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UNIVERSITY OF CALIFORNIA, SAN DIEGO

Leibniz on the Concept, Ontology, and Epistemology of Number

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Philosophy

by

Kyle Sereda

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2017

The Dissertation of Kyle Sereda is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

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VITA

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ABSTRACT OF THE DISSERTATION

Leibniz on the Concept, Ontology, and Epistemology of Number

by

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This dissertation concerns a topic that has been unduly neglected by historians of Early Modern philosophy and philosophers of mathematics alike: the highly original conception of number advanced by Gottfried Wilhelm Leibniz in the seventeenth and early eighteenth centuries. I aim to answer several questions regarding that conception, thereby illustrating its historical and philosophical importance: (1) How does Leibniz define the concept of number?; (2) Into which ontological category does Leibniz think numbers fall?; (3) Which sorts of numbers -- e.g. rational, irrational, complex -- does Leibniz think are conceptually legitimate, and to what extent does he realize that his own

definitions commit him to the acceptance of certain kinds of numbers as such?; and (4) How does Leibniz think we acquire knowledge about numbers?

In the course of answering these questions, I aim to show that Leibniz's conception of number is philosophically significant insofar as it unites the most productive aspects of earlier conceptions into one that goes a long way toward allowing him to accommodate numbers that had not been previously viewed as conceptually legitimate (e.g. irrationals and complex numbers); provides an original ontology of number as a certain kind of relation; and anticipates the core views of the logicist school in the philosophy of mathematics.

The dissertation is organized in a way that reflects the four core questions: I begin by discussing the intellectual climate in seventeenth-century mathematics in Chapter 1; I move on to an analysis of Leibniz's conceptual characterization of number in Chapter 2; I argue that this characterization is consistent with Leibniz's ontology of number (and explain the nature of that ontology) in Chapter 3; I discuss the scope of Leibniz's view of number in Chapter 4, arguing that he is committed to the existence of different sorts of non-rational numbers, while also delineating the conceptual and technical limitations of his views; I explain his epistemology of number in Chapter 5; and I close by arguing that his views -- conceptual, ontological, and epistemological -- anticipate those of the logicists in Chapter 6.

General Introduction

In philosophical circles, Leibniz is known primarily as a systematic metaphysician responsible for the theory that the world, at bottom, consists of mind-like entities and the results of their coordinated perceptions. In mathematical circles, he is known primarily as one of the founders the differential and integral calculus. Leibniz is not primarily known, in either circle -- in fact, he is barely recognized at all -- as a philosopher of mathematics, save for his well-documented and oft-debated views on the metaphysical and methodological status of infinitesimals and infinite numbers. The organizing principle of this work is that Leibniz is, and should be seen as, a philosopher with deeply sophisticated views on the subject matter of mathematics and the ontological and epistemological status of the objects under its purview. Furthermore, his views on the most basic questions in the philosophy of mathematics -- such as “what is a number?”, and “how do we acquire mathematical knowledge?” -- are of a piece both with his general metaphysical system and with his aforementioned views on the fundamental concepts of the calculus. Moreover, Leibniz's views on these most basic questions appear to arise in response to, and to be intended to improve upon, views of certain predecessors and contemporaries, and this intention should be viewed as at least a partial success.

The purpose of this work is to investigate, and propose answers to, several interrelated questions that make good on its organizing principle. The questions are as follows: (1) What are numbers, according to Leibniz? In other words, how does Leibniz define the concept of number? (2) What is the ontological status of numbers, once their

definition is understood? In other words, what kind of objects are they? (3) What sorts of numbers exist, according to Leibniz, and what sorts of numbers do not? Finally, (4) how does Leibniz think we acquire knowledge of numbers?

Each of these questions, it turns out, touches some or other widely known area of Leibniz studies, or of the philosophy of mathematics, or both. As will become evident, his answers to the first and second questions are intimately related to his metaphysical system and are intended -- as I say above, with partial success -- to improve upon the relevant views of those few mathematicians and philosophers who had explicitly proposed theses on the concept of number and the subject matter of arithmetic before and during the early modern period. His answers to the third question turn out to illuminate several aspects of Leibniz's philosophy of mathematics: first, the extent of Leibniz's anticipation of the modern conception of real number; second, the limitations of that anticipation insofar as Leibniz is ultimately unable to provide a mathematically rigorous characterization of the irrationals; and third, the deep conceptual difficulties that his conception of number encounters insofar as he seems unaware of his own commitments to the existence of numbers whose existence he explicitly denies. Finally, Leibniz's answer to the fourth question provides a case study that illuminates the harmony between his metaphysics and his epistemology, in addition to clarifying a deep scholarly confusion about his view on the role of different mental faculties in our acquisition of mathematical knowledge.

If there were one main claim that this work could be understood as attempting to establish, it would be that Leibniz's conception of number is philosophically fruitful along two broad dimensions. First, it is fruitful insofar as it unites and improves upon the most

productive aspects of earlier views into a view that allows him at least to *begin* formulating a conception of number that countenances the irrationals without reducing the concept of number to the concept of geometrical magnitude. Second, it is fruitful in that along the way, Leibniz also proposes a robust metaphysics and epistemology of number that anticipates later developments in the philosophy of mathematics -- particularly the views of the founding members of the logicist school.

This work is organized in the following way. I begin, in Chapter 1, by explaining the philosophical and mathematical situation with respect to the concept of number in the seventeenth century. During this time, Leibniz's predecessors and contemporaries attempt to expand and precisify this concept in response to developments in mathematics that require the field to admit non-integral numbers as conceptually legitimate, in opposition to the view received from (among others) Euclid and Aristotle. This chapter focuses on two opposing views, due to Barrow and Wallis, respectively, on the related questions of which numbers count as mathematically legitimate, and what kind of objects numbers might be.

In Chapter 2, I treat Leibniz's general *definition* of number and how it subsumes the positive rational numbers, while remaining agnostic on the question of Leibniz's *ontology* of number. This chapter describes Leibniz's conceptual characterization of number as a certain kind of *aggregate*, laying the groundwork for my subsequent exploration of the extent to which Leibniz's general definition of number might be able to accommodate -- as is his stated intention -- the more conceptually problematic case of irrational numbers. In Chapter 3, I show that Leibniz is committed to a view of numbers as a certain kind of *relation*, resolving the apparent inconsistency between this ontological

conception and his conceptual definition of the positive rationals as aggregates and establishing that Leibniz is best understood as a Platonist about numbers.

In Chapter 4, I leave the territory of the positive rationals and systematically treat Leibniz's views -- some of which he does not appear aware that he holds -- about non-rational and negative numbers, where non-rational numbers include irrationals, complex numbers, infinite cardinals, and infinitesimals. This chapter delineates the limitations of Leibniz's ability, given his account of number, to carry out his apparent intention to establish that irrational numbers exist. Additionally, I argue that Leibniz is committed to the existence of negative and complex numbers despite his adamant statements to the effect that such numbers are conceptually incoherent and so do not exist. Chapter 5 treats Leibniz's epistemology of mathematics; there, I have a positive goal and a negative goal: the positive goal is to establish what Leibniz's epistemological views are with respect to number, while the negative goal is to dispel a longstanding scholarly confusion about them, establishing in the process what his views cannot be. Finally, Chapter 6 explores the extent to which Leibniz's views on number -- conceptual, metaphysical, and epistemological -- can be seen as anticipating the views of the logicians in the late nineteenth and early twentieth centuries.

Chapter 1: The Status of Numbers in Leibniz's Time

1. Introduction

The goal of this chapter is to describe the status of the concept of number in Leibniz's time, providing a framework against which his account of number can be seen as unprecedented. Once this framework is in place, we will be able to see more clearly the precise way in which Leibniz combines various traditions in mathematical thought in order to formulate an account of number that moves beyond them and anticipates the modern conception of real number. My description of the relevant trends in mathematical thought is necessarily selective, focusing on the ideas that Leibniz either explicitly *rejects* or explicitly *adopts* in formulating his philosophy of arithmetic. I do not aim to provide an exhaustive summary of attempts to think about numbers from the Greeks to the early modern period. I do, however, aim to illuminate the relevant features of the intellectual landscape in which Leibniz's account of number, and his philosophy of mathematics in general, is situated.

I proceed in four stages: first, I describe the way in which Greek mathematicians and philosophers think of numbers, or at least the tradition in Greek mathematical thought that Leibniz's contemporaries and immediate predecessors inherit and grapple with. I focus on the views of Euclid and Aristotle, who encapsulate a more general trend in Greek mathematics. Next, I treat early modern thought about numbers along two related but distinct dimensions: first, the ways in which early modern mathematicians *define* number -- the way they delineate what counts as a genuine number -- second, the ways in which

early modern mathematicians and philosophers conceive of the *ontology* of number. Finally, I look ahead to subsequent chapters by providing a brief synopsis of the way these lines of thought inform Leibniz's own distinctive account of number.

2. Number in Greek Mathematical Thought

2.1. The Euclidean/Aristotelian Definition of Number. It has been well-documented -- for example in Klein (1968) -- that Euclid's definition of number, which also can be found in earlier authors such as Aristotle, plays a significant role in early modern mathematical thought, to the extent that much of the pioneering work in this period involves implicitly or explicitly rejecting it. Many mathematicians, for example the English algebraists profiled in Neal (2002), *implicitly* reject the definition by accepting fractional and even irrational solutions to various equations. A smaller group of mathematicians -- most prominently Stevin and Barrow -- *explicitly* reject it by formulating entirely novel definitions of number. I eventually show that none of these attempts belongs in the same category as Leibniz's, but before we examine them, we must first get clear on Euclid's definition, which is found at the beginning of Book VII of the *Elements*, a work that Dirk Struik duly notes is "next to the Bible, probably the most reproduced and studied book in the history of the Western world" (1987, 49). In Thomas Heath's standard translation, Definition 1 states that "a unit is that by virtue of which each of the things that exist is called one"; immediately following this, Definition 2 says that "a number is a multitude composed of units" (1956, 277).

For Euclid, then, the only genuine numbers are the positive integers. Heath notes in a footnote to Definition 2 that

[Euclid's] definition of a *number* is... only one out of many that are on record. Nicomachus [says] that it is "a defined multitude, or a collection of units, or a flow of quantity made up of units". Theon... says: "a number is a collection of units, or a progression of multitude beginning from a unit and a retrogression ceasing at a unit". According to Iamblichus the description "collection of units" was applied to the *how many*, i.e. to number, by Thales... while it was Eudoxus the Pythagorean who said that a number was "a defined multitude". (280)

In other words, Euclid's definition can be safely viewed as encapsulating the general Greek view of number. This is to be expected; the *Elements*, as a comprehensive textbook, is at least a partial compendium of Greek mathematical thought, in which Euclid "bring[s] together" various "discoveries of the recent past" (Struik 1987, 50). For Greek mathematicians, only the positive integers count as genuine numbers, and "our conception of real number [is] unknown" (ibid, 60). Further surviving examples of this conception of number can be found throughout Aristotle's corpus, as Heath notes later in the same footnote. For example, at *Metaphysics* 1039a, Aristotle notes, and seems to accept, what he labels the "popular assumption... that number is a combination of units" (2004). And at *Physics* 207b, he says that "any given number is a plurality of ones, a particular quantity of them" (2008). These are just two among many examples. For Aristotle and Euclid, and in Greek mathematics more generally, numbers are discrete collections of units; any magnitude not measurable by such a collection simply fails to be associated with a genuine number. As Struik notes, "a line segment did not always have a length", in the sense of a number indexed to a unit of measurement (1987, 60). One would expect, then, that a major episode in the history of mathematics might consist in the rejection of this

severely limited conception of number; this is precisely what occurs in the early modern period, and I discuss it shortly. But we must first address another aspect of Greek mathematical thought that was taken up by early modern mathematicians -- one that is equally relevant to Leibniz's eventual characterization of number.

2.2. Two Conceptions of Numerical Ontology in Greek Thought. In the first chapter of *Metaphysics M*, Aristotle describes a fundamental problem that still occupies a central place in the philosophy of mathematics:

It is necessary, if the objects of mathematics have being, that they are either in perceptible things as some say or separate from perceptible things (for there are those who think that); and if they have being in neither way, then either they just don't have being or they have being in some other way. Our controversy, then, turns not on whether they have being but on the mode of their having being. (2004, 1076a)

In other words, do mathematical objects exist independently of their concrete instances? Is there a "number three", for example, over and above the various collections of three objects in the world? Aristotle's answer is characteristic, again, of a trend in the philosophy of mathematics that still abides today, and that is taken up by the more philosophical early modern mathematicians: "the objects of mathematics are not substances to a greater degree than bodies nor prior in being to perceptible things" (ibid, 1077b). Aristotle's argument for this conclusion is famously complicated; its details need not concern us here. What matters is the position he adopts: that mathematical objects do not exist over and above concrete, perceptible things. Thus, the mathematical sciences do not study a particular kind of object; they study ordinary objects at a high level of abstraction, such that we ignore all their properties except the ones that are mathematically

relevant, like an object's being of a certain shape, or a collection's having so many discrete elements. Aristotle says that

Universal assertions in mathematics are not about separable entities which are beyond and apart from magnitudes and numbers. They are about these very things, only not qua such things as have magnitude and are divisible. So clearly there can be both assertions and demonstrations in connection with perceptible magnitudes, not, however, qua perceptible but qua their being of a certain sort. (2004, 1077b)

Mathematics studies shape and magnitude abstracted from perceptible, concrete things that have shape and magnitude. Another way to put this is to say that mathematics studies perceptible things, as Aristotle says, insofar as they are "of a certain sort" -- namely, insofar as they possess shape and magnitude, and *only* insofar as they have these features, i.e. excluding from consideration any other features that they might have. Mathematical statements, accordingly, are "separable from the question what each such thing is and what accidental features it has" (ibid), but they are not *about* a separate realm of abstract mathematical objects.

The view that mathematical statements do not reach out to a realm of mathematical objects is a straightforward denial of Plato's view that there *are* mathematical objects separate from concrete things, and that mathematics studies them, rather than studying ordinary objects at a high level of abstraction. Indeed, Books M and N of the *Metaphysics* contain some of Aristotle's most sustained and detailed attacks on Plato's general ontological scheme, at whose core lies the thesis that a realm of unchanging, intelligible *ante rem* universals -- the Forms -- underlies and makes possible the realm of fluctuating and unreliable perceptual experience. A particularly succinct exposition of Plato's views

on mathematical ontology can be found at *Republic* 510, where Socrates explains to Glaucon the metaphysical underpinnings of mathematical practice:

I think you know that students of geometry, calculation, and the like hypothesize the odd and the even, the various figures, the three kinds of angles, and other things akin to these in each of their investigations, as if they knew them. They make these their hypotheses and don't think it necessary to give any account of them, either to themselves or to others, as if they were clear to everyone. And going from these first principles through the remaining steps, they arrive in full agreement... You also know that, although they use visible figures and make claims about them, their thought isn't directed to them but to those other things that they are like. They make their claims for the sake of the square itself and the diagonal itself, not the diagonal they draw, and similarly with the others. These figures that they make and draw, of which shadows and reflections in water are images, they now in turn use as images, in seeking to see those others themselves that one cannot see except by means of thought. (1992, 510c-e)

Here Plato, through the mouth of Socrates, lays out the core of his philosophy of mathematics: mathematics studies objects “that one cannot see except by means of thought” -- such as “the square itself” and “the diagonal itself” -- of which diagrams and figures are images. The examples in this passage are geometrical, but the same line of thought can easily be carried over to arithmetic: any concrete representation of a number, for example three vertical lines drawn in the sand, is merely an image of that number *itself*, in this case three itself. Jacob Klein puts this point nicely:

[Geometers] draw certain figures and exhibit their properties; yet they do not intend the drawn figure itself but that which is imaged in this figure, e.g., the rectangle which is... accessible only to thinking... Similarly, logisticians have before their eyes the “odd” and the “even” in the shape of certain countable objects which they reflect on, but these reflections, being pursued in thought, are aimed not at these particular objects but at the “pure” numbers... (1968, 72)

Contra Aristotle, mathematical assertions, both geometrical and arithmetical, do not concern concrete things at a certain level of abstraction: they concern an entirely different kind of thing existing in a realm accessible only to thought.

This basic ontological debate is taken up in the early modern period by several important mathematicians and philosophers. In particular, the question whether *numbers* exist independently of *numbered things* finds expression in, and constitutes an integral part of, the sometimes heated debate over the relative priority of arithmetic and geometry. I will explain in due course the relationship between these questions; for now, it will suffice to note that they play a significant role in Leibniz's philosophy of mathematics in general and his account of number specifically.

3. Number in the Early Modern Period

3.1. The Definition of Number in the Early Modern Period. As Katherine Neal notes in her study of the treatment of number in early modern Britain (2002), the sixteenth and seventeenth centuries are characterized by the increasing acceptance and use of non-integral numbers in mathematical practice. For example, the three English algebraists she profiles -- Robert Recorde, William Oughtred, and Thomas Harriot -- all make wide-ranging use of fractions and irrational numbers in their manipulation and solution of many different kinds of algebraic problems. At the same time, though, "none of these mathematicians shows signs of being overwhelmed with ontological worries about the true nature of numbers" (78). Nor do they show signs of concern with the proper *definition* of number -- i.e. with the question of what counts as a genuine number in the first place,

aside from the ontological question of what numbers are, metaphysically speaking. Additionally, the pioneers of logarithms, Napier and Briggs, work freely with decimal expansions of irrational numbers and both treat number as a kind of continuum starting from zero. But for these practitioners, even more so than for the algebraists, we find a pronounced lack of concern with philosophical questions about the nature or the legitimacy of numbers falling outside the Euclidean definition. As Neal puts it, “there was no available foundation for the real numbers, but the utility of the logarithms necessitated that this lack of proper foundations be pushed aside” (114).

Some mathematicians, however, do explicitly concern themselves with the question of what counts as a genuine number. Simon Stevin, the sixteenth-century Dutchman, stands out as arguably the first person to offer a modern-looking *definition* of number, which seems on a cursory reading to put at least all positive numbers -- positive integers, rationals, and irrationals -- in the same conceptual category and to count them all as legitimate numbers. But we will soon see that the extent to which his definition anticipates the modern conception of real number is limited by his separation of numbers into two distinct conceptual kinds -- not, as one might expect, along the dimension of integral versus fractional, or rational versus irrational, but along an entirely different dimension.

