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Fixed Points and Coincidences in the Stable Homotopy Category

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FIXED POINTS AND COINCIDENCES IN THE STABLE HOMOTOPY CATEGORY

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Abstract

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by

Jonathan Chi

Given a topological space X and two self-maps $f, g : X \to X$, coincidence theory asks the question: which points $x \in X$ satisfy the property $f(x) = g(x)$? For g (or f) equal to the identity map, this question reduces to asking how many fixed points g (respectively f) has. In this manner, coincidence theory is a straight-forward generalization of fixed point theory. The classical approach to fixed point and coincidence theory examines the induced maps on the homology groups of the space in question. This thesis presents the classical Lefschetz number and the Lefschetz coincidence number as a foundation with the goal of generalizing these ideas to the stable homotopy category. Instead of passing directly from the category of "nice" topological spaces to graded $\mathbb{Q}$-vector spaces using rational homology, we generate spectra from these topological spaces first and then apply rational homology. This formulation of fixed point theory and coincidence theory in the stable homotopy category is carried out by using Atiyah duality with a nod towards Spanier-Whitehead duality, and traces in symmetric monoidal categories.

Dedicated To:

#fivebestfriends

You are my Rock, my Dwane, my Johnson.

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Chapter 1

Introduction

As a very basic example, consider an experiment where we compare rotations of the unit disk versus the unit circle in $\mathbb{R}^2$. We treat the function as the control and vary the space X. Let rotation through a fixed angle ψ be given by $r_\psi(r, \theta) = (r, \theta + \psi)$ where $\psi \in (0, 2\pi)$ and note that for any point on the unit circle, r_ψ has no fixed points. However, applying r_ψ to the unit disk shows that there is exactly one fixed point: the origin. This simple example show that, at least on some level, the question of analyzing fixed points of a general $f : X \to X$ may involve the topological properties of X itself.

Fixed point theory from the algebraic topology perspective is a well-established subject (think Brouwer's fixed point theorem), and appears in [4] as an application of the homology intersection product and orientation classes of manifolds. The definition of the Lefschetz number of f is the alternating sum of the level-wise traces of the induced homology maps of f.

$$L(f) = \sum_i (-1)^i tr_i(f_*)$$

The classical result for a finite simplicial complex is the Lefschetz-Hopf fixed point theorem which states that if the Lefschetz number of f is nonzero, then f has at least one fixed point. It was later established that $L(f)$ can also be expressed as the intersection product of the orientation class of the graph of f and the orientation class of the diagonal $\Delta \subset X \times X$.

More generally, the Lefschetz number for f and g, $L(f,g)$ is defined to be the intersection product of the orientation classes of the graphs of f and g. The reduction of coincidence theory to fixed point theory then follows by first considering the case where one of g or f is invertible. For example, if g is an invertible function, then we just compute the Lefschetz number for $g^{-1} \circ f$ instead and thus g and f have a coincidence point if and only if $g^{-1} \circ f$ has a fixed point. It is the goal of this thesis to apply these ideas in the stable homotopy category with a more categorical flavor.

Since its introduction by John Michael Boardman in 1969, the stable homotopy category has been a useful tool in algebraic topology. This category has a great deal of algebraic structure which has been well-documented in [23] with a more modern treatment in [8]. In this thesis, we describe the properties of a particular sub-category which inherits the symmetric monoidal structure and is closed under the operation of dualization. The symmetric monoidal structure and existence of dualizable objects plays an essential role in the definition of an abstract trace, which is then used to define the Lefschetz number. A collection of propositions cataloging the algebraic properties in the stable homotopy category appears in appendix A.2.6; the proofs of which are not novel, but original work by the author.

Chapter 2 establishes the proper setting in which to perform calculations by defining the stable homotopy category, **HoSpectra** referencing [23],[7], [18] as well as expository notes by [13]. To engage those that are new to the subject, we begin at the more concrete level of topological spaces and proceed through the layers of abstraction assuming a familiarity with basic category theory. In particular, we introduce the smash product which is consistent with the smash product of the category of pointed topological spaces and acts like the tensor product in the category of R-modules. We present some notable examples of stable objects and morphisms which will be used later in our computations.

An abstract trace can be defined for any symmetric monoidal category in which dualizable objects exist. So we focus our attention on the subcategory, $\mathbf{HoSpectra}^{\bigstar}$, of **HoSpectra** which is closed under dualization. After enumerating the formal algebraic structure of **HoSpectra**, chapter 3 presents the definition and properties of an abstract trace defined on morphisms in $\mathbf{HoSpectra}^{\bigstar}$. This abstract trace coincides with the traditional trace of a linear operator defined in linear algebra when considered in the category of finite dimensional vector spaces. Also in this section, we recall Atiyah's duality theorem and elements of Dold's proof of said theorem in order to gain access to rigorous descriptions of dual pairs and their evaluation and coevaluation maps. As an ancillary goal, chapters 1-3 attempt to unify the results of [2] [7] [13] [18].

In chapter 4, we reference [5],[18] and [19] for some classical results in fixed point and coincidence theory and extend these ideas to $\mathbf{HoSpectra}^{\bigstar}$.

Chapter 2

Preliminaries

2.1 Stable Homotopy Category

Starting at a more concrete level, let $\mathbf{Top}_+$ be the category consisting of compactly generated topological spaces with non-degenerate basepoint and basepoint preserving maps. From this category we take only those spaces that are homeomorphic to CW complexes; call this (full) sub-category $\mathbf{CW}_+$. In particular, $\mathbf{CW}_+$ contains all closed smooth manifolds (this is a consequence of Morse theory) and we will be mostly concerned with the finite CW complexes. Denote by $\overline{\mathbf{CW}}_+$ the subcategory of all spaces that are homeomorphic to finite CW complexes. Next define respective (naive) homotopy categories $\mathbf{HoTop}_+$ and $\mathbf{HoCW}_+$, to be the category with objects the same as $\mathbf{Top}_+$ and $\mathbf{CW}_+$, but morphisms the homotopy classes of maps between objects. A map in $\mathbf{Top}_+$ or $\mathbf{CW}_+$ is a homotopy equivalence if and only if it is an isomorphism in the corresponding homotopy category.

We will take the objects from $\mathbf{Top}_+$ and $\mathbf{CW}_+$ and construct new objects called *spectra*. See A.1 for a discussion on CW approximation.

Definition 2.1.1. *A* spectrum X *is a sequence of based spaces* $\{X_n\}, n \in \mathbb{Z}$ *together with "structure maps"* $\epsilon_n : \Sigma X_n \to X_{n+1}$. *We will use the notation* (X, ϵ) *or simply* X *to denote the spectrum along with its structure maps.*

In the above definition, Σ is the reduced suspension operation and the ϵ_n are continuous, basepoint preserving maps. When the spaces X_n are each CW-complexes, X is referred to as a *CW spectrum*.

Definition 2.1.2. *Given spectra* (X, ϵ) *and* (Y, ζ), *a* map $f : X \to Y$ *of degree* k *is a collection of functions* $f_n : X_n \to Y_{n-k}$ *such that the following diagram strictly commutes for all* $n \in \mathbb{Z}$.

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\Sigma f_n} & \Sigma Y_{n-k} \\ {\scriptstyle \epsilon_n}\downarrow & & \downarrow{\scriptstyle \zeta_{n-k}} \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n-k+1} \end{array}$$

The commutativity of this diagram is sometimes referred to as being compatible with (or respecting) the structure maps.

As a technical point, the currently defined functions between spectra do not always suspend or desuspend freely. For this reason and because we will be concerned with stable objects and morphisms, we modify the objects and maps slightly with the following definitions.

Definition 2.1.3. *A* cofinal subspectrum $X' \subseteq X$ *is a subspectrum of* X *that contains all of the cells of* X *after sufficiently many suspensions. Given spectrum* X *and* Y, *we now define a map* $f : X \to Y$ *as an equivalence class of functions coming out of the*

Category	Objects	Morphisms
$\mathbf{Top}_+$	Pointed spaces	pointed continuous maps
$\mathbf{CW}_+$	Pointed spaces $\cong$ CW complexes	pointed continuous maps
$\mathbf{HoCW}_+$	Pointed spaces $\cong$ CW complexes	homotopy classes of maps
Spectra	Spectra	structure-preserving maps
HoSpectra	Spectra	homotopy classes of spectra maps

$$\begin{tikzcd}
\overline{\mathbf{CW}}_+ \arrow[r, hook] & \mathbf{CW}_+ \arrow[r, hook] \arrow[d] & \mathbf{Top}_+ \arrow[r] \arrow[d] & \mathbf{Spectra} \arrow[d] \\
 & \mathbf{HoCW}_+ \arrow[r, hook] & \mathbf{HoTop}_+ \arrow[r] & \mathbf{HoSpectra}
\end{tikzcd}$$

cofinal subspectra X'. Two maps $f' : X' \to Y$ and $f'' : X'' \to Y$ are equivalent if they are equal on $X' \cap X''$.

Definition 2.1.4. *A* morphism *of spectra $\tilde{f} : X \to Y$ is a homotopy class of maps of spectra. The notation $[X, Y]$ will be used to denote the set of all morphisms between spectra X and Y, while $[X, Y]_n$ will denote only those morphisms of degree n.*

Let the *stable homotopy category* be denoted as **HoSpectra**. Take the objects of **HoSpectra** to be spectra, X and morphism sets $[X, Y]$. At this point, we will drop all the additional embellishments on our notation of objects and morphisms as they were previously only used to distinguish between steps in the construction. The objects of **HoSpectra** will be referred to as stable objects. The above figures describe the relationships between the categories discussed.

2.2 Notable Spectra and Additional Structure

In our definition of **HoSpectra**, we began with CW-complexes and created new objects from these spaces. We shall define some notable spectra in this section and describe some of the additional algebraic structure in **HoSpectra**. The first example is the sphere spectrum which plays the role of the identity object (with respect to the smash product which will be defined later).

Definition 2.2.1. *Let S^k be the k-dimensional sphere. Denote by S^∞ the* sphere spectrum*, with $\{S^\infty_k = S^k\}$ $k \in \mathbb{Z}$ and structure maps the usual homeomorphism:*

$$\iota_k : \Sigma S^k \cong S^{k+1}$$

In general, we can start with any topological space with nondegenerate basepoint X and construct a spectrum from X.

Definition 2.2.2. *Given $X \in \mathbf{Top}_+$, the suspension spectrum generated by X is denoted as $\Sigma^\infty X$ and defined by $\{X_k = \Sigma^k X\}$ $k \in \mathbb{Z}$ with structure maps the identity.*

$$id_k : \Sigma X_k \to X_{k+1}$$

This construction, along with the naturality requirements described in the previous section imply that there is a functor $\Sigma^\infty : \mathbf{Top}_+ \to \mathbf{HoSpectra}$ which creates a spectrum from any space with non-degenerate basepoint. The sphere spectrum can thus be identified with the suspension spectrum of S^0, $\Sigma^\infty S^0$.

A special case of suspension spectrum is that of a CW spectrum. The finite versions of these spectra will be the objects of focus because they are the dualizable objects in **HoSpectra**.

Definition 2.2.3. *A* CW-spectrum *is a sequence of based CW complexes $X := \{X_1, X_2, ...\}$ with inclusions $i_n : \Sigma X_n \to X_{n+1}$. A* cell *of a CW spectrum is a cell living in one of the component spaces, along with all of its suspensions living in higher component spaces.*

The CW spectrum is called finite *if each component space contains a finite number of cells.*

Recall the following theorems:

Theorem 2.2.4 (Hurewicz)**.** $\pi_n(S^n) \cong \mathbb{Z}$ *and* $\pi_m(S^n) = 0$ *for all* $m < n$.

Theorem 2.2.5 (Freudenthal Suspension)**.** *The suspension homomorphism*

$$S : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphism if $k < n-1$ *and an epimorphism if* $k = n-1$. *In other words, for a fixed* n, S *is* $n-1$ *connected.*

Corollary 2.2.6. $\pi_{n+k}(S^n)$ *depends only on* k *if* $n > k+1$.

Thus we may identify $[S^\infty, S^\infty]$ with the integers.

With these two examples of the sphere spectrum and suspension spectra, we can construct some familiar invariants. Recall that the traditional homotopy groups are $\pi_k(X) = [S^k, X]$, defined to be sets of homotopy classes of maps between S^k and X which are all abelian when $k \geq 2$. For further discussion on the group operation in $\pi_k(X)$ see appendices A.2.1 A.2.2, and A.2.3 for justification of the stable terminology.

Definition 2.2.7. *Let* $X \in \mathbf{Top}_+$ *and let* $\Sigma^\infty X$ *be the corresponding spectrum. The* stable homotopy groups *of* $\Sigma^\infty X$ *are* $\pi_*^S(\Sigma^\infty X) = [S^\infty, \Sigma^\infty X]$. *This follows from:*

$$\pi_k^S(\Sigma^\infty X) = [S^\infty, \Sigma^\infty X]_k = [\Sigma^k S^\infty, \Sigma^\infty X] = [S^\infty, \Sigma^\infty X]_{-k}$$

In **HoSpectra** there are two functors:

$$\Sigma : \mathbf{HoSpectra} \to \mathbf{HoSpectra}$$

$$\Omega : \mathbf{HoSpectra} \to \mathbf{HoSpectra}$$

called the *suspension* and *loopspace* functors. The notation is suggestive: these functors agree with the usual suspension and loopspace operations in $\mathbf{Top}_+$ and are categorical equivalences implying that in **HoSpectra** every object is isomorphic to the suspension of another object, and every object is also isomorphic to the loopspace of an other object. Concretely: suppose that X and Y are spectra. There are natural isomorphisms

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

This adjunction is analogous to the tensor-Hom adjunction in the category of finite dimensional vector spaces and gives rise to another type of spectrum.

Definition 2.2.8. *An* Ω-spectrum*,* Y *is a sequence of based spaces* $\{Y_n\}$ *whose adjoint structure maps are homotopy equivalences* $\omega_i : Y_i \to \Omega Y_{i+1}$*.*

A special example of an Ω-spectrum arises from the concept of an Eilenberg-MacLane space. Recall that if G is any Abelian group and $n > 1$, then there exists a topological space denoted as $K(G, n)$ known as the Eilenberg-MacLane Space associated to G with the particular property:

$$\pi_i(K(G, n)) = \begin{cases} 0 & i \neq n \\ G & i = n \end{cases}$$

Furthermore, $K(G, n)$ has the homotopy type of a CW-complex and is unique up to a weak homotopy equivalence. There are two relevant facts which pertain to the construction of a spectrum:

Theorem 2.2.9. *Let G be an abelian group.*

- $\pi_k(\Omega K(G, n)) \cong \pi_{k+1}(K(G, n))$.

- *The loop space of an Eilenberg-MacLane space is also an Eilenberg-MacLane space:* $\Omega K(G, n+1) = K(G, n)$.

