## Title

Twisted chimera states and multi-core spiral chimera states on a two-dimension torus

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# Chimera States in Nonlocal Phase-coupled Oscillators 

by<br>Jianbo Xie<br>A dissertation submitted in partial satisfaction of the<br>requirements for the degree of Doctor of Philosophy<br>in<br>Physics<br>in the<br>Graduate Division<br>of the<br>University of California, Berkeley<br>Committee in charge:<br>Professor Edgar Knobloch, Chair<br>Professor Jonathan S Wurtele<br>Associate Professor Jon S Wilkening

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# Chimera States in Nonlocal Phase-coupled Oscillators 

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Jianbo Xie

Abstract<br>Chimera States in Nonlocal Phase-coupled Oscillators<br>by<br>Jianbo Xie<br>Doctor of Philosophy in Physics<br>University of California, Berkeley<br>Professor Edgar Knobloch, Chair

This thesis studies systems of nonlocal phase-coupled oscillators. Various types of solutions have been discovered and analyzed with a combination of analytical and numerical methods. The work is motivated by recent interest in chimera states, in which domains of coherent and incoherent oscillators coexist. A major part focuses on the model equation

$$
\begin{equation*}
\frac{\partial \theta(x, t)}{\partial t}=\omega-\int G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y \tag{1}
\end{equation*}
$$

and its two-dimensional generalization.
In one-dimensional systems, the cases where $\omega$ is either a constant or space-dependent are investigated. When $\omega$ is a constant, we propose a class of coupling functions in which chimera states develop from random initial conditions. Several classes of chimera states have been found: (a) stationary multi-cluster states with evenly distributed coherent clusters, (b) stationary multi-cluster states with unevenly distributed clusters, and (c) a single cluster state traveling with a constant speed across the system. Traveling chimera states are also identified. A self-consistent continuum description of these states is provided and their stability properties analyzed through a combination of linear stability analysis and numerical simulation. When $\omega$ is space-dependent, two types of spatial inhomogeneity, localized and spatially periodic, are considered and their effects on the existence and properties of multicluster and traveling chimera states are explored. The inhomogeneity is found to break up splay states, to pin the chimera states to specific locations, and to trap traveling chimeras. Many of these states can be studied by constructing an evolution equation for a complex order parameter. Solutions of this equation are in good agreement with the results of numerical simulations.

The above mentioned solutions have counterparts in two-dimensional systems. However, some of these lose stability in two dimensions and hence cannot be obtained from numerical simulation with random initial conditions. In addition, some solutions which are unique in two dimensions have been found: (a) twisted chimera states with phase that varies uniformly
across the coherent domain, and (b) multi-core spiral wave chimera states with evenly distributed phase-randomized cores. Similar stability analysis as for one-dimensional systems is provided.

To my parents

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## Chapter 1

## Background

### 1.1 Overview of dynamical systems

## Introduction

Dynamical systems are systems that change with time. There are endless phenomena in our daily life and in scientific investigation that have dynamic aspects. Examples include boiling water, climate change, celestial motion, mechanical oscillators, chemical oscillations, and even population growth in ecology. In this wide range of problems, one is interested in problems such as whether the observed phenomena are stable, how the system changes from one state to another, and whether the system settles down to an equilibrium. Even though the behavior of these systems varies, viewed from the perspective of dynamics, they can be placed in the same mathematical framework.

Today, dynamics is an interdisciplinary subject. The basic topics of dynamical systems theory are broadly covered in many books $[1,2]$. In this and following sections, we briefly review some basic aspects of dynamical systems. In particular, we will cover equilibria and their stability analysis, and bifurcation theory. Before beginning, we should define what dynamical systems are mathematically. The two most common types of dynamical systems are maps and differential equations. They are classified depending on whether the behavior is viewed as discrete or continuous. In this thesis, we focus on the second type. Differential equations can be classified into ordinary (ODE) or partial (PDE).

One example of an ODE is the simple harmonic oscillator:

$$
\begin{align*}
\dot{x} & =v  \tag{1.1}\\
\dot{v} & =-\frac{k x}{m} \tag{1.2}
\end{align*}
$$

where $x$ and $v$ represent the displacement and the velocity of the oscillator and the dot represents the time derivative. Examples of PDEs include the famous heat equation, and the wave equation. Reaction-diffusion equations are another type relevant to pattern formation:

$$
\begin{equation*}
\partial_{t} \mathbf{u}=D \nabla^{2} \mathbf{u}+\mathbf{F}(\mathbf{u}) \tag{1.3}
\end{equation*}
$$

This type of equation is typically used to describe chemical reactions represented by the reaction term $\mathbf{F}(\mathbf{u})[3,4]$. The Swift-Hohenberg equation:

$$
\begin{equation*}
\partial_{t} u=r u-\left(1+\nabla^{2}\right)^{2} u+N(u) \tag{1.4}
\end{equation*}
$$

which is used to describe certain aspects of convection [5] is another example.
In this thesis, we will mainly focus on coupled ordinary differential equations. The general expression for an ODE is:

$$
\begin{equation*}
\dot{\mathbf{X}}=\mathbf{F}(\mathbf{X}, \mu), \quad \mathbf{X} \in \mathbb{R}^{n} \tag{1.5}
\end{equation*}
$$

Here $\mu$ is a parameter we can control and $n$ is the dimension of the system. This is an autonomous system which does not explicitly depend on $t$. Non-autonomous systems can be converted into the autonomous form by introducing an additional dimension: $\mathbf{X}_{n+1}=t$, $\dot{\mathbf{X}}_{n+1}=1$.

The space $\mathbb{R}^{n}$ is called the phase space of the system. In fact the structure of the phase space could be more general, and examples include cylindrical, spherical, or toroidal phase spaces. The solution $\mathbf{X}(t)$ forms a trajectory in the phase space. Under suitable conditions, one initial condition determines a unique trajectory $[6,7]$.

## Linear stability of equilibria

In this section, we describe the stability analysis for ordinary differential equations, largely following the discussion in Wiggins' book [2]. At this moment, we ignore the control parameter $\mu$ in Eq. (1.5) and consider the general system

$$
\begin{equation*}
\dot{\mathbf{X}}=\mathbf{F}(\mathbf{X}), \quad \mathbf{X} \in \mathbb{R}^{n} \tag{1.6}
\end{equation*}
$$

An equilibrium solution is a point $\mathbf{X}_{0}$ satisfying $\mathbf{F}\left(\mathbf{X}_{\mathbf{0}}\right)=0$, i.e., a solution which does not change over time.

To get some intuition, we consider a linear system as an example

$$
\begin{equation*}
\dot{\mathbf{X}}=\mathbf{A X} \tag{1.7}
\end{equation*}
$$

$\mathbf{X}_{\mathbf{0}}=\mathbf{0}$ is an equilibrium point for this system. At $t=0$, we give the system a perturbation $\mathbf{X}(\mathbf{0})$. This system can be solved exactly:

$$
\begin{equation*}
\mathbf{X}(t)=\mathbf{X}(\mathbf{0}) \exp (\mathbf{A} t) \tag{1.8}
\end{equation*}
$$

Since the eigenvectors of $\mathbf{A}$ span $\mathbb{R}^{n}, \mathbb{R}^{n}$ can be represented as direct sum of three subspaces based on the real part of the eigenvalues of $\mathbf{A}$ :

$$
\begin{align*}
& E^{s}=\operatorname{span}\left\{e_{1}, \cdots, e_{s}\right\}  \tag{1.9}\\
& E^{u}=\operatorname{span}\left\{e_{s+1}, \cdots, e_{s+u}\right\}  \tag{1.10}\\
& E^{c}=\operatorname{span}\left\{e_{s+u+1}, \cdots, e_{s+u+c}\right\} \tag{1.11}
\end{align*}
$$

Here, $s$ stands for "stable", and $\left\{e_{1}, \cdots, e_{s}\right\}$ are the eigenvectors corresponding to the eigenvalues of $\mathbf{A}$ having negative real part; $u$ stands for "unstable", and $\left\{e_{s+1}, \cdots, e_{s+u}\right\}$ are the eigenvectors corresponding to the eigenvalues of $\mathbf{A}$ having positive real part. Finally $c$ stands for "center", and $\left\{e_{s+u+1}, \cdots, e_{s+u+c}\right\}$ are the eigenvectors corresponding to the eigenvalues of $\mathbf{A}$ having zero real part (see [8] for more details). In addition, these three subspaces are invariant under the dynamics. If all the eigenvalues of $\mathbf{A}$ have negative real part, the equilibrium point is considered stable. If at least one of the eigenvalues of $\mathbf{A}$ has a positive real part, the equilibrium point is considered unstable. If $\mathbf{A}$ has eigenvalues on the imaginary axis, the case should receive special treatment.

We now turn to nonlinear systems. Using a change of coordinates, we can make $\mathbf{X}_{\mathbf{0}}=\mathbf{0}$ an equilibrium point for this system. Then the system can be written

$$
\begin{equation*}
\dot{\mathbf{X}}=\mathbf{D F}(\mathbf{0}) \mathbf{X}+\mathbf{N}(\mathbf{X}) \tag{1.12}
\end{equation*}
$$

where $\mathbf{N}(\mathbf{X})=O\left(\|\mathbf{X}\|^{2}\right)$.
Theorem 1 Suppose Eq. (1.12) is $C^{r}(r \geqslant 2)$, and $\mathbf{D F}(\mathbf{0})$ has $s$ eigenvalues with negative real part, $u$ eigenvalues with positive real part and $c$ eigenvalues with zero real part. Then the fixed point $\mathbf{X}=\mathbf{0}$ possesses a $C^{r} s$-dimensional local, invariant stable manifold, a $C^{r}$ $u$-dimensional local, invariant unstable manifold, and a $C^{r}$ c-dimensional local, invariant center manifold, each tangent to the respective eigenspace at $\mathbf{X}=\mathbf{0}$.

In the statement, the term "local" refers to the fact that the manifold is only defined in a neighborhood of the equilibrium point. As in the linear case, if all the eigenvalues of $\mathbf{D F}(\mathbf{0})$ have negative real part, the equilibrium point is considered locally stable. If at least one of the eigenvalues of $\mathbf{D F}(\mathbf{0})$ has a positive real part, the equilibrium point is considered locally unstable.

## Bifurcation theory

The reason we include $\mu$ in Eq. (1.5) is that in practical applications, many, if not most differential equations depend on some parameters. It is important to study the behavior of solutions and to examine their dependence on these parameters. It turns out that near some critical value, a small change in a parameter can have significant impact on the solution. Bifurcation theory of dynamical systems is a theory that refers to the study of changes in the qualitative structure of a system as the parameter changes. We now introduce the basic codimension-one bifurcations.

The saddle-node bifurcation is the most basic example. As the parameter varies, two fixed points move to each other, collide and then annihilate. To illustrate this, let's consider the normal form (a detailed discussion about normal forms is given in Appendix B) of saddle-node bifurcation:

$$
\begin{equation*}
\dot{x}=\mu+x^{2} \tag{1.13}
\end{equation*}
$$

where $\mu$ is the control parameter. When $\mu<0$, the system has two fixed points: $x_{0}^{+}=\sqrt{-\mu}$ (unstable) and $x_{0}^{-}=-\sqrt{-\mu}$ (stable). As $\mu$ increases, the distance between two fixed points decreases. They collide at the value of $\mu=0$ and annihilate when $\mu>0$. The bifurcation diagram is shown in Fig. 1.1(a).


Figure 1.1: (a) Illustration of the saddle-node bifurcation. (b) Illustration of the transcritical bifurcation. In both cases, solid lines represent stable fixed points while the dashed lines represent unstable fixed points.

Another basic bifurcation in the one-dimensional case is the transcritical bifurcation. In this bifurcation, one fixed point always exists but stability changes as the parameter changes. The normal form of a transcritical bifurcation is

$$
\begin{equation*}
\frac{d x}{d t}=\mu x-x^{2} \tag{1.14}
\end{equation*}
$$

The two fixed points for this system are $x_{0}=0$ and $x_{0}=\mu$. When $\mu<0$, the fixed point $x_{0}=0$ is stable while $x_{0}=\mu$ is unstable; When $\mu>0$, the fixed point $x_{0}=0$ is unstable while $x_{0}=\mu$ is stable. An exchange of stability has taken place between the two fixed points when the parameter $\mu$ equals 0 . The bifurcation diagram for the transcritical bifurcation is shown in Fig. 1.1(b).

Now we turn to the third bifurcation in one dimension: the pitchfork bifurcation, which is common in systems that have symmetry. There are normally two types of pitchfork bifurcation: supercritical and subcritical. The normal form of the supercritical case is

$$
\begin{equation*}
\frac{d x}{d t}=\mu x-x^{3} \tag{1.15}
\end{equation*}
$$

When $\mu<0, x_{0}=0$ is the only fixed point and it is stable. When $\mu=0, x_{0}=0$ is still stable but becomes weaker in the sense that the linearization vanishes. When $\mu>0$, two new fixed points $x_{0}= \pm \sqrt{\mu}$ are born and they are stable, while $x_{0}=0$ becomes unstable. Fig. 1.2(a) shows the bifurcation diagram for a supercritical pitchfork bifurcation.
(a)

(b)


Figure 1.2: (a) Illustration of the supercritical pitchfork bifurcation. (b) Illustration of the subcritical pitchfork bifurcation.

The normal form for the subcritical pitchfork bifurcation is

$$
\begin{equation*}
\frac{d x}{d t}=\mu x+x^{3} . \tag{1.16}
\end{equation*}
$$

The bifurcation diagram is shown in Fig. 1.2(b). In this case, two unstable fixed points are present when $\mu<0$. It should be mentioned that in real systems, we need to include higher order terms to suppress the explosive instability that takes place for $\mu>0$. One example that preserves the symmetry under $x \rightarrow-x$ is to add a term $-x^{5}$. Then the system becomes

$$
\begin{equation*}
\frac{d x}{d t}=\mu x+x^{3}-x^{5} \tag{1.17}
\end{equation*}
$$

The new feature due to the higher order term is that the unstable branches turn around and become stable at the critical value $\mu_{s}=-\frac{1}{4}$. In fact this is the saddle-node bifurcation that we discussed previously. In the range of $\mu_{s}<\mu<0, x_{0}=0$ is still stable and it coexists with the other two stable branches. The coexistence allows hysteresis as $\mu$ varies.

In dimension higher than one, the dynamics of the systems can be much richer. An important bifurcation in two-dimensional systems is the Hopf bifurcation. The normal form is

$$
\begin{gather*}
\dot{x}=\mu x-\omega y+(a x-b y)\left(x^{2}+y^{2}\right)  \tag{1.18}\\
\dot{y}=\omega x+\mu y+(b x+a y)\left(x^{2}+y^{2}\right) . \tag{1.19}
\end{gather*}
$$

Here, we regard $\mu$ as the control parameter while $\omega, a$ and $b$ are constants. The bifurcation occurs at $\mu=0$, where the eigenvalues of the linearized system cross through the imaginary axis. If we write it in polar coordinates, the system becomes

$$
\begin{align*}
& \dot{r}=\mu r+a r^{3},  \tag{1.20}\\
& \dot{\theta}=\omega+b r^{2} . \tag{1.21}
\end{align*}
$$

Eq. (1.20) has the same form as the pitchfork bifurcation except for the constraint $r \geqslant 0$. The sign of $a$ determines whether it is supercritical or subcritical. Let's assume $a$ is negative, so that the Hopf bifurcation is supercritical. As $\mu$ passes 0 , a nonzero stable fixed point for Eq. (1.20) is born. In the original two-dimensional system, it corresponds to a stable periodic orbit. We call this a limit cycle, which is defined as an isolated closed trajectory.

Fig. 1.3 shows some trajectories in phase space for $\mu<0, \mu=0$ and $\mu>0$. When $\mu<0$, the trajectories spiral towards the fixed point. When $\mu=0$, the fixed point is still stable but the speed of the trajectories towards the fixed point is much slower. When $\mu>0$, all trajectories spiral to the stable limit cycle.
(a)

(b)

(c)


Figure 1.3: Trajectories of Eq. (1.18) and Eq. (1.19) for (a) $\mu<0$; (b) $\mu=0$; and (c) $\mu>0$.

### 1.2 Limit cycle oscillator and the phase method

In the previous section, we saw that the supercritical Hopf bifurcation arises when a steady time-independent state loses stability and gives way to periodic motion (limit cycle) as the parameter of the system passes some critical value. Due to the periodic behavior, we call such systems limit cycle oscillators. Although the specific feature of the oscillators may vary from system to system, all systems behave in a similar manner close to the onset of oscillations. Mathematically, this is a consequence of a remarkable theorem called the center manifold theorem. And in fact, the application of Hopf bifurcation can be found in systems like chemical reactions, optics, biology and some other fields.

In the real world, the system could contain many or an infinite number of such limit cycle oscillators, and even worse, they could be coupled to each other. We know that, in general a single limit cycle motion cannot be solved analytically. It is even harder to deal with large numbers of coupled oscillators. Fortunately, there exists a fundamental theoretical technique called the phase reduction method, which is powerful for describing these oscillator systems. In this section, we focus on the cases where the perturbation or coupling is weak, largely following Kuramoto's book [16].

Let's start with a simple one-oscillator model to illustrate the perturbation idea. Consider a $n$-dimensional dynamical system

$$
\begin{equation*}
\dot{\mathbf{X}}=\mathbf{F}(\mathbf{X}) \tag{1.22}
\end{equation*}
$$

exhibiting a linearly stable periodic solution $\mathbf{X}_{0}(t)$ satisfying $\mathbf{X}_{0}(t)=\mathbf{X}_{0}(t+T)$ for any $t$. Here $T$ is the least period. If the system is perturbed by a small perturbation $\epsilon \mathbf{p}(\mathbf{X})$, it becomes

$$
\begin{equation*}
\dot{\mathbf{X}}=\mathbf{F}(\mathbf{X})+\epsilon \mathbf{p}(\mathbf{X}) . \tag{1.23}
\end{equation*}
$$

Periodic motion will persist but the period will deviate from $T$. We want to find the change in the period. Let $C$ denote the limit cycle of Eq. (1.22), and associate a phase $\phi$ to each point on $C$ such that

$$
\begin{equation*}
\dot{\phi}=1 \tag{1.24}
\end{equation*}
$$



Figure 1.4: Map between a limit cycle and a circle
Basically, it is a mapping from an arbitrary cycle to a circle (see Fig. 1.4). When the system is perturbed, the state point will no longer be on $C$, therefore it is desirable to extend the definition of $\phi$ to the vicinity of $C$. Assume a point $P$ on $C$ and another point $Q$ near $C$. If $P$ and $Q$ become infinitely close as $t$ goes to infinity, then we define $\phi(P)=\phi(Q)$. All the points mapped to the same $\phi$ constitute a $n-1$ dimensional hypersurface called an isochron, which was first introduced by Winfree [10]. In fact, the existence of isochrons is proved earlier but unknown to biologists. Mathematically, the isochron related to $P$ is the stable set of $P$. If no transverse Floquet multiplier of $C$ has absolute value one, the stable set is a transverse cross-section of $C$ and is a manifold diffeomorphic to Euclidean space. The proof of this statement can be found in some ODE books $[8,11]$, and the relation to isochrons is summarized in Guckenheimer's paper [12]. Thus we have Eq. (1.24) for all points on $C$ and in the vicinity of $C$.

On the other hand, we have identity

$$
\begin{equation*}
\dot{\phi}=\nabla_{X} \phi \cdot \dot{\mathbf{X}} \tag{1.25}
\end{equation*}
$$

Combined with Eq. (1.22), we have

$$
\begin{equation*}
\nabla_{X} \phi \cdot \mathbf{F}(\mathbf{X})=1 \tag{1.26}
\end{equation*}
$$

For the perturbed system,

$$
\begin{equation*}
\dot{\phi}=1+\epsilon \nabla_{X} \phi \cdot \mathbf{p}(\mathbf{X}) . \tag{1.27}
\end{equation*}
$$

The above deduction is exact. When the perturbation is small, $X$ can be approximated by $X_{0}(\phi)$. Thus we derived a closed form equation for the phase variable $\phi$ :

$$
\begin{equation*}
\dot{\phi}=1+\epsilon \Omega(\phi) \tag{1.28}
\end{equation*}
$$

where $\Omega(\phi)=\nabla_{X} \phi \cdot \mathbf{p}\left(\mathbf{X}_{\mathbf{0}}(\phi)\right)$. Here we call $\mathbf{Z}(\phi) \equiv \nabla_{X} \phi$ the sensitivity. Note that $\mathbf{Z}(\phi)$ and $\mathbf{p}\left(\mathbf{X}_{\mathbf{0}}(\phi)\right)$ are both $T$-periodic, so $\Omega(\phi)$ is also $T$-periodic. Introducing another phase variable $\theta=\phi-t$, we have

$$
\begin{equation*}
\dot{\theta}=\epsilon \Omega(t+\theta) . \tag{1.29}
\end{equation*}
$$

This shows that $\theta$ is a slow variable. This equation can be averaged as

$$
\begin{equation*}
\dot{\theta}=\epsilon \omega, \quad \omega=\frac{1}{T} \int_{0}^{T} \Omega(t) d t \tag{1.30}
\end{equation*}
$$

The above process is the phase reduction method for one oscillator. Now let's consider two oscillators which are coupled to each other:

$$
\begin{align*}
& \dot{\mathbf{X}}_{1}=\mathbf{F}_{1}\left(\mathbf{X}_{1}\right)+\epsilon \mathbf{V}\left(\mathbf{X}_{\mathbf{1}}, \mathbf{X}_{\mathbf{2}}\right),  \tag{1.31}\\
& \dot{\mathbf{X}}_{\mathbf{2}}=\mathbf{F}_{\mathbf{2}}\left(\mathbf{X}_{\mathbf{2}}\right)+\epsilon \mathbf{V}\left(\mathbf{X}_{\mathbf{2}}, \mathbf{X}_{\mathbf{1}}\right) \tag{1.32}
\end{align*}
$$

Assume that the oscillators are only slightly different from each other, i.e., $\mathbf{F}_{i}\left(\mathbf{X}_{i}\right)=$ $\mathbf{F}\left(\mathbf{X}_{i}\right)+\epsilon \mathbf{f}_{i}\left(\mathbf{X}_{i}\right)$. Considering $\mathbf{p}_{i}=\mathbf{V}\left(\mathbf{X}_{i}, \mathbf{X}_{i^{\prime}}\right)+\mathbf{f}_{i}\left(\mathbf{X}_{i}\right)$ as perturbations, we can derive coupled equations for $\phi_{1}$ and $\phi_{2}$ :

$$
\begin{align*}
& \dot{\phi}_{1}=1+\epsilon\left[\mathbf{Z}\left(\phi_{1}\right) \cdot\left(\mathbf{V}\left(\phi_{1}, \phi_{2}\right)+\mathbf{f}_{1}\left(\mathbf{X}_{0}\left(\phi_{1}\right)\right)\right)\right]  \tag{1.33}\\
& \dot{\phi_{2}}=1+\epsilon\left[\mathbf{Z}\left(\phi_{2}\right) \cdot\left(\mathbf{V}\left(\phi_{2}, \phi_{1}\right)+\mathbf{f}_{1}\left(\mathbf{X}_{0}\left(\phi_{2}\right)\right)\right)\right] \tag{1.34}
\end{align*}
$$

After the averaging process, as we have done in one-oscillator case, we obtain

$$
\begin{align*}
& \dot{\theta_{1}}=\omega_{1}+\Gamma\left(\theta_{1}-\theta_{2}\right)  \tag{1.35}\\
& \dot{\theta_{2}}=\omega_{2}+\Gamma\left(\theta_{2}-\theta_{1}\right) \tag{1.36}
\end{align*}
$$

where $\Gamma\left(\theta_{i}-\theta_{i^{\prime}}\right)=\frac{1}{T} \int_{0}^{T} \mathbf{Z}\left(\theta_{i}+t\right) \cdot \mathbf{V}\left(\theta_{i}+t, \theta_{i^{\prime}}+t\right) d t$ and $\omega_{i}=\frac{1}{T} \int_{0}^{T} \mathbf{Z}\left(\theta_{i}+t\right) \cdot \mathbf{f}_{i}\left(\mathbf{X}_{0}\left(\theta_{i}\right)\right) d t$.
This process can be easily extended to the $N$-oscillator case. We can derive the phase equation

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}+\sum_{j=1}^{N} \Gamma_{i j}\left(\theta_{i}-\theta_{j}\right) \tag{1.37}
\end{equation*}
$$

Eq. (1.37) is a phase-only model equation. By assuming weak coupling between oscillators, the phase reduction method neglects the amplitude. But we should point out that "weak coupling" is a sufficient but not a necessary condition for the averaging method to be
valid $[13,14]$. The phase reduction provides a very useful tool to approach problems such as synchronization. However, Eq. (1.37) is still formidable without further simplification of $\Gamma_{i j}$. Therefore, we assume that the couplings have a magnitude of $\frac{1}{N}$ and take a simple trigonometric function. Thus we write

$$
\begin{equation*}
\Gamma_{i j}\left(\theta_{i}-\theta_{j}\right)=-\frac{K}{N} G_{i j} \sin \left(\theta_{i}-\theta_{j}+\alpha\right) \tag{1.38}
\end{equation*}
$$

Here $K$ is a positive constant that represents the strength of couplings, $0 \leqslant|\alpha| \leqslant \frac{\pi}{2}$ indicates a phase lag and $G_{i j}=O(1)$ and indicate the network structure. This type of coupling function is known as a Kuramoto-type coupling and plays crucial rule in the study of synchronization phenomena in nature. We will discuss this topic in more detail in the next section.

### 1.3 Coupled oscillator systems and synchronization

In Section 1.2, we obtained the phase coupled oscillator model:

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}-\frac{K}{N} \sum_{j}^{N} G_{i j} \sin \left(\theta_{i}-\theta_{j}+\alpha\right) \tag{1.39}
\end{equation*}
$$

The original analysis of the synchronization behavior of this model was given by Kuramoto, assuming all-to-all coupling and zero phase lag [9, 16], i.e., $G_{i j}=1$ and $\alpha=0$. Parts of the discussion below follow closely the two review papers [22, 23].

## The Kuramoto Model

To get some intuition, we start with a simplified version with only two oscillators in the system:

$$
\begin{align*}
& \dot{\theta}_{1}=\omega_{1}-\frac{K}{2} \sin \left(\theta_{1}-\theta_{2}\right),  \tag{1.40}\\
& \dot{\theta}_{2}=\omega_{2}-\frac{K}{2} \sin \left(\theta_{2}-\theta_{1}\right) . \tag{1.41}
\end{align*}
$$

This simple model can describe the situation when two friends are running on the circular track with angular speed $\omega_{1}$ and $\omega_{2}$. Being friends, they would like to adjust their own speed to one another's. To analyze this mode, define $\Delta \theta=\theta_{1}-\theta_{2}$ and $\Delta \omega=\omega_{1}-\omega_{2}$. Subtracting Eq. (1.41) from Eq. (1.40), we get

$$
\begin{equation*}
\Delta \theta=\Delta \omega-K \sin (\Delta \theta) \tag{1.42}
\end{equation*}
$$

This is Adler's equation which can bee seen in many different branches of science and engineering. Without loss of generality, we assume $\Delta \omega>0$. When $\Delta \omega<K$, there exists a
stable fixed point $\theta^{*}$ satisfying $\sin \theta^{*}=\frac{\Delta \omega}{K}$ and $\cos \theta^{*}=\sqrt{1-\left(\frac{\Delta \omega}{K}\right)^{2}}$; when $\Delta \omega>K$, there is no fixed point and the period of oscillation can be found analytically:

$$
\begin{equation*}
T=\int_{0}^{2 \pi} \frac{d \theta}{\Delta \omega-K \sin \theta}=\frac{2 \pi}{\sqrt{(\Delta \omega)^{2}-K^{2}}} \tag{1.43}
\end{equation*}
$$

The SNIPER (saddle node infinite period) bifurcation occurs at $\Delta \omega=K$.
From this simple example, we conclude that when the coupling between the two oscillators is strong $(K>\Delta \omega)$, they will finally move with the same frequency, and be synchronized. Otherwise, they will move incoherently. This conclusion can be extended to the case with many oscillators. There is a competition between two mechanisms: the large coupling constant tends to synchronize the oscillators while the difference among the natural frequencies $\omega$ tends to de-synchronize the oscillators.

Now we briefly introduce the method that Kuramoto used to analyze his model. First, we define a complex order parameter:

$$
\begin{equation*}
R \exp (i \Theta)=\frac{1}{N} \sum_{j=1}^{N} \exp \left(i \theta_{j}\right) \tag{1.44}
\end{equation*}
$$

Here $0 \leqslant R(t) \leqslant 1$ measures the coherence of the oscillators and $\Theta(t)$ is the average phase, as shown in Fig. 1.5(a). Then, the Kuramoto model becomes

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}+K R \sin \left(\Theta-\theta_{i}\right) \tag{1.45}
\end{equation*}
$$

From this equation, we see the advantage of using the order parameter. The oscillators


Figure 1.5: (a) Illustration of the mean field quantities $R$ and $\Theta$.(b) Supercritical bifurcation for the Kuramoto model in a diagram showing $R$ versus $K$.
interact with each other only through the quantities $R$ and $\Theta$. Each phase $\theta_{i}$ is pulled towards the mean phase $\Theta$. It appears that the phases uncouple from each other. Another
important fact is that the effective coupling strength is $K R$, which is proportional to $R$. This indicates that as the oscillators become more coherent, the effective coupling increases, which tends to attract more oscillators to the group, hence further increasing the coherence. This process constitutes a positive feedback, which was first discovered by Winfree [10].

Kuramoto assumes a steady solution in which $R(t)$ is constant and $\Theta(t)$ rotates uniformly at a constant frequency. By going into a certain rotating frame, we can set $\Theta=0$ without loss of generality. Then the ODE becomes uncoupled:

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}-K R \sin \left(\theta_{i}\right) . \tag{1.46}
\end{equation*}
$$

As discussed in the two-oscillator example, this equation exhibits two types of solutions depending on $\left|\omega_{i}\right|$ and $K R$. When $\left|\omega_{i}\right|<K R$, the oscillator will settle down to a fixed point satisfying $\omega_{i}=K R \sin \left(\theta_{i}\right)$. This part of the oscillators are phase-locked. When $\left|\omega_{i}\right|>K R$, the oscillators drift in time. Their stationary density obeys $\rho v=C$, hence

$$
\begin{equation*}
\rho(\theta, \omega)=\frac{C}{|\omega-K R \sin (\theta)|} . \tag{1.47}
\end{equation*}
$$

The normalization condition implies $C=\frac{\sqrt{\omega^{2}-(K R)^{2}}}{2 \pi}$.
The constant order parameter must be consistent with Eq. (1.44). So we have

$$
\begin{equation*}
R=\langle\exp (i \theta)\rangle_{\text {lock }}+\langle\exp (i \theta)\rangle_{\text {drift }}, \tag{1.48}
\end{equation*}
$$

where the angular brackets represent average. Given the distribution of natural frequency $\omega$, we can derive a self-consistent equation from Eq. (1.48). For simplicity, we assume that $g(\omega)$ is unimodal and symmetric about its mean $\omega_{0}$, i.e., $g\left(\omega_{0}+\omega\right)=g\left(\omega_{0}-\omega\right)$. By rotating the frame, we can set $\omega_{0}=0$. With this symmetry, we have $\langle\sin (\theta)\rangle_{l o c k}=0$ and

$$
\begin{equation*}
\langle\exp (i \theta)\rangle_{\text {lock }}=\langle\cos (\theta)\rangle_{\text {lock }}=\int_{-K R}^{K R} \cos (\theta) g(\omega) d \omega . \tag{1.49}
\end{equation*}
$$

Here $\theta$ is a function of $\omega$ determined by $\omega=K R \sin (\theta)$. Change of variable gives

$$
\begin{equation*}
\langle\exp (i \theta)\rangle_{l o c k}=K R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos ^{2}(\theta) g(K R \sin (\theta)) d \theta \tag{1.50}
\end{equation*}
$$

This is an expression for the frequency-locked part of the oscillators. It turns out that the drifting part $\langle\exp (i \theta)\rangle_{\text {drift }}$ vanishes due to symmetry. So the self-consistent equation becomes

$$
\begin{equation*}
R=K R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos ^{2}(\theta) g(K R \sin (\theta)) d \theta \tag{1.51}
\end{equation*}
$$

There is a trivial solution $R=0$ for this equation, corresponding to the uniform incoherent state, $\rho=\frac{1}{2 \pi}$. It also has a second branch of solutions, corresponding to partially synchronized states, and satisfying

$$
\begin{equation*}
1=K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos ^{2}(\theta) g(K R \sin (\theta)) d \theta \tag{1.52}
\end{equation*}
$$

This branch bifurcates from $R=0$ at the critical value $K=K_{c}$. Here, $K_{c}=\frac{2}{\pi g(0)}$, which can be obtained by setting $R=0$ in Eq. (1.52). Near $K_{c}$, we can treat $R$ as a small parameter and expand Eq. (1.52) in powers of $R$. We find that near the onset, if $g^{\prime \prime}(0)<0$, which is the usual case, the bifurcation is supercritical and follows the scaling law

$$
\begin{equation*}
R \sim \sqrt{\frac{-16\left(K-K_{c}\right)}{\pi g^{\prime \prime}(0) K_{c}^{4}}} \tag{1.53}
\end{equation*}
$$

There is a special case in which the Eq. (1.52) can be solved exactly. When the natural frequency follows Cauchy distribution $g(\omega)=\frac{\gamma}{\pi\left(\gamma^{2}+\omega^{2}\right)}$, Eq. (1.52) gives

$$
\begin{equation*}
R=\sqrt{1-\frac{K_{c}}{K}} \tag{1.54}
\end{equation*}
$$

where $K_{c}=2 \gamma$.
The above process represents Kuramoto's original analysis of the phase transition from the incoherent state to a partially synchronized state as the coupling constant $K$ increases. The diagram is shown in Fig. 1.5(b). Eq. (1.54) can also be obtained by another method called the OA-Ansatz [46], which we will discuss later.

There remains the question whether the steady solutions are stable. Let $\rho(\theta, t, \omega)$ denote the fraction of oscillators that lie between $\theta$ and $\theta+d \theta$ at time $t$. The governing equation for $\rho$ is:

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}=-\frac{\partial}{\partial \theta}\left[\rho\left(\omega+K \int_{0}^{2 \pi} \int_{-\infty}^{\infty} \sin \left(\theta^{\prime}-\theta\right) \rho\left(\theta^{\prime}, t, \omega^{\prime}\right) g\left(\omega^{\prime}\right) d \omega^{\prime} d \theta^{\prime}\right)\right] \tag{1.55}
\end{equation*}
$$

$\rho=\frac{1}{2 \pi}$ corresponds to the uniform incoherent state. The stability of this state can be analyzed via studying the spectrum of the linearized operator. Strogatz and Mirollo showed that when $K<K_{c}$, the linearization about the incoherent state has a purely imaginary continuous spectrum. As $K$ increases, a real eigenvalue emerges from the continuous spectrum and moves into the right half plane [86].
J. D. Crawford has also made a great contribution to the stability analysis of the Kuramoto model. He derived and analyzed an equation for $r(t)$ - the deviation of $R(t)$ from the incoherent state [87]. The coefficients in the amplitude equation is finite if no noise exists. This result confirms the scaling law in Eq. (1.53). Inspired by the work of Daido [82, 83, 84], who investigated a generalized Kuramoto model in which more harmonics of the coupling functions are included, Crawford studied the amplitude equations for those models. He explained the reason why Daido's result differs from Kuramoto's: the scaling behavior near threshold is altered essentially when higher harmonics are included in the coupling function [88, 89].

## Variations of the Kuramoto Model

In the preceding subsection we have seen that some analytical results can be obtained with all-to-all coupling and many reasonable assumptions. Yet, one might ask if these results
can be extended when we modify the model. For example, the Kuramoto model with noise, time-delayed coupling, phase lag, or non-local coupling. Unfortunately, it is hard to derive analytical results in such cases.

In this subsection, we summarize some variations of the Kuramoto model discussed in the previous subsection [23], but without going into details.

The first generalization is known as the Kuramoto model with noise. The governing equation, after adding noise, is

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}-\frac{K}{N} \sum_{j=1}^{N} \sin \left(\theta_{i}-\theta_{j}\right)+\xi_{i}(t) \tag{1.56}
\end{equation*}
$$

Here $\xi$ 's are independent white noises satisfying

$$
\begin{equation*}
\left\langle\xi_{i}(t)\right\rangle=0, \quad\left\langle\xi_{i}(t) \xi_{j}\left(t^{\prime}\right)\right\rangle=2 D \delta\left(t-t^{\prime}\right) \delta_{i j} . \tag{1.57}
\end{equation*}
$$

It turns out that in this case the probability density $\rho(\theta, \omega, t)$ obeys a Fokker-Planck equation

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}=D \frac{\partial^{2} \rho}{\partial \theta^{2}}-\frac{\partial}{\partial \theta}(v \rho) \tag{1.58}
\end{equation*}
$$

where $v(\theta, \omega, t)=\omega+K R \sin (\Theta-\theta)$. With this equation, many problems such as the linear stability problem can be studied [85, 86, 87, 88, 89, 90, 91]. For example, the stability of the incoherent state is determined by the spectrum of the linearized operator about $\rho=\frac{1}{2 \pi}$. For $D>0$, the continuous spectrum is a vertical line segment in the left half plane, so the stability is determined by the point spectrum only. It turns out that the discrete spectrum is either a single point or empty, depending on the value of $K$. As before, there is a critical value $K_{c}$, corresponding to the onset of instability, given by

$$
\begin{equation*}
K_{c}=2\left[\int_{-\infty}^{\infty} \frac{D}{D^{2}+\omega^{2}} g(\omega) d \omega\right]^{-1} . \tag{1.59}
\end{equation*}
$$

In the special case when $g(\omega)=\frac{\gamma}{\pi\left(\gamma^{2}+\omega^{2}\right)}, K_{c}=2 \gamma+2 D$. This result is consistent with the case when $D=0$, which we have discussed in previous section. Apart from the incoherent state, Crawford also proved that the bifurcating branch of partially synchronized states is locally stable, near the threshold and in the presence of weak noise [87].

A second natural extension is the inclusion of time-delayed couplings. In many chemical and biological systems, signals propagate with finite speed and the time delay due to the transmission cannot be neglected. This time delay could substantially change the dynamical properties and make the behavior of the system much richer [92, 93, 94, 95, 96, 97, 98]. To simplify the problem, we assume the delay between each pair of oscillators is a constant. Then the system equation becomes

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}-\frac{K}{N} \sum_{j} \sin \left(\theta_{i}(t)-\theta_{j}(t-\tau)\right) \tag{1.60}
\end{equation*}
$$

This system becomes nontrivial even for few oscillators. We can also add noise to the system to make it more realistic. As in the standard Kuramoto model, in noisy systems there is a critical value of the coupling $K_{c}$, depending on the frequency distribution and delay, above which the incoherent solution is linearly unstable.

Another possible extension is to add an external field. Such a model can be written

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}-\frac{K}{N} \sum_{j=1}^{N} \sin \left(\theta_{i}-\theta_{j}\right)+\xi_{i}(t)+F_{i} \tag{1.61}
\end{equation*}
$$

where $F_{i}$ represents the external force. A commonly used force is $F_{i}=h \sin \left(\theta_{i}-\omega_{f} t\right)$, where $\omega_{f}$ is the frequency of the external force. By rotating the frame, we can set $\omega_{f}=0$ and shift the natural frequency distribution, $\omega_{i}$. Shinomoto and Kuramoto studied the case when $F_{i}=h \sin \left(\theta_{i}\right)$ and found two different regions in the phase diagram [99]. In one region observables are time-periodic and in another they are stationary and synchronized.

If we keep the all-to-all coupling and add the phase lag $\alpha$, the model becomes the so-called Sakaguchi-Kuramoto model:

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}-\frac{K}{N} \sum_{j}^{N} \sin \left(\theta_{i}-\theta_{j}+\alpha\right) . \tag{1.62}
\end{equation*}
$$

This model can be obtained by restricting the interaction function to the first Fourier component and is therefore quite general. Previously it was thought that when $|\alpha|<\frac{\pi}{2}$ and the frequency distribution $g(\omega)$ is unimodal, this model will transit from incoherence to partially synchronized state as the coupling constant $K$ increases. But recent studies [62,63] show that for certain frequency distributions, one can observe non-universal transitions, where incoherence may regain stability as the coupling constant increases. The authors present spectral analysis of complete incoherence and partially synchronized states, which extend the results of Mirollo and Strogatz, who studied the corresponding problem for the Kuramoto model. The phase lag plays an important role in the model when the coupling is nonlocal. The resulting model is the main focus of this thesis so we discuss it later.

One commonality in the above extensions is that they all keep the all-to-all coupling scheme, $G_{i j}=1$. $G_{i j}$ essentially indicates the structure of the network on which the oscillators are built. Complex networks with non-trivial topologies are themselves an interesting topic that has been studied for many years. When oscillators are built on networks, these two ingredients give rise to very rich dynamical behavior and make it a fascinating subject for inter-disciplinary research.

