Title
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Publication Date
2023

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Alexander Givental, Chair
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Spring 2023

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Abstract

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We prove a twisting theorem for nodal classes in permutation-equivariant quantum K-theory, and combine it with existing theorems of Givental [8] to obtain a twisting a theorem for general characteristic classes of the virtual tangent bundle of the moduli space of stable maps. Using this result, we develop complex cobordism-valued Gromov-Witten invariants defined via K-theory, and relate those invariants to K-theoretic ones via the quantization of suitable symplectic transformations. This procedure is a K-theoretic analogue of the quantum cobordism theory developed by Givental and Coates in [5]. Using the universality of cobordism theory, we give an example of these results in the context of "Hirzebruch K-theory", which is the cohomology theory determined by the Hirzebruch χ_{-y}-genus.

We then introduce Euler-theoretic Gromov-Witten invariants, which are based on the ordinary topological Euler characteristics of loci of curves. These invariants have useful enumerative properties. They are integer valued, there is a reliable way to remove boundary contributions, and when the moduli spaces are smooth orbifolds, have a concrete geometric intepretation. In this case, the invariants also encompass the ordinary (i.e. integer) topological Euler characteristic of $M_{g,n,d}(X)$ and its compactification.

We give a Wick-type formula computing these invariants in terms of twisted cohomological Gromov-Witten invariants, and we also show that they can be regarded as the limit as $y \to 1$ of the Hirzebruch-theoretic invariants introduced earlier in this work.

Specializing to the case of a point target space, our formulas yield similar results to the computation of the Euler characteristics of the moduli spaces of n-pointed curves, given by Bini-Harer.

To my parents, Rehana Huq and Sarosh Kuruvilla.

Contents

Acknowledgments

First and foremost, I would like to thank my advisor, Alexander Givental, for his support, guidance and patience.

My completion of this program would not have been possible without the friendship of those with whom I travelled this road. Particular thanks are owed to Yuan Yao, who I have learned much from, both about mathematics, and about how a mathematician should strive to live.

Part of the work in this thesis was supported by an NSF GRFP Fellowship. Related to this, I would like to thank Katrin Wehrheim, for providing vital assistance in my application for the aforementioned fellowship.

Chapter 1

Introduction and Summary of Results

Quantum K-theory, introduced by Givental and Lee, is a deformation of the K-ring of a manifold using K-theoretic Gromov-Witten invariants, which are certain holomorphic Euler characteristics of certain sheaves the Gromov-Witten moduli spaces $\overline{M}_{g,n,d}(X)$ (henceforth shortened to $X_{g,n,d}$).

For example, the invariant representing the count of degree-d genus g curves in X passing through n points would be:

$$\chi(X_{g,n,d}; \mathcal{O}^{vir} \otimes \bigotimes_i ev_i^* \mathcal{O}_{pt})$$

Givental generalized these invariants the *permutation-equivariant* context, which replaces the operation of taking the holomorphic Euler characteristic with taking the supertrace of some $h \in S_n$ on the sheaf cohomology.

As with quantum cohomology, there is a symplectic loop space formalism governing these invariants. Let $\mathcal{D}_X$ denote the generating function for these invariants, with contributions from disconnected curves. Givental situates this function (after a translation of the inputs known as the *dilaton shift*) as an *quantum state* in a *symplectic loop space* $\mathcal{K}^\infty$. The ingredients defining the data of $\mathcal{K}^\infty$, the symplectic form, the dilaton shift, and the choice of Lagrangian polarization, are in some sense derived from the formal group law computing the K-theoretic Euler class of the tensor product of two line bundles.

Various problems in cohomological Gromov-Witten theory are best understood in terms of *twisting classes*. Given some invertible multiplicative characteristic class C and some bundle V on $X_{g,n+1,d}$, twisted invariants are defined by modifying the virtual fundamental cycle $[X_{g,n,d}]^{vir}$ by capping it with $C(ft_* V)$.

Givental generalized this notion to K-theory, allowing the twisting class to be of the form $e^{\sum_{k \neq 0} \frac{\Psi^k}{k}(E_k)}$, where the E_k. must all be one of two types.

- Type I: $E_k = ft_* ev_{n+1}^* E$, for $E \in K^0(X)$.

- Type II: $E_k = ft_* f(L_{n+1})$, where f is a Laurent polynomial with coefficients pulled back from $K^0(X)$ satisfying $f(1) = 0$. (Changing $f(1)$ simply adds another twisting of Type I).

In the same work, Givental computed the operator relating $\mathcal{D}_X$ and $\mathcal{D}_X^{tw}$ for each type of twisting. The formulas themselves are complicated, but become conceptually simpler upon relating them to $\mathcal{K}^\infty$, where they essentially involve altering some of the symplectic data. For twistings of type I, the operator is a quantized version of a certain multiplication operator om $\mathcal{K}^\infty$, regarded as a symplectomprhism between $\mathcal{K}^\infty, \Omega$, and $\mathcal{K}^\infty, \Omega^{tw}$.

For twistings of type II, the operator is a translation on Fock spaces that is equivalent to modifying the dilaton shift.

We introduce a third type of twisting. We allow E_k to be $ft_* i_* f(L_+, L_-)$, where f is a symmetic Laurent polynomial, and i is the inclusion of the codimension-2 locus of nodes $\mathcal{Z}$ into $X_{g,n+1,d}$, L_+, L_- are the cotangent lines to each branch at the node.

Our twisting theorem is as follows:

Theorem 1. *For a twisting of type III*

$$\mathcal{D}_X^{tw} = \widehat{\Box}\mathcal{D}_X$$

Whwre $\widehat{\Box}$ is the quantization of a symplectic map $\Box$ that fixes $\mathcal{K}_+$ maps $\mathcal{K}_-$ to a different choice of negative subspace.

Remark. Twistings of different types can be applied in sequence, and the same theorems hold true, relating a twisted potential to a further-twisted one.

Remark. These twisting theorems easily specialize to the non S_n-equivariant case, relating the ordinary K-theoretic potential $\mathcal{D}_X^K$ to its twisted versions. The loop space in question is a direct summand of $\mathcal{K}^\infty$ denoted $\mathcal{K}$.

The virtual tangent bundle of the moduli space of stable maps has the following decomposition in K-theory:

$$T^{vir} = ft_* ev_{n+1}^*(TX - 1) + ft_*(1 - L_{n+1}^{-1}) - (ft_* \mathcal{O}_{\mathcal{Z}})^*$$

Roughly, the first factor corresponds to deformations of the stable map, the second comes from deforming the complex structure on the source curve, and the third comes from smoothing the nodes on the source curve. Each of the three twisting theorems governs classes of one of these 3 components, so taken together, we have twisting theorems for arbitrary characteristic classes of T^{vir}.

To any morphism of complex-oriented cohomology theories, there corresponds a Hirzebruch-Riemann-Roch formula that relates the pushforwards in each theory. MU^*, complex cobordism theory, is universal among such theories. The HRR theorem relating complex cobordism theory and K-theory looks as follows. There is an isomorphism $Ch_K : MU^*(X, \mathbb{Q}) \to$

$K^*(X) \otimes MU^*(pt, \mathbb{Q})$. If π denotes the map $X \to pt$, then the Hirzebruch-Riemann-Roch type theorem relating the MU^*-theoretic and K-theoretic pushforwards looks like:

$$\pi_* \alpha = \chi(X; Ch_K(\alpha) C(TX))$$

Where C is the universal multiplicative invertible characteristic class in K-theory. Specializing MU^* to any other theory recovers the corresponding Hirzebruch-Riemann-Roch theorem.

Using C as a twisting class, we can define MU^*-valued Gromov-Witten invariants of X by defining them via the right hand side of the Hirzebruch-Riemann-Roch formula, as K-theoretic invariants twisted by $C(T^{vir})$. Since there is no true Hirzebruch-Riemann-Roch formula of this kind, these invariants are not genuine MU^*-theoretic pushforwards on $X_{g,n,d}$ (as such things are not even defined).

Another verson of cobordism-theoretic invariants were defined by Coates and Givental, using cohomology rather than K-theory, and the theory was given the name *fake* quantum cobordism theory. Our invariants interact differentily with the orbifold structure of the moduli spaces, since they are defined K-theoreticlaly. Since K-theory is the cohomology theory corresponding to the multiplciative formal group law, we give the resulting theory the name *multiplicative quantum cobordism theory.*

We call the generating function for these invariants $\mathcal{D}_X^U$. If we denote the that converts inputs (and MU^*-theoretic characteristic classes of cotangent lines) to their K-theoretic counterparts by qCh_K, then we have the following theorem:

Theorem 2.

$$\mathrm{qCh_K}\langle D_X^U \rangle = \nabla \langle D_X^K \rangle$$

Here the operator ∇ corrects the symplectic form to one based on the MU^*-theoretic Poincare pairing, and modifies the dilaton shift and polarization to ones determined by the MU^*-theoretic formal group laws. The result is that the connection between moduli spaces of stable maps and formal group laws discovered in [5] persists in the quantum K-theoretic context.

A particularly interesting example of these kinds of invariants occur when considering the specialiation of C to the class C_y, the S^1-equivariant Euler class with equivariant parameter y. It is determined on line bundles by $C_y(L) = 1 - yL^{-1}$. When the moduli space is a manifold, such an invariant (with no inputs) computes the Hirzebruch χ_{-y}-genus, hence we call the twisted theory *Hirzebruch K-theory.*

For cases where the moduli space is a smooth orbifold, the limit as $y \mapsto 1$ of the Hirzebruch-theoretic correlator with no inputs is $\chi(X_{g,n,d})$. The ordinary (i.e. non-orbifold) topological Euler characteristic.

However, taking the limit of the entire formalism breaks down. To describe the what happens at the limit, we develop a new class of Gromov-Witten invariants we call *Euler Invariants.*

Roughly, they are defined as follows: Given $[f_i]$ bordism classes of maps $Y_i \to X$, Euler-theoretic correlators are integrals of the form:

$$\int_{IX_{g,n,d}} c(TIX_{g,n,d})(\prod_i ev_i^* f_{i*} c(T_{f_i})|_{IX_{g,n,d}}$$

These invariants represent (ordinary) Euler characteristics of the orbispace (or singular Deligne-Mumford stack) $M_f : X_{g,n,d} \times_{\prod ev_i} X^n \times_{\prod_i f_i} \prod_i Y_i$. In sufficiently nice situations, they can be literally geometrically interpreted as such.

Theorem 3. *If $X_{g,n,d}$ is a smooth orbifold of the correct expected dimension (i.e. $X = pt$ or $g = 0$ and X is convex), then the map $\prod_i f_i : \prod_i Y_i \to X^n$ can be moved by homotopy to a map F satisfying some transversality condition with respect to the evaluation maps (the precise notion we use will be explained in the relevant chapter), so that $M_F := X_{g,n,d} \times_{\prod ev_i} X^n \times_F \prod_i Y_i$ is a smooth almost complex orbifold. Doing this yields a geometric interpretation for the associated Euler-theoretic invariant, namely:*

$$\overline{\langle [f_1], \ldots, [f_n] \rangle}^E_{g,n,d} = \chi(M_F)$$

In particular, the invariants are integers in this case, and can be thought of as versions of Gromov-Witten invariants that ignore automorphisms, counting each curve with weight 1 regardless of the isotropy group.

We package these invariants (along with an S_n-equivariant generalization) into a generating function $\mathcal{D}_X^E$.

Another advantage of this theory is that the Euler characteristic is additive, so we can define invariants representing the contributions from smooth curves simply by subtracting boundary contributions. We call these invariants *incomplete* Euler-theoretic invariants, and denote their generating function by $\mathcal{E}_X$.

However thus defined, the invariants are not easy to compute, since they require integrating over $IX_{g,n,d}$, we give two theorems that provide means of computation.

Theorem 4. *Euler-theoretic invariants can be computed as limits of Hirzebruch-theoretic invariants for certain choices of input.*

By the twisting theorem for K-theory, this means the invariants can be expressed in terms of usual K-theoretic Gromov-Witten invariants of X.

However, they can also be computed via cohomological invariants of X. We introduce the potential $\mathcal{D}_X^{E,fake}$, which is the generating function for cohomological Gromov-Witten invariants for X, twisted by the total Chern class of the virtual tangent bundle. These have the same geometric interpreation as Euler-theoretic invariants, except the ordinary Euler characteristic is replaced with the orbifold one.

Theorem 5. *After some adjustment to the inputs, degrees, and Planck's constant, we have a Wick-type formula (given by summation over graphs) of the general form:*

$$\mathcal{D}_X^E = \left[exp(\sum_r \frac{\hbar^r r}{2}\nabla^r \bigotimes_M \mathcal{D}_{X\times B\mathbb{Z}_M}^{E,fake}\right] \tag{1.1}$$

For ∇^r an order-2 differential operator that plays the role of the propagator at each edge.

The potentials $\mathcal{D}_{X\times B\mathbb{Z}_M}^{E,fake}$ can be expressed in terms of $\mathcal{D}_X^{E,fake}$, so for the case when $X_{g,n,d}$ is smooth of the expected dimension, the formula expresses the (integer) ordinary Euler characteristics $\chi(X_{g,n,d})$ in terms of their (rational) orbifold counterparts.

Analogues of the above results also hold in the incomplete setting. Applying the resulting formula to the case $X = pt$ recovers a formula of Bini-Harer from [2] expressing $\chi(\mathcal{M}_{g,n})$ in terms of $\chi^{orb}(\mathcal{M}_{g,n})$.

Chapter 2

Quantum K-Theory and Twisting Classes

2.1 Basic Formulations

We recall the definition of permutation-equivariant K-theoretic Gromov-Witten invariants and the associated potentials, using the definitions introduced by Givental in [7].

For X a smooth projective variety, let $X_{g,n,d}$ be the moduli space of stable maps from n-pointed curves of genus g and degree d, with maps ev_i to X corresponding to evaluation at the ith marked point. Define $K = K^0(X) \otimes \Lambda$, for Λ some algebra equipped with Adams operations Ψ^k, extending the ones on $K^0(X)$.

Given q permutation $h \in S_n$ with $\ell_r(h)$ cycles of length r, with r ranging from 1 to s, h acts on $X_{g,n,d}$ by permuting the marked points.
For each r, given inputs $w_{r1}, \ldots, w_{r\ell_r}$ each Laurent polynomials in q of the form $\sum_{m\in\mathbb{Z}} \phi_m q^m$, for $\phi_m \in K$, associate to the input w_{rk} the bundle $W_{rk} = \bigotimes_{\alpha=1}^{r} \sum_m \mathrm{ev}^*_{\sigma_\alpha} \phi_m L^m_{\sigma_\alpha}$, where σ_α are the marked points permuted by the kth cycle of length r, and L_{σ_α} are the corresponding universal cotangent lines on $X_{g,n,d}$

Define the correlators of S_n-equivariant quantum K-theory to be (given a genus g, partition ℓ, and degree d), and inputs only in $K^0(X)[q, q^{-1}]$:
$\langle w_{11}, \ldots, w_{1\ell_1}, \ldots \rangle_{g,\ell,d} = \prod_r r^{-\ell_r} str_h H^*(X_{g,n,d}, \mathcal{O}^{vir}_{g,n,d} \bigotimes_{i=1}^{s} \otimes_{j=1}^{\ell_i} W_{ij})$.

Extend this to inputs in $K^0(X) \otimes \Lambda$ Ψ-linearly, i.e scaling w_{ri} by $\lambda \in \Lambda$ scales the potential by $\Psi^r(\lambda)$. The motivation for the choice of Ψ-linear scaling is if h permutes $V^{\otimes r}$ cyclically $tr_h(V^{\otimes r}) = \Psi^r(V)$.

Packaging these correlators together and normalizing appropriately gives the genus g

potential function

$$\mathcal{F}_g^X := \sum_d Q^d \sum_\ell \frac{1}{\prod_r \ell_r!} \langle t_1, \ldots, t_r, \ldots, \rangle_{g,\ell,d}$$

$\mathcal{F}_g^X$ represents all invariants coming from connected curves of genus g, in this formulation, inputs corresponding to cycles of length r are treated as the same, and have value t_r.

Note. We make a modification to $\mathcal{F}_0^X$ by adding $\frac{1}{2}(\Psi^2(t_2(1), 1)$, here () denotes the Poincare pairing on X. This contribution (in a sense that we shall describe later) accounts for curves with 2 marked points permuted by a 2 cycle.

The total potential $\mathcal{D}_X$ is defined to represent contributions from possibly disconnected curves. It is given by the formula

$$\mathcal{D}_X := e^{\sum_{g \geq 0}(\sum_{k>0} \hbar^{k(g-1)} \frac{\Psi^k}{k}(R_k(F_g)))}$$

Here the operator R_k takes $F(t_1, t_2, \ldots)$ to $F(t_k, t_{2k}, \ldots)$.

The justification for this choice of definition is as follows, the only nonzero contributions from disconnected curves occur when h cyclically permutes k curves of the same genus g, and h^k acts as the same permutation on each component. This is equivalent to computing tr_{h^k} on a single component, and then applying Ψ^r to the result. However the cycle lengths are all scaled by k, hence the operator R_k.

We will regard $\mathcal{D}_X$ and similar functions as quantum states in suitable symplectic vector spaces, using Givental's loop space formalism.

2.2 Symplectic Loop Space Formalism

Let $K = K^0(X) \otimes \Lambda$, where Λ is some algebra with an action of Adams operations[1].

Denote the K-theoretic Poncare pairing by $(a, b) := \chi(X; a \otimes b)$. Let $\mathcal{K}$ be the space of rational functions of q with coefficients in K, with poles allowed only at $0, \infty$, and roots of unity. $\mathcal{K}$ has the symplectic form $\Omega(f, g) = -[Res_{0,\infty}](\chi(f(q^{-1})g(q))\frac{dq}{q})$. Here $Res_{0,\infty}$ is the sum of the residues at 0 and ∞. $\mathcal{K}$ has a Lagrangian polarization given by $\mathcal{K}_+ = K[q^{\pm 1}]$, $\mathcal{K}_- = \{f : f(\infty) = 0, f(0) \neq \infty\}$.

Let the symplectic loop space $\mathcal{K}^\infty$ be defined as a K-module by $\prod_{r \in \mathbb{Z}_+} \mathcal{K}$. It is given a symplectic structure via the form $\Omega^\infty(f, g) = \bigoplus \frac{\Psi^r}{r}\Omega(f_r, g_r)$, and the polarization is similarly inherited from $\mathcal{K}$.

Remark. $\mathcal{K}_+^\infty$ represents legitimate inputs to the potential $\mathcal{D}_X$, however, there are many possible choices for complementary spaces, we describe a way of thinking about the choice of

[1] We also need to complete this space with respect to Novikov's variables Q^d, but we suppress this notation.

$\mathcal{K}_-^\infty$. Consider the expression $\frac{1}{1-L_1L_2}$, the reciprocal of the "inverse Euler class" of $L_1 \otimes L_2$ in K-theory. Regarding it as a symmetric tensor in $\mathcal{K} \otimes \mathcal{K}$ and dualizing it via the symplectic form gives a map $A : \mathcal{K} \to \mathcal{K}$ given by sending $f(q)$ to $Res_{0,\infty} \frac{f(q)}{q-x} dq$.

This map is the identity on $\mathcal{K}_+$, and has kernel $\mathcal{K}_-$. We will henceforth label different polarizations of $\mathcal{K}$ by expressions of this kind.

One can regard $\mathcal{K}^\infty$ as a phase space in classical mechanics, with $\mathcal{K}_+^\infty$ representing position coordinates and $\mathcal{K}_-^\infty$ representing momentum coordinates. The potential $\mathcal{D}_X$ is a function on $\mathcal{K}^\infty$ constant in the directon of $\mathcal{K}_-^\infty$, and hence defines a *quantum state* $\langle \mathcal{D}_X \rangle$ in the "Fock Space" of $\mathcal{K}^\infty$. We actually define the quantum state $\langle \mathcal{D}_X \rangle$ to be $\mathcal{D}_X$ shifted by $(1-q)1$ in each block, this translation is called the *dilaton shift*.

Given some infinitesemial symplectic transformation, it can be represented as a quadratic Hamiltonian $f(p,q)$, it induces an action on Fock space by a differential operator $\widehat{f}$, determined by the following quantization rules:

- $\widehat{q_\alpha q_\beta} = \hbar^{-r} q_\alpha q_\beta$
- $\widehat{q_\alpha p_\beta} = q_\alpha \partial_{q_\beta}$
- $\widehat{p_\alpha p_\beta} = \hbar^r \partial_{q_\alpha} \partial_{q_\beta}$

Given a linear symplectomorphism S, we can define the quantization $\widehat{S}$ as $exp(\widehat{log(S)})$. As we will see in the next chapter, complicated formulas relating two generating functions can be described as quantizations of relatively simple transformations on $\mathcal{K}^\infty$.

2.3 Twisting Classes

We can define "twisted"-K-theoretic potentials by tensoring $\mathcal{O}^{vir}$ with other classes from $K^0(X_{g,n,d})$.

These classes are expressions of the form $e^{\sum_{k \in \mathbb{Z}/0} \frac{\Psi^k}{k}(E_k)}$, where the E_ks must all be of the following 3 types:

- Type I: $E_k = \mathrm{ft}_* \mathrm{ev}^* V_k$, where $V_k \in K$.
- Type II:$E_k = \mathrm{ft}_* F_k(L)$, where L is the universal cotangent line and F_k is a polynomial with coefficients in $\mathrm{ev}^* K$, these are K-theoretic versions of Kabanov and Kimura's κ-classes.
- Type III: $E_k = \mathrm{ft}_* \mathrm{i}_* F_k(L_+, L_-)$, where $\mathrm{i} : \mathcal{Z} \to U_{g,n,d}$ is the inclusion of the codimension-2 locus of nodes, and F_k is a symmetric Laurent polynomial with coefficients pulled back by ev_{n+1} from K. L_+ and L_- are the cotangent branches to the node. (Note we could equivalently write $E_k = \mathrm{ft}(F_k(L_+, L_-) \otimes \mathrm{i}_* \mathcal{O}_{\mathcal{Z}})$)

Remark. To be somewhat more precise, for k a negative number, we define Ψ^k on line bundles by $\Psi^k(L) = \Psi^{-k}(L^{-1})$, and extend it to $K^0(X)$, we also requite some extension of these operations to Λ. The twisting classes must also be chosen to converge in the topology of Λ, which can be achieved by adding auxilliary variables to the coefficients. We will address the details of these requirements as they are needed in examples.

The generating functions $\mathcal{F}^g_X$ for twisted invariants are constructed exactly like the ones for untwisted ones, only with a change to $\mathcal{O}^{vir}$. However, there is a further modification necesary for the total descendant potentials. For a twisting class $\mathbf{T}$, the twisted potential is $\mathcal{D}^{\mathbf{T}}_X$ is given by the formula

$$\mathcal{D}_X := e^{\sum_{g\geq 0}(\sum_{k>0} \hbar^{k(g-1)} \frac{\Psi^k}{k}(R_k(\mathcal{F}^X_g)))}$$

Here the operator R_k takes $F(t_1, t_2, \dots)$ to $F(t_k, t_{2k}, \dots)$, but also changes E_k to E_{rk}.

The justification for this modification is the following: if we have r copies of M permuted cyclically with inputs labelled T, the corresponding contribution should be:

$$str_h H^*(M^r; \mathcal{O}_{M^r} \otimes e^{\sum_k \frac{\Psi^k}{k}(\sum_{i=1}^r pr_i^* E_k)\prod_i pr_i^* T})$$

However it was shown in [8] that this quantity is equal to

$$\Psi^r(\chi(M : \mathcal{O}_M \otimes \otimes e^{\sum_k \frac{\Psi^k}{k}(E_{rk})} T))$$

To describe the relationship between the twisted potentials $\mathcal{D}^{tw}_X$ and $\mathcal{D}_X$, we use the language of quantization. The potential $\mathcal{D}_X$ can be regarded as an element of the Fock space of $\mathcal{K}^\infty$, i.e. the space of lifts of function from $\mathcal{K}^\infty/\mathcal{K}^\infty_-$. It thus defines (after a translation of the inputs by $(1-q)\mathbf{1}$ called the *dilaton shift*) a "quantum state" $\langle \mathcal{D}_{\mathcal{X}} \rangle$.