In Definition 1 of his *Arithmetic* and its associated explanatory remarks, Stevin offers the following general characterization of number in terms of its relationship to quantity or magnitude:

Number is that by which is explained the quantity of each thing. As unity is [the] number by which the quantity of [something] is called one; and two [the number] by which we call it two; and one half [the number] by which we call it one half; and the square root of three the number by which we call it the square root of three, etc. (1958, 495)

What immediately stands out about this definition, in stark contrast to the one received from the Greeks, is Stevin's inclusion under it of every kind of what we now call positive real numbers -- he gives as examples one, two, one half, and the square root of three. For Stevin, every quantity is associated with a number. He goes on to make it explicit, "against the vulgar", that "number is not discontinuous quantity"; number is not to be conceived as a discrete collection of units (ibid, 501). Later on, he declares that "there are no absurd, irrational, irregular, inexplicable, or surd numbers" (ibid, 532). The early pages of the *Arithmetic* even contain an argument that directly associates the increasing arithmetical continuum with an increasing line, using the example of the number 60 as generated from zero by incremental addition, visualized in terms of the increase of a line (ibid, 499). And as Katherine Neal notes, for Stevin, lest we misunderstand him, "the relationship between number and magnitude [is] supposed to be more complex than the one being a name, or label, for the other" (2002, 35). The view that Stevin does *not* hold -- that numbers are nothing more than names of magnitudes -- is actually advanced unambiguously by Barrow, and we will have occasion to examine it shortly. Stevin's view, on the other hand, is that

[N]umber is something in magnitude, like humidity in water, for as [humidity] extends throughout all the water and each part of the water, so the number assigned to a given magnitude extends throughout all the magnitude and each part of the magnitude: as to one continuous water corresponds one continuous humidity, so to one continuous magnitude corresponds one continuous number..." (1958, 498)

I am attempting here to keep separate, as far as possible, definitional and ontological questions, and I return to the latter aspect of Stevin's view in the next section, comparing it with Barrow's.

It would appear, then, that Stevin offers a strikingly modern definition of number that places all numbers on the same conceptual footing. But careful attention to other sections of the *Arithmetic* reveals a fundamental tension in Stevin's conception of number that renders clear its limitations. To introduce this tension, let us return to the place where Stevin declares that there are no "irrational" numbers, in the sense that "irrational" (or "surd", "inexplicable", etc.) is an unfair misnomer for what is really just another subcategory of genuine number. Immediately after making this declaration, Stevin elaborates in the following way:

It is a very common thing among the authors of [arithmetical textbooks] to treat numbers like the square root of 8, and similar numbers, which they call absurd, irrational, irregular, inexplicable, surd, etc... But why?... It appears to me in the first place that a root is incommensurable with an Arithmetical number (like 3 or 4), therefore the square root of 8 is absurd, irrational, etc. But the conclusion is absurd... incommensurability does not cause the absurdity of the incommensurable terms... (1958, 532)

Here, in the same place where he argues against the very idea of a "surd" number, and argues for the inclusion of irrationals in the class of genuine numbers, he also hints at a basic distinction that prevents his conception of number from having the full generality that it at first appears to have. We see that he classifies numbers "like 3 or 4" as "arithmetical" numbers and seems to exclude irrational roots from this classification. Indeed, Stevin separates numbers into two distinct conceptual categories: the arithmetical numbers, which are those "that one explains without an adjective of magnitude" (1958, 504), and the geometrical numbers, which are numbers "explaining the value of a geometrical quantity" and which "have the name corresponding to the kind of quantity that [they] explain" (*ibid*, 528). Now, Stevin takes care to clarify that both kinds of

numbers are numbers properly so called, but at the same time, he thinks that only the positive integers fall under the former classification, and that roots fall under the latter. And while positive integers can also be geometrical numbers when they are being used to describe geometrical quantities, like squares, it does *not* appear to be the case that irrational roots can ever be arithmetical numbers; Stevin appears to have no way to make sense of irrational roots without recourse to geometrical concepts. He makes this clear in his explanation of the definition of arithmetical numbers:

Number has two kinds, of which one is explained with an adjective of magnitude, like square numbers, cubic numbers, roots, etc., which we call geometrical numbers... the other kind is simply explained without such an adjective, like one, two, three, three fifths, etc. We call such numbers arithmetical numbers, in distinction to the other kind. (ibid, 505)

So while the number nine, for example, is an arithmetical number but can also be understood as a geometrical number when used to describe a square with a side of three, the square root of two can *only* be understood as a geometrical number -- as a number describing the side of a square with an area of two. It cannot be understood without an “adjective of magnitude”. Interestingly, Stevin categorizes all positive rational numbers as arithmetical -- the only numbers that can’t be either arithmetical *or* geometrical are the irrational roots that are still nevertheless numbers. Ultimately, he does not quite succeed in putting irrationals in the same definitional category as integers and fractions. I would suggest that this is due to his excessively general initial definition of number as that by which the magnitude of things is explained. This definition appears extremely promising, but leaves Stevin with no choice but to characterize irrational roots as a different kind of number from integers and fractions. Scholars like Klein and Neal have noted the modern

feel of Stevin's treatment of number -- especially insofar as he associates number with a continuous line -- but in the process, they gloss over the fundamental conceptual distinction he ends up making between irrationals and other numbers. Stevin's treatment of number comes the closest to Leibniz's out of those we encounter here, but its initial appearance of generality, ironically, prevents it from being general enough to encompass all the positive real numbers.

Another mathematician who explicitly associates number with continuous quantity as a matter of definition, though differently from the way Stevin does, is Isaac Barrow. Barrow's characterization of number is inextricably tied up with his conception of the subject matter of mathematics, and as such cannot be treated fully in this section; he associates number with continuous magnitude as a matter of definition *and* as a matter of metaphysics, so that even his general definition of number contains a deeper claim about mathematical objects. Nonetheless, it is useful to examine his characterization of number while bracketing its metaphysical aspects as much as possible. In Lecture III of his *Mathematical Lectures*, Barrow says that number "is not something having a proper existence for itself, truly distinct from the magnitude which it signifies, but merely a certain mark or sign of magnitude itself considered in a definite way" (1860, 56). If numbers are mere signs of magnitude, then presumably every magnitude is associated with a number, and all numbers count as genuine numbers. This is indeed the case for Barrow, and much of Lecture III is devoted to explaining the way integers, fractions, and irrationals fit into this scheme -- in other words, to an explanation of the precise relationship between each kind of number and the magnitude it signifies. The first two

cases are quite simple: as Neal puts it, “integers [are] the symbol of magnitude that [is] composed of equal parts... if the magnitude A consist[s] of six equal parts, then A would be called six. Fractions [are] to be thought of as a sort of ratio, or comparison, between magnitudes composed of the same type of parts” (2002, 134). Barrow actually manages to illustrate the expressive role of integers and fractions at the same time in one of his examples. After claiming that a line composed of three equal parts is signified by the number three, he says that “if we should conceive that [a line] A is built up from seven equal lines (or that it can be divided into seven equal lines), [and] another line composed from ten of the same kind of parts... then line A is called seven, or seven of ten ($7/10$) parts of line B” (1860, 57). In other words, fractions signify ratios between magnitudes with a common measure. These need not be lines; lines simply provide Barrow with the easiest route to a perspicuous example.

The case of irrational numbers requires a separate exposition. How does Barrow conceive of the way irrationals signify magnitudes? He puts it in the following way:

Radicals or surd numbers are signs of any magnitude, showing distinctly that it is in whatever manner a mean in a proportion between any assumed homogeneous magnitude, composed equally, according to exigency, of an appropriate number, either whole or fractional, and a part of it serving in place of unity... as the second or square root of the number 3 denotes a mean proportional between whatever assumed magnitude and its triple. (ibid, 58)

Put a bit more clearly, irrational numbers signify a special kind of comparison between magnitudes: that which occurs when no common measure can be found between them, i.e. they cannot both be divided into the same kinds of parts. Barrow’s point is that any given magnitudes can always be compared, even if they lack a common unit of

measurement into which they can both be divided; and if all numbers are just signs of magnitude or of a particular comparison between magnitudes, then irrational numbers are just as admissible as integers and fractions. Indeed, as Neal puts it, “Barrow [is] quite distressed about the exclusion of surds from the realm of numbers... because when one is measuring or comparing magnitudes the results are more often irrational than rational” (2002, 134). Almost immediately after his presentation of irrationals, he characterizes the denial that they are numbers as “without merit” (1860, 59). When I discuss Barrow’s conception of mathematical ontology, I will have more to say about his reasons for admitting irrationals into the class of genuine numbers, and about the actual import of doing this given Barrow’s radical philosophy of mathematics.

The final definition of number that I discuss in this section is that of John Wallis, a contemporary and correspondent of Leibniz’s, and one of Barrow’s staunchest opponents. Wallis actually offers the *least* modern definition of those under consideration, at least in terms of how he delineates the class of genuine numbers. In Chapter IV of his *Mathesis Universalis*, he straightforwardly adopts the Greek definition, with one interesting conceptual improvement. He begins by saying that “unity is said to be truly the principle of number. But unity is that according to which anything is called one: moreover, a number is a multitude composed from units” (1695, 24). Wallis then explicitly cites the first two definitions of *Elements* VII as his source. But immediately following this, he argues against the Greek idea, which can be understood as a corollary of those definitions, that one is not a number. Incidentally, Stevin had also argued in his *Arithmetic* that one counts as a genuine number as against the Greek definition, but Stevin

approaches the concept of number at the outset in a radically different way from the Greeks, as has been noted. For Wallis, the idea that one is a number represents a much less significant conceptual modification of the Greek definition -- indeed, it is arguably the only such modification he wants to make. “Not unity”, Wallis argues, “but non-existence... or nothing has the same status with respect to numbers that a point has with respect to magnitude and a moment has with respect to time” (ibid, 25). Zero, in other words, is the beginning or generative principle of number in the way that a point is for magnitude and an instant is for time. This idea appears to have been in the air explicitly - - not just implicitly in mathematical practice -- during the seventeenth century; on the continent, we find a similar argument in Pascal’s “Of the Geometrical Spirit”:

[i]f we wish to take in numbers a comparison that represents with accuracy what we are considering in extension, this must be the relation of zero to numbers; for zero is not of the same kind as numbers, since, being multiplied, it cannot exceed them: so that it is the true indivisibility of number, as indivisibility is the true zero of extension. And a like one will be found between rest and motion, and between an instant and time... (1914, 436)

One, both Pascal and Wallis think, is not only the generative principle of number, but is itself a number just like the other positive integers -- even if, as Wallis thinks, number is a multitude composed of units. We can consider one as a multitude of units if we “understand by ‘multitude’ something looser” that is usually understood. In other words, the number one is the degenerate or limiting case of a multitude of units. Either way, one is a number.

Wallis also denies that fractions are genuine numbers. We see this in the first instance in the title of Chapter IV, where he asserts that “broken numbers [i.e. fractions]

are not true numbers” (1695, 24). He takes up this topic near the end of that chapter, where he says that “a fraction is not so much a number as an index of a ratio of numbers to each other. And therefore the smallest number is one, and not either zero or a fraction” (ibid, 27). Interestingly, Wallis thinks, as Barrow does, that fractions serve to indicate a relationship between magnitudes divided into the same parts. The argument of the section on fractions begins with a concession that it seems that “many things are given between zero and one”, but proceeds to claim that fractions

do not answer [the question] ‘how many?’ but ‘how much?’: for one does not say *how many* of time passes, but *how much* of time, or how many parts of time... Therefore this response pertains not so much to a discrete quantity, or number, but to a continuous quantity; just as an hour is taken to be a continuum divisible into parts; indeed a ratio of however many of these parts to the whole is expressed by numbers. (ibid)

Now, Wallis does think that fractions can be investigated without reference to any particular ratio between magnitudes, and indeed that they should be so investigated. In his *Arithmetica Infinitorum*, for example, he famously cements his place in the history of the calculus by investigating the sums of certain infinite series of fractions. But at the same time, he does not want to admit fractions into the category of genuine numbers. Much later in the *Mathesis Universalis*, he actually devotes a whole chapter to fractions -- Chapter XLI, near the conclusion of the work -- whose early paragraphs contain both a denial that they are genuine numbers and an explanation of their expressive role very similar to the one I have just translated. Katherine Neal notes that Wallis also denies that fractions are numbers in other works, for example in the *Treatise of Algebra* (2002, 154). She also notes that in the *Treatise of the Angle of Contact*, he states that arithmetic and algebra can legitimately operate on fractions and even irrational numbers without

reference to particular geometrical magnitudes, but “never clearly states that fractions and surds are numbers, merely that one could operate on them” (ibid). As has been noted, Wallis carries out exactly such a procedure in his contributions to the invention of the calculus. So there appears to be a deep tension in Wallis’ philosophy of arithmetic: on the one hand, only the positive integers are genuine numbers, but on the other hand, the mathematician can operate on abstract fractions and even abstract irrational numbers -- i.e. fractions and irrationals not explicitly grounded in any particular magnitude ratios. We will have occasion to examine this tension in more detail when we come to Wallis’ conception of mathematical objects, and of the relationship between arithmetic and geometry, in the next section.

3.2. The Metaphysics of Number in the Early Modern Period. Having discussed Wallis’ definition of number, we are now in a position to move on to an investigation of the way early modern mathematicians took up the other trend in Greek thought that I described at the beginning of this chapter -- the fundamental conflict over the ontological status of numbers exemplified by the positions of Plato and Aristotle. The question, we recall, is whether numbers exist independently of numbered things; I would like to suggest that in the early modern period, it finds new expression in the sometimes heated debate over the relationship between arithmetic and geometry. This debate, whose participants include Barrow, Wallis, and eventually Leibniz himself, is touched off by the dissemination of the techniques of the new symbolic algebra, pioneered and enshrined in Viète’s 1591 *Analytic Art*, and the new analytic geometry encapsulated in Descartes’

Geometry, published in 1637. Douglas Jesseph outlines the terms of the dispute succinctly in his (1999):

The advent of analytic methods provoked a philosophical debate on the question whether arithmetic or geometry was the genuinely foundational discipline in mathematics. Classical mathematicians distinguished discrete quantity (“number”) from continuous quantity (“magnitude”), declaring the former to be the object of arithmetic and the latter to be the proper object of geometry. Classically, then, geometry and arithmetic are distinct sciences with no common object, so there is no need to ask which is the more fundamental science. This situation changed with the development of analytic geometry. Many interpreted algebra as a kind of generalization of arithmetic... the basic principles of algebra were seen as deriving from arithmetic, and the prominence of algebraic methods in analytic geometry led some to conclude that geometry must, in some important sense, be based on arithmetic. (37)

One can quibble with Jesseph’s claim that classical mathematicians all treat geometry and arithmetic as “distinct sciences with no common object”, and see “no need to ask which is the more fundamental science”. Aristotle does claim, for example, that numbers are in some sense more precise than geometrical magnitudes: since the basic element of number is the unit, numbers measure numbered collections with a greater degree of precision than do geometrical magnitudes (*Metaphysics* 1052b). And some early modern mathematicians, such as Barrow, present themselves as arguing against earlier mathematicians who appear to take a stand on the question whether arithmetic or geometry is more fundamental. Barrow, in Lecture III of his *Mathematical Lectures*, represents the Hellenistic mathematician Nicomachus and the Renaissance mathematician Maurolico as claiming, in one way or another, that arithmetic is more fundamental than geometry. But the important point is that Descartes’ marriage of algebraic and arithmetical techniques with the study of geometrical magnitudes in the

Geometry raises questions about the relationship between arithmetic and geometry. Indeed, as Mancosu notes, the new method raises more general questions about the *interrelation* between the *three* fields, giving rise, as he says, to “endless discussions as to the status of algebra -- Is it an art or a science? -- and as to the relationship between algebra, arithmetic, and geometry” (1996, 85).

In the seventeenth century, the debate over the relationship between these fields is framed as one about the ultimate subject matter of mathematics, but it also necessarily involves deep questions about the nature of mathematical objects such as numbers and geometrical magnitudes. Inevitably, the answer one might give to the question of the ultimate subject matter of mathematics will presuppose or imply one thesis or another about the ontological status of whatever it is that mathematics ultimately studies. Perhaps it is fitting, then, that Descartes himself never quite articulates answers to the questions that interest us here -- the questions of the definitional characterization and the ontology of number. The *Geometry* itself contains no discussion of these questions, and when Descartes does address philosophical issues concerning number -- for example in Rules XII and XIV of the *Rules for the Direction of the Mind*, he is mainly interested in questions such as the proper role of the imagination in representing numbers in relation to numbered things (CSM 1:61). In general, his philosophy of mathematics, as documented, for example, in Sasaki’s comprehensive study (2003), is primarily concerned with the nature of mathematical demonstration, mathematical cognition, and the epistemology of mathematics. We will have occasion to remark on Leibniz’s rejection of certain Cartesian epistemological theses later in this work.

It is true that one finds remarks in the *Principles* to the effect that number is a mental entity: for example, “number [differs] from what is numbered, not in reality, but only in our conception” (2.8), and that “number, when considered abstractly or generally and not in created things, is but a mode of thinking; and the same is true of everything called universals” (1.58). Descartes appears to put himself in league with the medieval conceptualist tradition, which holds universals to be mental entities, but he does not expend much effort exploring these claims in detail. As far as this chapter is concerned, Descartes is best understood as igniting -- not directly participating in -- the debate about the ultimate subject matter of mathematics, and the related debate about mathematical ontology, that finds its most sustained expression in the positions adopted by Barrow and Wallis. Barrow couples his reduction of all mathematics to geometry with a position on the ontological status of numbers and even a position on the ontological status of geometrical magnitude itself, and Wallis adopts *his* subordination of geometry to arithmetic largely *because of* a position he holds on the nature of numbers. As we will see, the metaphysical views of Barrow and Wallis mirror in important ways the two sides of the Greek debate over the ontology of number and of mathematical objects more generally.