Here we would like to mention several special network structures that have received much attention. The first and simplest case is a $d$-dimensional lattice with the model equation

$$
\begin{equation*}
\dot{\theta}_{i}=\omega_{i}-K \sum_{(i, j)} \sin \left(\theta_{i}-\theta_{j}\right), \tag{1.63}
\end{equation*}
$$

where $(i, j)$ stands for nearest neighbor [100]. The network structure is simple, but the problem is actually quite difficult to analyze except in one dimension. Usually, global synchronization is rarely seen. However, phase locking or partial synchronization is observed more frequently. Another type is to build the oscillator system on a small world network [101, 102]. In this work, the statistical properties of the system are investigated. A third type of network structure is also $d$-dimensional in space $(d=1,2,3)$ but has nonlocal coupling. Nonlocal coupling is a coupling scheme that is between local and global. If we keep the phase lag $\alpha$, the model is particularly interesting and is our main focus in this thesis. In the one-dimensional case and the continuum limit, we can write the model equation in the form,

$$
\begin{equation*}
\frac{\partial \theta(x, t)}{\partial t}=\omega(x)-\int G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y \tag{1.64}
\end{equation*}
$$

We will discuss this model in more detail in the following chapters.

## Chapter 2

## Chimera states

### 2.1 A brief history of chimera states

As discussed in the preceding chapter, various cases of Eq. (1.39) have been studied for decades, but usually for a given natural frequency distribution $g(\omega)$. It was thought that the case of networks with identical oscillators is not very interesting. In 2002, Kuramoto and his colleagues reported a state of mixed synchronous and asynchronous behavior in a ring of identical coupled oscillators. The model equation they used is basically Eq. (1.64) with $\omega(x)=$ const. We rewrite the model equation here:

$$
\begin{equation*}
\frac{\partial \theta(x, t)}{\partial t}=\omega-\int G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y . \tag{2.1}
\end{equation*}
$$

Fig. (2.1) shows a snapshot of such a state. In the figure, the horizontal direction represents


Figure 2.1: 1-cluster chimera state.
space and the vertical direction represents the phase, $\theta$. Near $x=0$, the oscillators form a flat region that is phase-locked: the oscillators are nearly in phase and moving with the same frequency. Those oscillators that are not in phase are scattered over phase space and drift
in time relative to the phase-locked oscillators. They slow down when they pass near the coherent cluster, which is the reason why the oscillators appear more dense near the locked phase. The deeper reason will be clear when we carry out the self-consistent analysis later.

The existence of this structure was quite surprising. Generically, identical oscillator arrays settle into patterns in which all oscillators are synchronized, form smooth traveling waves or are completely incoherent. The coexistence of two seemingly incompatible patterns is so odd that the discovery was named a "chimera state" by Strogatz and Abrams [27]. The chimera is the mythological creature composed of a lion's head, a goat's body, and a serpent's tail.

Owing to its novelty, chimera states received a great deal of attention in the nonlinear dynamics community. How is the chimera state born in the sense of bifurcation theory? What are its dynamical stability properties? Under what condition do chimera states exist? Do they exist in higher dimensions and are they robust enough to be observed in experiment or nature? These questions naturally arise in the study of chimera states.

Chimera states motivate researchers in nonlinear dynamics community to explore systems with nonlocal coupling. Initially, it was thought that the nonlocal coupling was necessary for chimera states to exist. The nonlocal coupling regime might be more difficult to investigate compared with the nearest neighbor or global models, but it has important applications throughout physics [53], chemistry [54], biology [55], or even social science [56]. After focusing on the local and global interactions for many years, the intermediate case is the next regime to be explored.

In recent years, many of the questions mentioned above have been answered. Before discussing these works, we will review some important methodologies used frequently to investigate chimera states.

### 2.2 Methodology

## Self-consistent analysis

Kuramoto introduced a self-consistency method to his phase-coupled oscillator model [9, 16]; it is still one of the most commonly used methods to investigate chimera states. Before outlining his method, we notice that the constant frequency $\omega$ plays no essential role in the dynamics, and therefore we can set it to be 0 . Then the model equation can be written as

$$
\begin{equation*}
\frac{\partial \theta(x, t)}{\partial t}=-\int G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y \tag{2.2}
\end{equation*}
$$

First we seek a new reference frame which rotates with angular frequency $\Omega$ (to be determined later). In this frame the governing equation becomes

$$
\begin{equation*}
\frac{\partial \theta(x, t)}{\partial t}=\Omega-R(x, t) \sin [\theta(x, t)-\Theta(x, t)+\alpha] \tag{2.3}
\end{equation*}
$$

where we have redefined $\theta$ as $\theta+\Omega t$. Here, $R(x, t)$ and $\Theta(x, t)$ are called local order parameters and are defined by

$$
\begin{equation*}
R(x, t) \exp (i \Theta(x, t))=\int G(x-y) \exp (i \theta(y, t)) d y \tag{2.4}
\end{equation*}
$$

Note the analysis is very similar to the case of all-to-all coupling, except that the order parameters are defined as a weighted average of $\exp (i \theta)$. The oscillators interact with each other through $R(x, t)$ and $\Theta(x, t)$. Now we come to the key assumption that the chimera state is stationary in the large $N$ limit, which means that $R(x, t)$ and $\Theta(x, t)$ do not depend on time. With this assumption, the equations for the oscillators partially decouple:

$$
\begin{equation*}
\frac{\partial \theta(x, t)}{\partial t}=\Omega-R(x) \sin [\theta(x, t)-\Theta(x)+\alpha] . \tag{2.5}
\end{equation*}
$$

Oscillators with $R(x)>|\Omega|$ asymptotically approach to a stable fixed point $\theta^{\star}$ defined by

$$
\begin{equation*}
\Omega=R(x) \sin \left(\theta^{\star}(x)-\Theta(x)+\alpha\right) \tag{2.6}
\end{equation*}
$$

Oscillators with $R(x)<|\Omega|$ drift monotonically. According to the stationarity assumption, such oscillators must have a time-independent density $\rho(\theta)$ that can be determined from the conservation law, $\rho v=$ const, where $v=\Omega-R \sin (\theta-\Theta+\alpha)$. After normalization, we have

$$
\begin{equation*}
\rho(\theta)=\frac{\sqrt{\Omega^{2}-R^{2}}}{2 \pi|\Omega-R \sin (\theta-\Theta+\alpha)|} \tag{2.7}
\end{equation*}
$$

Now, with these conditions, we want to derive a self-consistency equation. For the phase locked oscillators,

$$
\begin{equation*}
\int G(x-y) \exp \left(i \theta^{\star}(y)\right) d y=\exp (-i \alpha) \int_{R(y)>\Omega} G(x-y) \exp \left(i \Theta^{\star}(y)\right) \frac{\sqrt{R^{2}-\Omega^{2}}+i \Omega}{R} d y \tag{2.8}
\end{equation*}
$$

For the drifting oscillators, Kuramoto and Battogtokh replace $\exp (i \theta(y))$ with its statistical average $\int_{-\pi}^{\pi} \exp (i \theta) \rho(\theta) d \theta$. By contour integration, we obtain

$$
\begin{equation*}
\int_{-\pi}^{\pi} \exp (i \theta) \rho(\theta) d \theta=\frac{i}{R}\left(\Omega-\sqrt{\Omega^{2}-R^{2}}\right) \exp (i \Theta-i \alpha) . \tag{2.9}
\end{equation*}
$$

For this phase-locked part of the oscillators, the integration domain consists of all $x$ satisfying $R(x)<|\Omega|$. In addition, making use of the identity

$$
\begin{equation*}
\sqrt{R^{2}-\Omega^{2}}+i \Omega=i\left(\Omega-\sqrt{\Omega^{2}-R^{2}}\right) \tag{2.10}
\end{equation*}
$$

allows us to combine the integrals for both the coherent part and the incoherent part. Finally we get the self-consistency equation

$$
\begin{equation*}
R(x) \exp (i \Theta(x))=\exp (i \beta) \int G(x-y) \exp (i \Theta(y)) \frac{\Omega-\sqrt{\Omega^{2}-R^{2}(y)}}{R(y)} \tag{2.11}
\end{equation*}
$$

where $\beta=\frac{\pi}{2}-\alpha$.

## The Ott-Antonsen Ansatz

Apart from the self-consistency analysis, the Ott-Antonson Ansatz is another useful technique in the study of phase-coupled oscillators. Ott and Antonsen showed, in their pioneering papers $[46,47]$, that for many systems of an infinite number of non-identical phase oscillators, a wide class of states can be reduced to a system with a low number of degrees of freedom. In this section, we will follow Ott and Antonsen's original paper [46] to introduce what is now known as the OA Ansatz to study the simplest Kuramoto model. Then we will show two examples of its application in the study of chimera states.

## OA Ansatz in the Kuramoto model

The Kuramoto model with all-to-all coupling is:

$$
\begin{equation*}
\frac{d \theta_{i}}{d t}=\omega_{i}-\frac{K}{N} \sum_{j}^{N} \sin \left(\theta_{i}-\theta_{j}\right) \tag{2.12}
\end{equation*}
$$

We define the order parameter $Z \equiv \frac{1}{N} \sum_{j} \exp \left(i \theta_{j}\right)$, so that $-\frac{K}{N} \sum_{j}^{N} \sin \left(\theta_{i}-\theta_{j}\right)=\operatorname{Im}\left[Z \exp \left(-i \theta_{i}\right)\right]$. Considering $N \rightarrow \infty$, the state of the oscillator system at time $t$ can be described by a continuous distribution function $f(\omega, \theta, t)$, where

$$
\begin{equation*}
\int_{0}^{2 \pi} f(\omega, \theta, t) d \theta=g(\omega) \tag{2.13}
\end{equation*}
$$

Here, $g(\omega)$ is the time-independent oscillator frequency distribution. The number of oscillators is conserved, so the distribution satisfies the following equations:

$$
\begin{align*}
& \frac{\partial f}{\partial t}+\frac{\partial}{\partial \theta}(f v)=0  \tag{2.14}\\
& v=\omega+\frac{K}{2 i}\left[\left(Z \exp (-i \theta)-Z^{*} \exp (i \theta)\right]\right.  \tag{2.15}\\
& Z=\int_{0}^{2 \pi} d \theta \int_{-\infty}^{\infty} d \omega f \exp (i \theta) \tag{2.16}
\end{align*}
$$

The following are the key steps of the OA Ansatz. First, expand $f(\omega, \theta, t)$ in a Fourier series in $\theta$ :

$$
\begin{equation*}
f(\omega, \theta, t)=\frac{g(\omega)}{2 \pi}\left\{1+\left[\sum_{n=1}^{\infty} f_{n}(\omega, t) \exp (-i n \theta)+c . c .\right]\right\} \tag{2.17}
\end{equation*}
$$

where c.c. represents the complex conjugate. We deviate from Ott and Antonsen's original paper by letting $f_{n}(\omega, t)$ to be the coefficient of $\exp (-i n \theta)$ for later convenience. Consider a restricted class of $f_{n}(\omega, t)$ such that $f_{n}(\omega, t)=[a(\omega, t)]^{n}$, where $|a(\omega, t)| \leqslant 1$ to avoid divergence of the Fourier series. One way to motivate this Ansatz is to note that both the
incoherent state and the partially synchronized state conform to $f_{n}=a^{n}$. Thus the Ansatz can be viewed as a family of distributions that connect these two states. Substituting this series into Eq. (2.14) we get

$$
\begin{align*}
& \frac{\partial a}{\partial t}=i \omega a-\frac{K}{2}\left(Z^{*} a^{2}-Z\right)  \tag{2.18}\\
& Z=\int_{-\infty}^{\infty} d \omega g(\omega) a(\omega, t) \tag{2.19}
\end{align*}
$$

Now let $g(\omega)$ be Lorentzian:

$$
\begin{equation*}
g(\omega)=\frac{\Delta}{\pi\left[\left(\omega-\omega_{0}\right)^{2}+\Delta^{2}\right]} . \tag{2.20}
\end{equation*}
$$

Without loss of generality, we set $\omega_{0}=0$ and $\Delta=1$. With a contour integral in the upper half $\omega$-plane, we have $Z(t)=a(i, t)$ and

$$
\begin{equation*}
\frac{d Z}{d t}=-Z-\frac{K}{2}\left(|Z|^{2} Z-Z\right) \tag{2.21}
\end{equation*}
$$

Using polar coordinates, $Z=R \exp (i \Theta)$, we have

$$
\begin{align*}
\frac{d R}{d t} & =-\frac{1}{2} K R^{3}-\left(1-\frac{1}{2} K\right) R  \tag{2.22}\\
\frac{d \Theta}{d t} & =0 \tag{2.23}
\end{align*}
$$

From Eq. (2.22), we conclude that when $K<K_{c}=2, R=0$ is a stable fixed point, corresponding to the stable incoherent state. A supercritical bifurcation occurs at $K=K_{c}$ as we increase $K: R=0$ becomes unstable and $R=\sqrt{1-\frac{K_{c}}{K}}$ becomes stable, corresponding to the partially synchronized state. Thus, the process recasts the result obtained by Kuramoto's self-consistent analysis introduced in Chapter 1.

The Ott-Antonsen Ansatz greatly simplifies the original infinite-dimensional problem by converting it to a two-dimensional system of ordinary differential equations. But we need to be cautious: to make the method work, several conditions should be verified. Notice that in the derivation, first we require $|a(\omega, t)| \leqslant 1$ to make the Fourier series converge, and second we require $a(\omega, t)$ can be analytically continued from the real line into the complex $\omega$-plane. Ott and Antonsen showed that when these specified conditions are satisfied for $t=0$, they will continue to be satisfied for $0<t<\infty$ [46]. And remarkably, they later showed that the order parameter dynamics obtained by restriction to the reduced manifold are, in fact, the only such attractors of the full system [47].

So far we have seen the power of OA Ansatz in Kuramoto model. In fact, this method can be extended to many other cases. First, the distribution is not restricted to Lorentzian distributions. For example, this method can be applied to $g(\omega)=\frac{\sqrt{2}}{\pi\left(\omega^{4}+1\right)}$, which has four
poles in total. In this case, the Kuramoto model can be reduced to four ordinary differential equations. Other generalizations include the Kuramoto model with external driving [105], communities of phase oscillators [32, 71], or oscillators on a ring [35, 37, 38, 39, 41, 60], etc. In the next sections, we introduce two examples where the OA Ansatz is used in the study of chimera states.

## OA Ansatz for a two-population phase model

In this section, we apply the Ott-Antonsen Ansatz to a two-population phase oscillator model, which is used as a simple model which yields a chimera state [32]. The model equations are

$$
\begin{equation*}
\frac{d \theta_{i}^{\sigma}}{d t}=\omega-\sum_{\sigma^{\prime}=1}^{2} \frac{K_{\sigma \sigma^{\prime}}}{N_{\sigma^{\prime}}} \sum_{j=1}^{N_{\sigma^{\prime}}} \sin \left(\theta_{i}^{\sigma}-\theta_{j}^{\sigma^{\prime}}+\alpha\right) \tag{2.24}
\end{equation*}
$$

Here $\sigma=1,2$ indicates the group number and $N_{\sigma}$ is the number of oscillators in group $\sigma$. To reveal the nonlocal property, let $K_{11}=K_{22}=\mu, K_{12}=K_{21}=\nu$ and $\mu>\nu$. For suitable parameter values, this system exhibits a chimera state one group is synchronized while the other is not. To explain the result, we consider the large $N$ limit. Similar to the Kuramoto model, we define $f^{\sigma}(\theta, t)$ as the probability density of oscillators in population $\sigma$. Conservation of probability gives

$$
\begin{equation*}
\frac{\partial f^{\sigma}}{\partial t}+\frac{\partial}{\partial \theta}\left(f^{\sigma} v^{\sigma}\right)=0 \tag{2.25}
\end{equation*}
$$

where $v^{\sigma}$ is the velocity expressed as

$$
\begin{equation*}
v^{\sigma}(\theta, t)=\omega-\sum_{\sigma^{\prime}=1}^{2} K_{\sigma \sigma^{\prime}} \int \sin \left(\theta-\theta^{\prime}+\alpha\right) f^{\sigma^{\prime}}\left(\theta^{\prime}, t\right) d \theta^{\prime} \tag{2.26}
\end{equation*}
$$

On the other hand, the order parameter in the large- $N$ limit is

$$
\begin{equation*}
Z_{\sigma}(t)=\sum_{\sigma^{\prime}=1}^{2} K_{\sigma \sigma^{\prime}} \int \exp \left(i \theta^{\prime}\right) f^{\sigma^{\prime}}\left(\theta^{\prime}, t\right) d \theta^{\prime} \tag{2.27}
\end{equation*}
$$

Then $v^{\sigma}(\theta, t)$ can be expressed in terms of $Z$ :

$$
\begin{equation*}
v^{\sigma}(\theta, t)=\omega+\frac{1}{2 i}\left(Z_{\sigma} \exp (-i \alpha) \exp (-i \theta)-Z_{\sigma}^{*} \exp (i \alpha) \exp (i \theta)\right) \tag{2.28}
\end{equation*}
$$

Assuming the OA Ansatz, we take the probability density to be of the form

$$
\begin{equation*}
f^{\sigma}(\theta, t)=\frac{1}{2 \pi}\left\{1+\left[\sum_{n=1}^{\infty} a_{\sigma}(t)^{n} \exp (-i n \theta)+c . c .\right]\right\} . \tag{2.29}
\end{equation*}
$$

Combining the above equations we finally arrive at:

$$
\begin{align*}
\dot{a_{1}} & =i \omega a_{1}-\frac{1}{2} a_{1}^{2}\left(K_{11} a_{1}^{*}+K_{12} a_{2}^{*}\right) \exp (i \alpha)+\frac{1}{2}\left(K_{11} a_{1}+K_{12} a_{2}\right) \exp (-i \alpha),  \tag{2.30}\\
\dot{a_{2}} & =i \omega a_{2}-\frac{1}{2} a_{2}^{2}\left(K_{22} a_{2}^{*}+K_{21} a_{1}^{*}\right) \exp (i \alpha)+\frac{1}{2}\left(K_{22} a_{2}+K_{21} a_{1}\right) \exp (-i \alpha) . \tag{2.31}
\end{align*}
$$

Now, let $a_{1}=R_{1} \exp \left(i \Theta_{1}\right)$ and $a_{2}=R_{2} \exp \left(i \Theta_{2}\right)$. A chimera state occurs when one of the groups is synchronized. Without loss of generality, let's assume the synchronized group is group one. Then we have $R_{1}=1$. Defining $R=R_{2}, \psi=\Theta_{1}-\Theta_{2}$, we can reduce the system to

$$
\begin{align*}
\dot{R} & =\frac{1-R^{2}}{2}[\mu R \cos (\alpha)+\nu \cos (\psi-\alpha)]  \tag{2.32}\\
\dot{\psi} & =\frac{1+R^{2}}{2 R}[\mu R \sin (\alpha)-\nu \sin (\psi-\alpha)]-\mu \sin (\alpha)-\nu R \sin (\psi+\alpha) \tag{2.33}
\end{align*}
$$

By analyzing this simple two-dimensional system, bifurcation properties of chimera states can be found.

## OA Ansatz for a ring of nonlocally coupled phase oscillators

In this section, we apply the OA Ansatz to a system of oscillators on a ring. The effective equation we will arrive at is very useful in the analysis of chimera states. We can recover the self-consistency equation of the previous section and we can also analyze the stability of the chimera state by investigating the spectrum of the linearized operator of the effective equation. The model equation of the system we investigated is:

$$
\begin{equation*}
\frac{\partial \theta(x, t)}{\partial t}=\omega(x)-\int_{-\pi}^{\pi} G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y \tag{2.34}
\end{equation*}
$$

where $\theta(x, t)$ is the phase of the oscillator located at position $x$. We suppose that an oscillator at position $x$ has intrinsic frequency $\omega(x)$, and assume that $\omega(x)$ is continuous; $G(x-y)$ is the coupling kernel. Introducing the probability density function $f(x, \omega, \theta, t)$ to characterize the state of the system, we have the continuity equation

$$
\begin{equation*}
\frac{\partial f}{\partial t}+\frac{\partial}{\partial \theta}(f v)=0 \tag{2.35}
\end{equation*}
$$

where $v(x, t)$ satisfies the relation

$$
\begin{equation*}
v(x, t)=\omega(x)-\int_{-\pi}^{\pi} G(x-y) f^{\prime}(y, t) d y \tag{2.36}
\end{equation*}
$$

and $f^{\prime}(y, t)=\int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \sin \left(\theta-\theta^{\prime}+\alpha\right) f\left(y, \omega, \theta^{\prime}, t\right) d \theta^{\prime} d \omega$. We also define the local order parameter

$$
\begin{equation*}
Z(x, t)=\int_{-\pi}^{\pi} G(x-y) \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \exp \left(i \theta^{\prime}\right) f\left(y, \omega, \theta^{\prime}, t\right) d \theta^{\prime} d \omega d y \tag{2.37}
\end{equation*}
$$

Then

$$
\begin{equation*}
v(x, t)=\omega(x)-\frac{1}{2 i}\left(Z^{*}(x, t) \exp (i(\theta+\alpha))-Z(x, t) \exp (-i(\theta+\alpha))\right] \tag{2.38}
\end{equation*}
$$

The above equations can be recast in a more convenient form using the OA Ansatz

$$
\begin{equation*}
f(x, \omega, \theta, t)=\frac{g(x, \omega)}{2 \pi}\left[1+\sum_{n=1}^{\infty} a^{n}(x, \omega, t) \exp (-i n \theta)+c . c .\right] . \tag{2.39}
\end{equation*}
$$

Here $g(x, \omega)$ represents the distribution of natural frequencies at each $x$. Matching terms proportional to different powers of $\exp (i \theta)$, we obtain

$$
\begin{equation*}
\frac{\partial a(x, \omega, t)}{\partial t}=i \omega a(x, \omega, t)+\frac{1}{2}\left[Z(x, t) \exp (-i \alpha)-Z^{*}(x, t) \exp (i \alpha) a^{2}\right] \tag{2.40}
\end{equation*}
$$

where the complex order parameter $Z(x, t)$ is given by

$$
\begin{equation*}
Z(x, t)=\int_{-\pi}^{\pi} G(x-y) \int_{-\infty}^{\infty} g(x, \omega) a(y, \omega, t) d \omega d y \tag{2.41}
\end{equation*}
$$

In the following we take

$$
\begin{equation*}
g(x, \omega)=\frac{D}{\pi\left((\omega-\omega(x))^{2}+D^{2}\right)} . \tag{2.42}
\end{equation*}
$$

Then, defining $z(x, t) \equiv a(x, \omega(x)+i D, t)$ and performing contour integration, we obtain

$$
\begin{equation*}
Z(x, t)=\int_{-\pi}^{\pi} G(x-y) z(y, t) d y \tag{2.43}
\end{equation*}
$$

For later convenience, we define a linear operator $K$ such that $K[z] \equiv \int_{-\pi}^{\pi} G(x-y) z(y) d y$. In the limit $D \rightarrow 0$ the distribution function $g$ reduces to a delta function. The corresponding quantity $z(x, t)$ satisfies

$$
\begin{equation*}
\frac{\partial z(x, t)}{\partial t}=i \omega(x) z+\frac{1}{2}\left[Z(x, t) \exp (-i \alpha)-Z^{*}(x, t) \exp (i \alpha) z^{2}\right] \tag{2.44}
\end{equation*}
$$

where $Z(x, t)$ is given by (2.43). Equations (2.43)-(2.44) constitute the required self-consistency description of the nonlocally coupled phase oscillator system with frequency distribution $\omega(x)$.

An important class of solutions of Eq. (2.44) consists of stationary rotating solutions, i.e., states of the form

$$
\begin{equation*}
z(x, t)=\tilde{z}(x) \exp (-i \Omega t) \tag{2.45}
\end{equation*}
$$

whose common frequency $\Omega$ satisfies the nonlinear eigenvalue relation

$$
\begin{equation*}
i[\Omega+\omega(x)] \tilde{z}+\frac{1}{2}\left[\exp (-i \alpha) \tilde{Z}(x)-\tilde{z}^{2} \exp (i \alpha) \tilde{Z}^{*}(x)\right]=0 \tag{2.46}
\end{equation*}
$$

Here $\tilde{z}(x)$ describes the spatial profile of the rotating solution and $\tilde{Z} \equiv \int_{-\pi}^{\pi} G(x-y) \tilde{z}(y) d y$.
Solving Eq. (2.46) as a quadratic equation in $\tilde{z}$ we obtain

$$
\begin{equation*}
\tilde{z}(x)=\exp (i \beta) \frac{\Omega+\omega(x)-\mu(x)}{\tilde{Z}^{*}(x)}=\frac{\exp (i \beta) \tilde{Z}(x)}{\Omega+\omega(x)+\mu(x)} \tag{2.47}
\end{equation*}
$$

The function $\mu$ is chosen to be $\left[(\Omega+\omega)^{2}-|\tilde{Z}|^{2}\right]^{1 / 2}$ when $|\Omega+\omega|>|\tilde{Z}|$ and $i\left[|\tilde{Z}|^{2}-(\Omega+\omega)^{2}\right]^{1 / 2}$ when $|\Omega+\omega|<|\tilde{Z}|$. This choice is dictated by stability considerations, and in particular the requirement that the essential spectrum of the linearized equation (with respect to the rotating solution) is either stable or neutrally stable (more detail will be provided in later chapters). The coherent (incoherent) region corresponds to the subdomain of $[-\pi, \pi]$ where $|\Omega+\omega(x)|$ falls below (above) $|\tilde{Z}(x)|$. Substitution of expression (2.47) into the definition of $\tilde{Z}(x)$ now leads to the self-consistency relation

$$
\begin{equation*}
\tilde{Z}(x)=\left\langle G(x-y) \exp (i \beta) \frac{\Omega+\omega(y)-\mu(y)}{\tilde{Z}^{*}(y)}\right\rangle . \tag{2.48}
\end{equation*}
$$

Here the bracket $\langle\cdot\rangle$ is defined as the integral over the interval $[-\pi, \pi]$, i.e.,

$$
\begin{equation*}
\langle u\rangle \equiv \int_{-\pi}^{\pi} u(y) d y \tag{2.49}
\end{equation*}
$$

If we write $\tilde{Z}(x)=R(x) \exp (i \Theta(x))$, and refer to $R(x)$ and $\Theta(x)$ as the amplitude and phase of the complex order parameter $\tilde{Z}(x)$, we can obtain

$$
\begin{equation*}
R(x) \exp (i \Theta(x))=\exp (i \beta) \int G(x-y) \exp (i \Theta(y)) \frac{\Omega+\omega(y)-\sqrt{(\Omega+\omega(y))^{2}-R^{2}(y)}}{R(y)} \tag{2.50}
\end{equation*}
$$

Setting $\omega(y)=0$ will recover the equation obtained from Kuramoto's original self-consistency argument.

Temporal stability of a stationary rotating solution is investigated via a linearization of Eq. (2.44). We write

$$
\begin{equation*}
z(x, t)=(\tilde{z}(x)+v(x, t)) \exp (-i \Omega t) \tag{2.51}
\end{equation*}
$$

where $|v(x, t)| \ll 1$, implying that $v$ is a small perturbation. By substituting Eq. (2.51) into Eq. (2.44) and neglecting high order terms, we obtain the linear evolution equation

$$
\begin{equation*}
v_{t}=L[v] \equiv i \mu(x) v+\frac{1}{2}\left[\exp (-i \alpha) V(x, t)-\exp (i \alpha) \tilde{z}^{2} V^{*}(x, t)\right] \tag{2.52}
\end{equation*}
$$

where $V(x, t)=K[v] \equiv \int_{-\pi}^{\pi} G(x-y) v(y, t) d y$, and $\mu(x) \equiv \omega+\Omega+i \exp (i \alpha) \tilde{z} \tilde{Z}^{*}$.
The stability of the rotating solution is determined by the spectrum of the linear operator L. As Eq. (2.52) contains complex conjugate terms, we should expand the eigenfunction as

$$
\begin{equation*}
v(x, t)=\exp (\lambda t) v_{1}(x)+\exp \left(\lambda^{*} t\right) v_{2}^{*}(x) \tag{2.53}
\end{equation*}
$$

leading to the eigenvalue problem

$$
\lambda\binom{v_{1}}{v_{2}}=\frac{1}{2}\left(\begin{array}{cc}
2 i \mu+\exp (-i \alpha) K & -\exp (i \alpha) \tilde{z}^{2} K  \tag{2.54}\\
-\exp (-i \alpha) \tilde{z}^{* 2} K & -2 i \mu^{*}+\exp (i \alpha) K
\end{array}\right)\binom{v_{1}}{v_{2}}
$$

Since $K$ is compact [41], its spectrum consists of two parts, a continuous spectrum given by $\left\{i \mu(x),-i \mu^{*}(x)\right\}$ with $x \in[-\pi, \pi]$ and a (possibly empty) point spectrum. The spectrum is in addition symmetric with respect to the real axis: if $\lambda$ is an eigenvalue with eigenvector $\left(v_{1}, v_{2}\right)^{T}$, then $\lambda^{*}$ is an eigenvalue with eigenvector $\left(v_{2}^{*}, v_{1}^{*}\right)^{T}$. The continuous spectrum is stable (negative) or neutrally stable (purely imaginary). Thus the stability of the chimera states is determined by the point spectrum $\lambda=\lambda_{p}$. For this purpose we rewrite Eq. (2.54) in the form

$$
\left(\begin{array}{cc}
2-\frac{\exp (-i \alpha) K}{\lambda_{p}-i \mu} & \frac{\exp (i \alpha) \tilde{z}^{2} K}{\lambda_{p}-i \mu}  \tag{2.55}\\
\frac{\exp \left(-i \alpha \tilde{z}^{*} 2\right.}{} & 2-\frac{\exp (i \alpha) K}{\lambda_{p}+i \mu^{*}}
\end{array}\right)\binom{v_{1}}{v_{2}+i \mu^{*}}=0 .
$$

The above approach is closest to that of Omel'chenko [39] who proved a number of general results about Eq. (2.1) for general coupling functions $G(x)$, while focusing on two simple choices, $G(x)=(2 \pi)^{-1}(1+A \cos x), 0<A<1[27,31]$, and $G(x)=(2 \pi r)^{-1}$ for $|x| \leq \pi r$ and zero otherwise [35, 41]. The detailed numerical computation of the point eigenvalues $\lambda_{p}$ is described in later chapters.

### 2.3 Recent progress

Fascinated by the unique property of chimera states, researchers have put much effort in this wonderful area of study. In this subsection, we summarize some of the relevant results, partially following Abrams' review paper [106].

## 1. Models that exhibit chimera states.

Chimera states were first discovered in a ring of nonlocally coupled phase oscillators, which is still the most popular system in which to study the properties of the chimera state $[26,27,31,33,35,37,38,39,41,60,76]$. In the model Kuramoto used, he adopts the exponential coupling function $G(x-y) \propto \exp (-\kappa|x-y|)$. This is a realistic assumption, but the self-consistency equations contain a large number of degrees of freedom and hence are hard to solve. Strogatz and Abrams studied a simpler model that made the analysis much easier without losing any qualitative property of the chimera state [27]. The choice of coupling function they use is $G(x-y)=\frac{1}{2 \pi}(1+A \cos (x-y))$. One reason to motivate this choice of coupling is that the right-hand side of Eq. (2.1) is a convolution integral, and trigonometric functions behave nicely under convolution. This choice of coupling function reduces the degrees of freedom of the self-consistency equations significantly. The self-consistency equations turn out to consist of four algebraic equations. With this simplification, the authors are able to discuss the birth and death of chimera states in the parameter space. Other
choices of coupling functions include piecewise constant functions [35, 41, 75], piecewise linear functions [38] and combinations of cosine functions [60]. Using these various coupling schemes, scientists have been able to find multi-cluster chimera states, or even traveling chimera states.

It is natural to extend the model from one dimension to higher dimensions. Chimera states have been reported on an infinite plane [64, 65, 67], a torus [68, 69] and a sphere [70]. An interesting result on two-dimensional systems is the existence of spiral chimeras which consist of an incoherent core surrounded by coherent rotating spiral arms. Spiral chimeras have also been found on a plane, a torus and the surface of a sphere.

For the models mentioned above, the network structures on which the systems are built have a clear spatial interpretation. A two-population phase model was provided in 2008 [32], and is now considered the simplest model exhibiting chimera states. Later this model was extended to a three-population network [71] and community-structured networks [72]. Apart from these simple network structures, Laing et al. considered a two-population system with randomly removed links and observed that chimera states are robust to small structure perturbations [36]. Yao et al. carried out a similar analysis on a ring of oscillators [73]. Zhu et al. analyzed an oscillator system built on a randomly generated Erdös-Rényi network and on scale-free networks [74].

Chimera states are mostly studied in phase oscillator systems. However, they can also be found in systems where the oscillators have an amplitude as well as phase. In fact, both one-dimensional and two-dimensional chimera states were first discovered in the nonlocally coupled complex Ginzburg-Landau equation by Kuramoto et al. [26, 64, 65, 66]. The equations they derived from a generic reaction-diffusion system are

$$
\begin{align*}
& \frac{\partial W}{\partial t}=\left(1+i \omega_{0}\right) W-(1+i b)|W|^{2} W+K(1+i a)(\bar{W}-W)  \tag{2.56}\\
& \bar{W}(\mathbf{x}, t)=\int G(\mathbf{x}-\mathbf{y}) W(\mathbf{y}, t) d \mathbf{y} \tag{2.57}
\end{align*}
$$

When the coupling is weak, these equations reduce to phase-coupled equations of the type discussed here. Sethia et al. [50] studied the nonlocal complex Ginzburg-Landau equation in the strong coupling limit. A new class of chimera states was found, where the incoherent regions display significant amplitude fluctuations. Other works include a paper by Laing, who extended the two-population phase oscillator model to a Stuart-Landau oscillator model, and showed that the expected bifurcations persist [51].

Chimera states have also been observed in systems of coupled iterated maps. Omel'chenko et al. [58] discussed a transition between coherent and incoherent dynamics in the system

$$
\begin{equation*}
z_{i}^{t+1}=f\left(z_{i}^{t}\right)+\frac{\sigma}{2 P} \sum_{j=i-P}^{i+P}\left(f\left(z_{j}^{t}\right)-f\left(z_{i}^{t}\right)\right) \tag{2.58}
\end{equation*}
$$

where the $z_{i}$ are real dynamical variables and $t$ is discrete time; $f(z)$ is a one-dimensional map such as the logistic map $f(z)=A z(1-z)$. Depending on the coupling strength $\sigma$ and coupling range $P$, they observed a combination of period-adding in space and period-doubling in
time. Later they showed that similar behavior exists in time-continuous nonlocally coupled chaotic systems such as Rössler systems [59].

## 2. Stability of chimera states

Temporal stability of a stationary solution is always of interest in the area of dynamics. Wolfrum et al. were the first to present a detailed analysis of the spectral properties of chimera states on a ring of oscillators [41]. They numerically computed the Lyapunov spectrum and studied its behavior for an increasing number of oscillators. They showed that as the system size grows, the hyperchaotic part of the spectrum tends to zero. This is consistent with their other work which numerically shows that chimera states are long-lived transients when the size of the system is finite and that the transience time grows exponentially with the system size [75].

The stability of chimera states can also be analyzed via the effective equation (2.44). It was shown that the continuous spectrum of the linearized operator has non-positive real part, so the stability is determined by the point spectrum only [39] (see Section 2.2). In most studies of chimera states, which explore the parameter region $\beta \approx 0$, there is no point spectrum to the right of the imaginary axis. In [60], we investigated the dependence of the real part of the point eigenvalues on the parameter $\beta$, and showed that in the system we investigated, chimera states lose stability at some critical value $\beta=\beta_{c}$. This is consistent with direct numerical simulation.

## 3. Experiment

The research we discussed thus far focused on numerical simulations. It is of interest to ask if chimera states are robust enough to exist in nature, or at least in the laboratory.

In 2012, two groups of researchers reported that chimera states were observed in experiment. The first group used coupled Belousov-Zhabotinsky oscillators to build a twopopulation model similar to [32], and observed various dynamical patterns including chimera states and full synchronization [54]. The second group observed chimeras in a system of coupled map lattices [77] as originally proposed in [58]. The critique of these two experiments is their reliance on the computer - they still look like a simulation [78]. In 2013, Martens et al. realized chimera states in a mechanical oscillator system [79] and successfully addressed this concern.

## Chapter 3

## Multi-cluster and traveling chimera states in identical nonlocal phase-coupled oscillators

### 3.1 Introduction

In Chapter 2, we introduced a system of identical phase-coupled oscillators with nolocal coupling. In the continuum limit, the model equation is written as

$$
\begin{equation*}
\frac{\partial \theta}{\partial t}=\omega-\int G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y . \tag{3.1}
\end{equation*}
$$

In this system, a surprising new state called a chimera state was discovered for the coupling function $G(y)=\frac{\kappa}{2} \exp (-\kappa|y|)$ [26]. These states consist of a domain or domains of coherent, phase-locked oscillators embedded in a background of incoherent oscillators, and resemble states consisting of laminar flow embedded within a turbulent state familiar from studies of plane Couette flow [28, 29, 30].

To obtain chimera states within the system (3.1), it is usually believed that two basic conditions are important [31]: the coupling should be nonlocal and the parameter $\alpha$ should be nonzero. In fact, the chimera states are mostly found when $\alpha \approx \frac{\pi}{2}$. These numerical observations inform many of the subsequent studies of the chimera state [32, 33, 34, 35, 36, 37, 38, 39]. In these papers, the coupling function $G(x)$ is chosen to be a nonnegative even function that decreases monotonically with $|x|$. In this chapter, we introduce a model with $G(x)$ that is even but relax, following [39], both the monotonicity requirement and the requirement that it be nonnegative. Our motivation for this generalization of the coupled oscillator problem comes from biology, and in particular neural systems [40], in which negative coupling at large separations is quite typical. With this coupling, we have found a much richer variety of chimera states, including a remarkable traveling chimera state. Some of these states, such as multi-cluster chimera states, are similar to those reported recently in [33, 37, 38]. Of these Ref. [33] reports the presence of multi-cluster chimera states in a particular time-delay

## CHAPTER 3. MULTI-CLUSTER AND TRAVELING CHIMERA STATES IN

 IDENTICAL NONLOCAL PHASE-COUPLED OSCILLATORSsystem while Refs. [37] and [38] report a 2-cluster state and an evenly spaced multi-cluster state, respectively. However, no analytical description of these states or of their stability properties has been provided.

We consider phase oscillators distributed uniformly on a one-dimensional ring of length $2 \pi$. The value of $\omega$ can be set to zero by going into a rotating frame. The model equation then takes the form

$$
\begin{equation*}
\frac{\partial \theta}{\partial t}=-\int_{-\pi}^{\pi} G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y \tag{3.2}
\end{equation*}
$$

with $\alpha \in[0, \pi / 2]$. As we did in previous chapters, we define $\beta \equiv \frac{\pi}{2}-\alpha$ for later convenience. Two families of coupling functions are considered:

$$
G_{n}^{(1)}(x) \equiv \cos (n x), \quad G_{n}^{(2)}(x) \equiv \cos (n x)+\cos [(n+1) x],
$$

where $n$ is an arbitrary positive integer. In Fig. 3.1, we show some profiles of $G_{n}^{(2)}(x)$;


Figure 3.1: Profile of $G_{n}^{(2)}(x)$ for (a) $n=1$; (b) $n=2$; (c) $n=3$.
the profiles of $G_{n}^{(1)}(x)$ is quite familiar. In each case the overall strength of the coupling has been set equal to unity by rescaling time. With this coupling, it follows that if $\theta(x, t)$ is a solution so is $\theta(-x, t)$. In view of the periodic boundary conditions this implies that Eq. (3.2) possesses $\mathrm{O}(2)$ symmetry [87]. In particular we are guaranteed the presence of reflection symmetric solutions; such solutions cannot drift in the $x$-direction and we refer to them as stationary states or standing waves. On the other hand solutions that break the symmetry $x \rightarrow-x$ are expected to drift in the $x$-direction, i.e., along the ring, and we refer to such solutions as traveling waves; for each solution that drifts to the right there is a solution that drifts to the left, obtained by reflection in $x$.

The approach we follow is summarized in Chapter 2. Our results for $G_{n}^{(1)}(x)$ resemble known results for $G(x)=(2 \pi)^{-1}(1+A \cos x), 0<A<1$, in that we identify both single and multi-cluster chimeras with evenly distributed clusters, indicating that the nonzero mean of $G(x)$ in [39] does not play a major role. On the other hand the situation changes dramatically when the coupling $G_{n}^{(2)}(x)$ is used instead. This coupling allows us to identify multi-cluster
chimeras with unevenly distributed clusters as well as two types of traveling structures: a traveling coherent state and a single-cluster traveling chimera state. To characterize these states we solve in each case a nonlinear integral equation for the complex order parameter describing the state in the continuum limit, and compare the result with extensive simulations using large numbers of oscillators that are necessary to reduce the effects of fluctuations due to finite oscillator number. In addition, we study the linear stability of these states, and examine their bifurcations as the parameter $\beta$ is varied. Some of these lead to hysteretic transitions to different states, while others lead to nearby stable states.