The effect of any such twistings on $\mathcal{D}_X$ is as follows: $\langle \mathcal{D}^{tw}_X \rangle = \nabla \langle \mathcal{D}_X \rangle$, where ∇ is some differential operator defined on the Fock space.

The following theorem, proven by Givental in [8], describes ∇ for twistings of type 1 and 2 in terms of the symplectic geometry of $\mathcal{K}^\infty$

Theorem 2.3.1. *For a twisting of type I, ∇ is the quantization of the transformation $\Phi : f_r \to \Phi_r f_r$, where Φ_r is the Euler-Maclaurin asymptotics of $e^{\sum_{k\neq 0} \frac{\frac{\Psi^{rk}}{k}(E^{rk})}{1-q^k}}$. Here Φ is regarded as a symplectomorphism from $\mathcal{K}^\infty$ with symplectic form governed by the twisted Poincare pairing $(a,b)^{tw,r} = \frac{\Psi^r}{r}\chi(a \otimes b \otimes e^{\sum \frac{\Psi^{rk}}{k}(E^{rk})})$ to $\mathcal{K}^\infty$ with the standard symplectic form.*
For a twisting of type II, the operator ∇ is the quantization of the translation that changes the dilaton shift from $v_r = \Psi^r(1-q)$ to $v_r = \Psi^r((1-q)e^{\sum_k \frac{\Psi^k(F^{rk}(q)-F^{rk}(1))}{k(1-q^k)}})$.

In the next chapter, we prove an analogous theorem for twistings of type III:

Theorem 2.3.2. *For a twisting of type III, the operator ∇ is a second-order differential operator representing change of the negative space of the polarization of $\mathcal{K}^\infty$, with the new polarization generated by the expression* $\frac{e^{\sum_k \frac{\Psi^k}{k}(F_{rk}(L_+,L_-)\otimes(1-L_+L_-)}}{1-L_+L_-}$.

To actually write down the operator relating the twisted and untwisted potential, we observe the following.

$\frac{e^{\sum_k \frac{\Psi^k}{k}(F_{rk}(L_+,L_-)\otimes(1-L_+L_-)}}{1-L_+L_-} - \frac{1}{1-L_+L_-}$ is in $\mathcal{K}_r^+ \otimes \mathcal{K}_r^+$, and as such represents a quadratic Hamiltonian of type $p_\alpha p_\beta$. Dualizing it using the symplectic form yields a map $A_- : \mathcal{K}_r^- \to \mathcal{K}_r^+$. The time-1/2 flow of this quadratic Hamiltonian is the symplectomorphism $(q,p) \mapsto (q + A_- p, p)$, sending the negatives space $q = 0$ to $q = A_- p$. Regarding the tensor as one living in $\mathcal{K}^r \otimes \mathcal{K}^r$, A_- can also be regarded as a map $A : \mathcal{K}^r \to \mathcal{K}^r$, sending (q,p) to $(Ap, 0)$.

This change of polarization is the desired one, since the map $\mathcal{K}_r \to \mathcal{K}_r$ given by dualizing $\frac{e^{\sum_k \frac{\Psi^k}{k}(F_{rk}(L_+,L_-)\otimes(1-L_+L_-)}}{1-L_+L_-}$ is $A - P$, where P is the identity on $\mathcal{K}_+$ and sends $\mathcal{K}_-$ to 0. Since we observe that $(A-P)(q,p)) = (A_- p, 0) - (q.0)$, the kernel of this map is $(A_- p, p)$, which is exactly the new negative space.

When calculating the new negative space, the $\frac{-1}{1-L_+L_-}$ term is ignored in some other parts of the literature, which does not change the outcome, since when the domain is restricted to $\mathcal{K}_r^-$, dualizing $\frac{1}{1-L_+L_-}$ yields the zero map.

Remark. We must once again address the case of the addition to $\mathcal{F}_0^X$. The term added is now $\frac{1}{8}(\Psi^2(t_2)(1), f(-1), f(-1))^{tw}$, where tw denotes the Poincare pairing on X twisted by the type-I twisting class, and f denotes the changed dilaton shift in the $r = 1$ coordinate. This term agrees with the correction in the untwisted case, since $\frac{1}{8}(\Psi^2(t_2)(1), 2, 2) = \frac{1}{2}(\Psi^2(t_2)(1), 1)$.

An important special case that warrants its own interpretation is when the twistings are a multiplicative (stable) characteristic class C applied to the virtual tangent bundle T.

Using the decomposition $T^{vir} = ft_* ev_{n+1}^*(TX - 1) + ft_*(1 - L_{n+1}^{-1}) - (ft_* \mathcal{O}_{\mathcal{Z}})^*$, we prove the following corollary:

Theorem 2.3.3. *Let C be an invertible multiplicative characteristic class defined by $C(V) = e^{\sum_{k\in\mathbb{Z}/0} \frac{\Psi^k}{k}}(s_k V)$, for coefficients $s_k \in \Lambda$. Twisting the (ordinary, not S_n-equivariant) K-theoretic potential by $C(T^{vir})$ amounts to the following changes (the symplectic loop space we consider is now just $\mathcal{K}$):*

Type I: The multiplication operator is the asymptotic expansion of $C((TX-1)/(1-q))$ (regarded as a function with poles at roots of unity), which scales the Poincare pairing by $C(TX-1)$.

Type II: The dilaton shift is $(1-q)/(C(q^{-1})$.

Type III: The new polarization is determined by the asymptotic expansion of the symmetric function $C^*(L_1 \otimes L_2)/C(1)(1 - L_1 L_2)$, where C^* denotes the characteristic class given by $C^*(V) = C(V^*)$.

2.4 Example: Hirzebruch K-Theory

Given a vector bundle V with the $\mathbb{C}^*$-action given by scaling the fibers, the equivariant K-theoretic Euler class is $C_y(V) = \sum_p (-y)^p \bigwedge^p(V^*)$. These invariants appear in [9], in the context of Grassmanians.

Complete Λ with respect to the variable y and declare $\Psi^r(y) = y^r$. The class $C_y(L) = 1 - L^{-1} = \sum_{k<0} \frac{\Psi^k}{k}(L/y)$.

Since $E_{rk} = E_k$ for any $r > 0$, the expression determining the twisting in the rth block is the same for each r. The resulting operator accomplishing this twisting is equivalent to making the following alterations to the symplectic space $\mathcal{K}^\infty$:

- Changing the K-theoretic Poincare pairing to the pairing $(a, b)_y = \frac{1}{1-y}\chi(X; C_y(T_X)ab)$.
- Changing the dilaton shift to $\frac{1-q}{1-yq}$ in each block.
- Changing the polarization to the one determined by $\frac{1}{(1-y)} \frac{1-yL_+L_-}{1-L_+L_-}$

We can calculate the negative polarization explicitly. The change of polarization map is:

$$f \mapsto -\operatorname{Res}_{0,\infty} \frac{1}{1-y} \frac{f(q)(1-yq^{-1}x)}{q-x} \frac{dq}{q},$$

which sends $f \in \mathcal{K}^\infty_-$ to $f + \frac{y}{1-y}f(0)$, so the new negative space is

$$\{f : f(\infty) = yf(0) \neq \infty\}.$$

Chapter 3

Proof of Nodal Twisting Theorem

Adelic Formula for $\mathcal{D}_X$

We recall the adelic formula for $\mathcal{D}_X$, which recasts the K-theoretic potential into purely cohomological terms. The proof of theorem 2.2 will rely heavily on this formula.

We define the adelic symplectic loop space $\underline{\mathcal{K}}^\infty = \bigoplus_{M \in \mathbb{Z}_+} \mathcal{K}^{fake}(X \times B\mathbb{Z}_M)$, where $\mathcal{K}^{fake}(X \times B\mathbb{Z}_M)$ denotes the loop space of the fake quantum K-theory of the orbifold $X \times B\mathbb{Z}_M$. Each summand splits as a direct sum of M sectors $\mathcal{K}_M^\zeta$ labelled by roots of unity ζ, each isomorphic to $K((q-1))$.

The symplectic structure on $\mathcal{K}^{fake}(X \times B\mathbb{Z}_M)$ comes from an additional twisting of fake quantum K-theory which we outline later, and is described as follows: The symplectic form Ω^{tw} pairs $\mathcal{K}_M^\zeta$ with $\mathcal{K}_M^{\zeta^{-1}}$ by $\Omega^{tw}(f,g) = \frac{1}{M}(f(q), g(q^{-1}))^{(r)}$, where $(\Psi^r a, \Psi^r b)^{(r)} = r\Psi^r(a,b)$, for (a,b) the usual Poincare pairing, and r is the index of $\langle\zeta\rangle$ in $\mathbb{Z}_M$. Let $m(\zeta) = \frac{M}{r(\zeta)}$ denote the primitive order of ζ.

Define the adelic potential $\underline{\mathcal{D}}_X$ to be $\bigotimes_M \mathcal{D}^{tw}_{X \times B\mathbb{Z}_M}$.

We can resum the component spaces according to r, to describe the adelic space as:

$$\bigoplus_\zeta \bigoplus_r \mathcal{K}_r^\zeta$$

Where the first sum is taken over all roots of unity.

After resumming, the symplectic form becomes

$$\underline{\Omega}^\infty(f,g) = \sum_\zeta \frac{1}{m(\zeta)} \sum_r Res_{q=1}(f_r^\zeta(q^{-1}), g_r^{\zeta^{-1}}(q))^{(r)} \frac{dq}{q}.$$

The adelic map $\Phi : (f_1, \dots) \mapsto \Psi^r(f_r(\frac{q^{\frac{1}{m}}}{\zeta}))$ defines a symplectic Ψ-linear transformation between $\mathcal{K}^\infty$ and $\underline{\mathcal{K}}^\infty$, which respects positive, but not negative polarizations, since an element of $\mathcal{K}_-^\infty$ will not be polar at every root of unity.

A result of [7] is that $\langle \mathcal{D}_X \rangle = \Phi^* e^{\hbar/2 \sum_{r,\zeta\eta\neq 1} \nabla_{r,\zeta,\eta}} \underline{\mathcal{D}}_X$, where $exp(\hbar/2 \sum_{r,\zeta\eta\neq 1} \underline{\nabla}_{r,\zeta,\eta})$ is the quantization of the rotation changing the standard polarization on $\underline{\mathcal{K}}^\infty$ to the uniform polarization, which is determined by the image of $\mathcal{K}_-{}^\infty$ under Φ.

This formula has the form of Wick's summation over graphs, and arises from the application of the Lefschetz-Kawasaki-Riemann-Roch theorem to $\mathcal{D}_X$. The theorem states that for $\mathcal{X}$ a orbifold, V an orbibundle, and h a discrete automorphism of $\mathcal{X}$ that lifts to V:

$$str_h(\mathcal{X}; V) = \chi^{fake}\big(I\mathcal{X}^h; \frac{tr_{\tilde{h}}(V)}{str_{\tilde{h}} N^*_{I\mathcal{X}^h|\mathcal{X}}}\big)$$

Here $\tilde{h}$ some lifting of h on each component of $I\mathcal{X}^h$, and $\chi^{fake}(A; V)$ is defined to be $\int_A \mathrm{ch}(V)\,\mathrm{td}(TA)$, i.e. the pushforward in fake K-theory. This theorem is consequence of the usual Kawasaki-Riemann-Roch theorem, which was shown by Tonita in [16] to hold for virtually smooth orbifolds.

We recall from [7] the following description of $IX^h_{g,n,d}$:

The total space itself corresponds to a moduli space of stable maps from curves $\mathcal{C}$ with a symmetry $\tilde{h}$ accomplishing the permutation h of marked points.

A connected component (henceforth referred to as a Kawasaki stratum) of this space is described by certain combinatorial data:

- A graph G dual to the quotient of the curve by the cyclic group generated by $\tilde{h}$.
- A positive integer M_v for each vertex v M_v representing the order of $\tilde{h}$ on the vertex v.
- The discrete characteristics (genus, degree) of the map on each irreducible component.
- A labelling of the vertices of G with eigenvalues of $\tilde{h}^r$ on the tangent lines to the branches at the ramification points of order r. These eigenvalues will be primitive mth roots of unity for $m = \frac{M_v}{r}$.
- A labeling of the edges of G (corresponding to nodes) with pairs of eigenvalues of $\tilde{h}^r$ on each branch to the node. We require that these eigenvalues not be inverse to each other (i.e. the node is unbalanced), so the node cannot be smoothed within the stratum.

After normalizing at the unbalanced nodes, each vertex represents a component of a Chen-Ruan moduli space of stable maps to the orbifold $X \times B\mathbb{Z}_M$. After doing this, the eigenvalue at a marked point or node also determines the sector of $I(X \times B\mathbb{Z}_M)$ in which the evaluation map at that marked point lands.

Thus the KRR formula relates a correlator to some $fake$ K-theoretic correlators of $X \times B\mathbb{Z}_M$, which are additionally twisted by the denominator terms. These account for the twistings of fake K-theory that appear in the adelic space formalism.

Marrying the vertices at edges involves the application of a propagator operator for each edge, which coincides with the change of polarization from the standard to the uniform polarization.

Twisted potentials

The exact same argument applies essentially verbatim to twisted potentials, with two differences. The vertex potentials are further twisted by the restriction of the twisting class (we label the resulting potentials $\mathcal{D}^{tw,\mathbf{E}}_{X\times B\mathbb{Z}_M}$). And, only in the case of type III twistings, the edge operators are modified as well.

Our strategy will thus be to begin with the twisted potential $\mathcal{D}^{\mathbf{T}}_X$, where $\mathbf{T}$ denotes a twisting of type III, we pass to the adelic potential $\underline{\mathcal{D}^{\mathbf{T}}_X}$, and analyze the vertex contributions coming from E to relate $\underline{\mathcal{D}^{\mathbf{T}}_X}$ and $\underline{\mathcal{D}_X}$. Then we use the adelic formula to convert that to a relationship between $\mathcal{D}^{\mathbf{T}}_X$ and $\mathcal{D}_X$, which will involve comparing the respective edge operators.

Rather than beginning with $\mathcal{D}_X$, we could take $\mathbf{T}$ to be the composition of $\mathbf{T_0}$, a twisting of type I and II, and $\mathbf{T_1}$, a twisting of type III. The resulting argument would give a relationship between $\mathcal{D}^{\mathbf{T}}_X$ and $\mathcal{D}^{\mathbf{T_0}}_X$, and is identical to the case where $\mathbf{T_0}$ is trivial, so we just work in the latter setting to minimize notation.

Vertex Contributions

Let $\widehat{\mathcal{M}}$ be a Kawasaki stratum with ambient moduli space $X_{g,n,d}$ (from which the twisting classes are inherited). Let $\mathcal{C}$ be the universal curve, and $\widehat{\mathcal{C}} = \mathcal{C}$ be the universal quotient curve by h. Let ft, ev, i denote the structure maps of $\mathcal{C}$ (the unitalicized such maps denote the ones coming from the ambient space $X_{g,n,d}$). Let the vertex and edge nodes of $\mathcal{C}$ be labelled Z_v, respectively, and label the cotangent branches by $L_\pm$. Any hatted version of the previously introduced notation refers to the corresponding construction on $\widehat{\mathcal{C}}$.

We have

$$\mathrm{ft}_*\mathrm{i}_*\mathrm{ev}^*V_k|_{\widehat{\mathcal{M}}} = ft_*i_*\mathcal{O}_{\mathcal{Z}_\mathcal{C}}F_k(L_+,L_-)Eu(N)),$$

for N some excess normal bundle bundle, and Eu the K-theoretic Euler class.

N_{Z_v} is trivial, since all vertex nodes can be smoothed within the stratum. This allows us to recast the nodal twisting restricted to 1-vertex strata solely in terms of the nodal loci of those strata.Let $\widehat{\mathcal{M}}$ now denote a stratum with one vertex and no edges.

If we denote the twisting class by S, the vertex potential is the cohomological potential of $X\times B\mathbb{Z}_M$, twisted by $\mathrm{ch}(tr_{\tilde{h}}(S|_{\widehat{\mathcal{M}}}))$, $\mathrm{td}(T\widehat{\mathcal{M}})$, and the denominator of the KRR formula, which contributes a class $\mathcal{O}^{tw}_{X\times B\mathbb{Z}_M}(T\widehat{\mathcal{M}})$. If V denotes the terms coming from the inputs in a particular twisted correlator, the contribution of of $\widehat{\mathcal{M}}$ into the Kawasaki-Riemann-Roch formula applied to that correlator is:

$$\chi^{fake}(\widehat{\mathcal{M}};\mathcal{O}^{vir}_{\widehat{\mathcal{M}}}\cdot tr_{\tilde{h}}(S\cdot V)\cdot \mathcal{O}^{tw}_{X\times B\mathbb{Z}_M}(T\widehat{\mathcal{M}})).$$

We will henceforth isolate the contribution of the locus of nodes with r copies on the covering curve, which we refer to as Z_r.

Differentiating the twisting class in E_k brings down the factor

$$\mathrm{ch}(\Delta^r_k) := \mathrm{ch}(tr_{\tilde{h}}\frac{\Psi^k}{k}(ft_*i_*F_k(L_+,L_-))).$$

Since taking the (genuine) K-theoretic pushforward from the quotient $\widehat{Z}_r = Z_r//\mathbb{Z}_M$ extracts $\mathbb{Z}_M$-invariants, we can rewrite this expression as $\mathrm{ch}(\sum_{\lambda^M=1}\lambda^k\frac{\Psi^k}{k}(\widehat{ft}_*\widehat{i}_*\widehat{F}_k(L_+L_-)\otimes \mathbb{C}_{\lambda^{-1}})$

Here $\widehat{F}$ is just F, with coefficients pulled back by $\widehat{ev}$. To simplify the expression, we make the following calculation:

Lemma 3.0.1. $\mathrm{ch}(\widehat{ft}_*\widehat{i}_*\widehat{ev}^*\alpha L_+^aL_-^b\otimes\mathbb{C}_{\lambda^{-1}})) = \begin{cases} 0 & \lambda^r \neq \zeta^{a-b} \\ \widehat{ft}_*\widehat{i}_*(\mathrm{ch}(\widehat{ev}^*\alpha L_+^aL_-^b)\,\mathrm{td}(L_+L_-^*)) & \lambda^r = \zeta^{b-a} \end{cases}$

Proof. We apply Toen's Grothendieck-Riemann-Roch theorem. The preimage of $\widehat{\mathcal{M}}$ in the inertia stack of Z_r is m copies of the node, labelled by elements of the automorphism group of the node, labelled by powers of $\tilde{h}^r$, which acts on $L_\pm$ by $\zeta^{\mp 1}$. Since L_+L_- is invariant under the $\tilde{h}^r$-action, the Todd class is invariant under $\tilde{h}^r$.

So the pushforward is equal to $\widehat{ft}_*\widehat{i}_*\,\mathrm{ch}(\sum_s \lambda^{-rs}\zeta^{(b-a)s}\widehat{ev}^*\alpha L_+^aL_-^b)\,\mathrm{td}(L_+L_-)$.

Since $\sum_{s=1}^m \lambda^{-rs}\zeta^{(b-a)s} = 0$ unless $\lambda^{-r}\zeta^{b-a}=1$, so $\lambda^r = \zeta^{b-a}$, in which case the result is:

$$\widehat{ft}_*\widehat{i}_*\,\mathrm{ch}(\widehat{ev}^*\alpha L_+^aL_-^b)\,\mathrm{td}(L_+L_-)$$

The factor m from the m copies is cancelled by the factor $\frac{1}{m}$ in the construction of the Chern character for orbifolds due to the size of the automorphism group at the node. □

Since the Chern character intertwines Adams operations and cohomological power operations (denoted here P^k), the contribution of the term $\widehat{ev}^*\alpha L_+^aL_-^b$ of $\widehat{F}_k$ to $\mathrm{ch}(\Delta_k^r)$ can be described as the following cohomological pushforward:

$$\sum_{\lambda^M=1,\lambda^r=\zeta^{b-a}} \lambda^k\frac{P^k}{k}\big((\widehat{ft}\circ\widehat{i})_*\,\mathrm{ch}(\widehat{ev}^*aL_+^aL_-^b\otimes\mathbb{C}_{\lambda^{-1}})\,\mathrm{td}(L_+L_-)\big)$$

If $\lambda^r = \zeta^{b-a}$ and λ is an Mth root of unity, we necessarily have that $r|M$. So we can relabel k as rl_0. Collecting the r terms corresponding to the eigenvalues with $\lambda^r = \zeta^{b-a}$ terms yields that the above expression is equal to:

$$\zeta^{l_0(b-a)}\frac{P^{rl_0}}{l_0}(\widehat{ft}_*\widehat{i}_*\,\mathrm{ch}(\widehat{ev}^*\alpha L_+^aL_-^b)\,\mathrm{td}(L_+L_-)$$

Since orbifold Gromov-Witten theory uses the cotangent lines $\widehat{L}_\pm$ on the quotient curve, we rewrite $L_\pm$ as $\widehat{L}_\pm^{\frac{1}{m}}$, which is valid in fake K-theory even though such a bundle may not exist genuinely. Pulling back $\Delta_{rl_0}^r$ to $\widehat{Z}_r$, renaming $\frac{\Psi^{l_0}}{l_0}(\widehat{F}_{l_0})$ to S_{l_0}, and reverting to the notation of fake K-theory yields:

$$\widehat{i}^*\widehat{ft}^*\Delta_{rl_0}^r = \Psi^r\big(\widehat{ev}^*S_{rl_0}(\zeta^{-1}\widehat{L}_+^{\frac{1}{m}},\zeta\widehat{L}_-^{\frac{1}{m}})(1-\widehat{L}_+^{\frac{l_0}{m}}\widehat{L}_-^{\frac{l_0}{m}})\big)$$

The factor $(1-\widehat{L}_+^{1/m}\widehat{L}_-^{1/m})$ occurs from pulling back $\widehat{i}_*\widehat{f}t_*\mathcal{O}_{\widehat{Z}_r}$, and is the K-theoretic Euler class of the normal bundle of $\widehat{Z}_r$ in $\widehat{\mathcal{M}}$.

To compute the correlator as an integral on $\widehat{Z}_r$, we use the general formula for a morphism $Y \to X$:

$$\chi^{fake}(X;V) = \chi^{fake}(Y; \frac{f^*(V)}{Eu(N_f)}).$$

If we label the twisting class, contributions from the KRR denominators, and correlator inputs together as B, we thus have:

$$\chi^{fake}(\widehat{\mathcal{M}}; \Delta_k^r \cdot B \cdot \mathcal{O}^{tw}_{X\times B\mathbb{Z}_M}) =$$

$$\chi^{fake}\big(\widehat{Z}_r; \frac{\Psi^r(S_{rl_0}(\zeta^{-1}\widehat{L}_+^{1/m}, \zeta\widehat{L}_-^{1/m})(1-\widehat{L}_+^{l_0/m}\widehat{L}_-^{l_0/m}))\cdot \widehat{i}^*\widehat{ft}^*(B\cdot\mathcal{O}^{tw}_{X\times B\mathbb{Z}_M})}{1-\widehat{L}_+^{1/m}\widehat{L}_-^{1/m}}\big).$$

Ungluing the nodes and integrating over the moduli spaces of component curves yields an order-2 recurrence relation on the correllators, in which the tensor $\frac{S_{rl_0}(\zeta^{-1}L_+,\zeta L_-)(1-L_+^{l_0}L_-^{l_0})}{1-L_+L_-}$ is split among the points that were unglued and inserted in the corresponding seats. However, since we need the virtual structure sheaves of the components to match $\mathcal{O}^{tw}$ (i.e. also include the KRR denominators), we must also replace the denominator $1-L_+L_-$ with $\Psi^r(1-L_+L_-)$. This follows from the explicit calculation of $\mathcal{O}^{tw}$ in [7], and accounts for the fact that deformations of a node on the quotient curve correspond to coherent deformations of the r preimages on the covering curve, whereas in general they can be deformed independently.

