A Note on Efficient Taxation

Theodore Bergstrom

The Journal of Political Economy, Volume 81, Issue 1 (Jan. - Feb., 1973), 187-191.

A Note on Efficient Taxation

Theodore Bergstrom

Washington University of Saint Louis

In an otherwise interesting article, Barlow (1970) commits a logical error which probably deserves correction. He considers a voting system in which the quantity of a public good supplied by a unit of government is the median of the quantities desired by its citizens. He argues that if the ratio of income elasticity of demand to price elasticity of demand exceeds the elasticity of tax share with respect to income, then the quantity supplied will be less than the Pareto-optimal quantity.[1] (To be more precise, we should say "less than that Pareto optimal quantity which is a Lindahl equilibrium quantity.")[2] This is not necessarily the case. Although under the circumstances which Barlow suggests, the quantity supplied will not in general be Pareto optimal, and certainly will not be a Lindahl equilibrium, this quantity may be either greater or less than the Lindahl equilibrium quantity. We will offer a counter-example to Barlow's claim and then attempt to rescue his result for certain very common types of income distributions.

Following Barlow, assume that the demand for a public good by a consumer with income Y_i is

$$Q_i = k Y_i{}^\alpha \, |P(Y_i)|^\beta,$$

where $P(Y_i)$ is the fraction of total tax revenue for the community which is paid by a consumer with income Y_i. Suppose that the tax schedule is such that $P(Y_i) = t Y_i{}^\gamma$ where t and γ are positive constants. Then $Q_i = k t^\beta Y_i{}^{\alpha+\beta\gamma}$. Since α, β, and γ are assumed to be constants, Q_i is a monotone function of Y_i and hence the median quantity demanded in a community is the quantity demanded by the citizen with the median income. If $\gamma = -(\alpha/\beta)$, all consumers demand the same quantity. This quantity is called the Lindahl equilibrium quantity, and the resulting allocation will be Pareto optimal (see Foley 1970; Bergstrom 1971). Barlow's claim

[1] This condition on elasticities is also discussed in Buchanan (1964) and in Bergstrom and Goodman (1971).

[2] Unless preferences are homothetic and identical, different Pareto-optimal quantities of public goods will in general correspond to different distributions of consumption of public goods. See Samuelson (1954).

is that if $\gamma < -(\alpha/\beta)$, less of the public good will be supplied than if $\gamma = -(\alpha/\beta)$.

In this system, whatever the value of γ, the median quantity demanded will be the quantity demanded by the consumer with the median income. Thus, to determine whether more is supplied when $\gamma = -(\alpha/\beta)$ than when $\gamma < -(\alpha/\beta)$, we need only to determine in which case the consumer with the median income demands more. It is clear from the demand function that if $\beta < 0$, then the consumer with the median income demands more the lower his tax share. It turns out that, depending on the income distribution, an increase in the progressivity of a tax could either increase or decrease the tax share of the consumer with median income.

Consider the following example. A community has three citizens with incomes of \$100, \$400, and \$441. Suppose that $\alpha = 1$ and $-(\alpha/\beta) = 1.5$. Assume that all local taxes are collected from these three citizens. If the tax schedule is $tY^\gamma = tY$, then the tax share of the consumer with median income is $400/(100 + 400 + 441) \cong .425$. If the Lindahl tax schedule $tY^{-(\beta/\alpha)} = tY^{1.5}$ is adopted, then his tax share is $400^{1.5}/(100^{1.5} + 400^{1.5} + 441^{1.5}) \cong .438$. In this example, although $\gamma < -(\beta/\alpha)$, the increase in the progressivity of taxation which results in Lindahl equilibrium would increase the tax share of the consumer with median income. Hence the Lindahl equilibrium quantity would be *less* than the quantity actually supplied.

In the example above, the income distribution was highly skewed to the right. We now offer a series of lemmas which serve to delineate some circumstances under which increases of progressivity of taxation result in an increase in the median quantity demanded.