I begin with an examination of Barrow’s views. As early as Lecture I of his *Mathematical Lectures*, Barrow dismisses what he represents as the ancient division of mathematics into the study of magnitude and multitude, locating the ultimate subject matter of mathematics in quantity (1860, 30). In Lecture II, he goes on to clarify what he means by this -- after all, we might think that multitude is a kind of quantity -- saying that

“no other quantity exists different from that which is called magnitude, or continuous quantity”, and that this is “the only object of mathematics” (ibid, 39). If the ultimate subject matter of mathematics is continuous quantity, then for Barrow, all branches of mathematics -- pure and applied -- are just so many ways of doing geometry. There is not even a place in Barrow’s framework for arithmetic as a distinct branch of pure mathematics; there is no point in studying the properties of numbers in an abstract setting, because numbers simply do not exist in themselves. We encountered this line of thought in the previous section when we saw Barrow define numbers as signs of magnitude without any proper existence of their own. As Mahoney points out, this is a multifaceted view. On the one hand, Barrow

[sees] no need to posit the existence of numbers independent of things counted: $2+2$ cannot be 4 unless the addends are two each of the same things and those things can be combined. The combinatorial properties of numbers are rooted in those of the objects being combined, not in the numbers themselves. (1990, 186-187)

Numbers, for Barrow, do not exist independently of numbered discrete collections. But mathematics is not the study of discrete collections -- it is the study of continuous quantity -- so there is a deeper view about the relationship between numbers and magnitude here. Fundamentally, numbers are signs of *continuous* magnitudes considered in certain ways; that is, “they enumerate collections of units precisely equal to one another or they measure magnitudes with reference to a common unit... [they] symbolize magnitudes conceived of as units, collections of units, or ratios of such collections” (ibid, 187-188). This is another way of putting Barrow’s exposition of the way that different kinds of numbers --

integers, fractions, and irrationals -- correspond to different ways of considering continuous magnitudes, some of which I translated in the last section.

The similarities and differences between Barrow's conception of number and that of Stevin, the first mathematician whose work we explored in the last section, are worth noting, and will further illuminate Barrow's metaphysical views. Barrow and Stevin both conceive of number in a continuous way: for both of them, all positive real numbers are genuine numbers on a conceptual par with one another because each signifies a magnitude. Barrow arguably has a more sophisticated view of the precise *manner* in which different kinds of positive real numbers signify magnitudes. But for Stevin, it appears that at least what he calls the "arithmetical" numbers do have a kind of existence as more than mere signs: recall his somewhat cryptic assertion that numbers are "in" magnitude like humidity is in water, and so can be conceived of as separated from magnitude, or "without an adjective of size". He never characterizes numbers as mere signs of magnitude the way Barrow does, and one of his major mathematical works -- indeed, the one that provides his definition of number -- is called *Arithmetic* and is largely devoted to the study of numbers by themselves. This distinguishes Stevin from Barrow despite Stevin's eventual inability to conceive of irrational numbers in particular as separate from magnitudes. For Barrow, on the other hand, all numbers have only what we might call a *formal* existence: there is really nothing to a given number over and above the way a particular combination of signs concisely expresses a particular continuous magnitude considered in a certain way. Jesseph puts this points quite perspicuously: in

Barrow's framework, "numbers... are mere symbols *whose content derives from their application to continuous geometric magnitude*" (1999, 39).

It is important to emphasize the extent to which Barrow's conception of the subject matter of mathematics strikes the modern ear as radical. Doing so will reinforce a key claim I made in the last section: that despite the revolutionary *appearance* of Barrow's characterization of number, such that all positive reals count as genuine numbers, Barrow's uncompromisingly reductionist take on the subject matter of mathematics ultimately makes this appearance a red herring. I think this point is best put by noting that Barrow's scheme rules out number theory -- taken in a very broad sense -- as a legitimate, distinct area of mathematical investigation; indeed, Barrow spends time in Lecture III pursuing various lines of thought to the effect that number-theoretic investigations are better carried out geometrically, for example when he urges that the study of infinite series of numbers will yield better results if we use geometrical figures like line segments rather than considering the numbers by themselves (this argument can be found in 1860, 48-49). In other words, if we consider the numbers in an infinite series as they truly are -- i.e. mere signs of magnitudes considered in a certain way -- then we see that the properties of the series can be determined by considering the relations between magnitudes signified by the terms. Mancosu notes that the reductionist strain in Barrow's mathematical thought goes all the way down to the grounding of continuous quantity itself, such that "the reality of geometrical entities is grounded in their material existence" (1996, 140). Mancosu provides his own translation of some remarks from Lecture V of the *Mathematical Lectures* on this point: Barrow declares that "all imaginable geometrical

figures are really inherent in every particle of matter; I say really inherent in fact and to the utmost perfection, though not apparent to the senses...” (Barrow 1860, 84, as quoted in Mancosu 1996, 140). So for Barrow, there are no numbers existing independently of numbered things, and there are also no geometrical magnitudes existing independently of the material constitution of the universe. An investigation of Barrow’s ontology of geometry would take us outside the scope of the present work, but I mention it by way of illustrating that Barrow’s overall position on mathematical ontology resembles, broadly speaking, the option provided to early modern mathematicians by Aristotle: that mathematics studies ordinary material objects at a high level of abstraction, or only insofar as they have certain features like magnitude and shape.

In many respects, Barrow’s conception of the subject matter of mathematics, and of mathematical ontology, is similar to that of Hobbes, who is remembered primarily as a philosopher rather than as a mathematician. Hobbes’ mathematical career involves several decades’ worth of debate with Wallis over whether Hobbes had satisfactorily solved certain ancient geometrical problems that are now known to be insoluble -- e.g. squaring the circle -- which is documented thoroughly in Jesseph’s (1999). Although Hobbes contributes nothing to the progress of mathematics proper, he does offer a comprehensive materialistic *philosophy* of mathematics which, like that of Barrow, reduces all of mathematics to geometry and grounds geometrical magnitudes in material reality. The most thorough description of this philosophy of mathematics can be found in *De Corpore*, where Hobbes “attempt[s] to show how all of mathematics can be interpreted as a science of body” such that “mathematics must found all of geometry upon the

principles of matter and motion” and “mathematical objects must be interpreted as bodies or as things produced by the motion of bodies” (Jesseph 1999, 74, 77). The precise details of Hobbes’ philosophy of mathematics involve us in broader questions about the architectonic of his whole philosophical system -- particularly the nature of his radical materialism -- and I will not attempt to document them here. I mention him to establish that Barrow is not the only well-known thinker of the period to identify mathematics with geometry and to ground geometrical objects in facts about the material universe. Thus the pre-Leibnizian debate about the ultimate subject matter and ontology of mathematics involves not just Barrow on one side and Wallis on the other, but Barrow and Hobbes on one side and Wallis on the other. Indeed, Barrow mentions aspects of Wallis’ positions by name repeatedly in his *Mathematical Lectures*, and Wallis is embroiled in an acrimonious controversy with Hobbes over Hobbes’ putative geometrical achievements for good portions of both men’s lives.

Barrow formulates his metaphysical view of numbers in direct and explicit opposition to those who claim that arithmetic is prior to geometry; one such person is Wallis. Wallis exemplifies, broadly speaking, the other line of thought that one might pursue in the debate inspired by the new analytic geometry. Mancosu suggests that Wallis is a “paradigm” for the “analytical mode of thought” (1996, 145), which subordinates geometry to arithmetic and tends to view algebra as a branch of mathematics in its own right, rather than just a very useful tool for the investigation of relations between magnitudes, as Barrow conceives of it. As I mentioned earlier, Wallis freely investigates numbers detached from any geometrical magnitudes they might represent, and considers

this method to be so reliable that he uses it without hesitation to study the infinite. For example, Wallis' infamous "method of induction" in the *Arithmetica Infinitorum* involves "stud[ying] infinite series by extending to them results obtained... in finite series and tr[ying] to apply the arithmetical results thus obtained to the solution of geometrical problems", as Mancosu succinctly puts it (1996, 144). Wallis' purely arithmetical study of series represents the exact opposite of Barrow's approach, which recommends, as we've just seen, that mathematicians study series geometrically by investigating the relations between magnitudes represented by the terms. Chapters XXIII and XXXV of Wallis' *Mathesis Universalis* contain, respectively, his attempts to prove the propositions of Books II and V of the *Elements* arithmetically, translating what appear to be relations between various geometrical magnitudes into relations between numbers considered abstractly.

The opening pages of the *Mathesis Universalis* describe Wallis' conception of mathematics as "all those arts or sciences that concern themselves with quantity in a special way, either continuous quantity or discrete quantity". For Wallis, unlike Barrow, there is not just one kind of quantity; accordingly, there is not just one branch of mathematics. Pure mathematics, says Wallis, is divided into two basic disciplines according to the kind of quantity treated; furthermore, one of these disciplines is more pure than the other:

I say that there are two pure mathematical disciplines: arithmetic and geometry, of which the one is about discrete quantity, or number; and the other about continuous quantity, or magnitude. And indeed of these the one is more, the other less pure: for the subject of arithmetic is more pure and more abstract than the subject of geometry; therefore it has more

universal speculations, which are equally applicable to geometrical things and to other things. (1695, 18)

Wallis thinks that arithmetic is conceptually and ontologically prior to geometry: arithmetic studies objects that are “more pure and more abstract” -- i.e. numbers -- than those studied in geometry -- i.e. continuous magnitudes. Since the statements of arithmetic concern a kind of abstract object, they have a wide scope of application: they can be applied to geometrical magnitudes, with the help of a unit of measurement; but they can also be applied to discrete collections in general, whatever those collections might consist of. In Chapter XI of the *Mathesis Universalis*, Wallis expounds this general idea with a specific example, translated by Jesseph in his (1999):

If someone asserts that a line of three feet added to a line of two feet makes a line five feet long, he asserts this because the numbers two and three added together make five... for the assertion of the equality of the number five with the numbers two and three taken together is a general assertion, applicable to other kinds of things... no less than to geometrical objects. For also two angels and three angels make five angels. (1695, 56 as quoted in Jesseph 1999, 38-39)

The contrast between Barrow and Wallis is instructive. We have seen that Barrow appears to characterize number in a quite modern way, admitting all positive reals into the class of genuine numbers. But we have also seen that his reductive conception of the subject matter of mathematics -- reducing numbers to geometric magnitudes and such magnitudes to the material constitution of the world -- turns numbers into mere combinations of marks and eliminates the study of numbers for its own sake. All mathematics is geometry, so there is no way legitimately to study the properties of numbers in the abstract: even the study of infinite series, so important to the development of the calculus, must be carried out geometrically. So the apparent novelty of Barrow's

conception of number is severely undercut by his conception of what mathematics studies. Wallis, by contrast, adheres to an apparently more antiquated conception of number that does not even admit fractions, much less surds, into the class of genuine numbers. Yet at the same time, his *metaphysical* view of number as a kind of abstract object with many different instantiations yields a conception of the subject matter of mathematics that frees the study of numbers from the need for a geometrical basis -- a basis in continuous magnitudes.

Furthermore, Wallis thinks that arithmetic can *study* fractions and surds even if they are not genuine numbers, and even if it is difficult to conceive of them in the abstract without initial reference to magnitudes. As many scholars have noticed (e.g. Klein 1968, Jesseph 1999, and Neal 2002), he makes this clear in Chapter XXXV of the *Mathesis Universalis*, which contains his attempt to render Book V of the *Elements* arithmetically. Book V contains Euclid's presentation of the geometrical theory of ratio or proportion, conceived as relations between line segments; Wallis thinks that such relations are better represented numerically, by fractions and irrational roots. Once we represent ratios in this way, we can study them arithmetically, without the "cumbersome and particularized form of expression" required when treating them as geometrical relations (Jesseph 1999, 148). And we can do this, again, even if fractions and surds do not belong to the class of genuine numbers. The point is that they can be *treated* as detached from any particular magnitude relations they might represent, once we understand them as representing relations between integers, which *are* genuine numbers and *do* exist independently of geometrical magnitudes. In sum, Barrow's apparently modern conception of number is undercut by

his reduction of all mathematics to geometry, and Wallis' apparently antiquated conception is enhanced by his metaphysical views and his attendant theory of the scope of arithmetic.

4. Conclusion: Setting the Stage

The purpose of this chapter has been to describe the relevant features of the intellectual climate surrounding the concept of number in Leibniz's time. This task has been necessary because one of the primary goals of this work is to cast Leibniz as formulating an account of number that unites the best features of the Barrow-style conception of mathematics and the Wallis-style conception of mathematics. As we will subsequently see, Leibniz proposes an account of number, in particular, that is designed to accommodate all of the positive real numbers -- in line with Barrow's goal -- *and* to liberate the study of number from the study of geometrical magnitude. The next two chapters exhibit the way Leibniz does this for the rational numbers, while Chapter 4 investigates Leibniz's prospects, given his general account of number, for admitting the irrationals into the class of genuine numbers without grounding them either ontologically or epistemically in geometry, or indeed in any notion of quantity at all.¹

¹ Though this chapter lays out a framework within which to understand Leibniz's account of number, there does not appear to be direct evidence that he is deliberately responding to either Wallis or Barrow in formulating his own views. The Leibniz-Wallis correspondence, for example, mostly concerns technical problems in the foundations of the differential and integral calculus, rather than debate over the concept of number or debate over which numbers should count as genuine numbers. Nonetheless, it is valuable to situate Leibniz's views on number within their intellectual context, if for no other reason than that doing so helps us further appreciate what is philosophically special about those views -- as those views do go a long way toward unifying the most promising aspects of Barrow's and Wallis', generating an account of number that defines numbers in purely conceptual terms, avoiding geometrical notions, but simultaneously allows for at least a large subset of what we now call the irrational numbers into the pantheon of genuine numbers.

Chapter 2: Leibniz on the Definition of the Rational Numbers

1. Introduction

Leibniz's account of number has gone largely unnoticed by scholars. Few treatments of Leibniz's thought make mention of it; the most prominent general work that addresses his conception of number is Russell (1937), but it merely mentions Leibniz's view, making no attempt to explain it. Furthermore, while there is a large literature on particular aspects of Leibniz's philosophy of mathematics -- for example, the nature of infinitesimals, the foundations of the calculus,² and the *analysis situs*³ -- this literature barely touches on Leibniz's conception of number.⁴ The aim of this chapter is to answer the first question proposed in my general introduction, with respect to the rational numbers. Chapter 3 addresses Leibniz's view on the ontological status of rational numbers, and Chapter 4 addresses the more problematic case of the extent to which Leibniz's definitional and ontological theses might -- or might not -- allow him to fulfill his apparent intention to subsume irrational numbers under his general account of number. Here, I first explore how Leibniz defines the positive integers, and then show how his definition of them generalizes to all positive rational numbers. In section 2, I

² E.g. Bos (1974); Ishiguro (1990); the chapters collected in Goldenbaum and Jesseph (2008); Mancosu (1996); Knobloch (2002).

³ E.g. De Risi (2007).

⁴ Exceptions are found in Grosholz and Yakira (1998) and De Risi (2007). The former attempts to answer what I call the "definitional" question in some detail; but it is based on only two texts, and cannot be correct for reasons that I describe below. The latter is quite cursory and does not make definitive, detailed interpretative claims about Leibniz's conception of number along any of the dimensions I outline in this introductory section, and so I will not be concerned with it.

begin by analyzing Leibniz's definition of “number in general” and his more specific definition of the integers as aggregates of unities. In doing so, I argue that the interpretation of Leibniz's definition of the integers found in Grosholz and Yakira (1998) cannot be correct on textual grounds. In section 3, I note an apparent inconsistency between Leibniz's aggregative definition of the integers and his more general definition of number, and I resolve that inconsistency. Finally, in section 4, I explain the way in which fractions fall under the aggregative definition and can also be rescued from a similar apparent inconsistency. The result is a generalization of Leibniz's internally consistent definition of number to all of the positive rationals.

2. “Number in General” and Positive Integers as Aggregates

2.1. The Aggregative Conception. Leibniz's earliest reflections on the concept of number are found in the *Dissertation on the Art of Combinations*, published in 1666. Such an early work must be read with caution, as Leibniz often changes his views over the course of his career. However, in this case, Leibniz advances several core theses on numbers and their relation to magnitude that do not change in their essentials. Concerning number, he writes: “The concept of unity is abstracted from the concept of one being, and the whole itself, abstracted from unities, or the totality, is called number. Quantity is therefore the number of parts” (L 76/GP IV 35). A number -- and here Leibniz seems to have specifically the positive integers in mind -- has in this early text the role of expressing the wholeness of a collection of beings considered as unities. The view that such an expression of the wholeness of a collection is a *relation* -- as I ultimately argue

Leibniz holds -- can also be extracted from the *Dissertation*. Immediately before the text just cited, Leibniz writes that “every relation is either one of union or one of harmony. In union the things between which there is this relation are called parts, and taken together with their union, a whole. This happens whenever we take many things simultaneously as one” (L 76/GP IV 35). Leibniz seems to claim here that the basis of the wholeness of a collection is a relation; if the positive integers provide such a basis, then they must be relations -- specifically, relations of union.

This conception of the positive integers remains largely constant throughout Leibniz's career, except that he later subsumes it under a more general conception of number that is designed to accommodate all of the positive real numbers. It is worth stressing that the *general* conception of number under which Leibniz eventually includes the positive integers is absent from the *Dissertation*. I note the view of the positive integers that appears in the *Dissertation* because this view itself remains unchanged in its conceptual core; it simply becomes a specific case of the general definition of number that Leibniz advances in his mature work, under which he intends to subsume all of the positive real numbers. Finally, it is worth noting that Leibniz also provides in the *Dissertation* an early indication of the importance of number within his metaphysical framework and within his general philosophy of mathematics: “[T]he Scholastics falsely believed that number arises only from the division of the continuum and cannot be applied to incorporeal beings. For number is a kind of metaphysical figure, as it were, which arises from the union of any beings whatever; for example, God, an angel, a man, and motion taken together are four” (L 76-77/GP IV 35).

Leibniz's subsequent writings on number build on the basic conception advanced in the *Dissertation*. In many texts, Leibniz says that the positive integers are a certain kind of *aggregate*. I have ordered these texts chronologically to the extent possible:

- (1) Number is a whole composed from unities. (A.VI.4.31; 1677)
- (2) Number is [that which is] homogeneous to unity. A whole number is that of which the aliquot part is unity, or a sum of unities. A fraction is a sum of aliquot parts of unity. (A.VI.4.421; 1680-84?)
- (3) Number is [that which is] homogeneous to unity, and so it can be compared with unity and added to or subtracted from it. And it is either an aggregate of unities, which is called an integer... or an aggregate of aliquot parts of unity, which is called a fraction. (GM VII 31, undated)
- (4) An integer is a whole collected from unities. (Grosholz and Yakira, 1998, 99; c.1700?)
- (5) An integer is a whole collected from unities as parts. (Grosholz and Yakira, 1998, 88; c.1700?)