This chapter is organized as follows. In Sections 3.2 and 3.3 we study the system (3.2) with the coupling functions $G_{n}^{(1)}(x)$ and $G_{n}^{(2)}(x)$, respectively. In both cases we describe a self-consistency analysis of the chimera states found, focusing on multi-cluster chimera and on their stability properties. Section 3.3 also reports our results on the two traveling states identified in our numerical simulations with $G_{n}^{(2)}(x)$ and formulates a nonlinear complexvalued eigenvalue problem for the drift speed and rotation frequency of these states. For the traveling coherent state the solution of this problem is in excellent agreement with the results of numerical simulations despite the episodic nature of the drift near the onset of the drift instability. A brief conclusion is provided in Section 3.4 together with directions for future work.

## $3.2 \quad G_{1}^{(1)}(x)$ coupling

In this section, we consider the case $G_{n}^{(1)}(x) \equiv \cos (n x)$. Here and elsewhere all numerical simulations are performed using a fourth-order Runge-Kutta method with time step $\delta t=$ 0.025 or $\delta t=0.01$. The ring is discretized into $N$ oscillators with $N$ ranging from 512 to 4096.

## Splay states

Synchronized states with

$$
\begin{equation*}
\theta(x, t)=-\Omega t+q x \tag{3.3}
\end{equation*}
$$

are referred to as splay states $[23,25]$ and form an important class of solutions to both locally and globally-coupled phase oscillator systems. Here we use $-\Omega$ instead of $\Omega$ to make the description of splay states consistent with the description of chimera states. In the present context the frequency $\Omega$ satisfies

$$
\begin{equation*}
\Omega=\int_{-\pi}^{\pi} G(y) \sin (q y+\alpha) d y \tag{3.4}
\end{equation*}
$$

and $q$ is an integer in order that periodic boundary conditions be satisfied. States of this type travel with speed $c=\Omega / q$, i.e., to the right if $\Omega>0$ and $q>0$ (positive slope) and to the left if $\Omega>0$ and $q<0$ (negative slope) and vice versa if $\Omega<0$. The fully synchronized

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Figure 3.2: Stable splay states with (a) $G_{1}^{(1)}(x) \equiv \cos (x)$ and (b) $G_{2}^{(1)}(x) \equiv \cos (2 x)$. In both cases $\beta=0.1$ and $N=512$. State (a) travels with speed $c=3.124$, while (b) travels with speed $c=1.563$, both towards the right.
state corresponds to the special splay state with $q=0$ and does not travel. Similar states exist in systems of nonidentical phase oscillators as well [42, 60].

To analyze the linear stability properties of these states, we follow [25] and let $\theta(x, t)=$ $-\Omega t+q x+\eta(x, t)$, where $|\eta| \ll 1$. The linearized equation takes the form

$$
\begin{equation*}
\frac{\partial \eta}{\partial t}=\int_{-\pi}^{\pi} G(x-y) \cos [q(x-y)+\alpha][\eta(y, t)-\eta(x, t)] d y \tag{3.5}
\end{equation*}
$$

Expanding Eq. (3.5) gives

$$
\begin{align*}
\frac{\partial \eta}{\partial t}= & \int_{-\pi}^{\pi} G(x-y) \cos [q(x-y)] \cos (\alpha) \eta(y, t) d y \\
& -\int_{-\pi}^{\pi} G(x-y) \sin [q(x-y)] \sin (\alpha) \eta(y, t) d y \\
& -\int_{-\pi}^{\pi} G(x-y) \cos [q(x-y)] \cos (\alpha) \eta(x, t) d y \tag{3.6}
\end{align*}
$$

which admits solutions of the form $\eta \sim \exp \left(\lambda_{m} t\right) \exp (i m x)$. Here $\lambda_{m}$ is the linear growth rate that can be expressed as

$$
\begin{equation*}
\lambda_{m}=\frac{1}{2}\left(\exp (-i \alpha) \hat{G}_{q+m}+\exp (i \alpha) \hat{G}_{q-m}\right)-\hat{G}_{q} \cos \alpha \tag{3.7}
\end{equation*}
$$

with $\hat{G}_{q} \equiv \int_{-\pi}^{\pi} G(y) \exp (i q y) d y$. This convention is adopted throughout this chapter as the definition of a Fourier coefficient. For the coupling function $G_{n}^{(1)}(x), \hat{G}_{q}=\pi\left(\delta_{n+q}+\delta_{n-q}\right)$. Here $\delta_{i}$ stands for Kronecker delta function which equals 1 when $i=0$ and equals 0 otherwise.

When $\alpha$ is equal to $\frac{\pi}{2}$, all the splay states are marginally stable. With a random incoherent initial condition, the system tends to remain incoherent as time evolves. When $\alpha<\frac{\pi}{2}$, the


Figure 3.3: Chimera states with (a) $G_{1}^{(1)}(x) \equiv \cos (x)$ and (b) $G_{2}^{(1)}(x) \equiv \cos (2 x)$ obtained from random initial conditions. In both cases $\beta=0.1$ and $N=512$.
splay states with $|q|=n$ become linearly stable while all the others become unstable. This is consistent with our numerical simulations. Snapshots of two different stable splay states are shown in Fig. 3.2. Starting from an unstable splay state with small random noise added or simply from a completely incoherent state with $R=0$ (see below) we find that the system always evolves into one of two attractors: a stable splay state with $|q|=n$ or a multi-cluster chimera state. The system is therefore bistable, with the final state selected by the initial condition chosen, cf. [33]. The properties of the multi-cluster chimera states are discussed in the next subsection.

## Multi-cluster chimera states

As mentioned above, multi-cluster chimera states are obtained in numerical simulations with the coupling function $G_{n}^{(1)}(x)$ and different values of $n$. Figures 3.3 and 3.4 show the results for $n=1,2,3$ and 4 and appropriate values of $\beta$, all at $t=5000$. In previous studies [26, 27, 32], chimera states were only observed when starting from carefully prepared initial conditions while here chimera states are easily obtained even from random initial conditions. This is also the case in the system with nonlinear nonlocal coupling described in [43]. This suggests the chimera states with the coupling function $G_{n}^{(1)}(x)$ have a larger basin of attraction than with the exponential coupling function used by Kuramoto and Battogtokh [26]. However, as in the previous studies, chimera states first appear when $\beta$ is small but nonzero, while for large $\beta$ the splay state is preferred and appears more and more frequently when the system is initialized using random initial conditions.

The chimeras shown in Figs. 3.3 and 3.4 are stationary in the sense that they do not display any organized or coherent motion. This is a consequence of the symmetry of these states under spatial reflection. Of course, owing to the incoherence of the surrounding oscillators and their finite number, each coherent cluster will undergo fluctuations in both its location and rotation frequency, although the clusters remain, on average, evenly spaced. These fluctuations are quite small for the simulations reported here and become even smaller


Figure 3.4: Chimera states with (a) $G_{3}^{(1)}(x) \equiv \cos (3 x)$ and (b) $G_{4}^{(1)}(x) \equiv \cos (4 x)$ obtained from random initial conditions. (a) $\beta=0.05$ and $N=512$; (b) $\beta=0.02$ and $N=1024$.
as the number $N$ of oscillators increases. Our simulations suggest that the length of the cluster and the positions of the bounding fronts execute zero-mean Brownian motion with standard deviation $\sigma(N)$ (see below).

For $G_{n}^{(1)}(x) \equiv \cos (n x)$, the number of coherent clusters in the chimera state is always $2 n$. In fact, if $\theta(x, t)$ is a solution for $G_{1}^{(1)}(x)$, then $\theta_{m}(x, t) \equiv \theta(m x, t)$ is a solution for $G_{m}^{(1)}(x)$. Thus each multi-cluster chimera with $G_{n}^{(1)}(x)$ is in fact a concatenation of $n$ single-cluster chimeras with $G_{1}^{(1)}(x)$.

To study these chimera states, we adopt the effective description as in Chapter 2. The effective equation for the current case can be written as:

$$
\begin{equation*}
\frac{\partial z(x, t)}{\partial t}=\frac{1}{2}\left[\exp (-i \alpha) Z(x, t)-\exp (i \alpha) z^{2}(x, t) Z^{*}(x, t)\right] \tag{3.8}
\end{equation*}
$$

where $Z(x, t) \equiv K z(x, t)$ and $K$ is a compact linear operator defined by

$$
\begin{equation*}
K v(x, t) \equiv \int_{-\pi}^{\pi} G(x-y) v(y, t) d y \tag{3.9}
\end{equation*}
$$

The variable $z(x, t)$ can be interpreted as local mean field [41], which can be defined as the local average of $\exp [i \theta(x, t)]$,

$$
\begin{equation*}
z(x, t) \equiv \lim _{\delta \rightarrow 0^{+}} \frac{1}{\delta} \int_{-\delta / 2}^{\delta / 2} \exp (i \theta(x+y, t)) d y \tag{3.10}
\end{equation*}
$$

The local mean field $z(x, t)$ effectively smooths out the phase in the incoherent region and yields a well-defined dynamical system.

The chimera states reported above belong to a special class of solutions of Eq. (3.8) referred to as standing waves [45]. Solutions belonging to this class are stationary in an appropriate rotating frame. We assume that this frame has angular frequency $-\Omega$ relative

(b)


Figure 3.5: (Color online) (a) The phase distribution $\theta$ in a chimera state with coupling $G_{1}^{(1)}(x) \equiv \cos (x)$. (b) The local order parameters $R$ (red dashed line) and $\Theta$ (blue dotted line) computed from Eq. (3.10) and the definitions $z(x, t) \equiv \exp (-i \Omega t) \tilde{z}(x)$ and $\tilde{Z} \equiv R(x) \exp (i \Theta(x))=K \tilde{z}$. The simulation was done with $\beta=0.1$ and $N=512$.
to the original frame. In this case the rotating wave takes the form $z(x, t) \equiv \exp (-i \Omega t) \tilde{z}(x)$ and Eq. (3.8) reduces to

$$
\begin{equation*}
i \Omega \tilde{z}(x)+\frac{1}{2}\left[\exp (-i \alpha) \tilde{Z}(x)-\tilde{z}^{2}(x) \exp (i \alpha) \tilde{Z}^{*}(x)\right]=0 \tag{3.11}
\end{equation*}
$$

where $\tilde{Z}(x) \equiv \exp (i \Omega t) Z(x, t)$. Solving Eq. (3.11) as a quadratic equation in $\tilde{z}$ we obtain [41]

$$
\begin{equation*}
\tilde{z}(x)=\exp (i \beta) \frac{\Omega-\mu(x)}{\tilde{Z}^{*}(x)}=\frac{\exp (i \beta) \tilde{Z}(x)}{\Omega+\mu(x)} \tag{3.12}
\end{equation*}
$$

where, for reasons explained in [41], $\mu(x)$ is equal to $\sqrt{\Omega^{2}-|\tilde{Z}(x)|^{2}}$ when $|\Omega|>|\tilde{Z}(x)|$ and $i \sqrt{|\tilde{Z}(x)|^{2}-\Omega^{2}}$ when $|\Omega|<|\tilde{Z}(x)|$. As explained below this choice of root also corresponds to solutions with a stable essential spectrum and hence to solutions that are potentially stable. Since $\tilde{Z}(x, t) \equiv K \tilde{z}(x, t)$, Eq. (3.12) is equivalent to the self-consistency relation [26]

$$
\begin{equation*}
\tilde{Z}(x) \equiv R(x) \exp (i \Theta(x))=\exp (i \beta) \int_{-\pi}^{\pi} G(x-y) \exp (i \Theta(y)) \frac{\Omega-\mu(y)}{R(y)} d y \tag{3.13}
\end{equation*}
$$

The $x$ dependence of the quantity $\mu$ arises from its dependence on the unknown function $\tilde{Z}(x)$. Here $R>0$ and $\Theta$ are real-valued functions of $x$ and play the role of local order parameters.

Equation (3.13) may be applied to our model (Fig. 3.5). Since Eq. (3.2) is invariant under (i) translation in $x$, and (ii) phase rotation (i.e., translation in $\theta$ ), it follows that if $R(x) \exp (i \Theta(x))$ is a solution of Eq. (3.13), then so is $R\left(x+x_{0}\right) \exp \left(i\left[\Theta\left(x+x_{0}\right)+\Theta_{0}\right]\right)$. Here $x_{0}$ and $\Theta_{0}$ are arbitrary real constants. Using this property, we pick $x_{0}$ and $\Theta_{0}$ such that the self-consistency relation for the $G_{n}^{(1)}(x)$ coupling takes the form

$$
\begin{equation*}
R \exp (i \Theta)=a \cos (n x)+i b \sin (n x) \tag{3.14}
\end{equation*}
$$

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where $a>0$ and $b$ are real constants satisfying

$$
\begin{align*}
a & =\exp (i \beta)\langle h(y) \exp (i \Theta(y)) \cos (n y)\rangle,  \tag{3.15}\\
i b & =\exp (i \beta)\langle h(y) \exp (i \Theta(y)) \sin (n y)\rangle . \tag{3.16}
\end{align*}
$$

The bracket $\langle\cdot\rangle$ is defined by $\langle f\rangle=\int_{-\pi}^{\pi} f(y) d y$.
This procedure can be applied to the local order parameters $R(x)$ and $\Theta(x)$ shown in Fig. 3.5(b). Moreover, since both $R$ and $\Theta$ are even functions of $x$ with respect to a suitable origin, it follows that $b=0$ except at phase discontinuities where $R=0$ (Fig. 3.5(b)), and hence that $R(x)=R_{0}|\cos (n x)|, R_{0}=a$. The self-consistency relation thus becomes

$$
\begin{equation*}
R_{0}^{2}=\exp (i \beta)\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \cos ^{2}(n y)}\right\rangle \tag{3.17}
\end{equation*}
$$

For any integer $n$, we have

$$
\begin{equation*}
\int_{-\pi}^{\pi} F(\cos (n y)) d y=\frac{1}{n} \int_{-n \pi}^{n \pi} F\left(\cos \left(y^{\prime}\right)\right) d y^{\prime}=\int_{-\pi}^{\pi} F(\cos (y)) d y \tag{3.18}
\end{equation*}
$$

for any integrable function $F$. Based on this observation, we can see that the solution of Eq. (3.17) is independent of $n$ provided $n$ is an integer. With a similar change of variable process, we can prove this is also true for a half-integer $n$.


Figure 3.6: (a) The quantities $R_{0}$ and $\Omega$, and (b) the coherent fraction $e$, all as functions of $\beta$ for the 2-cluster chimera state with $G_{1}^{(1)}(x)$ coupling (Fig. 3.5(a)).

Equation (3.17) can be regarded as two equations (real and imaginary parts) with two unknowns $a$ and $\Omega$. Solving these equations by numerical continuation with $n=1$ and $\beta=0.1$ as the starting point we can determine the dependence of $a, \Omega$ and the coherent fraction $e$ on $\beta$ (Fig. 3.6). The coherent fraction $e$ is defined as the ratio of the total length of coherent clusters to the spatial domain size, $2 \pi$. The starting values of $a$ and $\Omega$ are first obtained by temporal simulation and then corrected via the self-consistency relation (3.17). The plots indicate the 2-cluster chimera states are born from an incoherent state with $R>0$

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as $\beta$ increases from zero. Even though the self-consistency relation indicates the existence of chimera states for $0<\beta<\pi / 2$, such states are not necessarily stable or have a large enough basin of attraction to be observed in numerical simulations. For example, starting from the 2 -cluster chimera state in Fig. 3.5(a), we increased $\beta$ from 0.1 in small steps, each time evolving the system until it reached a steady chimera solution. This process failed for the first time at $\beta \approx 0.170 \pm 0.005$, where the solution evolved into a $|q|=1$ splay state. The calculation was repeated with four different choices of $N, N=512,1024,2048$ and 4096, with essentially identical results.

Linear stability of the chimera states can be studied by linearizing Eq. (3.8) about $\tilde{z}(x)$, as described in Chapter 2:

$$
\begin{equation*}
\frac{\partial v(x, t)}{\partial t}=i \mu v(x, t)+\frac{1}{2}\left[\exp (-i \alpha) V(x, t)-\tilde{z}^{2}(x, t) \exp (i \alpha) V^{*}(x, t)\right] \tag{3.19}
\end{equation*}
$$

Here $v(x, t)$ represents a small deviation from $\tilde{z}$ and $V(x, t) \equiv K v(x, t)$. This equation is solved by

$$
\begin{equation*}
v(x, t)=\exp (\lambda t) v_{1}(x)+\exp \left(\lambda^{*} t\right) v_{2}^{*}(x), \tag{3.20}
\end{equation*}
$$

leading to the eigenvalue problem

$$
\lambda\binom{v_{1}}{v_{2}}=\frac{1}{2}\left(\begin{array}{cc}
2 i \mu+\exp (-i \alpha) K & -\exp (i \alpha) \tilde{z}^{2} K  \tag{3.21}\\
-\exp (-i \alpha) \tilde{z}^{* 2} K & -2 i \mu^{*}+\exp (i \alpha) K
\end{array}\right)\binom{v_{1}}{v_{2}}
$$

The spectrum consists of two parts, a continuous spectrum given by $\left\{i \mu(x),-i \mu^{*}(x)\right\}$ with $x \in[-\pi, \pi]$ and a point spectrum. The continuous spectrum is stable or neutrally stable. Thus the stability of the chimera states is determined by the point spectrum. To compute unstable point eigenvalues $\lambda_{p}$, we rewrite Eq. (3.21) in the form

$$
\left(\begin{array}{cc}
2-\frac{\exp (-i \alpha) K}{\lambda_{p}-i \mu} & \frac{\exp (i \alpha) \tilde{z}^{2} K}{\lambda_{p}-i \mu}  \tag{3.22}\\
\frac{\exp (-i \alpha) \tilde{z}^{*}{ }^{2} K}{\lambda_{p}+i \mu^{*}} & 2-\frac{\exp (i \alpha) K}{\lambda_{p}+i \mu^{*}}
\end{array}\right)\binom{v_{1}}{v_{2}}=0
$$

and define $f \equiv \frac{1}{4} \frac{\exp (-i \alpha)}{\lambda_{p}-i \mu}, f^{*} \equiv \frac{1}{4} \frac{\exp (i \alpha)}{\lambda_{p}+i \mu^{*}}, g \equiv \frac{1}{4} \frac{\exp (i \alpha) \tilde{z}^{2}}{\lambda_{p}-i \mu}$, and $g^{*} \equiv \frac{1}{4} \frac{\exp (-i \alpha) \tilde{z}^{* 2}}{\lambda_{p}+i \mu^{*}}$. Note that $f^{*}$ and $g^{*}$ are not necessarily the complex conjugate of $f$ and $g$, as $\lambda_{p}$ can be complex. As suggested in [39], it is convenient to solve the eigenvalue problem using Fourier basis functions, especially so since the coupling function is sinusoidal. Equation (3.22) then takes the form

$$
\begin{equation*}
\sum_{m} \mathbf{B}_{l m} \hat{\mathbf{v}}_{m}=0 \tag{3.23}
\end{equation*}
$$

where

$$
\begin{aligned}
\mathbf{B}_{l m} & =\left(\begin{array}{cc}
\pi \delta_{l, m}-\hat{f}_{l-m} \hat{G}_{m} & \hat{g}_{l-m} \hat{G}_{m} \\
\hat{g}_{l-m}^{*} \hat{G}_{m} & \pi \delta_{l, m}-\hat{f}_{l-m}^{*} \hat{G}_{m}
\end{array}\right), \\
\hat{\mathbf{v}}_{m} & =\binom{\hat{v}_{1, m}}{\hat{v}_{2, m}},
\end{aligned}
$$

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and $\hat{f}_{l}, \hat{f}_{l}^{*}, \hat{g}_{l}, \hat{g}_{l}^{*}$ are the Fourier coefficients of $f, f^{*}, g$ and $g^{*}$, respectively; the latter are defined by $\hat{f}_{l}=\int_{-\pi}^{\pi} f \exp (i l x) d x$, etc.

The point eigenvalue $\lambda_{p}$ satisfies the condition $\operatorname{det}\left(\mathbf{B}\left(\lambda_{p}\right)\right)=0$. With the coupling function $G_{n}^{(1)}$ we obtain

$$
\operatorname{det}\left(\begin{array}{cccc}
1-\hat{f}_{0} & \hat{g}_{0} & -\hat{f}_{-2 n} & \hat{g}_{-2 n}  \tag{3.24}\\
\hat{g}_{0}^{*} & 1-\hat{f}_{0}^{*} & \hat{g}_{-2 n}^{*} & -\hat{f}_{-2 n}^{*} \\
-\hat{f}_{2 n} & \hat{g}_{2 n} & 1-\hat{f}_{0} & \hat{g}_{0} \\
\hat{g}_{2 n}^{*} & -\hat{f}_{2 n}^{*} & \hat{g}_{0}^{*} & 1-\hat{f}_{0}^{*}
\end{array}\right)=0
$$

or equivalently

$$
w(\lambda) \equiv \operatorname{det}\left(\begin{array}{cccc}
1-\hat{f}_{0} & -\hat{f}_{-2 n} & \hat{g}_{0} & \hat{g}_{-2 n}  \tag{3.25}\\
-\hat{f}_{2 n} & 1-\hat{f}_{0} & \hat{g}_{2 n} & \hat{g}_{0} \\
\hat{g}_{0}^{*} & \hat{g}_{-2 n}^{*} & 1-\hat{f}_{0}^{*} & -\hat{f}_{-2 n}^{*} \\
\hat{g}_{2 n}^{*} & \hat{g}_{0}^{*} & -\hat{f}_{2 n}^{*} & 1-\hat{f}_{0}^{*}
\end{array}\right)=0 .
$$

The resulting point eigenvalue is computed using continuation based on Newton's method. For the chimera state in Fig. 3.5(a) the calculation shows that an unstable real point eigenvalue emerges from the continuous spectrum as $\beta$ increases above $\beta \approx 0.17$, in agreement with the result from direct numerical simulation. To confirm that there is only one unstable point eigenvalue, we evaluate the integral $\frac{1}{2 \pi i} \oint w^{\prime}(\lambda) / w(\lambda) d \lambda$ for closed contours in the upper half of the complex $\lambda$ plane. Since the integral is consistently equal to 1 there are no additional unstable point eigenvalues. Figure 3.7(a) shows the spectrum of Eq. (3.21) when $\beta \approx 0.83$ while Fig. 3.7(b) shows the point eigenvalue $\lambda_{p}$ as a function of $\beta$. The corresponding eigenvector is shown in Fig. 3.8 and is symmetric under reflection $x \rightarrow-x$.


Figure 3.7: (a) Spectrum of the linearized operator in Eq. (3.21) for $G_{1}^{(1)}$ when $\beta \approx 0.83$. (b) Dependence of the point eigenvalue $\lambda_{p}$ on $\beta$.

As mentioned earlier, owing to the finite number of oscillators in the simulations, each coherent cluster in a multi-cluster chimera undergoes fluctuations in both its location and


Figure 3.8: The eigenvector $\left(v_{1}, v_{2}\right)$ of the unstable point eigenvalue with coupling $G_{1}^{(1)}(x)$ at threshold. Left panels: $\left|v_{1}(x)\right|$ and $\left|v_{2}(x)\right|$. Right panels: phase of $v_{1}(x)$ and $v_{2}(x)$. The phase jumps by $\pm \pi$ whenever the modulus vanishes.
rotation frequency, although the clusters remain on average evenly spaced. Here we examine the details of the associated fluctuations numerically for the case $n=1$ using several different values of $N: 256,362,512,724,1024,1448$, and 2048 . For each value of $N$, we collect data based on time simulation of a 2-cluster chimera state starting with three different but initial conditions with $R \approx 0$. The total simulation time for each run is 5000 . To avoid initial transients data points between $t=0$ and 1000 are excluded from the calculation of the statistics.

To obtain the location and rotation frequency of each coherent cluster, we compute the discrete version of the local order parameter $Z^{(N)} \equiv \frac{2 \pi}{N} \sum_{m=1}^{N} G\left(x-x_{m}\right) \exp \left(i \theta_{m}\right)$, where $x_{m}=\frac{2 \pi m}{N}$ and $\theta_{m}=\theta\left(x_{m}\right)$, and compare it with the general form of the local order parameter

$$
\begin{equation*}
\left\{a \cos \left[n\left(x-x_{0}\right)\right]+i b \sin \left[n\left(x-x_{0}\right)\right]\right\} \exp \left(-i \int \Omega d t^{\prime}\right) \tag{3.26}
\end{equation*}
$$

to obtain the real coefficients $x_{0}, a, b$, and $\Omega$, all of which fluctuate as time evolves. Figure 3.9 shows a sample plot of the position $x_{0}$ of one of the coherent clusters as a function of $t$ when $\beta=0.1$ and $N=256$. Because of the reflection symmetry of this state on average we expect the position $x_{0}$ to undergo fluctuations with zero mean, cf. [35].

The dynamics of $\dot{x}_{0}(t), \Omega(t)-\bar{\Omega}$, and $a(t)-\bar{a}$ are modeled well by a Gaussian white noise $\eta(t)$ satisfying $\overline{\eta(t) \eta\left(t^{\prime}\right)}=\sigma^{2} \delta\left(t-t^{\prime}\right)$. Figure 3.10 shows the dependence of the standard deviation $\sigma$ on $\log _{2} N$ for $\dot{x}_{0}(t), \Omega(t)-\bar{\Omega}$, and $a(t)-\bar{a}$. The results show that $\sigma$ scales as $N^{-0.523}$ for $\dot{x}_{0}, N^{-0.536}$ for $\Omega$ and $N^{-0.515}$ for $a$. For comparison, for a step-like coupling function and a 1-cluster chimera the corresponding result is $\sigma \sim N^{-0.845}$ [35].

## $3.3 G_{n}^{(2)}$ coupling

In this section we consider the case with the coupling function $G_{n}^{(2)}$.


Figure 3.9: The position $x_{0}$ of a coherent cluster as a function of $t$ in a chimera state obtained with the coupling $G_{1}^{(1)}(x)$ when $\beta=0.1$ and $N=256$.


Figure 3.10: The dependence of the standard deviation $\sigma$ of (a) $x_{0}$, (b) $\Omega-\bar{\Omega}$, and (c) $a-\bar{a}$ on $\log _{2} N$ when $\beta=0.1$.

## Splay states

Splay states with $G_{n}^{(2)}$ and their linear stability properties are determined as for $G_{n}^{(1)}$. The growth rate $\lambda_{m}$ for an eigenmode of the form $\exp (i m x)$ still satisfies the relation (3.7) but this time with

$$
\begin{equation*}
\hat{G}_{q}=\pi\left(\delta_{n+q}+\delta_{n-q}+\delta_{n+1+q}+\delta_{n+1-q}\right) . \tag{3.27}
\end{equation*}
$$

The $|q|=n, n+1$ splay states become linearly stable for nonzero $\beta$ while all the other splay states become unstable. Figure 3.11 shows the frequency $\Omega$ for the stable $q=3,4$ splay states for $G_{3}^{(2)}$ coupling as a function of $\beta$ together with their drift speeds $c=\Omega / q$. This speed is positive, implying that positive slope splay states travel to the right. As before the final state reached from random initial conditions can be either a splay state with $|q|=n$ or $n+1$, or a stationary multi-cluster chimera state. However, this time we have also identified a family of entirely new states that can also be reached from random initial conditions: traveling chimera states. Traveling coherent states are also present, as discussed further below.


Figure 3.11: (a) The dependence of speed $c$ on $\beta$ for $q=3$ (solid lines), and $q=4$ (dashed line) splay states with $G_{3}^{(2)}$ coupling. (b) The dependence of frequency $\Omega$ on $\beta$ for splay states with $G_{3}^{(2)}$ coupling, it is the same in both $q=3$ and $q=4$ cases.


Figure 3.12: Chimera states with (a) $G_{1}^{(2)} \equiv \cos (x)+\cos (2 x)$ and (b) $G_{2}^{(2)} \equiv \cos (2 x)+\cos (3 x)$ obtained from random initial conditions. In both cases $\beta=0.03$ and $N=512$.

## Multi-cluster chimera states

Here we report our results on stationary multi-cluster chimera states with the coupling $G_{n}^{(2)}$. Figures 3.12 and 3.13 show the phase distribution in multi-cluster chimera states obtained with $n=1,2,3$ and 4 . The figures reveal a total of $2 n+1$ coherent clusters in each case, distributed evenly across both the spatial domain and the phase $\theta$.

We examine the properties of these states in the case $n=1$. Figure 3.14 shows the position of one of the coherent clusters as a function of time. The plot indicates that the cluster remains on average stationary, at least for moderate times, although slow drift over very long times cannot be excluded. The apparent stationarity of the cluster permits us to employ a self-consistency analysis analogous to that leading to Eq. (3.13). With $G=G_{1}^{(2)}$ this equation yields

$$
\begin{equation*}
R \exp (i \Theta)=a \cos x+b \sin x+c \cos (2 x)+d \sin (2 x) \tag{3.28}
\end{equation*}
$$

## CHAPTER 3. MULTI-CLUSTER AND TRAVELING CHIMERA STATES IN IDENTICAL NONLOCAL PHASE-COUPLED OSCILLATORS



Figure 3.13: Chimera states with (a) $G_{3}^{(2)} \equiv \cos (3 x)+\cos (4 x)$ and (b) $G_{4}^{(2)} \equiv \cos (4 x)+\cos (5 x)$ obtained from random initial conditions. In both cases $\beta=0.03$ and $N=512$.


Figure 3.14: The position $x_{0}$ of a coherent cluster in the 3-cluster chimera state obtained with coupling $G_{1}^{(2)} \equiv \cos (x)+\cos (2 x)$ as a function of $t$ for $t \geq 5000$ when $\beta=0.1$ and $N=512$, starting from random initial conditions at $t=0$.
where $a, b, c$, and $d$ are complex numbers given by

$$
\begin{align*}
a & =\exp (i \beta)\langle h \exp (i \Theta) \cos y\rangle,  \tag{3.29}\\
b & =\exp (i \beta)\langle h \exp (i \Theta) \sin y\rangle,  \tag{3.30}\\
c & =\exp (i \beta)\langle h \exp (i \Theta) \cos 2 y\rangle,  \tag{3.31}\\
d & =\exp (i \beta)\langle h \exp (i \Theta) \sin 2 y\rangle, \tag{3.32}
\end{align*}
$$

with $h$ defined as $h(x) \equiv \frac{\Omega-\mu(x)}{R(x)}$. Translations in $x$ and $\theta$ allow us to fix two of the unknown variables in the self-consistency equation. With these conditions, we can solve for $a, b, c, d$ and the real quantity $\Omega$.

Instead of solving the self-consistency equation by brute force, Fig. 3.15 indicates that one can shift the coordinate so that $R(x)$ becomes an even function of $x$,

$$
\begin{equation*}
R(x)=R_{0}|\cos (3 x / 2)|, \quad R_{0}>0 \tag{3.33}
\end{equation*}
$$


(b)


Figure 3.15: (Color online) (a) The phase distribution $\theta(x)$ in a 3-cluster chimera state with coupling $G_{1}^{(2)} \equiv \cos (x)+\cos (2 x)$ and $\beta=0.05, N=512$ (Fig. 3.12(a)). (b) The local order parameters $R$ (red dashed line) and $\Theta$ (blue dotted line).
while $\Theta(x)$ consists of straight line segments with slope $\pm 1 / 2$ and phase jumps $\pm \pi$ whenever $R(x)$ touches zero (the case of positive slope is not shown in Fig. 3.15 but is also observed in the simulations). The even parity of $R(x)$ is consistent with the observation that the chimera is stationary. Combining these observations with phase translation, we conclude that the local order parameter for the multi-cluster chimera state takes the form

$$
\begin{equation*}
R \exp (i \Theta)=R_{0} \cos (3 x / 2) \exp ( \pm i x / 2) ; \tag{3.34}
\end{equation*}
$$

indeed, $\tilde{Z}(x)=R_{0} \cos [(2 n+1) x / 2] \exp \pm i x / 2$ for $n \geq 1$. Comparing Eq. (3.34) with Eq. (3.28), one obtains $a=c=R_{0} / 2$ and $b=-d=\mp i R_{0} / 2$. Substituting Eq. (3.34) into the self-consistency equation (3.13) we obtain four relations from the requirement that the coefficients of $\cos x, \sin x, \cos (2 x)$ and $\sin (2 x)$ all vanish. It turns out that these relations are all identical, leading to the final self-consistency requirement

$$
\begin{equation*}
R_{0}^{2}=\exp (i \beta)\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \cos ^{2}(3 y / 2)}\right\rangle \tag{3.35}
\end{equation*}
$$

This equation is of the form (3.17). Since the solutions of this equation are independent of $n$ when $n$ is an integer or half-integer, the solutions presented in Fig. 3.6 also describe the $\beta$-dependence of $R_{0}, \Omega$ and the coherent fraction $e$ for the 3-cluster chimera state with $G_{1}^{(2)}$ coupling. The multi-cluster chimera states are thus also born from the incoherent state as $\beta$ increases from zero. Direct numerical simulations show that this time the solution loses stability when $\beta$ reaches approximately $0.125 \pm 0.005$, a result obtained with $N=512$ and confirmed using $N=1024$, and 2048.

These results are in close agreement with a theoretical stability analysis based on Eq. (3.23). This time the calculation reveals a pair of unstable complex point eigenvalues that appear as $\beta$ increases above $\beta \approx 0.120$, followed by an unstable real point eigenvalue that emerges as $\beta$ reaches $\beta \approx 0.174$. The corresponding eigenvectors are again even, implying that neither instability results in drift. Numerical contour integration indicates that no other unstable point eigenvalues are present. Figure $3.16(\mathrm{a})$ shows the spectrum of Eq. (3.21) when $\beta \approx 0.83$ while Fig. 3.16(b) shows the unstable point eigenvalues $\lambda_{p}$ as a function of $\beta$. The results


Figure 3.16: (Color online) (a) Spectrum of the linearized operator in Eq. (3.21) for $G_{1}^{(2)}$ coupling when $\beta \approx 0.83$. (b) Dependence of the point eigenvalues on $\beta$. Black line: real point eigenvalue. Red (or gray) solid line: real part of the complex point eigenvalues. Red (or gray) dashed line: imaginary part of the complex point eigenvalues.


Figure 3.17: Eigenvectors of (a) the real unstable mode and (b) the Hopf mode at threshold (Fig. 3.16(b)). In each plot, the left panels correspond to $\left|v_{1}(x)\right|$ and $\left|v_{2}(x)\right|$ while the right panels show the phase of $v_{1}(x)$ and $v_{2}(x)$. The phase jumps by $\pm \pi$ whenever the modulus vanishes.
for $G_{3}^{(2)}$ are qualitatively similar, with the oscillatory instability appearing at $\beta \approx 0.129$ and the stationary instability located at $\beta \approx 0.176$.

There remains the question whether the multi-cluster chimera states with the coupling $G_{n}^{(2)}$ are the only nontrivial solutions of Eq. (3.13) other than the splay states. Our numerical simulations indicate that the answer is no. Figures 3.18 and 3.19 provide two examples of stationary but "exotic" chimera states. Figure 3.18 shows the results for $G_{1}^{(2)} \equiv \cos (x)+$ $\cos (2 x)$ while Fig. 3.19 shows the results for $G_{2}^{(2)} \equiv \cos (2 x)+\cos (3 x)$, both with $\beta=0.03$ and $N=512$. There are several differences between these states and the multi-cluster chimera states already discussed. First, the number of the clusters is not equal to $2 n+1$. Second, the coherent clusters in these states are distributed non-uniformly in space. As shown in the snapshots, these states consist of two pairs of coherent domains with the components of each


Figure 3.18: (Color online) (a) The phase distribution $\theta(x)$ in a 4-cluster chimera state with coupling $G_{1}^{(2)} \equiv \cos (x)+\cos (2 x)$ and $\beta=0.03, N=512$. (b) The local order parameters $R$ (red dashed line) and $\Theta$ (blue dotted line). (c,d) A related chimera state with order parameters $R$ and $-\Theta$.


Figure 3.19: (Color online) (a) The phase distribution $\theta(x)$ in a 4-cluster chimera state with coupling $G_{2}^{(2)} \equiv \cos (2 x)+\cos (3 x)$ and $\beta=0.03, N=512$. (b) The local order parameters $R$ (red dashed line) and $\Theta$ (blue dotted line).
pair closer to one another than the mean separation. Finally, the phase of the local order parameter is no longer linear in the spatial variable $x$ and exhibits oscillations. However, we can still shift the $x$-coordinate to make $R(x)$ even, implying that these exotic chimera states are stationary in space except for an overall rotation frequency $\Omega$ (and the presence of fluctuations due to finite size effects).

To understand the properties of the nonuniform multi-cluster chimera state in Fig. 3.18 we translate the state such that $R(0)=R(\pi)=0$. It follows from Eq. (3.28) that $a=c=0$,


Figure 3.20: (Color online) (a) The solution of the self-consistency conditions (3.36)-(3.37) as a function of $\beta$ ( $b_{r}$ : continuous black line, $d_{r}$ : continuous red line, $d_{i}$ : broken red line, $\Omega$ : broken black line). (b) The predicted order parameter $R(x)$ at $\beta=0.24$ (continuous line) in comparison with the line $R=\Omega$ (broken line) indicating that 4-cluster chimeras are present for $\beta \lesssim 0.24$; for $\beta \gtrsim 0.24$ only 2 -cluster chimeras are predicted.
leaving the following two consistency conditions:

$$
\begin{align*}
b \exp (-i \beta) & =\left\langle\frac{b \sin ^{2} x+d \sin x \sin 2 x}{\Omega+\mu(x)}\right\rangle  \tag{3.36}\\
d \exp (-i \beta) & =\left\langle\frac{b \sin x \sin 2 x+d \sin ^{2} 2 x}{\Omega+\mu(x)}\right\rangle \tag{3.37}
\end{align*}
$$

These equations constitute a pair of complex equations for the complex coefficients $b$ and $d$ and the unknown frequency $\Omega$. However, Fig. 3.18 also shows that $\Theta(\pi / 2)=0$ (modulo $\pi$ phase jumps) and we use this observation to deduce that $b_{i}=0$. In this case Eqs. (3.36) and (3.37) reduce to four real equations for $b_{r}, d_{r}, d_{i}$ and $\Omega$.

We begin by observing that a solution with order parameter $Z$ for $\left(b_{r}, d_{r}, d_{i}, \Omega\right)$ implies the presence of a solution with order parameter $Z^{*}$ for $\left(b_{r},-d_{r},-d_{i}, \Omega\right)$. This is a consequence of the invariance of the conditions (3.36)-(3.37) under $d \rightarrow-d$ corresponding to invariance with respect to the translation $x \rightarrow x+\pi$. Figure 3.18 compares a 4 -cluster chimera state with complex order parameter $Z$ with one with order parameter $Z^{*}$, both computed for $\beta=0.03$. Note also that a solution $\beta, b_{r}, d_{r}, d_{i}, \Omega$ implies the presence of a solution $-\beta, b_{r}, d_{r},-d_{i}, \Omega$. In view of these symmetries the self-consistency conditions have two solutions for $\beta=0.03$ : $b_{r}=2.1985, d_{r}=\mp 0.0073, d_{i}= \pm 1.8375, \Omega=2.4683$. A similar calculation for $G_{2}^{(2)}$ (Fig. 3.19) yields the result $b_{r}=-1.6529, d_{r}= \pm 0.00296, d_{i}= \pm 1.7256, \Omega=2.2426$, again for $\beta=0.03$. These results agree well with the measured order parameter in both cases.

Having established the value of the self-consistency analysis for the exotic chimera states we now use it as a predictive tool. In Fig. 3.20 we present a plot of the coefficients $\left(b_{r}, d_{r}, d_{i}, \Omega\right)$ as a function of the parameter $\beta$. The results show that at $\beta \approx 0.24$ the local minima of $R(x)$ touch the line $R=\Omega$ (Fig. 3.20(b)) and for larger $\beta$ dip below $R=\Omega$. At this point the quantity $\mu(x)$ in the consistency condition becomes pure imaginary and the


Figure 3.21: (Color online) (a) The computed phase distribution $\theta(x)$ at $\beta=0.24$ and (b) the corresponding order parameter $R(x)$ (red dashed line) and the associated phase $\Theta(x)$ (blue dotted line) for comparison with the prediction in Fig. 3.20.


Figure 3.22: (Color online) (a) The phase distribution $\theta(x)$ at $\beta=0.6$ and (b) the corresponding order parameter $R(x)$ (red dashed line) and the associated phase $\Theta(x)$ (blue dotted line) for comparison with Fig. 3.21.
solution ceases to exist. We have checked this prediction using numerical simulations. These indicate that for $\beta \gtrsim 0.24$ the 4 -cluster chimera state indeed disappears and that it does so by a pairwise merger of the clusters, forming a 2 -cluster chimera for $\beta \gtrsim 0.24$ (Fig. 3.21).

With increasing $\beta$ these two remaining clusters gradually grow in length (Fig. 3.22) but do not merge. Instead this 2 -cluster state loses stability at $\beta \approx 0.96$ where a real eigenvalue passes through zero. This prediction is consistent with direct numerical simulations provided a sufficiently large number of oscillators is used ( $\beta \approx 0.93$ when $N=512, \beta \approx 0.96$ when $N=2048$ ). The simulations reveal that this instability is responsible for a strongly hysteretic transition to the stationary fully coherent state shown in Fig. 3.23. Since the corresponding order parameters $R(x)$ and $\Theta(x)$ are both even with respect to the same point the coherent state must again be stationary. We therefore write

$$
\begin{equation*}
R \exp (i \Theta)=a \cos (x)+c \cos (2 x) \tag{3.38}
\end{equation*}
$$

and use the symmetries of the order parameter to set $a_{i}=0$. The self-consistency equations for this case are

$$
\begin{equation*}
a \exp (-i \beta)=\left\langle\frac{a \cos ^{2} x+c \cos x \cos 2 x}{\Omega+\mu(x)}\right\rangle, \tag{3.39}
\end{equation*}
$$


(b)


Figure 3.23: (Color online) (a) The phase distribution $\theta(x)$ and (b) the corresponding order parameter $R(x)$ (red dashed line) and the associated phase $\Theta(x)$ (blue dotted line) for the stationary coherent state with $G_{1}^{(2)}$ coupling present at $\beta=0.96$.