A more detailed account of how ungluing the nodes interacts with cohomological nodal twisting classes is given in [17] (see Proposition 3.9) for the case where F_k are constants, the addition of nonconstant terms does not alter the argument.

The differential operator determined from this recurrence adds a factor of $\hbar^r/2$, due to the symmetry between L_+ and L_-, and the genus reduction (one node on the quotient curve corresponds to r nodes on the covering curve).

So the potential $\mathcal{D}^{tw,\mathbf{E}}_{X\times B\mathbb{Z}_M}$ satisfies the same differential equation as $\underline{\nabla}_r\mathcal{D}^{tw}_{X\times B\mathbb{Z}_M}$, where $\underline{\nabla}_r$ corresponds to changing the polarization in the sectors of order r and eigenvalue ζ using the expression $\Psi^r\big(e^{\sum\frac{\Psi^l}{l}F_{rl}(\zeta^{-1}\widehat{L}_+^{\frac{1}{m}},\zeta\widehat{L}_-^{\frac{1}{m}})}\big)$.

$\mathbb{Z}_2$-symmetries

We now discuss the role of the anomalous term in the definition of the total descendant potential. Consider the case where h^r fixes a node but swaps the two branches. More precisely, the covering consists of three components, corresponding to the left branch, the right branch, and a contracted $\mathbb{C}P^1$ with two nodes. h^r acts by swapping the branches (and the nodes). Any order 2 automorphism of $\mathbb{C}P^1$ has two fixed points (e.g. consider $z\mapsto -z$, which fixes 0 and ∞), which become marked when passing to the quotient. So the quotient curve has a balanced node (with eigenvalues ± 1), and 2 marked points. However, the covering curve may not have those points marked, and hence the corresponding component would not be stable.

More precisely, the moduli space of quotient curves is the component of $\overline{\mathcal{M}}_{0,3,0}(X/\mathbb{Z}_2)$ where the first two marked points must have non-trivial orbifold structure. Thus the moduli space is the component of the Chen-Ruan space

$$\mathcal{M} := \overline{\mathcal{M}}_{0,3,0}(X \times B\mathbb{Z}_2)$$

corresponding to the sectors $(-1,-1,1)$ of $I(X \times B\mathbb{Z}_2)$.

There is only one such orbicurve, namely the quotient of $\mathbb{P}^1$ by the map $z \mapsto z^2$. The two orbifold points are at $0, \infty$, the nonorbifold point is at 1. Thus $\mathcal{M} \cong X \times B\mathbb{Z}_2$, and the universal cotangent line bundle is trivial.

The untwisted adelic formula compares the contibution of the covering curve $\mathbb{P}^1$ with the contribution of the quotient curve. However we have to deal with the possibility that the two marked points were not marked on the covering curve, making it unstable.

Formally, the adelic formula accounts for this do this into each of the inputs the expression $1-\zeta L^{-r/m}$), for ζ the eigenvalue and r the covering curve. value of the contribution is (since all cotangent line bundles are trivial):

$$\chi^{fake}(\mathcal{M}; \mathbb{T} \otimes (1-\zeta L_1^{1/2})(1-\zeta L_2^{1/2})\Psi^2(ev_3^* t_2) = \chi^{fake}(X \times B\mathbb{Z}_2; \mathbb{T}(2)(2)\Psi^2(t_2(1))$$

Here $\mathbb{T}$ is the twisting class from the Kawasaki-Riemann-Roch denominator. $\mathcal{M}$ parameterizes a certain class of $\mathbb{Z}_2$-fixed curves in an ambient moduli space, which in this case is $X_{0,4,0} \cong X \times \mathbb{P}^1$, with $\mathbb{Z}_2$ acting by cyclic permutation of a pair of marked points. The normal bundle of $\mathcal{M} \in X_{0,4,0}$ is just the restriction of the tangent bundle of $\mathbb{P}^1$, which is a trivial line bundle, on which h acts with eigenvalue -1. So the Kawasaki-Riemann-Roch denominator is $\frac{1}{1-(-1)1} = \frac{1}{2}$.

Thus the contribution of such curves is $\frac{1}{2}\chi^{fake}(X \times B\mathbb{Z}_2; (2)(2)\Psi^2(t_2(1))$. Changing from the Poincare pairing on $X \times B\mathbb{Z}_2$ to X adds another factor of $1/2$. This term appears in the adelic formula with coefficient $\frac{1}{3!} = \frac{1}{6}$, but there are three such choices of moduli space (depending on which marked point is allowed to have orbifold structure), so we get another factor of $\frac{1}{2}$, yielding a total contribution of:

$$\frac{1}{8}(\Psi^2(t_2)(1), 2, 2)$$

However, this contribution does not appear on the left side of the adelic formula, since the covering curve is not stable.

In the case of twisted potentials, the argument is essentially the same. The only changes are that the input $1-\zeta L^{1/2}$ is replaced $f(\zeta L)$, where f is the function representing the new dilaton shift in the $r = 1$ coordinate, and the Poincare pairing () is replaced by its twisted counterpart, since the type I component of the twisting class becomes $e^{\sum_k \frac{\Psi^k}{k}(ft_* ev^* V_k)} =$

$e^{\sum_k \frac{\Psi^k}{k}(V_k)}$. The modification to the dilaton shift comes from the fact that the restriction of the universal cotangent L_{n+1} is not trivialized on the points that are marked on the quotient curve but not the covering curve, so there is a correction term in each of those sectors.

Edge contributions

Recall that an edge in the graph of a Kawasaki stratum corresponds to an unbalanced node in the quotient curve corresponding to r nodes on the cover curve where $\tilde{h}^r$ acts on the tangent branches with eigenvalues ν_+, ν_-, which are respectively primitive m_+,m_-, roots of unity, let M be the order of h on the stratum, and let $m = \frac{M}{r}$.

Fixing a particular edge e_0, we perform the same procedure as the vertices to compute the contribution of the nodal locus Z_{e_0}. The Euler factor $Eu(N)$ in the previous section becomes $1 - L_+L_-$, since smoothing the edge node is normal to $\widehat{\mathcal{M}}$.

Differentiating in E_k as before brings out the term

$$\mathrm{ch}(\Delta_k^{e_0}) = \mathrm{ch}(tr_h \frac{\Psi^k}{k}(ft_* i_* F_k(L_+L_-)(1-L_+L_-)) = \mathrm{ch}(\sum_{\lambda^M=1} \lambda^k (\frac{\Psi^k}{k}(\widehat{ft}_* \widehat{i}_* \widehat{F}_k(L_+, L_-)(1-L_+L_-)).$$

The map $\widehat{ft} \circ \widehat{i}$ is an isomorphism on coarse spaces, since every point in $\widehat{\mathcal{M}}$ has a node corresponding to the edge. At the level of stacks, the automorphism group of the node is contracted to the identity, thus the (genuine) K-theoretic pushforward only extracts h^r invariants. The term $\widehat{ev}^* L_+^i L_-^j \otimes \mathbb{C}_{\lambda^{-1}}$ only has a nonzero contribution when $\lambda^r = \mu^{-i}\nu^{-j}$.

Thus if $k = rl_0$, then

$$\widehat{i}^* \widehat{ft}^* \Delta_k^e = \Psi^r(\widehat{ev}^* S^k(L_+\mu^{-1}, L_-\nu^{-1})(1 - \mu^{-1}\nu^{-1}L_+L_-).$$

This means that ungluing the edge nodes is done by applying the operator: $e^{\sum_{edges} r\Psi^r(\hbar/2\underline{\nabla}_{\mu,\nu})}$, where

$$\underline{\nabla}_{\mu,\nu} = \frac{e^{\sum_l \frac{\Psi^l}{l}(\widehat{F}_{rl}(\widehat{L}_+^{\frac{1}{m_+}}\mu^{-1}, \widehat{L}_-^{\frac{1}{m_-}}\nu^{-1})(1-\mu^{-1}\nu^{-1}\widehat{L}_+^{\frac{1}{m_+}}\widehat{L}_-^{\frac{1}{m_-}}))}(\phi^\alpha \otimes \phi_\alpha)}{1 - \mu^{-1}\nu^{-1}\widehat{L}_+^{\frac{1}{m_+}}\widehat{L}_-^{\frac{1}{m_-}}}.$$

The other ingredients here are the same as the ones calculated in [7]: The denominator is the contribution of the normal bundle of $\widehat{\mathcal{M}}$ in the denominator of Kawasaki-Riemann-Roch formula, ϕ^α, ϕ_α constitute a Poincare-dual basis of $K^0(X)$, which unglues the diagonal constraint at the nodes.

The resulting change of polarization on the adelic map pulls back to the one described in the theorem statement.

Chapter 4

Multiplicative Quantum Cobordism Theory

4.1 Complex-oriented cohomology theories and formal group laws

A complex-oriented cohomology theory is a generalized cohomology theory which admits Chern classes for complex vector bundles. For a theory A^*, a complex orientation is determined its value on the universal line bundle, which is an element u_A in $A^2(\mathbb{C}P^\infty)$. For the standard orientation of cohomology, this element is traditionally denoted z. In K-theory, we use $1-q^{-1}$, where q is the class of the universal line bundle itself.

A complex-oriented cohomology theory defines a formal group law by the rule

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

For ordinary cohomology theory, it is additive formal group law, for K-theory, the result is the multiplicative formal group law since $c_1^K(L_1 \otimes L_2) = 1 - L_1^{-1}L_2^{-1}$.

Complex cobordism theory is the cohomology theory defined by the Thom spectrum. It admits a tautological orientation u coming from the isomorphism between the Thom space of the universal line bundle and $\mathbb{C}P^\infty$. This orientation is universal in the following sense: a choice of complex orientation on a homotopy commutative ring spectrum A corresponds to a map $\phi : MU \to A$. Similarly, the formal group law associated with MU^* is the universal one, meaning that the coefficients of the defining power series are free generators of $\mathrm{MU}^*(pt)$, which is the ring of manifolds under complex cobordism. Over $\mathbb{Q}$, it a polynomial ring generated by $\mathbb{C}P^k$ in degree $-2k$.

The Chern-Dold character mentioned in the introduction is the isomorphism $U^*(X) \to H^*(X, U^*(pt))$ determined by sending $u \in U^2(\mathbb{C}P^\infty)$ to $u(z) \in H^*(\mathbb{C}P^\infty, U^*(pt))$, where $u(z)$ is the exponential of the cobordism-theoretic formal group law. Specializing to K-theory gives the series $1 - e^{-z}$, which is indeed an isomorphism from the additive to the multiplicative formal group.

The multiplicative Chern character $\mathrm{Ch}_{\mathrm{K}} : U^*(X) \to K^0(X) \otimes U^*(pt)$ is defined by $Ch \circ ch^{-1}$, and it is determined by the image $u \in U^2(\mathbb{C}P^n)$, which is a power series in $1 - q^{-1}$ which we denote by $u(1 - q^{-1})$.

As mentioned in the introduction Ch_{K} and the cobordism-theoretic pushforward map satisfy a Hirzebruch-Riemann-Roch formula, i.e. for π the map $X \to pt$, we have:

$$\pi_* \alpha = \chi(X; Ch_K(\alpha)\,\mathrm{Td}_K(TX)),$$

where the *multiplicative Todd class* Td_K is the universal stable exponential characteristic class in K-theory. It is defined on the universal line bundle by the formula $\frac{1-q^{-1}}{u(1-q^{-1})}$.

This is a consequence of a more general theorem of Dyer that gives a similar result between any two cohomology theories [6], but can be viewed more concretely as combination of the Hirzebruch-Riemann-Roch formula for U^* and the usual one relating pushforwards in K-theory and cohomology.

The logarithm of the formal group law of cobordism theory is given by Mischenko's formula as

$$z(u) = u + \sum_{n \geq 1} [\mathbb{C}P^n] \frac{u^{n+1}}{n+1}.$$

One can thus explicitly compute $u(1 - q^{-1})$ as the series inverse of $1 - e{-}z(u)$. There are some generators b_k of $U^*(pt)$ such that: $u(1 - q^{-1}) = 1 - q^{-1} + \sum_{k \geq 1} b_k (1 - q^{-1})^{k+1}$.

Similarly: $ln(\frac{1-q^{-1}}{u(1-q^{-1})}) = \sum_{k \geq 1} a_k (1 - q^{-1})^k$ for a different set of generators a_k. After completing with respect to this grading, $\sum_{k \geq 1} a_k (1 - q^{-1})^k$ can be rewritten as a series in q^{-k}, denoted $s(q) = \sum_{k \geq 0} c_k q^{-k}$. The $c_{k \geq 1}$ are independent in the completion of $U^*(pt)$, but c_0 is determined by the requirement $s(1) = 0$.

Since Td_K is multiplicative and Adams operations are additive, the formula for the multiplicative Todd class of a general bundle is:

$$\mathrm{Td}_K(\cdot) = e^{\sum_{k \geq 0} \frac{c_k}{k} \Psi^k(\cdot)}.$$

Here Ψ^0 is the rank operator. This is the universal K-theoretic characteristic class mentioned in the introduction, with the additional requirement of stability.

The stability requirement can be relaxed in the following way. Given a characteristic class C with $C(1) = t$ for t some unit, we can regard it as coming from a series $\frac{1-q^{-1}}{u(1-q^t)}$, where $u(1 - q^t)$ is a homomorphism the multiplicative group with orientation given by $(1 - q^{-t})$ instead of $(1 - q^{-1})$. This scales the logarithm $z(u)$ by a factor of $\frac{1}{t}$.

Using C and $u(1 - q^{-t})$ define new versions of Ch_{K} and Td_K, however, the resulting pushforwards have the same value as if we used the normalized version of C instead. To see this, apply the ordinary Riemann-Roch formula to rewrite $\chi(X; Ch_K(\alpha)\,\mathrm{Td}_K(TX))$ as an integral over X. The $\frac{1}{t}$ coefficients appearing from the expansion of $\mathrm{td}(\mathrm{Td}_K(TX))$ and $\mathrm{ch}(\mathrm{Ch}_{\mathrm{K}}(\alpha))$ cancel in the top degree. The same result is true in the orbifold setting, which can be shown by applying Kawasaki-Riemann-Roch and then considering top-degree terms on each stratum.

Keeping this in mind, the c_0 term in the exponential expression for Td_K can be ignored, provided we apply the above modifications consistently.

Cobordism-valued Gromov-Witten invariants

Define the algebra U to be $\widehat{U}^*(X)$, where the hat denotes completion by the grading introduced in the previous section, and further completion to ensure $u(1-q^{-1})$ is a Laurent polynomial in q (the latter may involve adding an additional variable).

Define $q(u)$ to be $e^{z(u)}$. The inputs to cobordism-theoretic correlators are drawn from $U[q(u)^{\pm}]$, regarded as a subalgebra of $U(u)$. This algebra contains u as well as $u^* := u(1-q(u))$, which represents the first U^*-theoretic Chern class of the dual to the universal line bundle.

For $\alpha_i \in U[q(u)^{\pm}]$, the cobordism theoretic correlators are defined via the right hand side of the Hirzebruch-Riemann-Roch formula, i.e.

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,d}^U = \chi(X_{g,n,d}; \mathcal{O}^{vir} \cdot \mathrm{Td}_K(T^{vir}) \prod_{i=1}^n ev_i^* \, \mathrm{Ch_K} \, \alpha_i(L_i)).$$

The genus g and total descendant potentials $\mathcal{F}_X^{g.U}$ and $\mathcal{D}_X^U$ are defined in the same way as for K-theory.

The loop space $\mathcal{U}$

We construct the space $\mathcal{U}$ in a similar manner to $\mathcal{K}$. As a U-module, $\mathcal{U}$ is defined as $U[q(u)^{\pm}]$ localized at $1-q(u)^m$ for each $m \in \mathbb{Z}_{\neq 0}$. The symplectic form is

$$\Omega^U(f,g) := Res_{q(u)=0,\infty}(f(u), g(u^*))^U dz(u).$$

$(,)^U$ denotes the cobordism-theoretic Poincare pairing.

As with $\mathcal{K}$, $\mathcal{U}_+$ is $U[q(u)^{\pm}]$, however the negative space is not the natural analogue of $\mathcal{K}_-$, consisting of functions holomorphic at 0 and vanishing at ∞. Rather, it is obtained from that space by dualizing the symmetric tensor $\frac{1}{c_1^U(L_1^* \otimes L_2^*)}$, and taking the image under the resulting linear map. With the above data, $\mathcal{D}_X^U$ defines a quantum state $\langle \mathcal{D}_X^U \rangle$ of $\mathcal{U}$ after a dilaton shift of u^*.

4.2 Formula for $\mathcal{D}_X^U$

We are now in the position to state the formula relating $\mathcal{D}_X^K$ and $\mathcal{D}_X^U$:

The *quantum multiplicative Chern character* $\mathrm{qCh_K}$, defined by extending $\mathrm{Ch_K}$ by $u \mapsto u(1-q^{-1})$, (equivalently $q(u) \mapsto q$), is a linear isomorphism from $\mathcal{U}$ to $\mathcal{K}$ (provided Λ is chosen to be $u^*(pt)$ completed appropriately). $\mathrm{qCh_K}$ is not a symplectomorphism, since it transforms the cobordism-theoretic Poincare pairing into the K-theoretic pairing with the insertion of $t \cdot \mathrm{Td}_K(TX)$. Furthermore, it does not identify dilaton shifts nor polarizations. Roughly, after correcting these discrepancies, $\mathrm{qCh_K}$ identifies the quantum states. More precisely, the following formula holds:

Theorem 4.2.1.

$$\mathrm{qCh_K}\langle D_X^U \rangle = \nabla \langle D_X^K \rangle$$

Where ∇ consists of 3 operators:

- *The quantization of the scalar multiplication by the asymptotic expansion of* $\mathrm{Td}_K(\frac{TX-1}{1-q})$, *which is regarded as a symplectomorphism from $\mathcal{K}$ with symplectic structure twisted by $t \cdot \mathrm{Td}_K(TX-1)$ to $\mathcal{K}$ with its original symplectic structure. Thus viewed, the quantization acts in the opposite direction.*

- *A translation operator on Fock space which changes the dilaton shift to from $1-q$ to* $\mathrm{qCh_K}(u^*) = u(1-q)$.

- *The quantization of a symplectomorphism of the form $(p, q) \mapsto (p, q+Sp)$, which leaves $\mathcal{K}_+$ unchanged and changes $\mathcal{K}_-$ into* $\mathrm{qCh_K}(\mathcal{U}_-)$.

$\mathrm{qCh_K}$ identifies the potentials $\mathcal{D}_X^U$ and $\mathcal{D}_X^{K,tw}$, where the twisting class is $\mathrm{Td}_K^U(T^{vir})$. We recall the decomposition of T^{vir} in $K^0(X_{g,n,d})$ proved in [4]:

$$T^{vir} = -\mathrm{ft}_*(L^{-1}-1) + \mathrm{ft}_*(\mathrm{ev}^*(T_X - 1)) - \mathrm{ft}_* i_* \mathcal{O}_{\mathcal{Z}}{}^*.$$

Thus twisting by $\mathrm{Td}_K(T^{vir})$ induces one twisting of each type.

By theorems 2.1, 2.2, and 2.3, $\langle \mathcal{D}_X^{tw} \rangle = \nabla' \langle \mathcal{D}_X \langle$, where ∇' is an operator that encodes a change of symplectic form, dilaton shift, and polarization.

So the formula is equivalent to showing that $\nabla = \nabla'$, and that $\mathrm{qCh_K}$ is a symplectomorphism, which respects dilaton shift and polarization, provided that the symplectic structure on $\mathcal{K}$ is the one determined by ∇.

Twisting by $\mathrm{Td}_K(T^{vir}) = e^{\sum_{k>0} \frac{\Psi^k}{k}(s_k T^{vir})}$, for some particular choices of s_k, results in three twistings, one of each type:

- Type I: $e^{\sum_{k<0} \frac{\Psi^k}{k}(s_k \mathrm{ft}_*(1-L^{-1}))}$

- Type II: $e^{\sum_{k<0} \frac{\Psi^k}{k}(s_k \mathrm{ft}_* ev^*(TX-1))}$

- Type III: $e^{\sum_{k<0} \frac{\Psi^k}{k}(s_k(-\mathrm{ft}_* i_* \mathcal{O}_{\mathcal{Z}})^*}$.

These result in the following changes:

- Multiplication operator and symplectic pairing: Since the twisting of type I is $\mathrm{Td}_K(\mathrm{ft}_*\mathrm{ev}^*(T_X - 1))$, the resulting multiplication operator is equivalent to changing the Poincare pairing into the following:

$$\frac{1}{\mathrm{Td}_K(1)} \mathrm{Res}_{q=0,\infty}\, \chi(X; f(q)g(q^{-1})\,\mathrm{Td}_K(T_X))\frac{dq}{q}.$$

  The operator itself is the asymptotic expansion of $\frac{\prod_{m\leq 0}\mathrm{Td}_K(\alpha_i q^m)}{\mathrm{Td}_K(q^m)}$. The residue operations on $\mathcal{U}$ and $\mathcal{K}$ themselves coincide since $\mathrm{qCh_K}(dz(u)) = \frac{1}{\mathrm{Td}_K(1)} d\log(q) = \frac{1}{\mathrm{Td}_K(1)}\frac{dq}{q}$.

- Dilaton shift: The dilaton shift changes to $(1-q)e^{\sum_{k<0}\frac{\Psi^k}{k}(s_k(1-q))}$, which is the asymptotic expansion of $(1-q)\,\mathrm{Td}_K(q^{-1}) = u(1-q) = \mathrm{qCh_K}(u^*)$.

- Change in polarization: The twisting of type III is by $\mathrm{Td}_K(-\mathrm{ft}_*\mathrm{i}_*\mathcal{O}_{\mathcal{Z}}{}^*) = 1/\,\mathrm{Td}_K^*(\mathrm{ft}_* i_*\mathcal{O}_{\mathcal{Z}})$. Here $\mathrm{Td}_K^*(V)$ denotes $\mathrm{Td}_K(V^*)$. So the expression determining the new polarization is:

$$\frac{1}{(\mathrm{Td}_K^*(1-L_+L_-)(1-L_+L_-)} = \frac{\mathrm{Td}_K(L_+L_-^*)}{\mathrm{Td}_K(1)(1-L+L_-)} = \mathrm{qCh_K}(\frac{1}{c_1^U(L_+^*L_-^*)})$$

4.3 Specialization and examples

Other cohomology theories

Over $\mathbb{Q}$, the universality of cobordism theory also holds for cohomology rings. Given a cohomology theory A, the specialization map $\phi : U^*(pt) \to A^*(pt)$ is given by sending $[\mathbb{C}P^n]$ to the pushforward to the point of the class $1 \in A^*(\mathbb{C}P^n)$. One recovers $A^*(X)$ by restriction of scalars from $U^*(X)$, which is exact over $\mathbb{Q}$.

In this way, one can in principle specialize the constructions of the previous section to any complex oriented cohomology theory, and thus define Gromov-Witten invariants valued in that theory. However, since we use a completed version of U^*, the map $\mathrm{qCh_K}$ and the class $\mathrm{Td}_K(T^{vir})$ will only be well-defined if ϕ factors through the completion, i.e. if $u_A(1-q^{-1})$ is actually a Laurent polynomial in q.