In (1), (4), and (5), Leibniz either defines number in general or the integers specifically as wholes composed of unities, mirroring his remarks in the *Dissertation*. The relational aspect of his view is not apparent here. In (2) and (3), he uses slightly different language, defining the integers as aggregates of unities. For ease of exposition, I refer to the definition of the positive integers in these texts as a definition of them as aggregates of unities. Leibniz seems to intend no difference between an aggregate of unities and a

“whole collected from unities”, as evidenced by the fact that he uses these terms interchangeably over a long stretch of years.

2.2. Grosholz and Yakira's Interpretation. Two of these texts (4 and 5) are drawn from Grosholz and Yakira's study; it is worth pausing to take account of their interpretation of Leibniz's conception of the integers in particular (though this chapter concerns all of the positive rationals), for the sake of eliminating it before I propose my own reading. Grosholz and Yakira outline an interpretation that takes into account only texts (4) and (5). The authors claim that text (5) in particular supports the thesis that Leibniz intends his definition of number to have an essentially geometric content, such that the “parts” Leibniz mentions are to be understood in the same way as the parts of a line. Their main textual evidence for the geometrical construal of “part” is Leibniz's definition of magnitude as a number of parts. They hold that in defining magnitude as such, Leibniz “associates a number with a geometrical entity” (1998, 80), and that he must think that “to understand what a whole number is, one must know not only that it can be composed of concatenated units, but also that it can be represented by line segments”, such that integers “are understood by analogy with relations among line segments” (1998, 81). My reconstruction of Leibniz's view of number, by contrast, will indicate no detour through geometrical notions, at least for the positive integers.

Grosholz and Yakira's interpretation can be shown to be inconsistent with other views Leibniz holds about mathematics. First, Leibniz holds that arithmetic is conceptually prior to geometry. Accordingly, he cannot be understood as intending his definitions of number to presuppose any geometrical content. Leibniz writes that

“geometry... or the science of extension is.... subordinated to arithmetic, since... there is repetition or multitude in extension...” (AG 251-252/GM VI 100). For Leibniz, arithmetic is the highest mathematical science, and geometry is subordinate to it. He writes:

There is an old saying according to which *God* created everything according to weight, measure, and number. But there are things which cannot be weighed, those namely which have no force or power. There are also things which have no parts and hence admit of no measure. But there is nothing which is not subordinate to number. Number is thus a basic metaphysical figure, as it were, and arithmetic is a kind of statics of the universe by which the powers of things are discovered. (L 221/GP VII 184)

Given this, it would be puzzling if Leibniz intended his definition of a basic arithmetical entity -- a positive integer -- to presuppose geometrical content.

Furthermore, when Leibniz says that arithmetic is prior to geometry, one specific claim he means to make is that geometrical magnitudes cannot be understood fully without recourse to number. This is because we cannot perform certain operations that yield understanding of magnitude if we do not first possess number concepts. Grosholz and Yakira's claim that Leibniz defines magnitude as a number of parts is true, but this is evidence *against* their interpretation. In defining magnitude as a number of parts, Leibniz defines magnitude *in terms of number*, rendering our understanding of magnitude dependent upon our understanding of number. He writes that magnitude is “measured by the number of determinate parts” (L 254/GM V 179); the “magnitude in a thing is represented by a number of parts” (GM VII 53). He holds that in order to acquire distinct knowledge of the size of a given geometrical magnitude, we must have recourse to number: “Precise distinctions amongst ideas of extension do not depend upon size: for we cannot distinctly recognize sizes without having recourse to whole numbers, or to

numbers which are known through whole ones; and so, where distinct knowledge of size is sought, we must leave continuous quantity and have recourse to discrete quantity” (RB 156). Given this, it is implausible to suppose that Leibniz intends to define number in a way that presupposes the concept of the division of a line into parts. Leibniz instead holds that in order to understand the magnitude of a line, we must understand how to measure it, and we must in turn understand numbers in order to do that.

It is worth elaborating upon this point by drawing on some key texts. Leibniz makes it clear in a number of texts that the way we are able to understand the size of any continuous magnitude is by conceiving it as a collection of unities (at least those magnitudes that can be so conceived -- I leave the status of irrational numbers and magnitudes measured by them for chapter 4), which is just to say that we are able to understand the size of a continuous magnitude by conceiving the magnitude in terms of an integer that specifies how many of a given unit of measure the magnitude contains. The integers allow us to recast the question “how much?” in terms of the question “how many?”, for any given continuous magnitude. Thus, we must first understand integer concepts in order to apply them in determining the size of a continuous magnitude. The size of some magnitude cannot be precisely determined unless some other magnitude is taken as a unity and then repeated until the original magnitude is exhausted. Leibniz writes:

The quantity of a thing, e.g. of the area ABCD (fig. 7) is expressed by a number, e.g. a multiple of four [*quaternarium*], when it has been assumed that some other thing, such as a square foot ACFG, is taken for a primary measure or real unity. For ABCD is four square feet. But if some other unity AHJK is assumed, which is a half-foot squared, then the quantity of the area ABCD would be 16. Thus, for the same quantity a different

number is produced, according to the unity that is assumed. And consequently quantity is not a definite number, but the material for a number, or an indefinite number that is made definite when a certain measure is assumed. (GM VII 30-31)

He elaborates in another text:

If a foot should be considered as unity, then a thumb will be $1/12$, a cubit will be $3/2$, and the armspan will be 6. If a thumb should be considered as unity, then a foot will be 12, a cubit will be 18, and the armspan will be 72. And in this manner the length of every straight line can indeed be represented by a whole number, if the measure has been drawn off several times, for example if a foot has been drawn off three times, and nothing is left over, then it will be ruled three feet in length. But if something remains when the measure, e.g. foot, has been drawn off as many times as can be done, for this too the thing to be measured will be able to be obtained by a definite part of a foot, for example tenths, which are drawn off afresh from this remainder as many times as can be done. (GM VII 36)

In sum, for Leibniz, we must understand the notion of a collection of unities -- that is, the notion of an integer -- in order to be able to measure continuous magnitudes, as that measurement consists in the conception of those magnitudes as collections of unities. Thus, Grosholz and Yakira's position cannot be correct. In section 4, I show that Leibniz's definition of fractions is conceptually parallel to his definition of the integers; as such, his definition of fractions cannot presuppose any geometrical content either.

3. The Consistency of the Aggregative Definition with the General Definition

3.1. An Apparent Inconsistency. With Grosholz and Yakira's claim about the content of Leibniz's definition of number eliminated, I turn back to the statements of that definition. Crucially, in texts (2) and (3), Leibniz offers more than a definition of the integers as aggregates of unities. He also defines “number in general” as “that which is

homogeneous to unity”⁵, and he subsumes various kinds of number under that definition. Concerning the positive integers, two questions arise here: first, is Leibniz *consistent* in offering both the general definition of number as “that which is homogeneous to unity” and the specific definition of the positive integers as aggregates of unities? Second, if he is consistent, what is the intended relationship between these definitions? A complete answer to these two questions will require an explanation of the phrase “that which is homogeneous to unity”, but one can discover a partial answer to the first question by carefully examining the above texts, without yet investigating the precise meaning of that phrase. It appears that Leibniz at least *intends* the two definitions to be consistent, and indeed to fit together in an unspecified manner. In texts (2) and (3), Leibniz characterizes the positive integers as aggregates of unities in the very next sentence after characterizing number in general as that which is homogeneous to unity. The structure of (2) and (3) is nearly the same: Leibniz defines “number in general”, and implies that integers and fractions fall under the general definition. This intent is slightly clearer in (3), where Leibniz links the definition of number in general with the definition of the integers (and of fractions) using “estque”, or “and it is...”, where the antecedent of the pronoun is clearly “numerus”. Accordingly, the text reads “number is [that which is] homogeneous to unity... And it [i.e. number] is either an aggregate of unities, which is called an integer, or an aggregate of aliquot parts of unity, which is called a fraction”. These texts do not

⁵ Translating “numerus est homogeneum unitatis” as “number is that which is homogeneous to unity” is required because Leibniz uses the neuter nominative singular “homogeneum” as the predicate for “numerus”. “Numerus” is masculine, so “homogeneum” must be acting as a substantive in the neuter nominative singular.

establish the precise nature of the relationship between the definitions, but it is clear that Leibniz intends to offer them as a coherent package, such that the definitions of the specific kinds of numbers are consistent with the definition of “number in general”.

In other texts, Leibniz is more explicit about his intention to include under his definition a larger class of numbers than the positive rationals. He appears to countenance at least a significantly large class of what we now call the positive *real* numbers -- including those numbers that Leibniz calls “surd” and “transcendental” -- under his general definition of number as “that which is homogeneous to unity”. Though I leave the question of what exactly that class contains for chapter 4, it is worth foregrounding it here. Leibniz evidently intends his general definition of number as that which is homogeneous to unity to *subsume* a variety of kinds of number -- integers, fractions, and at least some irrationals -- so that the definition of “number in general” and the definitions of these kinds of number are related as genus and species. This is suggested by (2) and (4), but it becomes more transparent in the following texts:

- (6) Number is [that which is] homogeneous to unity. And so not only integers are numbers but also fractions and surds. (A.VI.4.873, 1687?)
- (7) It is manifest that number in general -- integer, fraction, rational, surd, ordinal, transcendental -- can be defined by a general notion, as it is that which is homogeneous to unity, or that which is related to unity. (GM VII 24, 1714)
- (8) You may also define number in general, which comprehends integer, fraction, surd and transcendental. It is evidently nothing other than [that which is] homogeneous to unity. (LCW 173, 1715)

What (6) suggests, (7) and (8) explicitly state: Leibniz intends his general definition of number to be powerful enough to accommodate the integers, fractions, and at least some of the irrational numbers as instances, though as I have just noted, it is beyond the scope of this chapter to resolve the question of *which* irrationals Leibniz intends to countenance -- as that question can only be addressed by investigating what Leibniz writes about the numbers that he labels “surd” and “transcendental”. In these texts, he claims that all these numbers fall under the category picked out by “that which is homogeneous to unity”, though it is not yet clear how this is supposed to work. What is clear is that he intends the general definition to pick out a genus, number, which subsumes specific kinds of number as species.

It is beyond the scope of this chapter to resolve the issue of how Leibniz intends his account of number to accommodate irrational numbers. Instead, I focus on how exactly Leibniz’s definition of the positive integers relates to the definition of “number in general”. How are the positive integers, as aggregates of unities, a species of “that which is homogeneous to unity?” More fundamentally, what does it mean to be “homogeneous to unity?” Leibniz defines homogeneous things as “those which are similar or can be rendered similar by a transformation” (A.VI.4.872; identical or nearly identical language is found at A.VI.4.508 and GM VII 30). The concept of similarity is essential to Leibniz's definition of homogeneity; he offers multiple definitions of the term, and it is worth noting several of them. In the text from which (3) is taken, he defines similar things as those things “in which, considered by themselves, singly, it is not possible to find that by which they might be distinguished” (Nearly identical language is

found at A.VI.4.514). Another text reads “similar things are those which can be distinguished through themselves if they are together” (A.VI.4.155). Yet another reads “similar things are those which can be distinguished only by co-perception” (A.VI.4.508).

The common thread in these definitions is the idea of the indistinguishability of two things solely by the examination of the things in isolation from each other: to distinguish similar things, one needs to perceive them together. Leibniz's favorite example of this property -- discussed in tandem with his definition of similarity in many places -- is that of similar geometrical figures: for example, two differently sized triangles with the same ratios between their respective sides. According to Leibniz, the only way to discern the difference in size is to perceive the two triangles simultaneously, or to use a third figure as a measuring device by which to compare them. One cannot distinguish them merely by the examination of each figure by itself. These figures are also, according to the definitions just examined, *homogeneous*, because they are similar, fulfilling one of the sufficient conditions for homogeneity. But two things are also homogeneous when they are able to be *rendered similar* by a transformation.

Now, an aggregate of unities is intuitively not similar to unity. It would seem that an aggregate of unities can be distinguished from a unity conceptually, and two different aggregates of unities can be distinguished from each other, without the aid of simultaneous perception. Leibniz is never entirely clear about how this distinguishing is supposed to work, but he appears to accept that different aggregates of unities are not similar to one another, and presumably, this means that an aggregate of unities is also not similar to unity. He contrasts the case of distinct aggregates of unities with the case of distinct lines:

“But what of number? One is... not similar to another, for example four is not similar to three... [A]lthough three is not similar to four, but a line of three feet is similar to a line of four feet” (A.VI.4.933-34). Earlier in the same text, Leibniz provides some indication of why aggregates of unities are not similar to each other, in terms of yet another definition of similarity: “Similar things are those which... cannot be distinguished one by one through truths demonstrable about themselves; or those of which no different demonstrable predicates can be assigned. Thus every parabola is similar to every parabola, and every circle to every circle... Similar things are those of which all the internal predicates are the same...” (A.VI.4.931). In contrast to the truths demonstrable about different parabolas or circles, it seems clear both that two aggregates of unities can have different truths demonstrable about them, and that different truths are demonstrable about unity than are demonstrable about any aggregate of unities.

3.2. The Apparent Inconsistency Resolved. Despite the lack of similarity between unity and an aggregate of unities, the latter is homogeneous to the former because an aggregate of unities can be *rendered* similar to unity. It can be transformed into something that *cannot* be distinguished from unity: something that is only conceivable as a unity, and no longer as an aggregate. In text (3) above, Leibniz characterizes the homogeneity to unity borne by aggregates of unities as consisting in the fact that such aggregates can be “compared with unity and added to or subtracted from it” (GM VII 31). The required transformation, then, is a kind of subtraction: specifically, the successive removal of the constituents of an aggregate of unities until what remains is simply a unity -- and as such is indistinguishable from unity. Thus, an aggregate of unities falls under the category of

“that which is homogeneous to unity”, because it can be rendered similar to unity by a transformation. Accordingly, Leibniz's definition of the integers is both consistent with his general definition of number and related to it in a straightforward way.

Before closing this section, it must be noted that a selection of other texts might, at first glance, undercut the reconstruction proposed here. For example, Leibniz writes that a transformation is “a change that takes place in the original situation of the parts, none being added or removed” (A.VI.4.508); also that a transformation is “when from one thing it becomes another thing, no part having been added or removed” (A.VI.4.628). He also writes that numbers “cannot be rendered similar” (A.VI.4.933). The first two texts seem to suggest that the addition or removal of the constituents of an aggregate would not count as a transformation for Leibniz, and so it would be unclear how an aggregate of unities could be rendered similar to unity by a transformation. However, Leibniz clearly thinks that this is indeed the kind of transformation in virtue of which aggregates are homogeneous to unity: he says this explicitly in text (3), quoted earlier and partially reproduced in this paragraph. Aggregates of unities are homogeneous to unity because they can be “added to or subtracted from it”.

How are we to explain this apparent discrepancy? In one of the texts just quoted (A.VI.4.508), Leibniz is clearly restricting his discussion to transformations performed on continuous bodies: he begins by laying out several definitions (of “homogeneous things”, “similar things”, “equal things”, “congruent things”, and “transformation”), and then he provides a short discussion of the ways in which bodies can be transformed. This text, at least, has no bearing on whether Leibniz thinks transformations can involve the

addition or removal of parts, since he says nothing about the question of how aggregates might be transformed, limiting himself to transformations of continuous bodies. It is also worth noting that my text (3) is undated, whereas the two texts quoted in the previous paragraph are from 1682 and 1685, respectively. Perhaps Leibniz begins his career with a restricted notion of a transformation and later broadens it to include transformations that apply to aggregates. Finally, the text in which Leibniz says that numbers cannot be rendered similar directly conflicts with the significant amount of evidence presented in this chapter that Leibniz holds that aggregates of unities are homogeneous to unity. If aggregates of unities can be rendered similar to unity by removing their constituents, then they can also be rendered similar to each other by the same sort of transformation.

I close this section by noting that the discussion of the integers as aggregates of unities provides another way of ruling out the reading of Leibniz's conception of number in Grosholz and Yakira's study. Now that Leibniz's notion of homogeneity is understood, his remark in texts (2) and (5) that unities are the parts of integers can be cast in the proper light. While Grosholz and Yakira read Leibniz's use of "part" as essentially geometrical, so that his definition of the integers presupposes geometrical notions, the opposite is true: Leibniz understands the relation of part to whole in a general way that is detached from geometrical considerations. In several places, Leibniz defines "part" in terms of an ontological dependence relation that makes no reference to geometry and does not require that parts be understood in terms of continuous magnitude. For example, he writes that "a part is a homogeneous ingredient" (GM VII 19), and that "parts are homogeneous inexistents" (A.VI.4.932), where an "inexistent" is something that exists in something

else. A part, for Leibniz, is something that “is in” something else and is homogeneous to it. Parts, then, are a species of inexistents: *A* is a part of *B* if *A* exists in *B* and is homogeneous to *B*. It should be clear from the argument of this section that Leibniz's notion of homogeneity does not presuppose any geometrical content. His notion of “being in” [*inesse*] is similarly general: “We say that an entity is in or is an ingredient of something, if, when we posit the latter, we must also be understood, by this very fact and immediately... to have posited the entity as well” (GM VII 19). If parts are homogeneous inexistents, then Leibniz does not understand “part” geometrically, but rather in a broader way that includes *as one particular case* the way that segments are parts of a line, and as another particular case the way that unities are parts of integers⁶.

The relation between part and whole, understood in terms of *inesse*, is explained succinctly by Rutherford: “to say that parts 'are in' a whole... is to say that if the latter are supposed to exist, the existence of the former can immediately be asserted; conversely, if the former are supposed *not* to exist, it can immediately be asserted that the latter do not exist” (1990, 541). Given Leibniz's understanding of parthood in terms of *inesse*, line segments and unities both qualify as “parts” of their respective wholes in that they are both homogeneous inexistents of those wholes. They are inexistents in the sense just noted: without them, the wholes would not exist; and once the whole (line, integer) is posited, the parts (segments, unities) are thereby posited. It is worth reiterating that homogeneity is also only not a geometrical notion for Leibniz: it is the notion of one

⁶ The same point is made in Rutherford (1990). For detailed discussion of Leibniz's mereology, see also Di Bella (2005).

thing's being able to be rendered similar to another by a transformation, where the things in question can be non-geometrical entities such as aggregates or geometrical entities such as lines, and the transformations can involve the addition or removal of constituents (in the case of aggregates) or the lengthening or shortening of segments (in the case of lines). Unities are parts of integers in this sense, existing in aggregates and bearing the relation of homogeneity to them, since a unity can be transformed into an aggregate by the addition of other unities.