Proof. The first bullet point is proved by applying the π_* functor to the path space fibration: $\Omega K(G,n) \hookrightarrow P(K(G,n)) \to K(G,n)$ which yields a long exact sequence by the snake lemma:

$$\begin{tikzcd}
& & \cdots \arrow[dll] \\
\pi_{k+1}(\Omega K(G,n)) \arrow[r] & \pi_{k+1}(P(K(G,n))) \arrow[r] & \pi_{k+1}(K(G,n)) \arrow[dll] \\
\pi_{k}(\Omega K(G,n)) \arrow[r] & \pi_{k}(P(K(G,n))) \arrow[r] & \pi_{k}(K(G,n)) \arrow[dll] \\
\pi_{k-1}(\Omega K(G,n)) \arrow[r] & \pi_{k-1}(P(K(G,n))) \arrow[r] & \pi_{k}(K(G,n)) \arrow[dll] \\
\cdots & &
\end{tikzcd}$$

Since the path space is contractible, its homotopy groups are all trivial. Exactness implies the result. The second bullet point follows from the first and the existence and uniqueness of Eilenberg-MacLane spaces up to weak homotopy equivalence. $\square$

The corresponding *Eilenberg-MacLane spectrum*, typically denoted as HG, is defined to be the spectrum with n^{th} space $K(G,n)$. The structure maps

$$\omega_{n-1} : HG_{n-1} \to \Omega HG_n$$

are homotopy equivalences adjoint to $\sigma_{n-1} : \Sigma HG_{n-1} \to HG_n$ under the correspondence $[\Sigma E, F] \cong [E, \Omega F]$. These HG spectra are useful via the following special case of Brown Representability:

Theorem 2.2.10 (Brown Representability). *Let $\widetilde{H}^*(-;G)$ denote the reduced singular cohomology with coefficients in an abelian group G. Then for a space $X \in \mathbf{Top}_+$ there* *is a one to one correspondence*

$$\widetilde{H}^n(X;G) \leftrightarrow [X, K(G,n)]$$

In **HoSpectra**,

$$[\Sigma^\infty X_+, HG]_{-n} = [\Sigma^{\infty - n} X, HG]_0 \text{ or just}$$

$$\widetilde{H}^*(X;G) \cong [\Sigma^\infty X, HG]$$

and dually:

$$\widetilde{H}_n(X;G) \cong [S^\infty, HG \wedge X]_n = \pi_n(HG \wedge X) \text{ or just}$$

$$\widetilde{H}_*(X;G) \cong \pi_*(HG \wedge X).$$

That is, singular cohomology with coefficients in G is represented by the spectrum $K(G,n)$.

In the above theorem, we used the smash product which we define next and Spanier-Whitehead duality which is discussed in A.2.5 under the name $\mathbb{S}$-duality. Details of the smash product construction can be found in [1].

Theorem 2.2.11. *In* **HoSpectra**, *there exists a commutative, associative tensor product called the* smash product *and denoted by $\wedge$. With respect to the smash product, the sphere spectrum S^∞ acts as the identity object. If X, Y and Z are spectra, then the following isomorphisms are coherent and natural:*

1. $X \wedge Y \cong Y \wedge X$
2. $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$
3. $S^\infty \wedge X \cong X \cong X \wedge S^\infty$.

The last notable spectrum we examine is the function spectrum. Let X and Y be spectra, with $F(X,Y)$ the set of morphisms between X and Y. $F(X,Y)$ is the internal hom set in **HoSpectra** and via Brown representability, the next remark shows that $F(X,Y)$ is an object of **HoSpectra** which we call the *function spectrum.*

Remark 2.2.12. *Let W be a CW complex and $X=(X,\epsilon)$ a spectrum. Construct a new spectrum $(X\wedge W, \epsilon\wedge id_W)$ where $(X\wedge W)_n = X_n\wedge W$ and $\epsilon_n\wedge id_W : \Sigma X_n\wedge W \to X_{n+1}\wedge W$ are the structure maps. Then after showing that the functor $W\to[X\wedge W,Y]$ satisfies the Eilenberg-Steenrod axioms of a generalized cohomology theory, Brown representability 2.2.10 then implies that there exists a spectrum which represents this cohomology theory. The representing spectrum we're looking for is given by the adjunction:*

$$[X\wedge W, Y]\cong[W,F(X,Y)].$$

Proposition 2.2.13. *For X, Y and Z spectra, the function spectrum exhibits the following additional properties. These isomorphisms are coherent and natural.*

1. $[X\wedge Y, Z]\cong[X,F(Y,Z)]$
2. $F(S^\infty, X)\cong X$
3. $F(X\wedge Y,Z)\cong F(X,F(Y,Z))$

With the smash product and function spectrum defined, we can catalog some of the previous results with a more unified notational convention. These follow directly from the preceeding discussion.

Proposition 2.2.14. *Recall that Σ is the reduced suspension and Σ^∞ is the suspension functor which produces a stable object from a topological object. If X is a spectrum, then* *the following isomorphisms are natural:*

1. $\Sigma X = \Sigma^\infty(S^1) \wedge X$

2. $\Omega X = F(\Sigma^\infty(S^1), X)$

3. $\Sigma^\infty X = S^\infty \wedge X$

4. $\Omega^\infty X = F(S^\infty, X)$

5. $[\Sigma X, Y] \cong [X, \Omega Y]$

6. $\Sigma X \wedge Y \cong \Sigma(X \wedge Y) \cong X \wedge \Sigma Y$

7. $\Omega F(X, Y) \cong F(\Sigma X, Y) \cong F(X, \Omega Y)$.

The final property of **HoSpectra** that we mention in this section pertains to homotopy fiiber and cofiber sequences and allows us to call **HoSpectra** *triangulated*. The slogan “fiber and cofiber sequences are the same in the stable homotopy category” follows from these next remarks from [13].

Remark 2.2.15. *For every map between spectra $f : X \to Y$,*

- *There exists a homotopy cofiber $C(f)$ which is unique up to non-unique equivalence:*

$$X \xrightarrow{f} Y \longrightarrow C(f) \longrightarrow \Sigma X$$

- *There exists a homotopy fiber $F(f)$ which is unique up to non-unique equivalence:*

$$F(f) \longrightarrow X \xrightarrow{f} Y \longrightarrow \Sigma F(f)$$

- *There is a natural isomorphism between the cofiber and the suspension of the fiber:*

$$C(f) \cong \Sigma F(f)$$

The previous remark implies that a pair of spaces with any function between them generates a suspension spectrum:

Proposition 2.2.16. *Given a map between spaces X and Y in $\mathbf{Top}_+$,*

Proof. Let $f : X \to Y$ be a map of spaces. Via the cofiber sequence:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "i_0"] & C_f \arrow[dll, "q_0"] \\ \Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "i_1"] & C_{\Sigma f} \arrow[dll, "q_1"] \\ \Sigma^2 X \arrow[r, "\Sigma^2 f"] & \Sigma^2 Y \arrow[r, "i_2"] & C_{\Sigma^2 f} \arrow[dll, "q_2"] \\ \dots \end{tikzcd}$$

we extract the following sequence of spaces and structure maps which factor through the mapping cones:

$$\Sigma^\infty(X, Y) = \{\Sigma^k X, \Sigma^k Y)\}_{k \geq 0}$$

$$\begin{tikzcd} (\Sigma^k X, \Sigma^k Y) \arrow[rr, dashed] \arrow[dr, "i_k"] & & (\Sigma^{k+1} X, \Sigma^{k+1} Y) \\ & C_{\Sigma^k f} \arrow[ur, "q_k"] & \end{tikzcd}$$

which form a suspension spectrum. $\square$

In particular, pairs of the form (M, M') with subspace inclusion $M' \hookrightarrow M$ will be important.

In [23], Spanier describes a category rich with algebraic structure and equipped with a duality endo-functor which he called the suspension category or $\mathbb{S}$ category. It turns out that this category does not retain the idea of Brown Representability and is therefore lacking in a sense. However, it is a full subcategory on the suspension spectra of finite CW complexes inside the stable homotopy category. A more thorough description of the $\mathbb{S}$ category as well as a series of algebraic results are fully worked out in A.2.6 using a slightly adjusted version of Spanier's notation.

2.3 Duality in HoSpectra

The previous section establishes the stable homotopy category as a symmetric, monoidal, triangulated category. To define a suitable notion of trace for **HoSpectra**, this section defines the remaining ingredients: duality, evaluation and coevaluation. We first present a more categorical dual (the Spanier-Whitehead dual), and then give two more geometric notions of duality (Alexander duality and its extension Atiyah duality).

Definition 2.3.1. *Let X be a spectrum. A* dual *of X is a spectrum Y equipped with evaluation and coevaluation maps:*

$$\epsilon : X \wedge Y \to S^{\infty}$$

$$\eta : S^{\infty} \to X \wedge Y$$

which satisfy the following triangle identities:

- $(id_X \wedge \epsilon) \circ (\eta \wedge id_X) = id_X$
- $(\epsilon \wedge id_Y) \circ (id_Y \wedge \eta) = id_Y$

which can be visualized as:

$$\begin{tikzcd}
X = S^\infty \wedge X \arrow[rr, "id_X"] \arrow[dr, "\eta \wedge id_X"'] & & S^\infty \wedge X = X \\
& X \wedge Y \wedge X \arrow[ur, "id_X \wedge \epsilon"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
Y = Y \wedge S^\infty \arrow[rr, "id_X"] \arrow[dr, "id_Y \wedge \eta"'] & & S^\infty \wedge Y = Y \\
& Y \wedge X \wedge Y \arrow[ur, "\epsilon \wedge id_Y"'] &
\end{tikzcd}$$

Given a spectrum X, we say that X is *dualizable* if there exists a dual spectrum. If a dual exists it will be denoted as $X^\bigstar$. When a space is said to be dualizable, this means that the suspension spectrum of the space is dualizable in **HoSpectra**. The dual exhibits the following properties [7][23]:

Proposition 2.3.2. *1. If X is dualizable with dual $X^\bigstar$, then $X^\bigstar$ is also dualizable with dual X.*

2. If X is dualizable with dual $X^\bigstar$, then for all spectra Y, Z, the following composition is an isomorphism:

$$\eta^* \circ \wedge X : [Y \wedge X^\bigstar, Z] \to [Y \wedge X^\bigstar \wedge X, Z \wedge X] \to [Y, Z \wedge X]$$

3. If X is dualizable, then $X^\bigstar \cong F(X, S^\infty)$.

4. If X is a finite CW spectrum, then X is dualizable.

5. Suppose X and Y are dualizable with respective duals $X^\bigstar$, $Y^\bigstar$, evaluations ϵ_X, ϵ_Y and coevaluations η_X, η_Y. If $f \in [X, Y]$ is a morphism, then f has a dual morphism $f^\bigstar$ in$[Y^\bigstar, X^\bigstar]$ defined by:

$$Y^\bigstar = Y^\bigstar \wedge S^\infty \xrightarrow{\eta_X} Y^\bigstar \wedge X \wedge X^\bigstar \xrightarrow{f} Y^\bigstar \wedge Y \wedge X^\bigstar \xrightarrow{\epsilon_Y} S^\infty \wedge X^\bigstar = X^\bigstar$$

The previous proposition implies that there is a non-empty collection of objects in **HoSpectra** consisting of dualizable spectra. Indeed, S^∞ is trivially dualizable with $(S^\infty)^\bigstar = S^\infty$. We will denote the subcategory containing all dualizable spectra as **HoSpectra**$^\bigstar$. It is well-known that this subcategory is the stable homotopy category of finite CW spectra.

Furthermore, there exists a functor $D : \mathbf{HoSpectra}^\bigstar \to \mathbf{HoSpectra}^\bigstar$ which maps spectra to their duals and morphisms to their dual morphisms. The next theorem enumerates the defining properties of the duality functor, D, but in practice we will just use the $-^\bigstar$ notation for both morphisms and objects.

Theorem 2.3.3. *Given spectra $X, Y \in \mathbf{HoSpectra}^\bigstar$ and respective duals $X^\bigstar, Y^\bigstar$, there exists a unique map*

$$D : [X, Y] \to [Y^\bigstar, X^\bigstar]$$

with the following properties:

1. *If $i : X \hookrightarrow Y$ and $i' : Y^\bigstar \hookrightarrow X^\bigstar$ are inclusions of spectra, then $D(i) = i'$.*
2. *Given $f \in [X, Y], g \in [Y, Z]$ and duals $X^\bigstar, Y^\bigstar, Z^\bigstar$, we have the formula*

$$D(g \circ f) = D(f) \circ D(g)$$

3. *Considering X, Y as duals of $X^\bigstar$ and $Y^\bigstar$ respectively, we have maps*

$$[X, Y] \xrightarrow{D} \left[Y^\bigstar, X^\bigstar\right] \xrightarrow{D} [X, Y]$$

which compose to the identity.

4. D is a group homomorphism.

In the case of when X is the suspension spectrum of a closed smooth manifold, we have a concrete description of the its dual via theorems of Thom and Atiyah [2]. In our account, we develop Atiyah duality as an extension of Alexander duality. In **HoSpectra**, Atiyah duality is the composition of the Spanier-Whitehead duality A.2.6 with the Thom isomorphism.

Theorem 2.3.4 (Alexander)**.** *Let M be a finite CW complex embedded into S^n in such a way that $S^n \setminus M$ is homotopy equivalent to a finite CW complex. Then:*

$$M^{\bigstar} \cong \Sigma^{-(n-1)}(S^n \setminus M)$$

is an isomorphism in **HoSpectra$^{\bigstar}$***. Furthermore, we have the following results connecting the E-homology and E-cohomology for any spectrum E:*

$$E^k(M) \cong E_{n-k-1}(S^n \setminus M)$$

$$E_k(M) \cong E^{n-1-k}(S^n \setminus M).$$

In particular, if $E = H\mathbb{Q}$, we have the traditional results in rational (co)homology:

$$\widetilde{H}^k(M;\mathbb{Q}) \cong \widetilde{H}_{n-k-1}(S^n \setminus M;\mathbb{Q})$$

$$\widetilde{H}_k(M;\mathbb{Q}) \cong \widetilde{H}^{n-k-1}(S^n \setminus M;\mathbb{Q}).$$

To generalize this idea, we perform the Thom space construction with the premise that M is a closed, smooth manifold. Consider an embedding of M into S^n where n is sufficiently large to ensure that the embedding is not surjective. Since S^n is the one-point compactification of $\mathbb{R}^n$, we can make the identification $S^n \cup \{*\} \cong \mathbb{R}^n$.

