can examine the critical behavior by linearizing the RG flow equations about the fixed point $(\beta J, y)=(2 / \pi, 0)$. If we introduce the variables

$$
\begin{align*}
\Theta_{\lambda} & \equiv 2-\pi \beta_{\lambda} J  \tag{2.115}\\
Y_{\lambda} & \equiv 2 \pi y_{\lambda} \tag{2.116}
\end{align*}
$$

the fixed point expansion of the flow will be about the point $\left(\Theta_{\lambda}, Y_{\lambda}\right)=(0,0)$. Since $\Theta_{\lambda}$ and $Y_{\lambda}$ are small very small quantities near the fixed point, the linearized RG flow equations are

$$
\begin{align*}
\frac{d}{d \ln \lambda} \Theta_{\lambda} & =Y_{\lambda}^{2}  \tag{2.117}\\
\frac{d}{d \ln \lambda} Y_{\lambda}^{2} & =2 \Theta_{\lambda} Y_{\lambda}^{2} \tag{2.118}
\end{align*}
$$

By plotting these flow equations in the reduced parameter space, we see that there are 3 distinct regimes(see Figure 2.7). The substitution of $Y_{\lambda}= \pm \Theta_{\lambda}$ into equations (2.117) and (2.118) yields $Y_{\lambda}= \pm \Theta_{\lambda}$ as the equation for the separatrix, since the flow of $Y_{\lambda}$ and $\Theta_{\lambda}$ is at exactly the same rate.

### 2.8 Critical Temperature

In order to find the critical transition temperature, we must find the point where the initial conditions for the model intersect with the separatrix. Before any scaling $(\lambda=1)$, the vortex fugacity is given by (2.92), and the reduced variable $Y$ is initially

$$
\begin{align*}
Y & =2 \pi \exp \left[-\pi \beta J \ln \frac{a}{r_{0}}\right] \\
& =2 \pi \exp \left[(\Theta-2) \ln \frac{a}{r_{0}}\right] \tag{2.119}
\end{align*}
$$

where we have written it in terms of $\Theta$ given in (2.115). The condition for criticality is given when (2.119) intersects the separatrix(see Figure 2.7). Explicitly this condition is satisfied when $Y_{\mathrm{KT}}=-\Theta_{\mathrm{KT}}$, which yields an analytic expression for the KosterlitzThouless transition temperature,

$$
\begin{equation*}
\frac{2 \pi}{\Theta_{\mathrm{KT}}}\left(2^{3 / 2} e^{\gamma}\right)^{\Theta_{\mathrm{KT}}-2}=-1 \tag{2.120}
\end{equation*}
$$

where we have used (A.23). By solving this equation numerically we obtain

$$
\begin{equation*}
T_{\mathrm{KT}}=(1-0.0842) \frac{\pi J}{2 k_{B}} \tag{2.121}
\end{equation*}
$$

which should be compared with (2.8). The renormalization group treatment shows that the Kosterlitz-Thouless transition is about $8.4 \%$ lower than the temperature (2.8) which came about from a simple energy-entropy argument. This calculation is more accurate since it takes into account the screening effect of multiple vortices in the system.

### 2.9 Vortices which bind a Majorana Fermion

We now want to allow for the presence of Majorana fermions bound to vortex cores, and see how this might affect the Kosterlitz-Thouless transition temperature. In this section we are considering Majorana modes bound to full vortices, which is the case for spinless (or spin-polarized) 2D $p+i p$ superconductors. If we refer back to (2.47), we see that the energy of a system of normal vortices is

$$
\begin{equation*}
E_{\mathrm{V}}=-2 \pi J \sum_{r \neq r^{\prime}}^{\prime} m_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} m_{r^{\prime}}+\pi J \ln \frac{a}{r_{0}} \sum_{r} m_{r}^{2} \tag{2.122}
\end{equation*}
$$

The first term in (2.122) is the usual logarithmic interaction between vortex pairs and the second term is the chemical potential. Let us imagine that this energy describes a system of $2 n$ interacting vortices, or equivalently, $n$ pairs of vortices. At low temperatures, owing to the neutrality condition (2.43) and the energetic stability of vortices with low vorticity, we may assume that in each plaquette, $m_{r}=0, \pm 1$. This tells us that the term $\sum_{r} m_{r}^{2}$ simply counts the number of vortices, and can therefore be identified as equivalent to $2 n$.

We know that 2 Majorana operators can be combined to form a legitimate fermionic mode(see [1.10)], either occupied or unoccupied, which adds a 2 -fold degen-
eracy to the state of a system of 2 vortices. Physically then, the presence of Majorana modes manifests itself by generating entropy which increases the degeneracy of the ground state of vortices. Generalizing this argument to the case of $n$ vortex pairs, we can take the $2 n$ self-conjugate Majorana fermionic operators and combine them into $n$ complex fermionic operators, modes which again may either be filled or empty. Therefore, the ground state of a system of $2 n$ vortices acquires a $2^{n}$-fold degeneracy. This degeneracy can be written as

$$
\begin{equation*}
2^{n}=\exp [n \ln 2]=\exp \left[\frac{1}{2} \ln 2 \sum_{r} m_{r}^{2}\right] \tag{2.123}
\end{equation*}
$$

which follows from the proceeding discussion. From (2.47), we can now write the partition function for a system $2 n$ vortices that bind Majorana modes as

$$
\begin{align*}
Z_{\mathrm{V}} & =\sum_{\left\{m_{r}\right\}^{\prime}} 2^{n} \exp \left[2 \pi \beta J \sum_{r \neq r^{\prime}}^{\prime} m_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} m_{r^{\prime}}-\pi \beta J \ln \frac{a}{r_{0}} \sum_{r} m_{r}^{2}\right] \\
& =\sum_{\left\{m_{r}\right\}^{\prime}} \exp \left[2 \pi \beta J \sum_{r \neq r^{\prime}}^{\prime} m_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} m_{r^{\prime}}-\left(\pi \beta J \ln \frac{a}{r_{0}}-\frac{1}{2} \ln 2\right) \sum_{r} m_{r}^{2}\right] \tag{2.124}
\end{align*}
$$

The addition of Majorana modes shifts the energy spectrum for a collection vortices to

$$
E_{\mathrm{V}}=-2 \pi J \sum_{r \neq r^{\prime}}^{\prime} m_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} m_{r^{\prime}}+\left(\pi J \ln \frac{a}{r_{0}}-\frac{1}{2 \beta} \ln 2\right) \sum_{r} m_{r}^{2}
$$

which should be compared with (2.122). The energy of a dipole pair vortices of unit strength separated by a distance $r$ is

$$
\begin{equation*}
E_{\mathrm{pair}}=2 \pi J \ln \frac{r}{a}+2 \pi J \ln \frac{a}{r_{0}}-\frac{1}{\beta} \ln 2 \tag{2.125}
\end{equation*}
$$

where the second term is twice the chemical potential of a single vortex with a Majorana mode. Explicitly, this is

$$
\begin{align*}
\mu_{\mathrm{F}} & =\pi J \ln \frac{a}{r_{0}}-\frac{1}{2 \beta} \ln 2 \\
& =\pi J\left[\frac{1}{2}\left(3-\frac{1}{\pi \beta J}\right) \ln 2+2 \gamma\right] \tag{2.126}
\end{align*}
$$

which should be compared with (2.49). In order to calculate the effect that Majorana fermions have on the Kosterlitz-Thouless transition, we follow the same procedure as before, making the substitution $\mu \rightarrow \mu_{\mathrm{F}}$. The scaled variable $Y$ in (2.116) becomes

$$
\begin{align*}
Y^{\mathrm{F}} & =2 \pi \exp \left[-\beta \mu_{\mathrm{F}}\right] \\
& =2 \pi \exp \left[(\Theta-2) \ln \frac{a}{r_{0}}+\frac{1}{2} \ln 2\right] \tag{2.127}
\end{align*}
$$

and the condition for criticality, $Y_{\mathrm{KT}}^{\mathrm{F}}=-\Theta_{\mathrm{KT}}$, can be written

$$
\begin{equation*}
\sqrt{2} \frac{2 \pi}{\Theta_{\mathrm{KT}}}\left(2^{3 / 2} e^{\gamma}\right)^{\Theta_{\mathrm{KT}}-2}=-1 \tag{2.128}
\end{equation*}
$$

This analytic expression for the transition temperature is identical to (2.120) apart from a factor of $\sqrt{2}$ which we can see appearing in Figure 2.7. Solving numerically for the transition temperature, we find

$$
\begin{equation*}
T_{\mathrm{KT}}=(1-0.0842-.0222) \frac{\pi J}{2 k_{B}} \tag{2.129}
\end{equation*}
$$

which is to be compared with (2.121) and (2.8). Apart from RG corrections, the degeneracy in the ground state of vortices due to the presence of Majorana fermions lowers the Kosterlitz-Thouless transition temperature by a small additional percentage of approximately $2.2 \%$.


Figure 2.7: The intersection of $Y_{\mathrm{KT}}$ with the separatrix determines the critical temperature. The presence of Majorana fermions changes the effective fugacity to $Y_{\mathrm{KT}}^{\mathrm{F}}$ and so adds an additional correction to the Kosterlitz-Thouless transition temperature.

## Chapter 3

## Josephson Junction Arrays

### 3.1 Introduction

Josephson junction (JJ) arrays are convenient and controllable model systems that allow us to study the quantum mechanical and statistical properties of twodimensional systems on a macroscopic scale. The phase of the order parameter and the number of superconducting electrons on each junction are the conjugate dynamical variables which control the behavior of the array. In addition to the xy model, JJ arrays also provide a direct physical realization of the more general frustrated xy model. By exposing a planar JJ array to a transverse external magnetic field, a finite density of vortices, or circulating supercurrents, are induced. The interplay between the strength of the applied magnetic field, the mean separation of vortices, and the underlying periodic structure of the array gives rise to interesting physical phenomena.

Vortices play a central role in the low temperature 2D physics of JJ arrays. In JJ arrays we see both continuous symmetries and topological defects, which give rise to various domain wall and vortex-unbinding phase transitions. The simplest case is an ordinary Kosterlitz-Thouless transition[51], which can be observed experimentally in JJ
$\operatorname{arrays}[48,61,62]$.

The purpose of this chapter is to review some of the important aspects of conventional (s-wave) JJ arrays, before we turn to our results for chiral p-wave arrays. In Section 3.2 we start off by reviewing conventional JJ arrays and then in Section 3.3 we look at JJ arrays in a transverse external magnetic field, a system which can be described by the frustrated xy model. In Section 3.4 we review the fully frustrated case (where magnetic field is tuned to a strength of half a magnetic flux quantum per plaquette), as originally studied by T.C. Halsey[63]. In Section 3.5 we review the Coulomb gas representation[58] of the xy model. In Section 3.6 we review what is known about the phase transition of a fully frustrated xy model[64]. This includes calculating domain energies, both numerically and analytically. Numerically, we modify the calculation originally performed by C. Denniston[65] by introducing periodic boundary conditions, and analytically we derive an expression for domains that agrees with the results of S.E. Korshunov[66]. The major original results in this chapter are contained in Section 3.6.3, where we derive an expression for the ratio of fully frustrated to unfrustrated transitions for a conventional JJ array. Finally, in Section 3.7, following [61], we review vortex unbinding via applied supercurrent.

### 3.2 Conventional Josephson Junction Arrays

A large network of individual superconducting grains that interact via weak links known as Josephson junctions are known as Josephson junction (JJ) arrays (see Figure 3.1). Conventional Josephson junction arrays are arrays that are constructed out of ordinary s-wave superconducting grains. Before considering the properties of the entire array, let us first consider the properties of a single Josephson junction, that is,
two superconductors coupled by a weak link. The weak link refers to a weak coupling between the superconductors, a common example of which is a thin insulating barrier. B.D. Josephson first predicted in 1962 [67] that supercurrent will flow across a junction in the absence of any applied voltage, in accordance with the relation

$$
\begin{equation*}
I_{s}=I_{c} \sin (\Delta \theta) \tag{3.1}
\end{equation*}
$$

where $\Delta \theta$ is the difference in superconducting phase difference between the two coupled superconductors, and $I_{c}$ is the critical current through the barrier[3]. This current flow is quantum mechanical in nature, involving the tunneling of Cooper pairs through a potential barrier, and is known as the dc Josephson effect. In addition, Josephson also predicted that if a potential difference $V$ were applied across the junction, the phase difference $\Delta \theta$ would evolve linearly in time according to the relation

$$
\begin{equation*}
\frac{d(\Delta \theta)}{d t}=\frac{2 e V}{\hbar} \tag{3.2}
\end{equation*}
$$

where $e$ is the charge of the electron and $\hbar$ is Planck's constant. This is called the ac Josephson effect, and leads to an alternating current of amplitude $I_{c}$ and angular frequency $\omega=2 e V / \hbar$. Physically, the quantum of energy $\hbar \omega$ is the change in energy of a Cooper pair as it tunnels through the junction.

By calculating the electrical work done in changing the superconducting phase difference, it is easy to show that the potential energy of a single Josephson junction is

$$
\begin{equation*}
E_{J}(1-\cos (\Delta \theta)) \tag{3.3}
\end{equation*}
$$

where $E_{J}=\hbar I_{\mathrm{c}} / 2 e$ is the Josephson coupling energy. To determine the kinetic energy of the junction, we need also to consider the charging energy. This capacitive energy is


Figure 3.1: Each shaded square represents a small square piece of superconducting material, arranged to form a square lattice. Each superconducting grain is linked to its nearest neighbor by a Josephson junction, which are simply denoted as lines connecting each superconductor. This configuration is called a Josephson junction array.
the usual $1 / 2 C V^{2}$, where $C$ is the capacitance, $V$ is the electrical potential, and $Q$ is the charge of one of the superconducting islands. If we eliminate $V$ in terms of $\Delta \theta$ via (3.2), we can write the total classical energy of a Josephson junction as

$$
\begin{equation*}
H=-E_{J} \cos (\Delta \theta)+\frac{1}{2} C\left(\frac{\hbar}{2 e}\right)^{2}\left(\frac{d(\Delta \theta)}{d t}\right)^{2} \tag{3.4}
\end{equation*}
$$

The first and last terms are potential and kinetic energy-like terms respectively, and we have ignored any unimportant constant factors.

We can now rewrite (3.4) in terms of the charge $Q$, bearing in mind that phase difference and charge are not classical variables. The conventional number-phase commutation relation applied to a Josephson junction yields the uncertainty relation

$$
\begin{equation*}
\delta N \delta(\Delta \theta) \gtrsim 1 \tag{3.5}
\end{equation*}
$$

where $N$ is the number of superconducting electrons and $\Delta \theta=\theta_{2}-\theta_{1}$ is the phase
difference across the junction. This means that neither the number of superconducting electrons transferred across the junction nor the phase difference across the junction can be known exactly at any given time. In quantized form, $N$ becomes the operator $N \rightarrow i \delta / \delta(\Delta \theta)$. Since the charge of a Cooper pair is $2 e$, we know that $Q / 2 e=N$ and hence we can derive the Hamiltonian for a Josephson junction from (3.4) as

$$
\begin{equation*}
H=-E_{J} \cos (\Delta \theta)-4 E_{C} \frac{\partial^{2}}{\partial(\Delta \theta)^{2}} \tag{3.6}
\end{equation*}
$$

where we used $Q=C V$ and also identified the single electron charging energy as $E_{C}=$ $e^{2} / 2 C$.

We have seen that the number-phase uncertainty puts a constraint on the precise specification of phase and number of superconducting electrons in macroscopic superconductors. There are two different experimentally relevant limits of the model Hamiltonian (3.6), the semi-classical limit and the quantum limit. In the semi-classical limit $E_{J} \gg E_{C}$, which means we are justified in ignoring the second term in (3.6). The Ginzburg-Landau wavefunction is well-defined in terms of phase in the semi-classical limit. Typically this means that superconducting grains composing the junction are large enough that the charging energy is negligible. The charge of a single electron will be irrelevant, and therefore the number $N$ of superconducting electrons on a particular grain is not well-defined. Therefore, phase is a reasonably well-defined semi-classical variable.

The quantum limit arises when $E_{J} \sim E_{C}$, and the conjugate properties of number and phase become important. Modern microfabrication techniques allow superconducting grains to be small enough that the transfer of a single electron charge becomes important. In this limit, there will be quantum fluctuations even down to zero
temperature, where thermal fluctuations are completely absent. In the following we will be considering the semi-classical limit, where the superconducting grains are large enough such that the charging energy and hence quantum fluctuations can be ignored altogether.

We now proceed to finding the energy of a large network of semi-classical Josephson junctions. Consider an array of Josephson junctions arranged on a square lattice, where each superconductor is weakly coupled to its nearest neighbor (see Figure 3.1). The energy for the array will be a sum of the individual junction energies, and can be written as[68]

$$
\begin{equation*}
E=\sum_{\langle i j\rangle}-E_{J} \cos \left(\theta_{j}-\theta_{i}\right) \tag{3.7}
\end{equation*}
$$

where the sum is over nearest neighbors. This energy, up to an unimportant constant factor, is equivalent to the classical xy model. In this analogy however, $\theta_{i}$ does not represent physical lattice spins, but rather the superconducting phase of each individual superconducting grain $i$. Since the hamiltonian for a Josephson junction array is equivalent to the xy model, the ground state will be the state in which all superconducting phases are aligned. The overall $\mathrm{U}(1)$ symmetry due to the global rotation invariance of all phases allows for a Kosterlitz-Thouless phase transition, which has been observed experimentally [69].

### 3.3 JJ Array in an External Magnetic Field

If a uniform external magnetic field is applied in the transverse direction to the array, the Hamiltonian for the system can be written as $[70,64]$

$$
\begin{equation*}
H=-J \sum_{\langle i j\rangle} \cos \left(\theta_{j}-\theta_{i}-A_{i j}\right) \tag{3.8}
\end{equation*}
$$

where we have written the Josephson coupling energy as $J$ and $A_{i j}$ are link variables. This model is commonly known as the frustrated xy model, where the phase difference is now gauge-invariant. The $A_{i j}$ s are lattice bonds defined as the line integral of the vector potential $\mathbf{A}$,

$$
\begin{equation*}
A_{i j}=\frac{2 \pi}{\Phi_{0}} \int_{i}^{j} \mathbf{A} \cdot d \mathbf{l} \tag{3.9}
\end{equation*}
$$

Now if the applied magnetic field is uniform in the z-direction, perpendicular to the Josephson junction array which lies in the xy-plane, we can conveniently express the vector potential in the Landau gauge as

$$
\begin{equation*}
\mathbf{A}=B x \hat{\mathbf{y}} \tag{3.10}
\end{equation*}
$$

The link variables now take the simple form

$$
\begin{equation*}
A_{i j}=2 \pi f n \tag{3.11}
\end{equation*}
$$

where $n=0,1,2, \ldots$ is an integer which labels successive columns and

$$
\begin{equation*}
f=\frac{\Phi}{\Phi_{0}} \tag{3.12}
\end{equation*}
$$

is the frustration index. All horizontal link variables are zero. The parameter $f$ is controlled by the experimentalist who is applying the external field, and it is simply the ratio of magnetic flux threading a single plaquette to the magnetic flux quantum $\Phi_{0}$.

### 3.4 Fully Frustrated JJ Arrays

A fully frustrated Josephson junction array is one in which the frustration index is $f=1 / 2$. At $f=1 / 2$ the external magnetic field is tuned such that half of a superconducting flux quantum $\Phi_{0}$ threads each plaquette. The link variables become $A_{i j}=\pi n$, as shown in Figure 3.2.


Figure 3.2: The shaded boxes represent a small square piece of superconducting material, arranged to form a square lattice. Each superconducting grain is linked to its nearest neighbor by a Josephson junction. In the diagram we sketch the configuration of link variables $A_{i j}$ for a fully frustrated $(f=1 / 2)$ array. All horizontal bonds are zero while all vertical bonds alternate between 0 and $\pi$.

A Josephson junction array with zero applied magnetic field $(f=0)$ is said to be unfrustrated. The four nearest neighbor bonds are ferromagnetic in an unfrustrated array and a minimum energy of $-J$ per bond is achieved when all superconducting phases align. In a fully frustrated array, to each plaquette there corresponds three ferromagnetic bonds and one antiferromagnetic bond. Given the link variables $A_{i j}$ in

Figure 3.2 and the Hamiltonian (3.8) which describes the frustrated array, it is clear that no configuration of phases can exist to minimize each individual bond energy.


Figure 3.3: Ground state of a fully frustrated array. In 3.3(a), the superconducting phases $\theta$ alternate from winding clockwise in one plaquette to counterclockwise in the neighboring plaquette. This phase configuration represents a checkerboard groundstate, as in $3.3(\mathrm{~b})$. The 'crosses' label cells with counterclockwise winding of $\theta$ (these are the plaquettes which contain vortices), and empty cells correspond to clockwise winding.

All horizontal bonds in the array are ferromagnetic (prefer alignment of phase), and the vertical bonds alternate between antiferromagnetic and ferromagnetic. The antiferromagnetic bonds are the source of frustration in the array. The ground state of a frustrated Josephson junction array is found by minimizing the Hamiltonian (3.8), $\partial H / \partial \theta_{i}=0$. This yields a set of coupled equations,

$$
\begin{equation*}
\sum_{\left\langle j^{\prime}\right\rangle} \sin \left(\theta_{j^{\prime}}-\theta_{i}-A_{i j^{\prime}}\right)=0 \tag{3.13}
\end{equation*}
$$

where $j^{\prime}$ are the nearest neighbors of $i$. This constraint expresses Kirchhoff's law for supercurrent, namely conservation of supercurrent at each lattice site $i$. Halsey has demonstrated[63] that the ground state for $f=1 / 2$ is a periodic $2 \times 2$ unit cell that
resembles a checkerboard.

The checkerboard pattern arises from the supercurrent which flows in the ground state, where it circulates in alternating directions in neighboring plaquettes. Each gauge-invariant phase difference for each bond is $\pm \pi / 4$ in the ground state (see Figure 3.3 ). In Figure $3.3(\mathrm{~b})$, for each plaquette marked with a cross there is a single unit of flux quantum attached to it, half coming from the externally applied magnetic field, and half coming from the counterclockwise circulating supercurrent. The net flux of this circulating supercurrent is spread out over the effective penetration depth of the array, which can be on the order of the array size[48]. On the other hand, the flux coming from the external field is fully contained within a single plaquette. The plaquettes with no crosses are cells where the circulation of the supercurrent is in the clockwise direction. This generates half a flux quantum in the direction opposite to that of the externally applied magnetic field, and hence there is zero magnetic flux associated with these plaquettes.

### 3.5 The Coulomb Gas Representation

In studying the properties of the xy model, it is very useful to recast the system from superconducting phase degrees of freedom (which live on the sites of the lattice) into charge degrees of freedom (which live in individual plaquettes). This is known as the Coulomb gas representation, where phase variables are replaced by quantized charges that live on the dual lattice. In order to perform this transformation, the xy model must be studied within the Villain approximation[57], where the cosine potential is replaced by a periodic Gaussian potential,

$$
\begin{equation*}
\cos \left(\theta-\theta^{\prime}\right) \simeq 1-\sum_{m=-\infty}^{\infty} \frac{\left(\theta-\theta^{\prime}-2 \pi m\right)^{2}}{2} \tag{3.14}
\end{equation*}
$$

The Villain model is a very good approximation at low temperatures and naturally includes all $2 \pi$-periodic minima of the interaction by introducing a discrete field $m$ in addition to the continuous vector field $\theta$.

Through standard duality transformations, the partition function of the xy model can be separated into a Gaussian spin-wave part, and a 2D Coulomb gas of logarithmically interacting integer-charges, with the Hamiltonian[56]

$$
\begin{equation*}
H_{C}=2 \pi^{2} \tilde{J} \sum_{r, r^{\prime}} m_{r} V_{r-r^{\prime}} m_{r^{\prime}} \tag{3.15}
\end{equation*}
$$

where $\tilde{J}$ is a renormalized coupling constant. For $f=1 / 2$, it can be shown that $\tilde{J}=J / \sqrt{2}$, which is easily seen by calculating the excitation energy of vortices(see equation (3.54) and ref. [63] ) The $m_{r}$ variables are integer charges which prescribe discrete chiral order to each plaquette. They interact via the potential $V_{r-r^{\prime}}$ which is purely logarithmic as $\left|r-r^{\prime}\right| \rightarrow \infty$,

$$
\begin{equation*}
V_{r-r^{\prime}} \rightarrow \frac{1}{2 \pi} \ln \frac{\left|r-r^{\prime}\right|}{a} \tag{3.16}
\end{equation*}
$$

In addition, the ground state configuration obeys the neutral charge condition,

$$
\begin{equation*}
\sum_{r} m_{r}=0 \tag{3.17}
\end{equation*}
$$

When $f$ is non-zero the xy model becomes frustrated. When transformed into the Coulomb gas representation, the frustrated xy model becomes a fractionally-charged 2D Coulomb gas[71, 72, 73],

$$
\begin{equation*}
H_{C}=2 \pi^{2} \tilde{J} \sum_{r, r^{\prime}}\left(m_{r}+f\right) V_{r-r^{\prime}}\left(m_{r^{\prime}}+f\right) \tag{3.18}
\end{equation*}
$$

where the integer charges $m_{r}$ interact with each other and the with the background field $f$. The charges are constrained by the neutral condition

$$
\begin{equation*}
\sum_{r}\left(m_{r}+f\right)=0 \tag{3.19}
\end{equation*}
$$

We can now interpret Figure 3.3 as representing a lattice of interacting fractional charges, where squares with crosses represent charges of $+1 / 2$ and empty squares represent charges of $-1 / 2$.

The $f=1 / 2$ checkerboard ground state clearly preserves the neutrality condition because there is an equal number of $\pm$ charges. A vortex charge physically corresponds to the amount of circulating supercurrent around a particular plaquette. A plaquette with a cross in Figure 3.3 has a vorticity of $\gamma=2 \pi(m-f)$ with $m=1$, while empty plaquettes correspond to $m=0$. For general $f$, plaquettes containing vortices ( $m=1$ ) have a charge of $1-f$, while empty plaquettes have a charge of $-f$.

## $3.6 f=1 / 2$ Phase Transition

We have seen that the ground state of an unfrustrated array $(f=0)$ is continuously degenerate due to the $U(1)$ global rotational invariance of the phases. This continuous degeneracy permits the formation of vortices as topological defects. Vortices are the only type of topological excitations which are permitted in an unfrustrated array. At low temperatures all thermally excited vortices are tightly bound into vortex-antivortex pairs, and remain bound until the Kosterlitz-Thouless vortex unbinding transition.

In addition to a continuous ground state degeneracy, the ground state of a
fully frustrated array also possesses a discrete $Z_{2}$ degeneracy. This twofold discrete degeneracy arises because the Hamiltonian 3.8 is invariant under the simultaneous reversal of direction of all $\theta_{i}$ and $A_{i j}$. The $Z_{2}$ symmetry manifests itself physically as a reversal of the direction of all currents in the checkerboard pattern of the ground state. This changes clockwise supercurrent circulation to counterclockwise circulation and viceversa. The discrete degeneracy of the ground state breaks the $Z_{2}$ symmetry of the chiral order parameter $m_{r}$, and permits the existence of domain walls as an additional type of topological excitation.

Since the ground state of the fully frustrated array breaks an additional discrete degeneracy, one must consider the possibility of an additional phase transition related to the $Z_{2}$ symmetry. The symmetry of the chiral order parameter $m_{r}$ in the ground state is $U(1) \times Z_{2}$. The continuous $U(1)$ symmetry leads to the formation of vortices and a possible Kosterlitz-Thouless transition, while the discrete $Z_{2}$ symmetry leads to the formation of domains and thereby a possible transition domain wall proliferation. However, the vortices and domain walls are not entirely independent excitations. In the next few sections we will investigate the special properties of these domains and their interplay with vortex excitations, which turns out to be important in understanding the order of phase transitions in the fully frustrated XY model.

### 3.6.1 Domain Wall Transition

A domain wall is topological in nature because it cannot be undone by any smooth continuous deformation of the phases. Domains separate two different ground states, and are characterized by a finite energy $E_{\mathrm{dw}}$ per unit length. At low temperatures, domains that result from thermal fluctuations form closed loops of finite energy. For a frustrated xy model, there are two discrete ground states, and therefore only one type
of domain wall (see Figure 3.4). Therefore, the domain walls of the fully frustrated xy model resemble that of the Ising model.


Figure 3.4: A straight domain wall in a $f=1 / 2$ array.

To make this mapping to the Ising model formal, we first need to identify the relevant Ising variables in the $f=1 / 2$ xy model. In the Coulomb gas representation, the frustrated xy model is described by the order parameter $m_{r}$, which are equivalent to lattice charges which live on the dual lattice. For $f=1 / 2$ there is a regular alteration of positive and negative charges in the ground state which can be written as

$$
\begin{equation*}
m_{r}^{(0)}= \pm \frac{1}{2}(-1)^{x+y} \tag{3.20}
\end{equation*}
$$

where $x$ and $y$ are the plaquette coordinates in units of the lattice constant $a$. The $\pm$ describes the two-fold degeneracy of the checkerboard ground state, where every cell is charged as $\pm 1 / 2$ in an alternating pattern. It is useful to rescale $m_{r}$ by defining the cell chirality as $\sigma_{r}=2 m_{r}$, which then takes the value of $\pm 1$. Furthermore, one can also define the staggered chirality of each cell as

$$
\begin{equation*}
s_{r}=(-1)^{x+y} \sigma_{r} \tag{3.21}
\end{equation*}
$$

The staggered chirality $s_{r}$ turns out to be the relevant Ising spin. To see this, note that $s_{r}$ is either +1 or -1 within a particular ground state. At $T=0$, there is one uniform ground state and therefore $s_{r}$ will also be uniform, equivalent to the Ising ground state.

At finite temperature, domains separate regions of positive and negative staggered chirality, and the system looks identical to the Ising model. The fully frustrated xy model can now be reduced to a domain wall subsystem described by the Hamiltonian

$$
\begin{equation*}
H_{\mathrm{dw}}=-\frac{E_{\mathrm{dw}}}{2} \sum_{\left\langle r r^{\prime}\right\rangle} s_{r} s_{r^{\prime}} \tag{3.22}
\end{equation*}
$$

which is equivalent to the Ising model. The Ising model in two-dimensions on a square lattice has been solved exactly[74], and there exists a phase transition at

$$
\begin{equation*}
T_{\mathrm{dw}}=\frac{E_{\mathrm{dw}}}{\ln (1+\sqrt{2})} \tag{3.23}
\end{equation*}
$$

This transition involves the proliferation of domain walls and the destruction of long range order in $s_{r}$, and can actually be computed through duality relations[75] without even knowing the exact solution to the model (see Appendix C).

Using the numeric result $E_{\mathrm{dw}}=0.343 J$ for the domain energy, shown explicitly in the next section, the estimate for the domain wall proliferation temperature as first given by [64] is

$$
\begin{equation*}
T_{\mathrm{dw}}=0.389 \mathrm{~J} \tag{3.24}
\end{equation*}
$$

So far, we have ignored the interaction of kinks and corners on domain walls for the $f=1 / 2$ domains. The interaction of these defects are long-ranged and therefore turn out
to be important for understanding the sequence of the phase transition. In the following two subsections we calculate the domain energy both numerically by simulating the fully frustrated xy model, and analytically through studying the Coulomb gas representation derived from the Villain model.

### 3.6.1.1 Numeric Domain Energy

Starting with the fully frustrated xy model, we seek to calculate the energy per unit length of a topological domain excitation. In order to calculate the energy of a particular domain wall, we must know the phase of each individual site. This set of phases will correspond to the minima of (3.8) which satisfy equation (3.13). However, the minimization condition (3.13) makes no reference to the underlying pattern of vortices, and so we must additionally constrain the vortex lattice. By performing a constrained optimization of the Hamiltonian (3.8), we can uniquely determine (up to an overall uniform constant) the optimal set of phases, and hence calculate the domain energies.