4. Generalizing the Aggregative Definition to all Positive Rational Numbers

In the previous section, I noted that Leibniz characterizes fractions as aggregates of *parts* of unity, rather than as aggregates of unities. Now that his definition of the integers has been understood, and shown to be consistent with his general definition of number as “that which is homogeneous to unity”, his account of fractions becomes straightforward -- both in its conceptual details and in its consistency with the general definition.

Since aggregates of unities form whole numbers, it is not surprising that Leibniz defines fractions the way he does: whole numbers are larger than one, whereas fractions are smaller than one; and so fractions cannot, by Leibniz's own lights, be formed by the aggregation of unities. They must be formed by the aggregation of entities smaller than unity. Even fractions *larger* than one -- such as, for example, $5/3$ -- cannot be formed by the aggregation of unities, since fractions larger than one are not whole numbers. In the following, I treat these two cases in turn.

It should be noted at the outset that Leibniz does not hold unity itself to be “an aggregate of fractions into which it can be broken up” (S 129/GP II 268); fractions smaller than one represent the many different ways in which unity may be *conceived of* as broken up into parts, and the ways in which those parts may be aggregated to form numbers less than one. The parts of unity are “completely indefinite” (ibid): a unity can be conceived of as divided into any desired series of fractions, but it is not actually divided into any particular series of fractions. Leibniz writes:

[T]he unity in arithmetic... is a purely intellectual or ideal entity divisible into parts, as for example into fractions, which are not actually in unity itself (otherwise it would be reducible to minimal parts that are not present in numbers at all), but depends on how we have designated fractions (S 54-55/E 746)

Leibniz is also careful to note that the concept of a fraction presupposes the concept of an integer:

The analysis of necessities, which is that of essences, proceeds from the posterior by nature to the prior by nature, and it is in this sense that numbers are analyzed into unities.... It is true that the concept of numbers is finally resolvable into the concept of unity, which is not further analyzable and can be considered the primitive number.... When I say that unity is not further analyzable, I mean that it cannot have parts whose concept is simpler than it. Unity is divisible but not resolvable, for fractions, which are parts of unity, have less simple concepts than whole numbers, which are less simple than unity, since whole numbers always enter into the concepts of fractions. (L 664-65/GP III 583)

Integers, Leibniz says, always enter into the concepts of fractions: the fraction $3/5$, for example, presupposes the concept of 3 and the concept of 5; the fraction itself is formed when unity is conceived of as divided into 5 equal parts, and a part of that unity is specified by the aggregation of 3 of those equal parts. It is worth noting here that this line of thought illustrates that Leibniz intends fractions to be understood non-

geometrically, just as he intends the integers to be so understood, and in fact intends fractions to be presupposed in measurement similarly to how the integers are presupposed. As I argued earlier in this chapter, integer concepts are presupposed in the measurement of geometrical quantities; and this is equally true for measuring magnitudes in terms of a unit, and for distinctly specifying a part of a magnitude where the whole magnitude has been taken as a unity -- i.e. to specify a part of the magnitude expressed by a fraction. In order to mark out $3/5$ of a line, for example, we need to first count out 5 equal parts of the line. Only then can we specify a part of the line that consists of three of these equal parts.

The reason why fractions cannot be understood through geometrical notions is the same as the reason why integers cannot be: the definition of magnitude as a number of parts. This definition is what entails the need to understand integer concepts in order to measure magnitudes in terms of a unit, as I have explained. But the marking off of a number of equal parts using an integer is also required in order to be able to determine a specific part of the magnitude in relation to the whole -- in other words, to specify a ratio. One must have marked off five equal parts, and understood oneself as having divided the line into fifths, in order then to count off three of these as constituting $3/5$ of the line. This is why Leibniz says that the concepts of fractions are understood through the concepts of whole numbers. One has to possess the concept of an integer in order to divide the line in terms of that integer, and only when one has done this can one identify a part of the line that is expressed by a fraction with that integer as its denominator.

I close this section by treating improper fractions, or fractions greater than one. Now that fractions less than one have been understood as aggregates of parts of unity -- where “parts of unity” means, for Leibniz, that unity is conceived of as divided into an collection of equal parts -- fractions greater than one can also be so understood. Consider the fraction $5/3$: here, on Leibniz's account, unity is conceived of as divided into three equal parts, and then *five* of those equal parts are aggregated. This yields a number greater than one, but a number that is not an integer. Something similar can be said of the number $1\ 2/3$, which is equal to $5/3$ but is expressed differently. Here, on Leibniz's account, unity is conceived of as divided into two equal parts, and then two of those parts are added to unity itself. This, again, yields a number greater than one, but a number that is not whole.

A crucial outstanding question remains at this stage: how are fractions -- both those less than one and those greater than one -- homogeneous to unity? This question must be resolved for fractions for the same reason it needed to be resolved for the integers: Leibniz clearly intends to subsume his more specific definition of fractions under his general definition of number as “that which is homogeneous to unity”. Fortunately, we can use the conceptual tools developed earlier in this chapter in order to see how Leibniz is able to do this. Recall that to be homogeneous to unity is to be similar to unity or able to be rendered similar to unity by a transformation. In the case of the integers, we saw that the relevant transformation is subtraction: integers, as aggregates of unities, can be rendered similar to unity by the removal of their constituent unities until only unity remains. This renders a given integer similar to unity in the sense of indistinguishability

from unity, as explained above. For fractions smaller than one, the relevant transformation will be the opposite of subtraction -- i.e. addition. Here, unity is conceived of as divided into a given number of equal parts, and some of those parts are aggregated to form a number less than one. In order to render that aggregate similar to unity -- where we will have similarity again in the sense of indistinguishability -- we simply add the required additional number of equal parts, giving us unity, and rendering the original fraction indistinguishable from unity. Thus, fractions less than one straightforwardly qualify as homogeneous to unity. Fractions greater than one qualify in a similarly straightforward way: again, unity is conceived of as divided into a certain number of equal parts, but in this case, those parts are aggregated in a way that yields a number greater than one. In order to render such a number similar to unity, we simply remove the extra unities, rendering the aggregate indistinguishable from unity. At this stage, then, we can see that Leibniz has a definition of number that coherently and consistently yields all of the positive rational numbers: the integers, fractions less than one, and numbers greater than one that are not integers but can be expressed as fractions.

Chapter 3: Leibniz on the Ontology of the Rationals as Relations

1. Introduction

My goal in this chapter is to argue that Leibniz's definition of number -- explored in the previous chapter -- entails that at least the positive rationals have a well-defined place in Leibniz's metaphysics. Specifically, I argue that Leibniz conceives of these numbers as relations⁷; and that as relations, they have the ontological status of divine ideas, expressing certain kinds of possibilities and providing the basis for a class of necessary truths. Establishing the first of these points will require reconciling two sets of texts that appear to define numbers in different ways. One set of texts, analyzed in the previous chapter, leaves the impression that Leibniz conceives of the rationals as aggregates. Another set of texts -- the set I investigate here -- indicates that Leibniz conceives of numbers in general -- not only the rationals -- as examples of a distinctive type of relation.

I begin by providing a brief overview in section 2 of Leibniz's theory of relations. In section 3, I show how these texts can be reconciled, establishing that Leibniz ultimately conceives of the positive rationals as *relations* that provide the basis for the wholeness and size of *aggregates* of things taken as unities or parts of unity. I then turn to the question of the ontological status of the rationals for Leibniz, given that they are relations. In section 4, I argue that as a kind of relation, the rationals have a natural place in Leibniz's

⁷ Russell (1937, 14) notes this, but only in a cursory way and without attempting to explicate the content of Leibniz's relational conception of number.

ontological framework, expressing possibilities and grounding a class of necessary truths, as ideas in God's mind. In the course of this argument, I address the challenge my reconstruction faces from Mates (1986). The nominalist reading of Leibniz's metaphysics -- especially as formulated by Mates -- has been the subject of a good deal of critical scrutiny⁸, and it is not my goal to criticize this reading in detail. Instead, I only aim to show that Mates' reading does not represent a serious challenge to my interpretation of Leibniz's position on the ontological status of the rational numbers.

2. A Brief Primer on Leibniz's Theory of Relations

In the previous chapter, I explicated Leibniz's definition of the positive rational numbers as a certain kind of abstract aggregate. However, I also signaled that I would ultimately argue that this definition is equivalent, for Leibniz, to the view that these numbers are relations. In this chapter, I make that argument. However, before doing so, it is necessary to provide a brief exposition of Leibniz's general theory of relations. Leibniz writes much about relations throughout his career; in his fifth letter to Clarke, he makes remarks that reveal the content of his theory to a first approximation. When discussing the relation between a longer line L and a shorter line M, he first points out that “the ratio... may be considered in three several ways: as a ratio of the greater L to the lesser M; as a ratio of the lesser M to the greater L; and lastly as something abstracted from both, that is, the ratio between L and M without considering which is the antecedent or which the consequent” (LC 47/GP VII 401). In the first two cases, Leibniz says, L and

⁸.E.g. in Mondadori (1990); Hill (2008); and Ishiguro (1990).

M are respectively “the subject of that accident which philosophers call relation” (ibid), in that each line is said to have a relational accident that relates it to the other line. But in the third case,

It cannot be said that both of them... are the subject of such an accident; for if so, we should have an accident in two subjects, with one leg in one and the other in the other, which is contrary to the notion of accidents. Therefore we must say that this relation, in this third way of considering it, is indeed out of the subjects; but being neither a substance nor an accident, it must be a mere ideal thing... (ibid)

Because the created world consists only of substances and their individual accidents, relations -- conceived either as accidents with “legs” in multiple particular subjects or in a more abstract fashion without regard to any particular subjects -- must be mere “ideal things”. As Mugnai (1992) and (2012) demonstrates through an extensive marshaling of texts, Leibniz believes that relations as such are “mental beings and that they ‘result’ or ‘supervene’ when two or more things ‘are thought of simultaneously’” (2012, 182). For Leibniz, “no multiple inherence is admitted” in the world of substances and modifications (ibid, 184), so that relations are mental entities, arising in thought when the mind considers the appropriate individuals as related in a certain way. As Mugnai puts it, “it is precisely the polyadic nature” of relations that “reveals that they are ‘purely mental beings’” (ibid). For example, when the mind considers two blue things, insofar as they are both blue, it thinks of the two as related by the relation of similarity, and it apprehends the relational fact that the two things are similar.

This account naturally yields three “levels” at which one can talk about relations and relata: (1) individual things with their individual accidents, serving as the foundations of relations; (2) individual things, with the relevant individual accidents, considered by

the mind as related in a particular way; and (3) relations *in abstracto*, considered without regard to any particular individuals they might relate. For example, if we have two blue things, we can talk about them in sense (1), as individuals each of which has an individual accident of blue-ness; in sense (2) insofar as some mind thinks about them and their relevant individual accidents, and considers them as similar to one another insofar as they are both blue; and in sense (3), we can talk about *the relation of similarity* in the abstract, ignoring any particular similar things.

Of particular interest to us here is the ontological status of relations *in abstracto*: relations considered without regard to any particular relata, such as the relation of *similarity* in our example. For Leibniz, it may appear that relations depend entirely on the operations of created minds, since they arise in minds insofar as the appropriate subjects are thought as related in a certain way, so that certain relational facts about the subjects obtain. But Leibniz thinks relations would result even if no created mind were there to think about the subjects. In the *New Essays*, he makes multiple remarks that reveal the ultimate independence of relations and relational truths from the operations of the human intellect. For example, he writes that “although relations are the work of the understanding they are not baseless and unreal. The primordial understanding is the source of things...” (RB 146); and that concerning relations, “one can say that their reality, like that of eternal truths and of possibilities, comes from the Supreme Reason” (ibid, 227). He also says in the same work that “the reality of relations is dependent on mind... but they do not depend on the human mind, as there is a supreme intelligence that determines all of them from all time” (ibid, 265). In his commentary on a book by Aloys

Temmick, translated in Mugnai (1992), Leibniz says that “their reality [of relations] does not depend on our understanding -- they inhere without anyone being required to think of them. Their reality comes from the divine understanding...” (1992, 155). In other words, the relational truths about a given collection of individuals obtain objectively, independently of any human thought -- so relations do have a kind of reality. This is because individuals would still be thought of as united by relations in *God's* mind even if no finite mind did so in any given case. This thesis generalizes to the status of relations *in abstracto*. If God thinks of objects as related even if no human mind is present to do so, so that relational truths are grounded in the operations of the divine mind, then presumably the reality of relations *in abstracto* is also founded in the divine mind. This point is made especially clear when Leibniz compares the reality of relations to the reality of eternal truths and possibilities. Subsequently, when I explicate the sense in which numbers are relations, I will revisit what I have said in this section, expanding upon it in light of additional textual evidence to argue that numbers, as relations *in abstracto*, are in fact *expressions of possibilities* in God's mind.

3. The Relational Conception

It is not yet clear on what grounds Leibniz conceives of the rational numbers as *relations*. All that has been established thus far is that the definitions of the positive integers as aggregates of unities, and of fractions as aggregates of parts of unity, are consistent with, and indeed a species of, the definition of “number in general” as “that which is homogeneous to unity”. The aim of this section is to show that Leibniz should

be understood as holding that the rational numbers are indeed relations; furthermore, it turns out that his aggregative definitions of the rationals actually *entails* that they are relations.

In the previous chapter, I noted a text from the *Dissertation on the Art of Combinations* that seems to provide a version of the thesis that the integers are relations. There, Leibniz says that some relations are relations of union, which unite several things in a whole, and the way he characterizes the integers in the *Dissertation* seems to place them in the category of relations of union. However, in several places in his mature writings Leibniz more explicitly characterizes numbers in general as relations:

- (9) [N]umber or time are only orders or relations pertaining to the possibility and the eternal truths of things. (GP II 268-269, 1704)
- (10) Numbers... have the nature of relations. And to that extent in some way they can be called beings. (GP II 304, 1706)
- (11) Place and position, quantity -- such as number, proportion -- are nothing but relations, results from other things. (C 9, undated).

Limiting the focus to the rationals, this way of characterizing number appears to conflict with the aggregative definitions of the rationals: an aggregate of unities or parts of unity is not, at least intuitively, a relation. It is difficult, in the absence of further evidence, to see how Leibniz's claim that numbers are relations might be reconciled with his claim that the rationals are aggregates. *Prima facie*, a number cannot be both a relation and an aggregate.

Fortunately, supplementary evidence is forthcoming in the form of Leibniz's view of aggregates. In Leibniz's metaphysics, something can only be an aggregate on the basis of the relatedness of its constituents. An aggregate of unities, for example, is only a whole -- it is only *one thing* -- because of a certain relation among its constituents. This view is evident in its infancy in the *Dissertation*. It is straightforward to apply Leibniz's concept of relations of union to aggregates, since aggregates are composed of several things united in some way: what makes them a whole is a relation of union among their constituents. But Leibniz makes the view explicit in the following late remark from his comments on a book by Aloys Temmick: "Bare relations are not creatable things, and arise in the divine intellect alone... and such things are whatever results from posits, *such as the totality of an aggregate*" (my translation of a text found in Mugnai (1992, 156)). In the *New Essays*, Leibniz makes much the same point: "[The] unity of the idea of an aggregate is a very genuine one; but fundamentally we have to admit that this unity that collections have is merely a respect or relation" (RB 146). Aggregates, for Leibniz, are a kind of relational being: an aggregate only exists in virtue of a relation that provides the basis for uniting certain parts as a whole.

I now argue that in Leibniz's terms, if the rationals are aggregates, then they must also be relations. I treat the case of the integers first, and then run a parallel argument for fractions. If the positive integers are aggregates of unities, then several possibilities arise as to the meaning of that claim, for example:

- (1) A given integer is identical to a particular aggregate of concrete things taken as unities, e.g. {Leibniz, Spinoza, Locke}.

(2) A given integer is identical to the set of all aggregates of concrete things taken as unities that can be put in a one-to-one correspondence, e.g. all aggregates that can be put in a one-to-one correspondence with {Leibniz, Spinoza, Locke}.

(3) A given integer is identical to an aggregate of unities taken in abstraction, e.g. {unity, unity, unity}.

Option (1) is not a serious contender for a conception of number: it is not plausible to maintain that the number five is identical with a particular collection of concrete things. Even if it were plausible, there is no textual evidence that Leibniz ever considered this kind of view. Option (2) is more plausible, but there is again no evidence that it is Leibniz's view. In Leibnizian terms, the number three is not identical with either a particular aggregate or the collection of all aggregates that can be put in a one-to-one correspondence with a given aggregate {Leibniz, Spinoza, Locke}.

Thus, if Leibniz thinks of the integers as aggregates, then he must conceive of them in terms of option (3): as aggregates of unities taken in abstraction. However, (3) is equivalent, in Leibnizian terms, to the claim that the integers are relations. Consider the content of an aggregate of unities taken in abstraction: no particular aggregate is signified, only a general possibility of -- or basis for -- aggregation. In an aggregate of concrete things taken as unities, some determinate concrete aggreganda are unified by some relation. By contrast, in an aggregate of *abstract* unities, the aggreganda are merely unities taken in abstraction. Unities taken in abstraction are nothing more than placeholders for individual things; an aggregate of unities taken in abstraction, then, signifies nothing more than the possibility of taking individual things together in a certain

way. But it is relations, for Leibniz, that provide the basis for, or underlie, the possibility of aggregation -- the possibility of taking things together. Thus, an aggregate of unities taken in abstraction can only be understood as a relation. This line of thought is captured in Leibniz's remark in the *New Essays* that, "It may be that *dozen* and *score* are merely relations and exist only with respect to the understanding. The units are separate and the understanding takes them together, however scattered they may be" (RB 145). The taking together of things in an aggregate is nothing more than the consideration of a certain relation among them.

A parallel argument to the one establishing that an aggregate of unities is ultimately a relation can be used to establish that an aggregate of parts of unity is also ultimately a relation -- and hence that fractions both less than and greater than one are relations, since it was established in the previous chapter that both sorts of fraction are aggregates of parts of unity on Leibniz's view. When Leibniz defines a fraction as an aggregate of parts of unity, he might have one of three things in mind, in parallel to the options canvassed earlier for what he might mean in defining an integer as an aggregate of unities:

- (1) A given fraction is identical to a particular aggregate of concrete things taken as parts of some unity.
- (2) A given fraction is identical to a set of aggregates of concrete things taken as parts of some unity that can be put in 1-1 correspondence.
- (3) A given number is identical to an aggregate of parts of unity taken in abstraction.