We can also use the same framework to define invariants for algebraically-oriented theories. Levine and Morel's theory of algebraic cobordism outlined in [12] has the same universality properties among algebraic theories as MU^* does for complex oriented ones. The necessary Riemann-Roch theorems are due to Smirnov ([15]). So the formalism we have constructed works equally well in this context.

Example: the χ_{-y}-genus and Hirzebruch K-theory

The Hirzebruch χ_{-y}-genus is a polynomial deformation of the holomorphic Euler characteristic, it is defined on complex manifolds by

$$\chi_{-y}(C) = \chi(M; \sum_p (-y)^p \Omega_X^p),$$

with a y a formal variable.

The same definition extends to virtually smooth orbifolds $\mathcal{X}$ if we interpret $\Omega_{\mathcal{X}}$ to the the virtual cotangent bundle. Recall the class $C_y(V) := \sum_p (-y)^p (V^*)^p)$ can be rewritten as $e^{\sum_{k\leq 0} \frac{1}{y^k} \frac{\Psi^k}{k}(V)^k}$. We can treat this as if it were a multiplicative Todd class, but from K-theory to itself, with a different choice of complex orientation, we call this modification of K-theory *Hirzebruch K-theory.*

We can regard the K-theoretic Gromov-Witten invariants twisted by C_y introduced in Chapter 1 as Hirzebruch-theoretic Gromov-Witten invariants of X. If L is a line bundle then $C_y(L) = 1 - yL^*$, and $C_y(1) = 1 - y$. Treating C_y as a Todd class (i.e. ratio of two Euler classes) means, that Hirzebruch theory comes from the morphism of formal group laws given by the series $u(1-q^{-1}) = \frac{1-q^{-1}}{1-yq^{-1}}$. If we complete the base algebra with respect to y, $u(1-q^{-1})$ becomes a Laurent polynomial in q.

Using the formula from earlier, the transition to Hirzebruch K-theory has the following effects on the symplectic loop space:

- The multiplication operator from the type I part of the twisting changes the Poincare pairing to $(a, b) = \chi(X; a{\cdot}b{\cdot}\mathrm{Td}_y(TX))$, and scales the symplectic form by $\mathrm{Td}_y(1) = \frac{1}{1-y}$.
- The dilaton shift becomes $u(1-q) = \frac{1-q}{1-yq}$, demonstrating formal group inversion.
- The subsequent polarization changes to the one determined by $\frac{1}{1-y}u^*(L_+L_-) = \frac{1-yL_+L+_-}{1-L_+L_-}$.

These agree with the corresponding calculations in chapter 1 for $r = 1$.

One can interpet these invariants as computing the virtual $\chi_{-y}-$genera of loci inside $X_{g,n,d}$ satisfying some enumerative constraints. For genuinely smooth orbifolds, when $y = 1$, this version of the χ_{-y}-genus becomes the ordinary topological Euler characteristic of the coarse moduli space. So in cases where $X_{g,n,d}$ is genuinely smooth, which includes cases where X is homogeneous, in particular $\overline{\mathcal{M}}_{g,n}$. Applying the corresponding approach using the cohomologically defined invariants of [5] instead yields the orbifold Euler characteristic, which is a rational number given by a weighted count of simplices. This illustrates the general principle that multiplicative cobordism-theoretic invariants will have different relationships to the orbifold structure of $X_{g,n,d}$ than "fake" ones.

However, the symplectic formalism degenerates in this limit, so any computations must be done for a general y, and then specialized. A detailed discussion of the kind of invariants that thus result will be done in the next chapter.

Chapter 5

Euler-theoretic Gromov-Witten Invariants

5.1 Introduction

There are many different generalizations of the notion of Euler characteristics to orbifolds (a family of which are documented in [3]). They are given different names throughout the literature, to eliminate ambiguity, We will specify the two we use here below. The *ordinary Euler characteristic* of an orbifold $\mathcal{X}$ with coarse space X, is just $\chi(X)$. It is integral and behaves well with respect to covering maps, provided ones uses the degree of the map on coarse spaces, rather than the orbifold degree. We denote it by $\chi(X)$.

The other Euler characteristic we consider is the *orbifold Euler characterstic*. Given an effective orbifold $\mathcal{X}$ with underlying topological space $\mathcal{X}$, we can choose a presentation $\mathcal{X}$ as a quotient of a manifold M by an almost-free action of a Lie Group G.

An *orbifold triangulation* S on $\mathcal{X}$ is a triangulation on X such that each point in the interior of a simplex Δ has the same isotropy group G_Δ under the G-action, and that each simplex is small enough to be contained in a single orbifold chart of $\mathcal{X}$. Such a triangulation always exists, by a result of Illman [10].

Definition 5.1.1 ([13])**.** The *orbifold Euler Characteristic* $\chi^{orb}(\mathcal{X})$ is defined to be (for a choice of orbifold triangulation S):

$$\sum_{\Delta \in S} \frac{(-1)^{dim(\Delta)}}{|G_\Delta|}$$

By the same arguments as the ordinary cohomological Euler characterstic for manifolds, χ^{orb} is independent of the choice of triangulation.

The orbifold Euler characteristic is related to its ordinary Euler characteristic by the following theorem:

Theorem 5.1.2. *For an arbitrary orbifold $\mathcal{X}$:*
$\chi(\mathcal{X}) = \chi^{orb}(I\mathcal{X})$

Proof. For a simplex δ with isotropy group G_δ, it appears in C_{G_δ} components of the inertia orbifold, where C_{G_δ} is the class number, and in the component corresponding to class c is counted with multiplicity $\frac{1}{|c|}$. Thus the total contribution of preimages of δ in $I\mathcal{X}$ is $\sum_{c \in Cl(G_\delta)} \frac{1}{|c|} = 1$.

□

If $\mathcal{X}$ is a compact almost complex orbifold, the orbifold Euler characteristic satisfies the Poincare-Hopf and Chern-Gauss-Bonnet theorems.

Theorem 5.1.3 ([13])**.** *For V a vector field with isolated singularities on $\mathcal{X}$:*

$$\chi^{orb}(\mathcal{X}) = \sum_{\partial \in Sing(V)} Ind_p^{orb}(V)$$

(The orbifold index of a vector field at a point is its index in an orbifold chart containing that point, divided by the order of the isotropy group at that point).

Theorem 5.1.4 ([13])**.**

$$\chi^{orb}(\mathcal{X}) = \int_{\mathcal{X}} c_{top}(T\mathcal{X})$$

As a corollary, $\chi(\mathcal{X}) = \int_{IX} c_{top}(IX)$, and both Euler characteristics are invariant under deformations.

5.2 Euler characteristics and Gromov-Witten Invariants

Motivation: Computing Euler Characteristics of Fiber Products

For the rest of this text, we use the following notation:

Given a map f between orbifolds, If will denote the induced map between inertia orbifolds.

Given a bundle V. $c(V)$ denotes the total Chern class of that bundle.

Given a morphism $f : X \to Y$, the relative tangent bunlde T_f is $T_X - f^*T_Y$.

The motivation for the definition of the Gromov-Witten invariants we consider comes from the following theorem:

Theorem 5.2.1. *Given a fiber diagram:*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f"] \arrow[d, "j"] & Y \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "h"] & Z \end{tikzcd}$$

Where $\mathcal{X}$ is a compact almost-complex orbifold, Y and Z are compact almost-complex manifolds.

If $\mathcal{M}$ (which is a priori an orbispace), has the structure of a smooth orbifold, we have:

$$\chi(\mathcal{M}) = \int_{I\mathcal{X}} c(TI\mathcal{X})Ih^* g_* c(T_g)$$

Proof. Taking inertia orbifolds on all sides gives that:

$$\begin{tikzcd} I\mathcal{M} \arrow[r, "If"] \arrow[d, "j"] & Y \arrow[d, "g"] \\ I\mathcal{X} \arrow[r, "Ih"] & Z \end{tikzcd}$$

is also a fiber diagram.

Thus

$$Ih^* g_* c(T_g) = j_* If^* c(T_g) = Ij_* c(If^* T_g).$$

Another consequence of the above diagram being a fiber square is that $If^*T_g = T_{Ij}$, so we can rewrite

$$\int_{I\mathcal{X}} c(TI\mathcal{X})Ih^* g_* c(T_g) = \int_{I\mathcal{X}} c(TI\mathcal{X})Ij_*(T_{Ij}).$$

By the projection formula above is equal to:

$$\int_{I\mathcal{M}} c(Ij^*TI\mathcal{X})c(T_{Ij}).$$

Since $T_{Ij} = TI\mathcal{M} - Ij^*TI\mathcal{X}$, this integral is equal to

$$\int_{I\mathcal{M}} c(TI\mathcal{M}) = \chi(\mathcal{M}).$$

$\square$

Application to Gromov-Witten Moduli Spaces

Let $X_{g,n,d}$ be the moduli space of degree d maps from n-pointed stable curves to X. Let $f_i : Y_i \to X$ be (almost)-holomorphic maps from (almost)-complex manifolds Y_i. Consider the diagram:

$$\begin{tikzcd} M_f \arrow[r, "\alpha"] \arrow[d, "\beta"] & \prod_i Y_i \arrow[d, "\prod_i f_i"] \\ X_{g,n,d} \arrow[r, "\prod_i ev_i"] & X^n \end{tikzcd}$$

If $X_{g,n,d}$ and M_f are orbifolds, by the the theorem in the previous section we can conclude:

$$\chi(M) = \int_{IX_{g,n,d}} c(TIX_{g,n,d}) \prod_i Iev_i^* f_{i*} c(T_{f_i}).$$

Both quantities only depend on the complex bordism class of f_i, denoted $[f_i]$. (More precisely, they depend on the image of $[f_i]$ under the map $\varphi : MU^*(X) \to H^*(X)$ induced by the map $MU^*(pt) \to \mathbb{Z}$ sending a class of a manifold to its topological Euler characteristic).

5.3 Permutation-Equivariant Invariants

Based on this idea, we will define a theory of Gromov-Witten invariants encompassing the above Euler characteristics, and the Euler characteristics of their S_n-fixed loci.

Given an element $h \in S_n$, with cycle structure given by an integer vector ℓ, with ℓ_r cycles of length r. Take inputs $\alpha_1, \ldots, \alpha_{|\ell|} \in MU^*(X)$, such that for all α in the pth cycle of length q, they are all the same, equal to $\alpha_{p,q}$.

Define the S_n-equivariant Euler-theoretic correlator $\overline{\langle \alpha_{1,1}, \ldots, \alpha_{\ell_1,1}, \ldots, \alpha_{k,\ell_k} \rangle}^{E}_{g,\ell,d}$to be

$$\prod_r r^{-\ell_r} \int_{IX^h_{g,|\ell|,d}} c_{total}(T^{vir}) \prod_{j=1}^{k} \prod_{i=1}^{\ell_j} I\widehat{ev}^*_{i,j}(\varphi(\alpha_{i,j}))$$

Here $\widehat{ev}_{i,j}$ denotes the restriction of *any* of the evaluation maps from marked points permuted by the ith length-j cycle of h. All such resctrictions are equal since we restrict to the h-fixed locus. By construction these correlators are polyadditive.

We package these correlators into genus-g generating functions in the following manner:

$$\mathcal{F}^E_{g,X} := \sum_d Q^d \sum_\ell \frac{1}{\prod_r \ell_r!} \langle \mathbf{t_1}, \ldots, \mathbf{t_r}, \ldots \rangle^E_{g,\ell,d}$$

Here the input $\mathbf{t_r} = \sum \phi^a \mathbf{t_{a,r}}$ for ϕ^a running a basis of $MU^*(X)$ (so all cycles of length r receive the same input). As with K-theoretic invariants, we can extend the input algebra to some Λ equipped with Adams operations, but in this case these operations act trivially on $MU^*(X)$, and will not currently be relevant to the applications discussed in this work.

Assume we have h acting on some disconnected curve with components $C_1, \ldots, C_k$. The corresponding moduli space only has non-trivial h-fixed locus if h cyclically permutes the k

components, and h^b acts on each as a permutation of the marked points on that curve. The union of the curves also has Euler characteristic $k(2-2g)$.

Motivated by this fact, we define the total descendant potential $\mathcal{D}_X^E$ to be

$$exp(\sum_g \sum_k \hbar^{k(g-1)} \Psi^k(R_k(\mathcal{F}_{g,X}^E))/k)$$

. The R^k changes the input $(\mathbf{t_1}, \ldots, \mathbf{t_r}, \ldots)$ into $(\mathbf{t_k}, \ldots, \mathbf{t_{rk}}, \ldots)$, which are there to correct for the fact that the cycles of h are all k times longer than the corresponding cycle of h^k.

Remark. As in the K-theoretic case, we add to $\mathcal{F}_0^E$ an anomalous term, in this case $\frac{1}{4}(\Psi^2(t_2), 1)$, here $(,)$ denotes the Euler-theoretic Poincare pairing. There is an adelic type formula for these invariants, and this modification serves the same purpose as in the K-theoretic case. (As ever for Euler theoretic invariants, the Adams operations do not actually do anything here unless the coefficients are taken from some nontrivial $\Lambda-$algebra).

Remark. The inputs come from cobordism theory, but in the integral, they are specialized to the cohomology theory with the total Chern class playing the role of the Todd class. We will henceforce refer to this theory as *Euler Theory*.

Incomplete Invariants

An advantage of the Euler characteristic is that it is additive, so to define an integral representing contributions from the interior of the moduli space, we can simply subtract the corresponding Euler characteristics from the boundary. We call these "incomplete" Euler invariants. They are denoted without bars, i.e. the incomplete Euler correlator is denoted $\langle [f_1], \ldots, [f_n] \rangle_{g,\ell,d}^E$, and is "defined" by:

$$\langle [f_1], \ldots, [f_n] \rangle_{g,\ell,d}^E := \overline{\langle [f_1], \ldots, [f_n] \rangle}_{g,\ell,d}^E - \text{boundary contributions}$$

For a precise description of these correlators, we introduce the following notation: Write $t = \sum_a t_a \phi^a$, for ϕ^a a basis of $MU^*(X)$. Denote $\frac{\partial}{\partial t_{a,r}}$ by $\partial_{a,r}$, and let $g^{a,b}$ be the matrix of the specialization of the MU^*-theoretic Poincare pairing to Euler theory. Given a permutation h with cycle structure ℓ, boundary components of the associated moduli space $X_{g,n,d}^h$ arise from curves splitting into components along cycles of nodes, rather than individual nodes themselves.

A given codimension-d boundary stratum is a fiber product of lower-dimensional moduli spaces : $X_{g_1,|\ell_1|,d_1}^{h_1} \times_{ev} (X_{0,3,0})^{r_1} \times_{ev} X_{g_2,|\ell_2|,d_2}^{h_2} \times \ldots X_{g_k,|\ell_k|,d_k}^{h_k}$, together with an assignment I of the locations of the original marked points, and cycles corresponding to genus reductions. Call the set of such I $B(X_{g,|\ell|,d}^h))$, each $I \in B(X_{g,|\ell|,d}^h)$ determines a unique closed boundary stratum X_I, as well as an "open" boundary stratum M_I, obtained by replacing the moduli spaces in the fiber product with their uncompactified counterparts. The same statement is true at the level of inertia orbifolds, since passing to inertia orbifold preserves the fiber diagram, and $X_{0,3,0}$ is a manifold. In more concrete terms, an auotmorphism of a reducible curve fixing the marked points and nodes is nothing more than an automorphism of each of the components, fixing the marked points and attaching point. This means that the same procedure of "ungluing" boundary strata in the calculation of Gromov-Witten invariants applies in this setting.

Fixing such a stratum X_I, we denote the contribution to the correlator (with original inputs $\boldsymbol{\alpha} = (\alpha_{1,1}, \dots))$ by $C^E_{\boldsymbol{\alpha}}(X_i)$, which is defined by:

$C_{\boldsymbol{\alpha}}(X_I) := \prod_{i=1}^k \overline{\langle \alpha_I \rangle}^E_{g_i,n_i,d_i}$.

Here the insertion α_I into a correlator means that inputs are determined according to the assignment I, with inputs corresponding to a cycle of nodes of length r receiving $\sum r g^{a,b} \phi^a \otimes \phi^b$ (either as an individual inpout in the case of nodes corresponding to a genus reduction, or split between the two correlators for reducible nodes. We will omit the subscript $\boldsymbol{\alpha}$ when the inputs are obvious. Using this notation, we can formally define the incomplete correlators:

$$\langle \alpha_{1,1}, \dots \rangle^E_{g,n,d} := \sum_I (-1)^d C^E_\alpha(X_I).$$

The factor r in the input for nodes accounts for the normalization coefficient $\prod_r r^{-\ell_r}$, as each node contributes an additional cycle of length r which is not present in ℓ.

Remark. We can also rewrite these equations in terms of the "open" boundary strata, which will be useful later. Given the same labelling data, let M_I be the "open" part (i.e. ignoring the singular locus in each term in the fiber product) of X_I. Define $C(M_I)$ in the same way as $C(X_I)$, but with all complete correlators replaced with incomplete ones. Since the open boundaries do not overlap we can rewrite the previous equation as:

$$\langle \alpha_1, \dots, \alpha_n \rangle^E_{g,n,d} = \overline{\langle \alpha_1, \dots, \alpha_n \rangle}^E_{g,n,d} - \sum_I C^E(M_I).$$

Denote the total descendent potential of these invariants as $\mathcal{E}_X$.

Theorem 5.3.1.

$$\exp(-\bigoplus_r \frac{1}{2} r \hbar^r \sum_{a,b} g^{a,b} \partial_{a,r} \partial_{b,r}) \mathcal{D}^E_X = \mathcal{E}_X$$

We will first consider the case where $\mathbf{t_r} = 0$ for all $r \geq 1$, i.e. we consider the ordinary, rather than S_n-equivariant theory.

First note

$$\sum_{a,b} g^{ab} \hbar \partial_a \partial_b exp(\sum_g \hbar^{g-1} \mathcal{F}^E_{g,X}) = \\ \sum_{a,b} g^{ab} \mathcal{D}^E_X(\mathbf{t})(\hbar \sum \hbar^{g-1} \mathcal{F}^E_{g,X}(\mathbf{t}, \phi^a) \mathcal{F}^E_{g,X}(\mathbf{t}, \phi^b) + \mathcal{D}^E_X(\mathbf{t})(\sum \hbar^g) \mathcal{F}_{g,E}(\boldsymbol{t}, \phi^a, \phi^b) \tag{5.1}$$

This is precisely the generating function for invariants of disconnected curves, where the contribution from one component is replaced with the contribution from its codimension-1 boundary.

The placement of the extra $\hbar$ ensures that a product of correlators from curves with genus g_1 and g_2 each with one fixed insert has coefficient $\hbar^{g_1+g_2-1}$, so it is treated as coming from a degeneration of a curve with genus $g_1 + g_2$, and similarly, a correlator with 2 fixed inserts has coefficient $\hbar g$, so it is treated as coming from a genus $g+1$ curve. The factor $1/2$ accounts for the symmetry between the branches.

Similarly $\frac{1}{n!}(\frac{\hbar}{2}\partial_a\partial_b)^n$ accounts for contributions from the virtual codimension n boundary components, the $\frac{1}{n!}$ coefficient corrects for the order in which the insertions are taken.

The argument is essentially the same for the general case. In the case of invariants coming from a permutation h with cycle structue ℓ, the codimension-1 boundary strata of $X^h_{g,|\ell|,d}$ correspond to a cycle of marked points coinciding with another cycle. h restricts on each component to a symmetry h_1, h_2, with cycle structures $(\ell_1, C), (\ell_2, C)$, where C represents a single cycle of length r.

Splitting along such a cycle is accounted for entirely by the operator

$$\Delta := exp(-\frac{1}{2}\hbar^r r \sum_{a,b} g^{a,b}\partial_{a,r}\partial_{b,r}),$$

,the $\hbar^r$ to account for the change in Euler characteristic by r points, and the additional factor of r to correct the normalization term $\prod_r r^{-\ell_r}$, since the terms associated to (ℓ_1, C) and (ℓ_2, C) differ from the term associated to ℓ by a factor of r.

5.4 Geometric Interpretation

For this section, we provide a literal geometric interpretation of Euler-theoretic correlators in the case where $X_{g,n,d}$ is a smooth orbifold (i.e. $X = pt$ or $g = 0$ and X is convex). Consider a diagram of the form:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "h"] \arrow[d, "j"] & \prod_i Y_i \arrow[d, "F"] \\ X_{0,n,d} \arrow[r, "ev"] & X^n \end{tikzcd}$$

If $\mathcal{M}$ is an almost-complex orbifold, and h, d are smooth and almost holomorphic, then the Gromov-Witten invariant $\overline{\langle [f_1], \ldots, \rangle}^E_{0,n,d}$ computes $\chi(\mathcal{M})$. However, for general maps f_i, $\mathcal{M}$ will be an orbispace (or a singular Deligne-Mumford stack), and no such direct interpretation is possible.

Working in the smooth category, there is a notion of transversality introduced by Schmalz in [14] for maps from orbifolds to manifolds that gives sufficient conditions for $\mathcal{M}$ to be an orbifold.

Definition 5.4.1 ([14])**.** Given a map $f : \mathcal{X} \to M$, for $\mathcal{X}$ an orbifold and M a manifold, and Y a closed submanifold whose incluson into M is given by the embedding $i : Y \to M$, then f is *transverse* to Y if for a choice of orbifold charts U_i, the map $U_i \times Y \to M \times M$ is transverse to the diagonal in $M \times M$.

In [14], Schmalz also showed that this notion of transversality retains the expected properties:

- If f is transverse to Y, then $f^{-1}(Y)$ is a suborbifold of $\mathcal{X}$.
- Given f, M, Y, i as in the previous definition, one can deform the inclusion $i : Y \to M$ by homotopy to $i' : Y \to M$ such that f is transverse to Y'.
- If $f^{-1}(i(Y))$ is compact, we can choose i' so that property is preserved.

Using this notion, we will prove the following theorem:

Theorem 5.4.2. *If $X_{0,n,d}$ is a smooth orbifold, the correlator $\overline{\langle [f_1], \dots, \rangle}^E_{0,n,d}$ computes the Euler characteristic of the orbifold $X_{0,n,d} \times_{ev} X \times_{F'} \prod_i Y_i$, where F' is some smooth map homotopic of $\prod_i f_i$, chosen so that $F' \times ev$ is transverse to the diagonal.*

Proof. First, we deform $\Delta \in X \times X$ to Δ', so the map $\prod_i f_i \times ev$ is transverse to Δ'. However, we can regard Δ' as the graph of a map $g : X \to X$ that is homotopic to the identity, and the statement $f_i \times ev$ is transverse to Δ' becomes equivalent to $g \circ \prod_i f_i \times ev$ is transverse to Δ. We call $g \circ \prod_i f_i$ F'.

We now observe that transversality of the maps F', ev means that the preimage of the diagonal, which is $M' := X_{0,n,d} \times_{ev} X \times_{F'} \prod_i Y_i$, is indeed an orbifold. The maps h and j in the below diagram are smooth, since they are compositions of the inclusion of M' into the product, with projections onto each factor.

$$\begin{tikzcd} \mathcal{M}' \arrow[r, "h"] \arrow[d, "j"] & \prod_i Y_i \arrow[d, "F"] \\ X_{0,n,d} \arrow[r, "ev"] & X^n \end{tikzcd}$$

If all the maps in the above diagram were almost holomorphic, the result would follow immediately from the arguments in section 5.2 Unfortunately, we cannot necessarily guarantee that g (and thus F') will be almost holomorphic. However, there is a way around this, F' is almost holomorphic after giving X the almost-complex structure determined by pushing forward the original complex structure by dg. Since g is homotopic to the identity, this new complex structure is connected by some path to the original one, and thus determines the same Chern classes. Thus for the purposes of evaluating the correlator, we can treat all maps as if they were holomorphic, since the only integrals being taken are of Chern classes. We can also replace all the orbifolds in the diagram with their inertia orbifolds, thus allowing us to conclude:

$$\int_{IX_{0,n,d}} c(TIX_{0,n,d}) \prod f_{i*}c(T_{f_i}) = \int_{IX_{0,n,d}} c(TIX_{0,n,d}) \prod F'c(T_{F'}) = \int_{IM'} c(TIM') = \chi^(M')$$

□

Remark. The analogous result is true for the the permutation-equivariant case, $n = |\ell|$ for some partition ℓ, and the $[f_i]$ are chosen such that they are compatible with ℓ, then the corresponding permutation-equivariant correlator represents the Euler characteristic of the h-fixed locus of M_f, deformed as above. Since the fixed-point loci of a finite group acting on a smooth orbifold are also smooth orbifolds, the above calculation goes through essentially unchanged.