To formally set-up this problem, we label each gauge-invariant bonds as


Figure 3.5: The left picture is an arbitrary plaquette in which the vorticity is constrained [see (3.26)] to a specified vortex pattern by imposing a condition on the four gauge-invariant phases which surround the plaquette. The right picture represents the minimization condition [see (3.27)] imposed on each site in the array which conserves supercurrent. These two conditions uniquely determine the configuration of phases in the array for a specified domain wall configuration.

$$
\begin{equation*}
\gamma_{k}=\left(\theta_{j}-\theta_{i}-A_{i j}\right) \bmod (2 \pi) \tag{3.25}
\end{equation*}
$$

and restrict each $\gamma_{k}$ to the interval $(-\pi, \pi]$. Each $\gamma_{k}$ measures the gauge-invariant phase difference between two neighboring sites, and therefore it has an associated directionality. For convenience, each x-oriented and y-oriented $\gamma_{k}$-variable are defined to be positive along the x - and y-axis respectively. For an array with $N$ sites there will be $2 N$ gaugeinvariant bonds that need to be determined, and therefore we need $2 N$ independent equations to solve this problem completely.

In Figure 3.5, we show schematically the constraints that need to be imposed on every plaquette and every lattice site. The picture on the left represents a single plaquette in the array, whose vorticity must obey the following constraint,

$$
\begin{equation*}
\gamma_{i}-\gamma_{k}+\gamma_{j}-\gamma_{\ell}=2 \pi\left(m_{A}-f\right) \tag{3.26}
\end{equation*}
$$

The sum of all the gauge-invariant phase differences in a counterclockwise sense yields a restriction on the vorticity. For an arbitrary plaquette $A, m_{A}$ is an integer. For the ground states, as we have already seen, and hence also for the lowest energy domain excitations, $m_{A}$ will be either 0 or 1 . The parameter $m_{A}$ therefore measures the vortex occupation of a plaquette. By choosing a particular set of $m_{A}$, we fix the vortex lattice configuration while allowing individual phases to adjust to minimize the energy.

The picture on the right in Figure 3.5 pictures a single lattice site with four nearest-neighbor bonds. The condition that must be satisfied on each site is Kirchhoff's law for supercurrent, as expressed in (3.13). This conservation equation can be written explicitly as

$$
\begin{equation*}
\sin \gamma_{p}+\sin \gamma_{m}=\sin \gamma_{n}+\sin \gamma_{o} \tag{3.27}
\end{equation*}
$$

We can write (3.26) and (3.27) for each plaquette and lattice site respectively and obtain a set of $2 N$ equations. This system of equations can be solved numerically via the Levenberg-Marquardt Algorithm (LMA)[76]. The LMA numerically solves a system of non-linear equations by starting with an input of some initial guess of the parameters, and then iteratively approaches the solution by using a search direction that is a cross between the Gauss-Newton direction and the steepest descent direction.

In Figure 3.6 we show the array configuration used to solve this problem. The solid black line separates two distinct ground state vortex patterns, and as an initial guess for the gauge-invariant bonds, we simply choose the values corresponding to the two ground states that are being patched together. In the vertical direction, the array only has a height of $2 a$ (two lattice constants) and periodic boundary conditions are imposed. The reason a height of $2 a$ can be used is that the optimal phase configuration will be periodic in units of $2 a$ in a direction parallel to the domain wall, due to the underlying $2 a$ periodic nature of the vortex pattern. Moving in a direction perpendicular to the domain, the phase configuration will not be periodic because theses phases will adjust out to some characteristic distance in order to optimize the domain wall energy. This gives each domain wall a characteristic thickness, the distance to which ground state phases are disrupted due to the presence of a domain wall. We use the parameter $x$ (measured in units of $a$ ) to set the array size to a sufficiently large length. The accuracy of this numeric calculation of domain wall energy will increase as $x$ increases. We used arrays as large as $x \simeq 150$ and obtained a domain wall energy of $E_{\mathrm{dw}}=0.343284(5) J$, in agreement with Halsey's result[63].


Figure 3.6: This is an example of the array configuration we used to numerically compute domain wall energies. On each side of the dotted line, a single ground state phase configuration is chosen. Periodic boundary conditions are imposed in the vertical direction, and in the horizontal direction the length of the array is $x$.

### 3.6.1.2 Analytic Domain Energy

When the frustrated xy model is approximated by the Villain model and then written in the 2D Coulomb gas representation, the corresponding Hamiltonian has the form[56]

$$
\begin{equation*}
H=\sum_{r, r^{\prime}} m_{r} G_{r-r^{\prime}} m_{r^{\prime}} \tag{3.28}
\end{equation*}
$$

where the interaction $G_{r-r^{\prime}}$ is given by (setting the lattice constant $a$ to unit length)

$$
\begin{equation*}
G_{r-r^{\prime}}=\int \frac{d^{2} k}{(2 \pi)^{2}} G_{k} \exp \left[i \mathbf{k} \cdot\left(\mathbf{r}-\mathbf{r}^{\prime}\right)\right] \tag{3.29}
\end{equation*}
$$

and

$$
\begin{equation*}
G_{k}=\frac{\pi^{2} \tilde{J}}{2} \frac{1}{\sin ^{2} \frac{k_{x}}{2}+\sin ^{2} \frac{k_{y}}{2}} \tag{3.30}
\end{equation*}
$$

In the fully frustrated $(f=1 / 2)$ case, the vortex interaction at large distances in the Coulomb gas model $[64,63]$ is $\tilde{J}=J / \sqrt{2}$. Domain walls are topological excitations above the ground state, and so before we calculate the energy of domains, we must first calculate the energy of the ground state. The checkerboard ground state of alternating $+1 / 2$ and $-1 / 2$ vortex charges which live on the dual lattice can be written as

$$
\begin{equation*}
m_{r}^{(0)}= \pm \frac{1}{2}(-1)^{x+y}= \pm \frac{1}{2} \exp \left[i \mathbf{k}^{*} \cdot \mathbf{r}\right] \tag{3.31}
\end{equation*}
$$

where lengths are measured in units of the lattice constant and $\mathbf{k}^{*}= \pm(\pi, \pi)$. Following from (3.28) we can express the ground state energy as

$$
\begin{equation*}
H_{0}=\frac{\pi^{2} \tilde{J}}{2} \int_{-\pi}^{\pi} \frac{d^{2} k}{(2 \pi)^{2}} \frac{\left|\frac{1}{2} \sum_{r} \exp \left[i\left(\mathbf{k}-\mathbf{k}^{*}\right) \cdot \mathbf{r}\right]\right|^{2}}{\sin ^{2} \frac{k_{x}}{2}+\sin ^{2} \frac{k_{y}}{2}} \tag{3.32}
\end{equation*}
$$

In order to simplify this expression, we discretize the integral into a sum and use the completeness relation

$$
\begin{equation*}
\sum_{r} \exp [i \mathbf{k} \cdot \mathbf{r}]=N \delta_{\mathbf{k}, 0} \tag{3.33}
\end{equation*}
$$

where $N$ is the number of lattice sites. This allows us to write (3.32) as

$$
\begin{align*}
H_{0} & =\frac{\pi^{2} \tilde{J}}{8} \frac{1}{N} \sum_{\mathbf{k}} \frac{N^{2} \delta_{\mathbf{k}, \mathbf{k}^{*}}}{\sin ^{2} \frac{k_{x}}{2}+\sin ^{2} \frac{k_{y}}{2}} \\
& =\frac{\pi^{2} \tilde{J}}{8} \frac{N}{\sin ^{2} \frac{\pi}{2}+\sin ^{2} \frac{\pi}{2}} \\
& =\frac{\pi^{2} J}{16 \sqrt{2}} N \tag{3.34}
\end{align*}
$$

Now that we know that the ground state energy scales linearly with the number of sites $N=N_{x} N_{y}$ in the array, we can proceed to calculate the energy of a domain wall. In order to solve this problem, we first consider the interaction of two parallel domain walls a distance $L$ apart (see Figure 3.7). The basic approach will be to find the interaction energy of two parallel domains and then let the separation distance $L$ go to infinity. In Figure 3.7, shaded plaquettes contain a vortex charge of $+\frac{1}{2}$ while empty plaquettes contain a vortex charge of $-\frac{1}{2}$. This configuration of charges can be represented as


Figure 3.7: Two parallel domain walls in the fully frustrated xy model.

$$
\begin{equation*}
m_{r}=\frac{1}{2}(-1)^{y} m_{x} \tag{3.35}
\end{equation*}
$$

where

$$
m_{x}= \begin{cases}-(-1)^{x} & 1 \leq x \leq L \text { (Region B) }  \tag{3.36}\\ (-1)^{x} & \text { otherwise (Region A) }\end{cases}
$$

By plugging (3.35) into (3.32) we obtain

$$
\begin{equation*}
H=\frac{\pi^{2} \tilde{J}}{2} \int_{-\pi}^{\pi} \frac{d^{2} k}{(2 \pi)^{2}} \frac{1}{\sin ^{2} \frac{k_{x}}{2}+\sin ^{2} \frac{k_{y}}{2}}\left|\sum_{r} m_{r} \exp [i \mathbf{k} \cdot \mathbf{r}]\right|^{2} \tag{3.37}
\end{equation*}
$$

We can rewrite the last term with the absolute value as

$$
\begin{align*}
\sum_{r} m_{r} \exp [i \mathbf{k} \cdot \mathbf{r}] & =\frac{1}{2}\left(\sum_{y}(-1)^{y} \exp \left[i k_{y} y\right]\right)\left(\sum_{x} m_{x} \exp \left[i k_{x} x\right]\right) \\
& =\frac{1}{2}\left(\sum_{y} \exp \left[i\left(k_{y}-\pi\right) y\right]\right) \\
& =\frac{N_{y} \delta_{k_{y}, \pi}}{2} \sum_{x} m_{x} \exp \left[i k_{x} x\right] \tag{3.38}
\end{align*}
$$

which allows (3.37) to be written as

$$
\begin{equation*}
H=\frac{\pi^{2} \tilde{J} N_{y}}{8} \int_{-\pi}^{\pi} \frac{d k_{x}}{2 \pi} \frac{1}{1+\sin ^{2} \frac{k_{x}}{2}}\left|\sum_{x} m_{x} \exp \left[i k_{x} x\right]\right|^{2} \tag{3.39}
\end{equation*}
$$

The last term can be expressed as

$$
\begin{align*}
\sum_{x} m_{x} \exp \left[i k_{x} x\right] & =\left(\sum_{x \in A, C}-\sum_{x \in B}\right)(-1)^{x} \exp \left[i k_{x} x\right] \\
& =\left(\sum_{x}-2 \sum_{x \in B}\right)(-1)^{x} \exp \left[i k_{x} x\right] \\
& =N_{x} \delta_{k_{x}, \pi}-2 \sum_{x=1}^{L} \exp \left[i\left(k_{x}-\pi\right) x\right] \tag{3.40}
\end{align*}
$$

where we have used the completeness relation to rewrite the first term in the sum. The second term is simply a geometric sum, and so the square modulus of (3.40) is

$$
\begin{equation*}
\left|\sum_{x} m_{x} \exp [i k x]\right|^{2}=N_{x}^{2} \delta_{k_{x}, \pi}^{2}-4 N_{x} L \delta_{k_{x}, \pi}-4\left(\frac{1-\cos \left[\left(k_{x}-\pi\right) L\right]}{1-\cos \left[k_{x}-\pi\right]}\right) \tag{3.41}
\end{equation*}
$$

By plugging (3.41) into (3.39) and simplifying we obtain

$$
\begin{equation*}
H=\frac{\pi^{2} \tilde{J} N_{x} N_{y}}{16}+\pi^{2} \tilde{J} N_{y}\left[-\frac{1}{2} \int_{-\pi}^{\pi} \frac{d k_{x}}{2 \pi} \frac{1-\cos \left[\left(k_{x}-\pi\right) L\right]}{\left(1+\sin ^{2} \frac{k_{x}}{2}\right)\left(1-\cos \left[k_{x}-\pi\right]\right)}-\frac{L}{4}\right] \tag{3.42}
\end{equation*}
$$

The first term we recognize as the ground state energy and the second term is the interaction energy of two parallel domain walls. By making the substitution $q=k_{x}-\pi$ and simplifying, we can express the interaction energy as

$$
\begin{equation*}
H-H_{0}=\pi^{2} \tilde{J} N_{y}\left[\int_{0}^{2 \pi} \frac{d q}{2 \pi} \frac{1-\cos q L}{(3+\cos q)(1-\cos q)}-\frac{L}{4}\right] \tag{3.43}
\end{equation*}
$$

In order to find the energy for a single domain wall, we must take the limit of this expression as $L \rightarrow \infty$, divide by two because there are two walls, and divide by $N_{y}$ to obtain the energy per unit length,

$$
\begin{equation*}
E_{\mathrm{dw}}=\frac{\pi^{2} J}{2 \sqrt{2}} \lim _{L \rightarrow \infty}\left[\int_{0}^{2 \pi} \frac{d q}{2 \pi} \frac{1-\cos q L}{(3+\cos q)(1-\cos q)}-\frac{L}{4}\right] \tag{3.44}
\end{equation*}
$$

The limit converges to $1 / 8 \sqrt{2}$ and so the expression for the domain energy is simply

$$
\begin{equation*}
E_{\mathrm{dw}}=\frac{\pi^{2}}{32} J \simeq 0.308425 J \tag{3.45}
\end{equation*}
$$

This simple analytic expression agrees with the Korshunov's result[66]. It is also within $11 \%$ of the domain energy found numerically in the previous section, and the difference arises from the fact that our starting point in the analytic calculation was Villain's model, where the true cosine potential was replaced by an infinite sum of the periodic Gaussian minima.

### 3.6.2 Domain Wall Defects and Sequence of Transitions

For a fully frustrated xy model, there is excessive fractional vorticity bound to domain corners, which turns out to be very important in understanding the order of KT and domain wall transitions. Each corner on a $f=1 / 2$ domain behaves as a fractional vortex with a topological charge of $\pm 1 / 4$, whose pairwise interaction is 16 times weaker
than ordinary vortices. By combining two corners, one can form different types of kinks in the domain wall (see Figure 3.8). Simple kinks are composed of two corners of equal topological charge and therefore they have a net charge is $\pm 1 / 2$. Double kinks contain two corners of opposite charge and therefore have no net charge[64].

At very low temperatures, there will be a finite concentration of free double kinks, whereas all simple kinks will be bound into neutral pairs. The presence of free double kinks causes domain wall fluctuations to diverge at any finite temperature. However, since all kinks are neutral or bound in neutral pairs, there exists a well-defined phase gradient perpendicular to each domain wall. Thus phase fluctuations on either side of a domain remain coupled.

As the temperature increases, one would naively expect to see a simple KT transition involving fractional vortices of charge $\pm 1 / 4$. However, the additional constraint imposed on these fractional vortices is that they remain bound to domain wall defects. When $T<T_{\mathrm{dw}}$, all domains form closed loops and so a traditional two-dimensional KT transition of fractional vortices is forbidden. Rather, there will be a one-dimensional KT transition of simple kinks. Fractional vortices in one-dimension have an entropy which scales with the length $L$ of a domain wall as $\ln L$. The unbinding and dissociation of a gas of one-dimensional simple kinks occurs at a temperature $T_{\mathrm{k}}$, leading to a finite concentration of free simple kinks[77].

The proliferation of these free simple kinks of charge $\pm 1 / 2$ decouples phases across domain walls, leading to a loss of the effective phase stiffness in a direction perpendicular to domains. Once the temperature is raised above the kink-unbinding temperature $T_{\mathrm{k}}$, the rigidity for phase fluctuations on domain walls is destroyed. This is the mechanism which causes phase fluctuations on either side of a domain to decouple


Figure 3.8: (a) Double kink. (b) Simple kink. There are fractional vortices of charge $\pm 1 / 4$ bound to domain wall corners for $f=1 / 2$. The presence of excessive vorticity can be seen by averaging the chiralities of the four cells around each corner. Each $\pm$ pair of charges constitutes a dipole.
and become independent. Since the coherency of phase fluctuations is destroyed above $T_{\mathrm{k}}$, the proliferation of ordinary vortices at $T_{\mathrm{KT}}$ occurs automatically. In the Coulomb gas representation, we can understand this by noting that once kinks have dissociated, there will be a finite concentration of fractional vortices which are free to move about. These fractional vortices will screen the interaction of normal vortices that are separated by a domain wall. Therefore, all vortex-anitvortex pairs separated by a domain wall will immediately unbind. In effect, the dissociation of fractional domain wall defects initiates the dissociation of ordinary bulk vortices.

For $f=1 / 2$ domain walls the transition order $T_{\mathrm{k}}<T_{\mathrm{dw}}$ is allowed in principle, but $T_{\mathrm{k}}>T_{\mathrm{dw}}$ is forbidden. Starting at zero temperature, the kink-unbinding transition
can precede the lattice melting transition but not vice-versa, a fact demonstrated explicitly by Korshunov[64]. Below $T_{\mathrm{dw}}$, kinks are free to proliferate, which will in turn cause ordinary vortices to proliferate. However, as soon as the temperature exceeds $T_{\mathrm{dw}}$ in the thermodynamic limit, there will emerge infinite domain walls which destroy any long-range lattice ordering of the chiralities. The presence of infinite domains decouples the phase fluctuations on either side of a domain wall as the effective stiffness of the system with respect to the continuous twisting of the phase is lost. Therefore, the required order of transitions in a fully frustrated array is $T_{\mathrm{KT}} \leq T_{\mathrm{dw}}$. The existence of two transitions which satisfy this condition has been verified through various numerical methods $[78,79,80,81]$, most recently by Okumura et al[82].

### 3.6.3 Kosterlitz-Thouless Transition

We have seen that $T_{\mathrm{KT}} \leq T_{\mathrm{dw}}$ is a required condition for a fully frustrated array. In an unfrustrated array, no such condition exists because the KT transition is the only type of transition allowed. The only broken symmetry at low temperatures is the continuous $\mathrm{U}(1)$ symmetry, which is why there is only one phase transition (the KT transition) in an unfrustrated array. In this section, we are interested in calculating how the KT transition depends on the degree of frustration in the system. In general, one expects that the KT transition will be a function of the frustation index, which we will denote as $T_{\mathrm{KT}}^{\mathrm{f}}$. We will look specifically at the $f=0$ and $f=1 / 2$ cases, and calculate the expected factor for the ratio of transitions.

The simplest Kosterlitz-Thouless transition to analyze occurs in an unfrustrated array. The unbinding of vortices is the feature which defines a KT transition. The lowest energy excitations in an unfrustrated array are vortices, where the superconducting phase $\theta$ winds by an integer multiple of $2 \pi$ around a closed contour which
contains the vortex core. By expanding the cosine in (3.7) we find[50]

$$
\begin{equation*}
E_{\mathrm{V}}^{0}=\frac{J}{2} \sum_{\langle i j\rangle}\left(\theta_{i}-\theta_{j}\right)^{2} \tag{3.46}
\end{equation*}
$$

where $E_{0}$ is the ground state energy, and $E_{\mathrm{V}}^{0}$ is the excitation energy of a vortex for $f=0$. For a large system, the lattice spacing $a$ between spins is very small and it is convenient to write (3.46) in continuous notation as

$$
\begin{align*}
E_{\mathrm{V}}^{0} & \simeq \frac{J}{2} \int d^{2} r\left(\frac{\theta_{i}-\theta_{j}}{a}\right)^{2} \\
& \simeq \frac{J}{2} \int d^{2} r\left(\nabla \theta_{r}\right)^{2} \\
& =\pi J n^{2} \ln \frac{R}{a} \tag{3.47}
\end{align*}
$$

where $n$ is the vorticity or integer winding number of the vortex, $R$ is the radius of the sample, and $a$ is the lattice spacing.

The energy of a single vortex is logarithmically divergent with system size, and so in the thermodynamic limit vortices should not appear as thermal excitations. However, a system of two vortices has finite energy as long as the total vorticity vanishes. There is a simple energy-entropy argument known as the Kosterlitz-Thouless criterion[51] which determines the temperature at which vortex generation becomes important. A single vortex can live in roughly $R^{2} / a^{2}$ different locations, and so its entropy is

$$
\begin{equation*}
S_{\mathrm{V}}=2 k_{B} \ln \frac{R}{a} \tag{3.48}
\end{equation*}
$$

In thermal equilibrium, the Helmholtz free energy tends toward the global minima by minimizing the system's energy and maximizing it's entropy. The free energy for a single vortex is

$$
\begin{equation*}
F=E_{\mathrm{V}}^{0}-T S_{\mathrm{V}}=\left(\pi J n^{2}-2 k_{B} T\right) \ln \frac{R}{a} \tag{3.49}
\end{equation*}
$$

From this expression we see that when $T<\pi J / 2 k_{B}$ the free energy is positive and the existence of free unit vortices is highly improbable. In this temperature regime it is energetically very costly for single vortices to occur as thermal fluctuations, and they are therefore suppressed. Although free vortices are strongly suppressed in this low temperature regime, tightly bound vortices can form dipole pairs of zero net vorticity, since the energy cost is finite. When $T>\pi J / 2 k_{B}$, the free energy is negative and free vortices with $n= \pm 1$ begin to proliferate. This is known as the Kosterlitz-Thouless transition temperature, which for an unfrustrated conventional array occurs at

$$
\begin{equation*}
T_{\mathrm{KT}}^{0}=\frac{\pi J}{2 k_{B}} \tag{3.50}
\end{equation*}
$$

The proliferation of vortices in a fully frustrated array, where vortex-antivortex excitations occur on top of the checkerboard ground state, modify the KT transition temperature from that of the unfrustrated case. Our estimates of the transition temperature at $f=0$ have so far ignored the renormalization corrections. However, we are assuming that renormalization of $J$ at $f=1 / 2$ is the same as in the absence of frustration[64], and therefore the ratio of the transitions will be unaffected by this factor. In order to find the KT transition at $f=1 / 2$, consider small deviations about the checkerboard ground state. Let us denote the gauge-invariant phase difference as $\theta_{i j}=\theta_{j}-\theta_{i}-A_{i j}$ and small deviations $\delta \theta_{i j}$ about the ground state as

$$
\begin{equation*}
\theta_{i j}= \pm \frac{\pi}{4}+\delta \theta_{i j} \tag{3.51}
\end{equation*}
$$

Now we can expand each term in (3.8) as

$$
\begin{align*}
\cos \left(\theta_{j}-\theta_{i}-A_{i j}\right) & =\cos \left(\theta_{i j}\right) \\
& =\cos \left( \pm \frac{\pi}{4}+\delta \theta_{i j}\right) \\
& \simeq \cos \left(\frac{\pi}{4}\right) \mp \sin \left(\frac{\pi}{4}\right) \delta \theta_{i j}-\frac{1}{2} \cos \left(\frac{\pi}{4}\right)\left(\delta \theta_{i j}\right)^{2} \tag{3.52}
\end{align*}
$$

Now when we sum over the entire array, the linear terms cancel out and we get

$$
\begin{equation*}
H=-J \sum_{\langle i j\rangle} \frac{1}{\sqrt{2}}\left(1-\frac{\delta \theta_{i j}^{2}}{2}\right) \tag{3.53}
\end{equation*}
$$

By subtracting off the constant ground state energy, the energy of a vortex excitation in a fully frustrated array can be written as

$$
\begin{equation*}
E_{\mathrm{V}}^{1 / 2}=\frac{J}{2 \sqrt{2}} \sum_{\langle i j\rangle} \delta \theta_{i j}^{2} \tag{3.54}
\end{equation*}
$$

The procedure for calculating the KT transition now follows an equivalent energy-entropy argument which lead to (3.50). By taking the vortex energy to be

$$
\begin{equation*}
E_{\mathrm{V}}^{1 / 2}=\frac{\pi J}{\sqrt{2}} \ln \frac{R}{a} \tag{3.55}
\end{equation*}
$$

one would naively expect the KT transition temperature[63] to be

$$
\begin{equation*}
T_{\mathrm{KT}}^{1 / 2}=\frac{\pi J}{2^{3 / 2} k_{B}} \tag{3.56}
\end{equation*}
$$

which gives $T_{\mathrm{KT}}^{1 / 2}=1.111 J$ (in units where $k_{B}=1$ ). This however violates the condition $T_{\mathrm{KT}} \leq T_{\mathrm{dw}}$ because we have shown that $T_{\mathrm{dw}}=0.389 \mathrm{~J}$. The reason for the discrepancy is because we have completely ignored the presence of domain walls. As we have shown in the previous section, the proliferation of domain walls will simultaneously initiate the KT transition if it has not yet occurred. Since we would otherwise expect the KT
transition to occur at a higher temperature than the domain wall transition, the barecoupling in (3.56) must be replaced by an effective coupling $J^{\prime}$ which accounts for the interaction with domain walls.

This effective coupling must be chosen to lower the expected KT transition to the domain wall transition. This condition is $T_{\mathrm{KT}} \leq T_{\mathrm{dw}}$ and $J^{\prime}$ can be defined as

$$
\begin{equation*}
J^{\prime}=\frac{2 \sqrt{2}}{\pi} \frac{E_{\mathrm{dw}}}{\ln (1+\sqrt{2})} \tag{3.57}
\end{equation*}
$$

This leads to a theoretical ratio of

$$
\begin{equation*}
\frac{T_{\mathrm{KT}}^{1 / 2}}{T_{\mathrm{KT}}^{0}} \lesssim \frac{2}{\pi} \frac{E_{\mathrm{dw}} / J}{\ln (1+\sqrt{2})} \simeq 0.248 \tag{3.58}
\end{equation*}
$$

The lowering of the KT transition for $f=1 / 2$ compared with $f=0$ has been confirmed experimentally by J.E. Mooij et al.[83], and they measured a ratio of $T_{\mathrm{KT}}^{1 / 2} / T_{\mathrm{KT}}^{0} \simeq 0.3$. Monte Carlo simulations for unfrustrated[84] and fully frustrated xy models[82] yield a KT ratio of $T_{\mathrm{KT}}^{1 / 2} / T_{\mathrm{KT}}^{0} \simeq 0.495$.

### 3.7 Current-Induced Vortex Unbinding

The primary way the Kosterlitz-Thouless transition can be measured experimentally is by monitoring the current-voltage (IV) characteristics of the array. By applying a linear supercurrent to a 2D superconductor, the resulting voltage difference will vary as

$$
\begin{equation*}
V \sim I^{a(T)} \tag{3.59}
\end{equation*}
$$

where the exponent $a(T)$ is a function of temperature. The temperature at which $a(T)=$ 3 can be identified as the Kosterlitz-Thouless transition temperature. In particular, we
will now show following [61] that as the temperature in a 2D superconductor increases from below $T_{\mathrm{KT}}$ to above $T_{\mathrm{KT}}$, the exponent $a(T)$ will jump from 3 to 1 at $T_{\mathrm{KT}}$.

At any finite temperature, vortices arise as thermal fluctuations in 2D superconductors. In the presence of an applied supercurrent, vortices will experience a Lorentz force

$$
\begin{equation*}
F_{\mathrm{L}}=\frac{J_{s}}{c} \Phi_{0} \tag{3.60}
\end{equation*}
$$

where $\Phi_{0}=h c / 2 \pi$ is the superconducting flux quantum and $J_{\mathrm{s}}=n_{\mathrm{s}} e v_{\mathrm{s}}$ is the applied supercurrent, in terms of the velocity $v_{s}$ and number density $n_{s}$ of the superconducting electrons. The Lorentz force (3.60), which arises from the applied supercurrent interacting with the magnetic field of a vortex, tends to move vortices at a right angle to the direction of the current. Due to the current-induced steady, dissipative, net vortex motion, there will be a non-zero array resistance when there are free vortices present. Below the Kosterlitz-Thouless transition, all vortices are tightly bound together in pairs of zero net vorticity.

The Lorentz force on each vortex in a vortex-antivortex pair is equal and opposite, and therefore the net force is zero and there should be no net vortex motion. Without any net vortex drift, zero-resistivity, a major feature of superconductors, should remain. However, it turns out that an applied supercurrent will induce vortex unbinding because the Lorentz force will break some of the vortex pairs apart. Strictly speaking, zero resistance will only remain in the array, assuming there is no vortex pinning, when $J_{s}=0$. Any arbitrarily-sized current can break apart vortex pairs that are sufficiently separated. This yields a finite concentration of free vortices, which is in stable equilibrium via steady recombination and separation rates. In summary, we do expect a finite
array resistance below the KT transition because an applied supercurrent will dissociate certain vortex pairs and thereby lead to dissipative vortex motion.

As the current increases, a larger fraction of vortices will dissociate and therefore the array resistance will also increase. Larger currents are able to break apart vortex pairs that are more tightly bound. Therefore the resistance $R(I)$ will be an increasing function of current $I$. In order to determine the form of this function, we must first start with the energy per unit length of a 2D Pearl vortex-antivortex pair energy via (1.21),

$$
\begin{equation*}
E_{\mathrm{PV}}^{ \pm}=2 \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda^{2}} \ln \frac{r}{\xi} \tag{3.61}
\end{equation*}
$$

where $r$ is the separation distance and $\xi$ is the Ginzburg-Landau coherence length. The interaction is logarithmic and the coefficient can be written in terms of the vortex charge $q$ by identifying

$$
\begin{equation*}
q^{2}=\frac{\Phi_{0}^{2}}{8 \pi^{2} \lambda^{2}}=\frac{\pi \hbar^{2} \rho_{\mathrm{s}}}{2 m^{2}} \tag{3.62}
\end{equation*}
$$

where we have written this expression in terms of the definition of the penetration depth $\lambda$,

$$
\begin{equation*}
\lambda^{2}=\frac{m^{2} c^{2}}{4 \pi \rho_{\mathrm{s}}} \tag{3.63}
\end{equation*}
$$

The superfluid mass density is related to the number density via $\rho_{\mathrm{s}}=m n_{\mathrm{s}}$. We can now write the energy (per unit length) of a pair of 2 D vortices as

$$
\begin{equation*}
U_{0}(r)=q^{2} \ln \frac{r}{\xi} \tag{3.64}
\end{equation*}
$$

This expression is unrenormalized, and to include the effects of screening we would simply have to replace the vortex charge by its renormalized value, $q \rightarrow q_{\mathrm{R}}$.

In the presence of an applied supercurrent, the total energy of a vortexantivortex pair, orientated such that the line connecting them is perpendicular to the direction of the current, becomes

$$
\begin{equation*}
U(r)=U_{0}(r)-2 F_{\mathrm{L}} r \tag{3.65}
\end{equation*}
$$

where the second term is the potential energy on a vortex pair due to the Lorentz force. Using (3.60) and (3.62), this energy can be written as

$$
\begin{equation*}
U(r)=q^{2}\left[\ln \frac{r}{\xi}-\frac{2 m v_{\mathrm{s}} r}{\hbar}\right] \tag{3.66}
\end{equation*}
$$

This energy has a maxima at a critical separation distance of

$$
\begin{equation*}
r_{\mathrm{c}}=\frac{\hbar \rho_{\mathrm{s}} e}{2 m^{2} J_{\mathrm{s}}} \tag{3.67}
\end{equation*}
$$

where $J_{\mathrm{s}}=\rho_{\mathrm{s}} e v_{\mathrm{s}} / m$ is the applied supercurrent. Bound vortex pairs with a separation distance that meets or exceeds the critical separation $r_{\mathrm{c}}$ will unbind. Therefore, $r_{\mathrm{c}}$ is essentially the escape radius for bound vortex pairs. The energy at $r_{\mathrm{c}}$ is

$$
\begin{align*}
U\left(r_{\mathrm{c}}\right) & =q^{2}\left[\ln \frac{r_{\mathrm{c}}}{\xi}-1\right] \\
& \simeq-q^{2} \ln \frac{J_{\mathrm{s}}}{J_{0}} \tag{3.68}
\end{align*}
$$

where we have defined $J_{0}=\hbar \rho_{\mathrm{s}} e / 2 m^{2} \xi$ to be the Ginzburg-Landau critical current. The approximation in the first line above to ignore the constant term is justified when $r_{\mathrm{c}} \gg \xi$. This will be the case when the applied current $J_{\mathrm{s}}$ is very small or $\rho_{\mathrm{s}}$ is very large.