The first two options can be rejected for the same reasons that their counterparts were rejected above: (1) is not a plausible view of number, and there is no textual evidence that Leibniz ever considered (2), even if (2) may be more plausible than (1). Thus, in defining a fraction as an aggregate of parts of unity, Leibniz must have in mind an aggregate of parts of unity taken in abstraction. But the same considerations apply to an aggregate of parts of unity taken in abstraction that applied to an aggregate of unities taken in abstraction. Parts of unity taken in abstraction are nothing more than placeholders for things; thus, an aggregate of parts of unity taken in abstraction signifies nothing more than the possibility of taking things together in a certain way. But for Leibniz, relations provide the basis for, or underlie, the possibility of taking things together in a certain way. Therefore, an aggregate of parts of unity taken in abstraction can only be understood as a relation.

In sum, according to Leibniz, the integers are the relations that provide the basis for the wholeness, and express the size, of aggregates insofar as those aggregates are composed of unities, whereas fractions are the relations that provide the basis for the wholeness, and express the size, of aggregates insofar as those aggregates are composed of parts of unity. And so Leibniz turns out to have a unified account of all positive rational numbers as relations, though I will elaborate on the way in which fractions (both greater and less than one) are a different sort of relation from integers.

I have previously noted that the positive integers provide the basis for the measurement of continuous magnitudes by allowing us to conceive of those magnitudes in terms of collections of unities. The integers can now be understood more generally as

providing the basis for the counting of collections of things, expressing the wholeness of those collections and answering the question “how many?” with respect to them. In the continuous case, some magnitude is taken as a unity and repeated until the original magnitude has been re-cast as an aggregate of unities; in the case of the counting of individual things, the “measure” is just the notion of unity itself. Each thing is taken as a unity, and the resulting number counts the aggregate in terms of the question “how many unities?”. At bottom, then, the positive integers are those relations that express the *homogeneity to unity* possessed by aggregates of things taken as unities, signifying that such aggregates are composed of unities and can be reduced to unity by successively removing their constituents. This is not to identify a positive integer with a relation as exemplified by any *particular* aggregate. Positive integers, rather than being identified with some collection or other, are the relations of homogeneity that any collection may have to unity, expressing the size of the collection in terms of an answer to the question “how many?”.

As we have now seen repeatedly, a parallel account can be given on Leibniz's behalf of fractions both greater and less than one. If the integers can be understood as those relations that express the homogeneity to unity possessed by aggregates of things taken as unities, then fractions can be understood as those relations that express the homogeneity to unity of aggregates of things taken as parts of unity. An example of a continuous case and an example of a discrete case will both be instructive here. Consider an arbitrary line, designated as having a magnitude of one. If we take that line as a unity, and conceive of it as divided into five equal parts, then the fraction $3/5$ expresses the

homogeneity to unity possessed by those three equal parts of the line. Now consider an aggregate of five arbitrary things taken as unities: the fraction $3/5$ expresses the homogeneity to unity possessed by three of those unities taken as an aggregate. Interestingly, however, in both of these cases it can be seen (as I have noted previously) that the ability of the fraction to express the relevant sense of homogeneity to unity presupposes integer concepts. This is in line with Leibniz's explicit remarks (quoted in the previous chapter) that fractions conceptually presuppose integers, but it also indicates that fractions are a different sort of relation. Integers, on Leibniz's account, are a “first-order” relation, obtaining between aggregates of unities and unity itself. Fractions, in the final analysis, appear to be relations between integers, as integers are presupposed in the arbitrary division of a unity into equal parts. Thus, on Leibniz's account, fractions are relations between relations, and so may be labeled “second-order” relations. However, the crucial point is that both integers and fractions are relations, and so enjoy whatever ontological status Leibniz confers on that sort of entity. As I show in the next section, it turns out that the difference in order between integers and fractions makes no difference with respect to their ontological status, and that Leibniz has a unified account of the ontology of the positive rationals.

4. Positive Rationals as Divine Ideas

4.1. The Reality of the Rationals as Relations. Having answered the definitional question, I turn to the question of the ontological status of the rationals. If Leibniz conceives of these numbers as relations, then it is plausible to think that he confers the same

ontological status on them that he confers on other relations. But this leaves open several options for Leibniz: does he think of the rationals, as relations, along the lines of Platonic abstract objects? Does he think of them as some sort of mental entity? Does he even think that they have any reality at all? Are they part of his metaphysical picture of the world?

As noted earlier, Leibniz is clear throughout his corpus that relations have a particular ontological status: they are divine ideas, contents of God's mind. Given this, the rationals, as relations, should be understood as contents of God's mind. As such, they inhabit the top level of what is widely accepted to be Leibniz's "three-tiered" ontology, consisting of monads at the fundamental level, possessing the most inherent reality, phenomena (such as bodies) in the middle (grounded on monads), and ideal entities at the top⁹. Though ideal entities possess the least fundamental or inherent reality in Leibniz's ontology, given how Leibniz conceives of God's mind, these numbers should be understood along Platonistic lines, in the sense that they have timeless reality independent of the created world. The precise content of Leibniz's theory of relations has been the subject of much scholarship¹⁰, and we have had occasion to examine its relevant highlights in a previous section. For our purposes, a recapitulation of key texts will suffice to make the point that matters here. Following the terminology of Mugnai, relations *in abstracto*, or relations considered in the abstract, independently of any particular relata, are contents of the divine mind according to Leibniz. For example, Leibniz writes that "although relations are the work of the understanding they are not baseless and unreal.

⁹ The *locus classicus* for this interpretation of Leibniz's ontology (from which I borrow the phrases just quoted) is Hartz and Cover (1988)

¹⁰ In addition to Mugnai (1992), see also Mugnai (2012); Ishiguro (1990); and Mates (1986)

The primordial understanding is the source of things..." (RB 145); and that concerning relations, "one can say that their reality, like that of eternal truths and of possibilities, comes from the Supreme Reason" (RB 227). He also says in the same work that "the reality of relations is dependent on mind... but they do not depend on the human mind, as there is a supreme intelligence that determines all of them from all time" (RB 265). In his commentary on Aloys Temmick, Leibniz writes that "their reality [of relations] does not depend on our understanding -- they inhere without anyone being required to think of them. Their reality comes from the divine understanding..." (Mugnai 1992, 155).

Thus the positive rationals, as relations, have their ultimate ground in God's mind. In turn, God's mind, for Leibniz, is the realm where possibilities are expressed and the ground of necessary truths is provided. If the rationals are divine mental contents, then they express possibilities and form the basis of a class of necessary truths. That they occupy this place in Leibniz's metaphysics was telegraphed earlier by the excerpt from his letter to De Volder, where Leibniz says that numbers "pertain to the possibility and the eternal truths of things". But Leibniz is explicit in many texts that the realm of divine ideas is the realm of that which expresses possibility and provides the basis for necessary truths. The following passage represents a typical statement of Leibniz's conception of the realm of divine ideas:

Essences, truths, or objective realities of concepts do not depend either on the existence of subjects or on our thinking, but even if no one thinks about them and no examples of them existed, nevertheless in the region of ideas or truths, as I would say, i.e. objectively, it would remain true that these possibilities or essences actually exist, as do the eternal truths resulting from them... As in the region of eternal truths, or in the realm of ideas that exists objectively, there subsist unity, the circle, power, equality, heat, the rose, and other realities or forms or perfections, even if no individual beings exist, and these universals were not thought about. (S 185/A.II.1.392)

The contents of the divine mind have reality independent of the created world; they express possibilities; and they form the basis of eternal truths:

Neither... essences nor the so-called eternal truths pertaining to them are fictitious. Rather, they exist in a certain realm of ideas, so to speak, namely in God himself, the source of every essence and of the existence of the rest... It is necessary that eternal truths have their existence in a certain absolute or metaphysically necessary subject, that is, in God, through whom those things which would otherwise be imaginary are realized. (AG 151-152/GP VII 305)

Thus, the rationals, as relations, and so as inhabitants of the divine mind, have reality independent of the created world, express metaphysical possibility, and are the subject of necessary truths. The positive *integers* express the possible ways in which aggregates of things taken as unities can be homogeneous to unity, which is to say they express the possible sizes of such aggregates in terms of their composition out of things taken as unities. Positive *fractions* express the possible ways in which aggregates of things taken as *parts* of unity can be homogeneous to unity, which is to say they express the possible sizes of such aggregates in terms of their composition out of things taken as *parts* of unity. As inhabitants of the divine mind, the positive rational numbers also provide the basis for a class of necessary truths: the truths of arithmetic. Each positive integer is a first-order relation, and some of the truths of arithmetic concern relations between integers; these truths, for Leibniz, ultimately concern second-order relations -- in other words, relations between relations. Second-order relations, in this framework, express ways in which possible *groups of related things* can be with regard to each other, insofar as the given groups are united by particular first-order relations. So the eternal truth that $2+3=5$, for example, expresses a relation between 2 and 3, such that *any*

aggregate related to unity by the number 2 is related to *any aggregate related to unity by the number 3* in such a way that their combination yields an aggregate related to unity by the number 5. Now, as I have noted, fractions are also second-order relations, as they are ultimately relations between integers. Some of the truths of arithmetic concern relations between fractions; these truths, then, concern *third-order* relations. Thus, for example, the eternal truth that $3/5 + 1/5 = 4/5$ expresses a relation between $3/5$ and $1/5$, such that *any aggregate related to unity by the number $3/5$* is related to *any aggregate related to unity by the number $1/5$* in such a way that their combination yields an aggregate related to unity by the number $4/5$. But once again, in the case of fractions, integer concepts are presupposed, and so the relations *between* fractions can be called third-order -- and it is to these sorts of relations that the truths about fractions correspond.

4.2. Meeting the Nominalist Challenge. It seems, then, that Leibniz's answer to the ontological question is clear: though numbers are a kind of mental entity, they are the kind of mental entity that gives them the status of something like Platonic abstract objects. They are not merely contents of human minds, but of God's mind; as such, they exist timelessly and independently of the created world. This reconstruction, however, faces a challenge based on the reading of Leibniz's metaphysics proposed by Benson Mates. Mates characterizes Leibniz as a "nominalist," claiming that Leibniz denies a fundamental reality to anything other than concrete individual substances and their modifications: in other words, Leibniz excludes everything other than these entities from his fundamental picture of the world. According to Mates,

There can be little doubt that [Leibniz] was a nominalist... [in] the sense it bears in current Anglo-American philosophical discussion about so-called ontological commitment. According to this, a nominalist, as contrasted with a Platonist, is one who denies that there are abstract entities, asserts that only concrete individuals exist, and in consequence considers that all meaningful statements appearing to be about abstract entities must somehow be rephraseable as statements more clearly concerning concrete individuals only. (1986, 170)

On this interpretation, Leibniz rejects all manner of abstract entities, including those entities that are the subject of this chapter:

Leibniz would agree wholeheartedly with that notorious pronouncement of present-day nominalism: “We do not believe in abstract entities.” He does not believe in numbers, geometric figures, or other mathematical entities, nor does he accept abstractions like heat, light, justice, goodness, beauty, space or time, nor again does he allow any reality to metaphysical paraphernalia such as concepts, propositions, properties, possible objects, and so on. The only entities in his ontology are individuals-cum-accidents, and sometimes he even has his doubts about the accidents. (1986, 173)

For Mates, then, Leibniz’s variety of nominalism consists in two theses: first, the denial that abstract entities exist, and second, the thesis that statements about those entities can be rewritten in a way that refers only to concrete individuals. I address these in turn.

Mates’ evidence in favor of the first thesis consists in Leibniz’s oft-repeated remark that anything that is not a concrete individual substance or modification thereof has whatever *being* it has solely as a content of the divine mind¹¹. What is at issue here is the meaning of that thesis. Mates holds that it amounts to an elimination or rejection of whatever is identified with a divine idea. His interpretative strategy is to argue that such a move is intended to *reduce* everything that is not either an individual substance or a

¹¹ The thesis that everything besides individual substances and their individual accidents exists only in God’s mind appears repeatedly in Leibniz’s corpus. Mates cites GP VII 305/L 488 and GP VI 614-616/L 647-648 in particular.

modification thereof to certain divine *dispositions*. Mates proposes that “what [Leibniz] intends is not that there are two kinds of existence, namely, in the mind of God and out of the mind of God, but rather that statements purporting to be about these kinds of entities are only *compendia loquendi* for statements about God's capacities, intentions, and decrees” (1986, 177). In other words, talk of abstract objects, universals, or relations is just talk about certain of God's concepts or ideas, which in turn is nothing more than talk of God's mental dispositions. Crucially, for Mates, such a reduction amounts to an elimination of the entity from Leibniz's ontology. If divine ideas are ultimately dispositional, Mates' reasoning goes, then Leibniz must intend to eliminate from his fundamental ontology everything that he characterizes as a divine idea.

The rest of this section makes two claims: (1) Mates' proposed reduction lacks a textual basis, and in fact Leibniz holds the exact opposite view of the nature of divine ideas to that required by Mates' strategy; (2) even if Mates' proposed reduction had textual support, it still would not be the kind of *eliminative* reduction that his nominalist reading requires, and the abstract entities identified with divine ideas would still have timeless reality independent of the created world and would provide the basis for classes of necessary truths.

Mates' proposed reduction relies on Leibniz's well-known identification, for the *human* mind, of the *idea* of something with the *disposition to think about* that thing. For example, Leibniz writes: “In my opinion, namely, an idea consists, not in some act, but in the faculty of thinking, and we are said to have an idea of a thing even if we do not think of it, if only, on a given occasion, we can think of it” (L 206/GP VII 263). Our ideas

are not mental acts, but dispositions to perform mental acts. However, it is well-documented that Leibniz does not think *God's* ideas are the same kind of thing as *our* ideas. God's ideas *cannot* be identified with his dispositions to perform mental acts, since God *has* no dispositions and is in fact purely active. Mondadori puts the point as follows in his review of Mates' study:

[T]he view according to which 'having an idea at a given time does not require having an actual thought at that time, but only a disposition to think' cannot apply to divine ideas: for the (infinite) totality of God's thoughts includes 'all at once', as actually thought, everything that *can* be thought by an infinite understanding... Hence, we need not ascribe to God any (modally non-vacuous) dispositions to think; hence, divine ideas must be something other than dispositions to think; hence, they cannot be explained away by appealing to dispositions; hence, the reductive scheme put forth by Mates cannot be made to work, since it crucially depends on the claim that talk of ideas is in fact talk of dispositions to think. (1990, 622)

In short, God does not have any dispositions to think about anything: God is always (timelessly) thinking about everything, "all at once". Mondadori marshals a variety of texts that clearly establish this point; one that is particularly forceful reads: "God expresses everything perfectly, all at once, possible and existent, present and future" (GP IV 533, undated). In short, Leibniz cannot plausibly be understood as reducing divine ideas to divine dispositions, and so he cannot be understood as reducing to divine dispositions the abstract entities that he identifies with certain divine ideas.

However, even if Mates' proposal did have textual support, and Leibniz could be read as reducing divine ideas to divine mental dispositions, the reduction would lend little support to the nominalist reading. If divine ideas were reducible to divine dispositions, then those dispositions would still have all the features that Leibniz ascribes to the realm

of divine ideas in the passages analyzed earlier in this section. Those dispositions would have timeless reality independent of the created world and would provide the basis for classes of necessary truths; and so the abstract entities identified with those dispositions would still have these features. The place of relations *in abstracto*, and so the place of the rationals, in Leibniz's metaphysical framework remains the same regardless of whether the divine ideas are purely actual or are reduced to dispositions. Furthermore, although the contents of the divine mind are ideas, and so whatever entities Leibniz relegates to the divine mind are "mental entities" to that extent, this does not amount to a denial of their reality. This class of entities would exist even if there were no created world of individual substances, and even if no finite, created mind ever thought about them. Although it is certainly the case that these entities do not exist *in the same way that individual substances exist* -- as Ishiguro puts it, when we refer to relations, for example, we are not referring to "entities which are the basic constituents of the world in the manner that individual substances are" (1990, 140) -- it is a mistake to infer from this that they do not exist *at all*. Thus, it is evident that when Mates claims Leibniz does not have two senses of "existence" in mind -- one sense *in* God's mind and another sense *out of* God's mind -- he deviates considerably from the textual evidence. It seems that Leibniz does have exactly this in mind. In the created world of individual substances, one will not find any abstract entities; but one will find them in the divine mind, and one would find them there even if God had never created the actual world or any other world.

Mates' argument for his second thesis -- that Leibniz believes reference to abstract entities can be paraphrased out of discourse -- turns on Leibniz's various "rewriting

projects”: his lifelong efforts to reform language either by rewriting all Latin sentences in subject-predicate form or by inventing an artificial language to facilitate deduction and the discovery of new truths. The former is pertinent here. Mates writes that Leibniz's

nominalistic metaphysics provides the basis and motivation for much of what [he] says about language. If the real world consists exclusively of substances-with-accidents, it is natural to suppose that it could in principle be completely described by a set of propositions of 'A ist B' form, where A is the complete individual concept of a given substance, and B is a concept underwhich the substance falls at time t by virtue of one or more of its accidents. (1986, 178)

The claim is twofold: on the one hand, Leibniz's supposed nominalism must have motivated his efforts to rewrite Latin sentences in a nominalistically acceptable way, and on the other hand, the existence of such a rewriting project is evidence of his nominalism. His efforts to reduce all sentences to those of subject-predicate form reveals, to Mates, an intention to deny any reality to anything other than individual substances and their individual modifications, for the rewriting projects eliminate from discourse all reference to things like universals, both monadic and relational, and individual relational properties. Importantly, there's a further premise in Mates's reconstruction to the effect that the elimination from *discourse* of reference to such items tracks an elimination from *reality* of the items themselves: Leibniz's reduction of sentences involving these items reveals a thesis that they do not exist in any sense whatsoever.

A natural response to this line of thought would be to say that the elimination from *language* of all reference to abstract things only implies their elimination from *reality* if some further principle is explicitly adduced, to the effect that whatever is eliminated from the perfect language does not really exist. A recent article by Christopher Hill (2008)

takes up the task of spelling out this response, in the context of Leibniz's attempts to reduce sentences that explicitly refer to *relations* and relational properties. The idea, as seen in Mates' book, is that the effort to rewrite relational sentences as non-relational sentences entails or reveals an ontological commitment to the unreality of relations and relational properties. Hill challenges this thesis on two grounds: first, a philosopher's reduction of sentences mentioning entity X to sentences not mentioning X does not necessarily entail her *rejection* of X, unless there is *independent evidence* that the philosopher believes in a certain kind of *eliminative reduction principle*: namely, that if statements mentioning X can be rewritten as statements not mentioning X, then X is not real. Hill notes the pervasive acceptance, without much attempt at justification, of the thesis that Leibniz *does* believe in this sort of eliminative reduction principle, and not only in Mates, but in much of the literature concerning Leibniz's theory of relations. A similar point could be made about the reduction of sentences involving non-relational universals, like heat or redness: Leibniz's efforts to transform sentences mentioning such things into sentences that do not mention them only implies his outright *rejection* of such entities in the presence of the eliminative reduction principle, which it is merely assumed that Leibniz held.