The interpretation of these results are as follows. If f_i are all embeddings $Y_i \to X$, then the corresponding Euler-theoretic Gromov-Witten invariant computes the Euler characteristic of the locus of curves of ev_i maps into Y_i, for sufficiently generic deformation of $\prod f_i$. If $Y_i = pt$, these have the same enumerative interpretation as usual Gromov-Witten invariants. Passing to incomplete invariants subtracts the contributions from nodal curves, while analogous procedures are not possible in classical Gromov-Witten theory.

In the special cases we have used, the argument shows that the Euler-theoretic invariants are integer-valued. The same is true from Gromov-Witten invariants in these cases, however, we conjecture that Euler-theoretic invariants are always integers.

5.5 Fake Euler Theoretic Invariants

We can build the exact same formalism based on the orbifold Euler charactersitc characteristic, i.e. twisting the virtual fundamental class of $X_{g,n,d}$ by $c_{total}(TX^{vir}_{g,n,d})$, rather than using the tangent bundle to the inertia stack. We do so for now without permutation-equivariance, i.e. we define correlators:

$$\overline{\langle \alpha_1, \ldots, \alpha_n \rangle}^{E,fake}_{g,n,D} := \int_{X_{g,n,d}} c(TX^{vir}_{g,n,d}) \prod_i ev_i^* \alpha_i$$

We define the complete and incomplete correlators $\overline{\langle \alpha_1, \ldots, \alpha_n \rangle}^{E,fake}_{g,n,d}$ and $\langle \alpha_1, \ldots, \alpha_n \rangle^{E,fake}_{g,n,d}$ analogously to the Euler-theoretic case. That is:

$$\overline{\langle \alpha_1, \ldots, \alpha_n \rangle}^{E,fake}_{g,n,d} := \int_{[X_{g,n,d}]^{vir}} c(T^{vir}) \prod_i ev_i^* \phi(\alpha_i)$$

and

$$\langle \alpha_1, \ldots, \alpha_n \rangle^{E,fake}_{g,n,d} := \int_{[X_{g,n,d}]^{vir}} c(T^{vir}) \prod_i ev_i^* \phi(\alpha_i) - \text{boundary contributions}$$

Call the corresponding potentials $\mathcal{D}_X^{E,fake}$ and $\mathcal{E}_X^{fake}$, the operator subtracting the boundary contributions is the same as in the non-fake case, except only involving terms corresponding to $r = 1$:

$$\Delta^{E,fake} := exp(-\frac{\hbar}{2}\sum_{a,b} g^{a,b}\partial_a\partial_b).$$

The geometric interpretation for Euler-theoretic invariants discussed in the previous section also applies verbatim to fake Euler invariants, using the orbifold Euler characteristic rather than the ordinary one.

Unlike Euler theoretic invariants, fake Euler-theoretic invariants are actually already well-understood. $\mathcal{D}_X^{E,fake}$ is (after specializing the inputs from $MU^*(X)$), mothing more than the generating function for ordinary cohomological Gromov-Witten invariants, twisted by the total Chern class of the virtual tangent bundle, and is determined from the ordinary cohomological potential $\mathcal{D}_X$ by the cohomological twisting theorems, proven by Coates in [4].

The incomplete correlators also arise this way, and do not need a cumbersome definition. We remind that the virtual tangent bundle to $X_{g,n,d}$ has the form:

$$\pi_* ev_{n+1}^* T_X - \pi_* L_{n+1}^{-1} - (\pi_* i_* \mathcal{O}_{\mathcal{Z}})^*$$

.

The first two components form the part logarithmic with respect to the singular locus, denoted T_{log}^{vir}.

By Coates' theorem again, twisting by the total Chern class of the third component is equivalent to applying the operator $\Delta_{E^{fake}}^{-1}$ (see the Appendix for an explanation of this result). So omiting the third component from the twisting is equivalent to subtracting the boundary components, hence we can alternatively write incomplete fake Euler-theoretic correlators as:

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,D}^{E,fake} = \int_{X_{g,n,d}} c(T_{log}^{vir} X_{g,n,d}) \prod_i ev_i^* \varphi(\alpha_i)$$

Remark. Such a result is to be expected, as for D a normal crossings divisor in an orbifold $\mathcal{X}$:

$$\chi^{orb}(X - D) = \int_{\mathcal{X}} c(T\mathcal{X}_{log(D)}).$$

In the next chapter, we will see a corresponding formula for genuine incomplete invariants.

5.6 Target $X \times B\mathbb{Z}_M$

The adelic formula for Euler-theoretic invariants involves fake Euler-theoretic invariants for orbifolds $X \times B\mathbb{Z}_M$. For moduli spaces of stable maps to an orbifold $\mathcal{X}$, the evaluation maps actually land in the inertia orbifold $I\mathcal{X}$. However, given the lack of an appropriate notion of bordism theory for orbifolds, the same definitions do not carry over. Instead, for the case when the target space $\mathcal{X}$ is an orbifold, we avoid interpreting the notion of a bordism class of a map to $\mathcal{X}$, and define the potentials $\mathcal{D}_{\mathcal{X}}^{E,fake}$ to be the usual Gromov-Witten potential, but with inputs taken from $H^*(I\mathcal{X}) \otimes MU^*(pt)$, and virtual fundamental cycle twisted by the classes $c(T^{vir})$.

As in Chapter 3, $I(X \times B\mathbb{Z}_M)$ is a union of M sectors, each isomorphic to $X \times B\mathbb{Z}_M$, labelled by Mth roots of unity. The cohomology (with rational coefficients) is $\bigoplus_\zeta H^*(X)h_\zeta$, where h_ζ is the fundamental class of the sector labelled ζ.

These invariants are determined for in terms of those for X in the following way: instead of using sectors, we use a basis of characters $\chi : \mathbb{Z}_M \to \mathbb{C}^*$. Where the character χ corresponds to the cohomology ckass $\sum_\zeta \chi(\zeta)h_\zeta$. For some $\gamma = \sum_\zeta \gamma_\zeta h_\zeta$, we can express γ as $\sum_\chi \gamma_\chi \chi$ where:

$$\gamma_\zeta = \frac{1}{M}\sum_\chi \gamma_\chi \chi(\zeta)$$

and

$$\gamma_\chi = \sum_\zeta \chi(\zeta^{-1})$$

Theorem 5.6.1. *Using this basis, we can move from invariants of $X \times B\mathbb{Z}_M$ to invariants of X, via the formula*

$$\mathcal{D}^{E,fake}_{X\times B\mathbb{Z}_M}(\sum t_\chi \chi, \hbar) = \prod_\chi \mathcal{D}_X^{E,fake}(t_\chi, M^2\hbar)$$

This is an adapation of Jarvis-Kimura's formula from [11] for untwisted Gromov-Witten invariants of $X \times B\mathbb{Z}_M$, a proof in the twisted case is given in the appendix. It is important to note that this formula holds at the level of literal generating functions, not just quantum states (i.e. it is not just true up to scale factor).

5.7 Adelic Formulas for Euler Invariants

The inputs to the fake Euler theory of $X \times B\mathbb{Z}_M$ are labelled t_ζ for $\zeta^M = 1$ according to the sector in which the marked point lands. In the following argument, we consider the tensor product $\bigotimes_M \mathcal{D}^{E,fake}_{X\times B\mathbb{Z}_M}$. We relabel the t_ζ input to the potential of $X \times B\mathbb{Z}_M$ to $t_{\zeta,r}$, where $r = M/ord(\zeta)$.

We use the description of the inertia stack of $X_{g,n,d}$ introduced in [7] to prove the following "adelic formula" for the $\mathcal{D}_X^E$.

$$\mathcal{D}_X^E =$$

$$[exp(\sum_r \frac{\hbar^r r}{2}\Psi^r(\nabla^r_{\zeta,\eta}) \sum_{\zeta,\eta\neq 1} \bigotimes_M \mathcal{D}^{E,fake}_{X\times B\mathbb{Z}_M}(\frac{1}{\sqrt{\hbar^M}}(\frac{\Psi^M(\mathbf{t_M})}{\sqrt{\hbar^M}}h_1 + \sum_{\zeta^M=1,\zeta\neq 1}\frac{\Psi^r(\mathbf{t}_{\mathbf{r}(\zeta)})+1}{\sqrt{\hbar^{r(\zeta)}}}h_\zeta), \hbar^M, Q^M).$$

Where

$$\nabla^r_{\zeta,\eta} = \sum_{a,b} g^{ab}\partial_{\zeta,r,a}\partial_{\eta^{-1},r,b}$$

In addition:

$$\mathcal{E}_X = \Big[\bigotimes_M \mathcal{E}^{fake}_{X\times B\mathbb{Z}_M}\Big]|_{t_{\zeta,r}=\mathbf{t_r}} \tag{5.2}$$

Remark. The Ψ^r operations here play no real role, they are there to account for the fact that the S_n-equivariant correlators are Ψ-linear but the fake correlators are linear. So in applications where no nontrivial Λ-algebra structure is present, the Adams operations can safely be ignored.

Adelic Description of $IX_{g,n,d}$

A connected component (henceforth referred to as a Kawasaki stratum) of this space is described by certain combinatorial data:

- A graph G dual to $\widehat{\mathcal{C}}$, the quotient of $\mathcal{C}$ by the cyclic group generated by $\tilde{h}$.
- A positive integer M_v for each vertex v, denoting the number of preimages in $\mathcal{C}$ of the irreducible component corresponding to v.
- The discrete characteristics (genus $\widehat{g}_v$, degree $\widehat{d}_v$, number of marked points and edges $\widehat{n}_v$) of the map on each irreducible component.
- A labelling of the vertices of G with eigenvalues of $\tilde{h}^r$ on the tangent lines to any preimage of the ramification points of order r. These eigenvalues will be primitive mth roots of unity for $m = \frac{M_v}{r}$.
- A labeling of the edges of G (corresponding to nodes) with pairs of eigenvalues of $\tilde{h}^r$ on any branch at a preimage to the node. We require that these eigenvalues not be inverse to each other (i.e. the node is unbalanced), so the node cannot be smoothed within the stratum.

After normalizing at the unbalanced nodes, each vertex represents a component of a Chen-Ruan moduli space of stable maps to the orbifold $X \times B\mathbb{Z}_M$, given by taking the quotient of the stable map to X by $\tilde{h}$. After doing this, the eigenvalue at a marked point also determines the sector of $I(X \times B\mathbb{Z}_M)$ in which the evaluation map at that marked point lands.

In the reverse direction, given any collection of stable maps coming from the vertices of a graph G. The above data gives gives a ramified $\mathbb{Z}_{M_v}$-principal bundle over the curve $\widehat{C}_v$, for each vertex v.

By imposing a diagonal constraint at each edge, these bundles can be glued $\mathbb{Z}_{M_v}$-equivariantly. The resulting total space is a curve $\mathcal{C}$ with a map to X of degree $\sum_v M_v \widehat{d}_v$, and a symmetry $\tilde{h}$ given by a generator of $\mathbb{Z}_{lcm(M_v)}$.

Using this description we can rewrite the Euler-theoretic integral over $IX^h_{g,n,d}$, in terms of integrals over these Chen-Ruan spaces. The resulting formula takes the form of Wick's summation over graphs. As we will see, the vertex contributions end up being integrals in the fake Euler theory of $X \times B\mathbb{Z}_M$.

The justification for this technique is similar than other applications, albeit slightly simpler since the invariants are already defined as integrals on the inertia orbifold, so there is no need to invoke any kind of Kawasaki formula.

Vertex and Edge Contributions

Recall that the Euler theoretic correlator $\overline{\langle t_1, \ldots, \rangle}^E_{g,\ell,d}$ is equal to

$$\int_{IX^h_{g,\ell,d}} c(T^{vir} IX^h_{g,\ell,d}) \prod_i \widehat{ev}_i^* c(\mathbf{t})$$

We will compute this integral on each component of $IX_{g,n,d}$ as in Wick's formula, first by computing it on a stratum of a single vertex, and then computing the effect of joining two vertices via an edge.

Vertices

Let $\widehat{\mathcal{M}}$ be a Kawasaki stratum of $IX_{g,n,d}$ consisting of a single vertex with 0 edges. The contribution from this stratum to the above correlator is equal to:

$$\int_{\widehat{\mathcal{M}}} c(T^{vir}\widehat{\mathcal{M}}) \prod_i \widehat{ev}_i^* \mathbf{t}|_{\widehat{\mathcal{M}}}$$

$\widehat{\mathcal{M}}$ is a connected component Chen-Ruan moduli space of maps to $X \times B\mathbb{Z}_M$ (meaning the sectors in which each marked point lands are pretedeterined). In this guise, it has evaluation maps to $X \times B\mathbb{Z}_M$ which we abusively also denote $\widehat{ev}$. This is justified since restricting $\widehat{ev}^* \alpha_i$ to $\widehat{\mathcal{M}}$ is equivalent to pulling by $\alpha_i h_{\zeta_i}$ via the ith evaluation map on $\mathcal{M}$.

So the above integral is equal to:

$$\int_{\widehat{\mathcal{M}}} c(T^{vir}\widehat{\mathcal{M}}) \prod_i \widehat{ev}_i^* \Psi^r(\mathbf{t}) h_{\zeta(i)}$$

This is an invariant in the fake Euler-theory of $X \times B\mathbb{Z}_M$. (The marked points that do not appear as a result of the quotient have input 1 in their appropriate sector)

Thus the vertex contributions of Wick's formula are of the form $\bigotimes_M \mathcal{D}^{E,fake}_{X \times B\mathbb{Z}_M}$. After some adjustments to the inputs, Novikov's variables, and Planck's constant, which we delineate below:

Inputs

First, to reconcile the exponents of $\hbar$ in both sides of the formula, since $\hbar$ is weighted by half the Euler characteristic of the *covering* curve in $\mathcal{D}^E_X$, and it is based on the same invariant *quotient curve* in $\mathcal{D}^{E,fake}_{X\times B\mathbb{Z}_M}$. This procedure ends up being essentially the same as what is done in [7].

Namely, on a stratum given by a graph G where the component of the quotient curve has genus $\widehat{g}_v$, $\widehat{n}_v$ marked points, and $\tilde{h}$ has order M_v, we can apply the Riemann-Hurwitz to the quotient map to determine the Euler characteristic of the covering curve $\mathcal{C}$. (Each edge e has r_e preimages on $\mathcal{C}$, and the ith marked point on the component corresponding to v has r_i preimages on $\mathcal{C}$.)

The result is

$$\frac{-\chi(\mathcal{C})}{2} = \sum_v M_v(\widehat{g}_v - 1) - \sum M_v(\frac{\widehat{n}_v}{2}) - \sum_v\sum_i \frac{r_i}{2} + \sum_e r_e.$$

This means that the necessary steps to correct the exponents of $\hbar$ are as follows:

To account for the first two terms, replace $\hbar$ with $\hbar^{M_v}$ in each vertex potential, then divide each input by $\hbar^{M_v/2}$.

To address the remaining terms, add a factor of $\hbar^r$ at each edge, and divide each input by an additional factor of $\hbar^{r/2}$. In addition, to correct the degree of the maps we replace Q with Q^{M_v}.

Under the present accounting, a marked point on the quotient curve with eigenvalue ζ receives the input $\mathbf{t}_{\mathbf{r}(\zeta)}h_\zeta$ if it represents a cycle of r marked points on the covering curve. Otherwise, it receives an input of $1h_\zeta$, note that non-orbifold marked points ($\zeta = 1, r(\zeta) = M$) can only occur as images of the original points, so they do not ever received the input of 1.

To account for both of the above possibilities for orbifold marked points, and to perform the corrections to $\hbar$ discussed previously, the necessary vertex contributions must be:

$$\bigotimes_M \mathcal{D}^{E,fake}_{X\times B\mathbb{Z}_M}\Big(\frac{1}{\sqrt{\hbar^M}}\big(\frac{\mathbf{t_M}}{\sqrt{\hbar^M}}h_1 + \sum_{\zeta^M=1,\zeta\neq 1}\frac{\mathbf{t}_{\mathbf{r}(\zeta)}+1}{\sqrt{\hbar^{r(\zeta)}}}h_\zeta\big), \hbar^M, Q^M\Big)$$

Remark. The case of $\mathbb{Z}_2$ symmetries works essentially the same way as for K-theoretic invariants. THe adelic formula does not involve any further twisting classes, and the correction term for unmarked points is 1, so the contribution of a moduli space of orbicurves with three marked points coming from covering curves with a $\mathbb{Z}_2$ symmetry is equal to:

$$\int_{X\times B\mathbb{Z}_2}(\Psi_2(t_2), 1, 1) = \frac{1}{2}(\Psi^2(t_2(1)), 1, 1)$$

As before, there are three such moduli spaces, and they receive a coefficient of $1/6$ in the generating function, so the total contribution of such curves is $\frac{1}{4}(\Psi^2(t_2(1)), 1, 1)$.

Edge Contributions

Recall an edge of level r of a graph G connecting vertices v_+, v_- with labels η_+, η_-, represents an $r-$tuple of nodes on the covering curve permuted cyclically by $\tilde{h}$, with $\tilde{h}^r$ acting on each branch with eigenvalues ν_+ and ν_-, which are not mutually inverse. After normalization, the remnants of e represent marked points in the Chen-Ruan spaces $\mathcal{M}_+$ and $\mathcal{M}_-$, of sector ν_+ and ν_- respectively.

As we have discussed, at each edge we multiply by $\hbar^r$ to correctly account for the Euler characteristic of the covering curve. In addition, we need to enforce that evaluation maps $\widehat{ev}_+, \widehat{ev}_-$ at each branch send the marked points connected by e to the same class in $H^*(X)$, which regarded as the $\nu_\pm$ sector in $H^*(IX \times B\mathbb{Z}_{M_\pm})$.

To account for the fact that there are r ways to equivariantly glue the two covering curves of the vertices, and the symmetry between ν_+ and ν_-, we multiply by an additional factor of $r/2$. We also add a Ψ^r to correct for scale factors, since an edge corresponds to a cycle of r nodes on the covering curve.

This means the contribution for an edge e of r with eigenvalues $\nu_\pm$ is $\frac{r\hbar^r}{2}\Psi^r(\nabla_e^{\nu_+,\nu_-^{-1}})$, where Δ_e unglues the diagonal constraint between the sectors $\nu_+, \nu_-^{-1})$.

So the formula is, written naively:

$$exp(\sum_{edges} \frac{r\hbar^r}{2}\Psi^r(\nabla_{e,r})) \bigotimes_{vertices} \mathcal{D}^{E,fake}(v). \tag{5.3}$$

If e connects two vertices, $\nabla_{e,r}$ multiplies the two potentials and glues them along the ν, ν^{-1}. If e is a loop, Δ_e glues the two correlator series. Since we have the formula $e^F_{xy} = F_z F_y e^F + F_{xy} e^F$, applying a quadratic differential accounts for both kinds of insertion. For $r > 1$, the operator includes a factor of r to account for the r ways of choosing which point to label "1" when gluing the covering curves. Similarly, the operators receive a factor of $\hbar^r$ to correctly account for the Euler characteristic of the covering curve. If we denote by S the subspace of adelic inputs where $\Psi^r(\mathbf{t_r}) = t_{r,\zeta}$. The product of this procedure is:

$$\begin{gathered}\mathcal{D}^E_X(\mathbf{t}, \hbar, Q) = \\ [e^{\sum_r \frac{\hbar^r}{2}\Psi^r(\sum_{\nu_+\nu_-\neq 1}\sum_{a,b} g^{a,b}\partial_{\nu_+,r,a}\partial_{\nu_-^{-1},r,b})} \\ \bigotimes_M \mathcal{D}^{E,fake}_{X\times B\mathbb{Z}_M}(\frac{1}{\sqrt{\hbar^M}}(\frac{t_{M,1}}{\sqrt{\hbar^M}}h_1 + \sum_{\zeta^M=1,\zeta\neq 1}\frac{t_{r(\zeta),\zeta}+1}{\sqrt{\hbar^{r(\zeta)}}}h_\zeta), \hbar^M, Q^M)]|_S\end{gathered} \tag{5.4}$$

The formula suggests the following simplification. If we shift the Euler-theoretic inputs by 1, and the fake inputs by 1 in the identity component, the formula simply becomes:

$$\mathcal{D}_X^E(\mathbf{t},\hbar,Q) = \\ [e^{\sum_r \frac{\hbar^r}{2}\Psi^r(\sum_{\nu_+\nu_-\neq 1}\sum_{a,b} g^{a,b}\partial_{\nu_+,r,a}\partial_{\nu_-^{-1},r,b})} \\ \bigotimes_M \mathcal{D}^{E,fake}_{X\times B\mathbb{Z}_M}\big(\frac{1}{\sqrt{\hbar^M}}(\frac{t_{M,1}}{\sqrt{\hbar^M}}h_1 + \sum_{\zeta^M=1,\zeta\neq 1}\frac{t_{r(\zeta),\zeta}}{\sqrt{\hbar^{r(\zeta)}}}h_\zeta),\hbar^M,Q^M\big)]|_S \tag{5.5}$$

Incomplete Potentials

For this section, we ignore Adams operations for ease of reading, reinserting them where necessary is trivial. We can rewrite the term inside the exponential of the edge operator as the difference:

$$\frac{r\hbar^r}{2}\sum_{a,b} g^{a,b}\big((\sum_\zeta \partial_{\zeta,a})(\sum_\eta \partial_{\eta^{-1},b}) - \sum_\zeta \partial_{\zeta,a}\partial_{\zeta^{-1},b}\big)$$

.

The contributions of the second term of the difference do not involve any edges, since they only deal with balanced nodes. Restricted to a vertex of order M the operator is:

$$\Delta_M = exp(\sum_r -\frac{r\hbar^r}{2}\sum_{a,b} g^{a,b}\sum_{ord(\zeta)=M/r}\partial_{r,\zeta,a}\partial_{r,\zeta^{-1},b})$$

If we apply this to the vertex potential *before* correcting the inputs, we get:

$$\Delta_M = exp(\frac{\hbar}{2}\sum_{a,b} g^{a,b}\sum_{\zeta^M=1}\partial_{r(\zeta),\zeta,a}\partial_{r(\zeta),\zeta^{-1},b})$$

Since the input corrections account for the missing factor of r, and correct the exponent of $\hbar$.

Dividing the edge operator this ways leaves us with the formula:

$$\mathcal{D}_X^E(\mathbf{t},\hbar,Q) = exp(\sum_r \frac{r\hbar^r}{2}\sum_{a,b}(\sum_\zeta \partial_{r,\zeta,a})(\sum_\eta \partial_{r,\eta,b})\big(\bigotimes_M \Delta_M \bar{\mathcal{D}}^{E,fake}_{(X\times B\mathbb{Z}_M)}(\dots)\big) \tag{5.6}$$

The operator $exp(\sum_r r\frac{\hbar^r}{2}\sum_{a,b}(\sum_\zeta \partial_{r,\zeta,a})(\sum_\eta \partial_{r,\eta,b}))$ when applied *after* the restriction to the subspace $t_{\zeta,r} = \mathbf{t_r}$ takes the form:

$$\nabla := exp(\sum_r \hbar^r/2\sum_{a,b} g^{a,b}\partial_{a,r}\partial_{b,r})$$

.