The maximum vortex pair energy (3.68) allows us now to calculate the rate at which bound vortices escape and become free "particles". Only vortices with sufficient
separation will have enough energy to escape from their effective potential well. At any finite temperature $T<T_{\mathrm{KT}}$, a system of bound vortices will be described by the Boltzmann distribution function, and so the probability that a particular vortex-antivortex pair has an energy of $U\left(r_{\mathrm{c}}\right)$ is proportional to $\exp \left[-U\left(r_{\mathrm{c}}\right) / k_{B} T\right]$. The classical escape rate over a potential barrier will be proportional to this probability, and we can therefore write that the rate of production of free vortices scales as (ignoring the constant factors)

$$
\begin{equation*}
\Gamma_{\mathrm{V}} \sim \exp \left[-U\left(r_{\mathrm{c}}\right) / k_{B} T\right] \tag{3.69}
\end{equation*}
$$

Explicitly this can be written as

$$
\begin{equation*}
\Gamma_{\mathrm{V}} \sim\left(\frac{J_{\mathrm{S}}}{J_{0}}\right)^{q^{2} / k_{B} T} \tag{3.70}
\end{equation*}
$$

In order to determine the overall density of free vortices, $N_{\mathrm{F}}$, we must consider the rate of production and the rate of recombination of vortex pairs. The overall rate of change in the number of free vortices will be governed by the rate equation

$$
\begin{equation*}
\dot{N}_{\mathrm{F}}=\Gamma_{\mathrm{V}}-\alpha N_{\mathrm{F}}^{2} \tag{3.71}
\end{equation*}
$$

The first term on the left of (3.71) is the rate of production of free vortices, and the second term is the rate of recombination of free vortices. The recombination term scales as $N_{\mathrm{F}}^{2}$ because it is a 2-body process. In equilibrium the rate of change of free vortices, $\dot{N}_{\mathrm{F}}$, is zero and therefore $N_{\mathrm{F}}$ scales as

$$
\begin{equation*}
N_{\mathrm{F}} \sim \Gamma_{\mathrm{V}}^{1 / 2} \sim\left(\frac{J_{\mathrm{s}}}{J_{0}}\right)^{q^{2} / 2 k_{B} T} \tag{3.72}
\end{equation*}
$$

The resistance also turns out[85] to be directly proportional to the number of free vortices, $R \sim N_{\mathrm{F}}$. We can now write an expression for $R(I)$,

$$
\begin{equation*}
R(I) \sim I^{q^{2} / 2 k_{B} T} \tag{3.73}
\end{equation*}
$$

This follows from (3.72), and we replaced the applied supercurrent density $J_{\mathrm{s}}$ by the applied supercurrent $I$ since they are proportional to each other (drop the subscript s).

Now using $V=I R$, we find

$$
\begin{equation*}
V \sim I^{a(T)} \tag{3.74}
\end{equation*}
$$

where

$$
\begin{equation*}
a(T)=q^{2} / 2 k_{B} T+1 \tag{3.75}
\end{equation*}
$$

This can be expressed in terms of $\rho_{\mathrm{s}}$ via (3.62),

$$
\begin{equation*}
a(T)=\frac{\pi \hbar^{2}}{4 k_{B} m^{2}} \frac{\rho_{\mathrm{s}}(T)}{T}+1 \tag{3.76}
\end{equation*}
$$

As the Kosterlitz-Thouless transition is approached from below, we can utilize the result for the universal jump in superfluid stiffness,

$$
\begin{equation*}
\frac{\rho\left(T_{\mathrm{KT}}\right)}{T_{\mathrm{KT}}}=\frac{8 k_{B}}{\pi}\left(\frac{m}{\hbar}\right)^{2} \tag{3.77}
\end{equation*}
$$

which was derived in Section 2.6. Plugging (3.77) into (3.76) we obtain

$$
\begin{equation*}
a\left(T \rightarrow T_{\mathrm{KT}}^{-}\right)=3 \tag{3.78}
\end{equation*}
$$

If the KT transition temperature is approached from above we obtain

$$
\begin{equation*}
a\left(T \rightarrow T_{\mathrm{KT}}^{+}\right)=1 \tag{3.79}
\end{equation*}
$$

due to the universal jump in $\rho_{\mathrm{s}}\left(T_{\mathrm{KT}}\right)$. This makes sense since above $T_{\mathrm{KT}}$, the resistance which is proportional to the number of free vortices is independent of the strength of the applied supercurrent since all vortex pairs have dissociated. Strictly speaking, a sharp jump from 3 to 1 in $a(T)$ at $T_{\mathrm{KT}}$ only occurs in the thermodynamic limit. In any finite array there will be a finite width to the transition.

In summary, what we have shown is that a finite current $I$ tends to dissociate neutral bound vortex pairs just below $T_{\mathrm{KT}}$, giving rise to a nonlinear resistance, $V \sim$ $I^{a(T)}$, with $a\left(T_{\mathrm{KT}}\right)=3[61]$. The unbinding of vortex pairs above $T_{\mathrm{KT}}$ gives rise to a linear sheet resistance. The resistance jumps from nonlinear dependance just below $T_{\mathrm{KT}}, V \sim I^{3}$, to linear dependance above $T_{\mathrm{KT}}, V \sim I$. This jump is characteristic of the KT transition, and was first confirmed experimentally by Resnick et al.[86] in 1981.

## Chapter 4

## $p+i p$ Josephson Junction Arrays

### 4.1 Introduction

It has been discussed earlier that half-quantum vortices ( $\frac{1}{2} \mathrm{QVs}$ ) may be possible in $p+i p$ superconductors, and just recently they were discovered in annular-shaped mesoscopic samples of superconducting $\mathrm{Sr}_{2} \mathrm{RuO}_{4}[42]$. These excitations are exotic not only because they bind Majorana zero modes to their cores[39], but also because they are characterized by non-Abelian exchange statistics[41, 29, 39, 43]. Moreover, they are topologically stable against local, external perturbations[87] and thus offer an environment in which their non-trivial vortex statistics can potentially be both probed and exploited[88].
$\frac{1}{2} \mathrm{QVs}$ carry a flux of $\Phi_{0} / 2$, exactly half of the superconducting flux quantum $\Phi_{0}=h c / 2 e$. They are permitted in $p+i p$ SCs because triplet-pairing of Cooper pairs introduces a spin degree of freedom in addition to the orbital phase $\theta$ of the condensate wavefunction. The chiral order parameter, which characterizes the superconducting state, depends not only on $\theta$, but also on the symmetry direction $\hat{\mathbf{d}}$ of Cooper pair spin. A $\frac{1}{2} \mathrm{QV}$ simultaneously combines rotations of the director vector $\hat{\mathbf{d}}$ by $\pi$ and with the
phase $\theta$ by $\pi$ upon circulation of the vortex core, leaving the multicomponent order parameter single-valued. In a singlet SC , the existence of $\frac{1}{2} \mathrm{QVs}$ as topological defects is forbidden because of the requirement that the order parameter remain single-valued.

We suggest detecting the presence of $\frac{1}{2}$ QVs by measuring the ratio of KosterlitzThouless (KT) transitions in both unfrustrated and fully frustrated $p+i p$ Josephson junction arrays. By monitoring the nonlinear IV characteristics[61] in a JJ array, it is possible to directly measure the KT transition[83]. Although this transition does depend on array properties such as the effective penetration depth, we predict that the ratio of KT transitions at different frustrations in a JJ array will be material independent and thus a universal quantity. In p-wave SCs, both QVs and $\frac{1}{2} \mathrm{QVs}$ are legitimate topological excitations, and we argue that $\frac{1}{2} \mathrm{QVs}$ are energetically favorable in $p+i p \mathrm{JJ}$ arrays and should therefore drive a KT transition by proliferating at a lower temperature than QVs.

We propose two methods for detecting $\frac{1}{2}$ QVs. The first method involves measuring the ratio of KT transition temperatures between a fully frustrated and an unfrustrated $p+i p \mathrm{JJ}$ array, which is a universal quantity since it is independent of the microscopic properties of the array and is true for any system described by the frustrated xy model. If the KT transition is driven by $\frac{1}{2} \mathrm{QVs}$, we predict that there will be a significant deviation from the value of the ratio one would expect for ordinary QVs. The second method involves directly measuring the KT transition at $f=0$, which will depend on the microscopic parameters governing the array. The proliferation temperature of $\frac{1}{2} \mathrm{QVs}$ is expected theoretically to be lower than the proliferation of QVs by at least a factor of two. If the $K T$ transition was observed experimentally to occur at temperatures much lower than expected, we could infer that the transition was in fact driven by the proliferation of $\frac{1}{2} \mathrm{QVs}$.

In this chapter we will introduce $p+i p$ JJ arrays. In Section 4.2, we propose
a model to describe $p+i p$ JJ arrays. We then apply this model in Section 4.3 to fully frustrated $p+i p$ JJ arrays. This includes determining the ground state and calculating domain energies. In particular, we identify three topological defects in $p+i p \mathrm{JJ}$ arrays that arise as a result of the $U(1)$ winding symmetry, and then calculate transition ratios for fully frustrated and unfrustrated arrays. In Sections 4.4-4.5 we propose two different ways in which one could measure the effects of half-quantum vortices. One of these methods is independent of the array and depends only on the ratio of transitions, and the other method involves computing the specific parameters of the array.

## $4.2 \quad p+i p$ JJ Array Hamiltonian

If one were to construct a Josephson junction array out of p -wave superconducting material, half-quantum vortex excitations has to be taken into account. A phenomenological model which describes a $p+i p \mathrm{JJ}$ array is

$$
\begin{equation*}
H=-J\left(\frac{9-\alpha}{8}\right) \sum_{\langle i j\rangle} \cos \left(\theta_{j}-\theta_{i}-A_{i j}\right)\left[\cos \left(\phi_{i}-\phi_{j}\right)-\left(\frac{1-\alpha}{9-\alpha}\right) \cos 3\left(\phi_{i}-\phi_{j}\right)\right] \tag{4.1}
\end{equation*}
$$

where $J$ is the Josephson coupling energy [3] and $\alpha=\rho_{\mathrm{sp}} / \rho_{\mathrm{s}}$ is the ratio of the spin stiffness to the superfluid stiffness. The sum over $\langle i j\rangle$ denotes nearest neighbors only. The $A_{i j}$ links are lattice bonds defined as the line integral of the vector potential $\mathbf{A}$, which in the Landau gauge takes the form [see equation (3.11)]

$$
\begin{equation*}
A_{i j}=2 \pi f n \tag{4.2}
\end{equation*}
$$

where $n=0,1,2, \ldots$ is an index which labels successive columns and $f$ is the frustration index (see Section 3.3 for a complete explanation).

The first term in (4.1) is a cosine of the gauge-invariant superconducting phase difference, and this term on its own is simply the frustrated xy model [see equation (3.8)]. On its own this term is

$$
\begin{equation*}
H=-J \sum_{\langle i j\rangle} \cos \left(\theta_{j}-\theta_{i}-A_{i j}\right) \tag{4.3}
\end{equation*}
$$

and is used to describe conventional JJ arrays in transverse magnetic fields. To describe $p+i p \mathrm{JJ}$ arrays we must use the more general expression as in (4.1). The second term in (4.1) contains the sum of two cosines, which are functions of $\phi$, the 2 D polar angle of the complex vector $\mathbf{d}(\mathbf{k})$. This hamiltonian describes the physics of JJ arrays constructed of p-wave material through the use of two vector fields, $\theta$ and $\phi$, which are restricted to rotate in the xy-plane. If $\phi$ cannot vary from site to site (i.e. we are dealing with a conventional superconducting array), then all terms containing $\phi$ drop out of (4.1) and it reduces simply to (4.3).

The Hamiltonian (4.1) is invariant under the following transformations, $\theta_{i} \rightarrow$ $\theta_{i}+2 \pi, \phi_{i} \rightarrow \phi_{i}+2 \pi$, and $\theta_{i} \rightarrow \theta_{i}+\pi$ combined with $\phi_{i} \rightarrow \phi_{i}+\pi$. This means that various types of low energy topological excitations are allowed; full vortices in $\theta$, full vortices in $\phi$, and half vortices which involve simultaneous rotations of both $\theta$ and $\phi$. The two cosine terms in $\phi$ in (4.1) are $\pi$-periodic and are necessary in order to allow an asymmetry between the spin stiffness and the superfluid stiffness. If the stiffnesses are the same $(\alpha=1)$, the last cosine drops out completely. For each type of vortex allowed by the symmetries of (4.1), one must consider the possibility of a corresponding KT transition.

The low energy expansion of (4.1) in continuous notation is

$$
\begin{equation*}
H-H_{0} \simeq J \int d^{2} r{\frac{(\nabla \theta)^{2}}{2}}^{2}+J \alpha \int d^{2} r \frac{(\nabla \phi)^{2}}{2} \tag{4.4}
\end{equation*}
$$

where we have retained terms up to quadratic order. This form resembles the GinzburgLandau free energy for small fluctuations (twists) in the order parameter. Each neighboring interaction now has the form

$$
\begin{equation*}
\frac{\rho_{\mathrm{s}}}{2}(\nabla \theta)^{2}+\frac{\rho_{\mathrm{sp}}}{2}(\nabla \phi)^{2} \tag{4.5}
\end{equation*}
$$

since $\rho_{\mathrm{s}} \sim J$.

It is now clear why we chose the complicated factors of $\alpha$ in (4.1); it allows us to fix the superfluid stiffness $\rho_{\mathrm{s}}$ while allowing the spin stiffness $\rho_{\mathrm{sp}}$ to vary. In principle we should allow terms such as $\cos (5 \Delta \phi), \cos (7 \Delta \phi)$, and all higher order terms in order to keep $\rho_{\mathrm{s}}$ perfectly rigid. We have examined the simplest non-trivial case in (4.1) where $\rho_{\mathrm{s}} \neq \rho_{\mathrm{sp}}$. All higher-order terms in $\cos \Delta \phi$ will simply add small corrections to our analysis.

The Hamiltonian (4.1) can be understood as a model for a $p+i p$ JJ array as follows. A $p+i p \mathrm{JJ}$ array will have two degrees of freedom, $\theta$, which is the superconducting phase of the condensate, and $\phi$, which measures the 2D planar direction of the director vector $\hat{\mathbf{d}}$. Windings in either $\theta$ or $\phi$ cost energy, and relative windings between $\theta$ and $\phi$ also cost energy. The first cosine term is the frustrated xy model in $\theta$, and describes the energy cost of twisting $\theta$ from site to site and its coupling to the electromagnetic field. The next term is a sum of cosines that multiply the first cosine. Two cosines are needed in order to allow for a difference between the spin stiffness and superfluid stiffness. Each of these cosines must be $2 \pi$-periodic to account for vortices in $\phi$, and $\cos (3 \phi)$ is the first non-trivial term that can be added which satisfies this
condition. Also note that the spin sector does not couple to the electromagnetic field and so there is term for the vector potential as in the case for $\theta$.

To see that (4.1) has the right form, consider a low energy expansion with zero magnetic field, as in (4.4). Windings in the superfluid and spin sector are independent and add linearly. The total energy of a vortex excitation is just the sum of the winding energies within each sector. Hamiltonian (4.1) also has the right symmetry to describe the low energy physics of a $p+i p \mathrm{JJ}$ array. A product of the two cosine terms means that if the $\phi$-field is uniform, excitations only cost energy due to $\theta$-windings, and vice-versa. The three types of elementary vortex excitations are full vortices in $\theta$, full vortices in $\phi$, and half-vortices which combine rotations of both $\theta$ and $\phi$. The Hamiltonian (4.1) must be single-valued around any one of these vortex excitations, and it is easy to check that it indeed is.

It is useful to express the Hamiltonian (4.1) in a different form by defining the variables

$$
\begin{align*}
& u=\theta+\phi  \tag{4.6}\\
& v=\theta-\phi \tag{4.7}
\end{align*}
$$

which allows us to write

$$
\begin{align*}
& H=-\frac{\hat{J}}{2}\left\{\sum_{\langle i j\rangle} \cos \left(u_{j}-u_{i}-A_{i j}\right)+\sum_{\langle i j\rangle} \cos \left(v_{j}-v_{i}-A_{i j}\right)\right\} \\
& +\frac{\hat{J} \hat{\alpha}}{2}\left\{\sum_{\langle i j\rangle} \cos \left(2\left(u_{j}-u_{i}\right)-\left(v_{j}-v_{i}\right)-A_{i j}\right)+\sum_{\langle i j\rangle} \cos \left(2\left(v_{j}-v_{i}\right)-\left(u_{j}-u_{i}\right)-A_{i j}\right)\right\} \tag{4.8}
\end{align*}
$$

where $\hat{J}=(9-\alpha) J / 8$ and $\hat{\alpha}=(1-\alpha) /(9-\alpha)$. When the stiffnesses are equal $(\alpha=1)$, the second term in (4.8) vanishes since $\hat{\alpha}=0$, and the model reduces to

$$
\begin{equation*}
H(\alpha=1)=-\frac{J}{2}\left\{\sum_{\langle i j\rangle} \cos \left(u_{j}-u_{i}-A_{i j}\right)+\sum_{\langle i j\rangle} \cos \left(v_{j}-v_{i}-A_{i j}\right)\right\} \tag{4.9}
\end{equation*}
$$

Therefore, when the superfluid stiffness and the spin stiffness are equal, the Hamiltonian which describes the system can be thought of as the sum of two independent frustrated xy models in the $u$ and $v$ degrees of freedom, with half the coupling energy $(J \rightarrow J / 2)$.

### 4.3 Fully Frustrated $p+i p$ JJ Arrays

A $p+i p$ superconducting JJ array has two degrees of freedom, $\theta$ and $\phi$, as described by (4.1). The ground state for zero frustration is trivially uniform $\theta$ and $\phi$ over the entire array. For a fully frustrated $p+i p \mathrm{JJ}$ array, finding the ground state is a more complicated matter. When $\alpha=1$ the p -wave Hamiltonian reduces to two independent xy models, as in (4.9). We also know that the ground state for a conventional array at $f=1 / 2$ is the checkerboard pattern, which is doubly degenerate. Therefore, the degeneracy of the ground state at $\alpha=1$ is found by combining the twofold degeneracy in the $u$ and $v$ variables, which gives a total of $2 \times 2=4$ states. The four degenerate ground states at $\alpha=1$ are shown in Figure 4.1(a). Each of the four states corresponds to a checkerboard phase configuration in both $u$ and $v$. In order to determine which of these four states remain true ground states as $\alpha$ decreases, we have plotted the energy of each of these states as a function of $\alpha$ in Figure 4.1(b). It turns out that two states increase in energy while two decrease. The two states that decrease in energy and remain true ground states are the states with opposite checkerboard patterns in $u$ and $v$ as shown


Figure 4.1: (a) The four degenerate ground states at $\alpha=1$ and the two degenerate ground states when $\alpha<1$ in a fully frustrated $p+i p$ JJ array. Each state is drawn in terms of the independent $u$ and $v$ variables. Shaded plaquettes correspond to counterclockwise windings of $u / v$, while empty plaquettes correspond to clockwise windings. (b) The energy of the four degenerate ground states as a function of $\alpha$ (for a $2 \times 2$ cell). At $\alpha=1$ all four states are true ground states, but as $\alpha$ decreases, the states split into two branches, each containing two states. The lower branch of two states are the only remaining ground states out of the initial four states.
in Figure 4.1(a).

The ground state for a fully frustrated $p+i p$ JJ array when $\alpha<1$ can be understood by noting that the phase configuration of the ground state is the same as a conventional array, except the twisting is in 2 variables, $u$ and $v$. The checkerboard pattern is now due to the windings of $u$ and $v$, as in Figure 4.1(a). Shaded $u / v$ plaquettes correspond to counterclockwise $2 \pi$-winding in $u / v$, which means that the plaquette is occupied by either a $u$ or $v$ vortex. Consider the case in Figure 4.1(a) when $\alpha<1$ (note that Figure 4.1 (a) shows $u$ and $v$ patterns on separate lattices for clarity, but in reality there is one array and these patterns lay on top of each other). For $\alpha<1$, the $u$ and $v$ checkerboard vortex patterns do not overlap. This means that that every plaquette is occupied by either a $u$-vortex or a $v$-vortex. This is shown in Figure 4.2, where a
cross in every plaquette indicates that there is a vortex in every plaquette. In the next paragraph we will explain physically what it means to say that there is a $u$ or $v$ vortex in every plaquette.


Figure 4.2: The 'crosses' are uniformly distributed in the array, and represent the fact that there is a vortex in every single plaquette when $\alpha<1$ in a fully frustrated $p+i p$ JJ array. These vortices are full vortices in $u$ and $v$.

In Figure 4.2 , the crosses represent the fact that for $\alpha<1$, there is exactly one vortex per plaquette, where the vortices are $2 \pi$-windings in $u$ and $v$. From the relationship between $u, v, \theta$, and $\phi$, we can understand full vortices in $u$ and $v$ in terms of $\theta$ and $\phi$ windings. From the relations in Section 4.2 , it is a simple matter to show that full vortices in $u$ and $v$ correspond to half-vortices in $\theta$ and $\phi$. More specifically, a full vortex in $u$ where $u$ winds by $2 \pi$ corresponds to a half-vortex in $\theta$ and $\phi$, where $\theta$ and $\phi$ each wind in the direction by $\pi$. On the other hand, a full vortex in $v$ corresponds to opposite $\pi$-windings in $\theta$ and $\phi$.

The most convenient way to picture the $f=1 / 2$ ground state in a $p+i p \mathrm{JJ}$
array is as in Figure 4.3. Each shaded square represents a half-quantum vortex ( $\left.\frac{1}{2} \mathrm{QV}\right)$. The light shade corresponds to $\theta$ and $\phi$ winding by $\pi$ in the clockwise direction, whereas the dark shade corresponds to $\theta$ winding by $\pi$ and $\phi$ winding by $-\pi$. This means that the two shades of $\frac{1}{2} \mathrm{QVs}$ represent opposite windings in the spin sector. The $f=1 / 2$ ground state at $\alpha<1$ can therefore be thought of as a having a $\frac{1}{2} \mathrm{QV}$ in every plaquette, where the only thing that changes from cell to cell is the relative direction of winding in the spin sector. Two $\frac{1}{2} \mathrm{QVs}$ with opposite winding in the spin sector are topologically equivalent to a full quantum vortex (QV). The reason that pairs of $\frac{1}{2} \mathrm{QV}$ s with opposite spin windings do not coalesce into QVs is that when the stiffness is softer in the spin sector than in the superfluid sector $(\alpha<1)$, pairs of $\frac{1}{2}$ QVs with finite separation (of order of the penetration depth $\lambda$ ) are more energetically stable than a single $\mathrm{QV}[28]$.


Figure 4.3: The $f=1 / 2$ ground state when $\alpha<1$ in terms of $\theta$ and $\phi$. Each plaquette is occupied by a $\frac{1}{2} \mathrm{QV}$. In the light plaquettes, $\theta$ and $\phi$ wind by $\pi$ (counterclockwise direction), whereas in the dark plaquettes $\theta$ winds by $\pi$ and $\phi$ winds by $-\pi$.

### 4.3.1 Domain Walls

Similar to a fully frustrated conventional array, a fully frustrated $p+i p$ JJ array also supports domain walls as topological excitations. There is only one type of domain


Figure 4.4: The domain wall energy for a fully frustrated $p+i p \mathrm{JJ}$ array as a function of $\alpha$, the ratio between the spin stiffness and the superfluid stiffness.
wall due to the twofold degeneracy of the ground state. At $\alpha=1$, these domains will have the same energy as in the conventional case since the p-wave Hamiltonian decouples into two independent fully frustrated models with half the coupling constant [see equation (4.9)]. The domains can be thought of as topological excitations in both $u$ and $v$, which separate two ground states. For $\alpha<1$, a single type of domain wall can form between states 1 and 2 of the right panel of Figure 4.1(a).

In Figure 4.4 we have numerically calculated the domain wall energy as a function of $\alpha$. The energy is linear in $\alpha$ and can be fit to the line

$$
\begin{equation*}
E_{\mathrm{dw}}(\alpha)=(-0.1719978 \alpha+0.5152823) J \tag{4.10}
\end{equation*}
$$

The energy of the domain wall at $\alpha=1$ is $E_{\mathrm{dw}}(1)=0.3432845 \mathrm{~J}$, as in the conventional case. With this domain energy, we can calculate the domain wall proliferation temperature by using equation (3.23),

$$
\begin{equation*}
T_{\mathrm{dw}}=\frac{(-0.1719978 \alpha+0.5152823) J}{\ln (1+\sqrt{2})} \tag{4.11}
\end{equation*}
$$

This temperature will have to be compared with the proliferation temperature of vortices, which we proceed to calculate in the next few sections. In a $p+i p \mathrm{JJ}$ array there are three possible types of vortices; full vortices in $\theta$, full vortices in $\phi$, and half vortices which simultaneously combine half-integer windings in $\theta$ and $\phi$. Just as in the conventional case, the upper bound for vortex proliferation is the domain wall proliferation temperature.

At first sight it seems odd that the domain proliferation temperature (4.11) has a negative sign in front of $\alpha$. What this means is that as $\alpha$ decreases (the spin stiffness decreases relative to the superfluid stiffness), the model becomes effectively less frustrated. Recall in the conventional case that the $f=0$ transition is higher than the $f=1 / 2$ transition. Similarly, the transition temperature at $f=1 / 2$ in a $p+i p \mathrm{JJ}$ array will increase as the spin stiffness decreases. This appears to be counterintuitive because one would expect that as the relevant energy scale decreases (in this case $\rho_{\text {sp }}$ ), the transition temperature should also decrease. The reason it doesn't is because for $\alpha<1$ the model becomes effectively less frustrated. This can be seen in Section 4.3, where the number of degenerate ground states is cut in half when $\alpha<1$, just as one would expect for a system which is less frustrated.

### 4.3.2 Full Vortices in the Superconducting Phase

Vortices are topological point defect excitations above the $f=0$ and $f=1 / 2$ ground state of $p+i p$ JJ arrays. They come in three non-trivial varieties; windings by $2 \pi$ in $\theta$, windings by $2 \pi$ in $\phi$, and combined windings of $\theta$ and $\phi$ by $\pi$. We now proceed to calculate the KT transition temperature for each type of vortex structure. Let us
first consider full vortices in which $\theta_{i} \rightarrow \theta_{i}+2 \pi$ upon circulation around the vortex core. The gauge-invariant phase differences can be denoted as

$$
\begin{equation*}
\theta_{i j}=\theta_{j}-\theta_{i}-A_{i j} \tag{4.12}
\end{equation*}
$$

For fluctuations above the $f=0$ ground state, $\theta_{i j}=\delta \theta_{i j}$ and $\phi_{i j}=0$, since only the superconducting phase $\theta$ undergoes rotations in these types of vortex excitations. By performing a low energy expansion of (4.1) we find that

$$
\begin{align*}
E_{\mathrm{V}, \theta}^{0} & =J \sum_{\langle i j\rangle} \frac{\delta \theta_{i j}^{2}}{2} \\
& \simeq \pi J \ln \frac{R}{a} \tag{4.13}
\end{align*}
$$

where the second step follows from the continuum approximation

$$
\begin{equation*}
\sum_{\langle i j\rangle} \frac{\delta \theta_{i j}^{2}}{2} \simeq \pi \ln \frac{R}{a} \tag{4.14}
\end{equation*}
$$

where $a$ is the lattice constant and $R$ is the lateral size of the array. The sum in (4.14) has been converted to continuous polar coordinates and we have allowed the fluctuations $\theta_{i j}$ to accumulate to a net winding of $2 \pi$ in $\theta$.

As you can see, cumulative $2 \pi$-windings in the low energy fluctuations $\delta \theta_{j i}$ diverge logarithmically with the size of the system. Using the same energy-entropy argument that lead to (3.50), it is a simple matter to show that a KT transition of full superconducting vortices in an unfrustrated $p+i p \mathrm{JJ}$ array occurs at a temperature of

$$
\begin{equation*}
T_{\mathrm{KT}, \theta}^{0}=\frac{\pi J}{2 k_{B}} \tag{4.15}
\end{equation*}
$$

This result is in fact identical to (3.50), the transition temperature derived for a unfrustrated conventional array, which is to be expected. We are ignoring any renormalization corrections to this transition temperature because we are assuming that these corrections will equally modify the transition both with and without frustration. Since we are interested in the ratio of the $f=1 / 2$ and $f=0$ KT transitions, any renormalization modifications become irrelevant because they will cancel each other out.

Excitations about the ground state of a fully frustrated $(f=1 / 2) p+i p$ JJ array are of the form $\theta_{i j}=\delta \theta_{i j}$ and $\phi_{i j}= \pm \pi / 4$. This leads to a vortex excitation energy of

$$
\begin{align*}
E_{\mathrm{V}, \theta}^{1 / 2} & =\frac{J}{\sqrt{2}}\left(\frac{5-\alpha}{4}\right) \sum_{\langle i j\rangle} \frac{\delta \theta_{i j}^{2}}{2} \\
& =\frac{\pi J}{\sqrt{2}}\left(\frac{5-\alpha}{4}\right) \ln \frac{R}{a} \tag{4.16}
\end{align*}
$$

and a corresponding KT transition temperature

$$
\begin{equation*}
T_{\mathrm{KT}, \theta}^{1 / 2}=\frac{\pi J}{2^{3 / 2} k_{B}}\left(\frac{5-\alpha}{4}\right) \tag{4.17}
\end{equation*}
$$

Notice that when $\alpha=1$, this result reduces to the fully frustrated conventional case. So far we have completely ignored the influence of domain walls. Similar to the conventional case, it is impossible for a KT transition to occur at a higher temperature than the domain wall proliferation temperature. Therefore, the bare coupling in (4.17) needs to be replaced by an effective coupling $J^{\prime}$ which enforces this condition. However, there are three topologically distinct vortex excitations present in a $p+i p \mathrm{JJ}$ array, and so the effective coupling must rescale the vortex energy of the most expensive type of vortex at $f=1 / 2$ to ensure that the corresponding KT transition does not exceed the $Z_{2}$ domain wall transition. In the next two sections we will calculate the energy of the more exotic
spin vortices and half quantum vortices, and thereby determine which vortices have the highest vortex-antivortex unbinding transition.

### 4.3.3 Full Vortices in the Spin Sector

Full vortices in the spin sector involve $2 \pi$-windings of the $\mathbf{d}(\mathbf{k})$ vector around the vortex core. The $\mathbf{d}(\mathbf{k})$ vector angle difference between sites can be denoted as

$$
\begin{equation*}
\phi_{j i}=\phi_{j}-\phi_{i} \tag{4.18}
\end{equation*}
$$

Consider the excitations where the gauge-invariant phase does not fluctuate $\left(\theta_{i j}=0\right)$ and only the director vector undergoes $2 \pi$-windings through successive low energy fluctations $\phi_{j i}=\delta \phi_{j i}$. This type of topological excitation we will simply refer to as a spin vortex. By expanding (4.1) about small deviations $\delta \phi_{j i}$ from the unfrustrated ground state, the energy of a full vortex excitation in the spin sector works out to be

$$
\begin{align*}
E_{\mathrm{V}, \phi}^{0} & =J \alpha \sum_{\langle i j\rangle} \frac{\delta \phi_{j i}^{2}}{2} \\
& =\pi J \alpha \ln \frac{R}{a} \tag{4.19}
\end{align*}
$$

and the corresponding KT transition from the energy-entropy relations is

$$
\begin{equation*}
T_{\mathrm{KT}, \phi}^{0}=\frac{\pi J \alpha}{2 k_{B}} \tag{4.20}
\end{equation*}
$$

In the case of full frustration, deviations about the checkerboard ground state in $\phi$ are of the form $\phi_{j i}= \pm \frac{\pi}{4}+\delta \phi_{j i}$ while $\theta$ remains constant, $\theta_{j i}=0$. The excitation energy of such a vortex is

$$
\begin{align*}
E_{\mathrm{V}, \phi}^{1 / 2} & =\frac{J}{4 \sqrt{2}}(9-5 \alpha) \sum_{\langle i j\rangle} \frac{\delta \phi_{j i}^{2}}{2} \\
& =\frac{\pi J}{4 \sqrt{2}}(9-5 \alpha) \ln \frac{R}{a} \tag{4.21}
\end{align*}
$$

which leads to a KT transition temperature of

$$
\begin{equation*}
T_{\mathrm{KT}, \phi}^{1 / 2}=\frac{\pi J}{2 k_{B}}\left[\frac{1}{4 \sqrt{2}}(9-5 \alpha)\right] \tag{4.22}
\end{equation*}
$$

This result ignores the domain wall interaction and will have to be modified by $J \rightarrow J^{\prime}$.