Hill's challenge goes deeper, though, focusing on the rewritten Latin sentences themselves. Leibniz uses certain Latin connectives in rewriting relational sentences -- namely, *quatenus* and *et eo ipso*, which roughly translate to "insofar as" and "and by this very fact", respectively. These connectives have the function of eliminating explicit reference to relations; but as Hill points out, the connectives themselves *are*

implicitly relational; if they were not, i.e. if they did not imply a relation between the subjects they connect, then the rewritten sentence would not say the same thing as the original sentence, and the purported reduction would not be a reduction at all. To take one of Leibniz's favorite examples, “Paris is a lover, *et eo ipso* Helen is a loved one” fails to say the same thing as “Paris loves Helen” unless the connective implies some sort of relation between Paris's being a lover and Helen's being a loved one. In other words, the content of the connective “and by this very fact” is itself relational -- otherwise, the rewritten sentence would no longer be elliptical for “Paris loves Helen”.

Ultimately, it is clear that the rationals do have a kind of reality for Leibniz, and the reality they have resembles that which the Platonist confers upon numbers, though Leibniz's account differs in its details. As I have argued, the rationals are a kind of mental entity for Leibniz, but they are not the kind of mental entity that one would naturally think of -- indeed, they are just the sort of mental entity that a Platonist would envision. As contents of God's mind, the rationals have a robust ontological status, possessing reality independent of the created world, expressing possibilities for the aggregation of things in the created world, and providing the basis for a class of necessary truths.

1. Conclusion: The Philosophical Significance of Leibniz's Account of the Rationals

At this stage, one might ask what, if any, philosophical import Leibniz's account of the rationals as relations might have. It is worth concluding this chapter by briefly foregrounding an answer to this question, though I leave a full exposition of the philosophical significance of Leibniz's views for chapter 6.

His account appears to be philosophically significant along two major dimensions. First, his conception appears to contain elements of both the cardinal and ordinal conceptions of number, which is developed explicitly by later thinkers (such as, for example, Cantor), but whose roots one can see here. Leibniz's account simultaneously characterizes the rational numbers as cardinals and ordinals: as cardinals, in the way the rationals are relations that count the possible sizes of collections and answer the question “how many?”; as ordinals, in the way it puts at least the integers into something resembling an ordered set, given that each integer is related to its predecessor and to its successor by the operations of addition and subtraction. I leave a detailed treatment of this aspect of Leibniz's view for chapter 6, but it is worth foregrounding here.

The second, and perhaps the more major, philosophical contribution of Leibniz's view lies in its characterization of numbers (at least the rationals) in purely conceptual terms. As I have shown, his account presupposes no geometrical notions, and entails that numbers are conceptually prior to any such notions. This way of characterizing number bears a striking resemblance to that advanced by the logicians in the late 19th and early 20th centuries. As with the treatment of the cardinal and ordinal aspects of Leibniz's account, I leave the analysis of that resemblance for chapter 6

Chapter 4: Leibniz on Non-Rational Numbers

1. Introduction

In previous chapters, I have noted that my account of Leibniz's views on number does not straightforwardly apply to the case of irrational numbers -- numbers which, in contemporary terms, cannot be expressed as a ratio of two integers. In several passages quoted therein, Leibniz signals a desire to subsume under his account of number those numbers which he labels "surd" and "transcendental", but it is not immediately clear what he means by those terms. Additionally, Leibniz's views on the conceptual and ontological status of other sorts of numbers -- for example, negative and complex numbers -- remains unclear on the interpretation I have offered thus far. The purpose of this chapter is to explore two broad questions for four kinds of non-rational number (in addition to the case of negative rational numbers): irrationals, complex numbers, infinite cardinals, and infinitesimals. For each sort of non-rational number, that question bifurcates into two parts: (1) does Leibniz *intend* to admit a given sort of number into the class of genuine numbers? and (2) given that intent, to what extent is Leibniz *able* to admit (or exclude) that sort of number, given his general account of number?

It turns out that these two questions involve Leibniz, and any interpretation of his account of number, in a great deal of philosophical difficulty. On the one hand, Leibniz appears to intend to countenance at least some of what we now call irrational numbers, but it ultimately remains unclear which of these he is referring to. At the same time, Leibniz appears to be committed to the existence of *algebraic* irrationals without realizing

he is so committed, while his account of the concept of an irrational number is only able to provide a partial understanding of how any irrational number might qualify as homogeneous to unity. Even more intriguingly, Leibniz also appears to be committed to the existence of negative and complex numbers, despite his explicitly stated desire to *exclude* them from the class of numbers that he considers genuine. Finally, despite the difficulties involved in Leibniz's account of irrationals, negative numbers, and complex numbers, his general account of number appears to be consistent with his explicit rejection of infinite cardinals and infinitesimals -- even providing a way of understanding that rejection that is not advanced in the literature.

I proceed as follows. In the next section, I outline the difficulties presented by Leibniz's uses of various terms that are now technical terms corresponding to different sorts of irrational number. In section 3, I argue that Leibniz is committed to the existence of algebraic irrationals, but that his conceptual account of irrational numbers in terms of infinite series ultimately falls short of demonstrating precisely how they count as homogeneous to unity. In section 4, I argue that Leibniz is also committed to the existence of negative and complex numbers, despite his statements to the contrary. In section 5, I argue that Leibniz's general account of number helps us make further sense of his rejection of infinite cardinals and infinitesimals.

2. Different Kinds of Irrationals in Leibniz's Mathematical Writings

Before attempting any reconstruction of Leibniz's views on irrational numbers, we must survey the various ways he treats them in his mathematical work. It turns out

that what Leibniz means by terms such as “surd” and “transcendental” remains unclear. Furthermore, Leibniz uses other terms to refer to what we now call irrational numbers -- terms whose meaning remains nearly opaque.

In contemporary usage of the term “transcendental”, a transcendental number is a number that cannot be a root of any non-zero polynomial equation with rational coefficients. Leibniz appears to be aware of the difference between this kind of number and an irrational number that can be the root of such an equation: for example, Richard Arthur credits Leibniz with the discovery that pi is transcendental in the modern sense. But Leibniz also uses the word “transcendental” to refer to other mathematical concepts. For example, he uses it to refer to a difference between two kinds of *equation* or algebraic expression. It is not clear precisely what Leibniz has in mind here: for example, he writes that “the quadrature of the circle may require an expression of the kind I call ‘transcendental’ (A.III.1.203), and that “I call those [equations] transcendental which transcend all algebraic degrees” (GM IV 26). In another text, he speaks of a “geometry of transcendentals”, which has to do with “the quadratures of figures” and “the discovery of centers of gravity” (GM VII 12); this usage has the sense of mathematical procedures that transcend or “go beyond” the existing Cartesian methods for dealing with curves and rectification, whose inadequacy provides much of the inspiration for Leibniz's invention of the differential and integral calculus. The term “transcendental” [*transcendens*] occurs

regularly when Leibniz refers to the sorts of curves and rectification procedures that his calculus can treat, which are left out of Descartes' *Geometrie*.¹²

In other places, Leibniz explicitly refers to “transcendental quantities”, for example: “And transcendental quantities are granted, which I might call “more surd than surds” [*surdis surdiores*]” (GM VII 68). Leibniz seems to have in mind here certain expressions involving roots: in the *New Essays*, he distinguishes between “surd” and “super-surd”: surd numbers are “representable by means of an ordinary equation”, while “super-surd” numbers are only representable “by means of an extraordinary one which introduce[s] irrationals” or the unknown itself into the exponent” (RB 377). As Remnant and Bennett point out, Leibniz never makes himself entirely clear on what exactly distinguishes surd from “more-than-surd”/“super-surd”. Since he identifies “more-than-surds” with “transcendental quantities”, one can say that he never ultimately clarifies the general difference between irrational and transcendental numbers on his view. Remnant and Bennett refer to a passage in GM IV, a letter to Wallis in which Leibniz introduces the phrase “geometrice-irrationalia” [geometrically irrational] and says the following about this category:

I distinguish these from transcendentals, as genus from species. Thus I make two genera from these *geometrice-irrationalia*: some are of a definite degree, but irrational, of which the exponent is a surd number, such as [the root-2th root of the square root of 2], or the power of 2 whose exponent is $(1/\sqrt{2})$; and I call these *intercendentia*, because their degree falls between rational degrees: but they should be able to be called, in a stricter sense, geometrically (or if you prefer, algebraically) irrational.

¹² One representative example of this phenomenon is the text that Gerhardt titles “Inventorium Mathematicum” (GM VII, 12-17)

The others are truly of an indefinite degree, such as x^y : and I call these transcendental more properly. (GM IV 27-28)

Despite Leibniz's efforts to make distinctions among various kinds of irrational numbers in this passage, he ultimately renders his view of what constitutes a “transcendental quantity” even less clear, for he seems to identify them with expressions in which both base and exponent are variables. But it is unclear why such an expression need be treated as “of indefinite degree”: for example, if we substitute 2 for x , and 3 for y , we get 2^3 , which is not of “indefinite degree” and in fact yields the integer 8. Furthermore, this passage seems in conflict with the one just cited from the *New Essays*, where Leibniz identifies a broader class of numbers to be called “super-surds”, namely those which *either* have an irrational number as exponent *or* have “the unknown itself” as exponent. In the letter to Wallis, he identifies only the latter as “super-surd”, assuming that “super-surd” is synonymous with “transcendental”, which has just been shown.

In the texts where Leibniz defines number -- surveyed in earlier chapters -- he does not mention any of the distinctions just described, but does use the terms “surd” and “transcendental”. He does adduce familiar examples of what we now call algebraic irrational and transcendental numbers, such as the square root of 2 (which is algebraic irrational) and π (which is transcendental), but makes no effort to provide separate definitions for algebraic irrationals transcendentals. Furthermore, he does not even mention or any of the other sorts of numbers I have just surveyed. This leaves us puzzled at the outset when attempting to understand which sorts of irrationals Leibniz intends to admit, and to what extent his general account of number might allow him to admit them.

3. Leibniz on Algebraic and Other Irrationals

Given the results of the previous chapters, it is fairly easy to see how the rational numbers count as real for Leibniz, and it is similarly easy to see how he understands them as homogeneous to unity. Integers and fractions, as we have seen, qualify as homogeneous to unity because they can be rendered similar to unity by a transformation. Crucially, Leibniz explicitly lists the kinds of operations that count as “transformations” that yield numbers in the following text: “Numbers are... generated by operations which are either synthetic (addition, multiplication, raising to a power) or analytic (subtraction, division, extraction of roots)” (GM VII 208). Along these lines, Leibniz provides a deceptively simple division of integers, fractions, and irrationals into separate instances of the general category of that which is homogeneous to unity:

Number is that which is homogeneous to Unity, and so it can be compared with unity and added to or subtracted from it. And it is either an aggregate of unities, which is called an integer, like 2 (i.e. 1+1), likewise 3, 4 (i.e. 2+1 or 1+1+1), or an aggregate of several parts of unity, which is called a fraction... [N]umber is in some way determined through a relation to unity, which relations can indeed be infinite, but they are most often found through roots... For example, let there be the number 4... there is sought its square root... that is, the number which multiplied by itself makes 4; this number will be 2, and so since 2×2 ... is 4, $\sqrt{4}$ is 2. And in this case the root can be reduced to a common, or rational, number. But sometimes this reduction does not succeed. For example, if a number is sought, which multiplied by itself makes 2, this is not an integer (for otherwise, since it is necessary that it be less than 2, it would be unity, and unity multiplied by itself makes 1); nor is it a fraction, since any fraction multiplied by itself produces some other fraction, as $3/2$ produces $9/4$, or $2 + 1/4$. (GM VII 31)

Here, as we have seen previously, Leibniz defines the integers as aggregates of unities and fractions as aggregates of parts of unity. However, he also appears to define what are now called irrational numbers as those numbers which cannot be understood by

means of integers or fractions, though they are still “in some way determined through a relation to unity” -- though the precise extension of the class of numbers he intends to characterize this way shall remain unclear, given the results of the previous section.

Before discussing the way in which irrational numbers are so determined, it is worth noting the general structure of Leibniz's definition of number. Leibniz takes two basic notions -- the notions of unity and homogeneity -- and uses them in a recursive way to define the three different kinds of positive real number. The integers are understood as those numbers which bear the most basic relation to unity: they are simply those numbers that are intelligible using unity and the operation of addition (such as how the number 5 is intelligible as $(1+1+1+1+1)$). The notion of a fraction is built upon the notion of an integer: in characterizing a fraction in terms of a relation to unity, we first need to be able to conceive of unity as divided into an integral number of parts. For example, in order to render the fraction $3/5$ intelligible in terms of some operation on unity, we first need to be able to divide unity into five parts, and then to add three of those parts together. Leibniz understands the integers in a way that resembles Euclid's conception, defining them as aggregates of unities; of course, the difference between Leibniz's conception and Euclid's is that Leibniz holds that aggregates of unities are only one kind of number, while Euclid holds that they are the only kind of number. On Leibniz's view, integers qualify as homogeneous to unity simply because they are understood as collections of unities, which are generated from unity by addition and can be turned back into unity by subtraction. Every integer is intelligible in terms of the question “how many ones?” -- for example, as noted above, the number 5 can be resolved into $(1+1+1+1+1)$. Similarly, fractions qualify

as homogeneous to unity in a straightforward way. Generated from unity by division and addition, fractions are intelligible in terms of the question “how many equal parts of one?”: for example, the fraction $3/5$ can be resolved into $(1/5+1/5+1/5)$, and can be turned back into unity by further addition.

Now, recalling Leibniz's remark that numbers are generated by operations including “raising to a power” and “extraction of roots”, it is noteworthy that he appears to be committed to the view that all *algebraic irrational* numbers count as numbers. These are irrational numbers which can be solutions to algebraic equations, and they include all of the irrational roots, such as Leibniz's own example of the square root of 2. Although he writes that the relation to unity through which an irrational number is determined “may be infinite”, his definition of number appears to entail that regardless of the exact nature of its relation to unity, any n th root counts as a number. This is best illustrated by example. Consider the square root of 2. This number is generated by one of Leibniz's listed operations: the extraction of a square root, performed on the number 2. If anything that can be generated by one of Leibniz's operations counts as a number, then we can see that any root counts as a number, rational or not. The square root of 3, which is also irrational, is generated by the same operation -- the extraction of a square root -- performed on the number 3. Crucially, Leibniz does not limit the scope of root extraction merely to the extraction of square roots: he only notes the extraction of roots in general. Thus, he is committed to the view that, for example, the cube root of 2 is a number, as it is generated from 2 by the operation of extracting the cube root. The upshot is that before Leibniz says anything specific about *how* irrational roots are understood in terms of unity, he is already

committed to the claim that these numbers exist, contrary to the received view of number that is undergoing challenge in the seventeenth century.

However, there is a *prima facie* difference between rational numbers and irrational roots in terms of the relation of homogeneity they bear to unity. The number 2 is understood as homogeneous to unity because it can be turned into the number 1 directly, by the operation of subtraction. The number $3/5$ is so understood, again, because it can be turned into the number 1 directly, by the operations of addition or multiplication. Similarly, the numbers 2 and $3/5$ can be generated from the number 1 by addition and division. But the square root of 2 -- or any other irrational root, square or not -- cannot be transformed into the number 1 by means of these arithmetical operations, and cannot be generated from the number 1 except by means of an infinite series that has the root as its sum. Nonetheless, Leibniz appears to hold that these count as numbers because they can be generated from rationals by the extraction of roots, even though they are not directly intelligible in terms of unity in the same way that rationals are. This suggests that Leibniz holds that homogeneity is transitive: if to be a number is to be homogeneous to unity, and irrational roots count as numbers even though they cannot be directly generated from or directly transformed into unity, then they must count as numbers because they are homogeneous to numbers that *are* directly generable from and transformable into unity. This, again, is best illustrated by example. The square root of 2 cannot be generated from unity except by means of an infinite series; but it *can* be generated from a number that *is* generable from unity by simple addition -- to wit, the number 2. The square root of 2, in other words, is homogeneous to 2, because it is generable from and transformable into 2,

by the extraction of roots and by squaring, respectively. And 2 is homogeneous to unity because it is generable from and transformable into unity by addition and subtraction, respectively. Leibniz cannot claim that the square root of 2 is homogeneous to unity in the same direct way as the number 2; but since he clearly holds that the square root of 2 is a number, he must hold that homogeneity is transitive.

In order to see that Leibniz must endorse the transitivity of homogeneity for his definition of number to countenance irrational roots, we can run the following argument. This argument will show that given Leibniz's definition of number, together with the assumption that homogeneity is transitive, the n th root of any integer is also a number. Here, "x" stands for an integer:

- (1) Something is a number if and only if it is homogeneous to unity
- (2) A is homogeneous to B if and only if A can be rendered similar to B by a transformation
- (3) If A is homogeneous to B, and B is homogeneous to C, then A is homogeneous to C
- (4) The n th root of x can be rendered similar to x by the operation of raising to the n th power
- (5) Therefore, the n th root of x is homogeneous to x
- (6) x can be rendered similar to unity by the operation of subtraction
- (7) Therefore, x is homogeneous to unity
- (8) Therefore, the n th root of x is homogeneous to unity Therefore, the n th root of x is a number

Thus, the n th root of any integer counts as a number on Leibniz's conception, but only on the assumption that homogeneity is transitive.

Despite Leibniz's commitment to the existence of algebraic irrationals, he is unable to give a fully fleshed-out account of how any irrational number might be intelligible in terms of homogeneity to unity. It is one thing for him to be able to claim, given his definition, that a given kind of irrational number should be subsumed under his general account; it is quite another for him to specify what it means, precisely, to say that they are homogeneous to unity, beyond simply formulating a definition which entails that they are. In a passage quoted above, Leibniz makes the tantalizing remark that irrational numbers are determined through relations to unity that “may be infinite”. One may be inclined to interpret this remark through the lens of contemporary mathematics, in which every irrational number is the sum of an infinite series. Unfortunately, in the vast volumes of Leibniz's work on infinite series, he appears only to even begin carrying out this kind of analysis of irrational numbers in one case: that of π , which is transcendental, rather than algebraic. Nonetheless, it is worth examining this case in tandem with other evidence in attempting to determine how sophisticated Leibniz's understanding of irrational numbers might be.