Renaming ∇^{-1} to Δ and moving it to the other side of the formula gives:

$$\Delta\mathcal{D}_X = \bigotimes_M \Delta_M \mathcal{D}^{E,fake}_{X\times B\mathbb{Z}_M}(\dots). \tag{5.7}$$

The interpretation of this formula is as follows, Δ is the operator introduced earlier that subtracts the contributions from nodal curves to the Euler-theoretic integrals. Δ_M is the corresponding operator for fake Euler theory. So as a corollary (with the same input corrections), we obtain (for the same inputs as above):

$$\mathcal{E}_X(t,\hbar,Q) = \Big[\bigotimes_M \mathcal{E}^{fake}_{X\times B\mathbb{Z}_M}\big(\frac{1}{\sqrt{\hbar^M}}(\frac{t_{M,1}}{\sqrt{\hbar^M}}h_1 + \sum_{\zeta^M=1,\zeta\neq 1}\frac{t_{r(\zeta),\zeta}+1}{\sqrt{\hbar^{r(\zeta)}}}h_\zeta), \hbar^M, Q^M\big)\Big]_{\mathbf{t_r}=t_{r,\zeta}}$$

5.8 Example: Generating function for virtual Euler characteristics

If we set $t_r = 0$ for $r > 1$, and $t_1 = t[X]$, the generating function $\mathcal{E}_X$ becomes

$$exp(\sum_g \hbar^{g-1}\sum_n \frac{1}{n!}\chi^{vir}(\mathcal{M}_{g,n,d}(X)t^n)$$

In the adelic formula, for the term corresponding to M, inputs corresponding to primitive roots of unity receive an input of $t+1$, and other inputs receive a 1, except for the unit sector h_M, which receives nothing (except for $M = 1$), so the (not dilaton shifted) formula reads (Denoting the primitive M/rth roots of unity as $P(M,r)$):

$$\mathcal{E}_X = \bigotimes_M \mathcal{E}^{fake}_{X\times B\mathbb{Z}_M}\big(\frac{1}{\sqrt{\hbar^M}}(\sum_{\zeta\in P(1,M)}\frac{th_\zeta}{\sqrt{\hbar}} + \sum_{M>r\geq 1}\sum_{\zeta\in P(r,M)}\frac{t_{r(\zeta),\zeta}}{\sqrt{\hbar^{r(\zeta)}}}h_\zeta), \hbar^M, Q^M\big)\big)$$

Fix some generator h_0 of $\mathbb{Z}_M$. The input in the sector ζ only depends on $r(\zeta)$, hence the character coordinates only depend on the order of $\chi(h_0)$. Given a character χ, define the *level* of χ to be $M/ord(\chi(h_0))$. There are $\varphi(M/k)$ characters of level k.

Denote by $b(M,r,k)$ the sum of the kth powers of elements in $P(M,r)$. It is a classical result (see e.g. [1]), that:

$$b(M,r,k) = \frac{\varphi(M/r)}{\varphi(\frac{M/r}{(k,M/r)})}\mu(\frac{M/r}{(k,M/r)})$$

We observe that for a character χ of level k, we have: $\sum_{\zeta\in P(M,r)}\chi(\zeta^{-1}) = c(M,r,k)$

So the χ coordinate after changing the input into a basis of characteris is:

$$\frac{1}{M\sqrt{\hbar^M}}\sum_{M>r>=1} b(M,r,k)\frac{1}{\sqrt{\hbar^r}} + \frac{t}{\sqrt{\hbar}}b(M,1,k)$$

We will use the formula of Jarvis-Kimura to convert these potentials into ones for X. The conversion for characters of level k is the same for any such character, so the Mth term in the tensor product becomes:

$$\mathcal{E}_X(t,0,\dots) =$$
$$\bigotimes_M \prod_{k|M} \mathcal{E}_X^{fake}\big((\frac{1}{M\sqrt{\hbar^M}}(\frac{t}{\sqrt{\hbar}}b(M,1,k) + \sum_{M>r>=1} b(M,r,k)\frac{1}{\sqrt{\hbar^r}}), M^2\hbar^M, Q^M)\big)^{\varphi(M/k)}$$

Bini and Harer computed $\chi(\mathcal{M}_{g,n})$ and $\chi(\overline{\mathcal{M}}_{g,n})$ in terms of their orbifold Euler characteristics in [2], using the same combinatorial formalism of graph sums. The above formula, when specialized to $t = 1, X = pt$, is essentially the same as their formula (25) after applying a Mobius inversion.

Chapter 6

Limits from Hirzebruch theory

As promised in chapter 3, the Euler-theoretic invariants are the correct answer for what the limit at $y = 1$, should be for the Hirzebruch invariants. In this chapter, we will state and prove a precise version of this claim, both in the complete and incomplete contexts. These limiting formulas are also useful in understanding the Euler-theoretic invariants themselves, as we will demonstrate by using them to show that the invariants are integer valued.

Hirzebruch χ_{-y}−genus and Euler Characteristics

As observed classically by Hirzebruch, for manifolds $\lim_{y\to 1} \chi_{-y}(M) = \chi(M)$. A similar result holds in the case of orbifolds, using the ordinary Euler characteristic.

Before doing this, we prove a lemma that will be useful on many occasions.

Lemma 6.0.1.

$$\lim_{y\to 1} \frac{1 - ae^{(y-1)z}}{1 - yae^{(y-1)z}} = \begin{cases} \frac{z}{1+z} & \text{if } a = 1 \\ 1 & \text{otherwise} \end{cases}$$

Proof. The case $a \neq 1$ is obvious (even for me). For $a = 1$, we expand the numerator and denominator to yield:

$$\frac{-(y-1)z + \mathcal{O}(1-y)^2)}{1 - y - (y-1)z + O((1-y)^2)}$$

Cancelling a $1 - y$ gives:

$$\frac{z + O(1-y)}{1 + z + O((1-y))}$$

Taking the limit yields: $\frac{z}{1+z}$, as desired. □

Theorem 6.0.2. *If $\mathcal{X}$ is a virtually smooth orbifold then:*

$$\lim_{y\to 1} \chi(\mathcal{X}; \Lambda_{-y}(T^{vir,*}X)) = \chi^{vir}(\mathcal{X})$$

Proof. We begin with the case where $\mathcal{X}$ is smooth: We apply the Kawasaki-Riemann-Roch theorem, the left hand side becomes:

$$\int_{I\mathcal{X}} Td(I\mathcal{X})ch(tr(\Lambda_{-y}(i^*T^*\mathcal{X})))\frac{1}{ch(tr(\Lambda(N^*))}$$

Splitting up $i^*T\mathcal{X}$ into components tangent and normal to the stratum yields:

$$\int I\mathcal{X}Td(I\mathcal{X})ch(\Lambda_{-y}(T^*I\mathcal{X}))ch(tr(\frac{\Lambda_{-y}(N^*)}{\Lambda(N^*)}))$$

The trace is omitted from the cotangent term because the h-invariant part of $T\mathcal{X}$ is exactly the tangent bundle to the component of $I\mathcal{X}$ given by h,so no additional eigenvalues appear.

Denote by $T(V)$ the class $Td(V)ch(\Lambda_{-y}(V^*))$. It is the multiplicative characteristic class determined on line bundles by the formula $\frac{z(1-ye^{-z})}{1-e^{-z}}$, where z stands for the first Chern class of a line bundle.

The value of the integral does not change if we scale the class T by $\frac{1}{(1-y)}$, and simultaneously scale z by $(1-y)$ (this involves replacing the usual Chern character with $ch_y(L) = e^{(1-y)c_1(L)}$). The only terms that contribute to the integral are top-degree ones, and the changes cancel out in that case. After doing this, the integral looks like:

$$\int_{I\mathcal{X}} T_y(TI\mathcal{X})ch_y(tr(\frac{\Lambda_{-y}(N^*)}{\Lambda(N^*)}))$$

The normal bundle term is a product of terms of the form $\frac{1-y\zeta e^{(y-1)z}}{1-\zeta e^{(y-1)z}}$, for z Chern roots of the normal bundle, and ζ the corresponding eigenvalues of h. Since N is the normal bundle of the inclusion $I\mathcal{X}$ to $\mathcal{X}$, there are no eigenvalues equal to 1. So in the limit, the entire term becomes 1.

T_y is determined on line bundles by the formula $\frac{z(1-e^{(y-1)z})}{1-e^{(y-1)z}}$, which becomes in the limit $1+z$, using the lemma from earlier. The formula $1+z$ corresponds to the total Chern class, so the integral is equal to:

$$\int_{I\mathcal{X}} c(TI\mathcal{X}) = \chi^{orb}(I\mathcal{X}) = \chi(\mathcal{X})$$

This argument generalizes completely to the virtual case, using Tonita's virtual Kawasaki Riemann-Roch theorem. The main difference is the virtual tangent bundle is the difference of two bundles: $T^{vir} = E_1 - E_0$. This means that both the bundles $T^{vir}I\mathcal{X}$ and N are also such differences, being the fixed and moving parts of $E_1 - E_0$ respectively. The class $T_y(T^{vir}I\mathcal{X})$ becomes in the limit $c(E_1^{fix})/c(E_0^{fix}) = c(T^{vir}I\mathcal{X})$, and the normal bundle terms still become 1. □

Reminder of Hirzebruch Invariants

We can realize the Euler-theoretic correlators as limits of correlators from wquantum Hirzebruch theory. Quantum Hirzebruch theory is a variant of quantum K-theory, based on the Hirzebruch $\chi_{-y}-$genus. It is defined as follows:

Let the class $\Lambda_{-y}(V)$ be defined for bundles by $\sum_i (-y)^i \Lambda^i(V^*)$. Extend it multiplicatively to a characteristic class on virtual bundles. In this guise, it is determined by the series $1 - yq^{-1}$. On manifolds, the extension to virtual bundles is well-defined provided the coefficient ring is localized at $y-1$, since $(L-1)$ is a nilpotent class, and we can expand:

$$\frac{1}{1-yL^{-1}} = \frac{1}{1-y}\frac{1}{1-\frac{y}{1-y}L^{-1}} = \sum_{n\geq 0} \frac{y^n}{(1-y)^{n+1}}(L^{-1}-1)^n$$

However the case for orbibundles is slightly more complicated, as $(L-1)$ need not be nilpotent for a 1-dimensional orbibundle. Using the fact that Kawasaki's Chern character gives an isomorphism between $K^0(X)$ and $H^*(IX)$, we see that we the coefficients will be rational functions of y with poles at roots of unity, call the algebra of such functions A. Hirzebruch theory is S_n-equivariant K-theory, which the virtual structure sheaf $\mathcal{O}^{vir}$ tensored with $\Lambda_{-y}(T^{vir})$. Call a correlator from this theory $\langle \ldots \rangle^y_{g,\ell,d}$, the inputs are tensored with rational functions of y, and we impose that $\Psi^k(y) = y^k$.

We prove that the limit as $y \mapsto 1$ of these invariants (with specifically chosen inputs), recover Euler-theoretic invariants. For ease of reading, we first prove it for ordinary invariants, then address the changes required to adapt the proof to S_n-equivariant ones.

6.1 Ordinary Limit

Given inputs $V_i \in K^0(X) \otimes A$, the ordinary Hirzebruch-theoretic correlator is:

$$\langle V_1, \ldots, V_n \rangle^y_{g,n,d} = \chi^{vir}(X_{g,n,d}; \Lambda_{-y}(T^{vir}) \prod_i ev_i^* V_i).$$

Theorem 6.1.1.

$$lim_{y \mapsto 1} \langle f_{1*}\Lambda_{-y}(T_f), \ldots \rangle^y_{g,n,d} = \overline{\langle [f_1], \ldots, [f_n] \rangle}^E_{g,n,d}$$

Proof. We apply the Kawasaki-Riemann-Roch theorem (more precisely, the virtual version due to Tonita) to the right hand side, yielding the following:

$$\langle f_{1*}\Lambda_{-y}(T_f), \ldots \rangle^y_{g,n,d} = \int_{IX_{g,n,d}} Td(TIX_{g,n,d}) ch(tr(i^*\Lambda_{-y}*(TX_{g,n,d}) \prod_i ev_i^* f_{i*}\Lambda_{-y}(T_{f_i}))) \frac{1}{ch(tr(\Lambda N^*))}$$

(Here N is the normal bundle of the natural map $IX_{g,n,d} \to X_{g,n,d}$. The invariant part of $i^*TX_{g,n,d}$ is $TIX_{g,n,d}$, and the components with nontrivial eigenvalues form the normal bundle to i of the component. The trace operator does nothing to classes pulled back from X, since they are all necessarily g-invariant. Thus we can simplify the above expression as follows:

$$\langle f_{1*}\Lambda_{-y}(T_{f_1}), \dots \rangle^y_{g,n,d} = \int_{IX_{g,n,d}} Td(TIX_{g,n,d}) ch(\Lambda_{-y}*(TIX_{g,n,d})) \prod_i Iev_i^* ch(f_{i*}\Lambda_{-y}(T_{f_i})) \frac{ch(tr(\Lambda_{-y}(N))}{ch(tr(\Lambda N^*))}$$

We can then use the (ordinary) Grothendieck-Riemann-Roch theorem on each f_i to yield:

$$\langle f_{1*}\Lambda_{-y}(T_{f_1}), \dots \rangle^y_{g,n,d} =$$

$$\int_{IX_{g,n,d}} Td(TIX_{g,n,d}) ch(\Lambda_{-y} * (TIX_{g,n,d})) \prod_i Iev_i^* f_{i*} ch(\Lambda_{-y}(T_{f_i}) Td(T_{f_i})) \frac{ch(tr(\Lambda_{-y}(N))}{ch(tr(\Lambda N^*))}$$

The operation f_{i*} preserves cohomological degree, so we can make the same substitution as before (i.e. scale all Chern classes by $1-y$ and divide the result by $(1-y)^d$, for d the virtual dimension of the component of $IX_{g,n,d}$), without altering the result of the integral. Now we proceed exactly as in the lemma regarding ordinary orbifolds, yielding a result of:

$$\int_{IX_{g,n,d}} c(TIX_{g,n,d}) \prod_i Iev_i^* f_{i*} c(T_{f_i})$$

Which is equal to the desired correlator.

□

6.2 General Limit

Theorem 6.2.1. *Given maps $f_{i,j} : Y_{i,j} \to X$ compatible with the cycle structure ℓ:*

$$lim_{y \mapsto 1} \langle f_{1,1*}\Lambda_{-y}(T_{f_{1,1}}), \dots \rangle^y_{g,\ell,d} = \overline{\langle [f_{1,1}], \dots \rangle}^E_{g,\ell,d}$$

.

For inputs $V_{i,j}$, the Hirzebruch-theoretic correlator (ignoring the normalization factor) associated to $\mathfrak{g}, \ell, d$ is equal to (if we denote the ith cycle of length j by $C(i,j)$):

$$str_h(X_{g,|\ell|,d}; \Lambda_{-y}(T^{vir}) \bigotimes_{i,j} \prod_{\alpha \in C(i,j)} ev^*_{\sigma_\alpha} V_{i,j})$$

We proceed as in the first case, but apply the Lefschetz-Kawasaki-Riemann-Roch theorem from [7], which states that:

$$str_h(\mathcal{M}, V) = \int_{I\mathcal{M}^h} Td(I\mathcal{M}^h) ch\big(tr_h(V)\big) \frac{1}{tr_h \Lambda N^{h*}})$$

.

Here N^h is the normal bundle to $\mathcal{M}$ (not $\mathcal{M}^h$!).

Since each input $V_{i,j}$ appears j times, corresponding to the j marked points in the ith length-j cycle, h acts on these by cyclically permuting the copies. Each evaluation map in the cycle is the same on the h-fixed locus, we give the name $\widehat{ev}$ to any such map. We thus conclude:

$$tr_h(\bigotimes_\alpha ev^*_{\sigma_\alpha} V_{i,j}) = \widehat{ev}^*_{i,j}\Psi^j(V_{i,j})$$

Replacing the $V_{i,j}$ with the inputs from the theorem yields:

$$\int_{IX^h_{g,|\ell|,d}} Td(T^{vir}IX^h_{g,|\ell|,d})ch\Big(\prod_{i,j}\widehat{ev}^*_{i,j}\Psi^j(f_{i,j*}\Lambda_{-y}(T_{f_{ij}}))tr_h(\frac{\Lambda_{-y}(N^{h*})}{\Lambda(N^*)})\Big)$$

This is similar to the computation in the previous case replacing $X_{g,n,d}$ with its h-fixed locus, except the inputs have received Adams operations, and the normal bundle term is wrong.

However, as we have seen before, the terms from the normal bundle will become 1 in the limit. The geometric interpretation of this is that (for all inputs equal to 1) an S_n-equivariant Hirzebruch-theoretic correlator does not compute the (virtual) orbifold Hirzebruch χ_{-y}-genus of $X^h_{g,n,d}$, however it computes a quantity that has the same limit when $y \mapsto 1$.

The remaining change from the previous case is what happens to the inputs. So we need only consider what happens to a particular input, $V = f_*\Lambda_{-y}(T_f)$, coming from some cycle of length k. The corresponding term in the expression is $ch(\widehat{ev}^*\Psi^k(f_*\Lambda_{-y}(T_f)))$

We can apply Adams-Riemann-Roch on the target to this quantity to get:

$$ch(f_*(\Psi^k(\Lambda_{-y}(T_f))C_k(T_f)))$$

Here C_k is the Adams-Todd class, determined on a line bundle q by $\frac{1-q^{-1}}{1-q^{-k}}$.

Now we can apply Grothendieck-Riemann-Roch to rewrite this expression as:

$$f_*Td(T_f)ch(\Psi^k(\Lambda_{-y}(T_f))C_k(T_f))$$

, which is equal to $f_*(H(T_f))$, where $H(T_f)$ is the characteristic class determined by the expression: $\frac{(1-y^ke^{-kz})z}{1-e^{-kz}}$. Applying the same change of coordinates: i.e. scaling z by $(1-y)$ and dividing the characteristic class by $(1-y)$ gives:

$$\frac{(1-y^ke^{(y-1)kz})z}{1-e^{(y-1)kz}}$$

Expanding this expression yields:

$$\frac{1-y^k-y^k(k(y-1)z+\dots)}{-(y-1)kz+\dots}z = \frac{\frac{1-y^k}{1-y}-y^k(kz+\dots)}{-kz+\dots}z$$

The limit of this expression as $y \mapsto 1$ is $1+z$, the total Chern class, as before, so we are in exactly the same situation as case with ordinary invariants.

While it may seem surprising at first that Adams operations do not affect the limit here, it is essentially to be expected. If f_i was a map the point, the input term would compute $\chi_{-y}(Y_i)$. Applying an Adams operation to this only changes y to y^k, which does not change the limit. The general argument is essentially a relative version of the above observation.

6.3 Incomplete Invariants

The logarithmic part of the virtual tangent bundle to $X_{g,n,d}$ is given in $K-$theory by the formula $T_{log}^{vir} := ft_* ev^*(T_X - 1) - ft_*(L^{-1} - 1)$. In the genuinely smooth case, it is the dual to the bundle $\Omega(ft_* i_* \mathcal{O}_{\mathcal{Z}})$ of 1-forms with logarithmic poles at the normal crossings divisor given by the map $ft \circ i : \mathcal{Z} \to X_{g,n,d}$. We can define a variant of Hirzebruch theory using the logarithmic bundle instead of the usual virtual tangent bundle. We denote the correlators $\langle , \rangle^{y,log}$ and the generating function $\mathcal{D}_X^{y,log}$. These have the same relationship to incomplete Euler-theoretic correlators as the usual Hirzebruch-theoretic correlators do to complete ones.

Theorem 6.3.1. *If the inputs are taken to be the ones used in the complete limit:*

$$\lim_{y \to 1} \mathcal{D}_X^{y,log} = \mathcal{E}_X$$

One can argue similarly as to the previous case, by expressing $\chi(X_{g,n,d}; \Lambda_{-y}(\Omega^{vir}(i_* \mathcal{O}_{\mathcal{Z}}))$ in terms of holomorphic Euler characteristics on $X_{g,n,d}$ and each irreducible component of $\mathcal{Z}$, and then modifying these arguments for the S_n-equivariant case. However, a quicker (and perhaps more instructive) way to see this is to use the twisting theorems from the second chapter. The twisting class applied to each component of the logarithmic tangent bundle is $e^{\sum_{k<0} \frac{\Psi^k(V/y)}{k}}$. In comparison to the standard K-theoretic potential, the type I component scales the Poincare pairing in the rth block to $\frac{\Psi^r}{r(1-y^r)}(\chi(X; \Lambda_{-y}(T^*X) a \otimes b))$, the type II component alters the dilaton shift, which will not be relevant for this argument, as the translations are the same in both cases, and we can adjust the input so the dilaton shift does not appear in the correlators. Going from $\mathcal{D}_X^y$ to $\mathcal{D}_X^{y,log}$ is an additional nodal twisting, so the relationship between the two potentials is given by an operator Δ changing the polarization, after the dilaton shift and the Poincare pairing have been adjusted. For suitable choices of input, the limit of $\mathcal{D}_X^y$ is $\mathcal{D}_X^E$, we will show that the limit of Δ is the operator relating $\mathcal{D}_X^E$ and $\mathcal{E}_X$.

As we have calculated previously, the change to the polarization in the rth block is determined by expression $\frac{1-yL_+L_-}{(1-y)1-L_+L_-} = \frac{1}{1-L_+L_-} + \frac{y}{(1-y)}$. Thus, the corresponding differential operator is determined by the insertion of the symmetric tensor

$$\frac{1-yL_+L_-}{(1-y)(1-L_+L_-)} - \frac{1}{1-L_+L_-} = \frac{y}{1-y}.$$

However, taking the limit this way is not possible, since the symplectic form is based on the Poincare pairing of Hirzebruch theory scaled by $\frac{1}{(1-y)}$. If we modify the pairing to the "usual" Hirzebruch-theoretic Poincare pairing, the tensor is scaled by $(1-y)$, and becomes y. After the scaling, the rth block Poincare pairing (a,b) is $\Psi^r/r\chi(X:\Lambda_{-y}(T^*X)ab)$. Choose a,b of the form $a=\frac{1-\alpha^{-1}}{1-y\alpha^{-1}}$, $b=\frac{1-\beta^{-1}}{1-y\beta^{-1}}$, for α,β line bundles (we can find Poincare dual basis of the Hirzebruch-theory of X using elements of this form). Applying Adams-Riemann-Roch to the Poincare pairing $(a,b)_r$ result yields:

$$\frac{1}{r}\chi(X;\Lambda_{-y}(\Psi^r T^*X))\Psi^r(a)\Psi^r(b)Td_{\Psi^k}(TX))$$

The Adams-Todd class $Td_{\Psi^k}(TX)$ is determined on line bundles by $\frac{1-L^{-1}}{1-L^{-r}}$. Combining this with the Hirzebruch theoretic Todd class yields the class $\frac{1-y^r q^{-r}(1-q^{-1})}{1-q^{-r}}$. We can rewrite this as $(1-yq^{-1})\prod_{\zeta^r=1,\zeta\neq 1}\frac{1-\zeta yq^{-1}}{1-\zeta q^{-1}}$.

Applying the usual Hirzebruch-Riemann-Roch theorem, scaling the Todd classes as before, and taking the limit as $y\mapsto 1$ eliminates the terms containing ζs, and the result is:

$$\frac{1}{r}\int_X c(TX)\frac{c_1(\alpha)}{1+c_1(\alpha)}\frac{c_1(\beta)}{1+c_1(\beta)}$$

So the limit of the rth block Poincare pairing is the Euler-theoretic Poincare pairing, scaled by $\frac{1}{r}$.