Designating $\{*\}$ as the disjoint basepoint of M gives M_+ and yields an embedding of M in $\mathbb{R}^n$. By Alexander duality,

$$M_+^{\bigstar} \cong \Sigma^{-(n-1)}(\mathbb{R}^n \setminus M)$$

Using the cofiber sequence $\mathbb{R}^n \setminus M \hookrightarrow \mathbb{R}^n \twoheadrightarrow \Sigma(\mathbb{R}^n \setminus M)$ along with the fact that $\mathbb{R}^n$ is contractible, we have

$$M_+^{\bigstar} \cong \Sigma^{-n}(\mathbb{R}^n/(\mathbb{R}^n \setminus M)).$$

To form the Thom space of the normal bundle of M, take a tubular neighborhood of M in $\mathbb{R}^n$ and identify it with the normal disk bundle $D(\nu)$ which has a sphere bundle $Sph(\nu)$ as boundary [17](11.1). More precisely, for every point $x \in M$, attach a the Euclidean space $\mathbb{R}^p_x$ where p is the codimension of M in $\mathbb{R}^n$. The collection of all copies of $\mathbb{R}^p_x$ form the total space of the normal bundle $E(\nu)$ with the zero section $s : M \to E(\nu)$ as an embedding of M as a deformation retract of $E(\nu)$. Within each copy of $\mathbb{R}^p_x$, take the unit disk $D_x = \{v \in \mathbb{R}^p_x | \, \|v\| \leq 1\}$ and the unit sphere: $S_x = \{v \in \mathbb{R}^p_x | \, \|v\| = 1\}$. Then $D(\nu) = \bigsqcup_{x \in M} D_x$ and $Sph(\nu) = \bigsqcup_{x \in M} S_x$ are the disk and sphere bundles respectively over M:

$$\begin{tikzcd}
S^{p-1} \arrow[r, hook] \arrow[d] & D^p \arrow[r, hook] \arrow[d] & \mathbb{R}^p \arrow[d] \\
Sph(\nu) \arrow[r, hook] \arrow[dr] & D(\nu) \arrow[r, hook] \arrow[d] & E(\nu) \arrow[dl] \\
& M &
\end{tikzcd}$$

Thus

$$\mathbb{R}^n/(\mathbb{R}^n \setminus M) \cong D(\nu)/(D(\nu) \setminus M) \cong D(\nu)/Sph(\nu) \cong Th(\nu).$$

where the last equivalence is by the definiton of the Thom space. The Alexander's duality theorem now takes the following form [19]:

Theorem 2.3.5 (Atiyah)**.** *Let M be a closed smooth manifold. M_+ is dualizable with $M_+^\star \cong \Sigma^{-n} Th(\nu)$ where ν is the normal bundle of an embedding of M into $\mathbb{R}^n$.*

The associated *Thom spectrum* is the suspension spectrum of $Th(\nu)$. Therefore, in **HoSpectra**, we have the spectrum version:

Theorem 2.3.6. *Let M_+ be a closed smooth manifold with disjoint basepoint and associated suspension spectrum $\Sigma^\infty M_+$. Then $\Sigma^\infty M_+$ is dualizable with*

$$(\Sigma^\infty M_+)^\star \cong \Sigma^\infty Th(\nu).$$

In other words, the dual to a suspension spectrum of a smooth closed manifold is the suspension spectrum of the Thom space of the normal bundle.

This turns out to also follow by the definition of the Thom spectrum.

Definition 2.3.7. *Let ν be the normal vector bundle over M. The sequence of spaces $\{Th(\mathbb{R}^k \oplus \nu\}_{k \geq 0}$ is the* Thom spectrum *of ν. The Thom spectrum is a suspension spectrum by the homeomorphism $Th(\mathbb{R}^k \oplus \nu) \cong S^k \wedge Th(\nu) = \Sigma^k Th(\nu)$.*

By the Thom isomorphism theorem, we have a concrete description of the cohomology of the Thom space which plays a central role in applications. The theorem states that the cohomology of the Thom space of the normal bundle is isomorphic to the cohomology of the base manifold M with a uniform dimension shift by the rank of the normal bundle (p in this case). The isomorphism itself is defined by taking the cup product with the Thom class of the normal bundle. In particular, since smooth manifolds can be given the structure of a CW complex, the Thom space of the normal bundle inherits a CW structure as well and therefore the Thom spectrum is a CW spectrum.

Theorem 2.3.8 (Thom Isomorphism)**.** *Let ν be the normal bundle of an embedding of a closed, smooth m-dimensional manifold M into $\mathbb{R}^n$. For the rational homology theory $H\mathbb{Q}$, the Thom isomorphism takes the following form:*

$$H\mathbb{Q} \wedge Th(\nu) \cong H\mathbb{Q} \wedge \Sigma^{n-m} M_+$$

or, in more classical notation:

$$\widetilde{H}_i(Th(\nu); \mathbb{Q}) \cong \widetilde{H}_i(\Sigma^{n-m} M_+; \mathbb{Q}) \cong \widetilde{H}_{i-n+m}(M_+; \mathbb{Q})$$

We finish this section by defining a concrete description of the evaluation and coevaluation maps for the dual pair M_+, $\Sigma^{-n} Th(\nu)$ conceptually following [7], but with updated notation. The following lemmas will be applied in the definitions. The first states that excision induces an isomorphism in **HoSpectra**, while the second gives a convenient morphism to interpret the smash product more like a cartesian product. Recall Proposition 2.2.16 for the notion of the suspension spectrum of a pair.

Lemma 2.3.9 (Excision)**.** *Let V be a metrizable space with subspaces V', U such that $V \subset int(V') \cup int(U)$. Let $U' = V' \cap U$. Then $(U, U') \hookrightarrow (V, V')$ induces an isomorphism $\Sigma^\infty(U, U') \cong \Sigma^\infty(V, V')$ in* **HoSpectra***.*

Lemma 2.3.10. *If (V, V') and (U, U') are pairs of spaces with $V' \hookrightarrow V$ and $U' \hookrightarrow U$,* *then define:*

$$(V, V') \times (U, U') = (V \times U, V' \times U \cup V \times U').$$

There exists a canonical morphism in **HoSpectra***:*

$$\Sigma^\infty(V, V') \wedge \Sigma^\infty(U, U') \to \Sigma^\infty((V, V') \times (U, U'))$$

which is an isomorphism if V' and U' are open in V and U respectively.

Since we will be working in **HoSpectra**$^\bigstar$, it suffices to define $\Sigma^n\epsilon$ and $\Sigma^n\eta$ instead. Identify S^n with the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ which is essentially the Thom space of the normal bundle for the inclusion of the one point space into $\mathbb{R}^n$ as the origin. For ν the normal bundle of an embedding of M into $\mathbb{R}^n$, identify $Th(\nu)$ with the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus M)$ and M_+ with the pair $(M, \emptyset)$.

Definition 2.3.11. *Suspending the map ϵ n times, $\Sigma^n\epsilon$ is the dotted line morphism which makes the following diagram commute:*

$$\epsilon : \Sigma^{-n}(\mathbb{R}^n, \mathbb{R}^n \setminus M) \wedge (M, \emptyset) \to S^0$$

$$\Downarrow$$

$$\begin{tikzcd}
(\mathbb{R}^n, \mathbb{R}^n \setminus M) \wedge (M, \emptyset) \arrow[r, dashed, "\Sigma^n\epsilon"] \arrow[d, "lemma"] & \Sigma^n S^0 \arrow[d, leftrightarrow, "\cong"] \\
(\mathbb{R}^n, \mathbb{R}^n \setminus M) \times (M, \emptyset) \arrow[d, equal, "def."] & S^n \arrow[d, leftrightarrow, "\cong"] \\
(\mathbb{R}^n \times M, (\mathbb{R}^n \setminus M) \times M) \arrow[r, "difference"] & (\mathbb{R}^n, \mathbb{R}^n \setminus 0)
\end{tikzcd}$$

On the bottom row, the difference map is $(v, x) \mapsto v - x$. The result of this map is guaranteed to be non-zero since $v \in \mathbb{R}^n \setminus M$ and $x \in M$.

Let V be a neighborhood of $\mathbb{R}^n$ which retracts to M by $r : V \to M$, and let B be a closed ball in $\mathbb{R}^n$ centered at zero which contains M.

Definition 2.3.12. *Suspending the map η n times,*

$$\eta : S^0 \to (M, \emptyset) \wedge \Sigma^{-n}(\mathbb{R}^n, \mathbb{R}^n \setminus M)$$

$$\Downarrow$$

$$\begin{tikzcd}
\Sigma^n S^0 \arrow[rr, dashed, "\Sigma^n \eta"] \arrow[d, leftrightarrow, "\cong"] & & (M, \emptyset) \wedge (\mathbb{R}^n, \mathbb{R}^n \setminus M) \arrow[d, leftrightarrow, "lemma"] \\
(\mathbb{R}^n, \mathbb{R}^n \setminus 0) & & (M, \emptyset) \times (\mathbb{R}^n, \mathbb{R}^n \setminus M) \arrow[d, equal, "def."'] \\
(\mathbb{R}^n, \mathbb{R}^n \setminus B) \arrow[u, "\iota"] \arrow[r, "\subset"] & (\mathbb{R}^n, \mathbb{R}^n \setminus M) & (M \times R^n, M \times (\mathbb{R}^n \setminus M)) \\
& (V, V \setminus M) \arrow[u, "j"] \arrow[r, "diagonal"] & (V \times \mathbb{R}^n, V \times (\mathbb{R}^n \setminus M)) \arrow[u, "r \times id"']
\end{tikzcd}$$

The map ι is represented by a pair of homotopy equivalences

$$\mathbb{R}^n \simeq \mathbb{R}^n \text{ and } \mathbb{R}^n \setminus 0 \simeq \mathbb{R}^n \setminus B.$$

Therefore ι is an isomorphism in **HoSpectra** *and j is an isomorphism by the excision lemma 2.3.9. On the bottom row, the diagonal map is $v \mapsto (v, v)$ where the first factor projects onto the zero section of the normal bundle and the second factor points in the normal fiber direction.*

Chapter 3

Trace in HoSpectra$^\bigstar$

3.1 Trace

Within **HoSpectra**$^\bigstar$ it is now possible to define the trace of an endomorphism $f : X \to X$ for any dualizable spectrum X. The results of this section are the foundation of the extension from fixed point theory in $\mathbf{Top}_+$ to fixed point theory in **HoSpectra**. A general categorical account of trace for any symmetric monoidal category with dualizable objects can be found in section A.3.

Definition 3.1.1. *Let X be a dualizable object with dual $X^\bigstar$. For a morphism of spectra $f : X \to X$, the* trace *of f is the composite:*

$$S^\infty \xrightarrow{\eta} X \wedge X^\bigstar \xrightarrow{f \wedge id} X \wedge X^\bigstar \xrightarrow{\gamma} X^\bigstar \wedge X \xrightarrow{\epsilon} S^\infty$$

The trace exhibits the following properties.

Proposition 3.1.2. *Let $X, Y \in$* **HoSpectra**$^\bigstar$.

1. *(Constant on $[S^\infty, S^\infty]$) For any $f : S^\infty \to S^\infty$, $tr(f) = f$.*

2. *(Invariance under dualization) If* $f : X \to X$*, then* $tr(f^\star) = tr(f)$

3. *(Multiplicativity) If* $f : X \to X$*,* $g : Y \to Y$*, then* $tr(f \wedge g) = tr(f) \cdot tr(g)$.

4. *(Cyclicity) If* $f : X \to Y$ *and* $g : Y \to X$*, then* $tr(fg) = tr(gf) = tr(\gamma \circ (f \wedge g))$.

5. *The trace is independent of choice of dual, evaluation or coevaluation.*

Proofs for 3.1.2 can be found in [7] or in [18] using string diagrams.

Corollary 3.1.3. *For* $X \in \mathbf{HoSpectra}^\star$ *and any* $f : X \to X$*,* $tr(tr(f)) = tr(f)$.

Proof. Since $tr(f)$ is itself an element of $[S^\infty, S^\infty]$, this follows directly from Proposition 3.1.2. □

3.2 Strong Monoidal Functors

Recall from category theory:

Definition 3.2.1. *A* symmetric monoidal functor $F : \mathfrak{C} \to \mathfrak{D}$ *between symmetric monoidal categories consists of a functor* F *and natural transformations:*

$$\nabla_{\mathfrak{A},\mathfrak{B}} : F(A) \otimes_{\mathfrak{D}} F(B) \to F(A \otimes_{\mathfrak{C}} B)$$

$$\mathfrak{i} : I_{\mathfrak{D}} \to F(I_{\mathfrak{C}})$$

which make the following diagram commute for all A, B *objects in* $\mathfrak{C}$*:*

$$\begin{tikzcd} F(A) \otimes_{\mathfrak{D}} F(B) \arrow[r, "\gamma"] \arrow[d, "\nabla_{A,B}"] & F(B) \otimes_{\mathfrak{D}} F(A) \arrow[d, "\nabla_{B,A}"] \\ F(A \otimes_{\mathfrak{C}} B) \arrow[r, "F(\gamma)"] & F(B \otimes_{\mathfrak{C}} A) \end{tikzcd}$$

F is called lax symmetric monoidal *if it also satisfies the coherence axioms for associativity and unitality. F is called* strong symmetric monoidal *if $\nabla_{A,B}$ and $\mathfrak{i}$ are isomorphisms for all objects A, B in $\mathfrak{C}$.*

The subscript on the monoidal product will be omitted in the future since it should be obvious from context. The next lemma is an essential ingredient in the results that follow.

Lemma 3.2.2. *Let $F : \mathfrak{C} \to \mathfrak{D}$ be a strong monoidal functor. Then F preserves duals. More precisely, if $A \in \mathfrak{C}$ a dualizable object with dual $A^\bigstar$, then $F(A)$ is dualizable with dual $F(A)^\bigstar = F(A^\bigstar)$. Moreover, for any endomorphsim $f : A \to A$,*

$$F(tr(f)) = tr(F(f))$$

Proof. F is a strong monoidal functor: $F(I_\mathfrak{C}) = I_\mathfrak{D}$, and for $f : A \to B$, we have $F(f) : F(A) \to F(B)$, and $F(id_A) = id_{F(A)}$. The property $F(A \otimes A^\bigstar) \cong F(A) \otimes F(A^\bigstar)$ is equivalent to the existence of a morphism ϕ which makes the following diagram commute:

$$\begin{tikzcd}
F(I_\mathfrak{C}) \arrow[r, "F\eta"] \arrow[d, equal] & F(A \wedge A^\bigstar) \\
I_\mathfrak{D} \arrow[r, dashed, "\phi"] & F(A) \wedge F(A^\bigstar) \arrow[u, "\tau"]
\end{tikzcd}$$

where τ is a natural transformation compatible with the associativity and commutativity of $\otimes$. From the definition of the trace:

$$\begin{tikzcd}
F(I_\mathfrak{C}) \arrow[r, "F\eta"] \arrow[d, equal] & F(A \otimes A^\bigstar) \arrow[r, "F\gamma"] \arrow[d, leftrightarrow, "\cong"] & F(A^\bigstar 0 \otimes A) \arrow[r, "F(id_{A^\bigstar} \otimes f)"] \arrow[d, leftrightarrow, "\cong"] & F(A^\bigstar \otimes A) \arrow[r, "F\epsilon"] \arrow[d, leftrightarrow, "\cong"] & F(I_\mathfrak{C}) \arrow[d, equal] \\
I_\mathfrak{D} \arrow[r, dashed, "\exists\phi"] & F(A) \otimes F(A^\bigstar) \arrow[r, "\gamma"] & F(A^\bigstar) \otimes F(A) \arrow[r, "id_{F(A^\bigstar)} \otimes F(f)"] & F(A^\bigstar) \otimes F(A) \arrow[r, dashed, "\exists\psi"] & I_\mathfrak{D}.
\end{tikzcd}$$

The top row is $F(tr(f))$ while the bottom row is $tr(F(f))$, and since all the squares commute $F(tr(f)) = tr(F(f))$.

To show that $F(A^\bigstar)$ is dualizable, consider the maps:

$$\alpha : I_{\mathfrak{D}} \xrightarrow{\mathfrak{i}} F(I_{\mathfrak{C}}) \xrightarrow{F(\eta)} F(A \otimes A^\bigstar) \xrightarrow{\nabla^{-1}_{A,A^\bigstar}} F(A) \otimes F(A^\bigstar)$$

and

$$\beta : F(A^\bigstar) \otimes F(A) \xrightarrow{\nabla_{A,A^\bigstar}} F(A^\bigstar \otimes A) \xrightarrow{F(\epsilon)} F(I_{\mathfrak{C}}) \xrightarrow{\mathfrak{i}^{-1}} I_{\mathfrak{D}}$$

These are the candidates for the coevaluation and evaluation for $F(A)^\bigstar$. It remains to show that they satisfy the triangle identities given in 2.3.1, which is a direct calculation. The composition:

$$(id_{F(A^\bigstar)} \otimes \beta) \circ (\alpha \otimes id_{F(A^\bigstar)}) = (id_{F(A^\bigstar)} \otimes \mathfrak{i}) \circ (id_{F(A^\bigstar)} \otimes F(\epsilon)) \circ (id_{F(A^\bigstar)} \otimes \nabla_{A,A^\bigstar})$$
$$\circ (\nabla^{-1}_{A,A^\bigstar} \otimes id_{F(A^\bigstar)}) \circ (F(\eta) \otimes id_{F(A^\bigstar)}) \circ (\mathfrak{i} \otimes id_{F(A^\bigstar)})$$

collapses down from the center terms outward. Functoriality implies $F(\epsilon) \circ F(\eta) = F(\epsilon \circ \eta) = F(id_{I_{\mathfrak{C}}}) = id_{I_{\mathfrak{D}}}$, and we're left with just $id_{F(A^\bigstar)}$. The proof for the second triangle identity is similar. □

There are several monoidal functors that we consider. For the following, let $\overline{\mathbf{CW}_+}$ denote the category consisting of finite CW complexes.