Figure 3.24: (Color online) (a) The solution of the self-consistency conditions (3.39)-(3.40) as a function of $\beta$ ( $a_{r}$ : continuous black line, $c_{r}$ : continuous red line, $c_{i}$ : broken red line, $\Omega$ : broken black line). (b) The predicted order parameter $R(x)$ at $\beta=0.77$ (continuous line) in comparison with the line $R=\Omega$ (broken line).

$$
\begin{equation*}
c \exp (-i \beta)=\left\langle\frac{a \cos x \cos 2 x+c \cos ^{2} 2 x}{\Omega+\mu(x)}\right\rangle \tag{3.40}
\end{equation*}
$$

Figure 3.24(a) shows the solution of these equations as a function of $\beta$. Solutions exist for all values of $\beta$ but change their character dramatically below $\beta \approx 0.7644$. Figure 3.24(b) explains the reason for this change: at this value of $\beta$ the global minimum of the order parameter $R(x)$ touches the line $R=\Omega$ for the first time as $\beta$ decreases, and for $\beta \lesssim 0.7644$ the quantity $\mu(x)$ is no longer everywhere real. However, this time the consequences of this fact are different and are discussed in the next section.

## Traveling coherent states

Numerical simulations confirm the presence of a stable stationary coherent state for $G_{1}^{(2)}$ down to $\beta \approx 0.7644$ (a threshold value computed with $N=512$ oscillators) but reveal that for lower values of $\beta$ this state develops a small asymmetry (Fig. 3.25(b)) and begins to travel


Figure 3.25: (a) The phase distribution $\theta(x)$ at (a) $\beta=0.77$ (symmetric distribution), (b) $\beta=0.76$ (asymmetric distribution) and (c) $\beta=0.66$ (asymmetric distribution), all for $N=512$. The state in (b) oscillates in time while drifting to the left; state (c) travels to the left at constant speed. Reflected solutions travel to the right.
to the left. We have examined carefully the behavior of this state near the (nonhysteretic) transition at $\beta \approx 0.7644$. Figure $3.26(\mathrm{a})$ shows the position of the coherent state as a function of time when $\beta=0.762$. The figure reveals that the drift speed to the left is not constant in time but is accompanied by small amplitude oscillations. As $\beta$ increases and one approaches the threshold for this transition the motion takes on the characteristic of stick-slip motion, i.e., the coherent state spends longer and longer periods of time in a nearstationary state, interrupted by brief episodes of slip during which the phase decreases by $2 \pi$. In many if not most systems this type of behavior is associated with the presence of a sniper bifurcation, a saddle-node bifurcation of two equilibria on an invariant circle [1]. However, our detailed investigation of the origin of this behavior has failed to confirm the presence of this bifurcation. Instead, our self-consistency analysis indicates that the symmetric coherent state persists as $\beta$ decreases but loses stability below $\beta_{c} \approx 0.7644$, a threshold value that is in good agreement with the numerically determined threshold. At this parameter value a single real eigenvalue passes through zero, becoming positive for $\beta<\beta_{c}$. The associated eigenfunction is antisymmetric (not shown), indicating that this bifurcation should be a parity-breaking bifurcation leading to drift with a constant speed $c$ that varies with $\beta$ as $\left(\beta_{c}-\beta\right)^{1 / 2}$ [48].

To understand why this behavior is not observed we have examined the origin of the oscillations in the speed $c$. It turns out that these are associated with oscillations in the phase distribution (Fig. 3.27(a)) localized in the vicinity of a near-discontinuity in the distribution (compare Figs. 3.25(a) and (b)). The oscillators in this region periodically detrain and entrain, and it is the detraining events that are responsible for the observed episodic drift. We have checked that this is not a discreteness effect: the oscillation frequency at fixed $\beta$ remains unchanged when $N$ is increased from 512 to 1024 and 2048. However, as one approaches $\beta_{c}$ from below these episodes become more and more infrequent and in the vicinity of $\beta_{c}$ become nonperiodic. In contrast to the periodic motion, the observed nonperiodic motion is likely a consequence of intrinsic noise in the system whose effects are strongly amplified close to the transition at $\beta=\beta_{c}$. Indeed, sufficiently near the transition the sign of the phase slip begins


Figure 3.26: (a) The position of the coherent state as a function of time at $\beta=0.762$. (b) The time-averaged speed $\bar{c}$ of the coherent state as a function of $\beta$ as $\beta$ decreases. Note the abrupt decrease in speed at $\beta \approx 0.7570$ associated with the disappearance of the oscillations. (c) The time-averaged speed $\bar{c}$ of the coherent state as a function of $\beta$ as $\beta$ increases. Oscillations reappear at $\beta \approx 0.7595$ as shown in (d). The position $x$ of the coherent state is measured $\bmod 2 \pi$. All calculations are for $N=512$.
to fluctuate and the coherent state at times shifts to the left but at other times shifts to the right (not shown). Figure 3.26 (b) shows that as $\beta$ decreases the time-averaged speed $\bar{c}$ increases linearly with decreasing $\beta$ until $\beta \approx 0.7570$, where there is a hysteretic transition to an oscillation-free drift with a substantially lower speed. Figure 3.26 (c) shows that when $\beta$ is increased again the system remains in the oscillation-free state until $\beta \approx 0.7595$; at this point the oscillations reappear (Fig. 3.26(d)) and the speed jumps to a larger value. Evidently the oscillations increase the mean speed because $\bar{c}$ is dominated by the faster motion associated with time intervals corresponding to maximum asymmetry.

We believe that the discrepancy between the linear stability predictions and the observed behavior is a consequence of the fact that the former is based on the smoothed out order parameter $Z$, which is insensitive to the near-discontinuity in the slope of the phase distribution that appears to be responsible for the instability. For this reason the analysis is unable to capture the imaginary part of the eigenvalue, although it does correctly predict the onset of the instability. Moreover, because the order parameter profile is symmetric the unstable eigenfunction is necessarily antisymmetric, in contrast to the observed unstable mode which


Figure 3.27: Hidden line plots of the phase distribution $\theta$ as a function of time when (a) $\beta=0.762$ (oscillatory drift, Fig. $3.26(\mathrm{a})$ ) and (b) $\beta=0.755$ (constant drift), both for $N=512$.
is asymmetric but not antisymmetric. For these reasons a consistency analysis of the type used with success for stationary states does not appear to be appropriate for this type of drifting state.

As $\beta$ decreases further below the hysteresis region the speed $c$ continues to grow linearly but at $\beta \approx 0.646$ the coherent state itself undergoes a hysteretic transition to a stationary 2-cluster chimera of the type represented in Fig. 3.22(a). We have not investigated the origin of this instability in detail.

It turns out that the speed $c$ and the angular frequency $\Omega$ of the coherent state can both be computed theoretically and the predictions compared with measured values. For this purpose we suppose that the coherent state is stationary in the moving frame (Fig. 3.27 indicates that this is at best an approximation), i.e., we suppose that $z(x, t) \equiv u(\xi)$, where $\xi \equiv x-c t$, obtaining a complex nonlinear eigenvalue problem for the speed $c$ and the frequency $\Omega$ of the coherent state:

$$
\begin{equation*}
c \tilde{u}_{\xi}+i \Omega \tilde{u}+\frac{1}{2}\left[\exp (-i \alpha) \tilde{U}-\tilde{u}^{2} \exp (i \alpha) \tilde{U}^{*}\right]=0 . \tag{3.41}
\end{equation*}
$$

Here $\tilde{u}=u \exp i \Omega t$ and likewise for $\tilde{U}$. This equation is to be solved subject to periodic boundary conditions on $[-\pi, \pi]$. Figure 3.28 compares the solution of this eigenvalue problem (solid lines) with the measured values (open circles). The agreement is excellent. The inset in Fig. 3.28(a) shows that sufficiently near $\beta_{c}$ the speed $c$ varies as $\left(\beta_{c}-\beta\right)^{1 / 2}$, as expected of a parity-breaking bifurcation, while $\Omega$ is linear in $\beta$. Away from this region numerical fits yield $|c| \approx 0.82\left(\beta_{c}-\beta\right)$ and $|\Omega| \approx 1.03\left(\beta_{c}-\beta\right)$. Figure 3.29 performs a more rigorous test of the two procedures by comparing, for $\beta=0.66$, the details of the order parameter


Figure 3.28: Comparison of (a) the speed $c$ and (b) the frequency $\Omega$ obtained from the solution of the nonlinear eigenvalue problem (3.41) (solid lines) with measurements computed with $N=512$ oscillators (open circles), both as a function of $\beta$. The inset in (a) reveals the expected square root behavior near $\beta_{c} \approx 0.7644$. In contrast, the behavior of $\Omega$ is approximately linear everywhere.
profiles computed from simulation and from the nonlinear eigenvalue problem. Although the agreement is now less good, it is clear that the nonlinear eigenvalue problem captures the essential details of the order parameter profile and in particular of the asymmetry in the profile that is responsible for the presence of drift. As $\beta$ increases towards $\beta_{c}$ the agreement between the measured and predicted profiles improves dramatically, although a small residual discrepancy remains in regions of near-discontinuity, where the instantaneous profile exhibits localized oscillations (Fig. 3.25(b)).

## Traveling chimera states

Apart from the chimera states discussed in the previous subsections, which we consider as stationary in the large $N$ limit, we have also observed 1-cluster chimera states in which the coherent cluster drifts at constant speed in the $x$ direction as time evolves. Figure 3.30 shows a snapshot of such a traveling chimera state when the coupling function is $G_{2}^{(2)} \equiv$ $\cos (2 x)+\cos (3 x)$, while Fig. 3.31 shows examples for $G_{3}^{(2)} \equiv \cos (3 x)+\cos (4 x)$. The direction of motion is determined by the gradient of the phase in the coherent region: when the gradient is positive (left panels in Figs. 3.30 and 3.31), the cluster travels to the left; when it is negative, it travels to the right. However, the measured speeds are much smaller and in the opposite direction from the drift speeds of the $n=2$ and $n=3$ (Fig. 3.30) or $n=3$ and $n=4$ (Fig. 3.31) splay states, whose ghost-like signature is evident in the phase distribution in the figures, and in particular in Fig. 3.32(c). However, despite these differences, the phase gradient in the traveling chimera state with $G_{3}^{(2)}$ coupling is intermediate between the phase gradients associated with the competing $n=3$ and $n=4$ splay states (Figure 3.32(a,b)), and this is so for the $G_{2}^{(2)}$ coupling as well.

To determine the parameter dependence of the speed of these coherent structures we first


Figure 3.29: (Color online) Comparison of the phase and order parameter profiles from direct simulation (top panels) with those obtained from the nonlinear eigenvalue problem when $\beta=0.66$. The profiles are qualitatively similar modulo translation and overall phase rotation.


Figure 3.30: (a) A left-traveling 1-cluster chimera state. (b) A right-traveling 1-cluster chimera state. The simulation is done for $\beta=0.03$ with the coupling $G_{2}^{(2)} \equiv \cos (2 x)+\cos (3 x)$ and $N=512$.


Figure 3.31: (a) A left-traveling 1-cluster chimera state. (b) A right-traveling 1-cluster chimera state. The simulation is done for $\beta=0.03$ with the coupling $G_{3}^{(2)} \equiv \cos (3 x)+\cos (4 x)$ and $N=512$.


Figure 3.32: (a,b) Splay states with $n=3$ and $n=4$ for comparison with (c) the traveling chimera state with $G_{3}^{(2)} \equiv \cos (3 x)+\cos (4 x)$ coupling. (d) Instantaneous phase velocity of the state in (c). (e) Profile of the function $F$ used to track the position of the coherent structure.
need to be able to track their position. As shown in Fig. 3.32(d), the spatial profile of $\theta_{t}$ at a given instant in time consists of a flat part corresponding to the position of the coherent cluster. Thus the shift in the $\dot{\theta}$ pattern provides a unique indication of a shift in the location of the coherent cluster. Following [35] we pick a reference profile $f\left(x, x^{*}\right)=-\cos \left(x-x^{*}\right)$ and use the value of $x^{*}$ which minimizes the function

$$
\begin{equation*}
F\left(x^{*}\right):=\frac{1}{N} \sum_{k}^{N}\left(\dot{\theta}-f\left(x_{k}, x^{*}\right)\right)^{2} \tag{3.42}
\end{equation*}
$$

as the position of the coherent cluster. Figure $3.32(\mathrm{e})$ shows a snapshot of the function $F\left(x^{*}\right)$. Figure 3.33(a) shows the position of the coherent cluster in a periodic domain as a function of time determined using this method when $N=512, \beta=0.03$. The cluster moves to the right at an almost constant speed $c \approx 0.0077$. Figure $3.33(\mathrm{~b})$ shows that the speed $c$ is approximately independent of the number $N$ of oscillators, suggesting that the motion is an intrinsic property of the state and not an artifact of the finiteness of $N$. Finally, Fig. 3.33(c) shows that the speed $c$ increases with increasing $\beta$. As in the case of the stationary chimeras described above, the probability of obtaining a traveling chimera when the simulation starts from random initial conditions decreases as $\beta$ increases, i.e., the basin of attraction of this state, like those of the stationary states, appears to shrink as $\beta$ increases.

In fact, the traveling chimera state is more complex than suggested by the snapshots in Figs. 3.30 and 3.31 . This fact is clearly revealed only in a space-time plot of the instantaneous order parameter. Figure 3.34(a) shows the instantaneous modulus of the local order parameter $R(x, t)$ corresponding to Fig. 3.31(b), with time increasing upwards. The profile resembles a half-wavelength of a sinusoidal function (Fig. 3.35) and propagates with approximately constant speed to the right. However, the shape of the profile fluctuates in time, with smaller amplitude waves running on top of the translating bulk profile (Fig. 3.34(b)).


Figure 3.33: (a) The position of the coherent cluster as a function of time when $N=512$, $\beta=0.03$. (b) The dependence of the speed $c$ of the cluster on the oscillator number $N$ when $\beta=0.03$. (c) The dependence of the speed $c$ of the cluster on the parameter $\beta$ when $N=512$.

Altogether we have found for the coupling $G_{3}^{(2)}$ as many as four distinct stable states: splay states with $n=3$ and $n=4$, the stationary multi-cluster chimera state with 7 clusters


Figure 3.34: (a) The local order parameter $R(x, t)$ in a space-time plot for the traveling chimera in Fig. 3.31(b). (b) Zoom of (a) showing additional detail.


Figure 3.35: (Color online) (a) Snapshot of the phase distribution in a 1-cluster traveling chimera state with $G_{3}^{(2)} \equiv \cos (3 x)+\cos (4 x)$ coupling. (b) Local order parameters $R$ (red dashed line) and $\Theta$ (blue dotted line). The simulation is done with $\beta=0.03$ and $N=512$.
(Fig. 3.13(a)), and the 1-cluster traveling chimera (Fig. 3.31). Among these, the splay states are stable for all $0 \leq \beta<\frac{\pi}{2}$. The multi-cluster chimera is stable for small $\beta$ (it is observed in numerical simulations already when $\beta=0.001$ although the fraction of the oscillators in the coherent state is then very small). As $\beta$ increases this state loses stability when $\beta$ reaches a value between 0.115 and 0.12 and evolves to one or other of the splay states. The traveling chimera is stable in the interval $0.015 \lesssim \beta \lesssim 0.065$; below this range a linear instability takes it to the multicluster state while above this range it takes it into one or other of the splay states. However, despite considerable effort and simulations of very large oscillator systems to get accurate initial conditions, we have not succeeded in solving the corresponding nonlinear eigenvalue problem for the speed $c$ of this state. We attribute this failure to the fact, clearly visible in Fig. 3.34, that this state does not in fact drift as a rigid object - it is a time-dependent state even in the comoving frame.

### 3.4 Conclusion

In this chapter we have investigated, for the first time, the effects of nonlocal coupling of indefinite sign on a system of identical phase-coupled oscillators. We focused on the case with positive (attractive) coupling over small distances and negative (repulsive) coupling over large distances, as exemplified by the coupling functions $G_{n}^{(1)} \equiv \cos (n x)$ and $G_{n}^{(2)} \equiv \cos (n x)+$ $\cos [(n+1) x]$, and identified a variety of evenly spaced stationary multi-cluster chimera states without having to rely on the presence of time delay or parameter heterogeneity [33, 37, 38]. More significantly, we also found a class of chimeras with uneven separation, traveling coherent states and a novel traveling chimera state. In contrast to the earlier systems referred to above the chimera states in the system studied here have relatively large basins of attraction and no specially tailored initial conditions are required to obtain such states. In particular, robust chimeras are realized starting from random initial conditions.

We have given a fairly complete description of the different accessible states using a self-consistency analysis and showed that this type of analysis works well not only for stationary evenly spaced chimeras but also for the exotic unevenly spaced chimeras. In particular, we were able to show that the local order parameter profiles predicted by this type of analysis agree well with the phase distribution generated in direct numerical simulations of the coupled oscillator system. We used these results to make predictions for different types of transitions that these states may undergo, merger of coherent clusters and transition to a drifting state, and confirmed these predictions using numerical simulations. The self-consistency analysis predicted instability thresholds accurately but missed some crucial details, including the presence of localized small amplitude oscillations near the onset of the drift instability. Despite this failure the drift speed predicted from the solution of a nonlinear eigenvalue problem obtained from the self-consistency analysis was in excellent agreement with the numerically determined speed (3.28(a)), and likewise for the overall rotation frequency $\Omega$ of the phase distribution (3.28(b)). Stable (nonsplay) traveling states appear to require a significant contribution from both $n$ and $n+1$ modes in the coupling function $G_{n}^{(2)}$.

## CHAPTER 3. MULTI-CLUSTER AND TRAVELING CHIMERA STATES IN

 IDENTICAL NONLOCAL PHASE-COUPLED OSCILLATORSWe have checked, for example, that with $G(x)=\cos (3 x)+A \cos (4 x)$ the traveling chimera present at $A=1$ loses stability when $A \approx 0.95$; when this occurs the system evolves to a traveling splay state instead. It is of considerable interest to see to what extent the results from the system studied here carry over to more realistic oscillator systems, such as those studied in $[49,51,52,57]$, when these are coupled nonlocally with an attractive-repulsive coupling.

## Appendix

## 3.A Derivation of self-consistent equation (3.35)

In this appendix, we give the detailed derivation Eq. (3.35), which is the self-consistency equation for 3 -cluster chimera state. According to the discussion in the main text, we can shift the coordinate to make $R(x)$ even. Combining the numerical observation and phase translation, the local order parameter can be written as

$$
\begin{equation*}
R \exp (i \Theta)=R_{0} \cos (3 x / 2) e^{-i x / 2} \tag{3.43}
\end{equation*}
$$

From this expression, the parameters $a, b, c$ and $d$ can be obtained: $a=c=\frac{R_{0}}{2}$ and $b=$ $-d=\frac{i R_{0}}{2}$.

Substituting Eq. (3.43) into the self-consistency equation (3.29), we have

$$
\begin{align*}
\frac{R_{0}}{2} & =\exp (i \beta)\langle h \exp (i \Theta) \cos (y)\rangle \\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}(y)}}{R^{2}(y)} R(y) \exp (i \Theta) \cos (y)\right\rangle \\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}(y)}}{R_{0} \cos \left(\frac{3 y}{2}\right)} \exp \left(-\frac{i y}{2}\right) \cos (y)\right\rangle \\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}(y)}}{R_{0} \cos \left(\frac{3 y}{2}\right)} \exp \left(-i \frac{y}{2}\right) \frac{\exp (i y)+\exp (-i y)}{2}\right\rangle \\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}(y)}}{R_{0} \cos \left(\frac{3 y}{2}\right)} \frac{\exp \left(\frac{i y}{2}\right)+\exp \left(-\frac{3 i y}{2}\right)}{2}\right\rangle \\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}(y)}}{R_{0} \cos \left(\frac{3 y}{2}\right)} \frac{\cos \left(\frac{y}{2}\right)+i \sin \left(\frac{y}{2}\right)+\cos \left(\frac{3 y}{2}\right)-i \sin \left(\frac{3 y}{2}\right)}{2}\right\rangle \\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}(y)}}{R_{0} \cos \left(\frac{3 y}{2}\right)} \frac{\cos \left(\frac{y}{2}\right)+\cos \left(\frac{3 y}{2}\right)}{2}\right\rangle \tag{3.44}
\end{align*}
$$

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Here, the terms with $\sin \left(\frac{y}{2}\right)$ and $\sin \left(\frac{3 y}{2}\right)$ vanish because of their oddness. In addition, we notice that

$$
\begin{align*}
& \int_{-\pi}^{\pi} \cos ^{2 k-1}\left(\frac{3 y}{2}\right) \cos \left(\frac{y}{2}\right) d y \\
= & \frac{1}{2^{2 k}} \int_{-\pi}^{\pi}\left(\exp \left(\frac{3 i y}{2}\right)+\exp \left(-\frac{3 i y}{2}\right)\right)^{2 k-1}\left(\exp \left(\frac{i y}{2}\right)+\exp \left(-\frac{i y}{2}\right)\right) d y \\
= & 0 \tag{3.45}
\end{align*}
$$

for any positive integer $k$. We can obtain

$$
\begin{equation*}
\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}(y)}}{R_{0} \cos \left(\frac{3 y}{2}\right)} \cos \left(\frac{y}{2}\right)\right\rangle=0 . \tag{3.46}
\end{equation*}
$$

So the self-consistency equation (3.29) finally becomes

$$
\begin{equation*}
R_{0}^{2}=\exp (i \beta)\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \cos ^{2}\left(\frac{3 y}{2}\right)}\right\rangle \tag{3.47}
\end{equation*}
$$

Similar calculations can be carried out for Eqs. (3.30), (3.31), and (3.32). It turns out that they all lead to the same final equation.

## Chapter 4

## Chimera states in systems of nonlocal nonidentical phase-coupled oscillators

### 4.1 Introduction

In Chapter 1 we have derived a particular form of the phase-coupled oscillator model:

$$
\begin{equation*}
\frac{d \theta_{i}}{d t}=\omega_{i}-\frac{K}{N} \sum_{j=1}^{N} G_{i j} \sin \left(\theta_{i}-\theta_{j}+\alpha\right) \tag{4.1}
\end{equation*}
$$

where $\omega_{i}$ is the natural frequency of oscillator $i, G_{i j}$ represents the coupling between oscillators $i$ and $j, \alpha$ is a phase lag and $K$ is the overall coupling strength. A general treatment of this system is not easy, and two types of simplifications are commonly used. One of them is to assume global coupling among the oscillators, and assign the natural frequencies $\omega_{i}$ randomly and independently from some prespecified distribution [22, 23, 24]. The second tractable case arises when all the oscillators are assumed to be identical, and the coupling $G_{i j}$ is taken to be local, eg., nearest-neighbor coupling [25]. The intermediate case of nonlocal coupling is harder but the phenomena described by the resulting model are much richer.

In this chapter we suppose that the oscillators are arranged on a ring. In the continuum limit $N \rightarrow \infty$ the system is described by the nonlocal equation, cf. [14],

$$
\begin{equation*}
\frac{\partial \theta}{\partial t}=\omega(x)-\int G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y \tag{4.2}
\end{equation*}
$$

for the phase distribution $\theta(x, t)$. When the oscillators are identical ( $\omega$ is a fixed constant) and $G(x)=\frac{\kappa}{2} \exp (-\kappa|x|)$ this system admits a new type of state in which a fraction of the oscillators oscillate coherently (i.e., in phase) while the phases of the remaining oscillators remain incoherent [26]. In this chapter we think of this state, nowadays called a chimera state [27], as a localized structure embedded in a "turbulent" background. Subsequent studies of this unexpected state with different coupling functions $G(x)$ have identified a variety of

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 PHASE-COUPLED OSCILLATORSdifferent one-cluster and multi-cluster chimera states [31, 32, 33, 34, 35, 36, 37, 38, 39, 75, 76], consisting of clusters or groups of adjacent oscillators oscillating in phase with a common frequency. The clusters are almost stationary in space, although their position (and width) fluctuates under the influence of the incoherent oscillators on either side. In Chapter 3, a new type of chimera state was discovered, a traveling chimera state. In this state the leading edge plays the role of a synchronization front, which kicks oscillators into synchrony with the oscillators behind it, while the trailing front kicks oscillators out of synchrony; these two fronts travel with the same speed, forming a bound state.

It is natural to ask whether these states persist in the presence of spatial inhomogeneity, in particular, in the presence of spatial inhomogeneity in the natural frequency distribution $(\omega=\omega(x))$. It is known that in this case the phase-locked solution $\theta(x)=0$ present for identical oscillators with $\alpha=0$ perturbs to a unique phase-locked but spatially inhomogeneous solution $\theta(x, \alpha)$ as $\alpha$ increases from zero [14]. However, the persistence of the chimera state has only been studied in restricted settings, such as the two population model [34]. This model is perhaps the simplest model exhibiting chimera states [32] but includes no spatial structure. Consequently, we focus in this chapter on a ring of adjacent oscillators with a prescribed but spatially nonuniform frequency profile $\omega(x)$, where the continuous variable $x$ $(-\pi<x \leq \pi)$ represents position along the ring. To be specific, two classes of inhomogeneity are considered, a bump inhomogeneity $\omega(x)=\omega_{0} \exp (-\kappa|x|), \kappa>0$, and a periodic inhomogeneity $\omega(x)=\omega_{0} \cos (l x)$, where $l$ is a positive integer. In each case we follow Chapter 3 and study the coupling functions

$$
\begin{aligned}
G_{n}^{(1)}(x) & \equiv \cos (n x) \\
G_{n}^{(2)}(x) & \equiv \cos (n x)+\cos [(n+1) x]
\end{aligned}
$$

where $n$ is an arbitrary positive integer, focusing in each case on small values of the parameter $\omega_{0}$. These choices are motivated by biological systems in which coupling between nearby oscillators is often attractive while that between distant oscillators may be repelling [40]. There are several advantages to the use of these two types of coupling. The first is that these couplings allow us to obtain chimera states with random initial conditions. The second is that we can identify a large variety of new states for suitable parameter values, including (a) splay states, (b) stationary multi-cluster states with evenly distributed coherent clusters, (c) stationary multi-cluster states with unevenly distributed clusters, (d) a fully coherent state traveling with a constant speed, and (e) a single-cluster traveling chimera state (see Chapter $3)$.

In this chapter, we analyze the effect of the two types of inhomogeneity in $\omega$ on each of these states and describe the results in terms of the parameters $\omega_{0}$ and $\kappa$ or $l$ describing the strength and inverse length scale of the inhomogeneity, while varying the parameter $\beta \equiv \frac{\pi}{2}-\alpha$ representing the phase lag $\alpha$. In Section 4.2, we briefly review the notion of a local order parameter for studying chimera states and introduce the self-consistency equation for this quantity. In Sections 4.3, 4.4, and 4.5 we investigate, respectively, the effect of inhomogeneity on rotating states (including splay states and stationary chimera states), traveling coherent

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states and the traveling chimera state. We conclude in Section 4.6 with a brief summary of the results and directions for future research.

### 4.2 Effective equation

Equation (4.2) is widely used in studies of chimera states. An equivalent description can be obtained by constructing an equation for the local order parameter $z(x, t)$ defined as the local spatial average of $\exp [i \theta(x, t)]$,

$$
\begin{equation*}
z(x, t) \equiv \lim _{\delta \rightarrow 0^{+}} \frac{1}{\delta} \int_{-\delta / 2}^{\delta / 2} \exp (i \theta(x+y, t)) d y \tag{4.3}
\end{equation*}
$$

The evolution equation for $z$ then takes the form [41, 44, 45]

$$
\begin{equation*}
z_{t}=i \omega(x) z+\frac{1}{2}\left(\exp (-i \alpha) Z-\exp (i \alpha) z^{2} Z^{*}\right) \tag{4.4}
\end{equation*}
$$

where $Z(x, t) \equiv K[z](x, t)$ and $K$ is a compact linear operator defined via the relation

$$
\begin{equation*}
K[u](x, t) \equiv \int_{-\pi}^{\pi} G(x-y) u(y, t) d y \tag{4.5}
\end{equation*}
$$

A derivation of Eq. (4.4) based on the Ott-Antonsen Ansatz [46] is given in Section 2.2 of Chapter 2. Equation (4.4) can also be obtained directly from Eq. (4.2) using the change of variable

$$
\begin{equation*}
z(x, t) \equiv \exp [i \theta(x, t)] \tag{4.6}
\end{equation*}
$$

In Section 2.2, we also derived a self-consistency equation for stationary rotating solutions of the form

$$
\begin{equation*}
z(x, t)=\tilde{z}(x) \exp (-i \Omega t), \quad Z(x, t)=\tilde{Z}(x) \exp (-i \Omega t) \tag{4.7}
\end{equation*}
$$

For these states, the common frequency $\Omega$ and $\tilde{Z}$ satisfy

$$
\begin{equation*}
\tilde{Z}(x)=\left\langle G(x-y) \exp (i \beta) \frac{\Omega+\omega(y)-\mu(y)}{\tilde{Z}^{*}(y)}\right\rangle . \tag{4.8}
\end{equation*}
$$

Here the bracket $\langle\cdot\rangle$ is defined as the integral over the interval $[-\pi, \pi]$, i.e., $\langle u\rangle \equiv \int_{-\pi}^{\pi} u(y) d y$. This leads to Eq. (2.50) if we write

$$
\begin{equation*}
\tilde{Z}(x)=R(x) \exp [i \Theta(x)], \tag{4.9}
\end{equation*}
$$

where $R(x)$ and $\Theta(x)$ are referred as the amplitude and phase of the complex order parameter $\tilde{Z}(x)$.

### 4.3 Stationary rotating solutions

As shown in Chapter 3, Eq. (4.2) with constant natural frequency $\omega$ exhibits both stationary rotating solutions (splay states and stationary chimera states) and traveling solutions (traveling coherent states and traveling chimera states) for suitable coupling functions $G(x)$. In this section, we investigate the effect of spatial inhomogeneity in $\omega$ (i.e., $\omega=\omega(x)$ ) on the splay states and on stationary chimera states, focusing on the case $G_{n}^{(1)}(x)=\cos n x$ studied in Chapter 3. We find that when the inhomogeneity is sufficiently weak, the above solutions persist. However, as the magnitude of the inhomogeneity increases, new types of solutions are born. The origin and spatial structure of these new states can be understood with the help of the self-consistency relation (4.8). Since Eq. (4.8) is invariant under the transformation $z \rightarrow z \exp (i \phi)$ with $\phi$ an arbitrary real constant, the local order parameter $\tilde{Z}(x)$ for the coupling function $G(x)=\cos n x$ can be written as

$$
\begin{equation*}
\tilde{Z}(x)=a \cos n x+b \sin n x \tag{4.10}
\end{equation*}
$$

where $a$ is positive and $b \equiv b_{r}+i b_{i}$ with $b_{r}$ and $b_{i}$ both real. Substituting the Ansatz (4.10) into Eq. (4.8) we obtain the following pair of integral-algebraic equations

$$
\begin{align*}
& a \exp (-i \beta)=\left\langle\frac{\cos (n y)(\tilde{\Omega}(y)-\mu(y))}{a \cos n y+b^{*} \sin n y}\right\rangle  \tag{4.11}\\
& b \exp (-i \beta)=\left\langle\frac{\sin (n y)(\tilde{\Omega}(y)-\mu(y))}{a \cos n y+b^{*} \sin n y}\right\rangle \tag{4.12}
\end{align*}
$$

where $\mu(y)=\left(\tilde{\Omega}(y)^{2}-|a \cos (n y)+b \sin (n y)|^{2}\right)^{1 / 2}$ and $\tilde{\Omega}(y) \equiv \Omega+\omega(y)$. These equations may also be written in the more convenient form

$$
\begin{gather*}
\langle\tilde{\Omega}(y)-\mu(y)\rangle=\exp (-i \beta)\left(a^{2}+|b|^{2}\right)  \tag{4.13}\\
\left\langle\frac{\left(a^{2}-b^{2}\right) \sin (2 n y)-2 a b \cos (2 n y)}{\tilde{\Omega}(y)+\mu(y)}\right\rangle=0 \tag{4.14}
\end{gather*}
$$

In the following we consider two choices for the inhomogeneity $\omega(x)$, a bump $\omega(x)=$ $\omega_{0} \exp (-\kappa|x|), \kappa>0$, and a periodic inhomogeneity $\omega(x)=\omega_{0} \cos (l x)$ where $l$ is a positive integer. The resulting equations possess an important and useful symmetry, $x \rightarrow-x$, $b \rightarrow-b$. Moreover, for $\omega(x)$ satisfying $\omega(x+2 \pi / l)=\omega(x)$ with $l$ an integer, a solution $\tilde{Z}(x)$ of the self-consistency relation implies that $\tilde{Z}(x+2 m \pi / l)$ is also a solution. Here $m$ is an arbitrary integer.

In the following we use Eqs. (4.13) and (4.14) repeatedly to study the changes in both the splay states and the stationary chimera states as the magnitude $\omega_{0}>0$ of the inhomogeneity increases, and compare the resulting predictions with numerical simulation of $N=512$ oscillators evenly distributed in $[-\pi, \pi]$.

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 PHASE-COUPLED OSCILLATORS
## Bump inhomogeneity: $\omega=\omega_{0} \exp (-\kappa|x|)$

We now consider the case where $\omega(x)$ has a bump defect at $x=0$. For the spatial profile of the defect we pick $\omega(x)=\omega_{0} \exp (-\kappa|x|)$, where $\omega_{0}>0$ and $\kappa>0$ are parameters that can be varied.

## Effect on splay states

When $G(x)=\cos (n x)$ and $\omega$ is a constant, Eq. (4.2) exhibits so-called splay state solutions with $\theta(x, t)=q x-\Omega t$, where $\Omega$ is the overall rotation frequency and $q$ is an integer called the twist number. The phase in this type of solution drifts with speed $c=\Omega / q$ to the right but the order parameter is stationary. Consequently we think of the splay states as stationary rotating states. Linear stability analysis for $G(x)=\cos n x$ shows that the splay state is stable when $|q|=n$, a result that is easily confirmed in simulations starting from randomly distributed initial phases. In the following we consider the case $n=1$ and simulate $N=512$ oscillators evenly distributed in $[-\pi, \pi]$ for different values of $\omega_{0}$ and $\kappa$.

Figure 4.1 shows how the splay solutions change as $\omega_{0}$ increases. When $\omega_{0}$ becomes nonzero but remains small, the splay states persist but their phase $\theta(x, t)$ no longer varies uniformly in space. This effect may be seen in Fig. 4.1(a) upon enlargement. The figure shows that

$$
\begin{equation*}
\theta(x, t) \equiv \phi(x)-\Omega t \tag{4.15}
\end{equation*}
$$

where $\phi(x)$ is a continuous function of $x$ with $\phi(\pi)-\phi(-\pi)=2 \pi q$. When $q \neq 0$ we refer to this type of state as a near-splay state. As $\omega_{0}$ becomes larger an incoherent region appears in the vicinity of $x=0$, with width that increases with increasing $\omega_{0}$ (Figs. 4.1(b,c)). We refer to this type of state as a chimera splay state. As $\omega_{0}$ increases further and exceeds a second threshold, a new region of incoherence is born (Fig. 4.1(d)). Figure 4.2 provides additional information about the partially coherent near-splay state in Fig. 4.1(b). The figure shows the real order parameters $R(x)$ and $\Theta(x)$ together with $\bar{\theta}_{t}$, the local rotation frequency averaged over a long time interval (Fig. 4.2(d)), and reveals that in the coherent region the oscillation frequencies are identical (with $\bar{\theta}_{t}=-\Omega$ ), with an abrupt but continuous change in the frequency distribution within the incoherent region. Figure $4.2(\mathrm{~d})$ also reveals a slight asymmetry with respect to $x=0$, the bump maximum. This asymmetry becomes stronger and stronger as $\omega_{0}$ increases and is a consequence of the asymmetry introduced by the direction of travel, i.e., the sign of the frequency $\Omega$ in Eq. (4.15), as discussed further below.

These states and the transitions between them can be explained within the framework of the self-consistency analysis. To compute the solution branches and the transition thresholds, we numerically continue solutions of Eqs. (4.13) and (4.14) with respect to the parameter $\omega_{0}$. When $\omega_{0}=0$, the splay state (with positive slope) corresponds to $a=\pi, b_{r}=0$, $b_{i}=\pi$, with $\Omega=\pi$. The order parameter is therefore $R \exp (i \Theta)=\pi \exp ( \pm i x)$. We use this splay state as the starting point for continuation. As $\omega_{0}$ increases, a region of incoherence develops in the phase pattern in the vicinity of $x=0$. From the point of view of the


Figure 4.1: The phase distribution $\theta(x)$ for splay states observed with $G(x)=\cos (x), \omega(x)=$ $\omega_{0} \exp (-2|x|)$, and $\beta=0.05$. (a) $\omega_{0}=0.006$. (b) $\omega_{0}=0.02$. (c) $\omega_{0}=0.05$. (d) $\omega_{0}=0.1$. The states travel to the right $(\Omega>0)$. The inset in (a) shows an enlargement of the region near $x=0$.
self-consistency analysis, the incoherent region corresponds to the region where the natural frequency exceeds the amplitude $R(x)$ of the complex order parameter. The boundaries between the coherent and incoherent oscillators are thus determined by the relation $\Omega+$ $\omega(x)=\left|a \cos (x)+\left(b_{r}+i b_{i}\right) \sin (x)\right|$. For $\omega_{0}=0.02$, the left and right boundaries are thus $x_{l} \approx-0.4026$ and $x_{r} \approx 0.2724$, respectively.

These predictions are in good agreement with the values measured in direct numerical simulation (Fig. 4.2(d)). Figure $4.3(\mathrm{a})$ shows the overall frequency $\Omega$ obtained by numerical continuation of the solution of Eqs. (4.13) and (4.14) in the parameter $\omega_{0}$, while Figs. 4.3(b) and 4.3 (c) show the corresponding results for the fraction $e$ of the domain occupied by the coherent oscillators and the extent of the two intervals $x_{l} \leq x \leq x_{r}$ of incoherence, also as functions of $\omega_{0}$. From Fig. 4.3(c) we can see a clear transition at $\omega_{0} \approx 0.0063$ from a single domain-filling coherent state to a "splay state with one incoherent cluster" in $x_{l} \leq x \leq x_{r}$, followed by a subsequent transition at $\omega_{0} \approx 0.065$ from this state to a "splay state with two incoherent clusters." These transitions occur when the profiles of $\Omega+\omega(x)$ and $R(x)$ touch as $\omega_{0}$ increases and these points of tangency therefore correspond to the locations


Figure 4.2: (a) A snapshot of the phase distribution $\theta(x, t)$ for $G(x)=\cos (x)$ and $\omega(x)=$ $0.02 \exp (-2|x|)$. (b) The corresponding $R(x)$. (c) The corresponding $\Theta(x)$. (d) The oscillator frequency $\bar{\theta}_{t}$ averaged over the time interval $0<t<1000$ (solid line). In the coherent region $\bar{\theta}_{t}$ coincides with the global oscillation frequency $-\Omega$ (open circles). The inset in (a) shows an enlargement of the region near $x=0$. All simulations are done with $\beta=0.05$ and $N=512$.
where coherence is first lost. Figure 4.4 shows that tangencies between $\Omega+\omega(x)$ and $R(x)$ occur when $\omega_{0} \approx 0.0063$ and 0.065 , implying that intervals of incoherent oscillators appear first at $x=0$ (i.e., the bump maximum) and subsequently at $x \approx 1.83$, as $\omega_{0}$ increases. These predictions are in excellent agreement with the direct numerical simulations shown in Fig. 4.1. Moreover, the critical values of $\omega_{0}$ predicted by the self-consistency analysis are fully consistent with the simulation results when $\beta$ is increased quasi-statically (not shown). In each case we repeated the simulations for decreasing $\beta$ but found no evidence of hysteresis in these transitions.

## Effect on stationary chimera states

Chimera states with $2 n$ evenly distributed coherent clusters are readily observed when $G(x)=\cos (n x)$ and $\omega$ is a constant. These states persist when a bump is introduced into the frequency distribution $\omega(x)$. Figure 4.5 shows the phase distribution and the corresponding


Figure 4.3: (a) The overall frequency $\Omega$ and (b) the fraction $e$ of the domain occupied by the coherent oscillators as a function of the parameter $\omega_{0}$. (c) The two regions of incoherence $x_{l} \leq x \leq x_{r}$ that open up at $\omega \approx 0.0063$ and $\omega \approx 0.065$ corresponding to the transitions visible in panels (a) and (b). The calculation is for $\kappa=2, \beta=0.05$ and $N=512$.