Taking the limit as $y\mapsto 1$ of the tensor itself yields 1, which corresponds to the second-order differential operator $e^{\frac{\hbar^r}{2}r\sum_{a,b}g^{a,b}\partial_{t_a}\partial_{t_b}}$. The factor r comes from correcting for the factor $\frac{1}{r}$ by which the limit of the Hirzebruch-theoretic Poincare pairing differs from the usual Poincare pairing in Euler theory.

6.4 Example Calculations

We will include some examples illustrating how the limiting formulas work in practice. We will introduce the various complications in different sections.

$\chi(\bar{M}_{1,1})$

We begin with the simplest possible case $M=\bar{M}_{1,1}$ is a smooth orbifold, isomorphic to the weighted projective space $\mathbb{P}(4,6)$. Denote the line bundles corresponding to projective coordinates x_0,x_1 by L_0 and L_1. The tangent bundle to the moduli stack is isomorphic to $L_0\otimes L_1$ (for the same reason $T\mathbb{P}^1\cong\mathcal{O}(-2)$).

The only relevant correlator to compute is the orbifold χ_{-y} genus of $\bar{M}_{1,1}$ itself, which is equal to:

$$\chi(M;\Lambda_{-y}(T_M))=\chi(\mathbb{P}(4,6);\Lambda_{-y}(L_0^*+L_1^*-\mathcal{O}))$$

We can apply the Kawasaki-Riemann-Roch theorem, noting that the inertia stack of $\mathbb{P})(4,6)$ is to two copies of $\mathbb{P}(4,6)$ (corresponding to elements in the universal isotropy group of $\mathbb{Z}/2\mathbb{Z}$), 2 copies of $B\mathbb{Z}/4\mathbb{Z}$ (corresponding to the point 0 with isotropy group $\mathbb{Z}/4\mathbb{Z}$), and 4 copies of $B\mathbb{Z}/6\mathbb{Z}$ (corresponding to the point ∞ with isotropy group $\mathbb{Z}/6\mathbb{Z}$). If we let $x_i = c_1(L_i)$, the Chow ring of $\mathbb{P}(4,6)$ has presentation $\frac{\mathbb{C}[x_0,x_1]}{(x_0x_1, 3x_0-2x_1)}$:

Each copy of $\mathbb{P}(4,6)$ contributes the following integral to the Kawasaki-Riemann-Roch formula:

$$\int_{\mathbb{P}(4,6)} Td(TM)ch(\Lambda_{-y}(TM)) = \int_{\mathbb{P}(4,6)} Td(L_0)Td(L_1)ch(\Lambda_{-y}(TM))$$

Expanding the terms inside the characteristic classes gives:

$$\int_{\mathbb{P}(4,6)} \frac{(x_1+x_0)(1-ye^{-x_1-x_0}}{1-e^{-x_1-x_0}})$$

This expression can be simplified by observing that all terms with degree at least 2 vanish, yielding:

$$\int_{\mathbb{P}(4,6)} (1-y) + \frac{(1-y)}{2}(x_0+x_1) + y(x_0+x_1)$$

From this description alone it is obvious that setting $y = 1$ computes $\chi^{orb}(\mathbb{P}(4,6)) = deg(x_0 + x_1) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$, however we use this opportunity to demonstrate the effect of scaling the Chern classes. Replacing x_i with $(1-y)x_i$ and dividing the result by $(1-y)$ does not affect the value of the integral, but the expression inside it becomes:

$$1 + \frac{(1-y)}{2}(x_0+x_1) + y(x_0+x_1)$$

Now taking the limit $y \mapsto 1$ yields $1 + x_0 + x_1 = c(T\mathbb{P}(4,3))$, the total Chern class of the tangent bundle.

So the contribution from both of these components is $\frac{10}{12}$.

The remaining components are much simpler. Each stratum S_ζ is labelled by a root of unity ζ corresponding to an element of its isotropy group, which will be either $\mathbb{Z}/4$ or $\mathbb{Z}/6$. The components corresponding to $\zeta = \pm 1$ were discussed previously. S_ζ is either $B\mathbb{Z}_4$ or $B\mathbb{Z}_6$ depending on ζ. The normal bundle is the restriction of $L_0 + L_1$, and the isotropy group element acts on it by scaling by ζ^2. So the contribution to the integral is:

$$\int_{S_\zeta} \frac{1-y\zeta}{1-\zeta}$$

When $y = 1$, the integrand is 1, and the integral is either $\frac{1}{4}$ or $\frac{1}{6}$ depending on the size of the isotropy group. Adding up all contributions yields:

$$2(\frac{5}{12}) + 2(\frac{1}{4}) + 4(\frac{1}{6}) = 2 = \chi(\mathbb{P}(4.6))$$

S_n-equivariant case

Now consider $g = 0, n = 4, X = pt$. The moduli space $\overline{M}_{0,4} \cong \mathbb{P}^1$, by letting the first three points be $0, 1, \infty$, and having the 4th point parametrized by the cross-ratio $\lambda \in \mathbb{C}U\infty$. The action of the Klein 4-group generated by $(12)(34), (13)(24)$ inside S_4 preserves the cross-ratio, and thus fixes the entire space. The action of S_3, the permutations on (234), looks as follows:

- $(34) : \lambda \to \frac{1}{\lambda}$
- $(213) : \lambda \to \frac{1}{1-\lambda}$
- $(123) : \lambda \to \frac{\lambda-1}{\lambda}$
- $(231) : \lambda \to \frac{\lambda}{1-\lambda}$
- $(23) : \lambda \to 1 - \lambda$

Realizing S_4 as a semidirect product of these groups determines the action completely. We will choose $h = (34)$. Given some Y, let f be the map $Y \to pt$. We will compute the limit correlator $\langle f_*\Lambda_{-y}(T^*f), 1, 1\rangle^y_{0,\ell}$, where ℓ is the cycle structure of h, i.e. $[2, 1, 1]$. The corresponding Euler-theoretic correlator is $\frac{1}{2}$ times the Euler characteristic of the fiber product $Y \times \mathbb{P}^h$, so it has value $\frac{\chi(Y)}{2}$, since $\mathbb{P}^{1,h}$ is just the point $1 \in \mathbb{P}^1$.

Actually computing $\langle f_*\Lambda_{-y}(T^*f), 1, 1\rangle^y_{0,\ell}$ by the Lefschetz-Kawasaki-Riemann-Roch Formula yields:

$$\frac{1}{2}\,\mathrm{str}_h(\mathbb{P}^1; \Lambda_{-y}(\mathcal{O}(2))\prod_{i=3}^{4} ev_i^* f_*\Lambda_{-y}(TY)))$$

The fixed locus is the point $1 \in \mathbb{P}^1$, so applying the Lefschetz-Kawasaki-Riemann-Roch formula gives:

$$\int_{pt} ch(tr_h(\Lambda_{-y}(\mathcal{O}(2))(\widehat{ev} f_*\Lambda_{-y}(TY))^2)ch(tr_h(\frac{1}{\Lambda(\mathcal{O}(2))}))$$

h acts on the input by cyclic permutation of the two factors, so the trace operator becomes $\Psi^2(f_*\Lambda_{-y}(TY))$. The evaluation map is just the constant map from the point to itself. However since f_* is the map to a point, this expression is nothing more than $\Psi^2(\chi_{-y}(Y))$. h acts on the conormal bundle with eigenvalue -1.

So the integral becomes:

$$\Psi^2(\chi_{-y}(Y))\int_{pt}\frac{1+y}{2} = \Psi^2(\chi_{-y}(Y))(\frac{1+y}{4})$$

The limit of this quantity is $\chi(Y)$, as desired. The Adams operations only serve to replace y with y^2, which does not alter the limit. This is a special case of the Adams-Riemann-Roch argument given in the general proof of the limiting formula.

Logarithmic Corel;ators and Boundary Components

Now, we are interested in using the nodal twisting theorem to compute logarithmic Hirzebruch invariants and its limit as $y \mapsto 1$. Let us try calculating the logarithmic version of the same correlator as before. First directly, and then by use of the nodal twisting theorem. Here $i_*\mathcal{O}_{\mathcal{Z}}$ is just $\mathcal{O}_1+\mathcal{O}_0+\mathcal{O}_\infty$. So the value of the logarithmic correlator $\langle f_*\Lambda_{-y}(T^*Y),1,1\rangle^{y,log}_{0,[2,1,1]}$ is:

$$\int_{1\in\mathbb{P}^1} ch(tr_h(\Lambda_{-y}(\mathcal{O}(2)+\mathcal{O}_1+\mathcal{O}_0+\mathcal{O}_\infty)(\widehat{ev} f_*\Lambda_{-y}(T^*Y))^2)|_{pt} ch(tr_h(\frac{1}{\Lambda(\mathcal{O}(2))}))$$

The terms coming from $0,\infty$ simply become 0, as their restriction to the point 1 vanishes. The restriction of $\mathcal{O}_1$ to 1 is the Euler class of the conormal bundle, which is the cotangent space at 1, which is isomorphic to the pullback of $1-\mathcal{O}(2)$. Thus the integral is:

$$\int_{pt=1} ch(tr_h(\Lambda_{-y}(1)(\widehat{ev}^* f_*\Lambda_{-y}(T^*Y))^2)|_{pt}\frac{1}{2} = \frac{1-y}{2} tr_h((\widehat{ev}^* f_*\Lambda_{-y}(T^*Y))^2)) = \frac{1-y}{2}\Psi^2(\chi_{-y}(Y))$$

Since $\lambda_{-y}(1) = 1-y$, the limit of this quantity is equal to 0. This is as expected, since the fixed locus of h is contained inside the boundary of $\overline{M}_{0,4}$, so subtracting boundary contributions should kill its contribution.

We will also check that this calculation agrees with the result of applying the nodal twisting theorem, which states that the operator taking the $\mathcal{D}^y_{pt}$ to $\mathcal{D}^{y,log}_{pt}$ is $e^{\frac{-\hbar^r}{2}\frac{y}{1-y}\sum_r\sum_{a,b}\partial_{r,a}\partial_{r,b}}$.

The term of $\mathcal{D}^{y,log}_{pt}$ we're interested in is the one with coefficient $\hbar^{-1}t_2t_1^2$ (i.e. corresponding to contributions from genus 0 curves with 4 marked points with 1 cycle of length 2). Expanding the operator as $1-\frac{\hbar}{2}\frac{y}{1-y}\partial_{r,a}\partial_{r,b}+\dots$, we find that only the first two terms yield any contribution, and that only happens when $r=1$.

The first term is $\langle t_2,t_1,t_1,\rangle^y_{0,[2,1,1]}$, and the second term is $\frac{-y}{1-y}\sum_{a,b}\langle t_2,\phi^a\rangle_{0,[2,1]}\langle t_1,t_1,\phi^b\rangle^y_{0,[1,1,1]}$. Note that X is a point, and the Poincare pairing is $(a,b) := \chi(pt; a\otimes b\otimes\Lambda_{-y}(T_{pt}-1)) = \frac{1}{1-y}ab$. So a Poincare dual basis just consists of 1 and $1-y$. Thus the pole at $y=1$ in the second term gets cancelled, yielding a result of:

$$\rangle t_2,t_1,t_1,\langle^y_{0,[2,1,1]} - -y\langle t_2,1\rangle^y_{0,[2,1]}\langle t_1,t_1,1\rangle^y_{0,[1,1,1]}.$$

The left term is, as we calculated in the previous section $\frac{1}{2}\frac{1+y}{2}t_1^2tr_h(t_2)$. $\langle t_2,1\rangle^y_{0,[2,1]}$ is equal to $\frac{1}{2}tr_h(t_2)$, and $\langle t_1,t_1,1\rangle^y_{0,[1,1,1]} = t_1^2$. So the total value of the expression is:

$$\frac{1}{2}\frac{1+y}{2}t_1^2tr_h(t_2) - \frac{1}{2}tr_h(t_2)t_1^2 = \frac{1-y}{4}t_1^2tr_h(t_2)$$

This recovers the calculation we did by hand, and verifies that the corresponding incomplete Euler theoretic invariant is 0 for any input.

$$\overline{\langle t_2, t_2 \rangle}^E_{0,[2,2]} - \overline{\langle t_2, \phi_a \rangle}^E_{0,[2,1]} \overline{\langle t_2, \phi_b \rangle}^E_{0,[2,1]}$$

This is equal to $\langle t_1, t_2 \rangle^E_{0,[2,2]}$, as desired.

Chapter 7

Appendix

7.1 Background on Cohomological Gromov-Witten Theory

Ordinary Invariants

Givental's loop space formalism and the notion of twisting classes were first developed for the context of cohomological invariants. To better understand the "fake" side of Euler theory, we include some background about this in this section. Correlators for cohomological Gromov-Witten theory are defined as follows: For $\alpha_i \in H^*(X)[[z]]$, the correlator $\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,d} := \int_{[X_{g,n,d}]^{vir}} \prod_i ev_i^* \alpha_i(\psi_i))$, where $\psi_i = c_1(L_i)$. We denote the total descendant potential for this theory by $\mathcal{D}_X^{\mathcal{H}}$.

The loop space formalism for ordinary Gromov-Witten invariants looks as follows: Given some smooth projective target X, the loop space $\mathcal{H}_X$ is the vector space of formal Laurent series in a variable z with coefficients in $H^*(X)$. The symplectic form is $\Omega_X(f, g) := Res_{z=0}(f(z), g(-z))dz$, where () denotes the cohomological Poincare pairing. The positive space of the polarization consists of power series in z, and the negative space consists of power series in $1/z$ with no constant term.

The usual dilaton shift for the cohomological generating function is $-z$.

Orbifold Gromov-Witten Theory

When the target $\mathcal{X}$ is an orbifold, the moduli space $\mathcal{X}_{g,n,d}$ is the moduli of stable maps from n-pointed stable orbicurves C to $I\mathcal{X}$. The curves are allowed to have orbifold points at marked points and nodes, and the action of the isotropy group must be balanced at each node. $I\mathcal{X}$ is the union of sectors X_μ, and there is an involution map that sends a sector μ to its "inverse sector" μ^I. The orbifold type of a given marked point or node is determined by the sector μ it is sent to by the evaluation map, so the space $\mathcal{X}_{g,n,d}$ is the union of many components, corresspponding to the possible orbifold types of each marked point.

The universal family $\mathcal{U}_{g,n,d}$ consists of the components of $\mathcal{X}_{g,n+1,d}$, where the $n+1$st marked point is not an orbifold point.

The Poincare pairing on $H^*(I\mathcal{X})$ is $\bigoplus()_\mu$, where $()_\mu$ pairs X_μ with $X_{I\mu}$. The symplectic loop space $H_\mathcal{X}$ is $H^*(I\mathcal{X})[[z^\pm]]$, with symplectic form $\Omega_X(f,g) := Res_{z=0}(f(z), g(-z))dz$, using the Poincare pairing for $H^*(I\mathcal{X})$.

Twisting Classes

Similarly to what we have introduced in the K-theoretic context, there is a theory of twisting classes for cohomological invariants, first developed by Coates in [4], and developed for orbifold targets by Tonita in [17]. We will state the version for orbifold theory. A general multiplicative characteristic class in cohomology theory is described as follows:

$$\mathcal{C}(E) = exp\left(\sum_{k\geq 0} s_k ch_k E\right).$$

As with K-theory, we consider three types of twistings

- Type I: twistings by a class $\mathcal{C}(ft_*(ev^*_{n+1}E))$, where $E \in K^0(\mathcal{X})$.
- Type II: twistings by a class of the form:

$$\mathcal{C}\left(ft_*(f(L^{-1}_{n+1}) - f(1))\right),$$

  where L_{n+1} is the cotangent line bundle at the extra marked point on the universal curve, f is a polynomial with coefficients in $ev^*_{n+1}K^0(\mathcal{X})$ and 1 is the trivial line bundle.
- Type III: twistings by nodal classes $\mathcal{C}^\mu$ of the form:

$$\prod_\mu \mathcal{C}^\mu\left(ft_*(ev^*_{n+1}F_\mu \otimes i_{\mu *}\mathcal{O}_{\mathcal{Z}_\mu})\right),$$

  where $F_\mu \in K^0(\mathcal{X})$. $\mathcal{Z}_\mu$ is the locus of nodes with orbifold type μ, and i_μ is the corresponding inclusion into the universal family.

These effect of these twistings on the potential $\mathcal{D}_\mathcal{X}$ are given by the following theorems:

Theorem 7.1.1.

$$\langle \mathcal{D}^{tw}_\mathcal{X}\rangle = \widehat{\Delta}\langle \mathcal{D}_\mathcal{X}\rangle$$

Where

$$\Delta := exp\left(\sum_{k\geq 0} s_k \left(\sum_{j\geq 0}\frac{[A_j]_{k+1-j}z^{j-1}}{j!} + \frac{ch_k(E^{(0)})}{2}\right)\right),$$

Here $[A_j]$ are some elements in $H^*(I\mathcal{X})$ and $[A_j]_i$ denotes the degree i part. These elements act by multiplication.

We define the A_j as follows: Let m_μ be the order of a group element defining the sector μ of $I\mathcal{X}$. Restricted to X_μ, the bundle E used as input for the twisting class decomposes into characters E^l_μ, where the group element acts as ζ^l for $\zeta = e^{\frac{2\pi i}{m_\mu}}$.

Then

$$(A_j)_{|\mathcal{X}_\mu} := \sum_{l=0}^{l=r_\mu - 1} B_j(\frac{l}{r_\mu}) ch(E^{(l)}_\mu).$$

Where B_j denotes the jth Bernoulli polynomial.

Theorem 7.1.2. *The twisting by the classes of type II induces a translation of potentials:*

$$\langle \mathcal{D}^{tw}(\mathbf{t})\rangle = \langle \mathcal{D}\left(\mathbf{t} + z - z\mathcal{C}\left(-\frac{f(L_z^{-1}) - f(1)}{L_z - 1}\right)\right)\rangle$$

Where L_z be a line bundle with first Chern class z.

Theorem 7.1.3. *For a twisting of type III, the formula looks like:*

$$\rangle \mathcal{D}^{tw}_{\mathcal{X}} \langle = \widehat{\Delta} \rangle \mathcal{D}_{\mathcal{A},\mathcal{B}} \langle$$

Here the operator $\widehat{\Delta}$ is a quadratic Hamiltonian corresponding to a change of polarization, determined in the sector μ by the expression:

$$\frac{\Delta_{\mu *}\left(\mathcal{C}^\mu\left(\left((q^*F_\mu)^{(0)}_\mu \otimes (1 - L_z)\right) - 1\right)}{-\psi_+ - \psi_-} \in \in H^*(\mathcal{X}_\mu, \mathbb{Q})[\psi_+] \otimes H^*(\mathcal{X}_{\mu^I}, \mathbb{Q})[\psi_-].$$

q is the map $I\mathcal{X} \to X$, and $q^*F^{(0)}_\mu$ denotes the part of q^*F invariant under the action of the group element corresponding to the sector μ.

The twistings can be taken on top of one another, and the the same results hold relating twisted invariants to further-twisted ones.

Remark. Quantum states do not depend on scaling, so as stated, these theorems are only true projectively. In fact, as in K-theory, the third twisting theorems are true without the addition of any scale factors. The first and second twisting theorem do require scale factors. The scale factor from the first twisting theorem is 1 when $\mathcal{C}(1) = 1$, which is the only situation of our interest (see 4.2.1 in [19]). The second scale factor we will discus later.

Target $X \times B\mathbb{Z}_M$

The orbifold target spaces we consider in this work are all the form $X \times B\mathbb{Z}_M$. WThe actual target of the evaluation maps is $I(X \times B\mathbb{Z}_M)$, which is M disjoint copies of $X \times B\mathbb{Z}_M$ labelled by roots of unity ζ. The cohomology ring is thus $\bigoplus H^*(X)h_\zeta$, where h_ζ is the fundamental class on the ζth sector.

The Poincare pairing is $(\sum a_\zeta h_\zeta, \sum b_\zeta h_\zeta) = \frac{1}{M}\sum_\zeta (a_\zeta, b_{\zeta^{-1}})_X$, where $()_X$ is the Poincare pairing on X. If $\mathcal{H}, \Omega_X$, is the symplectic loop space coming from the cohomological GW theory of X, the loop space for $X \times B\mathbb{Z}_M$ is given by $\bigoplus_\zeta \mathcal{H}h_\zeta$. The symplectic form is given by $\frac{1}{M}\sum_\zeta \Omega_X(f_\zeta, g_\zeta)$, and the symplectic polarization is inherited from $\mathcal{H}$.

We can interpret the Gromov-Witten invariants of $X \times B\mathbb{Z}_M$ as invariants of X using a formula of Jarvis-Kimura [11], who proved that each connected component of $\overline{M}_{g,n,d}(X \times BG)$ is a virtual covering of $X_{g,n,d}$ with a prescribed degree, determined by the group G. This means, for ordinary cohomological Gromov-Witten invariants:

$$\langle \alpha_1 h_{\zeta_1}, \dots, \alpha_n h_{\zeta_n}\rangle_{g,n,d,X\times B\mathbb{Z}_M} = \begin{cases} M^{2g-1}\langle \alpha_1, \dots, \alpha_n\rangle_{g,n,d,X} & \prod_{i=1}^n \zeta_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Another way of interpreting this result is in the basis of characters. Rather than using elements of $\mathbb{Z}_M$, we break of $H^*(I(X \times B\mathbb{Z}_M))$ into $\bigoplus H^*(I)\chi$, for χ a character of $\mathbb{Z}_M$. Sector and character coordinates are related via Fourier transform: i.e. given some element f:

$$f_\zeta = \frac{1}{M}\sum_\chi f_\chi \chi(\zeta)$$

$$f_\chi = \sum_\zeta f_\zeta \chi(\zeta^{-1})$$

The translation of Jarvis-Kimura's result into the character basis is the following:

$$\langle t_1\chi_1, \dots, t_n\chi_n\rangle_{g,n,d}^{X\times B\mathbb{Z}_M} = \frac{M^{2g-1}}{M^n}\sum_{\prod_i \zeta_i = 1}\prod_i \chi_i(\zeta_i)\langle t_1, \dots, t_n\rangle_{g,n,d}^X$$

The coefficient $\sum_{\prod_i \zeta_i=1}\prod_i \chi_i(\zeta_i)$ is 0 unless all χ_is are equal, and then it is M^{n-1}. So the result is that:

$$\langle t_1\chi, \dots, t_n\chi\rangle_{g,n,d}^{X\times B\mathbb{Z}_M} = M^{2g-2}\langle t_1, \dots, t_n\rangle_{g,n,d}^X$$

And any other choices of inputs yield 0.

This observation allows us to identify $\mathcal{D}^{\mathcal{H}}_{X\times B\mathbb{Z}_M}(\sum t_\chi \chi, \hbar, Q)$ with $\bigotimes_\chi \mathcal{D}^{\mathcal{H}}_X(t_\chi, M^2\hbar, Q)$, at the level of generating functions. However, to make the formula amenable to twisting, we also need to identify these functions at the level of quantum states.

To draw conclusions about potentials, we need to also understand the dilaton shift, Poincare pairing and symplectic polarization, once written in the character basis. Resumming the sectorial expression for the Poincare pairing yields: $(f, g) = \frac{1}{M^2}\sum_\chi (f_\chi, g_\chi)_X$

The dilaton shift is $-z\sum_\chi \chi$, since the sum of all the characters is h_1, so it is equivalent to shifting each potential $D_X^{\mathcal{H}}$ by the correct value.

The positive and negative spaces are similarly just $\mathcal{H}_+$ and $\mathcal{H}_-$ in each coordinate χ.

So the quantum state $\langle \mathcal{D}^{\mathcal{H}}_{X\times B\mathbb{Z}_M}(\sum t_\chi \chi, \hbar, Q)\rangle$is the same as $\bigotimes_\chi \langle \mathcal{D}^{\mathcal{H}}_X(t_\chi, M^2\hbar, Q)\rangle$, provided that each factor in the tensor product is regarded as a quantum state in the loop space $\mathcal{H}_{X,\chi}$, which is the direct summand of $\mathcal{H}_{X\times B\mathbb{Z}_M}$ corresponding to the character χ. It is the same as $\mathcal{H}_x$, except the Poincare pairing is scaled by $\frac{1}{M^2}$.