- $\Sigma^\infty : \overline{\mathbf{CW}}_+ \to \mathbf{HoSpectra}^\bigstar$ is not a strong monoidal functor. Actually, that $F(X^\bigstar) = F(X)^\bigstar$ in the case where $F = \Sigma^\infty : \overline{\mathbf{CW}}_+ \to \mathbf{HoSpectra}$ and $A = M_+$, was shown from scratch in 2.3.6, but notice in 2.2.14 we don't have $\Sigma(X \wedge Y) \cong \Sigma X \wedge \Sigma Y$.

- D : $\mathbf{HoSpectra}^\bigstar \to \mathbf{HoSpectra}^\bigstar$ is strong monoidal. In 2.3.3 and A.2.5, we show that the duality functor D which takes a spectrum to its dual is strong monoidal (and contravariant).

- $\widetilde{H}_*(-;\mathbb{Q})$: $\mathbf{HoSpectra}^\bigstar \to \mathbf{GrMod}_\mathbb{Q}$ is strong monoidal. In [7], Dold and Puppe define a singular complex functor $\widetilde{S}$ and a homology functor H, and object in the stable category called (M,n) which is a chain complex with $(M,n)_q = M$ if $q = n$ and 0 otherwise (here M is an R-module). Taking $M = \mathbb{Q}$ ($\mathbb{Q}$ is itself a $\mathbb{Q}$-module) we see that $(\mathbb{Q},n)$ has the homotopy type of a $K(\mathbb{Q},n)$ space. Thus in **HoSpectra**, $(\mathbb{Q},n)$ is represented by the Eilenberg-MacLane spectrum $H\mathbb{Q}$. The singular complex functor $\widetilde{S}$ in [7] is just the functor $H\mathbb{Q}\wedge -$. It fits into an updated diagram with the homology with $\mathbb{Q}$ coefficients functor:

$$\begin{tikzcd}
\mathbf{Top}_+ \arrow[r] \arrow[d, "\widetilde{S}"] & \overline{\mathbf{CW}}_+ \arrow[r, "\Sigma^\infty"] & \mathbf{HoSpectra}^\bigstar \arrow[lld, "-\wedge H\mathbb{Q}"] \\
\partial - \mathbf{Mod}_\mathbb{Q} \arrow[d, "H_*(-;\mathbb{Q})"] & & \\
\mathbf{GrMod}_\mathbb{Q} & &
\end{tikzcd}$$

$\widetilde{H}_*(-;\mathbb{Q}) = H_*(-;\mathbb{Q})\circ\widetilde{S}$ is the reduced homology functor. This functor fits with our earlier notion of rational homology via 2.2.10: $\widetilde{H}_*(X;\mathbb{Q}) = \pi_*(X\wedge H\mathbb{Q})$. The functor $\widetilde{S}$ is strong monoidal and since $\mathbb{Q}$ is a field, we have Künneth isomorphisms :

$$\widetilde{H}_*(X;\mathbb{Q})\otimes\widetilde{H}_*(X^\bigstar;\mathbb{Q}) \cong \widetilde{H}_*(X\wedge X^\bigstar;\mathbb{Q})$$

We have the immediate corollary:

Corollary 3.2.3. *Let $f : M \to M$ be a continuous map closed smooth manifolds. Then $tr(\widetilde{H}_*(f;\mathbb{Q}) = \widetilde{H}_*(tr(f);\mathbb{Q})$.*

In other words, computing the trace either on the chain complex level or after passing to homology yields the same result.

Chapter 4

Applications: Fixed Point and Coincidence Theory in HoSpectra*

Corollary **??** indicates that the trace is a quantity well-suited for computations in **HoSpectra***. The applications we will analyze are fixed point theory and the generalization to coincidence theory. To be used in both, the Lefschetz number is a homotopy invariant defined using homology. The relationship between the trace of an endomorphism and fixed points is given by Lefschetz's Fixed Point Theorem, of which we give a stable version connecting the trace to the fixed point index. The following is adapted from [4].

4.1 Fixed Points

First, recall the definition of the Lefschetz number. Even though we compute with rational coefficients, $L(f)$ is always an integer:

Definition 4.1.1. *Let $f : K \to K$ be a map of a finite polyhedron to itself. The*

Lefschetz number *is defined as:*

$$L(f) = \sum_i (-1)^i tr_i(f_*)$$

where $tr(f)$ *is the sum of the entries in a matrix representation of* f*. If* $\{\alpha_j\}$ *is a basis for* $H_i(K;\mathbb{Q})$ *and* f_{jk} *is the matrix representation of* f *with respect to that basis, then* $tr_i(f_*) = \sum_j^{dim(H_i(K;\mathbb{Q}))} f_{jj}$.

Recall that the identity object in $\mathbf{Top}_+$ is the two point space, S^0, and that the rational homology functor is strong monoidal and comes with Künneth isomorphisms. These two facts give the generalization to smooth manifolds:

Definition 4.1.2. *For a closed, smooth manifold* M *the* Lefschetz number *of* f *is the trace of* $f_* = \widetilde{H}_*(f;\mathbb{Q})$*:*

$$\begin{tikzcd} \widetilde{H}_*(S^0;\mathbb{Q}) \arrow[r, "\eta_*"] & \widetilde{H}_*(M;\mathbb{Q}) \wedge \widetilde{H}_*(M^\bigstar;\mathbb{Q}) \arrow[r, "f_* \wedge id_*"] & \widetilde{H}_*(M;\mathbb{Q}) \wedge \widetilde{H}_*(M^\bigstar;\mathbb{Q}) \arrow[dl, "\gamma"] \\ & \widetilde{H}_*(M^\bigstar;\mathbb{Q}) \wedge \widetilde{H}_*(M;\mathbb{Q}) \arrow[r, "\epsilon_*"] & \widetilde{H}_*(S^0;\mathbb{Q}). \end{tikzcd}$$

By Atiyah duality and $\widetilde{H}_*(S^0;\mathbb{Q}) \cong H_0(\{pt\};\mathbb{Q}) \cong \mathbb{Q}$ the above diagram simplifies to:

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\eta_*"] & \widetilde{H}_*(M_+;\mathbb{Q}) \wedge \widetilde{H}_*(\Sigma^{-n}Th(\nu);\mathbb{Q}) \arrow[r, "f_* \wedge id_*"] & \widetilde{H}_*(M_+;\mathbb{Q}) \wedge \widetilde{H}_*(\Sigma^{-n}Th(\nu);\mathbb{Q}) \arrow[dl, "\gamma"] \\ & \widetilde{H}_*(\Sigma^{-n}Th(\nu);\mathbb{Q}) \wedge \widetilde{H}_*(M_+;\mathbb{Q}) \arrow[r, "\epsilon_*"] & \mathbb{Q} \end{tikzcd}$$

and by the correspondence $H_{k+1}(\Sigma X) \cong H_k(X)$ with $k \geq 1$ (proved by the Mayer-Vietoris homology sequence of the pair $(\Sigma X, X)$),

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\eta_*"] & \widetilde{H}_*(M_+;\mathbb{Q}) \wedge H_{*+n}(Th(\nu)) \arrow[r, "f_* \wedge id_*"] & \widetilde{H}_*(M_+;\mathbb{Q}) \wedge H_{*+n}(Th(\nu)) \arrow[dl, "\gamma"] \\ & H_{*+n}(Th(\nu)) \wedge \widetilde{H}_*(M_+;\mathbb{Q}) \arrow[r, "\epsilon_*"] & \mathbb{Q} \end{tikzcd}$$

3.2.3 then states that the Lefschetz number can be calculated either by applying rational homology first and then taking the trace, or by taking the trace first, then applying homology.

$$L(f) = \sum_i (-1)^i tr_i(f_*) = \sum_i (-1)^i [\epsilon_* \circ \gamma \circ (f_* \wedge id_*) \circ \eta_*]_i$$
$$= \sum_i H_i(\epsilon \circ \gamma \circ (f \wedge id) \circ \eta) = \sum_i H_i(tr_i(f)).$$

Theorem 4.1.3 (Lefschetz fixed point theorem). *Let $f : M \to M$ be a continuous map of a closed, smooth manifold to itself. If $L(f) \neq 0$, then f has at least one fixed point. Equivalently, if f has no fixed points, then $L(f) = 0$.*

An intuitive sketch of the proof begins with the consideration of a simplicial approximation of M, and the analysis of the induced map on the associated chain complex. If f is a fixed-point-free map, then in every dimension the induced map on the chain complex will move every simplex to a different simplex entirely. For each dimension, the linear map between chain groups will thus have all zeros in the diagonal (with respect to any basis). And since there are no contributions from any dimension, the sum of all the traces is also zero.

Suppose instead we start with a closed, smooth manifold M and take its suspension spectrum $\Sigma^\infty M_+$ with corresponding dual $\Sigma^{\infty - n} Th(\nu)$. If $\Sigma^\infty f$ is the corresponding map in $[\Sigma^\infty M_+, \Sigma^\infty M_+]$, then a generalization of 4.1.3 is obtained by examining the evaluation and coevaluation maps 2.3.11 2.3.12. In particular, if M is embedded in $\mathbb{R}^n$ then we look at the n^{th} stem of $\Sigma^\infty f$:

Corollary 4.1.4 (Lefschetz-Hopf). *The fixed point index of f is equal to the trace of $\widetilde{H}_*(f; \mathbb{Q})$ (i.e. the Lefschetz number of $\widetilde{H}_*(f; \mathbb{Q})$).*

Proof. Iterating **??** we have $\Sigma^n tr(f) = tr(\Sigma^n f)$. Let F be the fixed point set of f (assumed to be compact) and V a neighborhood of M which retracts by $r : V \to M$. Then $(V, V \backslash M) \subset (V, V \backslash F)$ and Δ extends to a map $(V, V \backslash F) \to (V \times \mathbb{R}^n, V \times (\mathbb{R}^n \backslash F))$. Applying the concrete definitions of $\Sigma^n \epsilon$ and $\Sigma^n \eta$ into the definition of the trace:

$$
\begin{tikzcd}
\Sigma^n S^0 \arrow[rr, dashed, "\Sigma^n \eta"] \arrow[d, leftrightarrow, "\cong"] & & (M,\emptyset) \wedge (\mathbb{R}^n, \mathbb{R}^n \setminus M) \arrow[r, "(\Sigma^n f \wedge id)\circ\gamma"] \arrow[d, leftrightarrow] & (\mathbb{R}^n, \mathbb{R}^n \setminus M) \wedge (M,\emptyset) \arrow[r, dashed, "\Sigma^n \epsilon"] \arrow[d] & \Sigma^n S^0 \arrow[d, leftrightarrow, "\cong"] \\
(\mathbb{R}^n, \mathbb{R}^n \setminus 0) & & (M,\emptyset) \times (\mathbb{R}^n, \mathbb{R}^n \setminus M) \arrow[d, equal] & (\mathbb{R}^n, \mathbb{R}^n \setminus M) \times (M,\emptyset) \arrow[d, equal] & S^n \arrow[d, leftrightarrow, "\cong"] \\
(\mathbb{R}^n, \mathbb{R}^n \setminus B) \arrow[u, "\iota"] \arrow[r, "\subset"] & (\mathbb{R}^n, \mathbb{R}^n \setminus M) & (M \times R^n, M \times (\mathbb{R}^n \setminus M)) & (\mathbb{R}^n \times M, (\mathbb{R}^n \setminus M) \times M) \arrow[r] & (\mathbb{R}^n, \mathbb{R}^n \setminus 0) \\
 & (V, V \setminus M) \arrow[u, "j"] \arrow[r, "\Delta"] \arrow[d, "\subset"] & (V \times \mathbb{R}^n, V \times (\mathbb{R}^n \setminus M)) \arrow[u, "r \times id"] \arrow[d, "\subset"] & & \\
 & (V, V \setminus F) \arrow[r, "\Delta"] & (V \times \mathbb{R}^n, V \times (\mathbb{R}^n \setminus F)) & &
\end{tikzcd}
$$

From $(V, V \setminus F)$ to $(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$, we chase an element $v \in (V, V \setminus F)$:

$$v \mapsto (v, v) \mapsto (rv, v) \mapsto (frv, v) \mapsto (v, frv) \mapsto v - frv.$$

Applying $\widetilde{H}_*(-; \mathbb{Q})$ to the above composition then yields a map: $\widetilde{H}_*(id - (\Sigma^n f) \circ r; \mathbb{Q}) : \widetilde{H}_*((V, V \setminus F); \mathbb{Q}) \to \widetilde{H}_*((R^n, R^n \setminus 0); \mathbb{Q}) \cong \mathbb{Q}$.

Recall a definition from [5]:

Definition 4.1.5. *Denote the fundamental class of $\Sigma^n S^0 = S^n$ by $[S^n]$ (which can also be identified with the fundamental class $[*]$ of $(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$). $[F]$ is the image of $[S^n]$ under the map $H_n(S^n) \to H_n(S^n, S^n \setminus F) \cong H_n(V, V \setminus F)$ and is characterized by the property that its image under the map $\zeta : H_n(V, V \setminus F) \to H_n(V, V \setminus p)$ is equal to $[p]$ for all $p \in F$.*

The fixed point index *of f, is the integer I_f defined by:*

$$\widetilde{H}_*((id - f); \mathbb{Q})[F] = (id - f)_*([F]) = I_f[*]$$

where $[F]$ is the fundamental class of the set F.

Consequently, picking v to be the image of the fundamental class in $H_n(S^n; \mathbb{Q})$ implies that the map $\widetilde{H}_*(id - (\Sigma^n f) \circ r; \mathbb{Q}) : \widetilde{H}_*((V, V \setminus F); \mathbb{Q}) \to \widetilde{H}_*((R^n, R^n \setminus 0); \mathbb{Q})$ is multiplication by the fixed point index of f. Finally,

$$\begin{aligned} I_f =& \widetilde{H}_n(tr(\Sigma^{\infty+n} f); \mathbb{Q}) \\ =& \widetilde{H}_n(\Sigma^n tr(\Sigma^\infty f); \mathbb{Q}) \\ \cong& \widetilde{H}_0(tr(\Sigma^\infty f); \mathbb{Q}) \\ =& \widetilde{H}_*(tr(\Sigma^\infty f); \mathbb{Q}) \\ =& tr \widetilde{H}_*(f; \mathbb{Q}). \end{aligned}$$

□

4.2 Coincidences

Definition 4.2.1. *A* coincidence point *of* $f, g : M \to N$ *is a point* $x \in M$ *such that* $f(x) = g(x)$*.*

Reiterating the central question: given two functions $f, g : M \to N$, how can we tell if there exist coincidence points? As with fixed point theory, the topology of M plays a role in laying down some criteria for solutions to this question. The reduction to looking for the fixed points of a single function f is given by taking g to be the identity function. For an idea of how [12] and [4] present solutions consider the simple case of $f, g : [0, 1] \to [0, 1]$. The number of fixed points of f is equivalent to counting the intersections of the graph of f in $[0, 1] \times [0, 1]$ and the diagonal

$$\Delta = \{(x, y) \in [0, 1] \times [0, 1] | y = x\}.$$

With this interpretation, the number of coincidence points of f and g is therefore the number of intersections of the graph of f and the graph of g. Each graph is considered as a submanifold of the codomain product manifold $N \times N$, both having fundamental class equal to the image of the fundamental class of the M under the respective induced homology maps. We identify the diagonal in $N \times N$,

$$\Delta = \{(f(x), g(x)) \in N \times N | f(x) = g(x)\}$$

with a copy of N embedded as a submanifold.