### 4.3.4 Half-Vortices

The most interesting type of topological excitation are $\frac{1}{2} \mathrm{QVs}$. These combine rotations of the superconducting phase $\theta$ by $\pi$ with rotations of $\mathbf{d}(\mathbf{k})$ by $\pi$ upon circulation of the vortex core, an overall action which leaves the order parameter single-valued. In an unfrustrated $p+i p$ array, the low energy fluctuations are $\theta_{i j}=\delta \theta_{i j}$ and $\phi_{i j}=\delta \phi_{i j}$ which allow us to expand (5.28) and write the energy of topological $\frac{1}{2} \mathrm{QVs}$ as

$$
\begin{align*}
E_{\mathrm{V},(\theta / \phi)}^{0} & =J \sum_{\langle i j\rangle} \frac{\delta \phi_{j i}^{2}}{2}+J \alpha \sum_{\langle i j\rangle} \frac{\delta \phi_{j i}^{2}}{2} \\
& =\frac{\pi J}{4}(1+\alpha) \ln \frac{R}{a} \tag{4.23}
\end{align*}
$$

In the line above we have switched to continuous notion which allows us to perform the integration and write

$$
\begin{equation*}
\sum_{\langle i j\rangle} \frac{\delta \theta_{j i}^{2}}{2}=\sum_{\langle i j\rangle} \frac{\delta \phi_{j i}^{2}}{2}=\frac{\pi}{4} \ln \frac{R}{a} \tag{4.24}
\end{equation*}
$$

since both $\theta$ and $\phi$ each wind by $\pi$ upon circulating the vortex core. This energy leads to an estimated KT transition temperature of

$$
\begin{equation*}
T_{\mathrm{KT},(\theta / \phi)}^{0}=\frac{\pi J}{8 k_{B}}(1+\alpha) \tag{4.25}
\end{equation*}
$$

for $\frac{1}{2}$ QVs. For a fully frustrated $p+i p$ JJ array, the ground state becomes a checkerboard in $\phi$, and fluctuations above the ground state are of the form $\theta_{j i}=\delta \theta_{j i}$ and $\phi_{j i}=$ $\pm \frac{\pi}{4}+\delta \phi_{j i}$. Now the energy of $\frac{1}{2} \mathrm{QV}$ excitations becomes

$$
\begin{align*}
E_{\mathrm{V},(\theta / \phi)}^{1 / 2} & =\frac{J}{\sqrt{2}}\left(\frac{5-\alpha}{4}\right) \sum_{\langle i j\rangle} \frac{\delta \theta_{j i}^{2}}{2}+\frac{J}{\sqrt{2}}\left(\frac{9-5 \alpha}{4}\right) \sum_{\langle i j\rangle} \frac{\delta \phi_{j i}^{2}}{2} \\
& =\frac{\pi J}{8 \sqrt{2}}(7-3 \alpha) \ln \frac{R}{a} \tag{4.26}
\end{align*}
$$

which leads to a KT transition temperature of

$$
\begin{equation*}
T_{\mathrm{KT},(\theta / \phi)}^{1 / 2}=\frac{\pi J}{8 k_{B}}\left[\frac{1}{2 \sqrt{2}}(7-3 \alpha)\right] \tag{4.27}
\end{equation*}
$$

As discussed, this first estimate for $f=1 / 2$ is naive because it ignores the influence of domain walls.

### 4.3.5 KT Transition

In a $p+i p$ Josephson junction array there are three different types of topologically distinct vortices. Only one of these types of vortices can lead to a KT transition, since once one type of vortex-antivortex pairs proliferate, the phases will scramble and destroy any remaining long-range order. Therefore the pairwise dissociation of a single type of vortex initiates the dissociation of all remaining vortex pairs. Additionally, the KT unbinding transition which does occur must happen at a temperature lower than the domain wall transition temperature. Similar to the fully frustrated conventional array, this is because the proliferation of fractional vortices bound to domain wall defects
decouples phase fluctuations across domain walls and hence initiates the dissociation of bulk vortices.

If we compare the energy of $\theta-, \phi-$, and $(\theta / \phi)$-vortices as in (4.16),(4.21), and (4.26) respectively, we find that $\phi$-vortices are the most energetically costly excitations at $f=1 / 2$. This means that the proliferation temperature of $\phi$-vortices, considered independent of all other excitations, would like to be higher than the proliferation of both $\theta$ - and $(\theta / \phi)$-vortices. However, we do not expect to observe subsequent KT transitions after the first KT transition occurs. The domain wall transition temperature $T_{\text {dw }}$ [see equation (4.11)] naturally occurs much lower than the bare KT transition temperatures [see (4.22),(4.17), and (4.27)], and so any possible KT transition will be renormalized so that the most energetically expensive vortices proliferate by the time $T_{\mathrm{dw}}$ is reached. Since $\phi$-vortices are the most expensive type of vortex at $f=1 / 2$, the effective coupling $J^{\prime}$ is chosen such that $\phi$-vortices necessarily proliferate at $T_{\mathrm{dw}}$,

$$
\begin{equation*}
J^{\prime}=\frac{8 \sqrt{2}}{\pi}\left(\frac{1}{9-5 \alpha}\right) \frac{E_{\mathrm{dw}}(\alpha) / J}{\ln (1+\sqrt{2})} \tag{4.28}
\end{equation*}
$$

where $E_{\mathrm{dw}}(\alpha)$ is given by (4.10). By making the prescription $J \rightarrow J^{\prime}$ in (4.22), we obtain the equality $T_{\mathrm{KT}, \phi}^{1 / 2}=T_{\mathrm{dw}}$. The ratio between $f=1 / 2$ and $f=0 \mathrm{KT}$ transitions for full spin vortices can now be written as

$$
\begin{equation*}
\Lambda_{\phi}=\left(\frac{T_{\mathrm{KT}}^{1 / 2}}{T_{\mathrm{KT}}^{0}}\right)_{\phi} \lesssim \frac{2}{\pi \alpha} \frac{E_{\mathrm{dw}}(\alpha) / J}{\ln (1+\sqrt{2})} \tag{4.29}
\end{equation*}
$$

Similarly, by replacing $J \rightarrow J^{\prime}$ in (4.17) and (4.27) we obtain the ratio

$$
\begin{equation*}
\Lambda_{\theta}=\left(\frac{T_{\mathrm{KT}}^{1 / 2}}{T_{\mathrm{KT}}^{0}}\right)_{\theta} \lesssim \frac{2}{\pi}\left(\frac{5-\alpha}{9-5 \alpha}\right) \frac{E_{\mathrm{dw}}(\alpha) / J}{\ln (1+\sqrt{2})} \tag{4.30}
\end{equation*}
$$

for full superconducting vortices ( $\theta$-vortices) and

$$
\begin{equation*}
\Lambda_{(\theta / \phi)}=\left(\frac{T_{\mathrm{KT}}^{1 / 2}}{T_{\mathrm{KT}}^{0}}\right)_{(\theta / \phi)} \lesssim \frac{4}{\pi}\left[\frac{(7-3 \alpha)}{(9-5 \alpha)(1+\alpha)}\right] \frac{E_{\mathrm{dw}}(\alpha) / J}{\ln (1+\sqrt{2})} \tag{4.31}
\end{equation*}
$$

for half quantum vortices $[(\theta / \phi)$-vortices]. If the proliferation of the $\theta$-vortices, $\phi$ vortices, and the $(\theta / \phi)$-vortices were independent of each other, there would be three KT transitions. However, in a $p+i p$ JJ array, since the proliferation of one type of vortex scrambles the phases and destroys order in the $\theta$ and $\phi$, there can be only one KT transition. Therefore, we expect to be able to observe only a single KT transition in a $p+i p \mathrm{JJ}$ array, the one which occurs at the lowest temperature. In the next section we will show how one could use the predicted transition ratios (4.29), (4.30), and (4.31) in order to detect $\frac{1}{2} \mathrm{QVs}$.

### 4.4 Array Independent Method of Detecting $\frac{1}{2}$ QVs

One possible way to detect $\frac{1}{2} \mathrm{QVs}$, independent of the microscopic array properties, would be to measure the KT transition temperature in a $p+i p$ Josephson junction array for two different frustrations, an unfrustrated ( $f=0$ ) array and a fully frustrated $(f=1 / 2)$ array. The ratio of these temperatures could then be compared with the theoretical predictions in (4.29), (4.30), and (4.31). Notice that these predictions don't depend on any microscopic array properties, but only on the material-dependent parameter $\alpha$ of p -wave superconductors.

In order to detect $\frac{1}{2} \mathrm{QVs}$, one must be able to measure the KT transition in a $p+i p$ JJ array. The Kosterlitz-Thouless proliferation temperature of QVs has been experimentally measured in conventional JJ arrays[86, 83], and in principal the proliferation of $\frac{1}{2} \mathrm{QVs}$ in $p+i p \mathrm{JJ}$ arrays can also be measured. The KT transition of spin vortices is more subtle, and it is not obvious that this transition could be detected


Figure 4.5: The ratio of Kosterlitz-Thouless transition temperatures for a fully frustrated array $(f=1 / 2)$ and an unfrustrated array $(f=0)$.
in any experimental measurements. However, if spin vortices or even normal vortices proliferate at a lower temperature than $\frac{1}{2} \mathrm{QVs}$, there will be no KT transition mediated by the proliferation of $\frac{1}{2} \mathrm{QVs}$. Therefore, $\frac{1}{2} \mathrm{QVs}$ must proliferate before full vortices in the phase and spin sector in order to drive the KT transition. For $f=0$ and $f=1 / 2$ the KT transition for $\frac{1}{2} \mathrm{QVs}$ is given by (4.25) and (4.27) respectively. If we compare this with the KT transition of normal QVs in (4.15) and (4.17), we find that $\frac{1}{2}$ QVs will always proliferate at a lower temperature. We can also compare the proliferation temperature of spin vortices with that of $\frac{1}{2} \mathrm{QVs}$. We find that $\alpha$ must satisfy

$$
\begin{equation*}
\frac{\rho_{\mathrm{sp}}}{\rho_{\mathrm{s}}} \geq \frac{1}{3} \tag{4.32}
\end{equation*}
$$

in order for $\frac{1}{2} \mathrm{QVs}$ to proliferate at a lower temperature than spin vortices. The lower limit is the bound for unfrustrated arrays. As long as $\alpha$ lies above this cutoff, $\frac{1}{2} \mathrm{QVs}$ will drive the Kosterlitz-Thouless transition. An example of a chiral p-wave superconductor that is believed to satisfy this property is strontium ruthenate $\left(\mathrm{Sr}_{2} \mathrm{RuO}_{4}\right)$, for which S.B Chung et al[28] estimate $\alpha \simeq 0.4$.

In Figure 4.5, we have plotted the ratio of KT transitions for $f=0$ and $f=1 / 2$ for full superconducting vortices ( QVs ), spin vortices, and $\frac{1}{2} \mathrm{QVs}$ in $p+i p$ arrays [see (4.30),(4.29), and (4.31) respectively]. By constructing a $p+i p$ Josephson junction array and simply measuring the KT transition for fully frustrated and unfrustrated arrays, the ratio can be taken and compared to this curve. If, for example, the ratio is measured and determined to be significantly greater than what would be expected for $\theta$-vortices, this is strong evidence that the transition was driven by $\frac{1}{2} \mathrm{QVs}$. Notice that $\Lambda_{\theta}$ is not flat in Figure 4.5. If the superfluid stiffness was completely fixed, then this line should be completely flat. The reason that it is not comes from the model Hamiltonian (4.1). The $\cos (3 \Delta \phi)$ term is a first order term which preserves the underlying symmetry while still allowing differing stiffnesses, i.e. situations where $\rho_{\mathrm{s}} \neq \rho_{\mathrm{sp}}$. The $\cos (3 \Delta \phi)$ is the most important term which allows for an asymmetry in the stiffnesses. However, in order to calculate higher order corrections to these results, one would have to include higher order terms in the Fourier expansion (i.e. $\cos (5 \Delta \phi), \cos (7 \Delta \phi)$, etc).

We should note that these curves can really only be trusted in the region $\alpha>1 / 3$, since below this cutoff spin vortices will proliferate before $\frac{1}{2} \mathrm{QVs}$ or QVs in $\theta$. If this is the case, then one will not even be able to observe the curves $\Lambda_{(\theta / \phi)}$ and $\Lambda_{\theta}$. If we are in the regime $\alpha>1 / 3$, we can rule out the possibility of observing an unbinding transition of $\phi$-vortices since the proliferation of these vortices will occur after $\frac{1}{2} \mathrm{QVs}$ and would not leave any signatures in the resistance measurements. This method also suggests a way of measuring $\alpha$, the ratio of neutral spin superfluid density (stiffness) $\rho_{\text {sp }}$ to the mass superfluid density $\rho_{\mathrm{s}}$.

### 4.5 Array Dependent Method of Detecting $\frac{1}{2}$ QVs

A second method of detecting $\frac{1}{2}$ QVs would be to directly measure and theoretically predict the KT transition in an unfrustrated $p+i p \mathrm{JJ}$ array. The predicted KT transition will depend on the microscopic parameters of the array, and by measuring these parameters independently of the KT transition, it should be possible to identify a transition driven by the proliferation of $\frac{1}{2} \mathrm{QVs}$. We now proceed to calculate the difference in KT transitions for QVs and $\frac{1}{2}$ QVs.

In a 2D superconducting thin film the energy of a pair of QVs with vorticity $\pm 1$ was first obtained by Pearl[45] and is of the form

$$
\begin{equation*}
E_{\text {pair }}^{\text {full }}=\frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}}\left[2 \ln \frac{\lambda_{\perp}}{\xi}+\pi\left\{N_{0}\left(\frac{r}{2 \lambda_{\perp}}\right)-H_{0}\left(\frac{r}{2 \lambda_{\perp}}\right)\right\}\right] \tag{4.33}
\end{equation*}
$$

where $N_{0}(x)$ and $H_{0}(x)$ are the Neumann and Struve functions respectively, $\xi$ is the core radius, and $\lambda_{\perp}=\lambda^{2} / d$ is Pearl's 2D transverse penetration depth defined in terms of the bulk penetration depth $\lambda$ and the film thickness $d$. The first term in (4.33) is twice the self-energy of a single vortex, and the second term is the magnetic interaction between the vortices. By comparing the energetics of QV pairs with $\frac{1}{2} \mathrm{QV}$ pairs one can determine which are more favorable at low temperatures. An isolated $\frac{1}{2} \mathrm{QV}$ costs a spin current energy that diverges logarithmically with system size due to the absence of screening of the spin current, $[16,28]$ but the energy of a pair of $\frac{1}{2} \mathrm{QVs}$ with opposite spin winding and vorticity is

$$
\begin{equation*}
E_{\text {pair }}^{\text {half }}=\frac{1}{2} \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}}\left[\ln \frac{\lambda_{\perp}}{\xi} \frac{\pi}{2}\left\{N_{0}\left(\frac{r}{2 \lambda_{\perp}}\right)-H_{0}\left(\frac{r}{2 \lambda_{\perp}}\right)\right\}+\alpha \ln \frac{r}{\xi}\right] \tag{4.34}
\end{equation*}
$$

where $\alpha=\rho_{\mathrm{sp}} / \rho_{\mathrm{s}}$. The last term in (4.34) is purely due to spin flow, which is absent in the case of QVs. In a thin superconducting film $\lambda_{\perp}$ is typically of the order of
centimeters, in which case one can have samples where the distance between vortices is always much less than the effective penetration depth[48]. In this regime, where $r \ll \lambda_{\perp}$, the interaction energy of $(4.33,4.34)$ reduces to a form that is logarithmic,

$$
\begin{equation*}
E_{\text {pair }}^{\text {full }}=2 \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}} \ln \frac{r}{\xi} \tag{4.35}
\end{equation*}
$$

for a pair of QVs and

$$
\begin{equation*}
E_{\text {pair }}^{\text {half }}=\frac{1}{2} \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}}(1+\alpha) \ln \frac{r}{\xi} \tag{4.36}
\end{equation*}
$$

for a pair of $\frac{1}{2}$ QVs. It is easy to verify that $\frac{1}{2} \mathrm{QVs}$ are energetically favorable as long as the condition $\alpha<3$ is satisfied. However, A.J. Leggett has shown[49] under very general conditions that $\alpha<1$, therefore pairs of $\frac{1}{2} \mathrm{QVs}$ with opposite windings are always energetically favorable in thin superconducting films.

The presence of $\frac{1}{2} \mathrm{QVs}$ should be detectable by an observable lowering of the KT transition temperature. The KT transition in $p+i p \mathrm{JJ}$ arrays is the temperature at which pairs of vortices, initially bound together in neutral pairs of zero total vorticity, unbind and begin to proliferate. In a $p+i p$ JJ array, which is capable of supporting the existence of $\frac{1}{2} \mathrm{QVs}$, we expect that the KT transition will be driven by $\frac{1}{2} \mathrm{QVs}$ because such a pair of tightly bound vortices is the most energetically stable configuration. By comparing (4.15) and (4.25) with (4.35) and (4.36) respectively, we can identify the Josephson coupling energy as

$$
\begin{equation*}
J=\frac{1}{\pi} \frac{\Phi_{0}^{2}}{16 \pi^{2} \lambda_{\perp}} \tag{4.37}
\end{equation*}
$$

By following the simple energy-entropy argument presented in Section 2.2, we can now express the Kosterlitz-Thouless transition temperature directly in terms of the micro-
scopic array parameters. For the proliferation of QVs this is

$$
\begin{equation*}
T_{\mathrm{KT}, \text { full }}^{0}=\frac{1}{k_{B}} \frac{\Phi_{0}^{2}}{32 \pi^{2} \lambda_{\perp}} \tag{4.38}
\end{equation*}
$$

and for $\frac{1}{2} \mathrm{QVs}$ it is

$$
\begin{equation*}
T_{\mathrm{KT}, \text { half }}^{0}=\frac{1}{4 k_{B}} \frac{\Phi_{0}^{2}}{32 \pi^{2} \lambda_{\perp}}\left(1+\frac{\rho_{s p}}{\rho_{s}}\right) . \tag{4.39}
\end{equation*}
$$

The presence of $\frac{1}{2} \mathrm{QVs}$ will be evident by an observable lowering of the KT transition. If one was to measure this transition and found that the actual critical temperature $T_{\mathrm{C}}$ of the array fell in the range

$$
\begin{equation*}
\frac{1}{4} T_{\mathrm{KT}, \text { full }}^{0}<T_{\mathrm{C}}<\frac{1}{2} T_{\mathrm{KT}, \text { full }}^{0} \tag{4.40}
\end{equation*}
$$

it would be reasonable to conclude that the KT transition was in fact driven by the proliferation of $\frac{1}{2} \mathrm{QVs}$.

The use of inequality (4.40) is only of practical use in testing for the presence of $\frac{1}{2} \mathrm{QVs}$ if (4.38) can be used to independently give a theoretical prediction for the KT transition of QVs in a particular $p+i p \mathrm{JJ}$ array. This would require the ability to have an independent way of measuring $\lambda_{\perp}(T)$, which can actually be experimentally achieved through either of the relations[89]

$$
\begin{equation*}
\lambda_{\perp}(T)=\frac{c^{2} L_{\mathrm{K} \square}(T)}{4 \pi} \simeq \frac{c \Phi_{0}}{8 \pi^{2} i_{\mathrm{C}}(T)} \tag{4.41}
\end{equation*}
$$

where $L_{\mathrm{K} \square}(T)$ is the kinetic inductance per square of the film and $i_{\mathrm{C}}(T)$ is the critical current.

## Chapter 5

## $p+i p \mathbf{J J}$ Array at $f=1 / 3$

### 5.1 Introduction

In this chapter we study the critical properties of a $p+i p$ Josephson junction array with a commensurate filling factor of $f=1 / 3$. The critical properties of a conventional Josephson junction array with a transverse magnetic field of $f=1 / 3$ was first studied by C. Denniston[90, 91]. His central results were that the relevant low energy excitations at $f=1 / 3$ are domains, and these domain walls determine the nature of the phase transition. We will extend these results to the case of a $p+i p \mathrm{JJ}$ array and show that the superconducting-resistive transition is strongly suppressed, if not altogether absent due to the presence of half-quantum vortices.

Before we present our results for the $p+i p$ case at $f=1 / 3$, we first review the simpler case of a conventional JJ array at $f=1 / 3$. We review the $f=1 / 3$ conventional JJ array in Section 5.2. T.C. Halsey[73] was the first one to calculate the ground state at $f=1 / 3$. In Section 5.3 we calculate the $f=1 / 3$ domain energy. Numerically, our analysis is similar to Denniston[90]. Analytically, we also derive expressions for the various $f=1 / 3$ conventional domains and show that they agree with our numeric
results. In Section 5.4 we apply the results of [90] and [64] in order to find the order of phase transitions and to analytically predict the $Z_{2}$ domain proliferation temperature. The rest of the chapter is devoted to the $p+i p$ case at $f=1 / 3$. In Section 5.5 we apply our model for $p+i p \mathrm{JJ}$ arrays to find the ground state at $f=1 / 3$. In Section 5.6 we calculate the energy of the relevant domains as a function of the ratio of the spin stiffness to the superfluid stiffness. We conclude the chapter with Sections 5.75.8, where we determine the sequence of possible phase transitions by mapping the low energy physics of the $p+i p \mathrm{JJ}$ array at $f=1 / 3$ onto an Ashkin-Teller model.

### 5.2 Conventional Array at $f=1 / 3$

The ground state of a frustrated Josephson junction array is found by minimizing Hamiltonian 3.8, $\partial H / \partial \theta_{i}=0$. This yields a set of coupled equations,

$$
\begin{equation*}
\sum_{\left\langle j^{\prime}\right\rangle} \sin \left(\theta_{j^{\prime}}-\theta_{i}-A_{i j^{\prime}}\right)=0 \tag{5.1}
\end{equation*}
$$

where $j^{\prime}$ are the nearest neighbors of $i$. This constraint expresses Kirchhoff's law for supercurrent, namely conservation of supercurrent at each lattice site $i$. At the commensurate value of $f=1 / 3$, one set of solutions found by $\operatorname{Halsey}[73]$ is the staircase state shown in Figure 5.1. These states correspond to two adjoining staircases of supercurrent, one flowing up and one flowing down the staircase, as in Figure 5.1. All arrows have the same magnitude, and correspond to a supercurrent of

$$
\begin{equation*}
I=I_{C} \sin \frac{\pi}{3}=\frac{\sqrt{3}}{2} I_{C} \tag{5.2}
\end{equation*}
$$

where $I_{C}$ is the critical current.


Figure 5.1: Ground state critical currents for a Josephson junction array on a square lattice and at a frustration of $f=1 / 3$. The superconducting islands are located at the corners of the square plaquettes, and the connecting lines are the tunnel junctions. Each arrow is of equal magnitude and represents the supercurrent direction. This pattern of staircase states repeats periodically across the entire array.

Halsey has shown[73] that this staircase state is the true ground state, and more generally that quasi one-dimensional staircase states, similar to that shown for $f=1 / 3$, are local minima of the frustrated xy Hamiltonian (3.8) for any value of of the frustration index $f$. Additionally, he shows that for $1 / 3 \leq f \leq 1 / 2$, these staircase states are indeed the true ground states for the array, whereas for arbitrary $f$ they are simply local minima. It is important to note, however, that the staircase state of Figure 5.1 is not unique, because there are three horizontal degrees of freedom for which we may shift the staircase, as well as three more staircase states with the opposite tilt. Therefore, for $f=1 / 3$, we find that there are a total of six degenerate ground states. In general, for any rational $f=p / q$, the gauge-invariant phase difference on any bond in the ground state is spatially periodic on a $q \times q$ unit cell, which leads to a $2 q$-fold degeneracy of the ground state[91].

In order to uniquely determine the state of the array, the gauge-invariant phase across each junction must be specified. These phase differences, however, cannot be
chosen arbitrarily. Not every conceivable set of phases is possible due to the constraint

$$
\begin{equation*}
\sum_{\square} A_{i j}=2 \pi f \tag{5.3}
\end{equation*}
$$

where the sum is directed around a single plaquette and imposed by the externally applied uniform magnetic field. The superconductivity in the array is not continuous, but consists of small discretized superconducting islands separated by insulating tunnel junctions and empty space. As such, there are no vortex cores[92], and therefore the only physically meaningful quantities are to talk about are the gauge-invariant phase differences between neighboring sites. A quantity of interest is therefore the amount of current that flows around a particular plaquette, and we define the plaquette vorticity as

$$
\begin{equation*}
\Gamma_{k}=\sum_{\square}\left(\theta_{j}-\theta_{i}-A_{i j}\right) \bmod (2 \pi) \tag{5.4}
\end{equation*}
$$

which lives on the dual lattice sites and each gauge-invariant link in the sum is restricted to the interval $(-\pi, \pi]$.

The ground state for $f=1 / 3$, shown in Figure 5.2 is expressed in terms of plaquette vorticity, and turns out to be sixfold degenerate. Empty plaquettes correspond to a vorticity of $\gamma=-2 \pi f$, while dark plaquettes correspond to regions of circulating supercurrent, with a net vorticity of $\gamma=2 \pi(1-f)$. Thus in the ground state, the vorticity of each plaquette follows the periodic staircase pattern depicted in Figure 5.2.

We can also interpret Figure 5.2 in terms of the Coulomb gas representation (see Section 3.5 for more details), where phase variables are replaced by quantized charges that live on the dual lattice. When $f$ is non-zero, this transformation yields a fractionally-charged 2D Coulomb gas


Figure 5.2: Fluxoid ground states patterns for $f=1 / 3$. Dark squares correspond to regions of circulating supercurrent, where the gauge-invariant phase winds by $2 \pi(1-f)$. Empty squares correspond to phase-windings of $-2 \pi f$

$$
\begin{equation*}
H_{C}=2 \pi^{2} \tilde{J} \sum_{r, r^{\prime}}\left(m_{r}+f\right) V_{r-r^{\prime}}\left(m_{r^{\prime}}+f\right) \tag{5.5}
\end{equation*}
$$

where the integer charges $m_{r}$ interact with each other and the with the background field $f$. The charges are constrained by the neutral condition

$$
\begin{equation*}
\sum_{r}\left(m_{r}+f\right)=0 \tag{5.6}
\end{equation*}
$$

We can now interpret Figure 5.2 as representing a lattice of interacting fractional charges, where the dark and white squares represent vortex charges of $1-f$ and $-f$ respectively, and preserve overall charge neutrality. For $f=1 / 3$ dark squares contain a vortex charge of $+2 / 3$ and empty squares $-1 / 3$. A vortex charge physically corresponds to the amount of circulating supercurrent around a particular plaquette. A shaded plaquette in Figure 5.2 has a vorticity of $\gamma=2 \pi(m-f)$ with $m=1$, while empty plaquettes correspond to $m=0$.

### 5.3 Domain Walls in a Conventional Array

A domain wall in the frustrated xy model is an energetic excitation that separates two different ground states. It is topological in nature because it cannot be undone by simultaneously rotating all phases or under any other smooth continuous transformation. For $f=1 / 3$, there are four different types of low energy domains walls that can be created, and they are shown in Figure 5.3. Herringbone walls are between states with diametrically opposed diagonal vortex stripes. Shift walls are between states of equivalent tilt and simply involve a horizontal shift of the vortex pattern across the domain wall. The energy of these domain walls depends on the domain wall type. We now proceed to calculate these domain energies both numerically and analytically. Table 5.1 compares the results of these calculations.

a)


Figure 5.3: Domain walls for $f=1 / 3$, marked with a dashed line. Types of domains: a)Herringbone-0, b)Herringbone-1, c)Shift-by-1, d)Shift-by-2

### 5.3.1 Numeric Calculation

Using the optimization method as discussed in Section 3.6.1.1, we numerically simulated the frustrated xy model at $f=1 / 3$ in order to calculate the energy of a num-
ber of topologically distinct domain walls. This method works through a constrained optimization of the superconducting phases which sit at each lattice site. The array we used (with periodic boundary conditions) to solve this problem for $f=1 / 3$ is shown in Figure 5.4. The solid black line separates two distinct ground state vortex patterns, and as an initial guess for the gauge-invariant bonds, we simply choose the values corresponding to the two ground states that are being patched together. In the vertical direction, the array only has a height of $3 a$ (three lattice constants) because we can take advantage of the $3 a$ periodic nature of the domain wall and impose periodic boundary conditions on the phases.

Moving in a direction perpendicular to the domain wall, the phase configuration will not be periodic because theses phases will adjust themselves out to some characteristic distance in order to optimize the domain wall energy. This gives each domain wall a characteristic thickness, the distance to which ground state phases are adjusted in order to minimize domain energy. We use the parameter $x$ (measured in units of $a$ ) to set the array size to a sufficiently large length. The accuracy of this numeric calculation of domain wall energy will increase as $x$ increases. In order to obtain the stabilized results presented in Table 5.1, we used arrays as large as $x \simeq 600$. In Table


Figure 5.4: This is an example of the array configuration we used to numerically compute domain wall energies in the $f=1 / 3$ frustrated xy model. On each side of the dotted line, a single ground state phase configuration is chosen. Periodic boundary conditions are imposed in the vertical direction, and in the horizontal direction the length of the array is $x$.
5.1 we have calculated the energy of four different types of domains and then compared this with an explicit analytic calculation where possible.

### 5.3.2 Analytic Calculation

When we express the frustrated xy model in the Coulomb gas representation, the corresponding Hamiltonian has the form[56]

$$
\begin{equation*}
H=\sum_{r, r^{\prime}} m_{r} G_{r-r^{\prime}} m_{r^{\prime}} \tag{5.7}
\end{equation*}
$$

where the interaction $G_{r-r^{\prime}}$ is given by (setting the lattice constant $a$ to unit length)

$$
\begin{equation*}
G_{r-r^{\prime}}=\int \frac{d^{2} k}{(2 \pi)^{2}} G_{k} \exp \left[i \mathbf{k} \cdot\left(\mathbf{r}-\mathbf{r}^{\prime}\right)\right] \tag{5.8}
\end{equation*}
$$

and

$$
\begin{equation*}
G_{k}=\frac{\pi^{2} \tilde{J}}{2} \frac{1}{\sin ^{2} \frac{k_{x}}{2}+\sin ^{2} \frac{k_{y}}{2}} \tag{5.9}
\end{equation*}
$$

In order to compare our analytic calculations in the Coulomb gas model with the numeric simulations of the frustrated xy model, we have to know how the coupling constant $J$ renormalizes at different values of the frustration parameter $f$. In the fully frustrated $(f=1 / 2)$ case, the renormalization of the coupling constant in the Coulomb gas model $[64,63]$ is $\tilde{J}=J / \sqrt{2}$. In order to calculate $\tilde{J}$ for $f=1 / 3$, we must allow for small fluctuations of the phase above the ground state. Each gauge-invariant phase difference in the ground state for $f=1 / 3$ is either zero or $\pi / 3$. In the $3 \times 3$ unit cell, there are 18 unique gauge-invariant bonds for each of the 6 degenerate states, 12 of which have a value of $\pi / 3$, while the remaining 6 are zero. Let us denote the gauge-invariant phase difference as $\theta_{j i}=\theta_{j}-\theta_{i}-A_{i j}$ and small deviations $\delta \theta_{j i}$ about the non-zero
ground state bonds as

$$
\begin{equation*}
\theta_{j i}= \pm \frac{\pi}{3}+\delta \theta_{j i} \tag{5.10}
\end{equation*}
$$

We can expand each $\pi / 3$ bond in the Hamiltonian as

$$
\begin{align*}
J \cos \left(\theta_{j}-\theta_{i}-A_{i j}\right) & =J \cos \left(\theta_{j i}\right) \\
& =J \cos \left( \pm \frac{\pi}{3}+\delta \theta_{j i}\right) \\
& \simeq J \cos \left(\frac{\pi}{3}\right) \mp J \sin \left(\frac{\pi}{3}\right) \delta \theta_{j i}-\frac{1}{2} J \cos \left(\frac{\pi}{3}\right)\left(\delta \theta_{j i}\right)^{2} \tag{5.11}
\end{align*}
$$

When a sum is performed over the entire array, the linear terms cancel out, leaving simply the first and last term. The first term is the ground state energy and the last term is the fluctuation energy. Therefore, the bonds with $\gamma=\pi / 3$ renormalize as $J \rightarrow J \cos \pi / 3$. In the Coulomb gas model, vortices at large distances will only be sensitive to the average effective coupling, hence for $f=1 / 3$ we calculate the average over the 18 unique bonds that are periodic in the ground state,

$$
\begin{equation*}
\tilde{J}=J\left(\frac{12}{18} \cos \pi / 3+\frac{6}{18} \cos 0\right)=\frac{2}{3} J \tag{5.12}
\end{equation*}
$$

Now that we have found an expression for the renormalization of J at $f=1 / 3$, we can proceed to calculate the domain wall energy of various domain types. Domain walls are topological excitations above the ground state, and so before we can calculate their energy in an infinite array, we must first obtain an analytic expression for the ground state energy at $f=1 / 3$ that scales with the array size. The ground state periodic configuration of $+2 / 3$ and $-1 / 3$ vortex charges can be expressed as

$$
\begin{equation*}
m_{r}^{(0)}=\frac{2}{3} \cos \left[\frac{2 \pi}{3}(x+y)\right] \tag{5.13}
\end{equation*}
$$

where $x$ and $y$ are integer distances (in units of $a$ ) that live on the sites of the dual lattice. The variable $m_{r}$ is an order parameter which assigns to each cell a chirality, and $m_{r}^{(0)}$ represents one of the ground states, which contains a periodic arrangement of $+2 / 3$ and $-1 / 3$ charges and is overall charge neutral (see Figure 5.5). By plugging (5.13) into (5.7) and summing over all sites, it is a simple matter to show that the ground state energy of the array is


Figure 5.5: Ground state of the array. Shaded and empty plaquettes contain vortex charges of $+2 / 3$ and $-1 / 3$ respectively.

$$
\begin{equation*}
H_{0}=\frac{4 \pi^{2} J}{81} N \tag{5.14}
\end{equation*}
$$

where $N=N_{x} N_{y}$ is the number of plaquettes (sites in the dual lattice, which becomes infinite in the thermodynamic limit).