Lemma 1

Consider a community with n citizens with incomes $Y_1, \ldots, Y_n$, where the median income is $\hat{Y}$. Let

$$f_r(Y_i) = \frac{Y_i^r}{\sum_{i=1}^{n} Y_i^r}$$

and

$$f_s(Y_i) = \frac{Y_i^s}{\sum_{i=1}^{n} Y_i^s},$$

where $r < s$. Then $f_s(Y) < f_r(Y)$ for all Y such that

$$Y < \frac{1}{n} \left[\sum_{i=1}^{n} Y_i^s \right]^{\frac{1}{s}}.$$

Proof:

Where $s > r$, the ratio $[f_s(Y)]/[f_r(Y)] = Y^{s-r}[(\Sigma Y_i^r)/(\Sigma Y_i^s)]$ is an increasing function of Y. Therefore if for some income Y^*, $[f_s(Y^*)]/[f_r(Y^*)] = 1$, it must be that for all $Y < Y^*$, $f_s(Y) < f_r(Y)$. If $[f_s(Y^*)]/[f_r(Y^*)] = 1$, then $Y^{*s-r} = [(1/n)\Sigma Y_i^s]/[(1/n\Sigma Y_i^r]$ and hence $Y^{*(s-r)/r} = |(1/n)\Sigma Y_i^s|^{1/r}/[(1/n)\Sigma Y_i^r]^{1/r}$. Now an expression of the form $|(1/n)\Sigma Y_i^r|^{1/r}$ is known as a mean of the order r. By a well-known theorem, sometimes called the inequality of generalized means, if $s > r$ then $|(1/n)\Sigma Y_i^r|^{1/r} \leqq |(1/n)\Sigma Y_i^s|^{1/s}$ (with strict inequality if not all the Y_i's are equal.) See Hardy, Littlewood, and Polya (1952, theorem 16).

Using this inequality, we have $Y^{*(s-r)/r} \geqq [(1/n)\Sigma Y_i^s]^{1/r}/[(1/n)\Sigma Y_i^s]^{1/s} = [1/n\Sigma Y_i^s]^{(s-r)/sr}$. Since $s > r > 0$, it follows that $Y^* \geqq [(1/n)\Sigma Y_i^s]^{1/s}$. Therefore if $Y < [(1/n)\Sigma Y_i^s]^{1/s}$, it must be that $Y < Y^*$ and hence that $f_s(Y) < f_r(Y)$. Q.E.D.

Corollary

If $s > 1$ and $s > r > 0$, then $f_s(\hat{Y}) < f_r(\hat{Y})$ whenever $\hat{Y} < \overline{Y} \equiv (1/n)\Sigma Y_i$.

Applying again the inequality of generalized means and the assumption that $s > 1$, we have $\hat{Y} < \overline{Y} < |(1/n)\Sigma Y_i^s|^{1/s}$. It follows from Lemma 1 that $f_s(\hat{Y}) < f_r(\hat{Y})$. Q.E.D.

If the only revenues in the community come from taxes paid by citizens, then when the tax schedule is of the form tY_i^γ, the tax share, $P(\hat{Y})$, of the consumer with the median income is $\hat{Y}^\gamma/(\Sigma Y_i^\gamma) = f_\gamma(\hat{Y})$. If $\gamma < -(\alpha/\beta)$, $-(\alpha/\beta) \geqq 1$; and if mean income exceeds median income, the corollary tells us that the tax share of the median consumer must be lowered and hence expenditures increased to reach Lindahl equilibrium.

We now consider the case where some revenue is raised from noncitizens. (This would be likely to happen where commercial and industrial property is taxed.) It will be assumed that the amount of revenue raised from noncitizens does not depend on the progressivity of the tax schedule for citizens but may depend on the quantity of the public good supplied. The proportion of total tax revenue which comes from citizens will then be a function, $r(Q)$. If the tax schedule for citizens is of the form tY^γ, then the tax share of a consumer with income Y is $f_\gamma(Y) \cdot r(Q)$. The median quantity demanded by citizens will remain equal to the quantity demanded by the citizen with the median income. This quantity will be a value of Q for which $k\hat{Y}^a [f_\gamma(\hat{Y}) r(Q)]^\beta = Q$ or $k\hat{Y}^a [f_\gamma(Y)]^\beta = g(Q)$ where $g(Q) \equiv Q[r(Q)]^{-\beta}$.