Figure 4.4: Comparison of $\Omega+\omega(x)$ and $R(x)$ at the critical values $\omega_{0}$ for the appearance of new regions of incoherence around (a) $x=0$ for $\omega_{0}=0.0063$ and (b) $x=1.83$ for $\omega_{0}=0.065$. The calculation is for $\kappa=2, \beta=0.05$ and $N=512$.


Figure 4.5: (a) A snapshot of the phase distribution $\theta(x, t)$ in a 2-cluster chimera state for $G(x)=\cos (x)$ and $\omega(x)=0.1 \exp (-2|x|)$. (b) The corresponding order parameter $R(x)$. (c) The corresponding order parameter $\Theta(x)$. Note that the oscillators in the two clusters oscillate with the same frequency but $\pi$ out of phase. The calculation is done with $\kappa=2$, $\beta=0.05$, and $N=512$.
local order parameters $R(x)$ and $\Theta(x)$ for $G(x)=\cos (x)$ when $\omega(x)=0.1 \exp (-2|x|)$. The figure shows that the two clusters persist, but are now always located near $x=-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This is a consequence of the fact that the presence of the bump breaks the translation invariance of the system. Figures $4.5(\mathrm{~b}, \mathrm{c})$ show that the local order parameter $\tilde{Z}(x)$ has the symmetry $\tilde{Z}(-x)=-\tilde{Z}(x)$. This symmetry implies $\tilde{Z}$ should take the form $R \exp (i \Theta)=$ $b \sin (x)$, where $b=b_{r}$ is real. The corresponding self-consistency equation takes the form

$$
\begin{equation*}
b^{2} \exp (-i \beta)=\left\langle\Omega+\omega(y)-\sqrt{(\Omega+\omega(y))^{2}-b^{2} \sin ^{2} y}\right\rangle . \tag{4.16}
\end{equation*}
$$

The result of numerical continuation of the solutions of Eq. (4.16) is shown in Fig. 4.6. The 2 -cluster chimera state persists to large values of $\omega_{0}$, with the size of the coherent clusters largely insensitive to the value of $\omega_{0}$. This prediction has been corroborated using direct simulation of Eq. (4.1) with $N=512$ oscillators.

When $\omega$ is constant, finite-size effects cause the phase pattern to fluctuate in location. In Chapter 3, we demonstrate that this fluctuation is well modeled by Brownian motion in which the variance is proportional to $t$, even though the original system is strictly deterministic. As mentioned above, when $\omega(x)$ is spatially dependent, the translation symmetry is broken and the coherent cluster has a preferred location. Figure 4.5 suggests that local maxima of the order parameter $R$ can be used to specify the location of coherent clusters. Consequently we plot in Fig. 4.7 the position $x_{0}(t)$ of the right coherent cluster as a function of time for three different values of the parameter $\kappa$. We see that the inhomogeneity pins the coherent cluster to a particular location, and that the cluster position executes apparently random oscillations about this preferred location, whose amplitude increases with increasing $\kappa$, i.e., with decreasing width of the bump. Figure 4.8 shows the standard deviation of the position $x_{0}(t)$ of the coherent cluster as a function of the parameter $\kappa$.

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Figure 4.6: The dependence of (a) $\Omega$ and the order parameter amplitude $b$ on $\omega_{0}$. (b) The fraction $e$ of coherent oscillators as a function of $\omega_{0}$. The calculation is done with $\kappa=2$, $\beta=0.05$, and $N=512$.


(c)


Figure 4.7: The position $x_{0}(t)$ of the coherent cluster as a function of time when $\omega_{0}=0.1$, $\beta=0.05, N=512$, and (a) $\kappa=10$, (b) $\kappa=6$, and (c) $\kappa=2$.

The behavior shown in Figs. 4.7 and 4.8 can be modeled using an Ornstein-Uhlenbeck process, i.e., a linear stochastic ordinary differential equation of the form

$$
\begin{equation*}
d x_{0}=\lambda\left(\mu-x_{0}\right) d t+\sigma d W_{t}, \tag{4.17}
\end{equation*}
$$

where $\lambda$ represents the strength of the attraction to the preferred location $\mu$, and $\sigma$ indicates the strength of the noise. Models of this type are expected to apply on an appropriate time scale only: the time increment $\Delta t$ between successive steps of the stochastic process must be large enough that the position of the cluster can be thought of as the result of a large number of pseudo-random events and hence normally distributed, but not so large that nonlinear effects become significant. Figure 4.9 reveals that for an appropriate interval of $\Delta t$ the fluctuations $x_{0}(t+\Delta t)-A x_{0}(t)-B$ are indeed normally distributed, thereby providing support for the applicability of Eq. (4.17) to the present system.

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Figure 4.8: The standard deviation of the position $x_{0}(t)$ as a function of the parameter $\kappa$ for $\omega_{0}=0.1, \beta=0.05$ and $N=512$.


Figure 4.9: The histogram of the residues $\varepsilon \equiv x_{0}(t+\Delta t)-A x_{0}(t)-B$ is well approximated by a normal distribution when $\Delta t=100$.

Equation (4.17) has the solution

$$
\begin{equation*}
x_{0}(t+\Delta t)=A x_{0}(t)+B+C N_{0,1}, \tag{4.18}
\end{equation*}
$$

where $A=\exp (-\lambda \Delta t), B=\mu(1-A)$ and $C=\sigma \sqrt{\frac{1-A^{2}}{2 \lambda}}$. To fit the parameters to the data in Fig. 4.7 we notice that the relationship between consecutive observations $x_{0}$ is linear with an i.i.d. error term $C N_{0,1}$, where $N_{0,1}$ denotes the normal distribution with zero mean and unit variance (Fig. 4.9) and $C$ is a constant. A least-squares fit to the data $\left(x_{0}(t), x_{0}(t+\Delta t)\right.$ ) gives the parameters $\lambda, \mu$, and $\sigma$. We find that for $\omega_{0}=0.1, \kappa=2$, and $\beta=0.05$, the choice $\Delta t=100$ works well and yields the empirical model parameters $\lambda \approx 0.016, \mu \approx-1.57$, and $\sigma \approx 0.02$, a result that is in good agreement with the simulation results for $\kappa=2$ summarized in Fig. 4.8.


Figure 4.10: A near-splay state for (a) $\omega(x)=0.005 \cos (x)$. Chimera splay states for (b) $\omega(x)=0.01 \cos (x)$, (c) $\omega(x)=0.1 \cos (x)$, and (d) $\omega(x)=0.2 \cos (x)$. In all cases $\beta=0.05$ and $N=512$.

## Periodic $\omega(x)$

In this section we consider the case $\omega(x)=\omega_{0} \cos (l x)$. When $\omega_{0}$ is small, the states present for $\omega_{0}=0$ persist, but with increasing $\omega_{0}$ one finds a variety of intricate dynamical behavior. In the following we set $G(x)=\cos x$ and $\beta \equiv \frac{\pi}{2}-\alpha=0.05$, with $\omega_{0}$ and $l$ as parameters to be varied.

## Splay states, near-splay states and chimera splay states

When $\omega(x)$ is a constant, splay states are observed in which the phase $\theta(x, t)$ varies linearly with $x$. When $\omega_{0}$ is nonzero but small, the splay states persist as the near-splay states described by Eq. (4.15); Fig. 4.10(a) shows an example of such a state when $\omega(x)=$ $0.005 \cos (x)$. As $\omega_{0}$ becomes larger, an incoherent region appears and it increases in width as $\omega_{0}$ increases further (e.g., Figs. $\left.4.10(\mathrm{~b})-4.10(\mathrm{c})\right)$. We refer to this type of state as a chimera splay state, as in the bump inhomogeneity case. For these solutions, the slope of the coherent regions is no longer constant but the oscillators continue to rotate with a constant overall frequency $\Omega$. This type of solution is also observed for other values of $l$. Fig. 4.11 shows examples of chimera splay states for $l=2$, while Fig. 4.12 shows chimera splay states for $l=3,4$, and 5 , with $l$ coherent clusters in each case.

To compute the solution branches and identify thresholds for additional transitions, we continue the solutions of Eqs. (4.13) and (4.14) with respect to the parameter $\omega_{0}$. Figures 4.13


Figure 4.11: A near-splay state for (a) $\omega(x)=0.1 \cos (2 x)$. Chimera splay states for (b) $\omega(x)=0.2 \cos (2 x)$, (c) $\omega(x)=0.3 \cos (2 x)$, and (d) $\omega(x)=0.4 \cos (2 x)$. In all cases $\beta=0.05$ and $N=512$.


Figure 4.12: Chimera splay states for $G(x)=\cos (x)$ and (a) $\omega(x)=0.2 \cos (3 x)$ (3-cluster state), (b) $\omega(x)=0.2 \cos (4 x)$ (4-cluster state), and (c) $\omega(x)=0.2 \cos (5 x)$ (5-cluster state). In all cases $\beta=0.05$ and $N=512$.


Figure 4.13: The dependence of (a) $\Omega$ and (b) the fraction $e$ of the domain occupied by the coherent oscillators on $\omega_{0}$, with $\omega(x)=\omega_{0} \cos (x)$ and $\beta=0.05$.
and 4.14 show the dependence of $\Omega$ and of the coherent fraction $e$ on $\omega_{0}$ when $l=1$ and 2 , respectively. The figures indicate that the coherent fraction $e$ falls below 1 at $\omega_{0} \approx 0.0075$ $(l=1)$ and $\omega_{0} \approx 0.16(l=2)$; for $l=3,4$, and 5 the corresponding transition takes place at $\omega_{0} \approx 0.0048$ (not shown). These values coincide with the parameter values at which an incoherent region emerges in numerical simulations. For example, Figs. 4.10(a) and 4.10(b) show that for $l=1$ the transition takes place for $\omega_{0}$ between 0.005 and 0.01 . The exact value of $\omega_{0}$ for this transition can be determined from Fig. 4.15(a) which shows that at the critical value of $\omega_{0}$ the profiles of $\Omega+\omega(x)$ and $R(x)$ touch; the location of the tangency corresponds to the location of the resulting incoherent region. Figure 4.16 shows the corresponding construction for $l=2$.

Subsequent transitions may take place as $\omega_{0}$ increases just as in the case of a bump inhomogeneity. As shown in Fig. 4.10 and $4.13(\mathrm{~b})$ this is the case for $l=1$, where a second transition takes place at $\omega_{0} \approx 0.13$, creating a second incoherent region. Figure $4.15(\mathrm{~b})$ shows that at this critical value of $\omega_{0}$ the profiles of $\Omega+\omega(x)$ and $R(x)$ also touch, allowing us to determine accurately the location of this second transition.

To understand the $l$-cluster chimera states $(l \geq 3)$ reported in Fig. 4.12, we computed the local order parameter $\tilde{Z}(x)$ and found that $R(x)$ is approximately constant while the phase $\Theta(x)$ varies at a constant rate, suggesting the Ansatz $\tilde{Z}(x)=a \exp (i x)$. With this Ansatz, Eqs. (4.13) and (4.14) reduce to a single equation,

$$
\begin{equation*}
2 \pi \Omega-\left\langle\sqrt{\left(\Omega+\omega_{0} \cos (l y)\right)^{2}-a^{2}}\right\rangle=2 \exp (-i \beta) a^{2} \tag{4.19}
\end{equation*}
$$

for all positive integers $l \geq 3$. Figure 4.17 shows the result of numerical continuation of a solution for $l=3$. The figure reveals no further transitions, indicating that the $l=3$ chimera state persists to large values of $\omega_{0}$. Numerical simulation shows that these states $(l=3,4$, $5)$ are stable and persist up to $\omega_{0}=1$.


Figure 4.14: The dependence of (a) $\Omega$ and (b) the fraction $e$ of the domain occupied by the coherent oscillators on $\omega_{0}$, with $\omega(x)=\omega_{0} \cos (2 x)$ and $\beta=0.05$.


Figure 4.15: Comparison of $\Omega+\omega(x)$ and $R(x)$ at the critical values $\omega_{0}$ for the appearance of a new region of incoherence around (a) $x=0$ for $\omega_{0}=0.0075$ and (b) $x=2.8$ for $\omega_{0}=0.13$. Parameters: $l=1$ and $\beta=0.05$.


Figure 4.16: Comparison of $\Omega+\omega(x)$ and $R(x)$ at the critical value $\omega_{0} \approx 0.16$. Parameters: $l=2$ and $\beta=0.05$.


Figure 4.17: Dependence of (a) $\Omega$ and (b) the coherent fraction $e$ on $\omega_{0}$, with $\omega(x)=$ $\omega_{0} \cos (3 x)$ and $\beta=0.05$.

## 1-cluster chimera states

Figure 4.18 shows an example of a 1-cluster chimera state when $\omega(x)=0.1 \cos (x)$; 1cluster chimera states of this type have not thus far been reported for $G(x)=\cos (x)$, $\omega_{0}=0$, where computations always result in 2 -cluster states. To understand the origin of this unexpected 1-cluster state we use Fig. 4.18(b) to conclude that $\tilde{Z}(x)$ is of the form $a \cos (x)$, and hence that Eqs. (4.13) and (4.14) reduce to the single equation

$$
\begin{equation*}
2 \pi \Omega-\left\langle\sqrt{(\Omega+\omega(y))^{2}-a^{2} \cos ^{2} y}\right\rangle=\exp (-i \beta) a^{2} \tag{4.20}
\end{equation*}
$$

Figure 4.19 shows the result of numerical continuation of the solution of this equation as a function of $\omega_{0}$. The figure reveals a transition at $\omega_{0}=0.0463$ (Figs. 4.19(a,b)). When

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Figure 4.18: (a) A snapshot of the phase distribution $\theta(x, t)$ in a 1-cluster chimera state for $G(x)=\cos (x)$ and $\omega(x)=0.1 \cos (x)$. (b) The corresponding order parameter $R(x)$. (c) The corresponding order parameter $\Theta(x)$. Panel (a) shows the presence of a nearly coherent region near $x=0$ with oscillators that oscillate $\pi$ out of phase with the coherent cluster. The calculation is done with $\beta=0.05$ and $N=512$.


Figure 4.19: (a) The quantities $a$ and $\Omega$, and (b) the coherent fraction $e$, all as functions of $\omega_{0}$ for the chimera state shown in Fig. 4.18. (c) $\Omega+\omega(x)$ and $R(x)$ as functions of $x$ when $\omega_{0}=0.0463$.
$\omega_{0}<0.0463$, the solution is a 2 -cluster chimera; as $\omega_{0}$ increases through $\omega_{0} \approx 0.0463$ the coherent region around $x=0$ disappears, leaving a single-cluster chimera state (Fig. 4.19(c)).

## 2-cluster chimera states

For $G(x)=\cos (x)$ and constant $\omega$, simulations always evolve into either a 2-cluster chimera state or a splay state. It is expected that when $\omega_{0}$ is small, the 2-cluster chimera is not destroyed. Figs. 4.20(a) and 4.20(b) gives examples of this type of state with $l=2$ and $l=3$ inhomogeneities, respectively. In both cases the clusters are located at specific locations selected by the inhomogeneity.

The order parameter profile with $l=3$ corresponding to the numerical solution in Fig. 4.20(b) has the same symmetry properties as that in Figs. 4.5(b,c). It follows that

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(b)


Figure 4.20: 2-cluster chimera states for $G(x)=\cos (x)$ and (a) $\omega(x)=0.1 \cos (2 x)$ and (b) $\omega(x)=0.1 \cos (3 x)$. In both cases $\beta=0.05$ and $N=512$.


Figure 4.21: The dependence of (a) $\Omega$ (solid line) and $b$ (dashed line), and (b) the coherent fraction $e$ on $\omega_{0}$ when $l=3$ and $\beta=0.05$.
we may set $a=0, b=b_{r}>0$ in the order parameter representation (4.10), resulting once again in the self-consistency relation (4.16). We have continued the solutions of this relation for $l=3$ using the simulation in Fig. 4.20(b) with $\omega_{0}=0.1$ to initialize continuation in $\omega_{0}$. Figure 4.21 shows the result of numerical continuation of the order parameter for this state. No further transitions are revealed. There are three preferred locations for the coherent clusters, related by the translation symmetry $x \rightarrow x+\frac{2 m \pi}{3}, m=0,1,2$, as shown in Fig. 4.22.

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Figure 4.22: The three possible locations of the 2-cluster chimera state when $\omega(x)=$ $0.1 \cos (3 x)$. The simulation is done with $\beta=0.05$ and $N=512$.


Figure 4.23: (a) A snapshot of the phase pattern in a traveling coherent state when $\omega$ is constant. (b) The position $x_{0}$ of this state as a function of time. The simulation is done for $G(x)=\cos (x)+\cos (2 x)$ with $\beta=0.75$ and $N=512$.

### 4.4 Traveling coherent solutions

We now turn to states with a spatially structured order parameter undergoing translation. With $G(x)=\cos (x)+\cos (2 x)$ and constant $\omega$ the system (4.2) exhibits a fully coherent but non-splay state that travels with a constant speed $c(\beta)$ when $0.646 \lesssim \beta \lesssim 0.7644$. Figure 4.23(a) shows a snapshot of this traveling coherent state while Fig. 4.23(b) shows its position $x_{0}$ as a function of time. The speed of travel is constant and can be obtained by solving a nonlinear eigenvalue problem (see Chapter 3). In this section, we investigate how the inhomogeneities $\omega(x)=\omega_{0} \exp (-\kappa|x|)$ and $\omega(x)=\omega_{0} \cos (l x)$ affect the dynamical behavior of this state.

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## Bump inhomogeneity: $\omega(x) \equiv \omega_{0} \exp (-\kappa|x|)$

In this section we perform two types of numerical experiments. In the first we fix $\beta$ and start with the traveling coherent solution for $\omega_{0}=0$. We then gradually increase $\omega_{0}$ at fixed values of $\kappa$ and $\beta$. In the second we fix $\omega_{0}>0, \kappa>0$, and vary $\beta$.

For each $0.646 \lesssim \beta \lesssim 0.7644$ we find that the coherent state continues to travel, albeit nonuniformly, until $\omega_{0}$ reaches a threshold value that depends on the values of $\beta$ and $\kappa$. The case $\beta=0.75$ provides an example. Figure $4.24(\mathrm{a})$ shows the traveling coherent state in the homogeneous case ( $\omega_{0}=0$ ) while Fig. $4.24(\mathrm{~b})$ shows the corresponding state in the presence of a frequency bump $\omega(x) \equiv \omega_{0} \exp (-\kappa|x|)$ with $\omega_{0}=0.04$ and $\kappa=2$. In this case the presence of inhomogeneity leads first to a periodic fluctuation in the magnitude of the drift speed followed by, as $\omega_{0}$ continues to increase, a transition to a new state in which the direction of the drift oscillates periodically (Fig. $4.24(\mathrm{~b})$ ). We refer to states of this type as direction-reversing waves, by analogy with similar behavior found in other systems supporting the presence of such waves [80, 81]. With increasing $\omega_{0}$ the reversals become localized in space (and possibly aperiodic; Fig. 4.24(c)) and then cease, leading to a stationary pinned structure at $\omega_{0}=0.12$ (Fig. $4.24(\mathrm{~d})$ ). Figure 4.25 shows the position $x_{0}$ of the maximum of the local order parameter $R(x, t)$ of the coherent state as a function of time for the cases in Fig. 4.24, showing the transition from translation to pinning as $\omega_{0}$ increases, via states that are reflection-symmetric on average. The final state is a steady reflection-symmetric pinned state aligned with the imposed inhomogeneity.

In fact, the dynamics of the present system may be more complicated than indicated above since a small group of oscillators located in regions where the order parameter undergoes rapid variation in space may lose coherence in a periodic fashion even when $\omega_{0}=0$ thereby providing a competing source of periodic oscillations in the magnitude of the drift speed. As documented in Chapter 3 this is the case when $0.7570<\beta<0.7644$. For $\beta=0.75$, however, the coherent state drifts uniformly when $\omega_{0}=0$ and this is therefore the case studied in greatest detail.

The profile of the pinned coherent state can also be determined from a self-consistency analysis. For $G(x)=\cos (x)+\cos (2 x)$, the local order parameter $\tilde{Z}(x)$ can be written in the form

$$
\begin{equation*}
\tilde{Z}(x)=a \cos (x)+b \sin (x)+c \cos (2 x)+d \sin (2 x) \tag{4.21}
\end{equation*}
$$

Owing to the reflection symmetry of the solution $b=d=0$; on applying a rotation in $\theta$ we may take $a$ to be real. The self-consistency equation then becomes

$$
\begin{align*}
a \exp (-i \beta) & =\left\langle\frac{\cos (y)(\Omega+\omega(y)-\mu(y))}{a \cos y+c^{*} \cos (2 y)}\right\rangle  \tag{4.22}\\
c \exp (-i \beta) & =\left\langle\frac{\cos (2 y)(\Omega+\omega(y)-\mu(y))}{a \cos y+c^{*} \cos (2 y)}\right\rangle \tag{4.23}
\end{align*}
$$

When $\omega_{0}=0.12, \kappa=2$, and $\beta=0.75$ the solution of these equations is $\Omega=1.9879$, $a=3.1259, c_{r}=-0.1007, c_{i}=-2.7300$, results that are consistent with the values obtained


Figure 4.24: Hidden line plot of the phase distribution $\theta(x, t)$ when (a) $\omega_{0}=0$. (b) $\omega_{0}=0.04$. (c) $\omega_{0}=0.08$. (d) $\omega_{0}=0.12$. In all cases, $\kappa=2, \beta=0.75$ and $N=512$.
from numerical simulation. For these parameter values, $\Omega+\omega(x)<|a \cos (x)+c \cos (2 x)|$ for $x \in(-\pi, \pi]$, indicating that all phases rotate with the same frequency $\Omega$. As we decrease $\omega_{0}$ to $\omega_{0} \approx 0.11$ the profiles $\Omega+\omega(x)$ and $|a \cos (x)+c \cos (2 x)|$ start to touch (Fig. 4.26) and for yet lower $\omega_{0}$ the stationary coherent state loses stability and begins to oscillate as described in the previous paragraph.

We have also conducted experiments at a fixed value of $\omega_{0}>0$ and $\kappa>0$ while changing $\beta$. For example, at fixed $\omega_{0}=0.12, \kappa=2$ and $\beta=0.75$ the system is in the pinned state shown in Fig. 4.24(a). Since the speed of the coherent state with $\omega_{0}=0$ gradually increases as $\beta$ decreases, we anticipate that a given inhomogeneity will find it harder and harder to pin the state as $\beta$ decreases. This is indeed the case, and we find that there is a critical value of $\beta$ at which the given inhomogeneity is no longer able to pin the structure, with depinning via back and forth oscillations of the structure [81]. For yet smaller values of $\beta$, this type of oscillation also loses stability and evolves into near-splay states. When we increase $\beta$ again we uncover hysteresis in each of these transitions. Figure 4.27 shows two distinct states at identical parameter values: $\omega_{0}=0.12, \kappa=2$ and $\beta=0.76$ generated using different protocols: Fig. 4.27(a) shows a direction-reversing state evolved from a traveling coherent state when we increase $\omega_{0}$ from 0 to 0.12 at $\beta=0.76$, while Fig. 4.27(b) shows a pinned state generated from the pinned state at $\beta=0.75$ when we change $\beta$ from 0.75 to 0.76 at fixed $\omega_{0}=0.12$.


Figure 4.25: The position $x_{0}$ of the coherent state in Fig. 4.24 as a function of time for (a) $\omega_{0}=0$, (b) $\omega_{0}=0.04$, (c) $\omega_{0}=0.08$, and (d) $\omega_{0}=0.12$. In all cases $\kappa=2, \beta=0.75$, and $N=512$.


Figure 4.26: The profiles of $\Omega+\omega(x)$ and $R=|a \cos (x)+c \cos (2 x)|$ for $\omega_{0}=0.11, \kappa=2$, and $\beta=0.75$.


Figure 4.27: Hidden line plot of the phase distribution $\theta(x, t)$ when $\omega_{0}=0.12$ and $\beta=0.76$. (a) Unpinned state obtained from a traveling coherent state with $\omega_{0}=0$ and $\beta=0.76$ on gradually increasing $\omega_{0}$ to 0.12 . (b) Pinned state obtained from a traveling coherent with $\omega_{0}=0$ and $\beta=0.75$ on gradually increasing $\omega_{0}$ to 0.12 , and then increasing $\beta$ to 0.76 . In both cases $\kappa=2$ and $N=512$.


Figure 4.28: (a) A snapshot of the phase distribution $\theta(x, t)$ of a coherent solution when $\omega_{0}=0.028$. (b) The position $x_{0}$ of the coherent solution as a function of time. In both cases $l=1, \beta=0.05$ and $N=512$.


Figure 4.29: (a) The mean angular velocity $\Omega$ and (b) the mean drift speed $\bar{c}$, both as functions of $\omega_{0}$ when $l=1, \beta=0.7$.

## Periodic inhomogeneity: $\omega(x) \equiv \omega_{0} \cos (l x)$

We now turn to the case $\omega(x) \equiv \omega_{0} \cos (l x)$. As in the bump inhomogeneity case, when $\beta$ is fixed, the traveling coherent state will continue to travel until $\omega_{0}$ reaches a certain threshold. However, we did not find a pinned state around $\beta=0.75$ as in the previous subsection. Instead we focus on the case $\beta=0.7$ for which the traveling coherent state has a reasonable speed when $\omega_{0}=0$ and take $l=1$. When $\omega_{0}$ is small, the state remains coherent (Fig. 4.28(a)) and continues to drift, albeit no longer with a uniform speed of propagation. Figure 4.28 (b) shows that the speed executes slow, small-amplitude oscillations about a well-defined mean value $\bar{c}\left(\omega_{0}\right)$ shown in Fig. 4.29(b); the corresponding time-averaged oscillation frequency $\Omega\left(\omega_{0}\right)$ is shown in Fig. 4.29(a). When $\omega_{0}$ is increased in sufficiently small increments the oscillations grow in amplitude but the solution continues to travel to the left until $\omega_{0} \approx 0.285$, where a hysteretic transition to a near-splay state takes place. Figure 4.29 (b) shows that prior to this transition the average speed first decreases as a consequence of the inhomogeneity, but then increases abruptly just before the transition owing to the loss of coherence on the part of a group of oscillators and the resulting abrupt increase in asymmetry of the order parameter.

### 4.5 Traveling chimera states

In addition to the states discussed in the previous sections, Eq. (4.2) with constant $\omega$ admits traveling one-cluster chimera states for $G(x)=G_{n}^{(2)}(x) \equiv \cos (n x)+\cos [(n+1) x]$ and appropriate values of the phase lag $\beta$. This state consists of a single coherent cluster that drifts through an incoherent background as time evolves at more or less a constant speed. Figure 4.30 (a) shows a snapshot of such a state when $n=3$. The direction of motion is determined by the gradient of the phase in the coherent region: the cluster travels to the


Figure 4.30: (a) A snapshot of the phase distribution $\theta(x, t)$ for a traveling chimera state in a spatially homogeneous system. (b) The position $x_{0}$ of the coherent cluster as a function of time. The simulation is done for $G(x) \equiv \cos (3 x)+\cos (4 x)$ with $\beta=0.03$ and $N=512$.
left when the gradient is positive and to the right when the gradient is negative. Figure 4.30 (b) shows the position $x_{0}$ of the coherent cluster as a function of time and confirms that the cluster moves to the right at an almost constant speed. In Chapter 3, we use numerical simulations to conclude that for $n=3$ the traveling chimera is stable in the interval $0.015 \lesssim \beta \lesssim 0.065$. We therefore focus on the effects of spatial inhomogeneity on the traveling chimera state when $\beta=0.03$. In fact the traveling chimera state is more complex than suggested in Fig. 4.30(a,b): unlike the states discussed in the previous sections, the profile of the local order parameter fluctuates in time, suggesting that the state does not drift strictly as a rigid object (see Chapter 3).

## Bump inhomogeneity: $\omega(x) \equiv \omega_{0} \exp (-\kappa|x|)$

In this section we investigate the effect of a bump-like inhomogeneity $\omega(x) \equiv \omega_{0} \exp (-\kappa|x|)$ on the motion of the traveling chimera state. Starting with the traveling chimera state for $\omega_{0}=0$, we increase $\omega_{0}$ in steps of $\Delta \omega_{0}=0.01$. To describe the motion of the coherent cluster, we follow the method in [35] and determine the instantaneous position $x_{0}$ of the cluster by minimizing the function $F\left(x^{*}\right)=\frac{1}{N} \sum_{k}^{N}\left[\theta_{t}\left(x_{k}, t\right)-f\left(x_{k}, x^{*}\right)\right]^{2}$, where $f\left(x, x^{*}\right)=-\cos \left(x-x^{*}\right)$ is a reference profile, and using the minimizer $x^{*}(t)$ as a proxy for $x_{0}(t)$. We find that even small $\omega_{0}$ suffices to stop a traveling chimera from moving: Fig. 4.31 shows that the threshold $\omega_{0} \approx 0.01$ for $\kappa=1$ and that it increases monotonically to $\omega_{0} \approx 0.03$ for $\kappa=10$. The resulting pinned state persists to values of $\omega_{0}$ as large as $\omega_{0}=1$.

Figures $4.32(\mathrm{a}, \mathrm{c}, \mathrm{e})$ show the position $x_{0}$ of the coherent cluster as a function of time obtained using the above procedure for $\omega_{0}=0.04,0.08,0.12$, respectively, i.e., in the pinned regime. The figures show that the equilibrium position of the coherent region is located


Figure 4.31: Dependence of the pinning threshold $\omega_{0}$ on $\kappa$ when $\beta=0.03$.


Figure 4.32: The position $x_{0}$ of the pinned coherent cluster in a traveling chimera state as a function of time when (a) $\omega_{0}=0.04$, (c) $\omega_{0}=0.08$, (e) $\omega_{0}=0.12$. The average rotation frequency $\bar{\theta}_{t}$ for (b) $\omega_{0}=0.04$, (d) $\omega_{0}=0.08$, (f) $\omega_{0}=0.12$. In all cases $\beta=0.03$ and $\kappa=10$, $N=512$.
farther from the position $x=0$ of the inhomogeneity peak as $\omega_{0}$ increases. The bump in $\omega(x)$ thus exerts a "repelling force" on the coherent cluster, whose strength increases with the height of the bump. We interpret this observation as follows. The coherent cluster can only survive when the frequency gradient is sufficiently small, and is therefore repelled by regions where $\omega(x)$ varies rapidly. In the present case this implies that the coherent cluster finds it easiest to survive in the wings of the bump inhomogeneity, and this position moves further from $x=0$ as $\omega_{0}$ increases. This interpretation is confirmed in Figs. 4.32(b,d,f) showing the average rotation frequency, $\bar{\theta}_{t}(x)$, of the oscillators. The plateau in the profile of $\bar{\theta}_{t}(x)$ indicates frequency locking and hence the location of the coherent cluster; the fluctuations in the position of the coherent cluster are smoothed out by the time-averaging.

## Periodic inhomogeneity: $\omega(x) \equiv \omega_{0} \cos (l x)$

We now turn to the effects of a periodic inhomogeneity $\omega(x) \equiv \omega_{0} \cos (l x)$. When $l=1$ and $\omega_{0}$ increases the coherent cluster initially travels with a non-constant speed but then becomes pinned in place; as in the case of the bump inhomogeneity quite small values of $\omega_{0}$ suffice to pin the coherent cluster in place (for $l=1$ the value $\omega_{0} \approx 0.003$ suffices). As shown in Figs. 4.33(a) and 4.33 (c) the position $x_{0}$ of the pinned cluster relative to the local maximum of the inhomogeneity (i.e., $x=0$ ) depends on the value of $\omega_{0}>0.003$. For $\omega_{0}<0.003$ (Fig. 4.34) the coherent cluster travels to the right and does so with a speed that is larger when $x_{0}(t)>0$ than when $x_{0}(t)<0$. This effect becomes more pronounced as $\omega_{0}$ increases. This is because the coherent structure is asymmetric, with a preferred direction of motion, and this asymmetry increases with $\omega_{0}$. Evidently, the speed of the synchronization front at the leading edge is enhanced when $\omega^{\prime}(x)<0$ but suppressed when $\omega^{\prime}(x)>0$ and likewise for the desynchronization front at the rear.

For $l>1$ we observe similar results. The coherent cluster is pinned in space already at small values of $\omega_{0}$. Since the inhomogeneous system has the discrete translation symmetry $x \rightarrow x+\frac{2 \pi}{l}$ the coherent cluster has $l$ possible preferred positions. Figure 4.35 shows an example for $l=2$. Panels (a) and (c) show snapshots of the phase distribution $\theta(x, t)$ for $\omega_{0}=0.01$ in the two preferred locations (separated by $\Delta x=\pi$ ), while panels (b) and (d) show the corresponding average rotation frequency $\bar{\theta}_{t}$.

### 4.6 Conclusion

In this chapter we have investigated a system of non-identical phase oscillators with nonlocal coupling, focusing on the effects of weak spatial inhomogeneity in an attempt to extend earlier results on identical oscillators to more realistic situations. Two types of inhomogeneity were considered, a bump inhomogeneity in the frequency distribution specified by $\omega(x)=\omega_{0} \exp (-\kappa|x|)$ and a periodic inhomogeneity specified by $\omega(x)=\omega_{0} \cos (l x)$. In each case we examined the effect of the amplitude $\omega_{0}$ of the inhomogeneity and its spatial scale $\kappa^{-1}\left(l^{-1}\right)$ on the properties of states known to be present in the homogeneous case $\omega_{0}=$


Figure 4.33: The position $x_{0}$ of the pinned coherent cluster in a traveling chimera state as a function of time when (a) $\omega_{0}=0.005$, (c) $\omega_{0}=0.01$. The average rotation frequency $\bar{\theta}_{t}$ for (b) $\omega_{0}=0.005$, (d) $\omega_{0}=0.01$. In all cases $\beta=0.03, l=1$ and $N=512$.


Figure 4.34: The position $x_{0}$ of the coherent cluster in a traveling chimera state as a function of time when (a) $\omega_{0}=0.001$, (b) $\omega_{0}=0.002$. In all cases $\beta=0.03, l=1$ and $N=512$.


Figure 4.35: (a,c) The two possible phase distributions $\theta(x, t)$ of pinned traveling chimera states when $\omega_{0}=0.01, \beta=0.03$ and $l=2$. (b,d) The corresponding average rotation frequencies $\bar{\theta}_{t}$. In both cases $N=512$.

0 , viz., splay states and stationary chimera states, traveling coherent states and traveling chimera states.

We have provided a fairly complete description of the effects of inhomogeneity on these states for the coupling functions $G(x)=\cos (x), \cos (x)+\cos (2 x)$ and $\cos (3 x)+\cos (4 x)$ employed in Chapter 3. Specifically, we found that as the amplitude of the inhomogeneity increased a splay state turned into a near-splay state, characterized by a nonuniform spatial phase gradient, followed by the appearance of a stationary incoherent region centered on the location of maximum inhomogeneity amplitude. With further increase in $\omega_{0}$ additional intervals of incoherence opened up, leading to states resembling the stationary multi-cluster chimera states also present in the homogeneous system. These transitions, like many of the transitions identified in this chapter, could be understood with the help of a self-consistency analysis based on the Ott-Antonsen Ansatz [46], as described in the Chapter 2. The effect of inhomogeneity on multi-cluster chimera states was found to be similar: the inhomogeneity trapped the coherent clusters in particular locations, and eroded their width as its amplitude $\omega_{0}$ increased, resulting in coalescence of incoherent regions with increasing $\omega_{0}$.

## CHAPTER 4. CHIMERA STATES IN SYSTEMS OF NONLOCAL NONIDENTICAL PHASE-COUPLED OSCILLATORS

More significant are the effects of inhomogeneity on traveling coherent and traveling chimera states. Here the inhomogeneity predictably pins the traveling structures but the details can be complex. Fig. 4.24 shows one such complex pinning transition that proceeds via an intermediate direction-reversing traveling wave. These waves are generated directly as a consequence of the inhomogeneity and would not be present otherwise, in contrast to homogeneous systems undergoing a symmetry-breaking Hopf bifurcation as described in [80]. Many of the pinning transitions described here are hysteretic as demonstrated in Fig. 4.27. The traveling chimera states are particularly fragile in this respect, with small amplitude inhomogeneities sufficient to arrest the motion of these states. In all these cases the coherent regions are found in regions of least inhomogeneity, an effect that translates into an effective repulsive interaction between the coherent cluster and the inhomogeneity.

In future work we propose to explore similar dynamics in systems of more realistic nonlocally coupled oscillators and compare the results with those for similar systems with a random frequency distribution.

## Chapter 5

## Twisted chimera states and multi-core spiral chimera states on a two-dimension torus

### 5.1 Introduction

In previous chapters, as well as many other previous works, chimera states were studied in one-dimensional systems. Rather less is known about chimera states in two-dimensional oscillator arrays and existing results are largely based on numerical simulations [64, 65, 68]. Both two- and three-dimensional arrays have been studied and the results have revealed new types of chimera states, in addition to the counterparts of the one-dimensional states already mentioned. In two dimensions these include spiral chimera states with an incoherent core while the oscillators outside the core oscillate coherently. In this chapter we consider the case of two-dimensional oscillator arrays with nonlocal coupling, with a view of studying, both numerically and analytically, different states of partial synchrony in two dimensions. Like earlier work, our work is restricted to arrays of phase-coupled oscillators with periodic boundary conditions in both directions, i.e., to phase oscillators on a flat torus.

First steps in this direction were taken by Kim et al. [103] in their study of the system

$$
\begin{equation*}
\frac{d \theta_{i j}}{d t}=\omega-\frac{K}{N(R)} \sum_{(m-i)^{2}+(n-j)^{2} \leqslant R^{2}}\left(\theta_{i j}-\theta_{m n}+\alpha\right) \tag{5.1}
\end{equation*}
$$

Here $\theta_{i j}$ denotes the phase of the oscillator at position $(i, j)$ on a two-dimensional periodic lattice, $K$ is the coupling strength, and $N(R)$ is the number of oscillators whose distance to $(i, j)$ is no larger than $R$. They found and investigated many interesting patterns that arise as $\alpha$ varies in the range $[-\pi, \pi]$, but did not observe chimera states. In 2012, Hagerstrom et al. created a physical realization of a two-dimensional iterated map with nonlocal coupling and periodic boundary conditions, and observed chimera states in an experiment [77]. Meanwhile, Omel'chenko et al. [68] presented a series of numerical experiments on the same system
as used in [103] but with a careful preparation of initial conditions. They reported chimera states of three types: a coherent spot, an incoherent spot, and different stripe patterns. In addition, they observed a stable configuration of four spirals. While Omel'chenko et al.'s work is largely based on numerical simulations, Panaggio and Abrams used asymptotic methods to derive conditions under which two-dimensional "spot" and "stripe" chimeras can exist on a two-dimensional periodic domain [69]. With their method, they also reveal an asymmetric chimera state; however, this state is unstable and appears in simulations only as a transient. In the work reported below additional two-dimensional chimera states, including states we call twisted chimera states and multi-core spiral wave chimera states, are observed and studied.

The system we consider consists of an oscillator array on a two-dimensional torus. The continuous version of the model leads to the following evolution equation for the phase $\theta(x, y, t)$ :

$$
\begin{equation*}
\frac{\partial \theta(x, y, t)}{\partial t}=-\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G\left(x-x^{\prime}, y-y^{\prime}\right) \sin \left(\theta(x, y, t)-\theta\left(x^{\prime}, y^{\prime}, t\right)+\alpha\right) d x^{\prime} d y^{\prime} \tag{5.2}
\end{equation*}
$$

To make the system analytically tractable, we assume $G$ can be decomposed as $G(x, y)=$ $G_{x}(x)+G_{y}(y)$, where $G_{x}$ and $G_{y}$ are functions representing the coupling in the $x$ and $y$ directions, respectively. Inspired by the work in Chapter 3, we choose $G_{x}$ and $G_{y}$ from the following two families:

$$
G_{n}^{(1)}(x) \equiv \cos (n x), \quad G_{n}^{(2)}(x) \equiv \cos (n x)+\cos [(n+1) x] .
$$

With these coupling functions, we observe chimera states on a two-dimensional torus with random initial conditions, including twisted chimeras and a variety of spiral wave chimeras. The simulations are carried out using a fourth-order Runge-Kutta method with time step $\delta t=0.025$ with both $x$ and $y$ directions discretized into $N$ uniform intervals with $N=256$ or $N=512$.

This chapter is organized as follows. In Section 5.2 we introduce the effective order parameter equation we will use to describe the phase patterns obtained in numerical simulations. These are described in Sections 5.3 and 5.4 for two different choices of the coupling function $G(x, y)$. These sections also relate the simulations to solutions of the order parameter equation and use the latter to determine the stability properties of the different states found. The chapter ends with a brief discussion in Section 5.5.