In order to analytically calculate the energy of a single herringbone-0 wall, we generalize the arguments of Korshunov for the $f=1 / 2$ case[66] by first calculating the interaction energy of two parallel herringbone-0 walls, and then let the domain wall


Figure 5.6: Two different neighboring ground states create two parallel herringbone-0 walls. The interaction energy is a function of the separation distance $3 L$, where $L$ is an integer.
separation distance go to infinity. In Figure 5.6 are sketched two parallel herringbone-0 walls a distance $3 L$ apart. The charge configurations $m_{r}$ can be expressed as

$$
m_{r}= \begin{cases}\frac{2 \pi}{3} \cos \frac{2 \pi}{3}(x-y), \quad 0 \leq x \leq 3 L-1 \text { (Region B) }  \tag{5.15}\\ \frac{2 \pi}{3} \cos \frac{2 \pi}{3}(x+y), & \text { otherwise (Region A) }\end{cases}
$$

which describes the vortex pattern shown in Figure 5.6. The energy of this configuration can be written as

$$
\begin{equation*}
H=\frac{\pi^{2} \tilde{J}}{2} \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{1}{\sin ^{2} \frac{k_{x}}{2}+\sin ^{2} \frac{k_{y}}{2}}\left|\sum_{r} m_{r} \exp [i \mathbf{k} \cdot \mathbf{r}]\right|^{2} \tag{5.16}
\end{equation*}
$$

In order to simplify, we begin by writing the modulus squared term as

$$
\begin{align*}
\sum_{r} m_{r} \exp [i \mathbf{k} \cdot \mathbf{r}] & =\sum_{r} m_{r}^{(0)} \exp [i \mathbf{k} \cdot \mathbf{r}]+\sum_{r \in B}\left[m_{r}-m_{r}^{(0)}\right] \exp [i \mathbf{k} \cdot \mathbf{r}] \\
& =\sum_{r} m_{r}^{(0)} \exp [i \mathbf{k} \cdot \mathbf{r}]+\sum_{r \in B} \frac{2}{3}\left[\cos \frac{2 \pi}{3}(x-y)-\cos \frac{2 \pi}{3}(x+y)\right] \exp [i \mathbf{k} \cdot \mathbf{r}] \\
& =\sum_{r} m_{r}^{(0)} \exp [i \mathbf{k} \cdot \mathbf{r}]+\sum_{r \in B} \frac{4}{3}\left(\sin \frac{2 \pi}{3} x\right)\left(\sin \frac{2 \pi}{3} y\right) \exp [i \mathbf{k} \cdot \mathbf{r}] \tag{5.17}
\end{align*}
$$

Each of these terms can be summed explicitly. The first term in (5.17) can be written as

$$
\begin{align*}
\sum_{r} m_{r}^{(0)} \exp [i \mathbf{k} \cdot \mathbf{r}]= & \frac{2}{3} \sum_{r} \cos \frac{2 \pi}{3}(x+y) \exp \left[i k_{x} x+i k_{y} y\right] \\
= & \frac{1}{3}\left[\sum_{r} \exp \left[i\left(k_{x}+\frac{2 \pi}{3}\right) x+i\left(k_{y}+\frac{2 \pi}{3}\right) y\right]+\right. \\
& \left.\sum_{r} \exp \left[i\left(k_{x}-\frac{2 \pi}{3}\right) x+i\left(k_{y}-\frac{2 \pi}{3}\right) y\right]\right] \\
= & \frac{N}{3}\left[\delta_{\mathbf{k}, 2 \pi / 3}+\delta_{\mathbf{k},-2 \pi / 3}\right] \tag{5.18}
\end{align*}
$$

where we have made use an identity of discrete Fourier transforms,

$$
\begin{equation*}
\sum_{r} \exp \left[i\left(\mathbf{k}-\mathbf{k}^{\prime}\right) \cdot \mathbf{r}\right]=N \delta_{\mathbf{k}, \mathbf{k}^{\prime}} \tag{5.19}
\end{equation*}
$$

The second term in (5.17) can be written as

$$
\begin{align*}
\sum_{r \in B} \frac{4}{3}\left(\sin \frac{2 \pi}{3} x\right)\left(\sin \frac{2 \pi}{3} y\right) \exp [i \mathbf{k} \cdot \mathbf{r}]= & \frac{4}{3}\left(\sum_{x \in B}\left(\sin \frac{2 \pi}{3} x\right) \exp \left[i k_{x} x\right]\right) \times \\
& \left(\sum_{y}\left(\sin \frac{2 \pi}{3} y\right) \exp \left[i k_{y} y\right]\right) \\
= & \frac{4}{3}\left(\frac{1}{2 i}\left(\delta_{k_{y},-2 \pi / 3}-\delta_{k_{y}, 2 \pi / 3}\right)\right)\left(\frac{1}{2 i}\left(\alpha_{+}-\alpha_{-}\right)\right) \\
= & -\frac{1}{3}\left(\delta_{k_{y},-2 \pi / 3}-\delta_{k_{y}, 2 \pi / 3}\right)\left(\alpha_{+}-\alpha_{-}\right) \tag{5.20}
\end{align*}
$$

where

$$
\begin{align*}
\alpha_{ \pm} & \equiv \sum_{x=0}^{3 L-1} \exp \left[i\left(k_{x} \pm \frac{2 \pi}{3}\right)\right] \\
& =\frac{1-\exp \left[i 3 L\left(k_{x} \pm \frac{2 \pi}{3}\right)\right]}{1-\exp \left[i\left(k_{x} \pm \frac{2 \pi}{3}\right)\right]} \tag{5.21}
\end{align*}
$$

By combining the results of $(5.18),(5.20)$, and (5.21), the square modulus of (5.17) can be written as

$$
\begin{align*}
& \left|\sum_{r} m_{r} \exp [i \mathbf{k} \cdot \mathbf{r}]\right|^{2}=\frac{N^{2}}{9}\left[\delta_{\mathbf{k}, 2 \pi / 3}+\delta_{\mathbf{k},-2 \pi / 3}\right]-\frac{N_{x} N_{y}^{2}}{9}\left[3 L\left(\delta_{k_{y}, 2 \pi / 3}+\delta_{k_{y},-2 \pi / 3}\right)+\right. \\
& \left.\sum_{x=0}^{3 L-1}\left(\exp \left[\frac{4 \pi i x}{3}\right] \delta_{k_{y}, 2 \pi / 3} \exp \left[-\frac{4 \pi i x}{3}\right] \delta_{k_{y},-2 \pi / 3}\right)\right]+\frac{N_{y}^{2}}{9}\left(\delta_{k_{y}, 2 \pi / 3}+\delta_{k_{y},-2 \pi / 3}\right) \times \\
& \quad\left[\frac{1-\cos \left[3 L\left(k_{x}+2 \pi / 3\right)\right]}{1-\cos \left[3 L\left(k_{x}+2 \pi / 3\right)\right]}+\frac{1-\cos \left[3 L\left(k_{x}-2 \pi / 3\right)\right]}{1-\cos \left[3 L\left(k_{x}-2 \pi / 3\right)\right]}-4 \frac{\sin ^{2}\left[3 k_{x} L / 2\right]}{1+2 \cos k_{x}}\right] \tag{5.22}
\end{align*}
$$

After substituting (5.22) into (5.16) we obtain, upon simplification, a closed analytic expression for the interaction energy of two Herringbone-0 domains,

$$
\begin{array}{r}
H-H_{0}=\frac{4}{9} \pi^{2} \tilde{J} N_{y}\left[2 \int_{-\pi}^{\pi} \frac{d k}{2 \pi} \frac{1-\cos [3 L(k+2 \pi / 3)]}{(1-\cos [k+2 \pi / 3])(5-2 \cos k)}\right. \\
\left.-4 \int_{-\pi}^{\pi} \frac{d k}{2 \pi} \frac{\sin ^{2}(3 k L / 2)}{(1+2 \cos k)(5-2 \cos k)}-L\right] \tag{5.23}
\end{array}
$$

where $H_{0}$ is the ground state energy (5.14). Therefore, the energy per unit length, in terms of the bare-coupling $J$, of a single Herringbone-0 wall is

$$
\begin{align*}
& E_{\mathrm{D}}^{\mathrm{H} 0}=\frac{4 \pi^{2} J}{27} \lim _{L \rightarrow \infty}\left[2 \int_{-\pi}^{\pi} \frac{d k}{2 \pi} \frac{1-\cos [3 L(k+2 \pi / 3)]}{(1-\cos [k+2 \pi / 3])(5-2 \cos k)}\right. \\
&\left.\quad-4 \int_{-\pi}^{\pi} \frac{d k}{2 \pi} \frac{\sin ^{2}(3 k L / 2)}{(1+2 \cos k)(5-2 \cos k)}-L\right] \tag{5.24}
\end{align*}
$$

This limit converges very quickly to the value 0.0363696 and becomes independent of $L$. Thus, the domain energy per unit length is $E_{\mathrm{D}}^{\mathrm{H} 0}=0.053178 J$, which is a $6.3 \%$ difference from the numeric result (see Table 5.1). A similar calculation for the energy per unit length of a single herringbone- 1 wall gives the result

Table 5.1: Domain energies (in units of $J$ ) for $f=1 / 3$, calculated both numerically and analytically within this section.

| Domain | Analytic | Numeric |
| :---: | :---: | :---: |
| Herringbone-0 | $.05317837(7)$ | $0.05673742(4)$ |
| Herringbone-1 | $0.372(2)$ | $0.333(7)$ |
| Shift-by-1 | - | $0.11419(9)$ |
| Shift-by-2 | - | $0.33333(2)$ |
| Shift-by-1 + Shift-by-2 | $0.47860(5)$ | $0.44753(1)$ |

$$
\begin{array}{r}
E_{\mathrm{D}}^{\mathrm{H} 1}=\frac{4 \pi^{2} J}{27} \lim _{L \rightarrow \infty}\left[2 \int_{-\pi}^{\pi} \frac{d k}{2 \pi} \frac{1-\cos [3 L(k+2 \pi / 3)]}{(1-\cos [k+2 \pi / 3])(5-2 \cos k)}\right. \\
 \tag{5.25}\\
\left.\quad+8 \int_{-\pi}^{\pi} \frac{d k}{2 \pi} \frac{\sin ^{2}(3 k L / 2)}{(1+2 \cos k)(5-2 \cos k)}-L\right]
\end{array}
$$

which numerically yields $E_{\mathrm{D}}^{\mathrm{H} 1}=0.37335 J$. For shift domain walls, this method cannot be used to calculate the individual domain energies because it is not possible to arrange two shift domains of the same type on top of a single ground state vortex lattice. However, we were able to calculate the interaction energy of a shift-by-1 and a shift-by-2 domain wall, using the same procedure outlined above. The final result is

$$
\begin{equation*}
E_{\text {Int }}^{\mathrm{S} 1-\mathrm{S} 2}=\frac{8 \pi^{2} J}{9} \lim _{L \rightarrow \infty}\left[\int_{-\pi}^{\pi} \frac{d k}{2 \pi} \frac{(2+\cos k) \sin ^{2}(3 k L / 2)}{(1-\sin [k-\pi / 6])(1+\sin [k+\pi / 6])(5-2 \cos k)}-L\right] \tag{5.26}
\end{equation*}
$$

This also turns out to converge very quickly to $0.478605 J$ and become independent of $L$. This energy is then simply the sum of two independent walls, a shift-by- 1 and shift-by- 2 wall. Table 5.1 compares this result with the numeric result, and the agreement is to within $6.5 \%$.

### 5.4 Phase Transition at $f=1 / 3$ in a Conventional Array

In the frustrated xy model at $f=1 / 3$, there are three different types of phase transitions which can occur in principle. This is a direct consequence of the structure of the order parameter $m_{r}$, which exhibits $U(1) \times Z_{3} \times Z_{2}$ symmetry. The $U(1)$ symmetry corresponds to the continuous rotation of all spins, and is associated with the well-known Kosterlitz-Thouless transition of proliferating vortex-antivortex pairs. The discrete $Z_{3} \times$ $Z_{2}$ symmetry is due to the sixfold degeneracy of the $f=1 / 3$ ground state. There are three discrete ways to shift the ground state vortex configuration $\left(Z_{3}\right.$ symmetry $)$ and two discrete diagonal vortex stripes $\left(Z_{2}\right.$ symmetry). A possible $Z_{3}$ transition originates from the proliferation of domain walls which separate the three types of shifted ground states. A third possible transition is related to $Z_{2}$ symmetry breaking, caused by fluctuating domains of oppositely tilted ground state vortex configurations.

These three transitions cannot occur in any arbitrary order. In fact, we will argue that instead of observing a sequence of three transitions at $f=1 / 3$, there will only be one transition related to the $Z_{2}$ symmetry of the vortex configuration. We will first demonstrate that there can be no $Z_{3}$ transition, and then show that a KosterlitzThouless transition is also not expected for $f=1 / 3$. The only remaining possibility is a $Z_{2}$ transition, and we will provide an estimate for this transition temperature.

### 5.4.1 $\quad Z_{3}$ Transition

It turns out that there will be no $Z_{3}$ transition in a conventional array at $f=1 / 3$. In order to show this, we need to consider in more detail the energetic and entropic factors of the various low-energy domain wall excitations that appear at low temperature. We have already seen that for the six degenerate ground states at
$f=1 / 3$, there is a discrete $Z_{3} \times Z_{2}$ symmetry. This is due to the fact that new ground states can be generated by shifting the diagonal stripes of vortices in a single ground state horizontally or vertically. This means the vortices along the diagonals sit on $q$ sublattices (for $f=p / q$ ) and can be associated with the members of a $Z_{q}$ group. Additionally, these diagonal stripes of vortices come in two discrete classes with opposing tilt, and so can be identified with members of a $Z_{2}$ group. As we have seen (see Table 5.1), domain walls formed between various states differ considerably in terms of energy. In particular, the low-energy domains fall into the two distinct categories of either herringbone or shift, each with differing topologies. Shift walls separate three members of $Z_{3}$ group. Herringbone walls separate the two members of a $Z_{2}$ group.

A shift wall can branch into other shift walls or into pairs of herringbone walls. Since shift walls differ considerably in terms of energy, bends in these walls are energetically unfavorable (see Figure 5.7). A bend in a shift-by-1 wall, for example, would turn the wall into a shift-by- 2 wall, which is energetically costly as it moves from a low domain energy to a high domain energy (nearly 3 times higher). For this reason, one typically finds simple kinks of 1 or 2 lattice constants in size for shift walls, while larger kinks are unstable[91]. The energy difference between two different shift walls will in turn reduce the entropy of a shift-by- 1 wall, since its movement and growth are restricted. Since shift walls change type when they bend, all closed domains composed of shift walls will contain both types of shift walls. Therefore, the effective energy of a shift-by-1 wall is about twice its actual energy (see Table 5.1), or about four times the cost of a single herringbone-0 domain wall. As opposed to shift walls, herringbone walls cannot branch into other herringbone walls. The no-branching characteristic of herringbone walls and their association with a $Z_{2}$ symmetry group makes these walls very similar to Ising domain walls.

One major difference between herringbone walls and normal Ising walls is that herringbone walls carry an extra vortex density at their corners. On a large distance scale, this accumulation of vorticity contributes a quadrupole moment to closed domains. However, the quadrupole interaction between closed herringbone domains doesn't affect any of the general properties of these domains or restrict their motion. This is because the entropic contribution to the quadrupole free energy scales logarithmically with domain separation distance and will always dominate over the pure quadrupole interaction energy [91].


Figure 5.7: Top panel: H0 domain wall will bend into a H0 domain wall, and cannot branch into other types. The basic building block for H 0 domains are $3 \times 3$ cells. Bottom panel: A S1 domain wall will change into a $S 2$ wall when bent, and can also branch into 2 H 0 walls.

The lowest energy stable domains for $f=1 / 3$ are the herringbone- 0 walls (see Table 5.1). A shift-by-1 (S1) wall can be viewed as two adjacent herringbone-0 (H0) walls. A single S 1 wall is more energetically costly than 2 H 0 walls, hence S 1 walls
are unstable and will decay into H 0 walls. When the temperature is high enough for domains to enter the system, only H0 domains will be present at large length scales. At low temperatures, on top of the ground state will sit fluctuating domains bounded by H 0 walls. These H 0 domains will be closed and will restore the $Z_{3}$ symmetry because ground states with any of the three types of shift are equally likely to appear. In principle the phase transition for $f=1 / 3$ could be due to the breaking of either the $Z_{2}$ or $Z_{3}$ symmetry. If a $Z_{3}$ transition were to occur at a lower temperature than the $Z_{2}$ transition, the overall tilt of the diagonal vortex stipes would be preserved and one would still expect to see a subsequent transition involving the $Z_{2}$ symmetry. However, if a $Z_{2}$ transition occurs at a lower temperature, then all memory of $Z_{3}$ order will be immediately lost. This is due to the fact that herringbone domains between oppositely tilted vortex stripes can be between states with any type of shift.

Although it is possible for a $Z_{3}$ transition to preempt a $Z_{2}$ transition (but not vice-versa), this will never occur in practice. The domains associated with $Z_{3}$ order are shift walls, which are unstable and much more energetically costly than the herringbone walls, which are associated with $Z_{2}$ order. The domain walls that first appear at low temperature will always be the herringbone type. As we have seen, once the temperature increases and herringbone domains begin to proliferate, all $Z_{3}$ symmetry is restored. Therefore, there will be no visible $Z_{3}$ transition in a conventional array at $f=1 / 3$.

### 5.4.2 Kosterlitz-Thouless Transition

We have already eliminated the possibility of a $Z_{3}$ transition, and so the only two remaining possible transitions are those related to the discrete $Z_{2}$ symmetry and the continuous $U(1)$ symmetry of the order parameter. The first type involves the melting of a regular ordered pattern of background charges. In the second type, excessive pairs
of logarithmically-interacting neutral vortices are thermally excited and appear above the background lattice in tightly bound pairs of zero net vorticity. At the well-defined Kosterlitz-Thouless transition temperature, bound vortices will unbind and begin to proliferate. The question to be addressed is whether or not the melting of the vortexlattice structure of charges will affect the proliferation of excessive charge defects.


Figure 5.8: There are fractional vortices of charge $\pm 1 / 9$ bound to domain wall corners for $f=1 / 3$. The presence of excessive vorticity can be seen by averaging the chiralities of the nine cells around each corner (the corner is centered on one cell and there are eight plaquettes surrounding each corner). Each $\pm$ pair of charges constitutes a dipole, which, as the temperature increases, will eventually unbind (see surrounding text for discussion).

For finite energy $Z_{2}$ domains, as is the case for both $f=1 / 2$ and $f=1 / 3$, we can imagine the scenario $T_{\mathrm{k}}<T_{\mathrm{dw}}$, but not $T_{\mathrm{k}}>T_{\mathrm{dw}}$. The kink-unbinding transition can preempt the lattice melting transition but not vice-versa, a fact demonstrated explicitly by Korshunov[64] for fully frustrated models. Below $T_{\text {dw }}$, kinks are free to proliferate, which will in turn cause ordinary vortices to proliferate. However, as soon as the temperature exceeds $T_{\mathrm{dw}}$ in the thermodynamic limit, there will emerge infinite domain walls which destroy any long-range lattice ordering of the chiralities. The presence of infinite domains decouples the phase fluctuations on either side of a domain wall as the effective stiffness of the system with respect to the continuous twisting of the
phase is lost.
It turns out that fractional vortices appear as domain wall defects not only for fully frustrated models, but also for smaller values of $f$. This feature of the frustrated xy model strongly suggests that the scenario $T_{\mathrm{KT}}>T_{\mathrm{dw}}$ is impossible for many other values of $f[66]$. In particular, for $f=1 / 3$, we find that topological defects of charge $\pm 1 / 9$ are bound to the corner of domain walls (see Figure 5.8 ). There is only one type of kink for $f=1 / 3$, a bound neutral pair of fractional vortices of charge $\pm 1 / 9$, whose separation grows in integer multiples of 3 lattice constants. As these kink defects begin to proliferate, the domain walls will grow diagonally across the lattice, and ordinary vortices will be forced to dissociate through the screening action of fractional vortices bound to domain walls. As we have seen, by the time infinite Ising-like domains emerge in the system, the effective phase stiffness across domains will vanish. Both $f=1 / 3$ and $f=1 / 2$ have a discrete ground state degeneracy, with finite energy Ising-like domains containing logarithmically-interacting fractional vortices bound to domain wall defects. Therefore, the screening mechanism that operates at $f=1 / 2$ will also be present at $f=1 / 3$, making the scenario $T_{\mathrm{KT}}>T_{\mathrm{dw}}$ impossible. There still remains the possibility that $T_{\mathrm{KT}}<T_{\mathrm{dw}}$, this is however very unlikely given that there were no clear signs of a separate KT transition present in the Monte Carlo simulations of Denniston[91].

### 5.4.3 $Z_{2}$ Transition

It turns out that the phase transition in a conventional array at $f=1 / 3$ involves the restoring of $Z_{2}$ symmetry through the proliferation of herringbone domains. It has also been shown to be Ising-like[91]. The proliferation of H0 domains necessarily entails that the $Z_{3}$ symmetry is also simultaneously restored, and there will be no subsequent phase transitions. At low temperatures, the array will only be populated
by small H0 domains. H0 domains must live on three sublattices of lattice constant $3 a$, which is enforced by the topology of the herringbone state. This means that basic building block of H 0 domains are $3 \times 3$ unit cells, with an effective lattice constant of $3 a$. These three sublattices are shifted with respect to each other by $(x, y)=(1,1)$, and H0 domains living on one sublattice are bound to and independent of the other two lattices. Each closed H0 domain has an effective domain energy which is three times larger then the bare H 0 domain energy, since these domains live on a sublattice with a lattice constant of $3 a$, and therefore must grow in lengths of $3 a$. Note that closed H0 domains are still energetically favorable over closed shift domains, because we pointed out earlier that closed shift domains cost nearly four times the energy of closed H0 domains.

As the temperature increases, the closed H0 domains will grow in size and additional closed fluctuating domains will begin to appear within them. To each of the three independent sublattices containing H0 domains, there correspond three additional and independent sublattices of lattice constant $3 a$ which also support H 0 domain wall fluctuations. This means there are a total of nine sublattices of lattice constant $3 a$, each occupying a site of the $3 \times 3$ unit cell of the original lattice. However, these nine sublattices are separated into three sets of three, since each set is localized over a particular ground state. The three sublattices and their respective H0 domains within are all bound to and contained within their respective closed H 0 domain sublattice. Each H0 domain on a particular sublattice is topologically equivalent and all the other H0 domains. They are permitted to cross, and so the set of domain configurations that begin to grow in the lattice behave as three identical and independent copies of the Ising model.

Each of these three Ising models, since they are identical and independent, will
have the same transition temperature. The transition temperature for a single 2D Ising model is (see Appendix C)

$$
\begin{equation*}
T_{\mathrm{dw}}=\frac{E_{\mathrm{dw}}}{\ln (1+\sqrt{2})} \tag{5.27}
\end{equation*}
$$

where $E_{\mathrm{dw}}$ is the effective domain wall energy. The relevant Ising domain energy $E_{\mathrm{dw}}$ for $f=1 / 3$ are H 0 walls of length $3 a$ and so the effective Ising domain energy is three times that of a H 0 domain, $E_{\mathrm{dw}}=3 E_{\mathrm{D}}^{\mathrm{H} 0}=3 \times 0.05673742 J=0.17021226 J$. This yields a critical temperature of $T_{\mathrm{C}}=0.1931 J$, above which the $Z_{2}$ symmetry is restored. This estimate for the transition temperature is found to be in excellent agreement with the estimate obtained by Denniston[90]. He found the critical temperature to be $T_{\mathrm{C}}=0.19 J[90]$, an analytic result that came from dividing the lattice into two separate regions separated by a solid-on-solid domain wall stretching across the entire array and calculating the temperature at which the fluctuations in this wall diverge. Both of these results are in remarkable agreement with the values $T_{\mathrm{C}}=(0.215-0.22) \mathrm{J}$ found in Monte Carlo simulations[93, 90]. In summary, out of the three transitions that can occur in principle at $f=1 / 3$ in a conventional array, there will be only one transition involving the $Z_{2}$ symmetry of the order parameter.

## $5.5 p+i p$ JJ Array at $f=1 / 3$

A model that can be used to describe a $p+i p$ Josephson junction array by allowing the spin stiffness to vary independently of the superfluid stiffness is

$$
\begin{equation*}
H=-J\left(\frac{9-\alpha}{8}\right) \sum_{\langle i j\rangle} \cos \left(\theta_{j}-\theta_{i}-A_{i j}\right)\left[\cos \left(\phi_{i}-\phi_{j}\right)-\left(\frac{1-\alpha}{9-\alpha}\right) \cos 3\left(\phi_{i}-\phi_{j}\right)\right] \tag{5.28}
\end{equation*}
$$

where $J$ is the Josephson coupling energy and $\alpha=\rho_{\mathrm{sp}} / \rho_{\mathrm{s}}$ is the ratio of the spin stiffness to the superfluid stiffness. The first cosine is a function of the gauge-invariant superconducting phase $\theta$, while $\phi$ in the other cosine terms represents the angle of the complex director vector $\mathbf{d}(\mathbf{k})$, which describes the direction of spin-triplet Cooper pairing[41]. The Hamiltonian (5.28) is invariant under the following transformations: $\theta_{i} \rightarrow \theta_{i}+2 \pi, \phi_{i} \rightarrow \phi_{i}+2 \pi$, and $\theta_{i} \rightarrow \theta_{i}+\pi$ combined with $\phi_{i} \rightarrow \phi_{i}+\pi$. The various types of low energy topological excitations that are allowed by this model are full vortices in $\theta$, full vortices in $\phi$, and half vortices which involve rotations of both $\theta$ and $\phi$. In order to find the ground state for this model with a frustration index of $f=1 / 3$, we take advantage of what is already known about the ground state in the frustrated xy model.

Since we already know that the ground state for $f=1 / 3$ with one degree of freedom $(\theta)$ is 6 -fold degenerate (see Figure 5.2), the Hamiltonian with two degrees of freedom and equal stiffness (which decouples $\theta$ and $\phi$ rotations) has $6 \times 6=36$ degenerate ground states (in order to see this explicitly we must rewrite $\theta$ and $\phi$ in terms of $u$ and $v$ as is described in Section 4.2). With 36 degenerate ground states, the symmetry of the order parameter will be $\left(U(1) \times Z_{3} \times Z_{2}\right)^{2}$. The interesting case is what happens to these 36 degenerate ground states as the spin stiffness is gradually made softer than the superfluid stiffness $(\alpha<1)$. In Figure 5.9 we plot the result of this calculation. We find that 12 states increase in energy, 12 states stay the same, and 12 states decrease as $\alpha$ is decreased. The lower branch in Figure 5.9 contains the 12 true ground states for $\alpha<1$.

### 5.6 Domains in a $p+i p$ JJ Array

In a $p+i p \mathrm{JJ}$ array where the spin stiffness and superfluid stiffness are unequal $(\alpha \neq 1)$, there are a total of 12 ground states, which means that there will be a large


Figure 5.9: The energy of the 36 degenerate ground states at $\alpha=1$ as a function of $\alpha$. At $\alpha=1$ all 36 states are true ground states, but as $\alpha$ decreases, the states split. 12 go up in energy, 12 stay the same, and 12 decrease. The lower branch of 12 states are now the only remaining ground states.
number of different domain walls. This can be checked explicitly by taking derivatives of the 36 possible states. The 12 ground states are shown in Figure 5.10 in terms of the two independent degrees of freedom $u$ and $v$. The shaded (empty) plaquettes in $u / v$ contain vortices with charge $+2 / 3(-1 / 3)$ respectively.

The 12 degenerate ground states at $\alpha<1$ have $Z_{2} \times Z_{2} \times Z_{3}$ symmetry. The reduction in symmetry results from the onset of $u / v$ interaction when $\alpha<1$. The remaining symmetry can be understood by analyzing the structure of the ground states pictured in Figure 5.10. The diagonal stripes of vortices in $u$ and $v$ are associated with a $Z_{2}$ symmetry group, while the three types of domain shifts correspond to $Z_{3}$ symmetry. The additional $Z_{2}$ symmetry group for $\alpha<1$ corresponds to the crossing of $u / v$ domains. In the ground state, either all $u$ diagonal stripes run on top of the $v$ diagonal stripes, or vice-versa. When domains are present, $u / v$ domains are permitted to cross, which corresponds to 2 discrete degrees of freedom.

We can also picture the $f=1 / 3$ ground state in terms of $\theta$ and $\phi$ directly (see
Figure 5.11). The ground state consists of two neighboring diagonal strips of $\frac{1}{2} \mathrm{QVs}$,


Figure 5.10: 12 degenerate ground states for $\alpha<1$ in terms of the $u$ and $v$ degrees of freedom. Each ground state has the same vortex tilt and there is a relative shift in the $u$ and $v$ pattern.
where the light and dark plaquettes correspond to opposite windings in the spin sector. Although topology permits pairs of $\frac{1}{2} \mathrm{QVs}$ with opposite spin winding to coalesce into full QVs, it turns out that pairs of $\frac{1}{2} \mathrm{QVs}$ are more energetically stable when the spin stiffness is softer than the superfluid stiffness $(\alpha<1)$. The $Z_{2} \times Z_{2} \times Z_{3}$ symmetry-breaking of the ground state can be understood in terms of the $\frac{1}{2} \mathrm{QVs}$. The first $Z_{2}$ corresponds to the tilt of the $\frac{1}{2} \mathrm{QV}$ diagonal strips, and the second $Z_{2}$ corresponds to the crossing of the positive and negative $\frac{1}{2} \mathrm{QV}$ diagonal strips. Finally, the $Z_{3}$ symmetry corresponds to the three discrete shift degrees of freedom of the ground state configuration.