Let $D : H_i(M^m) \to H^{m-i}(M)$ be the inverse Poincare duality map. Recall $- \bullet - : H_i(M) \otimes H_j(M) \to H_{i+j-n}(M)$ is the intersection product of homology classes defined by $a \bullet b = D^{-1}(D(b) \cup D(a)) = (D(b) \cup D(a)) \cap [M]$. Take $f, g : M \to N$ as usual and adopt the following notation: Γ_f and Γ_g are the graphs of f and g with orientation classes $[\Gamma_f]$ and $[\Gamma_g]$ respectively.

Definition 4.2.2. *The* Lefschetz coincidence number *of f and g is defined to be:*

$$L(f, g) = \epsilon_*([\Gamma_f] \bullet [\Gamma_g])$$

where $\epsilon_* : H_0(N \times N) \to \mathbb{Z}$ is the augmentation map taking on integer values.

Bredon's exposition of coincidence points generalized the Lefschetz number of a single map, by making use of the "shriek" or "transfer" map. From [4]:

Definition 4.2.3. *Let M and N be closed, oriented manifolds. If $f : M^m \to N^n$, then the homology* shriek f *is denoted by $f_!$ and defined by:*

$$f_! = D_M^{-1} \circ f^* \circ D_N : H_{n-p}(N) \to H^p(N) \to H^p(M) \to H_{m-p}(M).$$

We have the following computation theorem for the Lefschetz coincidence number:

Theorem 4.2.4 (Computational Lefschetz coincidence number formulae)**.** *In the following formulae, tr_i is the trace on the i^{th} degree (co)homology computed with $\mathbb{Q}$ or $\mathbb{Z}_p$ coefficients.*

1. $L(f, g) = \sum_i (-1)^i tr_i(f_* g_!)$
2. $L(f, g) = \sum_i (-1)^i tr_i(g_! f_*)$
3. $L(f, g) = (-1)^n L(g, f)$

[19] makes use of the computational theorem to reformulate the definition of the Lefschetz number of f and g:

Definition 4.2.5. *Given two maps $f, g : M \to N$, the* Lefschetz coincidence number *of f and g is given by:*

$$L(f,g) = \sum_i (-1)^i tr_i(g^* \circ (- \cap [M]) \circ (- \cap [N])^{-1} \circ f_*)$$

The trace tr_i is of the composition:

$$H_i(M;\mathbb{Q}) \xrightarrow{f_*} H_i(N;\mathbb{Q}) \xleftarrow[-\cap[N]]{} H^{m-i}(N;\mathbb{Q}) \xrightarrow{-\cap[M]} H^{m-i}(M;\mathbb{Q}) \xrightarrow{g^*} H_i(M;\mathbb{Q})$$

where $- \cap [N]$ and $- \cap [M]$ are the Poincare duality isomorphisms.

Along the same lines, the following theorem is a coincidence generalization of the Lefschetz fixed point theorem 4.1.3 coming from [12]and [19].

Theorem 4.2.6. *Suppose M and N are closed, smooth, orientable manifolds of the same dimension and $f, g : M \to N$. If f and g have no coincidence points, then the Lefschetz coincidence number of f and g is zero.*

Interpreting the above theorem in **HoSpectra**$^\bigstar$, [19] gives a proof of the next theorem following this definition:

Definition 4.2.7. *Define the notation $f \boxtimes g$ as the composition $M \to N \times N \to Th(\nu_\Delta)$ where the first map is $f \times g$ and second map is the Thom collapse map.*

Note that $f \boxtimes g$ will be homotopic to the constant map at the collapse point. Furthermore, if f and g have no coincidences (or are homotopic to maps without coincidences), then $\theta \circ (id \wedge (f \boxtimes g))$ will be homotopically trivial, and thus the homotopy class of the coincidence index of f and g relative to θ will be trivial.

Theorem 4.2.8. *Suppose M and N are closed, smooth manifolds. Let $\Delta \subset N \times N$ be the diagonal, with normal bundle ν_Δ and Thom space $Th(\nu_\Delta)$. Let*

$$\theta : Th(\nu_\Delta) \wedge K \to L \wedge M_+$$

be a stable map of spaces (or spectra) of K and L. If continuous maps $f, g : M \to N$ have no coincidences, then

$$\sum_i (-1)^i tr(\theta_* \circ (f \boxtimes g)_*)$$

is zero. The trace is of the composition:

$$\widetilde{H}_*(M_+ \wedge K; \mathbb{Q}) \xrightarrow{(f\boxtimes g)_*} \widetilde{H}_*(Th(\nu_\Delta) \wedge K; \mathbb{Q}) \xrightarrow{\theta_*} \widetilde{H}_*(L \wedge M_+; \mathbb{Q})$$

Definition 4.2.9. *The* coincidence index *of f and g relative to θ is the trace of the composition:*

$$K \wedge M_+ \xrightarrow{id\wedge f\boxtimes g} K \wedge Th(\nu_\Delta) \xrightarrow{\theta} M_+ \wedge L$$

The corresponding Lefschetz number *of f and g relative to θ is the trace of the above composition with the homology functor $\widetilde{H}_*(-;\mathbb{Q})$ applied:*

$$\begin{array}{ccc}
\widetilde{H}_*(K;\mathbb{Q}) \otimes \widetilde{H}_*(M_+;\mathbb{Q}) & \xrightarrow{id\otimes(f\boxtimes g)_*} & \widetilde{H}_*(K;\mathbb{Q}) \otimes \widetilde{H}_*(Th(\nu_\Delta);\mathbb{Q}) \\
& & \big\downarrow \theta_* \\
& & \widetilde{H}_*(M_+;\mathbb{Q}) \otimes \widetilde{H}_*(L;\mathbb{Q})
\end{array}$$

Recall that for dualizable objects and field coefficients, $\widetilde{H}_*(-;\mathbb{Q})$ is strong monoidal. Along with 3.2.2, we have that the map on rational homology induced by the coincidence index is equal to the Lefschetz number of f and g relative to θ.

$$\widetilde{H}_*((\theta \circ (id \wedge (f \boxtimes g))); \mathbb{Q}) = \theta_* \circ (id \otimes (f \boxtimes g)_*)$$

Appendix A

Appendix

A.1 Notes on CW-Approximation

For CW spectra, the "nicest" morphisms are those that preserve the homotopy type of the spectra. In the case that the homotopy type itself cannot be established, it is at least desirable to preserve the homotopy groups. This section is focused on the spectra derived from particular topological spaces, but we will start with some space-level definitions.

Definition A.1.1. *A map $f : X \to Y$ is called* n-connected *if the induced map on homotopy groups*

$$\pi_*(X, x) \to \pi_*(Y, f(x))$$

is, for all $x \in X$, an isomorphism in degrees $< n$ and an epimorphism in degree n.

Definition A.1.2. *$f : X \to Y$ is called a* weak homotopy equivalence *if it induces isomorphisms of the homotopy groups for all $n \geq 0$.*

$$\pi_n(X, x_0) \xrightarrow[\cong]{f_*} \pi_n(Y, y_0)$$

Hence a weak homotopy equivalence is an ∞-connected map. Weak homotopy equivalences have the additional advantage of preserving homology and cohomology groups according to the next proposition.

Proposition A.1.3. *A weak homotopy equivalence $f : X \to Y$ induces isomorphisms on the homology and cohomology groups:*

$$f_* : H_n(X, G) \cong H_n(Y, G) \text{ and}$$

$$f^* : H^n(Y, G) \cong H^*n(X, G).$$

for all $n \geq 0$.

More generally, weak homotopy equivalences behave very nicely under the functor $[X, \cdot]$:

Proposition A.1.4. *A weak homotopy equivalence $f : Y \to Z$ induces bijections*

$$[X, Y] \leftrightarrow [X, Z]$$

for any CW-complex X.

Weak homotopy equivalences play an important role in the approximation of a topological space by a CW-complex. We use a CW approximation $\hat{X}$ whose existence is given by the following propositions - the second being a consequence of the first.

Proposition A.1.5. *Let $f : A \to X$ be a continuous function between topological spaces. Then there exists for each $n \in \mathbb{N}$, a relative CW-complex $\hat{f} : A \hookrightarrow \hat{X}$ with an extension $\phi : \hat{X} \to X$ such that ϕ is n-connected.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "\hat{f}"'] & X \\ \hat{X} \arrow[ur, "\phi"'] & \end{tikzcd}$$

Proposition A.1.6. *Let A be a CW-complex and X a topological space. For every continuous function $f : A \to X$, there exists a relative CW-complex $\hat{f} : A \to \hat{X}$ that factors f followed by a weak homotopy equivalence.*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\hat{f}"'] & & X \\ & \hat{X} \arrow[ur, "\phi(WHE)"'] & \end{tikzcd}$$

The previous lemma is then iterated to give the spectrum version:

Proposition A.1.7. *For any suspension spectrum X, there exists a CW spectrum $\hat{X}$ and a homomorphism $\phi : \hat{X} \to X$ which is degree-wise a weak homotopy equivalence, and therefore a stable weak homotopy equivalence.*

We already know that there is an inclusion functor $\mathbf{CW}_+ \to \mathbf{Top}_+$. The previous propositions imply that there are functors in the opposite direction

$\mathbf{Top}_+ \to \mathbf{CW}_+$, $\mathbf{HoTop}_+ \to \mathbf{HoCW}_+$, the latter being an equivalence of categories.

The following theorem allows us to think of CW and cellular maps as being essentially equivalent.

Theorem A.1.8 (Cellular Approximation Theorem)**.** *Every map $f : X \to Y$ of CW-complexes is homotopic to a cellular map. If f is already a cellular map on a subcomplex $A \subseteq X$, then the homotopy may be taken to be stationary on A.*

A.2 Duality and $\mathbb{S}$-Theory: Connecting (co)Homotopy and (co)Homology

The goal of this appendix is to describe an instance of duality between two well-known theorems as well as the categorical setting in which this duality occurs. Briefly, the duality is first presented on the space-level with certain restrictions, and the passage to the category of suspension spectra yields a notationally cleaner duality depicted in a single commuative diagram. The final subsection is dedicated to exercises from Spanier's text [23] which catalog the algebraic properties in the category of spectra. We assume familiarity with integral homology and cohomology as well as a basic knowledge of category theory. Unless otherwise stated, omission of the coefficient group will imply that the (co)homology will be taken with integral coefficients.

A.2.1 Homotopy Groups and The Hurewicz Isomorphism Theorem

Let S^n be the standard n-sphere, Y be a space, $p \in S^n$ and $y \in Y$ designated base points. All maps considered in this section will be pointed maps. That is, if $f, g : S^n \to Y$ are maps, then $f(p) = g(p) = y$. Recall the one-point union construction for any pointed topological space (X, x): $X \vee X = X \times x \cup x \times X$.

Definition A.2.1. *Define $f + g : S^n \to Y$ to be the composite map*

$$\Delta \circ (f \vee g) \circ \Lambda : S^n \to S^n \vee S^n \to Y \vee Y \to Y$$

Here, Λ is the pinching map which takes an equator in S^n through p and collapses it to $p \times p \in S^n \times S^n$. Define $f \vee g$ component-wise $S^n \times p \cup p \times S^n \to Y \vee Y$.

Finally, the folding map Δ is applied. Δ is characterized by $\Delta(y, y') = \Delta(y', y) = y'$ for any $y' \in Y$.

It is a fact that this operation is well-defined on homotopy classes of maps. Thus,

Theorem A.2.2. *(Homotopy Groups)*

$\pi_n(Y, y) := ([S^n, Y], +)$ is a group. Further, an abelian group if $n > 1$.

The abelian group structure is established by a standard Eckmann-Hilton argument.

We now define the Hurewicz homomorphism and state a couple related results. Since $H_n(S^n) \cong \mathbb{Z}$, the choice of a generator $e \in H_n(S^n)$ determines a map called the Hurewicz homomorphism $\phi : \pi_n(Y) \to H_n(Y)$ by $\phi([f]) = f_*(e)$.

Theorem A.2.3. *(The Hurewicz Isomorphism Theorem)*

If Y is a space such that $\pi_i(Y) = 0$ for $i < n$, then

- *$\phi : \pi_i(Y, y) \to H_i(Y)$ is an isomorphism for $i \leq n$, and*
- *$\phi : \pi_{n+1}(Y, y) \to H_{n+1}(Y)$ is an epimorphism.*

For simply connected spaces, the Hurewicz theorem allows us to deduce the homotopy groups from the homology groups at least up to a fixed dimension. This is useful because in most cases, the homology groups are easy to calculate and, at least for finite dimensional spaces, have a dimension restriction.

Corollary A.2.4. *With the same assumptions on Y, $f, g : S^n \to Y$ are homotopic if and only if $f_*(e) = g_*(e)$.*

A.2.2 Cohomotopy Groups and The Hopf Classification Theorem

By reversing the direction of the maps from the previous section, we consider maps into S^n to define the cohomotopy groups of a space X. Let X be a space satisfying $dim(X) \leq 2n-2$. For $f, g : X \to S^n$ pointed maps, let (f, g) denote the product map

$$\begin{aligned} (f,g) :& X \to S^n \times S^n \\ & x \to (f(x), g(x)) \end{aligned}$$

By The Cellular Approximation Theorem, any map of CW-complexes may be homotoped to a cellular map; that is, $f(X^i) \subset (S^n)^i$ for all i. By the dimension restricton on X, (f, g) is homotopic to a cellular map with codomain $S^n \times S^n$. However, since $S^n \times S^n$ has no $2n-2$ skeleton, (f, g) is in fact homotopic to a cellular map h whose image lies in the n-skeleton of $S^n \times S^n$ (i.e., $(f, g) \simeq h : X \to S^n \vee S^n$). Following h by the folding map Δ, we arrive at an operation on the set $[X, S^n]$:

Definition A.2.5. *Let $[f]$ and $[g]$ be homotopy classes of maps in $[X, S^n]$. The element $[f] + [g]$ is represented by $\Delta \circ h$ where $h : X \to S^n \vee S^n$ is a cellular approximation of (f, g).*

Theorem A.2.6. *(Cohomotopy Groups)*

Let $dim(X) \leq 2n-2$. Then $\pi^n(X) := ([X, S^n], +)$ is an abelian group.

Again, since $H^n(S^n) \cong \mathbb{Z}$, the choice of a generator $s^* \in H^n(S^n)$ determines a homomorphism $\phi^* : \pi^n(X) \to H^n(X)$ by the formula $\phi^*([f]) = f^*s^*$ where $f : X \to S^n$, $[f] \in [X, S^n]$ and $f^* : H^*(S^n) \to H^*(X)$ is the induced map on cohomology. We now arrive at a theorem which can be thought of as dual to the Hurewicz Isomorphism theorem.