Lemma

If $r(Q)$ is differentiable, $r(Q) \geqq 0$; and if $\beta < 0$, then $g(Q)$ is a monotone increasing function whenever either (a) $(dr/dQ) > 0$ or $(b) - 1$

$< \beta < 0$ and the tax bill of any consumer increases when total expenditures increase.

Proof:

The sign of $[d\ g(Q)/dQ]$ is the same as the sign of $[d \log g(Q)/dQ] = (1/Q) - \beta(dr/dQ)[1/r(Q)]$. Hence, when $\beta < 0$ and $(dr/dQ) \geqq 0$, $[d\ g(Q)/dQ > 0$. The tax bill of a consumer with income Y is $f_\gamma(Y)r(Q)Q$. If the tax bill increases with Q, then $(d/dQ)[\log f_\gamma(Y)r(Q)Q] = (1/Q) + (dr/dQ)[1/r(Q)] > 0$. Hence $(dr/dQ)[1/r(Q)] > -(1/Q)$ and $-\beta(dr/dQ)\ [1/r(Q)] > \beta(1/Q)$. Substituting into the first equation of the proof, we have $(d/dQ) \log g(Q) > (1/Q)(1 + \beta) > 0$ whenever $-1 < \beta$. This establishes the lemma.

Where $g(Q)$ is a monotone increasing function, the equation $\hat{Y}^a [f_\gamma(\hat{Y})]^\beta = g\ (Q)$ has a unique solution, $\hat{Q} = g^{-1}\{\hat{Y}^a\ [f_\gamma(\hat{Y})]^\beta\}$ where g^{-1} is the inverse function of g. Since g^{-1} is an increasing function and $\beta < 0$, changes in γ which decrease $f_\gamma(\hat{Y})$ result in higher values of $\hat{Q}$. If we apply the corollary to Lemma 1, the following result is immediate.

Theorem

In a community with n consumers with incomes $(Y_1, \ldots, Y_n)$, let demands for a public good be of the form $Q_i = k_i Y_i^a [P(Y_i)]^\beta$, where $\beta < 0$, where $P(Y_i) = r(Q)[Y_i^\gamma/(\Sigma Y_i^\gamma)]$, and $r(Q)$ satisfies at least one of the conditions of Lemma 2. Then if $\gamma < -(a/\beta)$, $-(a/\beta) > 1$, and if the median income is less than the mean income, the median quantity demanded will be less than the Lindahl equilibrium quantity.

We compared estimated mean incomes with median incomes, using data from the 1960 Census for a random sample of 25 Michigan cities with populations exceeding 10,000. In each case, the mean income exceeds the median. (Furthermore, the mean housing value exceeds the median.)

In a recent paper (Bergstrom and Goodman 1971), Robert Goodman and I have estimated elasticities of demand for municipal expenditures (excluding education) in each of ten states. Our estimates indicate that the ratio of income elasticity to price elasticity exceeds unity and also exceeds the elasticity of tax share with respect to income. If these estimates are to be believed, then Theorem 1 suggests that municipal expenditures are in fact less than the Lindahl equilibrium levels.

References

Barlow, R. "Efficiency Aspects of Local School Finance." *J.P.E.* 78 (September/October 1970): 1028–40.

Bergstrom, T. "Collective Choice and the Lindahl Allocation Method." Mimeographed. 1971. Available from T. Bergstrom, Department of Economics, Washington University, St. Louis, Mo.

Bergstrom, T., and Goodman, R. "Private Demands for Public Goods." Mimeographed. 1971. Available from T. Bergstrom, Department of Economics, Washington University, St. Louis, Mo.

Buchanan, J. "Fiscal Institutions and Efficiency in Collective Outlay." *A.E.R.,* suppl. (May 1964).

Foley, D. "Lindahl's Solution and the Core of an Economy with Collective Goods." *Econometrica* 38 (January 1970).

Hardy, G.; Littlewood, H.; and Polya, H. *Inequalities.* 2d ed. London: Cambridge Univ. Press, 1952.

Samuelson, P. "The Pure Theory of Public Expenditures." *Rev. Econ. and Statis.* 36 (November 1954).