Since there are a total of 12 ground states, there is a rich structure of domain wall excitations permitted in the case of unequal stiffness. In order to classify the different types of domains, we will focus in particular on domains with the lowest energy. The low energy stable domains will be the relevant excitations necessary to behavior of


Figure 5.11: The $f=1 / 3$ ground state when $\alpha<1$ in terms of $\theta$ and $\phi$. There are two neighboring diagonal strips of $\frac{1}{2}$ QVs and one empty diagonal. In the light plaquettes, $\theta$ and $\phi$ wind by $\pi$ (counterclockwise direction), whereas in the dark plaquettes $\theta$ winds by $\pi$ and $\phi$ winds by $-\pi$.
a $p+i p$ Josephson junction array at finite temperature. We start out with the simple observation that all the domains formed in the $u$ and $v$ degrees of freedom are simply combinations of the domains which arise in a conventional array at $f=1 / 3$ (see Figure 5.3). Since there are two degrees of freedom in a $p+i p$ JJ array, domain walls involve kinks in both the $u$ and $v$ vortex configurations. By the very nature of the case, the low energy domains will be more numerous and complex in the $p+i p$ case than in the conventional s-wave case. In Figure 5.12 we sketch the simplest low energy domains, which are between the ground states of Figure 5.10 . Notice that there will be a domain in both $u$ and $v$, and these domains are subject to the constraint that both are either of the herringbone-type or of the shift-type. This is due to the fact that all 12 ground states have the same tilt in $u$ and $v$. Additionally, the $u / v$ patterns never overlap since the ground state vortex configurations in $u / v$ are always shifted with respect to one another.

The first interesting type of domain to consider is a co-linear herringbone-0 in both $u$ and $v$ (named $\mathrm{H} 0-\mathrm{H} 0[0]$ in Figure 5.12). These herringbone domains lie along
the same vertical line (they are co-linear domains). However, it is also possible to imagine more complex herringbone domains in $u$ and $v$, where the patterns are shifted with respect to each other. For example, herringbone domains in $u$ and $v$ shifted by any integer number of $n$ lattice constants ( $\mathrm{H} 0-\mathrm{H} 0[\mathrm{n}]$ ) are always possible to construct (see Figure 5.12 for some examples), but tend to become more energetically costly as $u / v$ herringbone patterns become increasingly separated. Other types of herringbone domains include pure herringbone-1 walls as well as mixtures between herringbone- 0 and herringbone-1 walls. When the wall is of shift-type, we can have pure S1 or S2 walls, or mixtures of S1, S2, and ground state (no shift) patterns.

The energy per unit length of the domain walls sketched in Figure 5.12 is calculated in Figure 5.13 as a function of $\alpha$. Let us first consider the H1-H1 domain. When $\alpha=1$, the domain energy is simply the sum of two independent herringbone- 1 walls with a coupling $J \rightarrow J / 2$. This fact follows immediately from equation (4.9). In fact, all domain energies at $\alpha=1$ can be understood as half the sum of two noninteracting $f=1 / 3$ domains. However, for $\alpha<1$, we pick up the interaction term in (5.28), and all domain wall energies begin to increase slowly in energy as a function of $\alpha$.

### 5.7 Phase Transition in a $p+i p$ JJ Array at $f=1 / 3$

When the spin stiffness is softer than the superconducting phase stiffness ( $\alpha<$ 1 ), the three domains with lowest energy are the $\mathrm{H} 0-\mathrm{H} 0[0], \mathrm{H} 0-\mathrm{H} 0[1]$, and the GS-S1 domains (see Figure 5.14). All other domain walls are much more costly when $\alpha<1$ and so we can safely ignore their presence. Therefore, we now focus our attention


Figure 5.12: Various types of stable domains. The tilted solid/dashed black lines represents the striped vortex pattern in $u$ and $v$ respectively. Each domain is labeled as $u-v$, which describes the type of domain in $u$ and $v$ respectively. The abbreviations are: GS=Ground State, H0=Herringbone-0, H1=Herringbone-1, S1=Shift-by-1, S2=Shift-by-2. For herringbone-type walls, the square brackets [n] further specify the distance between the kinks in $u$ and $v$. For example, $\mathrm{H} 0-\mathrm{HO}[2]$ is the domain where there is a herringbone wall in both $u$ and $v$, and they are separated by a distance of 2 lattice constants.
on understanding the constraints and symmetry properties of these walls. In terms of symmetry, the proliferation of $\mathrm{H} 0-\mathrm{H} 0[0]$ walls will preserve the $Z_{2}$ spontaneous symmetry breaking of $u / v$ positioning since these domains never cross (see Figure 5.12), but will restore the $Z_{2}$ tilt symmetry. On the other hand, $\mathrm{H} 0-\mathrm{H} 0[1]$ domain proliferation will restore both $Z_{2}$ tilt symmetry and the $Z_{2}$ symmetry of $u / v$ domain crossing. GS-S1 domain proliferation will restore the $Z_{2}$ symmetry of $u / v$ domain crossing as well as the


Figure 5.13: The energy of the lowest, non-trivial, stable domains. Each domain wall is a combination of domains in both $u$ and $v$, denoted as $u-v$. The most basic $u-v$ domain mixtures are combinations of herringbone-0 (H0), herringbone-1 (H1), shift-byone (S1), shift-by-two (S2), and ground state (GS). For $\alpha<1$ there are really only 3 relevant domains.
$Z_{3}$ shift symmetry.
Although $\mathrm{H} 0-\mathrm{H} 0[0], \mathrm{H} 0-\mathrm{H} 0[1]$, and GS-S1 domains are all energetically similar, we do not expect to see GS-S1 domains at low temperatures because there are additional constraints on these types of walls. Consider the parallel domains in Figure 5.15, where the top domain is of a GS-S1 wall and the bottom domain is a GS-S2 wall. It turns out that because of the topology of shift walls, closed domain loops composed of a single type of shift wall are prohibited. In general, any closed loop that one can form involving shift walls will be half GS-S1 and half GS-S2. From Figure 5.12, we see that GS-S2 domains are much more energetically costly, and therefore the effective minimum energy per unit length of a shift wall becomes (for $\alpha=1$ )


Figure 5.14: The 3 lowest energy domains which are a combination of walls in both $u$ and $v$.

$$
\begin{equation*}
\frac{1}{2}(\mathrm{GS}-\mathrm{S} 1+\mathrm{GS}-\mathrm{S} 2)=\frac{1}{2}\left(\frac{0.11419 J}{2}+\frac{0.33333 J}{2}\right)=0.11188 J \tag{5.29}
\end{equation*}
$$

where we have used Table 5.1. Herringbone walls obey a similar constraint, namely, any closed herringbone domain necessarily involves two types of herringbone walls. The lowest effective energy of a closed herringbone domain equally combines $\mathrm{H} 0-\mathrm{H} 0[0]$ and $\mathrm{H} 0-\mathrm{H} 0[1]$ domains, and is therefore (for $\alpha=1$ )

$$
\begin{equation*}
\frac{1}{2}(\mathrm{H} 0-\mathrm{H} 0[0]+\mathrm{H} 0-\mathrm{H} 0[1])=\frac{1}{2}(0.056737 J+0.056737 J)=0.056737 J \tag{5.30}
\end{equation*}
$$

Since $\mathrm{H} 0-\mathrm{H} 0[0]$ and $\mathrm{H} 0-\mathrm{H} 0[1]$ walls have similar energies to begin with, the effective herringbone domain energy is very close to the energy of each individual herringbonetype domain. Due to these constraints, the effective energy of shift-type loops is nearly twice that of herringbone-type loops and thus herringbone-type loops will be the most favorable type of domains at low temperature.


Figure 5.15: Oppositely faced shift domains are constrained to be different types. Therefore, any closed loop of shift domains necessarily contains an equal mixture of GS-S1 and GS-S2 domains.

At low temperatures, we can safely ignore shift-type loops, and the only relevant domain excitations will be $\mathrm{H} 0-\mathrm{H} 0[0]$ and $\mathrm{H} 0-\mathrm{H} 0[1]$ domains. We have already argued that closed herringbone-type loops involve an equal mixture of both $\mathrm{H} 0-\mathrm{H} 0[0]$ and $\mathrm{H} 0-\mathrm{H} 0[1]$ domains. However, it is possible to classify these domains in such a way that they become decoupled. As such, we are left with two intersecting domain loops, each composed of a single type of domain wall. In order to decouple the mixed $\mathrm{H} 0-\mathrm{H} 0[0]$ and $\mathrm{H} 0-\mathrm{H} 0[1]$ domain loops, we first observe that when a $\mathrm{H} 0-\mathrm{H} 0[\mathrm{n}]$ domains turns at a right angle, its degree n changes by $\pm 1$. At low temperatures and for $\alpha<1$, there are only two herringbone-type domains to choose from. Therefore, when bent by 90 degrees, a $\mathrm{H} 0-\mathrm{H} 0[0]$ will turn into a $\mathrm{H} 0-\mathrm{H} 0[1]$ and vice-versa.

Every closed herringbone-type loop is composed of two horizontal (vertical) sections of $\mathrm{H} 0-\mathrm{H} 0[0]$ and two vertical (horizontal) sections of $\mathrm{H} 0-\mathrm{H} 0[1]$. This is a constraint enforced by the topology of the vortex patterns in $u / v$ of the 12 degenerate ground states. We can successfully decouple these domains into two separate domains by prescribing a color to specific horizontal and vertical domain types. This provides a natural representation for our model as a two-color (flavor) intersecting loop model. A
loop model is simply a graphical model where closed domain loops are drawn along the bonds of the underlying lattice[94]. A two-loop model allows for two different types of loop bonds. The loops can intersect at a lattice vertex, but they cannot share bonds. In Figure 5.16 we identify horizontal (vertical) H0-H0[1] domains and vertical (horizontal) H0-H0[0] domains as being colored red (green). This prescription is permitted because $\mathrm{H} 0-\mathrm{H} 0[0]$ and $\mathrm{H} 0-\mathrm{H} 0[1]$ domains are very similar in energy and always appear together in pairs for closed loops. Now that $\mathrm{H} 0-\mathrm{H} 0[0]$ and $\mathrm{H} 0-\mathrm{H} 0[1]$ domains have been decoupled into two distinct kinds of loops, we can represent the low energy domains in a $p+i p \mathrm{JJ}$ array (left panel of Figure 5.16) as green and red intersecting loops.


Figure 5.16: (a) Vertical $\mathrm{H} 0-\mathrm{H} 0[0]$ and horizontal $\mathrm{H} 0-\mathrm{H} 0[1]$ domains are colored green, while horizontal $\mathrm{H} 0-\mathrm{H} 0[0]$ and vertical $\mathrm{H} 0-\mathrm{H} 0[1]$ domains are colored red. (b) The $\mathrm{H} 0-$ $\mathrm{H} 0[0]$ and $\mathrm{H} 0-\mathrm{H} 0[1]$ domains can be represented as two interacting colored loops.

### 5.7.1 Ashkin-Teller Model

As we have seen, the low energy $p+i p$ domain walls at $f=1 / 3$ are equivalent to a two-color loop model in that there are two distinct bond flavors (see Figure 5.16) which can intersect at any finite temperature. Therefore, the statistical mechanics of a $p+i p \mathrm{JJ}$ array at $\mathrm{f}=1 / 3$ can be understood in terms of a two-color loop model. We turn now to study the properties of a two-loop model and its representation as an Ashkin-Teller model. In writing the partition function for a two-color loop model, each
loop configuration is assigned a weight (probability measure) and then all possible loop configurations $\Gamma$ are summed over. The partition function for a two-color loop model can be written as[94]

$$
\begin{equation*}
Z=\sum_{\Gamma}\left(\frac{\lambda}{2}\right)^{b(\Gamma)} \tag{5.31}
\end{equation*}
$$

where $\lambda / 2$ is the partition weight of a single bond and $b(\Gamma)$ is the number of participating bonds for a given configuration $\Gamma$ of loops. The partition weight $\lambda / 2$ can be written in terms of the domain energies in the frustrated xy model as

$$
\begin{equation*}
\frac{\lambda}{2}=\mathrm{e}^{-\beta E_{\mathrm{D}}^{*}} \tag{5.32}
\end{equation*}
$$

where

$$
\begin{equation*}
E_{\mathrm{D}}^{*}=\frac{1}{2}\left(E_{\mathrm{D}}^{\mathrm{H} 0-\mathrm{H}[0]}+E_{\mathrm{D}}^{\mathrm{H} 0-\mathrm{H} 0[1]}\right) \tag{5.33}
\end{equation*}
$$

This is simply the effective domain energy for a herringbone-type loop as in (5.30). As we have seen, both loops have equal bond energies and thus equal partition weights. The partition function for a two-colored loop model is also related to the high temperature expansion of lattice spin systems[94]. Lattice spin systems generate domain loops and so are natural representations of various loop models. The high temperature expansion of a two component lattice spin partition function can be written as

$$
\begin{equation*}
Z=\operatorname{Tr} \prod_{\langle i j\rangle}\left[1+\frac{\lambda}{2}\left(\sigma_{i} \sigma_{j}+\tau_{i} \tau_{j}\right)\right] \tag{5.34}
\end{equation*}
$$

where $\sigma_{i}= \pm 1$ and $\tau_{i}= \pm 1$ are the discrete degrees of freedom which sit on each site of the square lattice. For a particular choice of $\lambda$, this model is equivalent to the AshkinTeller (AT) model[95], a model that describes two-interacting Ising models. The AT
model can be written as[96]

$$
\begin{equation*}
H_{\mathrm{AT}}=-\sum_{\langle i j\rangle}\left[J_{2}\left(\sigma_{i} \sigma_{j}+\tau_{i} \tau_{j}\right)+J_{4}\left(\sigma_{i} \sigma_{j} \tau_{i} \tau_{j}\right)\right] \tag{5.35}
\end{equation*}
$$

where $\sigma_{i}$ and $\tau_{i}$ are Ising spins and $J_{2}$ is the normal 2-spin Ising interaction and $J_{4}$ is a 4-spin interaction term. In order to show that the two-color loop model is equivalent to the AT model, we must choose $J_{2}$ and $J_{4}$ such that

$$
\begin{equation*}
\ln \left[1+\frac{\lambda}{2}\left(\sigma_{i} \sigma_{j}+\tau_{i} \tau_{j}\right)\right]=K_{2}\left(\sigma_{i} \sigma_{j}+\tau_{i} \tau_{j}\right)+K_{4}\left(\sigma_{i} \sigma_{j} \tau_{i} \tau_{j}\right) \tag{5.36}
\end{equation*}
$$

where $K_{2,4}=\beta J_{2,4}$. By re-summing the logarithm and noting that $\left(\sigma_{i}\right)^{2 n}=\left(\tau_{i}\right)^{2 n}=1$, it is a simple matter to show that the parameters $J_{2}$ and $J_{4}$ are related to $\lambda$ by

$$
\begin{align*}
& K_{2}=\frac{1}{2} \operatorname{arctanh}(\lambda)  \tag{5.37}\\
& K_{4}=\frac{1}{4} \ln \left(1-\lambda^{2}\right) \tag{5.38}
\end{align*}
$$

After eliminating $\lambda$ and simplifying we can write $K_{2}$ in terms of $K_{4}$ as

$$
\begin{equation*}
\cosh \left(2 K_{2}\right)=\mathrm{e}^{-2 K_{4}} \tag{5.39}
\end{equation*}
$$

This equation describes the trajectory of the AT couplings in parameter space. The phase diagram for the AT model is shown in Figure 5.17. The dashed line represents the parameter trajectories as in (5.39), and the solid lines are the phase boundaries for the AT model. Phase II is a completely disordered state, and in phase I there is ferromagnetic order in $\left\langle\sigma_{i}\right\rangle,\left\langle\tau_{i}\right\rangle$, and $\left\langle\sigma_{i} \tau_{i}\right\rangle$. In phase III there is ferromagnetic order in $\left\langle\sigma_{i} \tau_{i}\right\rangle$ but $\left\langle\sigma_{i}\right\rangle=\left\langle\tau_{i}\right\rangle=0$, and phase IV is the same as phase III except the $\left\langle\sigma_{i} \tau_{i}\right\rangle$ ordering is antiferromagnetic. Point B is a decoupling point where the critical line


Figure 5.17: Phase diagram for the Ashkin-Teller model. The dashed line is the $K_{2}-K_{4}$ trajectory of the original parameters in the 2-loop model.
bifurcates. At point $\mathrm{B}, K_{2}=K_{4}$ and the AT becomes equivalent to the 4 -state Potts model. The location of the line segment AB is know exactly through duality arguments, and the equation for this line is[96]

$$
\begin{equation*}
\sinh \left(2 K_{2}\right)=\mathrm{e}^{-2 K_{4}} \tag{5.40}
\end{equation*}
$$

This phase boundary AB turns out to be a line of continuously varying critical exponents. From Figure 5.17 we see that the trajectory of the parameters (dashed line) asymptotically approaches the boundary line between phases I and II but never crosses. At high temperatures (small $K_{2}$ ), the system is completely disordered. As the temperature is decreased ( $K_{2}$ increases), the system gets infinitesimally close to the phase boundary but never crosses it.

### 5.8 Sequence of Phase Transitions

When the superfluid stiffness and the spin stiffness are equal $(\alpha=1)$, the Hamiltonian for a $p+i p \mathrm{JJ}$ array reduces to two independent frustrated xy models. There are 36 degenerate ground states for this model and the symmetry of the chiral order parameter in $u / v$ is $\left(U(1) \times Z_{3} \times Z_{2}\right)^{2}$. When $\alpha<1$, the $u / v$ degrees of freedom begin to interact and the number of ground states is reduced to 12 . As we have seen, the structure of the order parameter is thereby reduced at $\alpha<1$ to $U(1) \times Z_{2} \times Z_{2} \times Z_{3}$. However, we have argued in the preceding sections that due to energetic factors and various entropic constraints on the domain walls, we do not expect to see separate transitions which break each symmetry.

At low temperatures, there are only two relevant domain walls, and the array model can be mapped onto the Ashkin-Teller model with two interacting Ising-like domains. This leaves us with a $Z_{2} \times Z_{2}$ symmetry. Due to the parameter couplings, this the $Z_{2} \times Z_{2}$ symmetry remains unbroken to zero temperature and there is no transition involving discrete symmetry-breaking. The only other possible transition would be the dissociation of neutral vortex pairs. The formation of vortices in $u$ and $v$ will tie together windings in both $\theta$ and $\phi$, and so in principle a Kosterlitz-Thouless transition between these vortices is possible. However, this possibility can be ruled out by the same reasoning used in the conventional $f=1 / 3$ case.

In a $p+i p \mathrm{JJ}$ array the $Z_{2} \times Z_{2}$ symmetry will remain unbroken down to zero temperature by the presence of two types of arbitrarily large domains. These are the herringbone-type domains discussed earlier, which just as in the conventional case, will bind fractional vortices to domain wall defects. The proliferation of these domains at any temperature guarantees that the fractional vortices bound to these
domains are never bound into neutral pairs. All fractional vortices will be unbound and free to proliferate. These free fractional vortices will screen the interaction between bulk vortices, thereby preventing bound vortex-antivortex pairs from forming. Since there is no effective domain wall stiffness, $u / v$ phase fluctuations across domain walls remain decoupled down to zero temperature. As we argued in the conventional case, a discrete $Z_{2}$ symmetry cannot preempt a Kosterlitz-Thouless transition. Since there is no discrete $Z_{2}$ symmetry breaking transition, there can also be no vortex unbinding transition. Therefore, we expect a $p+i p \mathrm{JJ}$ array at $f=1 / 3$ on a square lattice to be disordered at any finite temperature with no signs of a phase transition of any type. The array will remain in a resistive state even down to zero temperature.

## Chapter 6

## Screening Corrections to the KT

## Transition of $\frac{1}{2}$ QVs

### 6.1 Introduction

The ratio of KT transitions of $\frac{1}{2} \mathrm{QVs}$ in (4.31) has ignored any renormalization group corrections, which arise due to the screening of multiple vortices in the array. We explicitly calculate this correction and show that it is the same for unfrustrated ( $f=0$ ) and fully frustrated $(f=1 / 2) p+i p \mathrm{JJ}$ arrays. Therefore, the ratio between these two transitions is not affected. The procedure for the calculation is to first calculate the partition function for (4.1), then calculate the correlation between vortices, and then finally renormalize the distance to see how the relevant parameters in the model scale.

In this chapter we will describe the renormalization group procedure we used to handle half-quantum vortices. The idea is the same as the original Kosterlitz-Thouless RG procedure (see Chapter 2). The first step in Sections 6.2-6.3 is to calculate the vortex partition function. The RG equations derived in the chapter have some interesting consequences in terms of the universal jump in the superfluid stiffness and of
the current-voltage characteristics of the array (see Sections 6.5 and 6.6 respectively). The differences with an ordinary s-wave array arise because a new type of low energy excitation, namely a half-quantum vortex, is permitted to appear in a $p+i p \mathrm{JJ}$ array. In Sections 6.7-6.8 we calculate the RG flow equations and the transition temperature for the proliferation of half-quantum vortices. Finally, in Section 6.9, we calculate the effect of Majorana modes bound to half-quantum vortex cores on the Kosterlitz-Thouless transition temperature.

### 6.2 Vortex Partition Function for a $p+i p$ JJ Array

When the array is unfrustrated, all $A_{i j}=0$, and we can expand (4.1) about its minima

$$
\begin{aligned}
\theta_{j}-\theta_{i} & =2 \pi q_{j i}^{\theta}+\pi \Lambda_{j i}+\epsilon_{j i}^{\theta} \\
\phi_{j}-\phi_{i} & =2 \pi q_{j i}^{\phi}+\pi \Lambda_{j i}+\epsilon_{j i}^{\phi}
\end{aligned}
$$

where $\epsilon_{j i}^{\theta, \phi} \ll 1$ are small deviations about the ground state. The variables $q_{j i}^{\theta, \phi}$ can be any positive or negative integers, and this term captures the $2 \pi$-periodic minima of the array hamiltonian. The other minima of (4.1) is the non-trivial winding of both $\theta$ and $\phi$ by $\pi$, which physically corresponds to half-vortices. $\Lambda_{j i}$ is the link variable that describes half-vortex excitations by tying together $\pi$-rotations in both $\theta$ and $\phi$. The constraint on $\Lambda_{j i}$ is that it is non-vanishing at most once per plaquette and is restricted to the integer values of 0 or 1 .

By plugging the above expansion into (4.1), we can write

$$
\begin{align*}
H & =-J\left(\frac{9-\alpha}{8}\right) \sum_{\langle i j\rangle} \cos \left(\epsilon_{j i}^{\theta}\right)\left[\cos \left(\epsilon_{j i}^{\phi}\right)-\left(\frac{1-\alpha}{9-\alpha}\right) \cos \left(3 \epsilon_{j i}^{\phi}\right)\right] \\
& \simeq-J\left(\frac{9-\alpha}{8}\right) \sum_{\langle i j\rangle}\left[1-\frac{\left(\epsilon_{j i}^{\theta}\right)^{2}}{2}\right]\left[\left(1-\frac{\left(\epsilon_{j i}^{\theta}\right)^{2}}{2}\right)-\left(\frac{1-\alpha}{9-\alpha}\right)\left(1-\frac{9\left(\epsilon_{j i}^{\theta}\right)^{2}}{2}\right)\right] \\
& \simeq-J \sum_{\langle i j\rangle} 1+J \sum_{\langle i j\rangle} \frac{\left(\epsilon_{j i}^{\theta}\right)^{2}}{2}+J \alpha \sum_{\langle i j\rangle} \frac{\left(\epsilon_{j i}^{\phi}\right)^{2}}{2} \tag{6.1}
\end{align*}
$$

where we have retained terms up to quadratic order. This expansion has been performed for an unfrustrated array, and a fully frustrated array will simply renormalize $J$ to $\tilde{J}$ [see (3.54) for example]. However, the following calculations will remain the same and the final renormalization group ( RG ) corrections to the KT transitions for $f=0$ as quoted in (6.59) and (6.64) will turn out to be identical for $f=1 / 2$. Therefore, the ratio of KT transitions will be unaffected due to RG corrections.

The excitation energy for $f=0$ now follows from (6.1) as

$$
\begin{equation*}
H-H_{0}=\frac{J}{2} \sum_{\langle i j\rangle}\left(\theta_{j}-\theta_{i}-2 \pi q_{j i}^{\theta}-\pi \Lambda_{j i}\right)^{2}+\frac{J \alpha}{2} \sum_{\langle i j\rangle}\left(\phi_{j}-\phi_{i}-2 \pi q_{j i}^{\phi}-\pi \Lambda_{j i}\right)^{2} \tag{6.2}
\end{equation*}
$$

where $H_{0}$ is the ground state energy. The partition function can be written the Villain $\operatorname{model}[57,56]$ as

$$
\begin{gather*}
Z \propto \sum_{\left\{\Lambda_{j i}\right\}}\left[\left\{\sum_{\left\{q_{j i}^{\theta}\right\}} \int d \theta_{1} \cdots \int d \theta_{N} \exp \left[-\frac{\beta J}{2} \sum_{\langle i j\rangle}\left(\theta_{j}-\theta_{i}-2 \pi q_{j i}^{\theta}-\pi \Lambda_{j i}\right)^{2}\right]\right\} \times\right. \\
\left.\left\{\sum_{\left\{q_{j i}^{\phi}\right\}} \int d \phi_{1} \cdots \int d \phi_{N} \exp \left[-\frac{\beta J \alpha}{2} \sum_{\langle i j\rangle}\left(\phi_{j}-\phi_{i}-2 \pi q_{j i}^{\phi}-\pi \Lambda_{j i}\right)^{2}\right]\right\}\right] \tag{6.3}
\end{gather*}
$$

where the sum over $\{\cdots\}$ corresponds to a sum over all integers of every individual link. Written as such, we have included contributions from every $2 \pi$ - and $\pi$-periodic minima
in the array hamiltonian. Both bracketed terms in (6.3) have the same structure, and so both can be dealt with in the same manner. In order to rewrite (6.3) in a more useful form, we make use of the Poisson summation formula,

$$
\begin{equation*}
\sum_{q=-\infty}^{\infty} h(q)=\sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} d \tau h(\tau) e^{2 \pi i \ell \tau} \tag{6.4}
\end{equation*}
$$

which allows us to write the first bracketed term (involving $\theta$ ) as
$Z^{\theta}=\sum_{\left\{\ell_{j i}\right\}} \int d \theta_{1} \cdots \int d \theta_{N} \int_{-\infty}^{\infty} \prod_{\langle i j\rangle} d \tau_{j i} \exp \left[-\frac{\beta J}{2} \sum_{\langle i j\rangle}\left(\theta_{j}-\theta_{i}-2 \pi \tau_{j i}-\pi \Lambda_{j i}\right)^{2}+2 \pi i \ell_{j i} \tau_{j i}\right]$
The $\tau_{j i}$-integration can be factorized into a product of $2 N$ Gaussian integrals, which upon completing the square, yields the result (up to unimportant constant factors)

$$
\begin{equation*}
Z^{\theta} \propto \int d \theta_{1} \cdots \int d \theta_{N} \sum_{\left\{\ell_{j i}\right\}} \exp \left[-\sum_{\langle i j\rangle}\left(\frac{\ell_{j i}^{2}}{2 \beta J}-i \ell_{j i}\left(\theta_{j}-\theta_{i}-\pi \Lambda_{j i}\right)\right)\right] \tag{6.5}
\end{equation*}
$$

At this point it is useful to rewrite each link variable as a vector field, i.e. $\ell_{i j} \rightarrow \ell_{\mu}(r)$, where $r$ denotes the lattice site and the index $\mu=x, y$ directs the bond from the lattice site $r$ to either the nearest neighbor on the right (x-direction) or the top (y-direction). By this prescription, each lattice site $r$ has two lattice bonds $\ell_{\mu}(r)$ associated with it, which avoids double counting of bonds. We can now write (6.5) as
$Z^{\theta} \propto \int d \theta_{1} \cdots \int d \theta_{N} \sum_{\left\{\ell_{\mu}(r)\right\}} \exp \left[-\sum_{r, \mu}\left(\frac{\ell_{\mu}(r)^{2}}{2 \beta J}-i\left(\ell_{\mu}(r)-\ell_{\mu}(r-\mu)\right) \theta_{r}+i \pi \Lambda_{\mu}(r) \ell_{\mu}(r)\right)\right]$
where we have reorganized the sum in the exponent so that each term contains only a single $\theta_{r}$. We can now perform the $\theta_{r}$-integration to obtain the result

$$
\begin{equation*}
Z^{\theta} \propto \sum_{\left\{\ell_{\mu}(r)\right\}} \exp \left[-\sum_{r, \mu}\left(\frac{\ell_{\mu}^{2}(r)}{2 \beta J}+i \pi \Lambda_{\mu}(r) \ell_{\mu}(r)\right)\right] \prod_{r} \delta_{\left\{\sum_{\mu} \ell_{\mu}(r)-\ell_{\mu}(r-\mu)\right\}, 0} \tag{6.6}
\end{equation*}
$$

The product of $\delta$-functions enforces a zero-divergence constraint on the vector field $\ell_{\mu}(r)$, which can automatically be satisfied by introducing the dual scalar field $\eta_{r}$, whose twodimensional vector curl equals the components of $\ell_{\mu}(r)$. Explicitly, the relation is

$$
\begin{align*}
& \ell_{x}(r)=\eta_{r}-\eta_{r-y} \\
& \ell_{y}(r)=-\eta_{r}+\eta_{r-x} \tag{6.7}
\end{align*}
$$

We can now write (6.6) as

$$
\begin{equation*}
Z^{\theta} \propto \sum_{\left\{\eta_{r}\right\}} \exp \left[-\sum_{r, \mu}\left(\frac{1}{2 \beta J}\left(\eta_{r}-\eta_{r-\mu}\right)^{2}+i \pi \Lambda_{\mu}(r)\left(\eta_{r}-\eta_{r-\mu}\right)\right)\right] \tag{6.8}
\end{equation*}
$$

where we have used the fact that $e^{ \pm i \pi}=1$. The second term in the exponent of (6.8) can be rewritten as

$$
\begin{equation*}
\sum_{r, \mu} \Lambda_{\mu}(r)\left(\eta_{r}-\eta_{r-\mu}\right)=\sum_{r}\left[\sum_{\mu}\left(\Lambda_{\mu}(r)-\Lambda_{\mu}(r+\mu)\right)\right] \eta_{r} \tag{6.9}
\end{equation*}
$$

By defining

$$
\begin{equation*}
\delta_{r} \equiv \frac{1}{2} \sum_{\mu}\left(\Lambda_{\mu}(r)-\Lambda_{\mu}(r+\mu)\right) \tag{6.10}
\end{equation*}
$$

we can write (6.8) as

$$
\begin{equation*}
Z^{\theta} \propto \sum_{\left\{\eta_{r}\right\}} \exp \left[-\sum_{r, \mu} \frac{1}{2 \beta J}\left(\eta_{r}-\eta_{r-\mu}\right)^{2}-2 \pi i \sum_{r} \delta_{r} \eta_{r}\right] \tag{6.11}
\end{equation*}
$$

The Poisson summation formula (6.4) can again be utilized to write (6.11) as

$$
\begin{equation*}
Z^{\theta} \propto \int_{-\infty}^{\infty} \prod_{r} d \tau_{r} \sum_{\left\{m_{r}\right\}} \exp \left[-\frac{1}{2 \beta J} \sum_{r, \mu}\left(\triangle_{\mu} \tau_{r}\right)^{2}+2 \pi i \sum_{r}\left(m_{r}-\delta_{r}\right) \tau_{r}\right] \tag{6.12}
\end{equation*}
$$

After solving this integral by Fourier transform, we obtain

$$
\begin{equation*}
Z^{\theta}=Z_{\mathrm{SW}}^{\theta} Z_{\mathrm{V}}^{\theta} \tag{6.13}
\end{equation*}
$$

where the first term is the spin-wave contribution which has absorbed all the constant factors. The second term is the contribution due to logarithmically intertacting vortices, and is given by

$$
\begin{equation*}
Z_{\mathrm{V}}^{\theta}=\sum_{\left\{m_{r}\right\}^{\prime}} \exp \left[\pi \beta J \sum_{r, r^{\prime}} \tilde{m}_{r} \Gamma_{r-r^{\prime}} \tilde{m}_{r^{\prime}}\right] \tag{6.14}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{m}_{r}=m_{r}-\delta_{r} \tag{6.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma_{r}=\Gamma(\mathbf{r})=\int_{-\pi}^{\pi} \frac{d^{2} k}{2 \pi}\left(\frac{1-e^{i \mathbf{k} \cdot \mathbf{r} / a}}{4-2 \cos k_{x}-2 \cos k_{y}}\right) \tag{6.16}
\end{equation*}
$$

The sum over $\left\{m_{r}\right\}^{\prime}$ in (6.14) restricts all vorticities $m_{r}$ to sum to zero. This can be stated as the neutrality condition,

$$
\begin{equation*}
\sum_{r} m_{r}=0 \tag{6.17}
\end{equation*}
$$

which physically means that the lowest energy states of the system are arrangements of vortices in which the net vorticity of the system is zero. The total partition function for the array can be written as

$$
\begin{equation*}
Z=Z_{\mathrm{SW}}^{\theta} Z_{\mathrm{SW}}^{\phi} Z_{\mathrm{V}} \tag{6.18}
\end{equation*}
$$

where

$$
\begin{equation*}
Z_{\mathrm{V}}=\sum_{\left\{\delta_{r}\right\}} Z_{\mathrm{V}}^{\theta} Z_{\mathrm{V}}^{\phi} \tag{6.19}
\end{equation*}
$$

and

$$
\begin{equation*}
Z_{\mathrm{V}}^{\phi}=\sum_{\left\{n_{r}\right\}^{\prime}} \exp \left[\pi \beta J \alpha \sum_{r, r^{\prime}} \tilde{n}_{r} \Gamma_{r-r^{\prime}} \tilde{n}_{r^{\prime}}\right] \tag{6.20}
\end{equation*}
$$

The total partition function (6.18) factorizes into the product of two spin-wave partition functions in $\theta$ and $\phi$, and a vortex partition function which allows half-vortices, by combining rotations of $\theta$ and $\phi$. To write the vortex partition (6.19) in a more tractable form, we note that an excellent approximation for the $\Gamma(\mathbf{r})$ function is given by $[51,56]$

$$
\begin{equation*}
\Gamma_{r} \simeq \ln \frac{r}{a}+\ln \frac{a}{r_{0}} \tag{6.21}
\end{equation*}
$$

where

$$
\begin{equation*}
r_{0}=\frac{a}{2 \sqrt{2} e^{\gamma}} \tag{6.22}
\end{equation*}
$$

and $a$ is the lattice constant and $\gamma$ is Euler's constant. Using this approximation, we can write (6.19) as

$$
\begin{align*}
Z_{\mathrm{V}}=\sum_{\left\{\delta_{r}\right\}}\{ & {\left.\left[\sum_{\left\{m_{r}\right\}^{\prime}} \exp \left[2 \pi \beta J \sum_{r \neq r^{\prime}}^{\prime} \tilde{m}_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} \tilde{m}_{r^{\prime}}-\pi \beta J \ln \frac{a}{r_{0}} \sum_{r} \tilde{m}_{r}^{2}\right]\right]\right\} \times } \\
& \left\{\left[\sum_{\left\{n_{r}\right\}^{\prime}} \exp \left[2 \pi \beta J \alpha \sum_{r \neq r^{\prime}}^{\prime} \tilde{n}_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} \tilde{n}_{r^{\prime}}-\pi \beta J \alpha \ln \frac{a}{r_{0}} \sum_{r} \tilde{n}_{r}^{2}\right]\right]\right\} \tag{6.23}
\end{align*}
$$

where the prime on the sum in the exponent restricts it to unique pairings of $r$ and $r^{\prime}$. The sum of exponents describes the logarithmic interaction between vortices, the first
sum being vortices in $\theta$ and the second sum being vortices in $\phi$, where $\tilde{n}_{r}=n_{r}-\delta_{r}$. Since $m_{r}$ and $n_{r}$ can be any integers, and $\delta_{r}$ can be any integer or half-integer [see definition in (6.10)], it follows that $\tilde{m}_{r}$ and $\tilde{n}_{r}$ can now be integer and half-integer valued. This feature allows for the presence of half-vortices in $p+i p$ Josephson junction arrays.