### 5.2 Effective equation

As in the one-dimensional case, an equivalent description of Eq. (5.2) can be obtained by constructing an equation for the local order parameter $z(x, y, t)$ defined as the local spatial average of $\exp [i \theta(x, y, t)]$,

$$
\begin{equation*}
z(x, y, t) \equiv \lim _{\delta \rightarrow 0^{+}} \frac{1}{\delta^{2}} \int_{-\delta / 2}^{\delta / 2} \int_{-\delta / 2}^{\delta / 2} \exp \left[i \theta\left(x+x^{\prime}, y+y^{\prime}, t\right)\right] d x^{\prime} d y^{\prime} \tag{5.3}
\end{equation*}
$$

## CHAPTER 5. TWISTED CHIMERA STATES AND MULTI-CORE SPIRAL CHIMERA

 STATES ON A TWO-DIMENSION TORUSThe evolution equation for $z$ then takes the form

$$
\begin{equation*}
z_{t}=\frac{1}{2}\left(\exp (-i \alpha) Z-\exp (i \alpha) z^{2} Z^{*}\right) \tag{5.4}
\end{equation*}
$$

where $Z(x, y, t) \equiv K[z](x, y, t)$ and $K$ is a compact linear operator defined via the relation

$$
\begin{equation*}
K[u](x, y, t) \equiv \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G\left(x-x^{\prime}, y-y^{\prime}\right) u\left(x^{\prime}, y^{\prime}, t\right) d x^{\prime} d y^{\prime} \tag{5.5}
\end{equation*}
$$

A derivation of Eq. (5.4) based on the Ott-Antonsen Ansatz is very similar to the onedimensional case $[45,60]$ and is omitted.

Stationary rotating solutions of Eq. (5.4) take the form

$$
\begin{equation*}
z(x, y, t)=\tilde{z}(x, y) \exp (-i \Omega t) \tag{5.6}
\end{equation*}
$$

whose common frequency $\Omega$ satisfies the nonlinear nonlocal eigenvalue problem

$$
\begin{equation*}
i \Omega \tilde{z}+\frac{1}{2}\left[\exp (-i \alpha) \tilde{Z}(x, y)-\tilde{z}^{2} \exp (i \alpha) \tilde{Z}^{*}(x, y)\right]=0 \tag{5.7}
\end{equation*}
$$

Here $\tilde{z}(x, y)$ describes the spatial profile of the rotating solution and $\tilde{Z} \equiv K[\tilde{z}]$.
Solving Eq. (5.7) as a quadratic equation in $\tilde{z}$ we obtain

$$
\begin{equation*}
\tilde{z}(x, y)=\exp (i \beta) \frac{\Omega-\mu(x, y)}{\tilde{Z}^{*}(x, y)}=\frac{\exp (i \beta) \tilde{Z}(x, y)}{\Omega+\mu(x, y)} \tag{5.8}
\end{equation*}
$$

where $\beta \equiv(\pi / 2)-\alpha$ and $\mu(x, y)=\left[\Omega^{2}-|\tilde{Z}(x, y)|^{2}\right]^{1 / 2}$. Combining this equation and the relation between $\tilde{z}$ and $\tilde{Z}$ gives us the self-consistency equation

$$
\begin{equation*}
\tilde{Z}(x, y)=\exp (i \beta) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G\left(x-x^{\prime}, y-y^{\prime}\right) \frac{\Omega-\mu(x, y)}{\tilde{Z}^{*}(x, y)} d x^{\prime} d y^{\prime} \tag{5.9}
\end{equation*}
$$

As in the one-dimensional case, temporal stability of the stationary rotating solution is determined by the spectrum of the following linear operator:

$$
\begin{equation*}
v_{t}=L[v] \equiv i \mu(x, y) v+\frac{1}{2}\left[\exp (-i \alpha) V(x, y, t)-\exp (i \alpha) \tilde{z}^{2} V^{*}(x, y, t)\right] \tag{5.10}
\end{equation*}
$$

where $V(x, y, t) \equiv K[v]=\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G\left(x-x^{\prime}, y-y^{\prime}\right) v\left(x^{\prime}, y^{\prime}, t\right) d x^{\prime} d y^{\prime}$.

### 5.3 The case $G_{x}=G_{y}=G_{n}^{(1)}$

Here we consider the case where $G_{x}=\cos (n x)$ and $G_{y}=\cos (n y)$. In Chapter 3 we discussed the one-dimensional system with $G(x)=\cos (n x)$, where splay states and $2 n$ cluster chimera states are observed. In the current two-dimensional system, the counterparts

## CHAPTER 5. TWISTED CHIMERA STATES AND MULTI-CORE SPIRAL CHIMERA

 STATES ON A TWO-DIMENSION TORUSof splay states are still present, although no counterparts of $2 n$-cluster chimera states were observed: $2 n$-cluster chimera states are always unstable in two dimensions. This is because the 2 -cluster state is unstable (Sec. 5.3) and the stability calculation for the case $n>1$ is the same. However, in addition to the stable splay states, we have also observed two new types of chimera states. One is the twisted chimera state already mentioned, in which a coherent stripe with uniformly varying phase distribution coexists with an incoherent domain, while the other is a multi-core spiral wave chimera. These states are described below.

Before proceeding we wish to point out two facts about Eq. (5.2) with the present coupling. First, the system is invariant under the three reflections $(x, y) \rightarrow(-x, y),(x, y) \rightarrow$ $(x,-y)$, and $(x, y) \rightarrow(y, x)$, i.e., it is invariant under the group $D_{4}$ of rotations and reflections of a square. It follows that if $\theta(x, y, t)$ is a solution, then so are $\theta(-x, y, t), \theta(x,-y, t)$, and $\theta(y, x, t)$. Second, if $\int_{-\pi}^{\pi} G_{y}(y) d y=0$, then the following statement holds: if $\theta_{1}(x, t)$ is a solution for the one-dimensional system

$$
\begin{equation*}
\frac{\partial \theta}{\partial t}=-\int_{-\pi}^{\pi} G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y \tag{5.11}
\end{equation*}
$$

then $\theta_{2}(x, y, t) \equiv \theta_{1}(x, 2 \pi t)$ is a solution of Eq. (5.2). We say that the solution $\theta_{2}$ of the twodimensional system is inherited from the solution $\theta_{1}$ for the corresponding one-dimensional system.

## Splay states



Figure 5.1: (Color online) Snapshot of the phase pattern for splay states in two dimensions. (a) The phase distribution $\theta(x, y, t)$ for $G_{x}=G_{y}=G_{1}^{(1)}$. (b) The phase distribution $\theta(x, y, t)$ for $G_{x}=G_{y}=G_{2}^{1}$. The simulations are done with $\beta=0.05, N=256$ from a random initial condition. Colors indicate the phase of the oscillators.

Splay states with a phase that varies uniformly with respect to $x$ are observed in onedimensional systems. The inherited solution of the two-dimensional system is also observed

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 STATES ON A TWO-DIMENSION TORUSin numerical simulations. Figure 5.1 shows two examples in which the phase varies uniformly in the $x$ direction. Solutions in which the phase varies uniformly in the $y$ direction are also observed (not shown).

The splay state in Fig. 5.1 can be written in the form

$$
\begin{equation*}
\theta(x, y, t)=-\Omega t+q x \tag{5.12}
\end{equation*}
$$

where $-\Omega$ represents the overall rotation frequency and $q$ is the twist number. Substituting Eq. (5.12) into Eq. (5.2) we obtain a relation for the rotation frequency:

$$
\begin{equation*}
\Omega=\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(x, y) \sin (q x+\alpha) d x d y \tag{5.13}
\end{equation*}
$$

As in the one-dimensional case, states of this type travel with speed $c=\frac{\Omega}{q}$, i.e., to the right when both $\Omega$ and $q$ have the same sign and to the left when $\Omega$ and $q$ have opposite signs.

To investigate the linear stability of the splay states, we use the approach of Chapter 3 and suppose that $\theta(x, y, t)$ is perturbed to $\theta(x, y, t)+\epsilon \exp (\lambda t) \exp \left(i\left(k_{x} x+k_{y} y\right)\right)$ with $0<\epsilon \ll 1$. A straightforward calculation, using the fact that $G_{x}, G_{y}$ are both even and $\hat{G}_{y 0}=0$, leads to the following relation for the growth rate $\lambda$ of the perturbation:
$\lambda=\frac{1}{2}\left[\exp (i \alpha)\left(\hat{G}_{x q-k_{x}} \delta_{k_{y}, 0}+\delta_{k_{x}, q} \hat{G}_{y-k_{y}}\right)+\exp (-i \alpha)\left(\hat{G}_{x-k_{x}-q} \delta_{k_{y}, 0}+\delta_{k_{x},-q} \hat{G}_{y-k_{y}}\right)\right]-\hat{G}_{x q} \cos \alpha$,
where $\hat{u}_{k} \equiv \int_{-\pi}^{\pi} u(x) \exp (i k x) d x$. This relation shows that when $0<\alpha<\pi / 2$, the splay state is linearly stable when $|q|=n$ if $G_{x}=G_{y}=G_{n}^{(1)}$ and $|q|=n$ or $n+1$ if $G_{x}=G_{y}=G_{n}^{(2)}$.

## Two-cluster chimera states

As in one dimension the splay states may coexist with stable chimera states. However, no stable analogues of the $2 n$-cluster chimera states familiar from the one-dimensional problem have been observed in two dimensions since such states are now unstable. To demonstrate this we first compute such states using the self-consistency analysis and then determine their stability properties.

The local order parameter takes the form

$$
\begin{equation*}
\tilde{Z}(x, y)=R(x, y) \exp (i \Theta(x, y))=R_{0} \cos (n x) \tag{5.15}
\end{equation*}
$$

and the self-consistency relation for $R_{0}$ is

$$
\begin{equation*}
R_{0}^{2}=\exp (i \beta)\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \cos ^{2}(n x)}\right\rangle \tag{5.16}
\end{equation*}
$$

Here, the bracket $\langle\cdot\rangle$ is defined by $\langle f\rangle=\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f\left(x^{\prime}, y^{\prime}\right) d x^{\prime} d y^{\prime}$. Thus the self-consistency equation for $R_{0}$ has the same form as in one dimension (1D) except that the bracket represents


Figure 5.2: (Color online) Dependence of (a) $R_{0}$ and $\Omega$, and (b) the point eigenvalues $\lambda_{p}$ on the parameter $\beta$.
an integral over a torus rather than a ring. Therefore, if $\Omega^{\prime}$ and $R_{0}^{\prime}$ is the solution for 1D case, then $\Omega=2 \pi \Omega^{\prime}$ and $R_{0}=2 \pi R_{0}^{\prime}$ is the solution in two dimensions (2D). The solution of Eq. (5.16) is shown in Fig. 5.2(a). Since the right side of Eq. (5.16) is identical for all $n=k / 2, k \in \mathbb{N}$, it follows that the same solution describes all $2 n$-cluster chimera states, $n=1,2, \ldots$.

The linear stability of the $2 n$-cluster chimera states determined by the above analysis can be studied using Eq. (5.10). This equation is solved by

$$
\begin{equation*}
v(x, y, t)=\exp (\lambda t) v_{1}(x, y)+\exp \left(\lambda^{*} t\right) v_{2}^{*}(x, y) \tag{5.17}
\end{equation*}
$$

leading to the eigenvalue problem

$$
\lambda\binom{v_{1}}{v_{2}}=\frac{1}{2}\left(\begin{array}{cc}
2 i \mu+\exp (-i \alpha) K & -\exp (i \alpha) \tilde{z}^{2} K  \tag{5.18}\\
-\exp (-i \alpha) \tilde{z}^{* 2} K & -2 i \mu^{*}+\exp (i \alpha) K
\end{array}\right)\binom{v_{1}}{v_{2}}
$$

Since $K$ is compact, its spectrum consists of two parts, a continuous spectrum given by $\left\{i \mu(x, y),-i \mu^{*}(x, y)\right\}$ with $x, y \in[-\pi, \pi]$ and a (possibly empty) point spectrum. The spectrum is in addition symmetric with respect to the real axis: if $\lambda$ is an eigenvalue with eigenvector $\left(v_{1}, v_{2}\right)^{T}$, then $\lambda^{*}$ is an eigenvalue with eigenvector $\left(v_{2}^{*}, v_{1}^{*}\right)^{T}$. The continuous spectrum is stable (negative) or neutrally stable (purely imaginary). Thus the stability of the chimera states is determined by the point spectrum.

We can compute unstable point eigenvalues $\lambda_{p}$ numerically. For this purpose we rewrite Eq. (5.18) in the form

$$
\mathbf{L v} \equiv\left(\begin{array}{cc}
2-\frac{\exp (-i \alpha) K}{\lambda_{p}-i \mu} & \frac{\exp (i \alpha) \tilde{z}^{2} K}{\lambda_{p}-i \mu}  \tag{5.19}\\
\frac{\exp (-i \alpha) \tilde{z}^{*} K}{\lambda_{p}+i \mu^{*}} & 2-\frac{\exp (i \alpha) K}{\lambda_{p}+i \mu^{*}}
\end{array}\right)\binom{v_{1}}{v_{2}}=0
$$

and define $f \equiv \frac{1}{4} \frac{\exp (-i \alpha)}{\lambda_{p}-i \mu}, f^{*} \equiv \frac{1}{4} \frac{\exp (i \alpha)}{\lambda_{p}+i \mu^{*}}, g \equiv \frac{1}{4} \frac{\exp (i \alpha) \tilde{z}^{2}}{\lambda_{p}-i \mu}$, and $g^{*} \equiv \frac{1}{4} \frac{\exp (-i \alpha) \tilde{z}^{* 2}}{\lambda_{p}+i \mu^{*}}$. As suggested in [39, 60], it is convenient to solve the eigenvalue problem using Fourier basis functions,

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 STATES ON A TWO-DIMENSION TORUSespecially so since the coupling function is sinusoidal. Following the convention used in Chapter 3, we write

$$
\begin{equation*}
\mathbf{v}=\frac{1}{4 \pi^{2}} \sum_{m, n} \hat{\mathbf{v}}_{m n} \exp (-i m x) \exp (-i n y) \tag{5.20}
\end{equation*}
$$

In this basis, Eq. (5.19) takes the form

$$
\begin{equation*}
\sum_{m, n} \mathbf{B}_{k l, m n} \hat{\mathbf{v}}_{m n}=0 \tag{5.21}
\end{equation*}
$$

where $\hat{\mathbf{v}}_{m n}=\left(\hat{v}_{1, m n}, \hat{v}_{2, m n}\right)$,

$$
\mathbf{B}_{k l, m n}=\left(\begin{array}{cc}
2 \pi^{2} \delta_{k, m} \delta_{l, n}-\hat{f}_{k-m} l_{n-n} \hat{G}_{m n} & \hat{g}_{k-m} l-n \\
\hat{G}_{m n} \\
\hat{g}_{k-m l-n}^{*} \hat{G}_{m n} & 2 \pi^{2} \delta_{k, m} \delta_{l, n}-\hat{f}_{k-m l-n}^{*} \hat{G}_{m n}
\end{array}\right),
$$

and $\hat{f}_{l}, \hat{f}_{l}^{*}, \hat{g}_{l}, \hat{g}_{l}^{*}$ are the Fourier coefficients of $f, f^{*}, g$ and $g^{*}$, respectively; the latter are defined by $\hat{f}_{k l}=\langle f \exp (i k x) \exp (i l y)\rangle$, etc.

The point eigenvalue $\lambda_{p}$ satisfies the condition $\operatorname{det} \mathbf{B}\left(\lambda_{p}\right)=0$. We use $n=1$ as an example to calculate $\lambda_{p}$. In this case $\hat{G}_{m n}=2 \pi^{2}$ when $(m, n)=( \pm 1,0)$ and $(0, \pm 1)$, and 0 otherwise. Therefore, $\lambda_{p}$ satisfies

$$
\operatorname{det}\left(\begin{array}{cccccccc}
1-\hat{f}_{0,0} & \hat{g}_{0,0} & -\hat{f}_{-2,0} & \hat{g}_{-2,0} & -\hat{f}_{-1,1} & \hat{g}_{-1,1} & -\hat{f}_{-1,-1} & \hat{g}_{-1,-1}  \tag{5.22}\\
\hat{g}_{0,0}^{*} & 1-\hat{f}_{0,0}^{*} & \hat{g}_{-2,0}^{*} & -\hat{f}_{-2,0}^{*} & \hat{g}_{-1,1}^{*} & -\hat{f}_{-1,1}^{*} & \hat{g}_{-1,-1}^{*} & -\hat{f}_{-1,-1}^{*} \\
-\hat{f}_{2,0} & \hat{g}_{2,0} & 1-\hat{f}_{0,0} & \hat{g}_{0,0} & -\hat{f}_{1,1} & \hat{g}_{1,1} & -\hat{f}_{1,-1} & \hat{g}_{1,-1} \\
\hat{g}_{2,0}^{*} & -\hat{f}_{2,0}^{*} & \hat{g}_{0,0}^{*} & 1-\hat{f}_{0,0}^{*} & \hat{g}_{1,1}^{*} & -\hat{f}_{1,1}^{*} & \hat{g}_{1,-1}^{*} & -\hat{f}_{1,-1}^{*} \\
-\hat{f}_{1,-1} & \hat{g}_{1,-1} & -\hat{f}_{-1,-1} & \hat{g}_{-1,-1} & 1-\hat{f}_{0,0} & \hat{g}_{0,0} & -\hat{f}_{0,-2} & \hat{g}_{0,-2} \\
\hat{g}_{1,-1}^{*} & -\hat{f}_{1,-1}^{*} & \hat{g}_{-1,-1}^{*} & -\hat{f}_{-1,-1}^{*} & \hat{g}_{0,0}^{*} & 1-\hat{f}_{0,0}^{*} & \hat{g}_{0,-2}^{*} & -\hat{f}_{0,-2}^{*} \\
-\hat{f}_{1,1} & \hat{g}_{1,1}^{*} & -\hat{f}_{-1,1} & \hat{g}_{-1,1} & -\hat{f}_{0,2} & \hat{g}_{0,2} & 1-\hat{f}_{0,0} & \hat{g}_{0,0} \\
\hat{g}_{1,1}^{*} & -\hat{f}_{1,1}^{*} & \hat{g}_{-1,1}^{*} & -\hat{f}_{-1,1}^{*} & \hat{g}_{0,2}^{*} & -\hat{f}_{0,2}^{*} & \hat{g}_{0,0}^{*} & 1-\hat{f}_{0,0}^{*}
\end{array}\right)=0 .
$$

This equation applies for the coupling $G_{x}=G_{y}=G_{1}^{(1)}$ and will be used in later sections. For 2-cluster chimera states we have in addition $\hat{f}_{k l}=\hat{g}_{k l}=\hat{f}_{k l}^{*}=\hat{g}_{k l}^{*}=0$ whenever $l \neq 0$. Thus Eq. (5.22) reduces to

$$
\operatorname{det}\left(\begin{array}{cccc}
1-\hat{f}_{0,0} & \hat{g}_{0,0} & -\hat{f}_{-2,0} & \hat{g}_{-2,0}  \tag{5.23}\\
\hat{g}_{0,0}^{*} & 1-\hat{f}_{0,0}^{*} & \hat{g}_{-2,0}^{*} & -\hat{f}_{-2,0}^{*} \\
-\hat{f}_{2,0} & \hat{g}_{2,0} & 1-\hat{f}_{0,0} & \hat{g}_{0,0} \\
\hat{g}_{2,0}^{*} & -\hat{f}_{2,0}^{*} & \hat{g}_{0,0}^{*} & 1-\hat{f}_{0,0}^{*}
\end{array}\right)=0
$$

and

$$
\operatorname{det}\left(\begin{array}{cc}
1-\hat{f}_{0,0} & \hat{g}_{0,0}  \tag{5.24}\\
\hat{g}_{0,0}^{*} & 1-\hat{f}_{0,0}^{*}
\end{array}\right)=0
$$

One branch of the point spectrum comes from Eq. (5.23). The resulting point eigenvalue is computed using continuation based on Newton's method and shown in Fig. 5.2(b) (green solid line). Notice that

$$
\begin{equation*}
\hat{f}_{0,0}=\left\langle\frac{1}{4} \frac{\exp (-i \alpha)}{\lambda_{p}-i \sqrt{\Omega^{2}-R_{0}^{2} \cos ^{2} x}}\right\rangle=\int_{-\pi}^{\pi} \frac{1}{4} \frac{\exp (-i \alpha)}{\lambda_{p} /(2 \pi)-i \sqrt{\Omega^{\prime 2}-R_{0}^{\prime 2} \cos ^{2} x}} d x \tag{5.25}
\end{equation*}
$$

From this and similar relations we conclude that if $\lambda_{p}^{\prime}$ is an eigenvalue in 1 D , then $\lambda_{p}=2 \pi \lambda_{p}^{\prime}$ is an eigenvalue in 2D. Another branch of the point spectrum comes from Eq. (5.24) and is also shown in Fig. 5.2(b) (red dashed line). From the figure, we see that this point eigenvalue is positive for all $0 \leq \beta \leq \frac{\pi}{2}$, thereby explaining why we were unable to find $2 n$-cluster chimera states starting from random initial conditions.

## Twisted chimera states



Figure 5.3: (Color online) Snapshots of the phase patterns for twisted chimera states. (a) The phase distribution $\theta(x, y, t)$ for $G_{x}=G_{y}=G_{1}^{(1)}$. (b) The phase distribution $\theta(x, y, t)$ for $G_{x}=G_{y}=G_{2}^{(1)}$. The simulations are done with $\beta=0.05, N=256$ and random initial condition. Colors indicate the phase of the oscillators.

As mentioned above, twisted chimera states are obtained in numerical simulations with the coupling $G_{x}=\cos (n x), G_{y}=\cos (n y)$. Figure 5.3 provides examples for $n=1,2$ and $\beta=0.05$. In these states the coherent clusters form closed stripes on a torus. In addition, the phase varies uniformly along the stripes. We still use the term twist number to indicate the number of times the stripe wraps in either the $x$ or the $y$ direction. Numerical simulations suggest that this type of chimera state is one of the two most frequently observed states (the other being a splay state) when starting from random initial conditions with $\beta<\beta_{c} \approx 0.12$.


Figure 5.4: (Color online) Snapshot of the phase pattern for the 1:1 twisted chimera state. (a) The phase distribution $\theta(x, y)$. (b) The corresponding order parameter $R(x, y)$. (c) The corresponding order parameter $\Theta(x, y)$. The simulation is done with $G_{x}=\cos (x)$, $G_{y}=\cos (y), \beta=0.05, N=256$ from a random initial condition. In panels (a) and (c), the color indicates the phase of the oscillators; in (b), the color indicates the amplitude of the local order parameter $R(x, y)$.

To understand the properties of the twisted chimera states we invoke the self-consistency analysis. As in one-dimensional systems, $\tilde{Z}$ is interpreted as the local order parameter:

$$
\begin{equation*}
\tilde{Z}(x, y) \equiv R(x, y) \exp [i \Theta(x, y)]=\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G\left(x, y, x^{\prime}, y^{\prime}\right) \exp \left[i \theta\left(x^{\prime}, y^{\prime}\right)\right] d x^{\prime} d y^{\prime} \tag{5.26}
\end{equation*}
$$

Equation (5.9) can therefore be written in the form

$$
\begin{equation*}
R(x, y) \exp [i \Theta(x, y)]=\exp (i \beta) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G\left(x, y, x^{\prime}, y^{\prime}\right) \exp \left[i \Theta\left(x^{\prime}, y^{\prime}\right)\right] h\left(x^{\prime}, y^{\prime}\right) d x^{\prime} d y^{\prime} \tag{5.27}
\end{equation*}
$$

where

$$
\begin{equation*}
h\left(x^{\prime}, y^{\prime}\right) \equiv \frac{\Omega-\sqrt{\Omega^{2}-R^{2}\left(x^{\prime}, y^{\prime}\right)}}{R\left(x^{\prime}, y^{\prime}\right)} \tag{5.28}
\end{equation*}
$$

Figures $5.4(\mathrm{~b})$ and $5.4(\mathrm{c})$ show the pattern of $R(x, y)$ and $\Theta(x, y)$ for the $1: 1$ twisted chimera state obtained from the numerical simulation reported in Fig. 5.4(a). To understand the properties of these empirical order parameter fields we need to solve Eq. (5.27) for $R(x, y)$ and $\Theta(x, y)$. This formidable task can be simplified using appropriate symmetry relations valid for our choice of $G_{x}$ and $G_{y}$. We use the case $n=1$ to illustrate the process.

First, we notice that Eq. (5.27) can be written

$$
\begin{equation*}
R(x, y) \exp (i \Theta(x, y))=a \cos (x)+b \sin (x)+c \cos (y)+d \sin (y) \tag{5.29}
\end{equation*}
$$

where $a, b, c$, and $d$ are complex numbers given by

$$
\begin{equation*}
a=\left\langle\cos \left(x^{\prime}\right) \exp (i \beta) \exp \left(i \Theta\left(x^{\prime}, y^{\prime}\right)\right) h\left(x^{\prime}, y^{\prime}\right)\right\rangle \tag{5.30}
\end{equation*}
$$

$$
\begin{align*}
b & =\left\langle\sin \left(x^{\prime}\right) \exp (i \beta) \exp \left(i \Theta\left(x^{\prime}, y^{\prime}\right)\right) h\left(x^{\prime}, y^{\prime}\right)\right\rangle  \tag{5.31}\\
c & =\left\langle\cos \left(y^{\prime}\right) \exp (i \beta) \exp \left(i \Theta\left(x^{\prime}, y^{\prime}\right)\right) h\left(x^{\prime}, y^{\prime}\right)\right\rangle  \tag{5.32}\\
d & =\left\langle\sin \left(y^{\prime}\right) \exp (i \beta) \exp \left(i \Theta\left(x^{\prime}, y^{\prime}\right)\right) h\left(x^{\prime}, y^{\prime}\right)\right\rangle \tag{5.33}
\end{align*}
$$

Since Eq. (5.27) is invariant under translation in $x$ and $y$, we can choose the origin such that $R(x, y)$ is invariant under the transformation $(x, y) \rightarrow(y, x)$, and $\Theta(x, y)$ has a $\pi$ phase jump across the line $y=x$. This property implies

$$
\begin{equation*}
R(y, x) \exp (i \Theta(y, x))=R(x, y) \exp (i \Theta(x, y)+i \pi) \tag{5.34}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
a \cos (y)+b \sin (y)+c \cos (x)+d \sin (x)=-(a \cos (x)+b \sin (x)+c \cos (y)+d \sin (y)) . \tag{5.35}
\end{equation*}
$$

Matching the coefficients of the Fourier components gives $a=-c$ and $b=-d$.
In addition, we find from Fig. 5.4(c) that $\Theta(x, y)$ decreases linearly along the $(1,1)$ direction. This behavior is similar to the splay states with a twist number $q$. In Fig. 5.4(c), $q=-1$. From Fig. 5.4(b), we see that $R$ is constant along the $(1,1)$ direction. Using this property, we obtain

$$
\begin{equation*}
R(x+\delta, y+\delta) \exp (i \Theta(x+\delta, y+\delta))=R(x, y) \exp (i \Theta(x, y)+i \delta) \tag{5.36}
\end{equation*}
$$

where $\delta$ represents an arbitrary displacement in the $(1,1)$ direction. From the limit $\delta \rightarrow 0$ we obtain

$$
\begin{equation*}
-a \sin (x)+b \cos (x)-c \sin (y)+d \cos (y)=i(a \cos (x)+b \sin (x)+c \cos (y)+d \sin (y)), \tag{5.37}
\end{equation*}
$$

implying that $b=i a$ and $d=i c$.
The fact that Eq. (5.27) is also invariant under translations in $\theta$ allows us finally to choose $a$ to be real and positive. Writing $a=R_{0} / 2$, and substituting the expression $R \exp (i \Theta)=$ $a(\exp (i x)-\exp (i y))$ into Eqs. (5.30)-(5.33) leads to a single self-consistency requirement:

$$
\begin{equation*}
R_{0}^{2}=\exp (i \beta)\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \sin ^{2}\left(\frac{x-y}{2}\right)}\right\rangle \tag{5.38}
\end{equation*}
$$

To solve this equation, we find it convenient to use the coordinates $u \equiv y+x, v \equiv y-x$. Since the right side of Eq. (5.38) is $2 \pi$-periodic in both $x$ and $y$ directions, the integration domain can be changed from a square in ( $x, y$ ) coordinates (black solid lines in Fig. 5.5(a)) to a rectangle in $(u, v)$ coordinates (green dashed lines in Fig. 5.5(b)). In fact, for a general $2 \pi$-periodic function $F(x, y)$, we have

$$
\begin{equation*}
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(x, y) d x d y=\frac{1}{2} \int_{-2 \pi}^{2 \pi}\left(\int_{0}^{2 \pi} F\left(\frac{u-v}{2}, \frac{u+v}{2}\right) d v\right) d u \tag{5.39}
\end{equation*}
$$



Figure 5.5: (Color online) (a) Integration domain before changes of coordinates. (b) Integration domain after change of coordinates.

One additional change of variables now shows that Eq. (5.38) is identical to Eq. (5.16). Thus the dependence of $\Omega$ and $R_{0}$ on $\beta$ is identical to that shown in Fig. 5.2(a). When $\beta=0$, $\Omega=R_{0} \approx 14.346$, which is consistent with the value $\Omega=R_{0}=4 \pi^{2}-8 \pi$ obtained analytically from Eq. (5.38). As in the 1D case, the boundary between coherent and incoherent domains is determined by the relation $\Omega^{2}=R_{0}^{2} \sin ^{2}\left(\frac{x-y}{2}\right)$, implying that the coherent fraction (by area) is given by $r=1-2 \pi^{-1} \sin ^{-1}\left(\Omega / R_{0}\right)$. The dependence of $r$ on $\beta$ is shown in Fig. 5.6(a).


Figure 5.6: (Color online) Dependence of (a) the fraction $r$ of coherent oscillators and (b) the real part of the two point eigenvalues $\lambda_{p}$ on the parameter $\beta$.

The stability of the twisted chimera states can be analyzed in the same way as done for the $2 n$-cluster chimera states. When $n=1, \lambda_{p}$ is still determined by Eq. (5.22); in the new

## CHAPTER 5. TWISTED CHIMERA STATES AND MULTI-CORE SPIRAL CHIMERA

 STATES ON A TWO-DIMENSION TORUScoordinates $u$ and $v$, we have

$$
\begin{align*}
\mu & =\sqrt{\Omega^{2}-R_{0}^{2} \sin ^{2}(v / 2)}  \tag{5.40}\\
\tilde{Z} & =-i R_{0} \sin (v / 2) \exp (i u / 2)  \tag{5.41}\\
\tilde{z} & =-\frac{i \exp (i \beta) R_{0} \sin (v / 2) \exp (i u / 2)}{\Omega+\sqrt{\Omega^{2}-R_{0}^{2} \sin ^{2}(v / 2)}} \tag{5.42}
\end{align*}
$$

where

$$
\begin{array}{ll}
f=\frac{1}{4} \frac{\exp (-i \alpha)}{\lambda_{p}-i \sqrt{\Omega^{2}-R_{0}^{2} \sin ^{2}(v / 2)}}, & f^{*}=\frac{1}{4} \frac{\exp (i \alpha)}{\lambda_{p}+i{\sqrt{\Omega^{2}-R_{0}^{2} \sin ^{2}(v / 2)}}^{*}} \\
g=\frac{R_{0}^{2} \sin ^{2}(v / 2) \exp (i u) f}{\left(\Omega+\sqrt{\Omega^{2}-R_{0}^{2} \sin ^{2}(v / 2)}\right)^{2}}, & g^{*}=\frac{R_{0}^{2} \sin ^{2}(v / 2) \exp (-i u) f^{*}}{\left(\Omega+{\sqrt{\Omega^{2}-R_{0}^{2} \sin ^{2}(v / 2)}}^{*}\right)^{2}} . \tag{5.44}
\end{array}
$$

Here, we emphasize that the $*$ represents complex conjugate in the right hand side of the expressions, while $f^{*}$ and $g^{*}$ are not necessarily the complex conjugate of $f$ and $g$. Since many elements in Eq. (5.22) vanish the equation for the point eigenvalue $\lambda_{p}$ reduces to the three conditions

$$
\operatorname{det}\left(\begin{array}{cccc}
1-\hat{f}_{0,0} & -\hat{f}_{-1,1} & \hat{g}_{-2,0} & \hat{g}_{-1,-1}  \tag{5.45}\\
-\hat{f}_{1,-1} & 1-\hat{f}_{0,0} & \hat{g}_{-1,-1} & \hat{g}_{0,-2} \\
\hat{g}_{2,0}^{*} & \hat{g}_{1,1}^{*} & 1-\hat{f}_{0,0}^{*} & -\hat{f}_{1,-1}^{*} \\
\hat{g}_{1,1}^{*} & \hat{g}_{0,2}^{*} & -\hat{f}_{-1,1}^{*} & 1-\hat{f}_{0,0}^{*}
\end{array}\right)=0
$$

and

$$
\operatorname{det}\left(\begin{array}{cc}
1-\hat{f}_{0,0} & -\hat{f}_{-1,1}  \tag{5.46}\\
-\hat{f}_{1,-1} & 1-\hat{f}_{0,0}
\end{array}\right)=0, \quad \operatorname{det}\left(\begin{array}{cc}
1-\hat{f}_{0,0}^{*} & -\hat{f}_{-1,1}^{*} \\
-\hat{f}_{1,-1}^{*} & 1-\hat{f}_{0,0}^{*}
\end{array}\right)=0 .
$$

It turns out that Eq. (5.45) gives the same result as Eq. (5.23). This is not so surprising if we compare the elements of the two matrices in Eq. (5.23) and Eq. (5.45). These two matrices are almost identical except that part of the matrix elements in Eq. (5.45) have opposite sign as compared with the corresponding terms in Eq. (5.23). However, this does not affect the final determinant of the matrix. Equation (5.46) gives two additional branches of eigenvalues, which are complex conjugates of one another. Figure 5.6(b) shows the dependence of the real part of the two point eigenvalues $\lambda_{p}$ that result on the parameter $\beta$. As $\beta$ increases from 0 , the red (dashed) eigenvalue $\lambda_{p}$ becomes unstable first, at $\beta \approx 0.122$. This onset of instability is consistent with direct numerical simulation.

## Spiral wave chimeras

For larger values of $\beta$, stable configurations of multi-core spiral wave chimera states are found. In this state a core of incoherent oscillators is surrounded by spiral arms consisting of phase-locked oscillators. The arms rotate rigidly with a constant angular velocity. To pick out this rotation we define the time-averaged oscillation frequency $\bar{\theta}_{t}(x, y) \equiv$


Figure 5.7: (Color online) Snapshots of the phase patterns for a spiral wave chimera states. (a) The phase distribution $\theta(x, y)$ for $G_{x}=G_{y}=G_{1}^{(1)}$. (b) The phase distribution $\theta(x, y)$ for $G_{x}=G_{y}=G_{2}^{(1)}$. The upper left spirals rotate clockwise in both (a) and (b), with the direction of rotation alternating from core to core in both $x$ and $y$ directions. The phase patterns have the symmetry $D_{2}$ and not $D_{4}$. Simulations are done with $\beta=1, N=256$ and random initial conditions. Colors indicate the phase of the oscillators.
$\lim _{T \rightarrow \infty}[\theta(x, y, T)-\theta(x, y, 0)] / T$. From the self-consistency analysis, we find that $\bar{\theta}_{t}(x, y)=$ $-\Omega$ when $\Omega<R(x, y)$ (the coherent region outside the core) and $\bar{\theta}_{t}(x, y)=-\Omega+\sqrt{\Omega^{2}-R(x, y)^{2}}$ when $\Omega>R(x, y)$ (in the incoherent core). The former frequency corresponds to the angular velocity of the spiral arms, i.e., to the rotation rate of the spatial pattern; the latter frequency depends on the distance from the core (it vanishes at the core center where $R=0$ ) but because the core is incoherent the frequency cannot be identified with spatial rotation. Figure 5.7 shows the phase distribution in multi-core spiral chimeras obtained with the coupling functions $G_{1}^{(1)}$ and $G_{2}^{(1)}$. These figures reveal the presence of 4 and 16 incoherent cores, respectively, distributed evenly across both $x$ and $y$ directions. In addition, Figure 5.8 shows three examples of spiral wave chimeras with $G_{1}^{(1)}$ and different values of $\beta$. These results indicate that spiral wave chimeras can be present for $\alpha$ near $\pi / 2$, in contrast to earlier work in which such states were found only for $\alpha$ sufficiently close to zero [67]; see [68] for a more recent numerical study of a related system.

To understand the origin of spiral wave chimeras, we examine the properties of these states in the case $n=1$ using the self-consistency analysis. It turns out that spiral wave chimera states are also solutions of the self-consistency equations (5.30)-(5.33). To simplify the analysis, we translate one of the cores to the origin $(x, y)=(0,0)$ (cf. Fig. 5.9). Figure 5.9(c) shows that $\Theta(x, y)$ equals a constant, $\Theta(x, y)=\theta_{0}$, say, along the line segment between the point $(0,0)$ and the point $(\pi, 0)$. Invariance of Eq. (5.27) under translation in $\theta$ allow us to set $\theta_{0}=0$. With this choice, the local order parameters have the following symmetry:

$$
\begin{equation*}
(x, y) \rightarrow(y, x) \Longrightarrow(R, \Theta) \rightarrow\left(R, \frac{\pi}{2}-\Theta\right) \tag{5.47}
\end{equation*}
$$



Figure 5.8: (Color online) Snapshots of the phase patterns for spiral wave chimeras. (a) $\beta=1.5$. (b) $\beta=1$. (c) $\beta=0.5$. In all three panels, colors indicates the phase of the oscillators. The simulations are done with $G_{x}=G_{y}=G_{1}^{(1)}, N=256$ and random initial conditions.


Figure 5.9: (Color online)Snapshot of the phase pattern for a four-core spiral wave chimera state. (a) The phase distribution $\theta(x, y)$. (b) The corresponding order parameter $R(x, y)$. (c) The corresponding order parameter $\Theta(x, y)$. The simulation is done with $G_{x}=G_{y}=G_{1}^{(1)}$, $\beta=1, N=256$ and random initial conditions. In panels (a) and (c) colors indicate the phase of the oscillators; in (b) color indicates the amplitude of local order parameter $R(x, y)$.

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 STATES ON A TWO-DIMENSION TORUSThis symmetry property implies that

$$
\begin{equation*}
a \cos (y)+b \sin (y)+c \cos (x)+d \sin (x)=i\left(a^{*} \cos (x)+b^{*} \sin (x)+c^{*} \cos (y)+d^{*} \sin (y)\right) \tag{5.48}
\end{equation*}
$$

Matching coefficients leads to $a=i c^{*}$ and $d=i b^{*}$. In addition, Figs. 5.9(b,c) suggest the spiral wave chimera has a symmetry under rotation, viz.,

$$
\begin{equation*}
(x, y) \rightarrow(-y, x) \Longrightarrow(R, \Theta) \rightarrow\left(R, \Theta+\frac{\pi}{2}\right) \tag{5.49}
\end{equation*}
$$

corresponding to

$$
\begin{equation*}
a \cos (-y)+b \sin (-y)+c \cos (x)+d \sin (x)=i(a \cos (x)+b \sin (x)+c \cos (y)+d \sin (y)) . \tag{5.50}
\end{equation*}
$$

Matching coefficients leads to $a=i c$ and $d=i b$. Finally, reflection with respect to the $x$ axis,

$$
\begin{equation*}
(x, y) \rightarrow(x,-y) \Longrightarrow(R, \Theta) \rightarrow(R,-\Theta) \tag{5.51}
\end{equation*}
$$

requires that

$$
\begin{equation*}
a \cos (x)+b \sin (x)+c \cos (-y)+d \sin (-y)=a^{*} \cos (x)+b^{*} \sin (x)+c^{*} \cos (y)+d^{*} \sin (y) \tag{5.52}
\end{equation*}
$$

and hence implies $a=a^{*}, b=b^{*}, c=c^{*}$ and $d=-d^{*}$. Combining these relations we obtain

$$
\begin{equation*}
R \exp (i \Theta)=b(\sin (x)+i \sin (y)) \tag{5.53}
\end{equation*}
$$

Substituting this expression into Eqs. (5.30)-(5.33) we find that the expressions in Eqs. (5.30) and (5.32) vanish, while Eqs. (5.31) and (5.33) give

$$
\begin{equation*}
2 b^{2}=\exp (i \beta)\left\langle\Omega-\sqrt{\Omega^{2}-b^{2}\left(\sin ^{2} x+\sin ^{2} y\right)}\right\rangle . \tag{5.54}
\end{equation*}
$$

The right side of Eq. (5.54) remains unchanged when $\sin ^{2} x$ and $\sin ^{2} y$ are replaced by $\sin ^{2}(n x)$ and $\sin ^{2}(m y)$ for all $n, m \in \frac{1}{2} \mathbb{N}$. Solving Eq. (5.54) will give us the values of $\Omega$ and $b$. Notice that the azimuthal wave number $m$ of the spirals equals 1 . Therefore, $|\Omega|$ is also the angular frequency of the rotating spirals.

When $\beta=\frac{\pi}{2}$, we obtain $\Omega=0$ and $b=\left\langle\sqrt{\sin ^{2} x+\sin ^{2} y}\right\rangle / 2 \approx$ 18.91. Figure 5.10(a) shows the dependence of $b$ and $\Omega$ on $\beta$. As $\beta$ decreases from $\frac{\pi}{2}, b$ decreases, $\Omega$ increases and the area of incoherent cores increases. When $\beta=\beta_{c} \approx 0.38, b=\Omega$ and the incoherent cores touch each other (Fig. 5.11(b)). When $\beta$ decreases further, the incoherent domains reconnect and the coherent domains separate into four isolated islands (Fig. 5.11(c)).