Theorem A.2.7. *(Hopf Classification Theorem)*

If X is a complex and $H^i(X) = 0$ for $i > n$, then ϕ^* is an isomorphism for $i \geq n$.

Comparing the Hurewicz Theorem and the Hopf Classification Theorem, we see that the homotopy and homology groups have been exchanged for cohomotopy and cohomology groups and the lowest non-trivial dimension for homology is replaced with the highest nontrivial dimension in cohomology.

A.2.3 Freudenthal Suspension Homomorphism

This section will lay the foundation for a generalization of the duality presented in first and second sections. We seek to remove the dimension restriction and describe a duality functor which takes one theorem to the other within the appropriate category. Previously, we considered the Hom-like functors $[S^n, \cdot]$ and $[\cdot, S^n]$ to define the homotopy and cohomotopy groups respectively. Now, consider $[S^k, S^n]$ when $k \leq 2n - 2$. As groups, the cohomotopy groups and homotopy groups are the same. Recall the suspension construction:

Definition A.2.8. *Let X be a space. $\Sigma(X) = \Sigma X = X \times [a, b]/X \times \{a\} \cup X \times \{b\}$ is called the* suspension of *X.*

Definition A.2.9. *(Freudenthal suspension homomorphism)*

Let $S^{k+1} = \Sigma(S^k)$ with distinguished points a, b and $S^{n+1} = \Sigma(S^n)$ with distinguished points a', b'. Then for $f : S^k \to S^n$, define $\Sigma f : S^{k+1} \to S^{n+1}$ as the map with the following properties:

- $\Sigma f(a) = a'$ *and* $\Sigma f(b) = b'$ *(Distinguished points mapped to distinguished points)*

- $\Sigma f|_{S^k} = f$ *(Σf maps S^k to S^n via f)*

- *Any line segment* $\overline{a - x}$ *(respx.* $\overline{b - x}$*)* $(x \in S^k)$ *maps linearly to the line segment* $\overline{a' - f(x)}$ *(resp.* $\overline{b' - f(x)}$*).*

We have the following diagram:

$$\begin{array}{ccc} S^k & \xrightarrow{f} & S^n \\ \Sigma\downarrow & & \downarrow\Sigma \\ S^{k+1} & \xrightarrow{\Sigma f} & S^{n+1} \end{array}$$

The map $\Sigma : Map(S^k, S^n) \to Map(S^{k+1}, S^{n+1})$, $f \mapsto \Sigma f$ *induces a homomorphism*

$$S : [S^k, S^n] \to [S^{k+1}, S^{n+1}],$$

$$[f] \mapsto [\Sigma f].$$

Abusing the notation slightly, let $S : \pi_k(S^n) \to \pi_{k+1}(S^{n+1})$ *which we'll call the* Freudenthal suspension homomorphism

Theorem A.2.10. *For* $k \leq 2n - 2$, $S : \pi_k(S^n) \to \pi_{k+1}(S^{n+1})$ *is an isomorphism.*

The following corollary justifies the usage of the terminology '"stable homotopy groups," the term "stable" meaning that the invariant occurs in all higher dimensions in essentially the same way.

Corollary A.2.11. *Fix an integer* d *and consider the sequence:*

$$\pi_{n+d}(S^n) \xrightarrow{S} \pi_{n+1+d}(S^{n+1}) \xrightarrow{S} \ldots \xrightarrow{S} \pi_{m+d}(S^m) \xrightarrow{S} \pi_{m+1+d}(S^{m+1}) \longrightarrow \ldots$$

All maps S *except for a finite number at the beginning of the sequence are isomorphisms.*

We note that if the above construction is repeated for the cohomotopy groups then dually, S is also an isomorphism for the cohomotopy groups for the range of values for which $\pi^n(S^k)$ is defined.

Generalizing to the case of arbitrary spaces X and Y, we have the following theorem:

Theorem A.2.12. *(Freudenthal Suspension Theorem) If $\pi_i(Y) = 0$ for $i < n$ and X is a complex with $dim(X) \leq 2n - 2$, then $S : [X, Y] \to [\Sigma X, \Sigma Y]$ is a bijection. In particular, for all values of k for which the cohomotopy groups are defined, S is an isomorphism of $\pi^k(X)$ with $\pi^{k+1}(\Sigma X)$. However, $S : \pi_k(Y) \to \pi_{k+1}(\Sigma Y)$ is not always an isomorphism (specifically, it is not always surjective).*

Examining the previous theorems, we see that there are a couple obstacles to a direct exchange of homotopy groups for cohomotopy groups. First, there is a dimension restriction on the definition of cohomotopy groups. Second, for based topological spaces, the Freudenthal Suspension Theorem also has a dimension restriction (this is a consequence of the Homotopy Excision Theorem). The goal of the next section will be to define a category in which these restrictions are removed, and the suspension homomorphism is always an isomorphim.

A.2.4 The $\mathbb{S}$-Category (Suspension Category)

We define a category which can be constructed from the category of based CW complexes.

Definition A.2.13. *A* CW-spectrum *is a sequence of based CW complexes $X := \{X_1, X_2, ...\}$ with inclusions $i_n : S^1 \wedge X = \Sigma X_k \to X_{k+1}$.*

Definition A.2.14. *Let X, Y be CW complexes and consider the sequence:*

$$[X, Y] \xrightarrow{S} [\Sigma X, \Sigma Y] \xrightarrow{S} \left[\Sigma^2 X, \Sigma^2 Y\right] \xrightarrow{S} ...$$

Denote by $\{X, Y\} = \lim_k [\Sigma^k X, \Sigma^k Y]$ *the set of* $\mathbb{S}$*-maps from* X *to* Y*.*

Define $\mathbb{S}$ to be the category consisting of CW-spectra as objects and $\mathbb{S}$-maps as morphisms. Appropriate notions of homotopy can be defined:

Remark A.2.15. *Some comments about the set* $\{X, Y\}$*:*

- *If* $f : \Sigma^k X \to \Sigma^k Y$ *for some* $k \geq 0$*, then* $\{f\}$ *denotes the* $\mathbb{S}$*-map determined by* f*. We say that* $\{f\}$ *is represented by* f*.*
- *Clearly* $\{X, Y\} = \{\Sigma X, \Sigma Y\}$*. Therefore, by using* $\mathbb{S}$*-maps instead of just cellular maps,* $\mathbb{S}$ *is a category in which the Freudenthal suspension homomorphism,* S*, is always a one-to-one correspondence.*
- *For* k *sufficiently large,* $[\Sigma^k X, \Sigma^k Y]$ *is an abelian group under composition and* $S : [\Sigma^k X, \Sigma^k Y] \to [\Sigma^{k+1} X, \Sigma^{k+1} Y]$ *is an abelian group homomorphism. This implies that* $\{X, Y\}$ *is an abelian group for any spaces* X, Y*.*
- *Composition defines a bilinear pairing:*

$$\{X, Y\} \circ \{Y, Z\} \to \{X, Z\}$$

A.2.5 $\mathbb{S}$- Duality as an Extension of Alexander Duality

Back in the $\mathbf{Top}_+$ or $\mathbf{CW}_+$, begin by considering S^n where n is sufficiently large to accommodate the embedding $X \hookrightarrow S^n$.

Theorem A.2.16. *(Alexander Duality) Let* X *be a locally contractible subspace of* S^n *and let* Y *be the complement of* X *in* S^n*. Then* $\widetilde{H}_k(X) \cong \widetilde{H}^{n-k-1}(Y)$*.*

From now on, we work in the $\mathbb{S}$-category with notation adopted from [23].

Definition A.2.17. *$D_nX \subset S^n \setminus X$ is* n-dual *to X if D_nX is an $\mathbb{S}$-deformation retract of $S^n \setminus X$.*

Parsing out the definition , an n-dual requires that if $i : D_nX \hookrightarrow S^n \setminus X$ represents an $\mathbb{S}$-map $\{i\}$ and there exists an $r : S^n \setminus X \to D_nX$ such that $\{r \circ i\} \in \{D_nX, D_nX\}$ can be represented by the the identity id_{D_nX}. The notation is indicative of an emphasis on the dimension of the embedding.

Remark A.2.18. *Some properties of n-duals:*

- *Given any subpolyhedron X of S^n, there always exists an n-dual, D_nX, to X.*
- *If D_nX is n-dual to X then X is n-dual to D_nX.*
- *SD_nX is $n+1$-dual to X.*

The next theorem introduces the duality maps D_* and some of their defining properties.

Theorem A.2.19. *Given CW spectra X, Y and respective n-duals D_nX, D_nY, there exists a unique map*

$$D_n : \{X, Y\} \to \{D_nY \to D_nX\}$$

with the following properties:

1. *If $i : X \hookrightarrow Y$ and $i' : D_nY \hookrightarrow D_nX$, then $D_n\{i\} = \{i'\}$.*

2. *Given* $\{f\} \in \{X,Y\}, \{g\} \in \{Y,Z\}$ *and* n*-duals* D_nX, D_nY, D_nZ*, we have the formula*

$$D_n(\{g\} \circ \{f\}) = D_n\{f\} \circ D_n\{g\}$$

3. *Considering* X, Y *as* n*-duals of* D_nX *and* D_nY *respectively, we have maps*

$$\{X,Y\} \xrightarrow{D_n} \{D_nY, D_nX\} \xrightarrow{D_n} \{X,Y\}$$

which compose to the identity.

4. D_n *is a group homomorphism.*

5. *Taking* SD_nX *and* SD_nY *as* $n+1$*-dual to* X *and* Y *respectively, we have*

$$SD_n = D_{n+1}$$

6. *Taking* D_nX *and* D_nY *as* $n+1$*-dual to* SX *and* SY *respectively, we have*

$$D_{n+1}S = D_n.$$

7. *Let* $\mathfrak{D}_n$ *denote the Alexander duality isomorphism. We have the following commutative diagram:*

$$\begin{array}{ccc} H_k(X) & \xrightarrow{f_*} & H_k(Y) \\ \mathfrak{D}_n \downarrow & & \downarrow \mathfrak{D}_n \\ H^{n-k-1}(D_nX) & \xrightarrow{(D_nf)^*} & H^{n-k-1}(D_nY) \end{array}$$

Remark A.2.20. *Properties (1)-(4) state that* D_n *is a group isomorphism* $\{X,Y\} \cong \{D_nY, D_nX\}$ *with functorial properties. Properties (5)-(6) state that* D_n*, as an isomorphism, does not depend on* n *and is stable under the suspension map,* S*.*

Definition A.2.21. *Define the* $\mathbb{S}$-homotopy groups *of* X *as* $\pi_k(X) := \{S^k, X\}$ *and the* $\mathbb{S}$-cohomotopy groups *of* X *as* $\pi^k(Y) := \{X, S^k\}$*.*

Reformulating the previous results in the $\mathbb{S}$-category:

- By Alexander Duality, S^p is n-dual to S^{n-p-1}.

  This implies that $D_n : \pi_k(X) \to \pi^{n-k-1}(D_n X)$ is an isomorphism.

- The homomorphisms $\phi : \pi_k(X) \to H_k(X)$ (Hurewicz) and $\phi^* : \pi^k(Y) \to H^k(Y)$ (Hopf classification) fit into a commutative diagram:

$$\begin{tikzcd} \pi_k(X) \arrow[r, "\phi"] \arrow[d, "D_n"'] & H_k(X) \arrow[d, "\mathfrak{D}_n"] \\ \pi^{n-k-1}(D_n X) \arrow[r, "\phi^*"] & H^{n-k-1}(D_n X) \end{tikzcd}$$

  The Hurewicz theorem now reads:

  Theorem A.2.22. *If* $H_i(X) = 0$ *for* $i < n$*, then* $\phi : \pi_i(X) \to H_i(X)$ *is an isomorphism for* $i \leq n$*.*

  And the Hopf Classification theorem now reads:

  Theorem A.2.23. *If* $H^i(Y) = 0$ *for* $i > n$*, then* $\phi^* : \pi^i(Y) \to H^i(Y)$ *is an isomorphism for* $i \geq n$*.*

A.2.6 Algebraic Properties in the $\mathbb{S}$-Category

This section is devoted to presenting some exercises from Spanier's text on Algebraic Topology. Let $X^\star$ denote the dual of X. We begin by more carefully defining the morphism sets in $\mathbb{S}$.

Definition A.2.24. *Given X, Y CW-spectra in $\mathbb{S}$, define $\{X,Y\} = \varinjlim[\Sigma^k X, \Sigma^k Y]$ as a terminal object in the directed category with one morphism $[\Sigma^j X, \Sigma^j Y] \to \{X,Y\}$ for each integer j such that*

$$\begin{tikzcd} \left[\Sigma^j X, \Sigma^j Y\right] \arrow[d, "\Sigma^i"] \arrow[dr, "\rho_j"] & \\ & \{X,Y\} \\ \left[\Sigma^{i+j} X, \Sigma^{i+j} Y\right] \arrow[ur, "\rho_{i+j}"'] & \end{tikzcd}$$

commutes for all i, j.

This morphism set satisfies the following universal property: if there exists morphisms $\phi_j : [\Sigma^j X, \Sigma^j Y] \to Z$ for which each diagram

$$\begin{tikzcd} \left[\Sigma^j X, \Sigma^j Y\right] \arrow[d, "\Sigma^i"] \arrow[dr, "\phi_j"] & \\ & Z \\ \left[\Sigma^{i+j} X, \Sigma^{i+j} Y\right] \arrow[ur, "\phi_{i+j}"'] & \end{tikzcd}$$

commutes for all j, then there exists a unique morphism $\Phi : \{X,Y\} \to Z$ such that

$$\begin{tikzcd} \left[\Sigma^j X, \Sigma^j Y\right] \arrow[r, "\rho_j"] \arrow[rr, bend right, "\phi_j"'] & \{X,Y\} \arrow[r, "\Phi"] & Z \end{tikzcd}$$

commutes.

We will freely exploit the following associated identities:

1. $S^0 \wedge X = X \wedge S^0 = X$

2. $\Sigma^k S^n = S^k \wedge S^n = S^{k+n}$

3. $\Sigma^k X = S^k \wedge X$

4. $\{X,Y\}_{-n} = \lim_{k}[\Sigma^{k-n} X, \Sigma^k Y]$

5. $\{X,Y\}_q = \{\Sigma^q X, Y\} = \{X, \Sigma^{-q} Y\}$

This concrete definition of $\{X, Y\}$ allows us to give an alternate definition of the relationship between a spectrum and its dual.