### 6.3 Linked Correlations in $\theta$ and $\phi$

We now consider calculating the correlation in the superconducting phase $\theta$ between two sites, $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$. This correlation function can be written as

$$
\begin{equation*}
g\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\left\langle\cos \left(\theta_{o}-\theta_{n}\right)\right\rangle=\Re\left\langle e^{i p\left(\theta_{o}-\theta_{n}\right)}\right\rangle \tag{6.24}
\end{equation*}
$$

In the low-temperature spin-wave approximation, it can be shown that the correlation function (6.24) takes the form

$$
\begin{equation*}
g_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\exp \left[-\frac{1}{2 \pi \beta J} \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)\right] \tag{6.25}
\end{equation*}
$$

When we use the approximation given in (6.21), this becomes

$$
\begin{equation*}
g_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\left(\frac{r_{0}}{\left|\mathbf{r}_{o}-\mathbf{r}_{n}\right|}\right)^{1 / 2 \pi \beta J} \tag{6.26}
\end{equation*}
$$

which exhibits power law decay, and hence the phase correlation is quasi-long range.
If we consider a slightly higher temperature regime, vortices are generated in the array as thermal excitations, and the correlation function factorizes into two parts,

$$
\begin{equation*}
g\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=g_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) g_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \tag{6.27}
\end{equation*}
$$

The spin-wave contribution is identical to (6.26) and the vortex contribution takes the form


Figure 6.1: These two configurations depict the internal structure of the lowest energy vortex excitations. Overall, both diagrams represent the interaction of half-vortices with opposite net vorticity $\tilde{m}$, however the composition of the vorticity is not trivial.

$$
\begin{equation*}
g_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\exp \left[-\frac{\pi^{2}}{2}\left(\int_{1}^{\infty} d r r^{3}\left\langle\tilde{m}_{o} \tilde{m}_{r}\right\rangle_{Z_{\mathrm{V}}}\right) \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)\right] \tag{6.28}
\end{equation*}
$$

where $r$ has been scaled into units of $a$. The lowest non-vanishing contribution to $\left\langle\tilde{m}_{o} \tilde{m}_{r}\right\rangle_{Z_{\mathrm{V}}}$ can be calculated by first expanding it as

$$
\begin{align*}
\left\langle\tilde{m}_{o} \tilde{m}_{r}\right\rangle_{Z_{\mathrm{V}}} & =\left\langle\left(m_{o}-\delta_{o}\right)\left(m_{r}-\delta_{r}\right)\right\rangle_{Z_{\mathrm{V}}} \\
& =\left\langle m_{o} m_{r}\right\rangle_{Z_{\mathrm{V}}}-\left\langle m_{r} \delta_{o}\right\rangle_{Z_{\mathrm{V}}}-\left\langle m_{o} \delta_{r}\right\rangle_{Z_{\mathrm{V}}}-\left\langle\delta_{o} \delta_{r}\right\rangle_{Z_{\mathrm{V}}} \tag{6.29}
\end{align*}
$$

The largest non-vanishing terms will come from the logarithmic interaction of half-vortices as described by the vortex partition function (6.23). From Figure 6.1, we see that there are two configurations of interacting half-vortices, that correspond to the lowest excitation energies of the system. In this figure we have only outlined the composition for vortices $\tilde{m}_{r}$ (winding in $\theta$ ), but the exact same picture with $\tilde{m}_{r}$ replaced by $\tilde{n}_{r}$ would represent the lowest excitation energies for vortices in $\phi$. Note that the $\delta_{r}$
vorticity is what ties together rotations in $\theta$ and $\phi$ and hence allows for the existence of half-vortices. Summing together the two contributions from Figure 6.1, we approximate (6.29) as

$$
\begin{align*}
\left\langle\tilde{m}_{o} \tilde{m}_{r}\right\rangle_{Z_{\mathrm{V}}} & \simeq\left\langle m_{o} m_{r}\right\rangle_{Z_{\mathrm{V}}}^{0}-\left\langle m_{r} \delta_{o}\right\rangle_{Z_{\mathrm{V}}}-\left\langle m_{o} \delta_{r}\right\rangle_{Z_{\mathrm{V}}}+2\left\langle\delta_{o} \delta_{r}\right\rangle_{Z_{\mathrm{V}}} \\
& \simeq-\frac{1}{2} \exp \left[-\frac{\pi \beta J}{2}(1+\alpha) \ln \frac{r}{a}-\frac{\pi \beta J}{2}(1+\alpha) \ln \frac{a}{r_{0}}\right] \\
& =-\frac{1}{2} y_{\theta}^{1 / 2} y_{\phi}^{1 / 2}\left(\frac{a}{r}\right)^{\frac{\pi \beta J}{2}(1+\alpha)} \tag{6.30}
\end{align*}
$$

where

$$
\begin{equation*}
y_{\theta}=\exp \left[-\pi \beta J \ln \frac{a}{r_{0}}\right] \tag{6.31}
\end{equation*}
$$

is the unrenormalized vortex fugacity in $\theta$ and

$$
\begin{equation*}
y_{\phi}=\exp \left[-\pi \beta J \alpha \ln \frac{a}{r_{0}}\right] \tag{6.32}
\end{equation*}
$$

is the unrenormalized vortex fugacity in $\phi$. This allows us to write (6.28) as

$$
\begin{equation*}
g_{\mathrm{V}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\exp \left[-\frac{\pi^{2}}{4} y_{\theta}^{1 / 2} y_{\phi}^{1 / 2}\left(\int_{1}^{\infty} d r\left(\frac{1}{r}\right)^{\frac{\pi \beta J}{2}(1+\alpha)-3}\right) \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)\right] \tag{6.33}
\end{equation*}
$$

The complete correlation function (6.27) can now be written as

$$
\begin{equation*}
g\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\exp \left[\frac{1}{\pi \beta_{\mathrm{eff}} J} \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)\right] \tag{6.34}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{1}{\beta_{\mathrm{eff}} J}=\frac{1}{\beta J}+\frac{\pi^{3}}{2} y_{\theta}^{1 / 2} y_{\phi}^{1 / 2} \int_{a}^{\infty} d r\left(\frac{a}{r}\right)^{\frac{\pi \beta J}{2}(1+\alpha)-3} \tag{6.35}
\end{equation*}
$$

is the effective temperature and we have rescaled $r$ back into units of distance. This correlation can also be written conveniently as

$$
\begin{equation*}
g\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\left(\frac{r_{0}}{\left|\mathbf{r}_{o}-\mathbf{r}_{n}\right|}\right)^{p^{2} / 2 \pi \beta_{\mathrm{eff}} J} \tag{6.36}
\end{equation*}
$$

and we see that just as in the spin-wave approximation, the correlation decays as a power law and exhibits quasi-long range order. However, the proliferation of half-vortices has increased the effective temperature, causing the correlation to decrease more rapidly.

### 6.4 Renormalization

The parameters which control the appearance of half-vortices are the vortex fugacities $y_{\theta}$ and $y_{\phi}$ [see equations (6.31) and (6.32)] and the temperature $\beta$, both of which are dimensionless and are defined at a microscopic length scale. We can perform a real space renormalization by integrating out the short distance degrees of freedom. By scaling the cutoff length from $a \rightarrow \lambda a$ where $\lambda>1$, we can track how the parameters $y_{\lambda}$ and $\beta_{\lambda}$ scale as a function of $\lambda$ such that the correlation functions retain the same form throughout the re-scaling procedure.

Let us first consider breaking the integral in (6.35) into two parts, from $a$ to $\lambda a$, and from $\lambda a$ to $\infty$. By defining the dimensionless distance $x=r / a$, the integral becomes

$$
\begin{aligned}
\int_{1}^{\infty} d x\left(\frac{1}{x}\right)^{2 \pi \beta J-3} & =\int_{1}^{\lambda} d x x^{3-\frac{\pi \beta J}{2}(1+\alpha)}+\int_{\lambda}^{\infty} d x x^{3-\frac{\pi \beta J}{2}(1+\alpha)} \\
& =\frac{\lambda^{4-\frac{\pi \beta J}{2}(1+\alpha)}-1}{4-\frac{\pi \beta J}{2}(1+\alpha)}+\lambda^{4-\frac{\pi \beta J}{2}(1+\alpha)} \int_{1}^{\infty} d x x^{3-\frac{\pi \beta J}{2}(1+\alpha)}(6.37)
\end{aligned}
$$

In the limit of infinitesimal scaling, $\lambda \rightarrow 1$, we can approximate $\ln \lambda \simeq \lambda-1$ and (6.37)
can be written as

$$
\begin{equation*}
\ln \lambda+\left(1+\left(4-\frac{\pi \beta J}{2}(1+\alpha)\right) \ln \lambda\right) \int_{1}^{\infty} d x x^{3-\frac{\pi \beta J}{2}(1+\alpha)} \tag{6.38}
\end{equation*}
$$

such that (6.35) now becomes

$$
\begin{align*}
& \frac{1}{\beta_{\mathrm{eff}} J}= \frac{1}{\beta J}+ \\
& \frac{\pi^{3}}{2} y_{\theta}^{1 / 2} y_{\phi}^{1 / 2} \ln \lambda+  \tag{6.39}\\
& \frac{\pi^{3}}{2} y_{\theta}^{1 / 2} y_{\phi}^{1 / 2}\left(1+\left(4-\frac{\pi \beta J}{2}(1+\alpha)\right) \ln \lambda\right) \int_{1}^{\infty} d x x^{3-\frac{\pi \beta J}{2}(1+\alpha)}
\end{align*}
$$

In order to retain the original form of the correlation function (6.34) in the rescaled dimensions, we make the following prescriptions;

$$
\begin{equation*}
\frac{1}{\beta_{\lambda} J}=\frac{1}{\beta J}+\frac{\pi^{3}}{2} y_{\theta, \lambda}^{1 / 2} y_{\phi, \lambda}^{1 / 2} \ln \lambda \tag{6.40}
\end{equation*}
$$

for the renormalized temperature at the distance scale $\lambda a$, and

$$
\begin{equation*}
y_{\theta, \lambda}^{1 / 2} y_{\phi, \lambda}^{1 / 2}=y_{\theta}^{1 / 2} y_{\phi}^{1 / 2}\left(1+\left(4-\frac{\pi \beta_{\lambda} J}{2}(1+\alpha)\right) \ln \lambda\right) \tag{6.41}
\end{equation*}
$$

for the renormalized vortex fugacities. The fugacities $y_{\theta, \phi}$ are very small at low temperatures, we may approximate (6.41) as

$$
\begin{aligned}
y_{\theta, \lambda}^{1 / 2} y_{\phi, \lambda}^{1 / 2} & =y_{\theta}^{1 / 2}\left(1+\left(4-\frac{\pi \beta_{\lambda} J}{2}(1+\alpha)\right) \ln \lambda\right)^{1 / 2} y_{\phi}^{1 / 2}\left(1+\left(4-\frac{\pi \beta_{\lambda} J}{2}(1+\alpha)\right) \ln \lambda\right)^{1 / 2} \\
& \simeq y_{\theta}^{1 / 2}\left(1+\left(2-\frac{\pi \beta_{\lambda} J}{4}(1+\alpha)\right) \ln \lambda\right) y_{\phi}^{1 / 2}\left(1+\left(2-\frac{\pi \beta_{\lambda} J}{4}(1+\alpha)\right) \ln \lambda\right)
\end{aligned}
$$

This allows us to make the prescription

$$
\begin{align*}
y_{\theta, \lambda}^{1 / 2} & =y_{\theta}^{1 / 2}\left(1+\left(2-\frac{\pi \beta_{\lambda} J}{4}(1+\alpha)\right) \ln \lambda\right)  \tag{6.42}\\
y_{\phi, \lambda}^{1 / 2} & =y_{\phi}^{1 / 2}\left(1+\left(2-\frac{\pi \beta_{\lambda} J}{4}(1+\alpha)\right) \ln \lambda\right) \tag{6.43}
\end{align*}
$$

In the limit $\lambda \rightarrow 1$ we can approximate $\ln \lambda \simeq \Delta \lambda / \lambda$, and the expressions (6.40), (6.42), and (6.43) can expressed in differential form

$$
\begin{align*}
\frac{d}{d \ell} y_{\theta, \lambda}^{1 / 2} & =\left(2-\frac{\pi \beta_{\lambda} J}{4}(1+\alpha)\right) y_{\theta, \lambda}^{1 / 2}  \tag{6.44}\\
\frac{d}{d \ell} y_{\phi, \lambda}^{1 / 2} & =\left(2-\frac{\pi \beta_{\lambda} J}{4}(1+\alpha)\right) y_{\phi, \lambda}^{1 / 2}  \tag{6.45}\\
\frac{d}{d \ell} \frac{1}{\beta_{\lambda} J} & =\frac{\pi^{3}}{2} y_{\theta, \lambda}^{1 / 2} y_{\phi, \lambda}^{1 / 2} \tag{6.46}
\end{align*}
$$

where $\ell \equiv \ln \lambda$. These mathematical equations describe the renormalization group flow in the parameter space of $y_{\theta}, y_{\phi}$ and $\beta$.

### 6.5 The Universal Jump in Superfluid Stiffness with HalfQuantum Vortices

Just as is the case in a conventional JJ array, a $p+i p \mathrm{JJ}$ array, or any $2 D p+i p$ superconductor for that matter, will undergo a universal jump in the superfluid stiffness at the Kosterlitz-Thouless transition. In conventional s-wave superconductors this jump is mediated by the proliferation of full-quantum vortices. In the $p+i p$ case however, since half-quantum vortices are expected to proliferate at a temperature lower than full-quantum vortices, the universal jump in stiffness will be mediated by proliferating half-quantum vortices.

In order to show this, our argument will parallel the argument presented in Section 2.6. Consider the RG flow equations as in (6.44), (6.45), and (6.46). The RG
flow described by these equations looks very similar to the conventional s-wave case, the only difference being that there are two fugacity terms instead of just one. As in Section 2.6, we find that below that KT transition temperature the fugacities will renormalize to zero as the short length scales are integrated out,

$$
\begin{equation*}
y_{\theta, \mathrm{R}}\left(T \leq T_{\mathrm{KT}}\right)=y_{\phi, \mathrm{R}}\left(T \leq T_{\mathrm{KT}}\right)=0 \tag{6.47}
\end{equation*}
$$

As we approach the critical temperature from below, the coupling $K=\beta J$ renormalizes to

$$
\begin{equation*}
K_{\mathrm{R}}^{-1}\left(T_{\mathrm{KT}}\right)=\lim _{T \rightarrow T_{\mathrm{KT}}^{-}} K_{\mathrm{R}}^{-1}(T)=\frac{\pi}{8}(1+\alpha) \tag{6.48}
\end{equation*}
$$

Above the critical temperature the flow is away from the fixed point and the fugacities will become very large while the coupling $K$ renormalizes to zero,

$$
\begin{equation*}
K_{\mathrm{R}}\left(T_{\mathrm{KT}}\right)=\lim _{T \rightarrow T_{\mathrm{KT}}^{+}} K(T)=0 \tag{6.49}
\end{equation*}
$$

Making use of relation (2.113), we can write (6.48) as

$$
\begin{equation*}
\frac{\rho\left(T_{\mathrm{KT}}\right)}{T_{\mathrm{KT}}}=\frac{32 k_{B}}{\pi} \frac{1}{1+\alpha}\left(\frac{m}{\hbar}\right)^{2} \tag{6.50}
\end{equation*}
$$

This is universal jump in the superfluid stiffness when half-quantum vortices appear in the system. It is interesting to compare this equation to (2.114) which describes the conventional case. Notice that when $\alpha=1$, i.e. the spin and superfluid stiffness are equal, this jump is twice as large as it is in the conventional case. When $\alpha<1$, which is the more realistic situation for real chiral p-wave superconductors, the jump is even larger, up to four times the conventional case.

### 6.6 Current-Voltage Characteristics with $\frac{1}{2}$ QVs

In Section 3.7 we explained that in 2D superconductors, the voltage varies with current as $V \sim I^{a(T)}$ where

$$
\begin{equation*}
a(T)=\frac{\pi \hbar^{2}}{4 k_{B} m^{2}} \frac{\rho_{\mathbf{s}}(T)}{T}+1 \tag{6.51}
\end{equation*}
$$

In conventional Josephson junction arrays, the exponent $a(T)$ is 3 at the KosterlitzThouless transition, and is 1 above the transition. The sudden change from non-linear to linear behavior in the current-voltage characteristics at $T_{\mathrm{KT}}$ has been one of the traditional signatures used to experimentally locate the KT transition[86]. In a $p+$ $i p$ array, the non-linear dependance of the IV characteristics should differ from the conventional case due to the presence of half-quantum vortices. If we plug (6.50), the universal jump condition for the superfluid stiffness in 2D $p+i p$ superconductors, into the previous expression we obtain

$$
\begin{equation*}
a\left(T_{\mathrm{KT}}\right)=\frac{8}{1+\alpha}+1 \tag{6.52}
\end{equation*}
$$

This is a very interesting result because it predicts that the nonlinear IV characteristics in the $p+i p$ case differ significantly from the conventional $s$-wave case. Above $T_{\mathrm{KT}}$ the exponent $a(T)=1$, similar to the conventional case. However, below the KT transition the nonlinear dependance will be a function of $\alpha$, the ratio of the superfluid stiffness to the spin stiffness. If we take $\alpha=1$, then $a\left(T_{\mathrm{KT}}\right)=5$, whereas if $\alpha=0$, then $a\left(T_{\mathrm{KT}}\right)=9$.

### 6.7 Linearized RG Flow

The fixed points for this RG flow are $y_{\theta}=0, y_{\phi}=0$, and $2-\frac{\pi \beta_{\lambda} J}{4}(1+\alpha)=0$. We can examine the critical behavior by linearizing the RG equations about the fixed point $\left(\beta J, y_{\theta}, y_{\phi}\right)=\left(\frac{8}{\pi(1+\alpha)}, 0,0\right)$. If we introduce the variables

$$
\begin{align*}
\Theta_{\lambda} & \equiv 2-\frac{\pi \beta_{\lambda} J}{4}(1+\alpha)  \tag{6.53}\\
Y_{\lambda} & \equiv \pi\left(\frac{8}{1+\alpha}\right)^{1 / 2} y_{\theta, \lambda}^{1 / 4} y_{\phi, \lambda}^{1 / 4} \tag{6.54}
\end{align*}
$$

the fixed point expansion of the flow will be about the point $\left(\Theta_{\lambda}, Y_{\lambda}\right)=(0,0)$. Since $\Theta_{\lambda}$ and $Y_{\lambda}$ are very small quantities near the fixed point, the linearized $R G$ flow equations reduce to

$$
\begin{align*}
\frac{d}{d \ln \lambda} \Theta_{\lambda} & =Y_{\lambda}^{2}  \tag{6.55}\\
\frac{d}{d \ln \lambda} Y_{\lambda}^{2} & =2 \Theta_{\lambda} Y_{\lambda}^{2} \tag{6.56}
\end{align*}
$$

where (6.56) follows from multiplying (6.44) and (6.45) by $y_{\theta, \lambda}^{1 / 2}$ and $y_{\phi, \lambda}^{1 / 2}$ respectively and then adding. By plotting these flow equations in the reduced parameter space, we see that there are three distinct regimes(see Figure 6.2). The line separating the various regimes is called the separatrix. By substituting $Y_{\lambda}= \pm \Theta_{\lambda}$ into equations (6.55) and (6.56), we see that that $Y_{\lambda}= \pm \Theta_{\lambda}$ is the equation for the separatrix, since $Y_{\lambda}$ and $\Theta_{\lambda}$ scale at exactly the same rate.

### 6.8 Critical Temperature

In order to find the critical transition temperature, we must find the point where the initial conditions for the model intersect with the separatrix. Before any
scaling ( $\lambda=1$ ), the vortex fugacities are given by (6.31) and (6.32), and the reduced variable $Y$ is initially

$$
\begin{align*}
Y & =\pi\left(\frac{8}{1+\alpha}\right)^{1 / 2} \exp \left[-\frac{\pi \beta J}{4}(1+\alpha) \ln \frac{a}{r_{0}}\right] \\
& =\pi\left(\frac{8}{1+\alpha}\right)^{1 / 2} \exp \left[(\Theta-2) \ln \frac{a}{r_{0}}\right] \tag{6.57}
\end{align*}
$$

where we have written it in terms of $\Theta$ defined in (6.53). The condition for criticality is given when (6.57) intersects the separatrix(see Figure 6.2). Explicitly this condition is satisfied when $Y_{\mathrm{KT}}=-\Theta_{\mathrm{KT}}$, which yields the analytic expression for the KosterlitzThouless transition temperature,

$$
\begin{equation*}
\pi\left(\frac{8}{1+\alpha}\right)^{1 / 2}\left(\sqrt{8} e^{\gamma}\right)^{\Theta_{\mathrm{KT}}-2}=-\Theta_{\mathrm{KT}} \tag{6.58}
\end{equation*}
$$

where we have used (6.22). The analytic solution to this equation is

$$
\begin{equation*}
\Theta_{\mathrm{KT}}=-\frac{W\left(\frac{\pi \exp [-2 \gamma]}{\sqrt{8(1+\alpha)}}\left(\frac{3}{2} \ln 2+\gamma\right)\right)}{\frac{3}{2} \ln 2+\gamma} \tag{6.59}
\end{equation*}
$$

where $W(x)$ is the Lambert $W$-Function $[97]$ and $\gamma$ is Euler's constant.

### 6.9 Renormalization Correction due to Majorana Fermions

We now want to allow for the presence of Majorana fermions bound to vortex cores, and see how this might affect the Kosterlitz-Thouless transition temperature. If we refer back to (6.23), we see that the energy of a system of vortices is

$$
\begin{equation*}
E_{\mathrm{V}}=-2 \pi J \sum_{r \neq r^{\prime}}^{\prime}\left(\tilde{m}_{r} \tilde{m}_{r^{\prime}}+\alpha \tilde{n}_{r} \tilde{n}_{r^{\prime}}\right) \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a}+\pi J \ln \frac{a}{r_{0}} \sum_{r}\left(\tilde{m}_{r}^{2}+\alpha \tilde{n}_{r}^{2}\right) \tag{6.60}
\end{equation*}
$$

The first term in (6.60) is the usual logarithmic interaction between vortex pairs[51] and the second term is the chemical potential. Let us imagine that this energy describes a system of $2 n$ interacting vortices, or equivalently, $n$ pairs of vortices. At low temperatures, owing to the neutrality condition (6.17) and the energetic stability of vortices with low vorticity, we may assume that in each plaquette, $\tilde{m}_{r}=0, \pm 1 / 2$. This tells us that the term $\sum_{r} \tilde{m}_{r}^{2}$ can be identified as $n / 2$.

2 Majorana operators can be combined to form a legitimate fermionic mode, either occupied or unoccupied, which adds a 2 -fold degeneracy to the state of a system of 2 vortices. The presence of Majorana modes manifests itself by adding an additional entropy piece to the ground state of vortex pairs. Generalizing this argument to the case of $n$ vortex pairs, we can take the $2 n$ self-conjugate Majorana fermionic operators and combine them into $n$ complex fermionic operators, modes which again may either be filled or empty. Therefore, the ground state of a system of $2 n$ vortices acquires a $2^{n}$-fold degeneracy. This degeneracy factor can be written as

$$
\begin{equation*}
2^{n}=\exp [n \ln 2]=\exp \left[2 \ln 2 \sum_{r} \tilde{m}_{r}^{2}\right] \tag{6.61}
\end{equation*}
$$

which follows from the preceding discussion.
An additional ground state degeneracy to consider for a system of half-quantum vortices is the relative winding between $\theta$ and $\phi$. These degrees of freedom don't necessarily have to wind in the same direction. For every half-quantum vortex where $\theta$ and $\phi$ wind in the same direction, one also has to consider the possibility of $\theta$ and $\phi$ winding in the opposite direction. However, it turns out that for materials with half-quantum vortices appearing as the emergent low energy excitation, relative windings between $\theta$ and $\phi$ tend to destabilize the superconducting state[16]. Therefore, we need not consider
opposite windings between $\theta$ and $\phi$ as an additional source of entropy as it breaks the ground state degeneracy. Opposite winding is energetically more costly than like winding and is therefore not the preferred type of winding in half-quantum vortices. The only scenario where a relative winding degeneracy would have to be considered is the case of equal stiffness $(\alpha=1)$. In this case we would have to deal with vortices $u$ and $v$ which represent two decoupled frustrated xy models [see (4.9)]. A $u$ vortex represents a half-quantum vortex with like $\pi$-windings in $\theta$ and $\phi$, while a $v$ vortex is a half-quantum vortex with opposite $\pi$-windings in $\theta$ and $\phi$.

Returning to Majorana modes, we can write the vortex partition function (6.23) for a system of $2 n$ vortices that bind Majorana modes as

$$
\begin{aligned}
& Z_{\mathrm{V}}=\sum_{\left\{\delta_{r}\right\}}\left\{\left[\sum_{\left\{m_{r}\right\}^{\prime}} 2^{n} \exp \left[2 \pi \beta J \sum_{r \neq r^{\prime}}{ }^{\prime} \tilde{m}_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} \tilde{m}_{r^{\prime}}-\pi \beta J \ln \frac{a}{r_{0}} \sum_{r} \tilde{m}_{r}^{2}\right]\right]\right\} \times \\
& \left\{\left[\sum_{\left\{n_{r}\right\}^{\prime}} \exp \left[2 \pi \beta J \alpha \sum_{r \neq r^{\prime}}^{\prime} \tilde{n}_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} \tilde{n}_{r^{\prime}}-\pi \beta J \alpha \ln \frac{a}{r_{0}} \sum_{r} \tilde{n}_{r}^{2}\right]\right]\right\} \\
& =\sum_{\left\{\delta_{r}\right\}}\left\{\left[\sum_{\left\{m_{r}\right\}^{\prime}} \exp \left[2 \pi \beta J \sum_{r \neq r^{\prime}}^{\prime} \tilde{m}_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} \tilde{m}_{r^{\prime}}-\left(\pi \beta J \ln \frac{a}{r_{0}}-2 \ln 2\right) \sum_{r} \tilde{m}_{r}^{2}\right]\right]\right\} \times \\
& \left\{\left[\sum_{\left\{n_{r}\right\}^{\prime}} \exp \left[2 \pi \beta J \alpha \sum_{r \neq r^{\prime}}^{\prime} \tilde{n}_{r} \ln \frac{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}{a} \tilde{n}_{r^{\prime}}-\pi \beta J \alpha \ln \frac{a}{r_{0}} \sum_{r} \tilde{n}_{r}^{2}\right]\right]\right\}
\end{aligned}
$$

Thus we can account for Majorana modes by simply subtracting the term $2 \ln 2$ from $\pi \beta J \ln \frac{a}{r_{0}}$. By tracking the result of this additional term throughout the proceeding renormalization group calculation, we find that the only change is the initial vortex fugacity in $\theta$ (6.31), which gets corrected to

$$
\begin{equation*}
y_{\theta}^{\mathrm{F}}=\exp \left[-\pi \beta J \ln \frac{a}{r_{0}}-2 \ln 2\right] \tag{6.62}
\end{equation*}
$$



Figure 6.2: The intersection of $Y_{\mathrm{KT}}$ with the separatrix determines the critical temperature. The presence of Majorana fermions changes the effective fugacity to $Y_{\mathrm{KT}}^{\mathrm{F}}$ and so adds an additional correction to the Kosterlitz-Thouless transition temperature.
where F denotes fermion. The condition for criticality now becomes [use same steps as in equations (6.33)-(6.59)]

$$
\begin{equation*}
\frac{4 \pi}{(1+\alpha)^{1 / 2}}\left(\sqrt{8} e^{\gamma}\right)^{\Theta_{\mathrm{KT}}-2}=-\Theta_{\mathrm{KT}} \tag{6.63}
\end{equation*}
$$

The analytic solution to this equation is

$$
\begin{equation*}
\Theta_{\mathrm{KT}}=-\frac{W\left(\frac{\pi \exp [-2 \gamma]}{2 \sqrt{1+\alpha}}\left(\frac{3}{2} \ln 2+\gamma\right)\right)}{\frac{3}{2} \ln 2+\gamma} \tag{6.64}
\end{equation*}
$$

If we compare (6.59) with (6.64), we find that the presence of Majorana modes has added an additional correction to the Kosterlitz-Thouless transition temperature (see Figure 6.2). Notice that this RG correction depends on the parameter $\alpha$, the ratio of the spin stiffness to the superfluid stiffness. The unrenormalized Kosterlitz-Thouless transition temperature $T_{\mathrm{KT}}$ for an unfrustrated array, which follows from a simple energy-entropy argument is

$$
\begin{equation*}
T_{\mathrm{KT}}=\frac{\pi J}{8 k_{B}}(1+\alpha) \tag{6.65}
\end{equation*}
$$

The corrected transition temperature $T_{\mathrm{KT}}^{\mathrm{R}}$ which takes into account renormalization follows directly from (6.59) and (6.64). For comparison, we calculate this correction explicitly for $\alpha=1,0.7,0.4$ which yields

$$
\begin{aligned}
T_{\mathrm{KT}}^{\mathrm{R}}(\alpha=1) & =(1-.0842-.0222) T_{\mathrm{KT}} \\
T_{\mathrm{KT}}^{\mathrm{R}}(\alpha=0.7) & =(1-.0891-.0230) T_{\mathrm{KT}} \\
T_{\mathrm{KT}}^{\mathrm{R}}(\alpha=0.4) & =(1-.0980-0.0241) T_{\mathrm{KT}}
\end{aligned}
$$

Consider for example the case where $\alpha=1$. The first correction of $8.42 \%$ is strictly an RG correction due to vortex screening. The second correction piece of $2.22 \%$ is due to the presence of Majorana fermions. Overall, the renormalization calculation is more accurate since it accounts for the additional entropy generated by Majorana fermions and also corrects for the screening effects of multiple vortices in the system. Note that the corrections become larger as $\alpha$ decreases. Strontium Ruthenate is expected to have $\alpha \simeq 0.4[28]$.