The stability of spiral wave chimera states can be analyzed in the same way as for the $2 n$ cluster chimeras and twisted chimeras. When $n=1$, the point eigenvalue $\lambda_{p}$ is determined by Eq. (5.22). In the current case, $f$ and $f^{*}$ are even with respect to the $x$ and $y$ axes, while


Figure 5.10: (Color online) Dependence on the parameter $\beta$ of (a) $b, \Omega$, (b) the fraction $r$ of incoherent oscillators, and (c) unstable point eigenvalues $\lambda_{p}$, computed from Eq. (5.66) (green solid line) and Eq. (5.67) (red dashed line).


Figure 5.11: (Color online) Snapshots of the phase patterns for spiral wave chimeras showing localized regions of coherence embedded in an incoherent background. (a) $\beta=0.4$. (b) $\beta=0.38$. (c) $\beta=0.36$. In all three panels, colors indicate the phase of the oscillators. The simulation is done with $G_{x}=G_{y}=G_{1}^{(1)}, N=256$ when $\beta$ is gradually decreased.
$g$ and $g^{*}$ are not. We separate $g$ and $g^{*}$ into even and odd parts, $g=g_{e}+g_{o}$ and $g^{*}=g_{e}^{*}+g_{o}^{*}$, where

$$
\begin{array}{ll}
g_{e}=\frac{b^{2} f\left(\sin ^{2} y-\sin ^{2} x\right)}{\left(\Omega+\sqrt{\Omega^{2}-b^{2}\left(\sin ^{2} x+\sin ^{2} y\right)}\right)^{2}}, \quad g_{o}=\frac{-2 i b^{2} f \sin x \sin y}{\left(\Omega+\sqrt{\Omega^{2}-b^{2}\left(\sin ^{2} x+\sin ^{2} y\right)}\right)^{2}}(5.55) \\
\left.g_{e}^{*}=\frac{b^{2} f^{*}\left(\sin ^{2} y-\sin ^{2} x\right)}{\left(\Omega+\sqrt{\Omega^{2}-b^{2}\left(\sin ^{2} x+\sin ^{2} y\right)^{*}}\right)^{2}}, \quad g_{o}^{*}=\frac{2 i b^{2} f^{*} \sin x \sin y}{\left(\Omega+{\sqrt{\Omega^{2}-b^{2}\left(\sin ^{2} x+\sin ^{2} y\right)}}^{*}\right)^{2}} 5.56\right)
\end{array}
$$

We then have

$$
\begin{align*}
& \hat{f}_{1, \pm 1}=\hat{f}_{-1, \pm 1}=\hat{f}_{1, \pm 1}^{*}=\hat{f}_{-1, \pm 1}^{*}=\hat{g}_{0,0}=\hat{g}_{0,0}^{*}=0,  \tag{5.57}\\
& \hat{f}_{0} \equiv \hat{f}_{0,0}=\langle f\rangle \tag{5.58}
\end{align*}
$$

$$
\begin{align*}
& \hat{f}_{0}^{*} \equiv \hat{f}_{0,0}^{*}=\left\langle f^{*}\right\rangle  \tag{5.59}\\
& \hat{f}_{2} \equiv \hat{f}_{0, \pm 2}=\hat{f}_{ \pm 2,0}=\langle f \cos (2 x)\rangle  \tag{5.60}\\
& \hat{f}_{2}^{*} \equiv \hat{f}_{0, \pm 2}^{*}=\hat{f}_{ \pm 2,0}^{*}=\left\langle f^{*} \cos (2 x)\right\rangle  \tag{5.61}\\
& \hat{g}_{2} \equiv \hat{g}_{0, \pm 2}=\hat{g}_{ \pm 2,0}=\left\langle g_{e} \cos (2 x)\right\rangle  \tag{5.62}\\
& \hat{g}_{1} \equiv \hat{g}_{1,1}=\hat{g}_{-1,-1}=-\hat{g}_{1,-1}=-\hat{g}_{-1,1}=\left\langle g_{o}(-\sin x \sin y)\right\rangle  \tag{5.63}\\
& \hat{g}_{2}^{*} \equiv \hat{g}_{0, \pm 2}^{*}=\hat{g}_{ \pm 2,0}^{*}=\left\langle g_{e}^{*} \cos (2 x)\right\rangle  \tag{5.64}\\
& \hat{g}_{1}^{*} \equiv \hat{g}_{1,1}^{*}=\hat{g}_{-1,-1}^{*}=-\hat{g}_{1,-1}^{*}=-\hat{g}_{-1,1}^{*}=\left\langle g_{o}^{*}(-\sin x \sin y)\right\rangle \tag{5.65}
\end{align*}
$$

the determinant in Eq. (5.22) therefore factors into three factors:

$$
\begin{align*}
& \operatorname{det}\left(\begin{array}{cc}
1-\hat{f}_{0}-\hat{f}_{2} & \hat{g}_{2} \\
\hat{g}_{2}^{*} & 1-\hat{f}_{0}^{*}-\hat{f}_{2}^{*}
\end{array}\right)=0  \tag{5.66}\\
& \operatorname{det}\left(\begin{array}{cc}
1-\hat{f}_{0}+\hat{f}_{2} & -2 \hat{g}_{1}-\hat{g}_{2} \\
-2 \hat{g}_{1}^{*}-\hat{g}_{2}^{*} & 1-\hat{f}_{0}^{*}+\hat{f}_{2}^{*}
\end{array}\right)=0  \tag{5.67}\\
& \operatorname{det}\left(\begin{array}{cc}
1-\hat{f}_{0}+\hat{f}_{2} & 2 \hat{g}_{1}-\hat{g}_{2} \\
2 \hat{g}_{1}^{*}-\hat{g}_{2}^{*} & 1-\hat{f}_{0}^{*}+\hat{f}_{2}^{*}
\end{array}\right)=0 \tag{5.68}
\end{align*}
$$

Solving these three equations determines the number of point eigenvalues and their dependence on $\beta$, just as the case of $2 n$-cluster chimeras and the twisted chimeras. We plot the unstable eigenvalues in Fig. 5.10(c). The green solid line is computed from Eq. (5.66) while the red dashed line is computed from Eq. (5.67); Eq. (5.68) does not provide unstable eigenvalues. Figure 5.10 (c) indicates an onset of instability already at $\beta \approx 0.46$ as $\beta$ decreases. However, in numerical simulations we find that the spiral state remains stable down to $\beta=0.349$. We do not understand the origin of this discrepancy but note that the unstable mode for $0.349 \lesssim \beta \lesssim 0.46$ is reflection-symmetric in both modulus and phase (Fig. 5.12), in contrast to the states in Fig. 5.11, suggesting that the unstable eigenvalue (red dashed line) is likely spurious. In contrast, the onset of the instability from Eq. (5.66) ( $\beta \approx 0.344$, green solid line) agrees quite well with the numerically determined onset of instability, $\beta \approx 0.349$.

### 5.4 The case $G_{x}=G_{y}=G_{n}^{(2)}$

In this section, we consider the coupling function with $G_{x}=\cos (n x)+\cos [(n+1) x]$ and $G_{y}=\cos (n x)+\cos [(n+1) x]$. Much richer dynamics are observed.

## Inherited solutions

Splay states with $G_{x}=\cos (n x)+\cos [(n+1) x]$ and $G_{y}=\cos (n x)+\cos [(n+1) x]$ are also observed (not shown). Of these only the splay states with twist number $|q|=n$ and $|q|=n+1$ are linearly stable.


Figure 5.12: (Color online) (a) Modulus and (b) phase of the eigenvector $v(x, y)$ when $\beta \approx 0.41$. The corresponding eigenvalue is $\lambda_{p} \approx 0.717$ as computed from Eq. (5.67).


Figure 5.13: (Color online) Snapshot of the phase pattern for (a) a 3-cluster chimera state, and (b) a 4-cluster chimera state. The simulation is done for $G_{x}=G_{y}=G_{1}^{(2)}$, with $\beta=0.05$, $N=256$, starting from a random initial condition.

Apart from splay states, many of the states observed in one-dimensional systems [60] have counterparts that are stable in two dimensions. Examples of 1D-like 3-cluster and 4-cluster chimeras are shown in Fig. 5.13, while a traveling coherent state and a traveling chimera state are shown in Fig. 5.14.

## Twisted chimera states

When $G_{x}=\cos (n x)+\cos [(n+1) x]$ and $\left.G_{y}=\cos (n y)+\cos [(n+1) y)\right]$, three types of twisted chimera states are observed, $n: n$ twisted chimeras, $n+1: n+1$ twisted chimeras and $n: n+1$ twisted chimeras. In the following, we use the case $n=1$ to illustrate these states and the associated self-consistency analysis. The 1:1 twisted chimera looks identical to Fig. 5.4, while Figs. 5.15 and 5.16 show the phase distribution and the local order parameter for a 2:2 and a 1:2 twisted chimera, respectively.


Figure 5.14: (Color online) (a) Snapshot of the phase pattern in a right-traveling coherent state. The simulation is done for $G_{x}=G_{y}=G_{1}^{(2)}$, with $\beta=0.7$ and $N=256$. (b) Snapshot of the phase pattern for a right-traveling chimera state. The simulation is done for $G_{x}=G_{y}=G_{3}^{(2)}$, with $\beta=0.03$ and $N=256$.


Figure 5.15: (Color online) (a) Snapshot of the phase distribution in a $2: 2$ twisted chimera state. (b) The corresponding order parameter $R$. (c) The corresponding order parameter $\Theta$. The simulation is done for $G_{x}=G_{y}=G_{1}^{(2)}$, with $\beta=0.05, N=256$ and random initial conditions.

## CHAPTER 5. TWISTED CHIMERA STATES AND MULTI-CORE SPIRAL CHIMERA

 STATES ON A TWO-DIMENSION TORUSWe discuss the 2:2 twisted chimera in detail. In this case, two closed stripes of coherence are distributed uniformly on the torus. The self-consistency analysis can, once again, be simplified using the observed symmetry properties of this state. First, the local order parameter is written as

$$
\begin{equation*}
\tilde{Z}(x, y)=\Sigma_{m=1}^{2}\left\{a_{m} \cos (m x)+b_{m} \sin (m x)+c_{m} \cos (m y)+d_{m} \sin (m y)\right\}, \tag{5.69}
\end{equation*}
$$

with the coefficients satisfying the self-consistency relations

$$
\begin{align*}
a_{m} & =\left\langle\cos \left(m x^{\prime}\right) \exp (i \beta) \exp \left(i \Theta\left(x^{\prime}, y^{\prime}\right)\right) h\left(x^{\prime}, y^{\prime}\right)\right\rangle,  \tag{5.70}\\
b_{m} & =\left\langle\sin \left(m x^{\prime}\right) \exp (i \beta) \exp \left(i \Theta\left(x^{\prime}, y^{\prime}\right)\right) h\left(x^{\prime}, y^{\prime}\right)\right\rangle,  \tag{5.71}\\
c_{m} & =\left\langle\cos \left(m y^{\prime}\right) \exp (i \beta) \exp \left(i \Theta\left(x^{\prime}, y^{\prime}\right)\right) h\left(x^{\prime}, y^{\prime}\right)\right\rangle,  \tag{5.72}\\
d_{m} & =\left\langle\sin \left(m y^{\prime}\right) \exp (i \beta) \exp \left(i \Theta\left(x^{\prime}, y^{\prime}\right)\right) h\left(x^{\prime}, y^{\prime}\right)\right\rangle, \tag{5.73}
\end{align*}
$$

and $h$ defined as in Eq. (5.28). From Figs. 5.15(b) and (c), we observe that the local order parameters have the symmetry

$$
\begin{equation*}
(x, y) \rightarrow(x+\delta, y+\delta) \Longrightarrow(R, \Theta) \rightarrow(R, \Theta+2 \delta) \tag{5.74}
\end{equation*}
$$

Thus

$$
\begin{equation*}
R(x+\delta, y+\delta) \exp (i \Theta(x+\delta, y+\delta))=R(x, y) \exp (i \Theta(x, y)+2 i \delta) \tag{5.75}
\end{equation*}
$$

which implies $a_{1}=b_{1}=c_{1}=d_{1}=0, b_{2}=i a_{2}$ and $d_{2}=i c_{2}$. The symmetry

$$
\begin{equation*}
(x, y) \rightarrow(y, x) \Longrightarrow(R, \Theta) \rightarrow(R, \Theta+\pi) \tag{5.76}
\end{equation*}
$$

leads to the relations $a_{2}=-c_{2}$ and $d_{2}=-b_{2}$. Combining these results we obtain $\tilde{Z}(x, y)=$ $a_{2}(\exp (2 i x)-\exp (2 i y))$. Translation symmetry in $\theta$ allows us to set $a_{2}=\frac{R_{0}}{2}$, where $R_{0}$ is a positive real number. The final self-consistency equation is thus

$$
\begin{equation*}
R_{0}^{2}=\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \sin ^{2}(x-y)}\right\rangle \tag{5.77}
\end{equation*}
$$

which is also equivalent to Eq. (5.16) after a change of variables.
The 1:2 twisted chimera shown in Fig. 5.16 can be analyzed similarly. In this case, the local order parameter has the symmetry

$$
\begin{equation*}
(x, y) \rightarrow(x+2 \delta, y+\delta) \Longrightarrow(R, \Theta) \rightarrow(R, \Theta+2 \delta) \tag{5.78}
\end{equation*}
$$

implying that

$$
\begin{equation*}
R(x+2 \delta, y+\delta) \exp (i \Theta(x+2 \delta, y+\delta))=R(x, y) \exp (i \Theta(x, y)+2 i \delta) \tag{5.79}
\end{equation*}
$$

This condition is equivalent to $c_{1}=d_{1}=a_{2}=b_{2}=0, b_{1}=i a_{1}$ and $d_{2}=i c_{2}$. The symmetry

$$
\begin{equation*}
(x, y) \rightarrow(2 y, x / 2) \Longrightarrow(R, \Theta) \rightarrow(R, \Theta+\pi) \tag{5.80}
\end{equation*}
$$



Figure 5.16: (Color online) (a) Snapshot of the phase distribution in a 1:2 chimera state. (b) The corresponding order parameter $R$. (c) The corresponding order parameter $\Theta$. The simulation is done for $G_{x}=G_{y}=G_{1}^{(2)}$, with $\beta=0.05, N=256$.
likewise yields $a_{1}=-c_{2}$ and $b_{1}=-d_{2}$ so that $\tilde{Z}(x, y)=a_{1}(\exp (i x)-\exp (2 i y))$. With $a_{1}=R_{0} / 2$, we obtain the self-consistency equation

$$
\begin{equation*}
R_{0}^{2}=\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \sin ^{2}(x / 2-y)}\right\rangle \tag{5.81}
\end{equation*}
$$

This condition is again equivalent to Eq. (5.16) after an appropriate change of variables. The 1:1 chimera states have local order parameter $\tilde{Z}=\frac{R_{0}}{2}(\exp (i x)-\exp (i y))$ and the selfconsistency equation is the same as Eq. (5.38). From the derivation of Eq. (5.38) (for a $1: 1$ twisted chimera), Eq. (5.77) (for a $2: 2$ twisted chimera) and Eq. (5.81) (for a $1: 2$ twisted chimera) for the case $G_{x}=G_{y}=G_{1}^{(2)}$, we see that even if multiple Fourier components are included in the coupling function, some of them may not contribute to the final results owing to the symmetries of the solution. However, the stability properties may be affected. For the three types of twisted chimera states discussed above, the stability is determined by the point eigenvalues satisfying $\operatorname{det} \mathbf{B}\left(\lambda_{p}\right)=0$, where $\mathbf{B}\left(\lambda_{p}\right)$ is defined in Eq. (5.21). In the present case, the corresponding matrices $\mathbf{B}$ are $16 \times 16$ matrices. It turns out that these share the same point eigenvalues as the 1:1 chimera state for $G_{x}=G_{y}=G_{1}^{(1)}$, which are determined by Eqs. (5.45) and (5.46). Additional point eigenvalues may be present. However, these additional eigenvalues do not affect the stability and hence the theory predicts that all three types of twisted chimera lose stability at $\beta \approx 0.122$. This value agrees reasonably well with the instability threshold $\beta=0.13 \pm 0.01$ obtained from direct numerical simulations.

## Spiral wave chimeras

Multi-core spiral wave states are also observed for $G_{x}=\cos (n x)+\cos [(n+1) x]$ and $G_{y}=\cos (n y)+\cos [(n+1) y]$ and some examples are given in Fig. 5.17. The states shown in Figs. 5.17 (b,c) are similar to the spiral wave chimera states for $G_{x}=\cos (n x)$ and $G_{y}=$ $\cos (n y)$; in particular the spiral arms rotate with constant angular velocity given by the


Figure 5.17: (Color online)Snapshots of the phase pattern in spiral wave chimeras. (a) 4 intermittently incoherent cores; (b) 8 incoherent cores; (c) 16 incoherent cores. The simulation is done for $G_{x}=G_{y}=G_{1}^{(2)}$, with $\beta=1.2, N=256$ and random initial condition.
time-averaged oscillator frequency $\bar{\theta}_{t}(x, y)$ introduced in section 5.3. The order parameters of these states can also be reduced to simpler form by appropriate symmetry arguments and it turns out that the order parameters for the solutions in Figs. 5.17(b,c) take the form $\tilde{Z}(x, y)=b_{2}(\sin (k x)+i \sin (l y))$ with $k=2$ and $l=1,2$. The final self-consistency equation is

$$
\begin{equation*}
2 b_{2}^{2}=\exp (i \beta)\left\langle\Omega-\sqrt{\Omega^{2}-b_{2}^{2}\left(\sin ^{2}(k x)+\sin ^{2}(l y)\right)}\right\rangle \tag{5.82}
\end{equation*}
$$

and is equivalent to Eq. (5.54).
The state shown in Fig. 5.17(a) is particularly interesting. Unlike the other two spiral wave states shown in Figs. $5.17(\mathrm{~b}, \mathrm{c})$ the cores of the state in Fig. 5.17(a) are incoherent only intermittently; these partially or intermittently incoherent regions are not stationary but rotate in the same sense as the coherent state to the outside but rather more slowly. We illustrate this behavior using an example for $\beta=1.4$. For this value of $\beta$, a snapshot of the phase pattern at a particular instant is shown in panel (a) of Fig. 5.18. We should emphasize that no persistently incoherent cores have as yet developed. Analysis of snapshots of the phase distribution along a slice through one of the cores reveals the presence of a group of oscillators in the core region that alternately detrain and then entrain as the incoherent oscillators precess around the core (Fig. 5.19). The figure suggests that the detraining/entraining process is approximately periodic (i.e., that the precession takes place at a constant angular speed), much as is the case for a similar process observed in onedimensional oscillator arrays in Chapter 3. The resulting repeated detraining and entraining at a fixed location implies that the phase distribution in the core is not time-independent and hence that a self-consistency analysis based on the assumption that the order parameters are time-independent must necessarily fail as soon as the number of oscillators undergoing this process becomes substantial.

We performed a numerical experiment, starting from a 4 -core spiral wave at $\beta=1.57$ and decreasing $\beta$ gradually. Figure 5.18 shows snapshots of the states for different values of


Figure 5.18: (Color online) Snapshots of chimera states for $G_{x}=G_{y}=G_{1}^{(2)}, N=256$, obtained by gradually decreasing $\beta$. (a) $\beta=1.4$. (b) $\beta=1.2$. (c) $\beta=1.118$. (d) $\beta=1.117$. (e) $\beta=1$.1. (f) $\beta=0.8$. (g) $\beta=0.4$. (h) $\beta=0.06$. (i) $\beta=0.025$.
$\beta$. When $\beta \gtrsim 1.118$, the partially incoherent cores exhibit crescent moon phase structure with $m=1$ azimuthal wave number. As $\beta$ decreases the mode number $m$ increases from $m=1$ to $m=2$ and once the fully incoherent cores are born at $\beta \approx 1.117$ the patterns have three arms (Fig. 5.18(d)). As $\beta$ is further decreased, the fully incoherent cores grow in extent and swallow the arms one by one (Fig. 5.18(f)). In contrast, the azimuthal wave number for the coherent domain remains $m=1$ as $\beta$ decreases and the arms rotate clockwise in the upper right and bottom left panels and counter-clockwise in the upper left and bottom right panels. The arms of the partially coherent or incoherent cores rotate in the same sense, but with a substantially slower angular velocity than in the coherent domain.

When $\beta$ is decreased yet further some oscillators again detrain for a certain time interval before entraining again. As a result the snapshot shown, for example, in Fig. 5.18(g) fluctu-


Figure 5.19: Snapshots of the phase pattern $\theta\left(x, y_{0}, t\right)$ for $G_{x}=G_{y}=G_{1}^{(2)}$, with $N=256$ and $\beta=1$.4. (a) $t=0$. (b) $t=5$. (c) $t=10$. (d) $t=15$. (e) $t=20$. (f) $t=25$. Here, panel (a) is a slice through the upper two cores of Fig. 5.18(a). Note the significant detraining of some oscillators in panels (c) and (e).
ates in time. The intermittent incoherence is located along a vein-like structure surrounding the cores and moves from one side of the incoherent core to another side as time progresses. However, the size of the incoherent core appears to fluctuate only as a consequence of the finite number of oscillators used in the simulations. When $\beta$ is decreased further, the incoherent regions begin to merge at $\beta \approx 0.06$ (Fig. 5.18(h)) and incoherence invades a larger and larger portion of the two-torus for yet smaller values of $\beta$ (Fig. 5.18(i)). Finally, the state loses stability and evolves into a twisted chimera state at $\beta \approx 0.024$. The process described above exhibits some hysteresis - if $\beta$ is increased from small values the transitions reflected in Fig. 5.18 do not occur at identical $\beta$ values. Figure 5.20 shows snapshots of the states for different values of $\beta$ when $\beta$ is gradually increased. The fully incoherent cores disappear at $\beta \approx 1.25$, a value that differs from the value $\beta \approx 1.117$ for the corresponding transition when $\beta$ is gradually decreased.

Figure 5.21 shows plots of the time-averaged frequency $\bar{\theta}_{t}(x, y)$ for different values of $\beta$ and provides more information about the regions of detraining/entraining oscillators. Compared with the completely phase-random region, where $\bar{\theta}_{t}$ gradually increases (in absolute value) from the center of an incoherent core towards the coherent region, $\bar{\theta}_{t}$ in the detraining/entraining regions varies in a step-wise fashion, and not necessarily monotonically (Fig. $5.21(\mathrm{a}, \mathrm{b}))$. The vein-like structure surrounding the cores for certain values of $\beta$ also


Figure 5.20: (Color online) Snapshots of chimera states for $G_{x}=G_{y}=G_{1}^{(2)}, N=256$ and (a) $\beta=1.12$. (b) $\beta=1.2$. (c) $\beta=1.25$. The phase patterns are got when $\beta$ is increased gradually.
becomes clear (e.g., Fig. 5.21(g)). Compared with Figure 5.18, we conclude that, while the reflection symmetry about the axis through the centers of incoherent cores is broken at any given time $t$, it is retained in time-averaged quantities like the mean rotation frequency $\bar{\theta}_{t}(x, y)$.

To confirm that the non-monotonic behavior is not a finite size effect, we plot in Fig. 5.22 the same figure as Fig. $5.21(\mathrm{a}, \mathrm{b})$ but with $N=512$. In addition, we plot the behavior of $\bar{\theta}_{t}$ as a function of the distance $d$ to the center of the core for different values of $\beta$ (Fig. 5.23). As the cores are not exactly circular, we pick two directions as representative. In Fig. 5.23(a), we pick a horizontal cut through a core, and denote the distance to the center of the core as $d_{h}$. In Fig. 5.23(b), we pick instead a diagonal cut through a core, and denote the distance to the center as $d_{d}$. The behavior of $\bar{\theta}_{t}$ in these two cases is quite similar: $\bar{\theta}_{t}$ varies in general in a step-wise fashion, but not necessarity monotonically. When $\beta=0.8$, the core is completely incoherent and the variation of $\bar{\theta}_{t}$ is similar to the case described in 5.3. Apart from nonmonotonic behavior, we also noticed that in a certain range of $\beta$ the oscillator frequency at the center of a core remains nonzero (e.g., Fig. 5.21(a)-(e)). To see in detail how the phase oscillates in the core area, we investigate the dynamical behavior for particular oscillators when $\beta=1.1$. It turns out that even in the center of a fully coherent core the phase oscillates with a nonzero frequency, in contrast to the spiral wave chimeras described in 5.3. We plot $\theta$ as a function of $t$ for different distances $d_{h}$ from the core (horizontal cut) in Fig. 5.24. Figure $5.24(\mathrm{a})$ confirms that $\bar{\theta}_{t} \neq 0$ in the center of the core, and reveals in addition that the frequency fluctuates on a characteristic fluctuation timescale. Figures 5.24(b,c) show the behavior of oscillators in the partially incoherent region farther from the core center. The figures reveal that the oscillators at this location undergo episodes when their frequencies fluctuate; between these episodes the oscillation frequency remains constant. We identify these episodes (which appear to recur periodically) with the detraining/entraining events mentioned above and conclude that at this location the spatial phase pattern undergoes intermittent rigid rotation. The duration of the detraining/entraining events decreases as


Figure 5.21: (Color online) Snapshots of the oscillator frequency $\bar{\theta}_{t}(x, y)$ averaged over the time interval $0 \leq t \leq 250$ for $G_{x}=G_{y}=G_{1}^{(2)}$, and $N=256$. (a) $\beta=1.4$. (b) $\beta=1.2$. (c) $\beta=1.118$. (d) $\beta=1.117$. (e) $\beta=1.1$. (f) $\beta=0.8$. (g) $\beta=0.4$. (h) $\beta=0.06$. (i) $\beta=0.025$.
$d_{h}$ increases and they are absent in the fully coherent region farther outside (Fig. 5.24(d)). At the same time the mean oscillation frequency $\bar{\theta}_{t}(x, y)$ gradually increases to its value $\Omega$ characterizing the fully coherent region.

To gain a partial understanding of the above results we note that in certain intervals of $\beta(0.8 \gtrsim \beta \gtrsim 0.45)$, the multi-core spiral wave states in Fig. 5.18 are similar to the states found with $G_{x}=\cos (x)$ and $G_{y}=\cos (y)$ for which the self-consistency analysis works. Since Eq. (5.27) is invariant under translations in $x$ and $y$ directions, we choose the origin of coordinates such that $R(x, y)$ and $\Theta(x, y)$ are invariant under the pair of reflections $(x, y) \rightarrow(-x, y)$ and $(x, y) \rightarrow(x,-y)$. These symmetries allow us to write

$$
\begin{equation*}
R \exp (i \Theta)=a_{1} \cos (x)+c_{1} \cos (y)+a_{2} \cos (2 x)+c_{2} \cos (2 y) . \tag{5.83}
\end{equation*}
$$

When $\beta$ is around 0.8 , as in the 4 -core spiral wave chimera state of the case $G_{x}=\cos (x)$


Figure 5.22: (Color online) Snapshots of the oscillator frequency $\bar{\theta}_{t}(x, y)$ averaged over the time interval $0 \leq t \leq 250$ for $G_{x}=G_{y}=G_{1}^{(2)}$, and $N=512$. (a) $\beta=1.4$, (b) $\beta=1.2$, confirming the presence of nonmonotonic rotation profiles in the partially incoherent cores.


Figure 5.23: (Color online) The dependence of $\bar{\theta}_{t}$ as a function of distance to the center of the cores for different values of $\beta$ (color online). (a) $d_{h}$ represent the horizontal distance. (b) $d_{d}$ represent the diagonal distance. Notice that the curve for $\beta=1.118$ and $\beta=1.117$ almost overlap.


Figure 5.24: (Color online) The phase $\theta\left(x=d_{h}, y=0, t\right)$ as a function of $t$ for $\beta=1.1$ and different values of $d_{h}$. (a) $d_{h}=0$. (b) $d_{h}=\pi / 8$. (c) $d_{h}=3 \pi / 16$. (d) $d_{h}=\pi / 4$.
and $G_{y}=\cos (y)$, we can assume that the local order parameter $R$ is invariant under the transformation $(x, y) \rightarrow(y, x)$, at least approximately. Alternatively, we can solve $a_{1}, c_{1}$, $a_{2}$ and $c_{2}$ directly without the use of this symmetry argument. The result is $\Omega \approx 12.284$, $a_{1} \approx 17.533, c_{1} \approx 17.533 i, a_{2}=0$, and $c_{2}=0$. For comparison, in view of the narrow spiral structures along the core edge (Fig. 5.18(f)) one can not expect accurate agreement between the simulation and the theory at this value of $\beta$. In fact the theory appears to work better for smaller $\beta$. For example, when $\beta=0.7, a_{1} \approx 17.11$ from the self-consistency equation, compared with the value $a_{1} \approx 17.15$ from simulation. However, we emphasize here that the self-consistency requirement must be regarded as an approximation that is valid only in a certain range of $\beta$. In other regimes the self-consistency equation fails to account for the more exotic chimera states observed there. This is for the following reasons:

- The self-consistency analysis requires that the incoherent cores have completely random phases. Figures. 5.18(a)-(e) show that this is not the case.
- The self-consistency analysis requires that the local order parameter $R$ vanishes at the phase singularity point at the core center $\left(x_{0}, y_{0}\right)$. It therefore predicts that $\partial \theta\left(x_{0}, y_{0}, t\right) / \partial t=\Omega$ in the rotating frame, or equivalently that $\bar{\theta}_{t}\left(x_{0}, y_{0}\right)=0$ in the original frame. However, numerical calculations show that this is not the case (e.g., Fig. 5.21(a)-(e))

These reasons are similar to those that were found to apply to phase-coupled oscillators in one-dimensional arrays in the presence of intermittent detraining. In one dimension it was possible to confirm that these oscillations were intrinsic to the system and hence not an artifact of a finite oscillator number [60]. It is considerably harder to confirm that this continues to be the case in two dimensions, although we believe that this is in fact the case - computations with an $512 \times 512$ oscillator array yield results very similar to those obtained for a $256 \times 256$ array (Fig. 5.22).

### 5.5 Discussion and conclusions

In this chapter we have investigated a two-dimensional system of identical phase-coupled oscillators with nonlocal coupling of indefinite sign. We focused on the case when coupling function $G(x, y)$ can be decomposed into $G(x, y)=G_{x}(x)+G_{y}(y)$. As in Chapter 3, $G_{x}$ and $G_{y}$ are chosen from two families of coupling functions $G_{n}^{(1)}(x) \equiv \cos (n x)$ and $G_{n}^{(2)}(x) \equiv$ $\cos (n x)+\cos [(n+1) x]$. We found that the stationary solutions in the corresponding onedimensional system (5.11) have counterparts in the two-dimensional system, although their stability properties may differ owing to possible longitudinal instability modes. For example, $2 n$-cluster chimera states were shown to be unstable in 2 D for all values of $\beta$, while they are stable for certain range of $\beta$ in 1D systems. In addition to these expected states we also found a class of twisted chimera states in which the phase varies uniformly along a closed coherent stripe, and a class of spiral wave chimera states in which cores of incoherent oscillators are surrounded by coherent spiral arms. With the coupling functions we have used, these chimeras are quite common and robust in the sense that they can be obtained starting from random initial conditions.

We have obtained a fairly complete description of the above-mentioned stationary phase patterns using a self-consistency analysis based on Eq. (5.9). Symmetry arguments allowed us to reduce Eq. (5.9) to several algebraic equations. The local order parameters obtained using this approach agree well with those generated from direct simulations. In addition the loss of stability of these states with respect to $\beta$ was found to be predicted accurately via a linearized equation of Eq. (5.4) describes the evolution of local mean field. The results obtained are consistent with direct simulations when $\beta$ is changed gradually (5.2(b),5.6(b)). A key input into the analysis was the $D_{4}$ symmetry of the coupling function $G_{n}^{(1)}$. It would be of interest to perform a similar study of other $D_{4}$ symmetric coupling functions in order to determine whether these support similar states. We note that none of our states exhibit the $D_{4}$ symmetry of the coupling function, and surmise that all such states are unstable.

More exotic spiral wave chimera states were found to be present for the coupling function $G_{x}=G_{y}=G_{n}^{(2)}$ and their properties were investigated both as $\beta$ is gradually decreased and when it is gradually increased. Many of the remarkable states identified with this coupling exhibit intermittent detraining and entraining on the part of a subset of the oscillators similar to that found in one-dimensional oscillator arrays [60]. The presence of this intermittency renders the self-consistency analysis unusable and a full understanding of these states remains
a challenge. However, despite the absence of instantaneous $D_{4}$ symmetry, time-averaged quantities such as the time-averaged oscillation frequency $\bar{\theta}_{t}(x, y)$ were found to be accurately $D_{4}$ symmetric, suggesting that the results on "symmetry on average" obtained by Golubitsky and colleagues $[112,113]$ for chaotic maps with symmetry could be extended to the nonlocal systems studied here.

This chapter provides additional insight into chimera states in systems with dimension higher than one. It shows that spiral wave chimeras are natural objects in two dimensions and that these can be stable in a large range of the parameter $\beta$ (or $\alpha$ ). In future work we will study the existence and properties of states of this type in more realistic systems, including coupled arrays of Landau-Stuart oscillators.

## Appendix

## 5.A Derivation of self-consistency equations

In this chapter, we used symmetry arguments to simplify the expressions of local order parameter and reduce the self-consistency equation (5.9) for many cases. We use 1:1 twisting chimera as an example to illustrate the idea. In this case, the symmetry arguments give

$$
\begin{equation*}
R \exp (i \Theta)=a(\exp (i x)-\exp (i y))=2 a i \sin \left(\frac{x-y}{2}\right) \exp \left(i \frac{x+y}{2}\right) \tag{5.84}
\end{equation*}
$$

Substituting this expression into Eq. (5.30), we have

$$
\begin{align*}
a & =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}}}{R} \exp (i \Theta) \cos (x)\right\rangle  \tag{5.85}\\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}}}{4 a^{2} \sin ^{2}\left(\frac{x-y}{2}\right)} 2 a i \sin \left(\frac{x-y}{2}\right) \exp \left(i \frac{x+y}{2}\right) \cos (x)\right\rangle  \tag{5.86}\\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}}}{2 a \sin \left(\frac{x-y}{2}\right)} i \exp \left(i \frac{x+y}{2}\right) \cos (x)\right\rangle  \tag{5.87}\\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}}}{2 a \sin \left(\frac{x-y}{2}\right)} i \exp \left(i \frac{x+y}{2}\right)\left(\frac{\exp (i x)+\exp (-i x)}{2}\right)\right\rangle  \tag{5.88}\\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}}}{4 a \sin \left(\frac{x-y}{2}\right)} i\left(\exp \left(i \frac{3 x+y}{2}\right)+\exp \left(-i \frac{x-y}{2}\right)\right)\right\rangle  \tag{5.89}\\
& =\exp (i \beta)\left\langle\frac{\Omega-\sqrt{\Omega^{2}-R^{2}}}{4 a}\right\rangle \tag{5.90}
\end{align*}
$$

## CHAPTER 5. TWISTED CHIMERA STATES AND MULTI-CORE SPIRAL CHIMERA

 STATES ON A TWO-DIMENSION TORUSThe final self-consistency equation is:

$$
\begin{equation*}
4 a^{2}=\exp (i \beta)\left\langle\Omega-\sqrt{\Omega^{2}-4 a^{2} \sin ^{2}\left(\frac{x-y}{2}\right)}\right\rangle . \tag{5.91}
\end{equation*}
$$

Setting $a \equiv \frac{R_{0}}{2}$ gives us Eq. (5.38). Similar process can be done for Eq. (5.31), (5.32) and (5.33). The equations either vanish or give the same result as Eq. (5.38).

## Chapter 6

## Conclusion

This thesis is dedicated to a study of chimera states in nonlocally phase-coupled oscillator systems. The results of Chapter 3 appeared in [60]; the result of Chapter 4 appeared in [61]. We summarize here some of the major discoveries detailed in the previous chapters and outline some ongoing and future work.

### 6.1 Summary of results

In Chapter 3 we studied the effects of nonlocal coupling of indefinite sign on a system of identical phase-coupled oscillators. In these models, chimera states can develop from random initial conditions. In fact, chimera states and splay states are two most frequently observed states in simulations, suggesting that chimeras have a large basin of attraction. Several classes of states have been found: (a) splay states, (b) stationary multicluster states with evenly distributed coherent clusters, (d) stationary multi-cluster states with unevenly distributed clusters, (d) a single cluster state traveling with a constant speed across the system, and (e) traveling chimera states. The splay states are well understood. We derived an analytical expression for the overall rotation frequency and the conditions for stability. For multi-cluster chimera states, a self-consistent continuum description was provided. Using this description the stability properties of multi-cluster chimera states could be analyzed through a linear stability analysis and the prediction compared with numerical simulation. In addition, finite size effects were also investigated. The motion of a coherent cluster was found to follow a Brownian motion with the noise term modeling the effect of the incoherent oscillation outside the cluster. Fully coherent traveling states were analyzed through a nonlinear eigenvalue equation for the speed of travel. The traveling speed computed from the nonlinear eigenvalue problem was found to be consistent with direct simulation. In simulations, hysteresis behavior was observed in a certain parameter region due to the fact that some oscillators periodically detrain and entrain. However we were unable to solve the corresponding eigenvalue problem for the traveling chimera states. We attribute this failure to the fact that this state does not in fact drift as a rigid object.

In Chapter 4 we studied systems similar to those in Chapter 3 but with inhomogeneity introduced through the dependence of the oscillator frequency on its location. Two types of spatial inhomogeneity, localized and spatially periodic, were considered. We explored the effects of these two sources of inhomogeneity on the existence and properties of each of the states discussed in Chapter 3. The results were described in terms of the parameters $\omega_{0}$ and $\kappa$ or $l$ describing the strength and inverse length scale of the inhomogeneity. When $\omega_{0}$ is small, the splay states persist but their phase varies nonuniformly in space. Further increasing $\omega_{0}$ breaks up splay states, creating new regions of incoherence, while multi-cluster chimera states were found to be pinned to specific locations. The motion of a coherent cluster was modeled using an Ornstein-Uhlenbeck process. The traveling coherent states and traveling chimera states were found to undergo a trapping transition when $\omega_{0}$ exceeded some critical threshold. Many of these states would be studied using effective equations for a complex order parameter. Solutions of this equation are in good agreement with the results of numerical simulations.

In Chapter 5 we studied nonlocal phase oscillator systems on a two-dimensional torus. We assumed that the coupling functions in the $x$ and $y$ directions decouple and chose them from the family of functions used in Chapters 3 and 4. Each stationary solution discussed in Chapter 3 has a corresponding solution in two dimensions. However, some of these lose stability because some eigenmodes related to the $y$ direction are unstable. In addition to these states inherited from one-dimensional systems, two types of new chimera states were discovered. One is the twisted chimera state, in which coherent clusters form closed stripes on a torus. In addition, the phase varies uniformly along the stripe. The other is a multi-core spiral wave chimera state, in which cores of incoherent oscillators are surrounded by spiral arms consisting of phase-locked oscillators. Analysis based on self-consistency equations was provided.

### 6.2 Ongoing work

In this section, we describe two ongoing projects we are currently studying. In the first, we use the same model equation as in Chapter 3 but with different coupling functions. We investigate how changes in the coupling scheme affect the solutions. In the second, a timedependent natural frequency is used. Alternating chimera states and other new states are discovered.

## Varying the coupling function

In this subsection, we still consider the model equation

$$
\begin{equation*}
\frac{\partial \theta}{\partial t}=-\int_{-\pi}^{\pi} G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y \tag{6.1}
\end{equation*}
$$

which we studied in Chapter 3. However, here we consider the coupling function $G(x)=$ $\cos (n x)+\gamma \cos [(n+1) x]$. There are two ways to motivate this kind of coupling. First,
$G(x)=\cos (n x)+\cos [(n+1) x]$ is so special that we would like to see how the solutions will change if the coupling functions deviate from this form. We expect the qualitative behavior should remain when the deviation is small. But how the solutions change if the deviation becomes larger is an interesting question. Second, we observed traveling solutions when $\gamma=1$ (e.g., traveling chimera states exist for $G(x)=\cos (3 x)+\gamma \cos (4 x)$ and suitable parameters), but did not find them when $\gamma=0$. We would like to know what happens as $\gamma$ decreases from 1 to 0 . This section focuses on how the traveling solutions (both traveling chimera states and traveling coherent states) behave when $\gamma$ deviates from 1.


Figure 6.1: A snapshot of the phase pattern of a traveling chimera state. Simulation is done with $G(x)=\cos (3 x)+\cos (4 x), \beta=0.03$ and $N=512$.

We consider first the case $G(x)=\cos (3 x)+\gamma \cos (4 x)$. For suitable values of $\beta$ (where $\beta=\frac{\pi}{2}-\alpha$ ) and $\gamma$, traveling chimera states have been discovered. Fig. 6.1 shows an example of such a traveling state for $\gamma=1$ and $\beta=0.03$, consisting of a traveling coherent cluster with a nearly constant slope embedded in an incoherent background. These states travel along the $x$ direction at a nearly constant speed. When the slope of the coherent cluster is positive (negative), this cluster travels to the left (right). Fig. 6.2(a) shows the dependence of the position $x_{0}$ of the coherent cluster on time $t$. The definition of $x_{0}$ is given in Chapter 3 and [35]. As described in Chapter 3, a complete description of these states is still lacking. Here, we perform some numerical experiments to investigate how this traveling chimera state behaves as we decrease $\gamma$ gradually from $\gamma=1$.