Fake Euler-Theoretic Invariants for $X \times B\mathbb{Z}_M$

As discussed earlier, we define the fake Euler-theoretic potentials $\mathcal{D}^{E,fake}_{\mathcal{X}}$ to be the usual Gromov-Witten potential, but with inputs taken from $H^*(I\mathcal{X}) \otimes MU^*(pt)$, and virtual fundamental cycle twisted by the classes $c(T^{vir})$. The same decomposition theorem for T^{vir} holds in the orbifold case, so we have:

$$T^{vir} = ft_* ev^*_{n+1}(T\mathcal{X} - 1) + ft_*(1 - L^{-1}_{n+1}) - (ft_*\mathcal{O}_{\mathcal{Z}})^*$$

The main difference to note here is that $\mathcal{Z}$ is the union of all $\mathcal{Z}_\mu$.

Theorem 7.1.4. *A Jarvis-Kimura formula holds in this context as well:*

$$\mathcal{D}^{E,fake}_{X\times B\mathbb{Z}_M}(\sum_\chi t_\chi \chi, \hbar) = \prod_\chi \mathcal{D}^{E,fake}_X(t_\chi, M^2\hbar)$$

We will prove this by using Tonita's twisting theorem for orbifold cohomological Gromov-Witten theory. Twisting by the class $c(T^{vir})$ will decompose into twisting transformations in each character. We will find that these twistings correspond to the twistings that change $\mathcal{D}^{\mathcal{H}}_X$ into $\mathcal{D}^{E,fake}_X$.

- A multiplication operator, the same in each sector, since $T(X \times B\mathbb{Z}_M)$ has no interaction with the orbifold structure of $X \times B\mathbb{Z}_M$, all the E^l_μs are equal to TX if $l = 0$ and 0 otherwise. The corresponding operator, restricted to a single sector is equal to $sqrt(c(TX))$ multiplied by the Euler-Maclaurin asymptotics of the infinite product $\prod_{m=1}^\infty c(TX \otimes L_z^{-m})$. The effect is to scale the Poincare pairing in each sector from the cohomological pairing to the Euler-theoretic one. Each operator is identical to the multiplication operator involved in changing from the non-orbifold potential $\mathcal{D}_X$ to $\mathcal{D}^{E,fake}_X$ (this is the same operator as above, but with $M = 1$).

- A change in the dilaton shift in the unit sector from $-z$ to:

$$-zc(-\frac{1-L_z^{-1}}{L_z - 1}) = \frac{-z}{c(L_z^{-1}} = \frac{-z}{1-z}$$

in the unit sector. Changing from sector to character coordinates becomes $\sum_\chi \frac{z}{z-1}\chi$, matching with the dilaton shift of the fake potential of X in each character.

- A change in the polarization in each sector, given by the expansion of the symmetric tensor
$$\frac{\Delta_{\mu*}\left(c\left((1-L_z)^*\right)-1\right)}{-\psi_+-\psi_-}$$
Expanding this yields:
$$\frac{1-\psi_+-\psi_--1}{-\psi_+-\psi_-}=1$$
Let ϕ^α, be a basis for Euler theory of X. The inputs to the fake Euler theory of $X\times B\mathbb{Z}_M$ can be expressed as $\sum_{\zeta,\alpha} t_{\alpha,\zeta}\phi^\alpha h_\zeta$, and let $p_0^{\alpha,\zeta}$ be the Darboux coordinate dual to $t_{\alpha,\zeta}$, using the unscaled Poincare pairing from X.

  The tensor 1 corresponds to the quadratic Hamiltonian $\sum_{\alpha,\beta,\zeta}\frac{M}{2}g^{\alpha,\beta}p_0^{\alpha,\zeta}p_0^{\beta,\zeta^{-1}}$, the M is to account for the difference between the sectorial pairing and the one on X (dividing the symplectic form by a constant results in multiplying the corresponding Darboux coordinate).

  Changing to character coordinates gives: $\frac{M^2}{2}\sum_{\alpha,\beta,\chi} g^{\alpha,\beta}p_0^{\alpha,\chi}p_0^{\beta,\chi}$. The factor M^2 accounts for the difference between the character pairing and the one on X.

  Restricting to a fixed χ, this Hamiltonian, when quantized, corresponds to the operator $e^{\frac{M^2\hbar}{2}\sum_{\alpha,\beta}\partial_{t_\alpha}\partial_{t_\beta}}$, which is the operator corresponding to the twisting when $M=1$, except $\hbar$ is scaled by M^2, correctly matching the adjustment to the potential $\mathcal{D}_X^{\mathcal{H}}$.

- A multiplication operator, the same in each sector, since $T(X\times B\mathbb{Z}_M)$ has no interaction with the orbifold structure of $X\times B\mathbb{Z}_M$, all the E_μ^ls are equal to TX if $l=0$ and 0 otherwise. The corresponding operator, restricted to a single sector is equal to $sqrt(c(TX))$ multiplied by the Euler-Maclaurin asymptotics of the infinite product $\prod_{m=1}^\infty c(TX\otimes L_z^{-m})$. The effect is to scale the Poincare pairing in each sector from the cohomological pairing to the Euler-theoretic one. Each operator is identical to the multiplication operator involved in changing from the non-orbifold potential $\mathcal{D}_X$ to $\mathcal{D}_X^{E,fake}$ (this is the same operator as above, but with $M=1$).

- A change in the dilaton shift in the unit sector from $-z$ to:
$$-zc(-\frac{1-L_z^{-1}}{L_z-1})=\frac{-z}{c(L_z^{-1}}=\frac{-z}{1-z}$$
in the unit sector. Changing from sector to character coordinates becomes $\sum_\chi \frac{z}{z-1}\chi$, matching with the dilaton shift of the fake potential of X in each character.

- A change in the polarization in each sector, given by the expansion of the symmetric tensor
$$\frac{\Delta_{\mu*}\left(c\left((1-L_z)^*\right)-1\right)}{-\psi_+-\psi_-}$$
Expanding this yields:
$$\frac{1-\psi_+-\psi_--1}{-\psi_+-\psi_-}=1$$
Let ϕ^α, be a basis for Euler theory of X. The inputs to the fake Euler theory of $X\times B\mathbb{Z}_M$ can be expressed as $\sum_{\zeta,\alpha} t_{\alpha,\zeta}\phi^\alpha h_\zeta$, and let $p_0^{\alpha,\zeta}$ be the Darboux coordinate dual to $t_{\alpha,\zeta}$, using the unscaled Poincare pairing from X.

  The tensor 1 corresponds to the quadratic Hamiltonian $\sum_{\alpha,\beta,\zeta}\frac{M}{2}g^{\alpha,\beta}p_0^{\alpha,\zeta}p_0^{\beta,\zeta^{-1}}$, the M is to account for the difference between the sectorial pairing and the one on X (dividing the symplectic form by a constant results in multiplying the corresponding Darboux coordinate).

  Changing to character coordinates gives: $\frac{M^2}{2}\sum_{\alpha,\beta,\chi}g^{\alpha,\beta}p_0^{\alpha,\chi}p_0^{\beta,\chi}$. The factor M^2 accounts for the difference between the character pairing and the one on X.

  Restricting to a fixed χ, this Hamiltonian, when quantized, corresponds to the operator $e^{\frac{M^2\hbar}{2}\sum_{\alpha,\beta}\partial_{t_\alpha}\partial_{t_\beta}}$, which is the operator corresponding to the twisting when $M=1$, except $\hbar$ is scaled by M^2, correctly matching the adjustment to the potential $\mathcal{D}_X^{\mathcal{H}}$.

The result is that splitting up the twisting operators into their character terms have the effect of twisting $\prod_\chi \mathcal{D}_X^H$ into $\prod_\chi \mathcal{D}_X^{E,fake}$, so the Jarvis-Kimura formula remains true in this context, at the level of quantum states.

Unfortunately, the above argument actually omits an important detail. The twistings of type II can also potentially scale the potentials, so we must also check that these scale factors are correct. The correction term is calculated as follows.

It is the exponential of:

$$\int_{[(X\times B\mathbb{Z}_M)_{1,1,0}]^{vir,tw}}[f(e^{-\psi_1})-f(1))Td(L_1^*)]_1 C(\frac{f(L^{-1})-f(1)}{L-1})$$

Here the virtual class is twisted by the type I and III twistings, but not the type II one. $[A]_1$ denotes the degree 1 part of A. A_1 is a multiple of ψ_1, so the nodal twisting term does not contribute, since all terms are killed in the product.

So the integral we are interested in computing is:

$$\int_{[(X\times B\mathbb{Z}_M)_{1,1,0}]^{vir}} ft_*ev^*c(T_X)[f(e^{-\psi_1})-f(1))Td(L_1^*)]_1 C(\frac{f(L^{-1})-f(1)}{L-1})$$

Since $f(L^{-1})=L^{-1}-1$, and $Td(L_1^*)=\frac{-1}{2}\psi_1$ all the terms except for the twisting class contribute $\frac{-1}{2}\psi_1$. (Term of higher degrees do not affect the integral for dimension reasons)

So we can rewrite the integral as:

$$\frac{-1}{2}\int_{[(X\times B\mathbb{Z}_M)_{1,1,0}]^{vir}} ft_*ev^*c(T_X)\psi$$

Using techniques from [18]] we can rewrite this integral as an integral on $\overline{M}_{1,1}\times II(X\times B\mathbb{Z}_M)$. For an orbifold $\mathcal{X}$, let $\mathcal{X}_2$ be the space of (x,H), where H is a subgroup of $Aut(x)$ with at most 2 generators.

We have two morphisms $\pi_1 : \mathcal{X}_{1,1,0} \to \mathcal{X}_2 \times \overline{M}_{1,1}$, and $\pi_2 : \overline{M}_{1,1}\times II\mathcal{X}$. These are both etale of the same degree. The first map is given by taking monodromy of admissible covers, the second is given by sending the two group elements g_1, g_2 to the subgroup they generate. We also have

$$\pi_{1*}[\mathcal{X}_{1,1,0}]^{vir} = \pi_{2*}(c_{top}(E^*\boxtimes TII\mathcal{X}))$$

Thus, we can calculate integrals on our moduli space by instead working on $\overline{M}_{1,1}\times II\mathcal{X}$. By another result from [18], the twisting term $ft_*ev^*T(X\times B\mathbb{Z}_M)$ is expressible as $\pi_1^*((1-E^*)\boxtimes T(X\times B\mathbb{Z}_M)_2$, since the tangent bundle is invariant under the isotropy action.

Thus we can write the integral (by pushing forward by π_1 and using the projection formula to pull back by π_2):

$$\frac{-1}{2}\int_{\overline{M}_{1,1}\times II(X\times B\mathbb{Z}_M)} c(1-E^*\boxtimes TII(X\times B\mathbb{Z}_M))c_{top}(E^*\boxtimes TII(X\times B\mathbb{Z}_M))\psi_1$$

The only nonzero contribution to this integral is:

$$\frac{-1}{2}\int_{\overline{M}_{1,1}\times II(X\times B\mathbb{Z}_M)} \psi_1 c_{top}(TII(X\times B\mathbb{Z}_M)) = -\frac{M}{48}\chi(X)$$

This scale factor splits up as $\prod_1^M e^{-\frac{\chi(X)}{48}}$, so it matches with the scale factor for each character (which is obtained by the same process but setting $M=1$).

7.2 Modifying Type III Cohomological Twistings

The nodal twisting classes in K-theory are slightly more general than analogues of the ones introduced in cohomology, they allow additional functions $F(L_+,L_-)$ to modify the structure sheaf of the locus of nodes. We need a slightly stronger version of Tonita's theorem for twistings of type 3.

Theorem 7.2.1. *If the allowed twistings are expanded to include classes of the form:*

$$\mathcal{C}(i_{\mu *}\pi_{\mu^*}F(L_+,L_-),$$

where F is a symmetric Laurent polynomial with coefficients in $ev^*K^*(\mathcal{X})$, then the theorem for twistings of type III still holds in this setting, with the operator $\widehat{\Delta}$ being the quadratic

Hamiltonian corresponding to a change of polarization, determined in the sector μ by the expression:

$$\frac{\Delta_{\mu*}\left(\mathcal{C}^{\mu}\left(\left(q^* F_\mu(L_z, L_z)\right)^{(0)}_\mu \otimes (1 - L_z)\right) - 1\right)}{-\psi_+ - \psi_-} \in \in H^*(\mathcal{X}_\mu, \mathbb{Q})[\psi_+] \otimes H^*(\mathcal{X}_{\mu^I}, \mathbb{Q})[\psi_-].$$

The proof of this theorem is essentially identical to the case of normal twistings of type III, and the argument is given in [17]. It relies on understanding how the twisting classes pull back under the forgetting map. We do not repeat the entire argument, but we include the necessary pullback results for the modified twistings. Carrying through the proof requires two lemmas about how the twisting classes pull back under the forgetting maps. Given some function F as above, denote F_i to be the function with coefficients pulled back by ev_i rather than by the universal marked point ev_{n+1}.

Lemma 7.2.2.
i):

$$ft^*_{n+1} i_{\mu*} ft_{n+1*} F(L_+, L_-)\mathcal{O}_{\mathcal{Z}_\mu} = ft_{n+2*} i_{\mu*}(F_{n+1}(L_+, L_-)\mathcal{O}_{\mathcal{Z}_{\mu+1}}) -$$
$$\sum_{\text{marked points } i \text{ of orbifold type } \mu} F(1, L_i)\sigma_{i*} D_i - i_{\mu*}(F(L_+, L_-)\mathcal{O}_{\mathcal{Z}_\mu}$$

ii):
If p_1 and p_2 are the projections onto each f]component in the locus of irreducible nodes then we have:

$$(\pi \circ i)^*(ft_{n+1*} i_{\mu*}(F(L_+, L_-) \otimes \mathcal{O}_{\mathcal{Z}_\mu}) = p_1^*(ft_* i_{\mu*}(F(L_+, L_-) \otimes \mathcal{O}_{\mathcal{Z}_\mu})) +$$
$$p_2^*(ft_* i_{\mu*}(F(L_+, L_-) \otimes \mathcal{O}_{\mathcal{Z}_\mu})) + (F(L_+, L_-) \otimes (1 - L_+ L_-))$$

We begin by proving part i). Following the notation in [17], we call the $n+1$st marked point $\circ$, and the $n+2$nd marked point $\bullet$. By forgetting both marked points in different orders, we have the following fiber diagram:

$$\begin{array}{ccccc} ft_\bullet^{-1}(\mathcal{Z}_{\circ,\mu}) & \xrightarrow{i_\mu} & \mathcal{X}_{g,n+\circ+\bullet,d} & \xrightarrow{ft_\circ} & \mathcal{X}_{g,n+\bullet,d} \\ {\scriptstyle ft_\bullet}\downarrow & & {\scriptstyle ft_\bullet}\downarrow & & {\scriptstyle ft_\bullet}\downarrow \\ \mathcal{Z}_{\circ,\mu} & \xrightarrow{i_\mu} & \mathcal{X}_{g,n+\circ,d} & \xrightarrow{ft_\circ} & \mathcal{X}_{g,n,d}. \end{array}$$

In this notation, the quantity we are interested in computing can be expressed as $ft_\bullet * ft_{\circ*} i_{\mu*}(F_\circ \otimes \mathcal{O}_{\mathcal{Z}_\mu})$. Since the diagram commutes, we can rewrite this in a slightly more tractable way as $ft_{\circ,*} i_{\mu*} ft^*_\bullet(F_\circ \otimes \mathcal{O}_{\mathcal{Z}_\mu})$.

The term inside the pushforwards can be expressed as

$$F_\circ(ft^*_\bullet L_+, ft^*_\bullet L_-) \otimes \mathcal{O}_{ft_\bullet^{-1}(\mathcal{Z}_{\circ,\mu})}$$

However $ft_\bullet^* L_+$ is just L_+, since the marked point $\circ$ remains the node.

$\mathcal{Z}_{\circ,\mu}$ is a union of products of lower-dimensional moduli spaces glued together along $\circ$. The preimage of such a component under $ft_\bullet$ is a union of three components $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$, where the pint $\bullet$ lands on the first factor, the contracted rational component containing the node, and the second factor, respectively. More precisely, given a component $\mathcal{Z}_{\circ,\mu}$ of the form $\mathcal{X}_{n_1+1,g_1,d_1} \times_{ev} X_{0,2+\circ,0} \times_{ev} \mathcal{X}_{n_2+1,g_2,d_2}$. we have:

- The corresponding component of $\mathcal{Z}_1$ is

$$\mathcal{X}_{n_1+1+\bullet,g_1,d_1} \times_{ev} X_{0,2+\circ,0} \times_{ev} \mathcal{X}_{n_2+1,g_2,d_2}$$

- The corresponding component of $\mathcal{Z}_2$ is

$$\mathcal{X}_{n_1+1,g_1,d_1} \times_{ev} X_{0,2+\circ+\bullet,0} \times_{ev} \mathcal{X}_{n_2+1,g_2,d_2}$$

- The corresponding component of $\mathcal{Z}_3$ is

$$\mathcal{X}_{n_1+1,g_1,d_1} \times_{ev} X_{0,2+\circ,0} \times_{ev} \mathcal{X}_{n_2+1+\bullet,g_2,d_2}$$

They intersect along components $\mathcal{Z}_{12}, \mathcal{Z}_{23}$, where $\bullet$ is the only marked point on a contracted component of the first or second factor respectively (i.e. the components corresponding to the boundary of $\mathcal{X}_{2+\circ+\bullet}$ corresponding to $\bullet$ colliding with a marked point that is not $\circ$). Using the same notation as before, components of $\mathcal{Z}_{12}$ and $\mathcal{Z}_{23}$ can be described as:

$$(\mathcal{X}_{n_1+1,g_1,d_1} \times_{ev} X_{0,2+\bullet,0}) \times_{ev} X_{0,2+\circ,0} \times_{ev} \mathcal{X}_{n_2+1,g_2,d_2}$$

and:

$$\mathcal{X}_{n_1+1,g_1,d_1} \times_{ev} X_{0,2+\circ,0} \times_{ev} (X_{0,2+\bullet,0} \times_{ev} \mathcal{X}_{n_2+1,g_2,d_2}),$$

respectively.

We can thus write $\mathcal{O}_{ft_\bullet^{-1}(\mathcal{Z}_{\circ,\mu})} = \mathcal{O}_{\mathcal{Z}_1} + \mathcal{O}_{\mathcal{Z}_2} + \mathcal{O}_{\mathcal{Z}_3} - \mathcal{O}_{\mathcal{Z}_12} - \mathcal{O}_{\mathcal{Z}_13}$.

We first process the term $F_\circ(ft_\bullet^* L_+, ft_\bullet^* L_-) \otimes (\mathcal{O}_{\mathcal{Z}_1} + \mathcal{O}_{\mathcal{Z}_3})$. $\mathcal{Z}_1 \cup \mathcal{Z}_3$ is the locus of nodes of type μ in the universal family $X_{g,n+\circ+\bullet,d}$, except the nodes corresponding to strata $\mathcal{X}_{g,n,d} \times_{ev} \mathcal{X}_{0,3,0} \times_{ev} \mathcal{X}_{0,3,0}$. These are mapped isomorphically by the forgetting map to D_i, for i a marked point of orbifold type μ.

So the pushforward of these components of $F_\circ(ft_\bullet^* L_+, ft_\bullet^* L_-) \otimes \mathcal{O}_{ft_\bullet^{-1}(\mathcal{Z}_{\circ,\mu})}$ is:

$$ft_{\bullet *} i_{\mu *} F_\circ(L_+, L_-) \mathcal{O}_{\mathcal{Z}_\mu} - \sum_i F_\circ(L_+, L_-) \sigma_{i*} \mathcal{O}_{D_i}$$

Note now that on D_i, $L_+ = L_i, L_- = 1$ (or vice versa), so we get a final result of:

$$ft_{\bullet *} i_{\mu *} F_\circ(L_+, L_-) \mathcal{O}_{\mathcal{Z}_\mu} - \sum_i F_\circ(L_i, 1) \sigma_{i*} \mathcal{O}_{D_i}$$

Now, to process the remaining terms. Each of $\mathcal{Z}_{12}$, $\mathcal{Z}_{13}$ are sent isomorphically to $\mathcal{Z}_{\bullet,\mu}$. On these loci, the evaluation maps $\circ$ and $\bullet$ coincide, so L_+ is also the normal bundle to one branch of $\mathcal{Z}_\bullet$. $\mathcal{Z}_2$ is a $\mathbb{P}^1$ bundle over $\mathcal{Z}_{\bullet,\mu}$, as the marked point $\circ$ can lie anywhere on the contracted component $\mathcal{X}_{0,4,0}$. Since L_+, L_- are pulled back from the base (since $\circ = \bullet$), the pushforward is just multiplication by the Euler characteristic of $\mathcal{O}_{\mathbb{P}^1}$, which is 1. So the total contribution of $\mathcal{O}_{\mathcal{Z}_2} - \mathcal{O}_{\mathcal{Z}_{12}} - \mathcal{O}_{\mathcal{Z}_{23}}$ is just $i_{\mu *} F_\bullet(L_+, L_-)\mathcal{O}_{\mathcal{Z}_\mu}$.

Totalling the above two contributions gives the desired result.
Now to prove ii).

We first state a preliminary lemma:

Lemma 7.2.3.

$$i^* ft_{n+2*} i_{\mu *}(F_{n+1}(L_+, L_-)\mathcal{O}_{\mathcal{Z}_{\mu+1}}) = p_1^* ft_* i_*(F_{n+1}(L_+, L_-) \otimes \mathcal{O}_{\mathcal{Z}_\mu}) +$$

$$p_2^* ft_* i_*(F_{n+1}(L_+, L_-) \otimes \mathcal{O}_{\mathcal{Z}_\mu}) + (2 - L_+ - L_-)F(L_+, L_-)$$

The proof of this is identical to that of Lemma 3.8 in [17], which has the terms F_{n+1} replaced with constants. The addition of L_+, L_- changes essentially nothing.

Using this lemma, we compute $(ft_{n+1} \circ i)^* i_{\mu *} ft_{n+1*} \mathcal{O}_{\mathcal{Z}_\mu}$. By part i) this is equal to:

$$i^*(ft_{n+2*} i_{\mu *}(F_{n+1}(L_+, L_-)\mathcal{O}_{\mathcal{Z}_{\mu+1}}) - \sum_i F(1, L_i)\sigma_{i*} D_i - i_{\mu *}(F(L_+, L_-)\mathcal{O}_{\mathcal{Z}_\mu})$$

The first term is $p_1^* ft_* i_*(F_{n+1}(L_+, L_-) \otimes \mathcal{O}_{\mathcal{Z}_\mu}) + p_2^* ft_* i_*(F_{n+1}(L_+, L_-) \otimes \mathcal{O}_{\mathcal{Z}_\mu}) + (2 - L_+ - L_-)F(L_+, L_-)$ by the previous lemma, the terms supported on the divisors D_i vanish, and the final term is $i^* i_{\mu *} F(L_+, L_-) = i^* i_{\mu *} F(L_+, L_-)(1 - L_+)(1 - L_-)$, since $(1 - L_+)(1 - L_-)$ is the normal bundle of $\mathcal{Z}$ inside the universal family.

Adding these two contributions together yields:

$$p_1^*(ft_* i_{\mu *}(F(L_+, L_-) \otimes \mathcal{O}_{\mathcal{Z}_\mu}))+$$
$$p_2^*(ft_* i_{\mu *}(F(L_+, L_-) \otimes \mathcal{O}_{\mathcal{Z}_\mu})) + (F(L_+, L_-) \otimes (1 - L_+ L_-)),$$

as desired.

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