Definition A.2.25. *An* n-duality *is an element* $u \in \{X^\star \wedge X, S^0\}_{-n}$ *such that the maps determined by* u*:*

$$D_u : \{S^0, X^\star\}_q \to \{X, S^0\}_{q-n} \cong \{S^q \wedge X, S^0\}_{-n}$$

$$\{\alpha\} \mapsto u \circ (\{\alpha\} \wedge \{id_X\})$$

$$D^u : \{S^0, X\}_q \to \{X^\star, S^0\}_{q-n} \cong \{X^\star \wedge S^q, S^0\}_{-n}$$

$$\{\beta\} \mapsto u \circ (\{id_{X^\star}\} \wedge \{\beta\})$$

are isomorphisms. X *and* $X^\star$ *are* n-dual *if there exists a* $u \in \{X^\star \wedge X, S^0\}_{-n}$ *which is an* n*-duality.*

Because $\{S^0, X\}_q$ is the q^{th} stem of the stable homotopy groups of X and $\{X, S^0\}_{q-n}$ is the $(q-n)^{th}$ stem of the stable cohomotopy groups of X, the alternate defintion reformulates an n-duality in terms of the induced (co)homotopy maps. Were an Eilenberg-MacLane spectrum HG fixed, smashing with the spectrum X yields the homology or cohomology with coefficients in G in the following way [1]:

Definition A.2.26. *Let* X *be a spectrum,* HG *and Eilenberg-MacLane Spectrum. Then the* generalized reduced HG-homology of X *is* $\{S^0, HG \wedge X\}$*, and the* generalized reduced HG-cohomology of X *is* $\{HG \wedge X, S^0\}$*.*

In [9], the above definition comes in the form of a space-level theorem:

Theorem A.2.27. *Fix an Abelian group* G*, then for all CW complexes* X *and all* $n > 0$

there exist natural bijections:

$$T : [X, K(G,n)] \to H^n(X, G)$$

$$[f] \mapsto f^*(u)$$

where u is the fundamental cohomology class in $H^n(K(G,n), G)$.

Given $\{\alpha\} \in \{S^0, X^\bigstar\}_q$, a concrete description of the dual map

$$D_u(\{\alpha\}) \in \{\Sigma^{q-n} X, S^0\} = \{X, S^0\}_{q-n}$$

is given by the following diagram:

$$\begin{tikzcd}
\Sigma^{q-n} X \arrow[d, equal] \arrow[ddddrrr, bend left, "D_u(\{\alpha\})"] & & & \\
S^{q-n} \wedge X \arrow[d, equal] & & & \\
S^0 \wedge S^{q-n} \wedge X \arrow[r, "\alpha \wedge id \wedge id"] & \Sigma^{-q} X^\bigstar \wedge S^{q-n} \wedge X \arrow[d, equal] & & \\
& X^\bigstar \wedge S^{-n} \wedge X \arrow[d, equal] & & \\
& S^{-n} \wedge X^\bigstar \wedge X \arrow[r, "id \wedge u"] & S^{-n} \wedge S^n \arrow[r, equal] & S^0
\end{tikzcd}$$

We have the following results:

Proposition A.2.28. *Let X, Y be spectra with closed subspectra A, B respectively. There exist pairings*

$$\{(X,A),(Y,B)\}_p \otimes H_q(X,A) \to H_{p+q}(Y,B)$$

$$\{\alpha\} \otimes z \mapsto \Sigma^{-k}(\alpha_*(\Sigma^{p+k} z))$$

$$\{(X,A),(Y,B)\}_p \otimes H^r(Y,B) \to H_{r-p}(X,A)$$

$$\{\alpha\} \otimes u \mapsto \Sigma^{-k-p}(\alpha^*(\Sigma^k u))$$

Proof. The spectrum map $\{\alpha\} \in \{X, Y\}_p$ can be represented by a space-level map $\alpha : \Sigma^{k+p}(X, A) \to \Sigma^k(Y, B)$, with induced degree-preserving maps

$$\alpha^* : H^*(\Sigma^k(Y, B)) \to H^*(\Sigma^{k+p}(X, A)) \text{ and } \alpha_* : H_*(\Sigma^{k+p}(X, A)) \to H_*(\Sigma^k(Y, B)).$$

We make use of the following suspension isomorphisms:

$$\widetilde{H}^*(X, A) \cong \widetilde{H}^{*+i}(\Sigma^i(X, A))$$

$$\widetilde{H}_*(X, A) \cong \widetilde{H}_{*+i}(\Sigma^i(X, A))$$

Given $\{\alpha\} \in \{(X, A), (Y, B)\}_p$, pick an element $z \in H_q(X, A)$ and follow it through the diagram:

$$\begin{array}{ccccccc} H_q(X, A) & \xrightarrow{\Sigma^{p+k}} & H_{q+p+k}(\Sigma^{p+k}(X, A)) & \xrightarrow{\alpha_*} & H_{q+p+k}(\Sigma^k(Y, B)) & \xrightarrow{(\Sigma^k)^{-1}} & H_{q+p}(Y, B) \\ z & \longmapsto & \Sigma^{p+k} z & \longmapsto & \alpha_*(\Sigma^{p+k} z) & \longmapsto & \Sigma^{-k}(\alpha_*(\Sigma^{p+k} z)) \end{array}$$

It is worth noting that $(\Sigma^k)^{-1} = \Sigma^{-k}$ which is formal "desuspension". Since we are interested only in stable objects, we allow the formal manipulation of the dimension of spectra because this essentially amounts to reindexing the spectrum. The cohomology case follows a similar argument. Pick an element $u \in H^r(Y, B)$:

$$\begin{array}{ccccccc} H^r(Y, B) & \xrightarrow{\Sigma^k} & H^{r+k}(\Sigma^k(Y, B)) & \xrightarrow{\alpha^*} & H_{r+k}(\Sigma^{k+p}(X, A)) & \xrightarrow{(\Sigma^{k+p})^{-1}} & H^{r-p}(X, A) \\ u & \longmapsto & \Sigma^k u & \longmapsto & \alpha^*(\Sigma^k u) & \longmapsto & \Sigma^{-k-p}(\alpha^*(\Sigma^k u)) \end{array}$$

□

Proposition A.2.29. *If $f : S^p \wedge S^q \to S^{p+q}$ is a homeomorphism, then $\{f\} \in \{S^p \wedge S^q, S^0\}_{-p-q}$ is an $n = p + q$-duality.*

Proof. Recall:

$$\{S^p \wedge S^q, S^0\}_{-p-q} = \{S^{p+q}, S^0\}_{-p-q} = \{S^0, S^0\} = \varinjlim[\Sigma^k S^0, \Sigma^k S^0] = \varinjlim[S^k, S^k] = \varinjlim \pi_k(S^k) = \mathbb{Z}$$

We prove that the induced maps of $\{f\}$: D_f and D^f are isomorphisms of $\{S^0, S^p\}_k \to \{S^q, S^0\}_{k-(p+q)}$ and $\{S^0, S^q\}_k \to \{S^p, S^0\}_{k-(p+q)}$ respectively.

Let $\{\alpha\}$ be an element of $\{S^0, S^p\}_k$. Since f is a homeomorphism, the spectrum map $\{f\}$ is an isomorphism. This is due to the fact that f represents the map $\{f\}$ and f^{-1} is represented by the map $\{f^{-1}\}$, implying that $f \circ f^{-1} = id_{S^q}$ can be represented by the map $\{f\} \circ \{f^{-1}\} = \{id_{S^q}\}$. Further, since $\{f\} \in \{S^p \wedge S^q, S^0\}_{-p-q} \cong \mathbb{Z}$, the map $\{f\}$ amounts to multiplication by either 1 or -1 (the only invertible elements in $\mathbb{Z}$). Then by definition, $D_f(\{\alpha\}) = \{f\} \circ (\{\alpha\} \wedge 1_{S^q}) = \pm\{\alpha\} \wedge 1_{S^q}$ showing that this correspondence is obvously an isomorphism. A similar argument shows that D^f is also an isomorphism. □

Proposition A.2.30. *If $u \in \{X^\star \wedge X, S^0\}_{-n}$ is an n-duality and $u' \in \{X \wedge X^\star, S^0\}$ is the image of u under the homeomorphism $T : X^\star \wedge X \to X \wedge X^\star$, then u' is also an n-duality.*

Proof. Observe that for any $\{\alpha\} \in \{S^0, X\}_q$:

$$\begin{aligned} D_{u'}(\{\alpha\}) &= u' \circ (\{\alpha\} \wedge id_{X^\star}) \\ &= \{T\} \circ u \circ (\{\alpha\} \wedge id_{X^\star}) \\ &= \{T\} \circ D^u(\{\alpha\}). \end{aligned}$$

Since T is a homeomorphism, and D^u is an isomorphism, we have that $\{T\} \circ D^u = D_{u'}$ is an isomorphism. A similar argument gives that $D_{u'}$ is an isomorphism. Thus u' is an n-duality. $\square$

Proposition A.2.31. *If $u \in \{X^\star \wedge X, S^0\}_{-n}$ is an n-duality, then for any Z, Y there exist isomorphisms:*

$$D_u : \{Y, Z \wedge X^\star\}_q \cong \{Y \wedge X, Z\}_{q-n}$$

$$D^u : \{Y, X \wedge Z\}_q \cong \{X^\star \wedge Y, Z\}_{q-n}$$

such that for all $\{\alpha\} \in \{Y, Z \wedge X^\star\}_q$ and $\{\beta\} \in \{Y, X \wedge Z\}_q$

$$D_u(\{\alpha\}) = (id_Z \wedge u) \circ (\{\alpha\} \wedge id_X)$$

$$D^u(\{\beta\}) = (u \wedge id_Z) \circ (id_{X^\star} \wedge \{\beta\}).$$

Proof. Suppose Z is a sphere and Y is an m-dimensional CW-complex. Then we have cofibrations of the form:

$$Y^{m-1} \hookrightarrow Y^m \twoheadrightarrow Y^m/Y^{m-1} = \bigvee_{i=0}^{k} S_i^m$$

Apply the contravariant functor $\{\cdot, Z \wedge X^\star\}$ to the cofibration sequence to get the long exact sequence:

$$\cdots \longrightarrow \{\bigvee_{i=1}^{k} S_i^m, Z \wedge X^\star\}_q \longrightarrow \{Y^m, Z \wedge X^\star\}_q \longrightarrow \{Y^{m-1}, Z \wedge X^\star\}_q \longrightarrow \cdots$$

which, under the assumption that Z is a sphere, is equivalent to:

$$\cdots \longrightarrow \{\bigvee_{i=1}^{k} S_i^m, S^l \wedge X^\star\}_q \longrightarrow \{Y^m, S^l \wedge X^\star\}_q \longrightarrow \{Y^{m-1}, S^l \wedge X^\star\}_q \longrightarrow \cdots$$

By induction on m, our induction hypothesis guarantees an isomorphism of the groups $\{Y^{m-1}, S^l \wedge X^\star\}_q$ and $\{Y^m \wedge X, S^l\}_{q-n}$. Suppose $k = 1$:

$$
\begin{tikzcd}
\cdots \arrow[r] & \{S^m, S^l \wedge X^\bigstar\}_q \arrow[r] \arrow[d, "\cong"] & \{Y^m, S^l \wedge X^\bigstar\}_q \arrow[r] \arrow[d] & \{Y^{m-1}, S^l \wedge X^\bigstar\}_q \arrow[r] \arrow[d, "\cong"] & \{S^m, S^l \wedge X^\bigstar\}_{q+1} \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & \{S^m \wedge X, S^l\}_{q-n} \arrow[r] & \{Y^m \wedge X, S^l\}_{q-n} \arrow[r] & \{Y^{m-1} \wedge X, S^l\}_{q-n} \arrow[r] & \{S^m \wedge X, S^l\}_{q+1} \arrow[r] & \cdots
\end{tikzcd}
$$

From left to right: the first and fourth vertical arrows are the given duality isomorphism D_u, the third is by the induction hypothesis. By the Five Lemma, we establish an isomorphism for the second vertical arrow. Thus $\{Y^m, S^l \wedge X^\bigstar\}_q \cong \{Y^m \wedge X, S^l\}_{q-n}$.

Assume the isomorphism has been established for $j = k$, we now show that it is also true for $j = k+1$. Since $\bigvee_{i=1}^{k+1} S_i^m \cong \bigvee_{i=1}^{k} S_i^m \vee S^m$, the group $\{\bigvee_{i=1}^{k} S_i^m \vee S^m, Z \wedge X^\bigstar\}$ splits as $\{\bigvee_{i=1}^{k} S_i^m, Z \wedge X^\bigstar\} \oplus \{S^m, Z \wedge X^\bigstar\}$.

$$
\begin{tikzcd}
\cdots \arrow[r] & \{\bigvee_{i=1}^{k+1} S_i^m, S^l \wedge X^\bigstar\}_q \arrow[r] \arrow[d, "\cong"] & \{Y^m, S^l \wedge X^\bigstar\}_q \arrow[r] \arrow[dd] & \{Y^{m-1}, S^l \wedge X^\bigstar\}_q \arrow[dd] \arrow[ddddl] \\
& \{\bigvee_{i=1}^{k} S_i^m, S^l \wedge X^\bigstar\}_q \oplus \{S^m, S^l \wedge X^\bigstar\} \arrow[d, "\cong"] & & \\
\cdots \arrow[r] & \{\bigvee_{i=1}^{k} S_i^m \wedge X, S^l\}_{q-n} \oplus \{S^m \wedge X, S^l\}_{q-n} \arrow[r] \arrow[d, "\cong"] & \{Y^m \wedge X, S^l\}_{q-n} \arrow[r] & \{Y^{m-1} \wedge X, S^l\}_{q-n} \arrow[ddddl] \\
& \{\bigvee_{i=1}^{k+1} S_i^m \wedge X, S^l\}_{q-n} & & \\
& & \{\bigvee_{i=1}^{k+1} S_i^m, S^l \wedge X^\bigstar\}_{q+1} \arrow[r] \arrow[d, "\cong"] & \cdots \\
& & \{\bigvee_{i=1}^{k} S_i^m, S^l \wedge X^\bigstar\}_{q+1} \oplus \{S^m, S^l \wedge X^\bigstar\} \arrow[d, "\cong"] & \\
& & \{\bigvee_{i=1}^{k} S_i^m \wedge X, S^l\}_{q+1-n} \oplus \{S^m \wedge X, S^l\}_{q+1-n} \arrow[r] \arrow[d, "\cong"] & \cdots \\
& & \{\bigvee_{i=1}^{k+1} S_i^m \wedge X, S^l\}_{q+1-n} &
\end{tikzcd}
$$

The isomorphisms

$$\{\bigvee_{i=1}^{k} S_i^m, S^l \wedge X^\bigstar\}_q \oplus \{S^m, S^l \wedge X^\bigstar\} \text{ and}$$
$$\{\bigvee_{i=1}^{k} S_i^m \wedge X, S^l\}_{q-n} \oplus \{S^m \wedge X, S^l\}_{q-n}$$

are given by the $k+1$-fold component-wise map $D_u \oplus \cdots \oplus D_u$. Again, the Five Lemma establishes the vertical arrow in the second column as an isomorphism, and we have the existence of the isomorphism

$$\widetilde{D}_u : \{Y, Z \wedge X^\bigstar\}_q \to \{Y \wedge X, Z\}_{q-n}$$

whenever Z is a sphere.