As the value of $\gamma$ decreases, the average drift speed of the coherent cluster decreases. In Fig. 6.2, we show the dependence of $x_{0}$ on $t$ for $\gamma=1,0.99,0.98,0.97,0.96,0.95$. Surprisingly, when $\gamma \approx 0.97$, the coherent cluster switches traveling direction. Figs. 6.2(d,e,f) show that when $\gamma<0.97$, the coherent cluster with a positive slope travels to the right. As $\gamma$ is decreased further, this state loses stability at $\gamma \approx 0.94$ and evolves into a splay state.

To see more detail about how the average velocity of the coherent cluster changes with $\gamma$, we plot the dependence of the average velocity $\bar{c}$ on $\gamma$ in Fig. 6.3. The velocity is averaged over a time interval $\Delta t=5000$. From the plots of $x_{0}$ vs $t$ (Fig. 6.2), this measurement


Figure 6.2: Position of the coherent cluster $x_{0}$ vs time $t$ for (a) $\gamma=1$, (b) $\gamma=0.99$, (c) $\gamma=0.98$, (d) $\gamma=0.97$, (e) $\gamma=0.96$, and (f) $\gamma=0.95$.


Figure 6.3: Dependence of the mean speed of the traveling chimera state on $\gamma$.
maybe be inaccurate as the velocity is not constant. However, we can clearly see that the drift direction reverses as $\gamma$ passes through a critical value $\gamma=\gamma_{c}$.

The traveling coherent state discovered in a system with $G(x)=\cos (x)+\cos (2 x)$ provides another example for understanding how the profiles of the coupling functions affect the solutions. Fig. 6.4 shows a phase pattern for $G(x)=\cos (x)+\cos (2 x)$ and $\beta=0.75$.


Figure 6.4: A snapshot of the phase pattern of a traveling coherent state. Simulation is done with $G(x)=\cos (x)+\cos (2 x), \beta=0.75$ and $N=512$.

We now include the parameter $\gamma$, and assume $G(x)=\cos (x)+\gamma \cos (2 x)$. We performed a simulation starting with the solution shown in Fig. 6.4 and gradually decreased $\gamma$ in steps of size $\delta \gamma=0.01$. The speed of the traveling coherent state decreases very rapidly as $\gamma$ decreases. It becomes zero when $\gamma=0.98$ and remains zero as $\gamma$ is decreased further. Figs. 6.5 (a,b,c) show hidden line plots of the phase patterns for $\gamma=0.99, \gamma=0.98$ and $\gamma=0.92$. This fully coherent state loses stability and evolves to a splay state at $\gamma=0.91$.

To look into the detail, we performed another numerical experiment. Starting with $\gamma=0.99$, we decreased $\gamma$ gradually in steps of $\delta \gamma=0.001$. The hidden line plots of the phase patterns show that the speed becomes zero when $\gamma \approx 0.984$. Figs. 6.6(b,c) show the hidden line plots when $\gamma$ is near the critical value (i.e., for $\gamma=0.984$ and $\gamma=0.983$ ). However, when $\gamma \approx 0.985$, the motion of the phase profile exhibits an oscillation (Fig. 6.6(a)). The reason is that some oscillators periodically detrain from the coherent cluster thereby breaking the reflection symmetry of the phase profile in a time-periodic fashion [80]. This is verified in Fig. 6.7.

For the traveling coherent states, we can compute the speed of travel by solving a nonlinear eigenvalue problem. As described in Chapter 3, we suppose that the coherent state is stationary in the moving frame, i.e., we suppose that $z(x, t) \equiv u(\xi)$, where $\xi \equiv x-c t$. Thus we obtain a complex nonlinear eigenvalue problem for the speed $c$ and the frequency $\Omega$ of


Figure 6.5: Hidden line plots of the phase pattern for (a) $\gamma=0.99$, (b) $\gamma=0.98$, and (c) $\gamma=0.92$.


Figure 6.6: Hidden line plots of the phase pattern for (a) $\gamma=0.985$, (b) $\gamma=0.984$, and (c) $\gamma=0.983$.


Figure 6.7: A phase pattern of a traveling coherent state for $G(x)=\cos (x)+\gamma \cos (2 x)$ when $\gamma=0.985$. Simulation is done with $\beta=0.75$ and $N=512$. At some values of $x$ (e.g., $x \approx 0.27$ or 2.21 ), oscillators deviate from coherent profile and create a discontinuity.
the coherent state:

$$
\begin{equation*}
c \tilde{u}_{\xi}+i \Omega \tilde{u}+\frac{1}{2}\left[e^{-i \alpha} \tilde{U}-\tilde{u}^{2} e^{i \alpha} \tilde{U}^{*}\right]=0 \tag{6.2}
\end{equation*}
$$

Here $\tilde{u}=u \exp i \Omega t$ and likewise for $\tilde{U}$. This equation is to be solved subject to periodic boundary conditions on $[-\pi, \pi]$. Fig. 6.8 shows the dependence of the speed $c$ and $\Omega$ on $\gamma$. The inset of Fig. 6.8(a) reveals that the speed $c$ varies as $\left(\gamma-\gamma_{c}\right)^{\frac{1}{2}}$, where $\gamma_{c} \approx 0.98307$.


Figure 6.8: (a) Dependence of speed of travel on $\gamma$. (b) Dependence of $\Omega$ on $\gamma$.

To compare the speed calculated from the nonlinear eigenvalue equation and the speed from direct simulation, we plot these two speeds in the same figure, as shown in Fig. 6.9. We can see that they match very well when $\gamma<0.982$ or $\gamma>0.991$. For the values of $\gamma$
in between, due to the periodic entrainment and detrainment, the state is no longer a fully coherent state. In addition, because the reflection symmetry is broken, first one way, and then the other, in a periodic fashion, the speed oscillates in time [80]. In addition to Fig. 6.7, Fig. 6.9(b) provides another example when $\gamma=0.99$. This oscillatory effect was investigated in Chapter 3. In this region of $\gamma$, hysteresis behavior was observed. So the solution of the nonlinear equation does not match very well with the results from simulation.


Figure 6.9: (a) The dependence of the speed $c$ on $\gamma$. The circles are the speeds from the nonlinear eigenvalue equation and the ' + ' represents the speed from simulation. (b) The dependence of the position of the traveling coherent state on time $t$ for a solution with $G(x)=\cos (x)+0.99 \cos (2 x)$; note the small-amplitude oscillations in $x_{0}(t)$.

## Time-dependent systems

Possible applications of chimera states range across various disciplines. An important example is uni-hemispheric sleep in various types of mammals and birds. In these phenomena, the neurons in the active part of the brain show desynchronized behavior while the neurons in the inactive part are synchronized. An interesting fact is that the synchronization is known to alternate between the hemispheres. This alternating behavior was reproduced in systems with periodic external forcing [105], in two groups of oscillators with heterogeneity in natural frequency [36], and in oscillatory media with nonlinear global coupling [104].

Here, we investigate systems with time-varying natural frequencies, in which we observed chimera states with alternating behavior. Moreover, we can control the position of the chimera by controlling the natural frequencies. The model equation we are using is

$$
\begin{equation*}
\frac{\partial \theta}{\partial t}=\omega(x, t)-\int_{-\pi}^{\pi} G(x-y) \sin [\theta(x, t)-\theta(y, t)+\alpha] d y . \tag{6.3}
\end{equation*}
$$

In contrast to Eq. (3.1), the natural frequencies in Eq. (6.3) are space and time-dependent. In this section, we focus on two types of $\omega(x, t)$. The first one is $\omega(x, t)=\omega(x) \cos (2 \pi f t)$, where
$f$ is a constant frequency. The second one is $\omega(x, t)=\omega(x-v t)$, where $v$ is a constant speed. We refer to the first case as the standing wave case and the second one as the traveling wave case. For the coupling function $G$, we adopt $G(x)=\cos (x)$ as a specific example. We present some numerical results using a 4th order Runge-Kutta method with simulation parameters $d t=0.01$ and $N=512$.

## Standing wave case

In this subsection, we consider the case when $\omega(x, t)=\omega_{0} \cos (x) \cos (2 \pi f t)$. When the frequency $f$ is small, the system can be treated as quasi-static. We assume $\omega_{0}=O(1)$, $f=O(\epsilon)$, and introduce a slow time scale $\tau=\epsilon t$. Expanding $\theta=\theta_{0}+\epsilon \theta_{1}+\cdots$, where $\theta_{j} \equiv \theta_{j}(x, t, \tau), j=0,1, \cdots$, and substituting it into Eq. (6.3), we obtain the leading order of the phase equation

$$
\begin{equation*}
\frac{\partial \theta_{0}}{\partial t}=\omega_{0}(\tau) \cos (x)-\int_{-\pi}^{\pi} G(x-y) \sin \left[\theta_{0}(x, t)-\theta_{0}(y, t)+\alpha\right] d y \tag{6.4}
\end{equation*}
$$

This equation describes the evolution of $\theta_{0}(x, t, \tau)$ on the fast time scale $t$; at this order the slow time $\tau$ is simply a parameter. Thus the leading order system corresponds to the nonlocal phase-coupled oscillator system with spatial inhomogeneity, which we have studied in Chapter 4. In this quasistatic regime the solution is therefore as given in Chapter 4 but with parameters that vary on the slow time scale $\tau$ as obtained from the self-consistency analysis with $\omega_{0}=\omega_{0}(\tau)$.


Figure 6.10: Snapshots of the phase pattern for (a) $t=1500$, (b) $t=1750$, and (c) $t=2000$. Simulation is done with $G(x)=\cos (x), \omega(x, t)=0.1 \cos (x) \cos (2 \pi f t), f=0.001, N=512$ and random initial conditions.

Recall that several different states are observed when $\omega(x)=\omega_{0} \cos (x)$. When $\omega_{0}=0$, we observe 2 -cluster chimera states and splay states, depending on initial conditions. As we increase $\omega_{0}$, 1-cluster chimera states and chimera splay states are observed. In the case where $\omega(x, t)=\omega_{0} \cos (2 \pi f t) \cos (x)$, with $f=O(\epsilon)$, the system exhibits near periodic behavior with period $T=\frac{1}{f}$. Figure $6.10(\mathrm{a})$ shows a snapshot of the phase pattern for $t=1500$, where $\omega(x, t)=-\omega_{0} \cos (x)$. This is a 1 -cluster chimera state with a coherent
cluster located around $x=0$, which is consistent with our previous result in Chapter 4. When $t=1750, \omega(x, t)=0$ and we observe a 2-cluster chimera state (Fig. 6.10(b)). When $t=2000, \omega(x, t)=+\omega_{0} \cos (x)$ and a 1-cluster chimera state with a coherent cluster located near $x=\pi$ is observed (Fig. 6.10(c)).

The oscillation between the chimera splay states and the near-splay states is shown in Fig. 6.11. The state becomes fully coherent at time $t=1775$, slightly after the time when $\omega(x, t)=0(t=1750)$.


Figure 6.11: Snapshots of the phase pattern for (a) $t=1500$, (b) $t=1775$, (c) $t=2000$. Simulation is done with $G(x)=\cos (x), \omega(x, t)=0.1 \cos (x) \cos (2 \pi f t), f=0.001, N=512$ and random initial conditions.

When $f=O(1)$, we observe 2-cluster chimera states and near splay states (Fig. 6.12). For certain values of $f$ (e.g., $f=0.1$ ), the position of the coherent cluster is still pinned (Fig. 6.12(a)). However, when $f$ is larger (e.g., $f=1$ ), the location of the coherent cluster moves chaotically, just as in the case when $\omega_{0}=0$. When the frequency $f$ is large, one may expect that the oscillation in $\omega$ averages out.


Figure 6.12: (a) A snapshot of a chimera state. (b) A snapshot of a near-splay state. Simulation is done with $G(x)=\cos (x), \omega(x, t)=0.1 \cos (x) \cos (2 \pi f t), f=0.1$ and $N=512$.

## Traveling wave case

We now consider the case with $\omega(x, t)=\omega_{0} \cos (x-v t)$. In the following simulations $\omega_{0}=0.1$ is used. Similar to the standing wave case, when $v$ is small, this system can be treated as quasi-static. Fig. 6.13(a) shows a snapshot of a 1-cluster chimera state with $v=0.01$. Fig. 6.13(b) shows the dependence of the location of the coherent cluster on time, where the location is defined as the maximum of the local order parameter $R$ ( $R$ is as defined in Chapter 2). The coherent cluster moves with an average speed $c \approx 0.01$.


Figure 6.13: (a) A snapshot of a chimera state with $G(x)=\cos (x), \omega(x, t)=0.1 \cos (x-v t)$, $v=0.01$ and $N=512$. (b) Dependence of the position $x_{0}$ on time $t$.

The behavior of the chimera splay states is similar to the 1-cluster chimera states. The coherent region moves with an average speed $c \approx 0.01$, as shown in Fig. 6.14.


Figure 6.14: (a) A snapshot of a chimera splay state with $G(x)=\cos (x), \omega(x, t)=0.1 \cos (x-$ $v t), v=0.01$, and $N=512$. (b) Dependence of the position $x_{0}$ on time $t$.

When $v=0.1$, we observe a new traveling wave solution. Fig. 6.15(a) shows a snapshot of the phase distribution, and Fig. 6.15(b) is a position-vs-time plot, showing that the speed
of travel $c=v$.


Figure 6.15: (a) A snapshot of traveling wave solution with $G(x)=\cos (x), \omega(x, t)=$ $0.1 \cos (x-v t), v=0.1$ and $N=512$. (b) Dependence of the position $x_{0}$ on time $t$.

We have shown some preliminary numerical results for nonlocal phase-coupled oscillator systems with time-dependent natural frequencies. However, this work is in its beginning stages and some of the results are not well understood. Future study will focus on higherorder approximations from asymptotic expansions and the explanation of the constant-speed traveling wave solution. Another direction we can explore is time-dependent systems with other types of coupling. As described in previous chapters, interesting states, e.g., traveling chimera states, exist for particular coupling and suitable parameters. It is worth seeing if there are new and interesting phenomena when a time-dependent natural frequency term is introduced.

### 6.3 Future work

We have investigated chimera states in one and two dimensions. Arrays of nonlocally coupled oscillators in three-dimensions have not been studied. In general, three-dimensional systems are difficult to simulate. However, with the methods discussed in Appendix C, the time complexity is significantly reduced if some specific types of coupling functions are chosen. Therefore, it is possible to explore three-dimensional systems by feasible simulations. We expect to discover some new types of solutions.

Another direction we could explore is using different coupling functions in our model equations. In Chapter 3, we allow the coupling to be negative and discovered many interesting solutions. In the coupling functions we have used so far, the interaction is positive when oscillators are close to each other. It is interesting to see what happens when this is not the case. For example, $G(x)=-\cos (x)$ is a reasonable choice. In this case, the model equation shares the same stationary solutions with the equation studied in Chapter 3. However, the
stability of these solutions changes. It is worth seeing if there exist some stationary solutions that we missed.

One aspect we would like to mention is that we used periodic boundary conditions for all model equations. In the real world, it is hard to build a system on a ring or a torus. Open boundary conditions are more realistic. It is natural to ask how the boundary conditions affect the solutions. For example, in two-dimensional systems, only an even number of incoherent cores can exist for topological reasons. With open boundary conditions, this restriction does not apply.

A study of these topics will help us gain a better understanding of chimera states in general, and their occurrence in physics, chemistry and biology.

## Appendix A

## Linear stability in infinite dimensions

In this appendix, we briefly review some notions about the spectrum of linear operators and discuss their relation to stability analysis. More details can be found in [107, 108]. In Chapter 1, we reviewed the linear stability theory of finite-dimensional ordinary differential equations. We would like to extend the results to the infinite-dimensional case, which will be important in analyzing the stability properties of PDEs. In particular, it has application in the stability analysis of the chimera states (see Section 2.2 in Chapter 2). We now consider a system governed by

$$
\begin{equation*}
\frac{\partial u(x, t)}{\partial t}=F(u) \equiv L u+N(u) \tag{A.1}
\end{equation*}
$$

where $u$ is some variable of interest (e.g., an order parameter), $L$ is a linear operator on a Banach space and $N(u)$ represents the nonlinear part. Assume $u_{0}(x)$ is a stationary solution, i.e., $F\left(u_{0}(x)\right)=0$. The temporal stability of $u_{0}$ is investigated via a linearization of Eq. (A.1). Writing $u=u_{0}+v$, we can get

$$
\begin{equation*}
v_{t}=D F\left(u_{0}\right) v+O\left(|v|^{2}\right) \tag{A.2}
\end{equation*}
$$

When $v$ is a small perturbation, the evolution near the solution $u_{0}$ is dominated by $v_{t}=$ $D F\left(u_{0}\right) v$. Thus, the linear equation is of the form:

$$
\begin{equation*}
v_{t}=L v \tag{A.3}
\end{equation*}
$$

where we have now incorporated the linear terms resulting from $N(u)$ into $L$. In Chapter 1 , we saw that in the finite-dimensional case, the stability of a hyperbolic equilibrium depends strictly on the eigenvalues of the linear operator. The infinite-dimensional case is less straightforward. However, the stability is still closely related to the spectrum of $L$.

## A. 1 Spectrum of linear operators

Let $X$ be the Banach space on which $L$ is defined. Assume $L$ has a domain $D(L) \subset X$ and is closed (for every sequence $\left\{u_{n}\right\}$ in $D(L)$ converging to $u$ and $L u_{n}$ converging to $v$,
one has $L u=v$. ) and densely defined $(D(L)$ is dense in $X)$. These conditions are required for much of the theory. As they hold in most reasonable settings, we will try to avoid the technicalities associated with these issues.

The resolvent set of $L$ is defined as:

$$
\begin{equation*}
\operatorname{res}(L)=\{\lambda \in \mathbb{C}: L-\lambda \text { has a bounded inverse defined on } X\} . \tag{A.4}
\end{equation*}
$$

For $\lambda$ in the resolvent set, the inverse $(L-\lambda)^{-1}$ is called the resolvent operator. The spectrum of $L$, denoted by $\Sigma(L)$, is defined as:

$$
\begin{equation*}
\Sigma(L)=\mathbb{C} \backslash \operatorname{res}(L) \tag{A.5}
\end{equation*}
$$

Here, we would like to point out the fact that the spectrum is not the set of all eigenvalues in infinite-dimensional case. A complex number $\lambda$ is called an eigenvalue of $L$ if the operator $L-\lambda$ has a nontrivial null space in $X$. In finite dimensions, the only way for $L-\lambda$ to fail to have a bounded inverse is if $\lambda$ is an eigenvalue. In infinite dimensions, there are multiple ways for the resolvent operator to fail to be bounded on all of $X$. Therefore, it is useful to divide the spectrum according to how $L-\lambda$ fails to have a bounded inverse.

The essential spectrum of $L$, denoted $\Sigma(L)_{\text {ess }}$, is the set of all complex numbers $\lambda$ such that $L-\lambda$ is not a Fredholm operator. Here, an operator is said to be Fredholm if its range is closed and the dimensions of its kernel and the cokernel are finite. The complement of the essential spectrum is called the point spectrum, so $\Sigma_{p}(L)=\Sigma(L) \backslash \Sigma_{\text {ess }}(L)$. A number $\lambda$ is in the point spectrum if it is an isolated eigenvalue of finite multiplicity. But we need to be careful with this statement. As the eigenvalues can be embeded in the essential spectrum, the point spectrum is not equivalent to the set of eigenvalues.

Dividing the spectrum into the point and essential parts is useful when computing the spectrum, as shown in Chapters 2 and 3 . We now turn our attention to the relation between the spectrum of $L$ and the stability of certain stationary solutions. We are interested in the object $T(t)=\exp (L t) . T(t)$ is a semigroup with generator $L$, and $\exp (L t) u_{0}$ is the solution of the linear equation (A.3) with initial condition $v(x, 0)=u_{0}(x)$. The growth bound of $T(t)$ is defined as

$$
\begin{equation*}
\eta_{0}(L)=\inf \{\eta: \exists M(\eta)<\infty \text { such that }\|T(t)\| \leq M(\eta) \exp (\eta t) \forall t \geq 0\} \tag{A.6}
\end{equation*}
$$

In finite dimensions, $\eta_{0}$ is directly connected with the spectrum of $L$. As we mentioned before, it is a bit more tricky in infinite dimensions. If we define the spectral bound of a linear operator $L$ as

$$
\begin{equation*}
s(L)=\sup \{\operatorname{Re}(\lambda): \lambda \in \Sigma(L)\} \tag{A.7}
\end{equation*}
$$

then we can only prove $s(L) \leq \eta_{0}(L)$. However, for a large class of operators, one can show that $s(L)=\eta_{0}(L)$. Thus we build a close connection between the stability and the spectrum of $L$. A natural question is: How do we compute the spectrum of a linear operator? This is a nontrivial question. For examples such as the Laplacian, one may explicitly determine the
spectrum. But in general, no such straightforward solutions can be found. In Chapter 2, we have the linear evolution equation (2.52)

$$
\begin{equation*}
v_{t}=L v \equiv i \mu(x) v+\frac{1}{2}\left[\exp (-i \alpha) V(x, t)-\exp (i \alpha) \tilde{z}^{2} V^{*}(x, t)\right] \tag{A.8}
\end{equation*}
$$

where $\alpha$ is a constant, $z$ is the stationary solution of original equation, $\mu(x)$ is a term related to $z$ (See Chapter 2), and

$$
\begin{equation*}
V(x, t)=K[v] \equiv \int_{-\pi}^{\pi} G(x-y) v(y, t) d y \tag{A.9}
\end{equation*}
$$

The spectrum of this linear operator $L$ is discussed in Chapter 2 and 3 . What we would like to point out is that for a general coupling function $G$, the point spectrum $\lambda_{p}$ is impractical to determine, even numerically. We were able to obtain an exact equation for $\lambda_{p}$ in previous chapters because we only included a few Fourier components in the coupling function.

## Appendix B

## Normal form theory

In Chapter 1, we used normal forms to analyze some basic bifurcations. In this appendix, we briefly introduce the concept of a normal form, largely following Wiggins' book [2] and Crawford's paper [109].

The basic idea behind normal form theory is finding a series of coordinate transformations which algebraically simplify the equations of a dynamical system as much as possible. First, we assume we have a system

$$
\begin{equation*}
\dot{\mathbf{X}}=\mathbf{J X}+\mathbf{F}_{2}(\mathbf{X})+\mathbf{F}_{3}(\mathbf{X})+\cdots+\mathbf{F}_{r-1}(\mathbf{X})+O\left(|\mathbf{X}|^{r}\right), \quad \mathbf{X} \in \mathbb{R}^{n} \tag{B.1}
\end{equation*}
$$

Here, we have made a transformation such that the fixed point is at the origin and $\mathbf{J}$ is in the Jordan canonical form of the linear part. $\mathbf{F}_{i}$ represents the terms including $i$ th order polynomials of $\mathbf{X}_{i}, i=1,2, \cdots, n$. We next introduce the coordinate transform

$$
\begin{equation*}
\mathbf{X}=\mathbf{Y}+\mathbf{h}_{2}(\mathbf{Y}) \tag{B.2}
\end{equation*}
$$

where $\mathbf{h}_{2}$ is of second order in $\mathbf{Y}$. For $\mathbf{Y}$ sufficiently small, Eq. (B.1) and Eq. (B.2) give

$$
\begin{equation*}
\dot{\mathbf{Y}}=\mathbf{J Y}+\mathbf{J h}_{2}(\mathbf{Y})-\mathbf{D h}_{2}(\mathbf{Y}) \mathbf{J} \mathbf{Y}+\mathbf{F}_{2}(\mathbf{Y})+\tilde{\mathbf{F}}_{3}(\mathbf{Y})+\cdots+\tilde{\mathbf{F}}_{r}(\mathbf{Y})+O\left(|\mathbf{Y}|^{r}\right) \tag{B.3}
\end{equation*}
$$

We will chose $\mathbf{h}_{2}$ to simplify $O\left(|\mathbf{Y}|^{2}\right)$ as much as possible. We define a linear operator:

$$
\begin{equation*}
\mathbf{h}_{2}(\mathbf{Y}) \longrightarrow-\mathbf{J h}_{2}(\mathbf{Y})+\mathbf{D h}_{2}(\mathbf{Y}) \mathbf{J Y} \tag{B.4}
\end{equation*}
$$

Here $\mathbf{h}_{2}(\mathbf{Y})$ can be viewed as an element in $H_{2}$, where $H_{k}$ is the space of vector-valued homogeneous polynomials of degree $k$. As an example, consider basis on $\mathbb{R}^{2}$, and $k=2$,

$$
\begin{equation*}
H_{2}=\operatorname{Span}\left\{\binom{x^{2}}{0},\binom{x y}{0},\binom{y^{2}}{0},\binom{0}{x^{2}},\binom{0}{x y},\binom{0}{y^{2}}\right\} . \tag{B.5}
\end{equation*}
$$

Similarly, this operator can be defined from $H_{k}$ to $H_{k}$ :

$$
\begin{equation*}
L_{J}^{(k)}\left(\mathbf{h}_{k}(\mathbf{Y})\right) \equiv-\mathbf{J h}_{k}(\mathbf{Y})+\mathbf{D h}_{k}(\mathbf{Y}) \mathbf{J} \mathbf{Y} \tag{B.6}
\end{equation*}
$$

It is clear that $H_{2}$ is a vector space with finite dimension. From linear algebra, $H_{2}$ can be represented as

$$
\begin{equation*}
H_{2}=L_{J}^{(2)}\left(H_{2}\right) \oplus G_{2}, \tag{B.7}
\end{equation*}
$$

where $L_{J}^{(2)}\left(H_{2}\right)$ is the range of $L_{J}^{(2)}$, and $G_{2}$ is its complementary space. If $\mathbf{F}_{2} \in L_{J}^{(2)}\left(H_{2}\right)$, we can chose $\mathbf{h}_{2}$ satisfying

$$
\begin{equation*}
-\mathbf{J h}_{2}(\mathbf{Y})+\mathbf{D h}_{2}(\mathbf{Y}) \mathbf{J Y}=\mathbf{F}_{2}(\mathbf{Y}) \tag{B.8}
\end{equation*}
$$

Then all $O\left(|\mathbf{Y}|^{2}\right)$ terms can be eliminated from Eq. (B.3). In general, all $O\left(|\mathbf{Y}|^{2}\right)$ terms in $L_{J}^{(2)}\left(H_{2}\right)$ can be eliminated and only those are in $G_{2}$ remain. These terms are called "resonance" terms for reasons explained in Section 19.12 of [2]. Thus Eq. (B.3) can be reduced to

$$
\begin{equation*}
\dot{\mathbf{Y}}=\mathbf{J Y}+\mathbf{F}_{2}^{r}(\mathbf{Y})+\tilde{\mathbf{F}}_{3}(\mathbf{Y})+\cdots+\tilde{\mathbf{F}}_{r}(\mathbf{Y})+O\left(|\mathbf{Y}|^{r}\right) \tag{B.9}
\end{equation*}
$$

where $\mathbf{F}_{2}^{r}(\mathbf{Y})$ represent resonance terms.
This procedure can be iterated so that we have the following Normal form theorem.
Theorem 2 By a sequence of analytic coordinate changes (B.1) can be transformed into

$$
\begin{equation*}
\dot{\mathbf{Y}}=\mathbf{J} \mathbf{Y}+\mathbf{F}_{2}^{r}(\mathbf{Y})+\mathbf{F}_{3}^{r}(\mathbf{Y})+\cdots+\mathbf{F}_{r-1}^{r}(\mathbf{Y})+O\left(|\mathbf{Y}|^{r}\right), \quad \mathbf{Y} \in \mathbb{R}^{n} \tag{B.10}
\end{equation*}
$$

where $\mathbf{F}_{r-1}^{r}(\mathbf{Y}) \in G_{k}, 2 \leqslant r-1$, and $G_{k}$ is a space complementary to $L_{J}^{(k)}\left(H_{k}\right)$.
The normal form techniques can be extended to systems such as

$$
\begin{equation*}
\dot{\mathbf{X}}=\mathbf{F}(\mathbf{X}, \mu) \tag{B.11}
\end{equation*}
$$

where $\mu$ is a control parameter. A similar procedure as for the systems with no parameters can be followed. We illustrate this idea by deriving the normal form for the Hopf bifurcation in detail.

Suppose $\mathbf{X} \in \mathbb{R}^{2}$ and $\mathbf{X}_{0}=\mathbf{0}$ is a fixed point of Eq. (B.11). $\mathbf{D F}(\mathbf{0}, \mu)$ has two eigenvalues $\lambda(\mu)$ and $\bar{\lambda}(\mu)$. For the Hopf bifurcation, we assume $\lambda(0)=i \omega(0)$. In this two-dimensional system, we introduce $z=x+i y$ to make expressions more compact. Thus the equations for $x$ and $y$ can be transformed to equations for $z$ and $\bar{z}$. Similar to Eq. (B.1), we have the equation for $z$ :

$$
\begin{equation*}
\dot{z}=\lambda z+F_{2}(z, \bar{z})+F_{3}(z, \bar{z})+\cdots+F_{r-1}(z, \bar{z})+O\left(|z|^{r},|\bar{z}|^{r}\right) \tag{B.12}
\end{equation*}
$$

The equation for $\bar{z}$ is simply the complex conjugate of Eq. (B.12). We need keep in mind that $\lambda$ and $F_{i}$ depend on the parameter $\mu$.

We now introduce the coordinate transformation

$$
\begin{equation*}
z \longrightarrow z+h_{2}(z, \bar{z}) \tag{B.13}
\end{equation*}
$$

where $h_{2}(z, \bar{z})$ is the second-order polynomial in $z$ and $\bar{z}$ with coefficients depending on $\mu$. Under this transformation, we have

$$
\begin{equation*}
\dot{z}+\frac{\partial h_{2}}{\partial z} \dot{z}+\frac{\partial h_{2}}{\partial \bar{z}} \dot{\bar{z}}=\lambda\left(z+h_{2}(z)\right)+F_{2}(z, \bar{z})+O(3) . \tag{B.14}
\end{equation*}
$$

Here we have used $O(3)$ to denote terms of order higher than 2. Similarly, we will use $O(r)$ to denote terms of order higher than $r-1$. Notice that

$$
\begin{equation*}
\dot{\bar{z}}=\bar{\lambda} \bar{z}+\bar{F}_{2}+O(3) \tag{B.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(1+\frac{\partial h_{2}}{\partial z}\right)^{-1}=1-\frac{\partial h_{2}}{\partial z}+O(2) \tag{B.16}
\end{equation*}
$$

implying Eq. (B.14) becomes

$$
\begin{equation*}
\dot{z}=\lambda z-\lambda \frac{\partial h_{2}}{\partial z} z-\bar{\lambda} \frac{\partial h_{2}}{\partial \bar{z}} \bar{z}+\lambda h_{2}+F_{2}+O(3) \tag{B.17}
\end{equation*}
$$

We define a linear operator

$$
\begin{equation*}
L^{(2)}: \quad h_{2} \longrightarrow-h_{2}+\left(\lambda \frac{\partial h_{2}}{\partial z} \bar{z}+\bar{\lambda} \frac{\partial h_{2}}{\partial \bar{z}} \bar{z}\right) \tag{B.18}
\end{equation*}
$$

Now $h_{2}$ is an element in $H_{2}=\operatorname{Span}\left(z^{2}, z \bar{z}, \bar{z}^{2}\right)$. The matrix representation of $L^{(2)}$ in this basis is

$$
\left(\begin{array}{ccc}
\lambda & 0 & 0  \tag{B.19}\\
0 & \bar{\lambda} & 0 \\
0 & 0 & -\lambda+2 \bar{\lambda}
\end{array}\right)
$$

For $\mu=0$, we have $\lambda \neq 0$ and $\lambda+\bar{\lambda}=0$. So for sufficiently small $\mu$, this matrix is invertible, which means that $L^{(2)}\left(H_{2}\right)=H_{2}$. Therefore, all second order terms in $z$ and $\bar{z}$ can be eliminated and Eq. (B.17) becomes

$$
\begin{equation*}
\dot{z}=\lambda z+F_{3}(z, \bar{z})+O(4) \tag{B.20}
\end{equation*}
$$

Similarly, by introducing the coordinate transform

$$
\begin{equation*}
z \longrightarrow z+h_{3}(z, \bar{z}) \tag{B.21}
\end{equation*}
$$

Eq. (B.20) becomes

$$
\begin{equation*}
\dot{z}=\lambda z-\lambda \frac{\partial h_{3}}{\partial z} z-\bar{\lambda} \frac{\partial h_{3}}{\partial \bar{z}} \bar{z}+\lambda h_{3}+F_{3}+O(4) \tag{B.22}
\end{equation*}
$$

We define a linear operator

$$
\begin{equation*}
L^{(3)}: \quad h_{3} \longrightarrow-h_{3}+\left(\lambda \frac{\partial h_{3}}{\partial z} \bar{z}+\bar{\lambda} \frac{\partial h_{3}}{\partial \bar{z}} \bar{z}\right) \tag{B.23}
\end{equation*}
$$

where $h_{3}$ is an element in $H_{3}=\operatorname{Span}\left(z^{3}, z^{2} \bar{z}, z \bar{z}^{2}, \bar{z}^{3}\right)$. In this basis, the matrix representation of $L^{(3)}$ is

$$
\left(\begin{array}{cccc}
2 \lambda & 0 & 0 & 0  \tag{B.24}\\
0 & \lambda+\bar{\lambda} & 0 & 0 \\
0 & 0 & 2 \bar{\lambda} & 0 \\
0 & 0 & 0 & -\lambda+3 \bar{\lambda}
\end{array}\right)
$$

When $\mu=0$, the second diagonal element in (B.24) is zero, but the remaining diagonal elements are nonzero. Therefore, third order terms that are not of the form $z^{2} \bar{z}$ can be eliminated. Thus Eq. (B.22) becomes

$$
\begin{equation*}
\dot{z}=\lambda z+c z^{2} \bar{z}+O(4) \tag{B.25}
\end{equation*}
$$

where $c$ may possibly depend on $\mu$. Choosing $\lambda=\mu+i \omega, c=a+i b$ and neglecting the terms of order higher than 3, we obtain the simplest normal form for the Hopf bifurcation

$$
\begin{gather*}
\dot{x}=\mu x-\omega y+(a x-b y)\left(x^{2}+y^{2}\right)  \tag{B.26}\\
\dot{y}=\omega x+\mu y+(b x+a y)\left(x^{2}+y^{2}\right) \tag{B.27}
\end{gather*}
$$

which is used in Chapter 1.

## Appendix C

## Numerical methods

In this appendix, we summarize the numerical methods which are frequently used in this thesis.

## C. 1 Time stepping for simulations

The model equations we investigated in this thesis are integral-differential equations. In the one-dimensional case, they are of the form

$$
\begin{equation*}
\frac{\partial \theta(x, t)}{\partial t}=\omega-\int_{-\pi}^{\pi} G(x-y) \sin (\theta(x, t)-\theta(y, t)+\alpha) d y \tag{C.1}
\end{equation*}
$$

where $\omega$ is constant for chapter 3 , and $\omega=\omega(x)$ for chapter 4 . The discretized version of Eq. (C.1) is

$$
\begin{equation*}
\frac{d \theta_{i}}{d t}=\omega_{i}-\frac{2 \pi}{N} \sum_{j}^{N} G\left(\frac{2 \pi|i-j|}{N}\right) \sin \left(\theta_{i}-\theta_{j}+\alpha\right) \tag{C.2}
\end{equation*}
$$

where $N$ is the number of oscillators on a ring. This is a set of $N$ nonlinearly coupled ordinary differential equations. To numerically simulate them, we adopt the fourth-order Runge-Kutta method [110], or "RK4" for short. Generally, the initial value problem is specified by:

$$
\begin{equation*}
\dot{y}=f(t, y), \quad y(0)=y_{0} . \tag{C.3}
\end{equation*}
$$

Here, $y$ and $f$ could be vectors, but we use the scalar case to illustrate the idea. Choosing a time step size $h$, the iterative relations are

$$
\begin{align*}
& y_{n+1}=y_{n}+\frac{h}{6}\left(k_{1}+2 k_{2}+2 k_{3}+k_{4}\right)  \tag{C.4}\\
& t_{n+1}=t_{n}+h \tag{C.5}
\end{align*}
$$

Here, the $k_{1}, k_{2}, k_{3}$ and $k_{4}$ are computed from

$$
\begin{align*}
k_{1} & =f\left(t_{n}, y_{n}\right),  \tag{C.6}\\
k_{2} & =f\left(t_{n}+\frac{h}{2}, y_{n}+\frac{1}{2} k_{1} h\right),  \tag{C.7}\\
k_{3} & =f\left(t_{n}+\frac{h}{2}, y_{n}+\frac{1}{2} k_{2} h\right),  \tag{C.8}\\
k_{4} & =f\left(t_{n}+h, y_{n}+k_{3} h\right) . \tag{C.9}
\end{align*}
$$

This time stepping process is quite standard. We now focus on each time step. To compute $k_{1}, k_{2}, k_{3}$ and $k_{4}$, we need evaluate $f(t, y)$, which in our case is $\omega_{i}-\frac{2 \pi}{N} \sum_{j}^{N} G\left(\frac{2 \pi|i-j|}{N}\right) \sin \left(\theta_{i}-\right.$ $\left.\theta_{j}+\alpha\right)$. For a general coupling kernel $G(x)$, the complexity is $O\left(N^{2}\right)$, requiring a large amount of computation resources. For the coupling $G(x)=\cos (n x)$, we have

$$
\begin{align*}
& \sum_{j}^{N} G\left(\frac{2 \pi|i-j|}{N}\right) \sin \left(\theta_{i}-\theta_{j}+\alpha\right) \\
= & \operatorname{Im}\left[\operatorname { e x p } ( i \alpha ) \left(\cos \left(\frac{2 i \pi}{N}\right) \exp \left(i \theta_{i}\right) \sum_{j}^{N} \cos \left(\frac{2 j \pi}{N}\right) \exp \left(-i \theta_{j}\right)\right.\right. \\
& \left.\left.+\sin \left(\frac{2 i \pi}{N}\right) \exp \left(i \theta_{i}\right) \sum_{j}^{N} \sin \left(\frac{2 j \pi}{N}\right) \exp \left(-i \theta_{j}\right)\right)\right] . \tag{C.10}
\end{align*}
$$

By rearranging terms, we can reduce the complexity of each time step from $O\left(N^{2}\right)$ to $O(N)$, making the computation much more efficient. A similar process may be used for two-dimensional systems.

## C. 2 Newton's method for the self-consistency equation

In this thesis, we frequently used a self-consistency analysis. For the special class of coupling functions we have studied, we can reduce the self-consistency equations to lowdimensional algebraic equations. We use Newton's method to solve these equations.

Consider a continuously differentiable function $F: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$. Here, $k$ is the number of equations, as well as the number of unknown variables. Newton's method tells us that the zeros of $F$ can be found by the following iteration:

$$
\begin{equation*}
X_{n+1}=X_{n}-J\left(X_{n}\right)^{-1} F\left(X_{n}\right) \tag{C.11}
\end{equation*}
$$

where $J$ is the Jacobian matrix at $X_{n}$. Sometimes, $J$ is unavailable or too expensive to compute, in which case a family of Quasi-Newton methods is used instead. One example
is to use $\left(F\left(X_{n}^{i}+h\right)-F\left(X_{n}^{j}\right)\right) / 2 h$ as an approximation for the elements of the Jacobian matrix.

There are two points we mention here. First, Newton's method can only be used to find one of the roots of $F$. We need to choose the initial value $X_{0}$ carefully to make the iteration process converge to the root we want. In solving the self-consistency equations, the initial values are obtained from direct numerical simulations. Second, even when reasonable initial values are used, Newton's method may converge to some other roots that we are not interested in. One way to deal with this issue is to use the iteration scheme

$$
\begin{equation*}
X_{n+1}=X_{n}-\alpha J\left(X_{n}\right)^{-1} F\left(X_{n}\right) \tag{C.12}
\end{equation*}
$$

where $\alpha \in[0,1]$ is a real number.
In self-consistency equations, complex numbers are involved. However, we can separate the real and the imaginary parts. For example, consider the self-consistency equation (3.17)

$$
\begin{equation*}
R_{0}^{2}=\exp (i \beta)\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \cos ^{2}(n y)}\right\rangle \tag{C.13}
\end{equation*}
$$

We can define

$$
\begin{align*}
& F_{1}\left(R_{0}, \Omega\right)=\operatorname{Re}\left(\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \cos ^{2}(n y)}\right\rangle\right)-R_{0}^{2} \exp (-i \beta)  \tag{C.14}\\
& F_{2}\left(R_{0}, \Omega\right)=\operatorname{Im}\left(\left\langle\Omega-\sqrt{\Omega^{2}-R_{0}^{2} \cos ^{2}(n y)}\right\rangle\right)-R_{0}^{2} \exp (-i \beta) \tag{C.15}
\end{align*}
$$

Then $F\left(R_{0}, \Omega\right)$ is a function from $\mathbb{R}^{2}$ to $\mathbb{R}^{2}$. A Quasi-Newton method can be used to solve $R_{0}$ and $\Omega$, as described in Chapter 3.

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