## Chapter 7

## Summary

In summary, we have discussed many properties of half-quantum vortices in $p+i p \mathrm{JJ}$ arrays. In these concluding paragraphs we will briefly summarize the most important results of this work. In Section 1.7.3 we showed that a pair of half-quantum Pearl vortices would interact logarithmically on length scales $\xi \ll r \ll \lambda_{\perp}$. Although this logarithmic form does not persist up to arbitrarily large scales, it will lead to a crossover which cannot be distinguished from a true Kosterlitz-Thouless phase transition since JJ arrays are typically much smaller than the transverse penetration depth $\lambda_{\perp}$. We found in Section 4.3.4 that a KT transition mediated by the proliferation of half-quantum vortices will be at least two times lower than the transition temperature for that of full vortices (if the spin stiffness equals the superfluid stiffness). In Chapter 4 we found that the ratio of Kosterlitz-Thouless transitions, fully frustrated to unfrustrated in a $p+i p$ JJ array, can deviate significantly (up to approximately 3 times) from the conventional value if half-quantum vortices are present. This ratio, as in (4.31), provides a way to experimentally measure the ratio of spin stiffness to superfluid stiffness.

In Chapter 5 we found that there is no superconducting-resistive transition in
a $p+i p \mathrm{JJ}$ array at $f=1 / 3$ down to zero temperature. The array is in a resistive state down to zero temperature because of the presence of half-quantum vortices. We have also calculated the effect of Majorana modes on half-quantum vortices, which turns out to be quite small. In Section 2.9 we found that Majoranas bound to full vortices in spinless chiral p-wave superconductors will change the KT transition temperature by $2.2 \%$. In a spinful chiral p-wave superconductor, where Majoranas bind to half-quantum vortices (see Section 6.9) the effect is just as small (although now it is dependent on the ratio of the stiffnesses). Another feature which we calculated was the universal jump in stiffness in the presence of half-quantum vortices. In Sections 6.5-6.6 we found that this jump is much larger [see (6.50)] and therefore the IV characteristics will differ drastically from the conventional case.

In this thesis we studied the phase transitions and some of the interesting properties of $p+i p$ Josephson junction arrays. The most exciting possibility is the observance of half-quantum vortices in these arrays. At present, no one has constructed a Josephson junction array out of a chiral p-wave superconductor, but in principal this could be done out of materials such as strontium ruthenate. A $p+i p$ Josephson junction array is therefore interesting because it offers a controllable 2D system where half-quantum vortices could arise as the emergent low energy degrees of freedom. The excitement around half-quantum vortices, apart from fundamental physics itself, is the existence of bound Majorana modes that could potentially be utilized for topological quantum computation.

## Appendix A

## Low-Temperature Spin-Spin

## Correlations in the XY Model

The ground state of the xy model is the state in which all spins point in the same direction. Lattice spins are correlated over a distance (correlation length) of infinite extent. As the temperature increases, excitations arise which tend to disorder the system. The lowest energy excitations are spin-waves, in which spins vary little from site to site, but become completely disordered over large distances. In this appendix we follow [52] in calculating the low-T spin-spin correlations.

To calculate the correlation between spins in the low temperature limit (spinwave approximation), we take $\theta_{i}-\theta_{j} \ll 1$ and $0<\theta_{i}<2 \pi[52]$. After expanding the cosine, the partition function (2.2) becomes

$$
\begin{align*}
\cos \left(\theta_{i}-\theta_{j}\right) & =1-\frac{\epsilon^{2}}{2} \\
& =1-\frac{\left(\theta_{i}-\theta_{j}\right)^{2}}{2} \tag{A.1}
\end{align*}
$$

We assume that the lattice is periodic, with $N$ sites and $2 N$ lattice bonds, allowing us
to write the partition function (2.2) as

$$
\begin{equation*}
Z=e^{2 N \beta J} \int d \theta_{1} \cdots \int d \theta_{N} \exp \left[-\frac{\beta J}{2} \sum_{\langle i j\rangle}\left(\theta_{i}-\theta_{j}\right)^{2}\right] \tag{A.2}
\end{equation*}
$$

The sum in the exponent can be written as

$$
\begin{equation*}
\sum_{\langle i j\rangle}\left(\theta_{i}-\theta_{j}\right)^{2}=\sum_{r, \mu}\left(\theta_{r}-\theta_{r-\mu}\right)^{2} \tag{A.3}
\end{equation*}
$$

where $r$ denotes a lattice site and the index $\mu=x, y$ has the length of the lattice spacing $a$ and is directed either to the right (x-direction) or the top (y-direction). By this prescription, the sum counts every site $r$ along with 2 of its nearest nearest neighbors, which avoids double counting.

In order to express (A.3) in a more useful form we take a discrete Fourier transform of the lattice spin variables $\theta_{r}$. Formally, let us consider some arbitrary variable $\varphi_{r}$ defined on each lattice site. For a discrete 2 D lattice with $N$ sites subject to periodic boundary conditions, $\varphi_{r}$ can be Fourier transformed into $\varphi_{k}$ via

$$
\begin{equation*}
\varphi_{k}=\sum_{r} \varphi_{r} e^{-i \mathbf{k} \cdot \mathbf{r}} \tag{A.4}
\end{equation*}
$$

where the sum is over all lattice sites $r$. The inverse transform is

$$
\begin{equation*}
\varphi_{r}=\frac{a^{2}}{(2 \pi)^{2}} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d^{2} k \varphi_{k} e^{i \mathbf{k} \cdot \mathbf{r}} \tag{A.5}
\end{equation*}
$$

where $a$ is the lattice constant. Substituting (A.5) into (A.4) gives the resolution of the delta function,

$$
\begin{equation*}
\sum_{r} e^{i \mathbf{p} \cdot \mathbf{r}}=\frac{(2 \pi)^{2}}{a^{2}} \delta(\mathbf{p}) \tag{A.6}
\end{equation*}
$$

With these relations in place, it is a simple matter to show (A.3) can be written as

$$
\begin{equation*}
\sum_{r, \mu}\left(\theta_{r}-\theta_{r-\mu}\right)^{2}=\sum_{r, r^{\prime}} \theta_{r} \theta_{r^{\prime}} \frac{1}{(2 \pi)^{2}} \int d^{2} k e^{i \mathbf{k} \cdot\left(\mathbf{r}-\mathbf{r}^{\prime}\right) / a}\left\{4-2 \cos k_{x}-2 \cos k_{y}\right\} \tag{A.7}
\end{equation*}
$$

where $k a \rightarrow k$ is now a dimensionless variable, and the region of integration is $-\pi<$ $k_{x}, k_{y}<\pi$.

Now let us define the matrix

$$
\begin{equation*}
A_{r, r^{\prime}}=\frac{\beta J}{(2 \pi)^{2}} \int d^{2} k e^{i \mathbf{k} \cdot\left(\mathbf{r}-\mathbf{r}^{\prime}\right) / a}\left\{4-2 \cos k_{x}-2 \cos k_{y}\right\} \tag{A.8}
\end{equation*}
$$

such that by using (A.7), the partition function (A.2) can be written as

$$
\begin{align*}
Z & =e^{2 N \beta J} \int d \theta_{1} \cdots \int d \theta_{N} \exp \left[-\frac{1}{2} \sum_{r, r^{\prime}} \theta_{r} A_{r, r^{\prime}} \theta_{r^{\prime}}\right] \\
& =e^{2 N \beta J} \int d \boldsymbol{\Theta} \exp \left[-\frac{1}{2} \boldsymbol{\Theta}^{T} \mathbf{A} \boldsymbol{\Theta}\right] \tag{A.9}
\end{align*}
$$

where we have denoted the lattice vector

$$
\begin{equation*}
\boldsymbol{\Theta} \equiv\left(\theta_{1}, \ldots, \theta_{N}\right) \tag{A.10}
\end{equation*}
$$

The partition function is now written simply in terms of the $N \times N$ matrix $\mathbf{A}$ which is sandwiched in between the N -dimensional lattice vector $\boldsymbol{\Theta}$ and its transpose. The integral in (A.9) is a simple Gaussian and is given by[98]

$$
\begin{equation*}
\int d \boldsymbol{\Theta} \exp \left[-\frac{1}{2} \boldsymbol{\Theta}^{T} \mathbf{A} \boldsymbol{\Theta}\right]=(2 \pi)^{N / 2}(\operatorname{det} \mathbf{A})^{-1 / 2} \tag{A.11}
\end{equation*}
$$

which allows us to write the partition function (A.9) as

$$
\begin{equation*}
Z=\left(\sqrt{2 \pi} e^{2 \beta J}\right)^{N}(\operatorname{det} \mathbf{A})^{-1 / 2} \tag{A.12}
\end{equation*}
$$

Using this form of the partition function, we can calculate the correlation between two arbitrary spins $\mathbf{S}_{o}$ and $\mathbf{S}_{n}$ via the form

$$
\begin{align*}
g_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) & =\left\langle\mathbf{S}_{o} \cdot \mathbf{S}_{n}\right\rangle \\
& =\left\langle\cos \left(\theta_{o}-\theta_{n}\right)\right\rangle \\
& =\Re\left\langle e^{i\left(\theta_{o}-\theta_{n}\right)}\right\rangle \tag{A.13}
\end{align*}
$$

It is easier to use the complex exponent and then simply take the real part at the end of the calculation.

Due to the translational invariance of the system, the correlation function depends only on the difference of the coordinates $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$. The thermal average of (A.13) is with respect to (A.12) and takes the explicit form

$$
\begin{equation*}
g_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\frac{1}{Z} \int d \boldsymbol{\Theta} \exp \left[-\frac{1}{2} \boldsymbol{\Theta}^{T} \mathbf{A} \boldsymbol{\Theta}+i\left(\theta_{o}-\theta_{n}\right)\right] \tag{A.14}
\end{equation*}
$$

In order to evaluate this integral we use the Gaussian identity[98]

$$
\begin{equation*}
\int d \boldsymbol{\Theta} \exp \left[-\frac{1}{2} \boldsymbol{\Theta}^{T} \mathbf{A} \boldsymbol{\Theta}+i \mathbf{j}^{T} \boldsymbol{\Theta}\right]=(2 \pi)^{N / 2}(\operatorname{det} \mathbf{A})^{-1 / 2} \exp \left[-\frac{1}{2} \mathbf{j}^{T} \mathbf{A}^{-1} \mathbf{j}\right] \tag{A.15}
\end{equation*}
$$

The correlation function (A.14) can be written in an form equivalent to (A.15) if we make the prescription

$$
\begin{equation*}
\mathbf{j}=\left(j_{1} \cdots j_{o} \cdots j_{n} \cdots j_{N}\right)=(0 \cdots 1 \cdots-1 \cdots 0) \tag{A.16}
\end{equation*}
$$

Now the inverse of the matrix $\mathbf{A}$ defined in (A.8) is

$$
\begin{equation*}
A_{r, r^{\prime}}^{-1}=\frac{1}{(\beta J} \int d^{2} q e^{i \mathbf{k} \cdot\left(\mathbf{r}-\mathbf{r}^{\prime}\right) / a}\left\{\frac{1}{4-2 \cos k_{x}-2 \cos k_{y}}\right\} \tag{A.17}
\end{equation*}
$$

This can be explicitly verified by simply evaluating the product

$$
\begin{equation*}
\sum_{r^{\prime}} A_{r, r^{\prime}} A_{r^{\prime}, r}^{-1}=1_{r} \tag{A.18}
\end{equation*}
$$

The matrix $\mathbf{A}$ and its inverse $\mathbf{A}^{-1}$ are symmetric, $A_{\alpha, \beta}^{-1}=A_{\beta, \alpha}^{-1}$, and all diagonal components are the same, $A_{\alpha, \alpha}^{-1}=A_{\beta, \beta}^{-1}$. Now using (A.12),(A.15), and (A.16) we can write the correlation function (A.14) as

$$
\begin{align*}
g_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) & =\exp \left[-\frac{1}{2}\left(A_{o o}^{-1}+A_{n n}^{-1}-A_{o n}^{-1}-A_{n o}^{-1}\right)\right] \\
& =\exp \left[-\left(A_{o o}^{-1}-A_{o n}^{-1}\right)\right] \tag{A.19}
\end{align*}
$$

By defining the function $\Gamma(\mathbf{r})$

$$
\begin{equation*}
\Gamma(\mathbf{r})=\int_{-\pi}^{\pi} \frac{d^{2} k}{2 \pi}\left(\frac{1-e^{i \mathbf{k} \cdot \mathbf{r} / a}}{4-2 \cos k_{x}-2 \cos k_{y}}\right) \tag{A.20}
\end{equation*}
$$

we can write (A.19) as

$$
\begin{equation*}
g_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\exp \left[-\frac{1}{2 \pi \beta J} \Gamma\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)\right] \tag{A.21}
\end{equation*}
$$

The only approximation made in obtaining (A.21) for the low-temperature correlation function was when we replaced the cosine potential by its Gaussian form. In order to write (A.21) in a more useful form, we make use of the following approximation[51, $56]$ for the $\Gamma(\mathbf{r})$ function

$$
\begin{equation*}
\Gamma(\mathbf{r}) \simeq \ln \frac{r}{a}+\ln \frac{a}{r_{0}} \tag{A.22}
\end{equation*}
$$

where

$$
\begin{equation*}
r_{0}=\frac{a}{2 \sqrt{2} e^{\gamma}} \tag{A.23}
\end{equation*}
$$

and $a$ is the lattice constant, $r=|\mathbf{r}|$, and $\gamma$ is Euler's constant. The function $\Gamma(\mathbf{r})$ is isotropic, depending only absolute distances, which physically is to be expected. This approximation to $\Gamma(\mathbf{r})$ is a very good estimate down to the distance scale $|\mathbf{r}|=a$, but fails below this distance, since we see from (A.20) that $\Gamma(0)=0$. Physically, this doesn't create an issue since $a$ is the smallest distance possible between spins which are living on lattice sites.

Using the approximate form (A.22) we can write (A.21) as

$$
\begin{equation*}
g_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\exp \left[-\frac{1}{2 \pi \beta J} \ln \left(\frac{r_{0}}{\mathbf{r}_{o}-\mathbf{r}_{n}}\right)\right] \tag{A.24}
\end{equation*}
$$

or more simply as

$$
\begin{equation*}
g_{\mathrm{SW}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)=\left(\frac{r_{0}}{\left|\mathbf{r}_{o}-\mathbf{r}_{n}\right|}\right)^{1 / 2 \pi \beta J} \tag{A.25}
\end{equation*}
$$

The correlation between spins decreases with increasing separation as a power law, and the system is said to possess quasi-long range order[50]. The correlation function also decreases with increasing temperature, which is a result of thermal disorder. It is important to remember that (A.25) is only valid at low temperatures in the spin-wave approximation, where $0<\theta_{i}<2 \pi$.

## Appendix B

## XY Spin-Spin Correlations in the

## High-Temperature Limit

In the limit of high temperature, the correlation function has a radically different behavior. Following [53], we perform a high temperature expansion, $\beta J \ll 1$, which allows us to write the exponential in the partition function (2.2) as

$$
\begin{equation*}
\exp \left[\beta J \sum_{\langle i j\rangle} \cos \left(\theta_{i}-\theta_{j}\right)\right]=\sum_{m=0}^{\infty} \frac{(\beta J)^{m}}{m!}\left\{\sum_{\langle i j\rangle} \cos \left(\theta_{i}-\theta_{j}\right)\right\}^{m} \tag{B.1}
\end{equation*}
$$

The cosine term can be written as

$$
\begin{equation*}
\left\{\sum_{\langle i j\rangle} \cos \left(\theta_{i}-\theta_{j}\right)\right\}^{m}=\frac{1}{2^{m}}\left\{\sum_{\langle i j\rangle}\left(e^{i\left(\theta_{i}-\theta_{j}\right)}+e^{i\left(\theta_{j}-\theta_{i}\right)}\right)\right\}^{m} \tag{B.2}
\end{equation*}
$$

and the correlation between two spins is

$$
\begin{align*}
g_{\mathrm{HT}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right)= & \left\langle\cos \left(\theta_{o}-\theta_{n}\right)\right\rangle \\
= & \frac{1}{2 Z} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\beta J}{2}\right)^{m} \times \\
& \operatorname{Tr}\left[\left(e^{i\left(\theta_{o}-\theta_{n}\right)}+e^{i\left(\theta_{n}-\theta_{o}\right)}\right)\left\{\sum_{\langle i j\rangle}\left(e^{i\left(\theta_{i}-\theta_{j}\right)}+e^{i\left(\theta_{j}-\theta_{i}\right)}\right)\right\}^{m}\right] \tag{B.3}
\end{align*}
$$

In this expression $Z$ is given by $(2.2), \cos \left(\theta_{o}-\theta_{n}\right)$ has been rewritten as a sum of exponentials, and the trace is the N -dimensional integral

$$
\begin{equation*}
\operatorname{Tr}=\int d \theta_{1} \cdots \int d \theta_{N} \tag{B.4}
\end{equation*}
$$

Even though we have performed a high temperature expansion to arrive at (B.3), further approximations must be made as its form is still quite intractable. The term $\beta J$ is very small, therefore the largest contribution to the sum over $m$ in (B.3) will come from the lowest non-vanishing term. It is useful to think of $e^{i\left(\theta_{i}-\theta_{j}\right)}$ as a lattice bond directed from site $i$ to site $j$. Let us consider the trace in (B.3), which is a sum of infinitely many terms containing integrals over products of lattice bonds. Each term will be non-vanishing if the following conditions are met: every site must have either no lattice bonds attached to it, or it must have both an incoming and an outgoing lattice bond attached to it. We can understand this as follows; an arbitrary site with just one incoming/outgoing bond contributes the factor

$$
\begin{equation*}
\int_{0}^{2 \pi} d \theta_{i} e^{ \pm i \theta_{A}}=0 \tag{B.5}
\end{equation*}
$$

to a particular term, causing it to vanish (see Figure B.1). On the other hand, a site with either no bonds or both an incoming/outgoing bond contributes a finite factor


Figure B.1: Site A has only one attached lattice bond, and therefore such terms in (B.3) are zero. Site B is attached to both an incoming and outgoing bond, and therefore yield a finite contribution.

$$
\begin{equation*}
\int_{0}^{2 \pi} d \theta_{i} e^{i\left(\theta_{B}-\theta_{B}\right)}=2 \pi \tag{B.6}
\end{equation*}
$$

which leads to an effective contribution for a particular term.

Now the cosine term connecting sites $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$ in the correlation function (B.3) adds an extra lattice bond factor $e^{i\left(\theta_{o}-\theta_{n}\right)}$ to each term in the sum, and therefore we conclude that only diagrams which connect sites $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$ lead to finite contributions.

The dominant contribution to (B.3) will come from the shortest path connecting sites $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$ (see Figure B.2). We take, for simplicity, sites $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$ to lie along the same principal axis of the system. The distance between these sites is then $\left|\mathbf{r}_{o}-\mathbf{r}_{n}\right|=a m$, and the dominant contribution to the correlation function is

$$
\begin{equation*}
g_{\mathrm{HT}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \sim\left(\frac{\beta J}{2}\right)^{m}=\left(\frac{\beta J}{2}\right)^{\left|\mathbf{r}_{o}-\mathbf{r}_{n}\right| / a} \tag{B.7}
\end{equation*}
$$

where we have ignored all the unimportant constant factors. We expect this result to


Figure B.2: This path corresponds to the leading order term in the correlation function. When $\mathbf{r}_{o}$ and $\mathbf{r}_{n}$ are well-separated, the correlation function should be isotropic and not depend on the relative positions of the coordinates.
also hold whenever $\left|\mathbf{r}_{o}-\mathbf{r}_{n}\right| \gg a$, as physically one expects the correlation function to be isotropic, depending only on absolute distances and not on angles. This result can be expressed as

$$
\begin{equation*}
g_{\mathrm{HT}}\left(\mathbf{r}_{o}-\mathbf{r}_{n}\right) \sim e^{-\left|\mathbf{r}_{o}-\mathbf{r}_{n}\right| / \xi} \tag{B.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\xi=\frac{a}{\ln (2 / \beta J)} \tag{B.9}
\end{equation*}
$$

is the correlation length.

Although the expression for the correlation function (B.8) in the high temperature limit is not exact, it captures the essential behavior. It tells us that the leading order behavior of the correlation between spins decreases exponentially with increasing distance. The characteristic length scale over which correlations decrease is the correlation length, and we can see from (B.9) that it decreases with increasing temperature.

A more precise calculation would simply add unimportant constant factors to (B.8) without changing the basic result.

## Appendix C

## Domain Wall Proliferation

## Temperature in the 2D Ising

## Model

The two-dimensional Ising model is described by the Hamiltonian

$$
\begin{equation*}
H=-J \sum_{\langle i j\rangle} \sigma_{i} \sigma_{j} \tag{C.1}
\end{equation*}
$$

where $\sigma_{i}= \pm 1$ and the sum is over all nearest neighbors. This two-dimensional, square lattice of nearest neighbor interacting spins has a finite temperature phase transition. This model was solved completely for the first time by L. Onsager in 1944[74], however the precise location of the critical point had already been discovered a few years earlier by H. Kramers and G. Wannier[75]. A simplified version of their argument will be made following [96], which allows us to find the critical temperature of a 2D isotropic Ising model on a square lattice.

The partition function for an Ising model with $N$ lattice sites is

$$
\begin{equation*}
Z_{N}=\sum_{\sigma_{i}} \exp \left[K \sum_{\langle i j\rangle} \sigma_{i} \sigma_{j}\right] \tag{C.2}
\end{equation*}
$$

where $K=\beta J$ and the sum over $\sigma_{i}$ denotes all possible spin configurations. In order to determine the critical point of this model, we will first expand the partition function in the low-temperature regime, and then in the high-temperature regime, and then through duality arguments[96] try and locate where low-temperature behavior crosses over to high-temperature behavior.

The partition function $Z_{N}$ can be graphically represented in the low-temperature limit as follows. Consider a particular set of lattice spins denoted as $\sigma_{i}$, nearly all spins will be aligned in the extreme low-temperature regime since if $J>0$ the Ising model is ferromagnetic. If we let $r$ be the number of anti-aligned nearest neighbor spins, and $M$ be the total number of bonds in the lattice (horizontal and vertical bonds), then the summand of (C.2) is simply

$$
\begin{equation*}
\exp [K(M-2 r)] \tag{C.3}
\end{equation*}
$$

There are $M$ ferromagnetic bonds (like spins) with individual energies of $-J$ and $r$ antiferromagnetic bond (unlike nearest neighbor spins), which cost an additional energy of $2 J$. In order to represent this graphically, let us consider the dual lattice $\zeta_{D}$. The dual lattice $\zeta_{D}$ is defined by drawing points in the center of the plaquettes of the original lattice $\zeta$, and connecting each of these points with nearest neighbors. Now unlike nearest neighbor pairs on $\zeta$ can be represented by shaded bonds on $\zeta_{D}$ while like pairs remain unshaded. Therefore, any given spin configuration can be represented graphically on the dual lattice $\zeta_{D}$, as in Figure C.1. The low-temperature graphical prescription on $\zeta_{D}$ is as follows: if adjacent spins are different then shade the corresponding bond on $\zeta_{D}$, if
they are the same then do nothing; do this for all nearest neighbor spins.

| + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
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Figure C.1: Up-spin and down-spin domains in the Ising model. In the low-temperature limit, the lattice background will be largely uniform, with small regions of closed domains of finite extent. In this figure, the $\pm$ spins represent the original lattice $\zeta$, and the domain walls live on the dual lattice $\zeta_{D}$. In the high-temperature limit, domain walls live on the regular lattice $\zeta$ (see text for more discussion).

In Figure C. 1 we are left with a set of polygons which represents a particular spin configuration at low-temperatures. Notice that each particular set of polygons is two-fold degenerate since all spins can be simultaneously reversed, which would simply reverse up-spin and down-spin domains. Also note that domains must be closed, since the number of shaded lines ending at a lattice point must be even. This follows from the fact that on a square lattice there will always be an even number of spin changes around any corner. The low-temperature partition function can now be written as

$$
\begin{equation*}
\left.Z_{N}=2 \mathrm{e}^{K M} \sum_{P} \exp [-2 K r)\right] \tag{C.4}
\end{equation*}
$$

where the sum is over all polygons $P$. The factor of 2 represents the previously dis-
cussed fact that every set of polygons is two-fold degenerate. Equation (C.4) is an exact expression but useful in the low temperature limit $(K \gg 1)$ because the dominant terms in this expression will be when $r=0$, i.e. there are no domains at all.

To perform a high-temperature expansion of (C.2) we can first write

$$
\begin{equation*}
\exp \left[K \sigma_{i} \sigma_{j}\right]=\cosh K+\sigma_{i} \sigma_{j} \sinh K \tag{C.5}
\end{equation*}
$$

since $\sigma_{i} \sigma_{j}= \pm 1$. By writing the sum of exponents as a product of exponentials and using (C.5), we can express the partition function (C.2) as

$$
\begin{align*}
Z_{N} & =\sum_{\sigma_{i}} \prod_{\langle i j\rangle}\left(\cosh K+\sigma_{i} \sigma_{j} \sinh K\right) \\
& =(\cosh K)^{M} \sum_{\sigma_{i}} \prod_{\langle i j\rangle}\left(1+\alpha \sigma_{i} \sigma_{j}\right) \tag{C.6}
\end{align*}
$$

where $\alpha=\tanh K$. If we were to expand the product in (C.6) there would be a total of $M$ factors. These factors can be represented graphically by the following prescription: draw a shaded line on the $i j$-bond if the factor contains $\alpha \sigma_{i} \sigma_{j}$, do nothing if it simply contains 1. This gives us a direct correspondence between terms in the expansion and line configurations. Every term in the expansion will be of the form

$$
\begin{equation*}
\alpha^{r} \sigma_{1}^{n_{1}} \sigma_{2}^{n_{2}} \sigma_{3}^{n_{3}} \cdots \sigma_{N}^{n_{N}} \tag{C.7}
\end{equation*}
$$

where $r$ is the total number of lines and $n_{i}$ is the number of lines that are attached to site $i$. The partition function will contain $M$ such factors which are summed over all possible spin configurations $\sigma_{i}$. Since $\sigma_{i}= \pm 1$, the only factors that survive this sum will be terms in which $n_{1}, n_{2}, \ldots, n_{N}$ are even. For each even term, the sum will yield $2^{N} \alpha^{r}$. We can now account for all such terms and collectively write the partition function as

$$
\begin{equation*}
Z_{N}=2^{N}(\cosh K)^{M} \sum_{P} \alpha^{r} \tag{C.8}
\end{equation*}
$$

where the sum is over all sets of polygons P . This high-temperature expansion for the partition function is also exact.

Notice that both the low and high-temperature partition functions in (C.4) and (C.8) respectively are very similar in form. Although both are sums over sets of all possible polygons, the low-temperature expression lives on the dual lattice $\zeta_{D}$ and the high-temperature expression lives on the original lattice $\zeta$. The difference in boundaries between $\zeta$ and $\zeta_{D}$ will vanish in the thermodynamic limit and so we can consider the sum over polygons as equivalent. Also, $M \rightarrow 2 N$ in the thermodynamic limit. The free energy per site is given by (renormalized in terms of $k_{B} T$ )

$$
\begin{equation*}
f=-\frac{F}{k_{B} T}=\lim _{N \rightarrow \infty} \frac{1}{N} \ln Z_{N} \tag{C.9}
\end{equation*}
$$

Since the low and high-temperature expressions for $Z_{N}$ were exact, the expressions in each case will be equal to each other. In the low-temperature case we can plug (C.4) into (C.9) which yields

$$
\begin{equation*}
f_{L}=2 K+\psi\left(\mathrm{e}^{-2 K}\right) \tag{C.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi(x)=\lim _{N \rightarrow \infty} \frac{1}{N}\left(x^{r}\right) \tag{C.11}
\end{equation*}
$$

The high temperature free energy follows from plugging (C.8) into (C.9) which yields

$$
\begin{equation*}
f_{H}=2 \ln (2 \cosh K)+\psi\left(\tanh K^{r}\right) \tag{C.12}
\end{equation*}
$$

where $\psi(x)$ is also given by (C.11). The two expressions for the free energy in (C.10) and (C.12) are exact and therefore equivalent, $f_{L}=f_{H}=F(K)$. We can eliminate the function $\psi(x)$ by replacing $K$ by $K^{*}$ in (C.12) where

$$
\begin{equation*}
\tanh K^{*}=\mathrm{e}^{-2 K} \tag{C.13}
\end{equation*}
$$

Note from this relation that if $K^{*}$ is large then $K$ is small, and vice-versa. We can then interpret this as a duality relation which provides a mapping from a high-temperature regime into a low-temperature regime. If we now subtract (C.10) from (C.12) the function $\psi(x)$ cancels out and we are left with the relation

$$
\begin{equation*}
F\left(K^{*}\right)=F(K)+2 K-2 \ln \left(2 \cosh K^{*}\right) \tag{C.14}
\end{equation*}
$$

This makes the relation between the high and low-temperature free energies explicit.
We can rewrite (C.13) and (C.14) through the use of hyperbolic trigonometric identities as

$$
\begin{equation*}
F(K)=F\left(K^{*}\right)+\ln \left(2 \sinh 2 K^{*}\right) \tag{C.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\sinh 2 K \sinh 2 K^{*}=1 \tag{C.16}
\end{equation*}
$$

The free energy will be non-analytic at a critical point. Let us say that the point of non-analyticity occurs at $K=K_{c}$, then according to (C.15) the free energy will also be non-analytic at $K^{*}=K_{c}^{*}$. Suppose that there is only one critical point, such that $K_{c}=K_{c}^{*}$. Therefore, it follows from (C.16) that the critical transition temperature will be given by

$$
\begin{equation*}
\left(\sinh 2 K_{c}\right)^{2}=1 \tag{C.17}
\end{equation*}
$$

Using the quadratic formula to solve for $K_{c}=k_{B} T_{c} J$, it is easy to show that the critical transition occurs at

$$
\begin{equation*}
k_{B} T_{c}=\frac{E_{\mathrm{dw}}}{\ln (1+\sqrt{2})} \tag{C.18}
\end{equation*}
$$

where $E_{\mathrm{dw}}=2 J$ is the energy per unit length of a domain wall. This duality argument provides the exact location of the critical point, under the assumption that there is only one critical, which later proved to be entirely correct[74].

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