Now, suppose Y is a sphere and Z is an m-dimensional CW-complex. Applying the covariant functor $\{Y \wedge X^\bigstar, \cdot\}$ to an analogous cofibration sequence for Z gives a similar diagram and argument establishing the isomorphism

$$\widetilde{D}^u : \{Y, Z \wedge X\}_q \to \{Y \wedge X^\bigstar, Z\}_{q-n}$$

whenever Y is a sphere. Combining these two results gives the result when Y and Z are arbitrary. □

Proposition A.2.32. *Given n-dualities $u \in \{X^\bigstar \wedge X, S^0\}_{-n}$ and $v \in \{Y^\bigstar \wedge Y, S^0\}_{-n}$, define the isomorphism*

$$D(u,v) : \{X, Y\}_q \cong \{Y^\bigstar, X^\bigstar\}_q$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\{X, Y\}_q \arrow[rr, "D(u,v)"] \arrow[dr, "D_v"'] & & \{Y^\bigstar, X^\bigstar\}_q \arrow[dl, "D_u"] \\
& \{Y^\bigstar \wedge X, S^0\}_{q-n} &
\end{tikzcd}$$

If u' is the image of u under the homeomorphism $X^\bigstar \wedge X \to X \wedge X^\bigstar$, and v' is the image of v under the homeomorphism $Y^\bigstar \wedge Y \to Y \wedge Y^\bigstar$, then

$$D(v', u') = D(u, v)^{-1} : \{Y^\bigstar, X^\bigstar\}_q \cong \{X, Y\}_q.$$

Proof. From proposition A.2.29, we know that switching the order of the smash product for an n-duality yeilds an n-duality: $D_v = D^{v'}$, $D^v = D_{v'}$, $D_u = D^{u'}$ and $D^u = D_{u'}$. Thus $(D_u)^{-1} = (D^{u'})^{-1}$, $(D^u)^{-1} = (D_{u'})^{-1}$, etc.

By definition, $D(u, v) = D_u^{-1} \circ D^v$. Taking the inverse formally gives the result:

$$\begin{aligned}(D(u, v))^{-1} &= (D_u^{-1} \circ D^v)^{-1} \\ &= (D^v)^{-1} \circ (D_u^{-1})^{-1} \\ &= D_{v'}^{-1} \circ D_u \\ &= D_{v'}^{-1} \circ D^{u'} = D(v', u').\end{aligned}$$

□

Proposition A.2.33. *If $u \in \{X^\bigstar \wedge X, S^0\}_{-n}$, $v \in \{Y^\bigstar \wedge Y, S^0\}_{-n}$ and $w \in \{Z^\bigstar \wedge Z, S^0\}_{-n}$ are n-dualities and $\{\alpha\} \in \{X, Y\}_p$, $\{\beta\} \in \{Y, Z\}_q$, then in $\{Z^\bigstar, X^\bigstar\}_{p+q}$,*

$$D(u, w)(\{\beta\} \circ \{\alpha\}) = (D(u, v)\{\alpha\}) \circ (D(v, w)\{\beta\})$$

Proof. Using the definition of $D(\cdot, \cdot)$ and the isomorphism $\{X, Z\} \cong \{X \wedge Y, Z \wedge Y\}$ given by $\{f\} \mapsto \{f\} \wedge \{1_Y\}$, consider the diagrams:

$$\begin{tikzcd} \{X, Z\} \cong \{X \wedge Y, Z \wedge Y\} \arrow[rr, "D(u,w)"] \arrow[dr, "D^w"'] & & \{Z^\bigstar \wedge Y, X^\bigstar \wedge Y\} \cong \{Z^\bigstar, X^\bigstar\} \\ & \{Z^\bigstar \wedge X \wedge Y, Y\} \arrow[ur, "D_u^{-1}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} \{X,Y\} \arrow[rr, "D(u,v)"] \arrow[dr, "D_v"'] & & \{Y^\bigstar, X^\bigstar\} \\ & \{Y^\bigstar \wedge X, S^0\} \arrow[ur, "D_u^{-1}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} \{Y,Z\} \arrow[rr, "D(v,w)"] \arrow[dr, "D_w"'] & & \{Z^\bigstar, Y^\bigstar\} \\ & \{Z^\bigstar \wedge Y, S^0\} \arrow[ur, "D_v^{-1}"'] & \end{tikzcd}$$

which show that $D(u,v)\{\alpha\} \in \{Y^\bigstar, X^\bigstar\}_p,\ D(v,w)\{\beta\} \in \{Z^\bigstar, Y^\bigstar\}_q$, therefore

$D(u,v)\{\alpha\} \circ D(v,w)\{\beta\} \in \{Z^\bigstar, Y^\bigstar\}_{p+q}$. □

Assume that $f : X^\bigstar \wedge X \to S^n,\ g : Y^\bigstar \wedge Y \to S^n$ are such that $\{f\}$, $\{g\}$ are n-dualitites. Further, let $\alpha : X \to Y$ and $\beta : Y^\bigstar \to X^\bigstar$ be maps satisfying

$$f \circ (\beta \wedge id_X) \simeq g \circ (id_{Y^\bigstar} \wedge \alpha) : Y^\bigstar \wedge X \to S^n$$

(which implies that $D(\{f\},\{g\})\{\alpha\} = \{\beta\}$).

Let C_α and C_β be the mapping cones of α and β respectively, and consider the exact sequences:

$$X \xrightarrow{\alpha} Y \xrightarrow{i} C_\alpha \xrightarrow{k} \Sigma X \xrightarrow{\Sigma\alpha} \Sigma Y$$

$$Y^\bigstar \xrightarrow{\beta} X^\bigstar \xrightarrow{i'} C_\beta \xrightarrow{k'} \Sigma Y^\bigstar \xrightarrow{\Sigma\beta} \Sigma X^\bigstar$$

Proposition A.2.34. *There exists a map $h : C_\beta \wedge C_\alpha \to S^{n+1}$ such that the following rectangles are homotopy commutative:*

$$\begin{tikzcd} X^\bigstar \wedge C_\alpha \arrow[r, "id\wedge k"] \arrow[d, "i'\wedge id"] & X^\bigstar \wedge \Sigma X \arrow[r, leftrightarrow] & \Sigma(X^\bigstar \wedge X) \arrow[d, "\Sigma f"] \\ C_\beta \wedge C_\alpha \arrow[rr, "h"] & & S^{n+1} \end{tikzcd}$$

$$\begin{array}{ccc} C_\beta \wedge Y & \xrightarrow{\quad id \wedge i \quad} & C_{beta} \wedge C_\alpha \\ \big\downarrow{\scriptstyle k' \wedge id} & & \big\downarrow{\scriptstyle h} \\ \Sigma Y^\bigstar \wedge Y \leftrightarrow \Sigma(Y^\bigstar \wedge Y) & \xrightarrow{\Sigma g} & S^{n+1} \end{array}$$

Furthermore, $\{h\} \in \{C_\beta \wedge C_\alpha\}_{-n-1}$ *is an* $n+1$*-duality.*

Proposition A.2.35. *For any* X*, there exists an integer* n *for which there exists a space* $X^\bigstar$ *and an* n*-duality* $u \in \{X^\bigstar \wedge X, S^0\}_{-n}$*.*

A.3 Duality and Trace for Symmetric Monoidal Categories

This appendix takes a generalized categorical approach to the definition of trace. We review some of the major results by following the account of [7].

Definition A.3.1. *A* symmetric monoidal category*,* $\mathfrak{C}$*, is a category with identity object* I *and bifunctor* $\otimes : (A, B) \mapsto A \otimes B$ *which satisfy the following:*

1. $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$
2. $I \otimes A \cong A \otimes I \cong A$
3. $\gamma : A \otimes B \cong B \otimes A$

Definition A.3.2. *An object* $A \in \mathfrak{C}$ *is* weakly dualizable *with* weak dual $A^\bigstar$ *if the object* $A^\bigstar$ *represents the functor*

$$X \mapsto Hom(X \otimes A, I)$$

I.e. there is a bijection

$$Hom(X \otimes A, I) \cong Hom(X, A^\bigstar)$$

In particular, we have that the identity object I is dualizable with $I^\bigstar = I$. In addition to dualizing objects in $\mathfrak{C}$, we also define dual morphisms.

Definition A.3.3. *Let A, B be dualizable objects with respective duals $A^\bigstar, B^\bigstar$. If $f : A \to B$ is a morphism, then there exists a contravariant functor*

$$D : \mathfrak{C}^\bigstar \longrightarrow \mathfrak{C}$$

$$A \longmapsto A^\bigstar$$

$$(f : A \to B) \mapsto (f^\bigstar : B^\bigstar \to A^\bigstar)$$

The map $f^\bigstar$ is referred to as the dual morphism *of f. The functor D satisfies the following commutative square:*

$$\begin{array}{ccc} B^\bigstar \otimes A & \xrightarrow{id \otimes f} & B^\bigstar \otimes B \\ {\scriptstyle f^\bigstar \otimes id}\downarrow & & \downarrow{\scriptstyle \epsilon_{B^\bigstar}} \\ A^\bigstar \otimes A & \xrightarrow{\epsilon_A} & I. \end{array}$$

Definition A.3.4. *If $A^\bigstar$ is the weak dual of A, then there exists a morphism called the* evaluation

$$\epsilon = \epsilon_A : A^\bigstar \otimes A \to I$$

Definition A.3.5. *Suppose objects $A, B, B \otimes A$ are all weakly dualizable. Then define $\mu = \mu_{AB} : A^\bigstar \otimes B^\bigstar \to (B \otimes A)^\bigstar$ to be the map corresponding to the composition:*

$$A^\bigstar \otimes B^\bigstar \otimes B \otimes A \xrightarrow{id \otimes \epsilon_B \otimes id} A^\bigstar \otimes I \otimes A \cong A^\bigstar \otimes A \xrightarrow{\epsilon_A} I$$

under the bijection: $Hom((A^\bigstar \otimes B^\bigstar \otimes (B \otimes A), I) \cong Hom(A^\bigstar \otimes B^\bigstar, (B \otimes A)^\bigstar)$.

Definition A.3.6. *An object A is* strongly dualizable *if:*

$$A \cong (A^\bigstar)^\bigstar \text{ and } A \otimes A^\bigstar \cong (A \otimes A^\bigstar)^\bigstar$$

(i.e. $\mu_{A,A^\bigstar}$ is an isomorphism). In this case, $A^\bigstar$ is called the strong dual *of A.*

For the purposes of this discussion, we will restrict our attention to only those objects of $\mathfrak{C}$ which are strongly dualizable and will from now on omit the qualifier "strongly." Let $\mathfrak{C}^\bigstar$ denote the subcategory of $\mathfrak{C}$ consisting of all the dualizable objects.One advantage to working only with dualizable objects is that they come equipped with a map which is dual to the evaluation.

Definition A.3.7. *Let $A \in \mathfrak{C}$. Define the* coevaluation $\eta = \eta_A : I \to A \otimes A^\bigstar$ *to be the composition*

$$I = I^\bigstar \xrightarrow{\epsilon^\bigstar} (A^\bigstar \otimes A)^\bigstar \xrightarrow[\cong]{\mu^{-1}} A^\bigstar \otimes (A^\bigstar)^\bigstar \xrightarrow[\cong]{id \otimes \delta^{-1}} A^\bigstar \otimes A \xrightarrow[\cong]{\gamma} A \otimes A^\bigstar$$

where $\gamma : A^\bigstar \otimes A \to A^\bigstar \otimes A$ is the switching morphism and $\delta : A \to (A^\bigstar)^\bigstar$ is the morphism corresponding to $\epsilon_A \circ \gamma : A \otimes A^\bigstar \to I$ under the bijection

$$Hom(A \otimes A^\bigstar, I) \cong Hom(A, (A^\bigstar)^\bigstar).$$

Equivalently,

Definition A.3.8. *An object $A \in \mathfrak{C}$ is* dualizable *if there exists an object $A^\bigstar$ and maps $\epsilon : A^\bigstar \otimes A \to I$ and $\eta : I \to A \otimes A^\bigstar$ satisfying the triangle identities:*

- $(id_A \otimes \epsilon) \circ (\eta \otimes id_A) = id_A$
- $(\epsilon \otimes id_{A^\bigstar}) \circ (id_{A^\bigstar} \otimes \eta) = id_{A^\bigstar}$

which can be visualized as:

$$\begin{tikzcd} A = I \otimes A \arrow[rr, "id_A"] \arrow[dr, "\eta \otimes id_A"'] & & I \otimes A = A \\ & A \otimes A^\bigstar \otimes A \arrow[ur, "id_A \otimes \epsilon"'] & \end{tikzcd}$$

$$
\begin{tikzcd}
A^\bigstar = A^\bigstar \otimes I \arrow[rr, "id_A"] \arrow[dr, "id_{A^\bigstar} \otimes \eta"'] & & I \otimes A^\bigstar = A^\bigstar \\
& A^\bigstar \otimes A \otimes A^\bigstar \arrow[ur, "\epsilon \otimes id_{A^\bigstar}"'] &
\end{tikzcd}
$$

Thus in any symmetric monoidal category, we can define a trace operation.

Definition A.3.9. *Let A be a dualizable object with evaluation $\epsilon : A^\bigstar \otimes A \to I$ and coevaluation $\eta : I \to A \otimes A^\bigstar$. If $f : A \to A$ is an endomorphism, then the* trace of f, *$tr(f)$ is defined as the composition:*

$$I \xrightarrow{\eta} A \otimes A^\bigstar \xrightarrow{f \otimes id} A \otimes A^\bigstar \xrightarrow[\cong]{\gamma} A^\bigstar \otimes A \xrightarrow{\epsilon} I$$

More generally, for a morphism $f : Q \otimes A \to B \otimes P$ where P, Q are objects in $\mathfrak{C}$ and A, B objects in $\mathfrak{C}^\bigstar$, there is a dual morphism $f^\bigstar$ which can be represented by $D(f)$ defined by the composition:

$$f^\bigstar : B^\bigstar \otimes Q \xrightarrow{id \otimes id \otimes \eta} B^\bigstar \otimes Q \otimes A \otimes A^\bigstar \xrightarrow{id \otimes f \otimes id} B^\bigstar \otimes B \otimes P \otimes A^\bigstar \xrightarrow{\epsilon \otimes id \otimes id} P \otimes A^\bigstar$$

If $P = Q = I$ and $A = B$, $f^\bigstar \in Hom(Q \otimes A, B \otimes P)$ reduces to a morphism in $Hom(B^\bigstar, A^\bigstar)$ as we had before.

Definition A.3.10. *The* twisted trace *of a morphism $f : Q \otimes A \to A \otimes P$ is the composition:*

$$Q \xrightarrow{id \otimes \eta} Q \otimes A \otimes A^\bigstar \xrightarrow{f \otimes id} A \otimes P \otimes A^\bigstar \xrightarrow{\gamma} A^\bigstar \otimes A \otimes P \xrightarrow{\epsilon} P$$

When $Q = P = I$, the twisted trace reduces to the original trace.

The twisted trace exhibits the following properties:

Proposition A.3.11. *1. (Invariance under dualization) $tr(f) = tr(\gamma \circ f^\bigstar \circ \gamma)$.*

2. (Naturality in P and Q) If $g : Q' \to Q$ and $h : P \to P'$, then

$$h \circ tr(f) \circ g = tr((id_A \otimes h) \circ f \circ (g \otimes id_A)).$$

3. *(Fixed point property) Let* $\delta : A \to A \otimes P$ *be a map and* $h : P \to P$ *satisfies* $(f \otimes h) = \delta \circ f)$*. Then* $h \circ tr(\delta \circ f) = tr(\delta \circ f)$*. In particular, if* $P = A$ *and* $h = f$*, we have* $tr((f \otimes f) \circ \delta) = tf(\delta \circ f)$*. This is an analogue of the diagonal map construction in coincidence theory.*

4. *(Constant on* $Hom(Q \otimes I, I \otimes P)$*) For all* $f : Q \otimes I \to I \otimes P$*,* $tr(f) = f$*.*

5. *(Cyclicity) For* $f : Q \otimes A \to B \otimes P$ *and* $g : K \otimes B \to A \otimes L$*,*

$$tr((g \otimes id_P) \circ (id_K \otimes f)) = tr(\gamma \circ (f \otimes id_L) \circ (id_Q \otimes g) \circ \gamma).$$

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