At the outset, it is worth noting that π is not among the numbers whose existence is entailed by Leibniz's definition of number together with the assumption that homogeneity is transitive. π is not generable from any number via any of the operations Leibniz lists; as a transcendental number, π is not the solution to any algebraic equation. Nonetheless, π represents the case in which Leibniz gives his clearest *direct* expression

of an irrational number via an infinite series. In particular, Leibniz is noted for discovering an infinite series whose sum is $\pi/4$ -- the so-called "Leibniz Series". For Leibniz, this number bears an "infinite" relation to unity in the sense that it is only expressible in terms of unity as the sum of an infinite series of rational numbers: $(1 - 1/3 + 1/5 - 1/7 + 1/9\dots)$. In this series, $\pi/4$ is generated from 1, and so is understood in terms of unity, in the sense that it is the sum of an infinite series of rational numbers beginning with 1.

Interestingly, Leibniz appears to have a similar conception of the way in which the square root of 2 is related to unity, though he expresses it less directly. In the course of discussing the relation between the diagonal of a square and the side, where the side is stipulated to be of length 1, Leibniz writes: "since therefore we have defined number as that which is homogeneous to unity, and indeed there must be some number, of which this [i.e. the diagonal] is the relation to unity... there must be a number by which the quantity of [the diagonal] itself is expressed, which is said to be $\sqrt{2}$ ". Just as π is the number that expresses the relation between a circle's circumference and its diameter, where the diameter is 1, the square root of 2 is the number that expresses the relation between a square's diagonal and its side, where the side is length 1. In both of these cases, just as the diagonal is incommensurable with the side, and the circumference with the diameter, the *numbers* that express the diagonal and the side are incommensurable with 1, in the sense that they cannot be generated from 1 by any finite sum of rational numbers. What is particularly suggestive about these cases is that Leibniz actually *defines* irrational numbers *in general* -- without restricting himself to any specific examples -- as those numbers that are "incommensurable with unity" (GM VII 73; A.VI.4.419). In turn,

Leibniz appears to hold that “incommensurable with X” just means “only expressible in terms of X by means of an infinite series”. He writes:

Proportion is containment of a smaller quantity in a larger or of an equal in an equal. It is shown by displaying the relation of the numbers through the analysis of both terms into common quantities. This analysis is either finite or infinite. If it is finite, it is said to be the discovery of a common measure or a commensuration; and the proportion is expressible, for it is reduced to congruence with respect to the same repeated measure... But if the analysis proceeds to infinity and never attains completion then the proportion is unexpressible, one which has an infinite number of quotients, but in such a way that there is always something that remains, a new remainder that furnishes a new quotient. Moreover, the analysis continued yields an infinite series... (AG 98-99/C 1-2)

Here, Leibniz suggests that he understands incommensurability in terms of infinite series. In turn, if he defines irrational numbers as those numbers which are incommensurable with unity, then it is reasonable to believe that he understands these numbers as intelligible in terms of unity only by means of infinite series. However, as I have noted, Leibniz does not explicitly provide infinite series for irrational numbers other than π in his mathematical writings. Thus, although there is some evidence to suggest that he understands other irrational numbers in terms of unity in the way that he understands π , it is not possible to establish for certain that Leibniz has more than a general conception of irrationals as related to unity by means of infinite series, rather than a sophisticated account of how to analyze particular irrational numbers by means of infinite series of rationals.

Ultimately, on Leibniz's view, it appears that irrational numbers -- both those that we now call algebraic and those that we now call transcendental -- are supposed to qualify as homogeneous to unity even though the way they are related to unity is “infinite,”

suggesting a conception of irrational numbers on which they qualify as homogeneous to unity by way of some sort of infinite process. However, Leibniz lacks the technical tools to give a mathematically acceptable account of what that infinite process involves. Nonetheless, it is worth exploring the extent to which Leibniz does provide a conceptual account of what we now call irrational numbers in these terms.

First, it is worth exploring exactly why irrationals cannot be understood in terms of unity by means of a finite process *in Leibnizian terms*. They cannot be so understood because they do not qualify as expressing proper aggregates, given Leibniz's conception of the latter. In the previous chapter, it was established that Leibniz conceives of the *rational* numbers as relations that unite the constituents of actual or possible aggregates insofar as the latter are composed of unities or parts of unity, where these relations express the wholeness and size of those aggregates by answering the question “how many?”. Positive integers express the wholeness and size of aggregates of unities; fractions express the wholeness and size of aggregates of *parts* of unity, where parts of unity are specified relative to a chosen unit object. *Prima facie*, a problem arises for Leibniz in subsuming irrational numbers under this analysis. The aggregates, actual or possible, that are united by *rational* numbers are all *finite*: every positive integer and fraction expresses the size of an aggregate whose constituents can be finitely enumerated. But it is the defining characteristic of *irrational* numbers that they cannot do this. Irrational numbers cannot be resolved into any finite combination of operations on rational numbers.

For Leibniz, the foregoing amounts to the claim that irrational numbers cannot be relations that express aggregates, since he repeatedly denies that there are any infinite

aggregates. The aggregate expressed by an infinite series would presumably be infinite, and so it would fail to be a genuine aggregate. Leibniz writes that “an infinity of things is not one whole, i.e... there is no aggregate of them” (A.VI.3.504/LOC 101). He rejects infinite aggregates for the specific reason that they violate the part-whole axiom. There are several instances of this line of argument Leibniz's corpus. One particularly lucid example is the following:

[I]f the infinite number of all unities, or what is the same thing, the infinite number of all numbers, is a whole, it will follow that one of its parts is equal to it; which is absurd. I will show the force of this consequence as follows. The number of all square numbers is a part of the number of all numbers: but any number is the root of some square number, for if it is multiplied into itself, it makes a square number. But the same number cannot be the root of different squares, nor can the same square have different roots. Therefore there are as many numbers as there are square numbers, that is, the number of numbers is equal to the number of squares, the whole to the part, which is absurd. (A.VI.3.98/LOC 13)

Another instance worth noting occurs in Leibniz's 1672 notes on Galileo's *Two New Sciences*:

[Galileo] thinks that one infinity is not only not greater than another infinity, but not greater than a finite quantity. And the demonstration is worth noting: Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo... and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole. (A.VI.3.168/LOC 9)

Leibniz points out that if a number expresses how many positive integers there are, that same number will express how many squares there are -- in modern terms, the set of all positive integers is equinumerous with one of its proper subsets, a violation of the axiom

that the part is smaller than the whole. For Leibniz, working before set theory, such a conclusion is absurd, implying the incoherence of supposing that there is a number of all positive integers. But the violation of the part-whole axiom applies to infinite aggregates in a general sense: infinite aggregates cannot be genuine wholes because they are equinumerous with some of their proper parts.

Given Leibniz's denial of infinite aggregates, one would not be out of line in objecting that he cannot subsume irrational numbers under his account precisely because accommodating irrational numbers would require him to accept the infinite aggregates that they express: namely, those that correspond to infinite series. However, even though irrational numbers do not express aggregates, Leibniz still appears to conceive of them as intelligible in terms of unity: specifically, they are so intelligible by means of infinite series of rational numbers. For Leibniz, an infinite series provides us with an intelligible conception of an irrational number in the same way that, for example, the finite series $(1+1+1+1+1)$ gives us an intelligible conception of the number 5. In both cases, we understand the number in terms of some operation on unity. What Leibniz seems to have in mind is an account of how irrational numbers are *generated from* unity using a series of transformations -- indeed, an infinite process involving adding together rational numbers according to a rule. Ultimately, irrationals qualify as homogeneous to unity for Leibniz because they come from unity by means of an intelligible infinite process. Unfortunately, as I show presently, Leibniz cannot give a satisfactory account of the precise nature of that infinite process. At this point, it must be noted that certain features of Leibniz's philosophy of mathematics initially cast doubt on any interpretation of his

view of irrational numbers that invokes infinite series. My suggestion that Leibniz conceives of irrational numbers in terms of infinite series must be squared with a view that he appears to hold about infinite series themselves. Squaring his conception of irrationals with his view of infinite series will further illuminate the former.

The first step in reconciling these views will be to present Leibniz's *apparent* conception of infinite series and their sums. A representative example of his apparent view of infinite series occurs in a discussion of the series whose sum is $\pi/4$. Here, he writes:

We must still investigate whether... the square is to the circle as 1 to $1/1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + \dots$. For when we say 'etc.', '...', or 'to infinity', the last number is not really understood to be the greatest of the numbers, for there isn't one, but it is still understood to be infinite. But seeing as the series is not bounded, how can this be the case? For something must be added, even if it is assumed to be an infinite number, so that it must be said that this is not rigorously true. And seeing as the circle is nothing, this series will of course also be nothing (A.VI.3.502/LOC 97).

Leibniz here asks what it means to say that an infinite series -- in particular, the kind of series we understand as summing to an irrational number -- goes "to infinity." Levey helpfully elaborates on the question Leibniz grapples with. Leibniz, Levey writes, recognizes that

'something must be added' to the infinite series in order to calculate its sum. That 'something' does not in fact occur in the series. The mathematical conception of the infinite series smuggles in a fictional terminus under its interpretation of the rider 'etc.' or 'to infinity'... Since that series does not in fact contain a last term, the proposition involving its measured form engages directly (if tacitly) in a fiction, and so must be said not to be rigorously true" (1998, 80).

Leibniz thus raises the question: what does it mean to say that an infinite series -- in particular, the kind of series we understand as convergent upon a real number -- goes “to infinity”? His remark about the circle at the end of this passage connects the question with a view that lies at the heart of Leibniz's calculus. Richard Arthur explains the link by way of a passage from *De Quadratura Arithmetica*:

This conclusion, that there is no last number in an infinite series, not even an infinite one, is very much in keeping with the new interpretation of the calculus that Leibniz develops... In his first comprehensive treatise on the calculus, written between the fall of 1675 and the summer of 1676, Leibniz writes that his readers 'will sense how much the field has been opened up when they correctly perceive this one thing, that every curvilinear figure is nothing but a polygon with an infinite number of sides, of an infinitely small magnitude'" (LOC Iv)

The passage that Arthur quotes occurs in the context of a scholium in which Leibniz reflects on “what we have said thus far about infinities and infinitely small things” (A.VII.6.585), referring to the subject of the treatise -- the approximation of a circle by polygons with increasing numbers of sides. In the course of these reflections, Leibniz explicitly states that “it does not matter whether these quantities might exist in the nature of things, for it is sufficient that they be introduced by means of a fiction” (ibid). For Leibniz, both the infinitely-sided polygon and the sum of an infinite series are *fictions* or *fictitious entities*. In “Infinite Numbers”, he says this about the circle:

The circle -- as a polygon greater than any assignable, as if that were possible -- is a fictive entity, *and so are other things of that kind*. So when something is said about the circle we understand it to be true of any polygon such that there is some polygon in which the error is less than any assigned amount *a*, and another polygon in which the error is less than any other definite assigned amount *b*. However, there will not be a polygon in which this error is less than all assignable amounts *a* and *b* at once, even if it can be said that polygons somehow approach such an entity in order. *And so if certain polygons are able to increase according to some*

law, and something is true of them the more they increase, our mind imagines some ultimate polygon; and whatever it sees becoming more and more so in the individual polygons, it declares to be perfectly so in this ultimate one. And even though this ultimate polygon does not exist in the nature of things, one can still give an expression for it, for the sake of abbreviation of expressions. (A.VI.3.498/LOC 89, emphasis mine)

“Other things of that kind” include the limit of an infinite series of numbers. Indeed, for Leibniz, the issues described in this passage arise whenever one seeks to expound an irrational ratio by means of a series. He notes the same issue for the numerical expression of the ratio between a square and its diagonal in the same text: “Diagonal to square is a certain ratio, since the diagonal is a line, a real quantity, and the side is too. If this is to be expounded by means of numbers, there will also be a need for infinite numbers -- indeed, for all numbers in general. But to say all numbers is to say nothing; and for this reason this ratio also means nothing, *unless it is something as close as desired*” (A.VI.3.502-503/LOC 99, emphasis mine).

Leibniz's talk of the circle as a kind of “ideal limit” -- to use Arthur's term -- of a series of polygons with ever more sides is directly analogous to talk of a given number as an ideal limit of a series of ever-diminishing numbers. Leibniz's gloss of the meaning of talk of the circle, as being elliptical for talk of polygons in which the error is less than any assigned amount, also parallels his gloss of the meaning of talk about the properties of infinite series, as Arthur notes. The latter gloss can also be found in “Infinite Numbers”: “Whenever it is said that a certain infinite series has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like” (A.VI.3.503/LOC 99).

Thus, it initially appears that for Leibniz, the sum of an infinite series is fraught with such conceptual difficulty that he is forced to claim that it can only be understood as something ideal, which the mind conceives by extrapolating from features of the series. So the convergent series that one might think exhibit irrational numbers face the problem of explaining what it means to say that they have a sum -- to say, in other words, that they are infinite but somehow bounded or limited. Without an answer to this question, Leibniz's intention to confer the status of real numbers upon irrationals appears to be incoherent, as irrationals just are such sums. Thus, an objection to my interpretation might be put as follows: Leibniz cannot maintain both that irrational numbers are to be understood as sums of infinite series *and* that such numbers exist, as he appears to hold that the sums of infinite series do not exist.

Ultimately, however, it turns out that Leibniz's considered view of infinite series yields both a coherent interpretation of his view that at least some irrational numbers are intelligible in terms of a relation to unity, *and* a direct illustration of the limitations of that view. At the very least, if Leibniz *does* understand irrationals in terms of unity by means of infinite series, he is not inconsistent in doing so. First, consider again a passage just cited: "Whenever it is said that a certain infinite series has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like" (A.VI.3.503/LOC 99). The fact that the finite series, and the terms therein, are generated by the same "rule" is what guarantees that the infinite series with the same rule will have a sum, *despite* the fact that the series "goes to infinity." Indeed, Leibniz actually

appears to *reject* the notion that an infinite series gets its sum by means of an infinitieth term. The idea that infinite series operate according to a rule is crucial for Leibniz; this guarantees that infinite series are conceivable to the human mind despite the fact that, strictly speaking, they go on without end, lacking a final term.

Leibniz encapsulates his view that infinite series are intelligible despite “going to infinity” in his Principle of Continuity. He formulates the principle differently in different texts, but the version that is relevant here can be found in a text on the calculus. There, Leibniz advances the following proposition:

Proposito quocunque transitu continuo in aliquem terminum desinente, liceat ratiocinationem communem instituere, qua ultimus terminus comprehendatur. (In any proposed continuous transition ending in some terminus, it should be permissible to institute a common reasoning, in which the final terminus may also be included.) (HODC 40)

Shortly after this, Leibniz adduces as an example the consideration of a parabola as an ellipse that is stretched out to the extent that one of its foci “vanishes or becomes impossible” (“evanescat seu fiat impossibilis”) (41). In this case, he says, “it is permissible, from our postulate, to include the parabola with the ellipse in one reasoning” (41). This is, presumably, structurally similar to considering the circle “in the same reasoning” with a polygon whose number of sides is continually increased, and similar to considering the sum of an infinite series “in the same reasoning” with the terms that add up to it. Later in the text, Leibniz indirectly implies that the consideration of the circle as the limit of a sequence of polygons with increasing numbers of sides is an example of his principle at work (42). These are far from unintelligible or empty notions, and the mind employs them in order to conceptualize the “end” of infinite processes that gradually

converge upon a terminus in accordance with some intelligible principle. Leibniz elaborates on these points in another text on the calculus:

[W]hat I call the *law of continuity*... has long served me as a principle of discovery... Some years ago I published an example of this in the *Nouvelles de le republique des lettres*, in which I take equality as a particular case of inequality, rest as a special case of motion, parallelism as a case of convergence, etc... Although it is not at all rigorously true that rest is a kind of motion or that equality is a kind of inequality, any more than it is true that a circle is a kind of regular polygon, it can be said, nevertheless, that rest, equality, and the circle terminate the motions, the inequalities, and the regular polygons which arrive at them by a continuous change and vanish in them. And although these terminations are excluded, that is, are not included in any rigorous sense in the variables which they limit, they nevertheless have the same properties as if they were included in the series, in accordance with the language of infinities and infinitesimals, which takes the circle, for example, as a regular polygon with an infinite number of sides. Otherwise the law of continuity would be violated, namely, that since we can move from polygons to a circle by a continuous change and without making a leap, it is also necessary not to make a leap in passing from the properties of a polygon to those of a circle. (GP IV 106/L 546)

According to this principle, then, we may take the sum of an infinite series -- the “final terminus” of the series -- because we do so “in the same reasoning” with which we take the terms that add up to it. The human mind, Leibniz appears to think, is able to understand the sum of an infinite series because it understands the intelligible rule that generates all of the terms in the series. Leibniz does not actually hold that the sum of an infinite series expresses the value of a completed infinite collection with a final term; indeed, the point of his Principle of Continuity is to say that the addition of a final term (which would produce a completed infinite collection) is not required for the understanding of the series' sum. Rather than “going to infinity” in the sense of having an infinite number of terms, the series “goes to infinity” in the sense that *despite lacking an*

infinitieth term, it has a sum that can be determined via the rule of the series. Each term is generated by the same rule, and the difference between them diminishes arbitrarily as the series grows. The sum of the whole series, rather than being the result of adding an *infinitieth term*, is simply the value extrapolated from the diminishing differences between the terms. In contrast to the view Leibniz initially appears to hold, the sum of an infinite series has a perfectly intelligible status in his philosophy of mathematics, and does not undermine his inclusion of irrationals in the pantheon of conceptually coherent, genuine numbers. Thus, to whatever extent he does understand irrationals in terms of infinite series, his understanding is consistent with his considered view of the conceptual status of such series.

Ultimately, however, Leibniz's account of infinite series in terms of the Principle of Continuity illustrates how limited his account of irrational numbers turns out to be. First, Leibniz lacks a rigorous account of how infinite series might have sums. Despite giving primacy to the notion of a rule in his Principle of Continuity, Leibniz never provides a method of determining how any given irrational number might be generated from unity by means of a rule. Though he appears to understand at least some irrational numbers (though it is worth stressing again the unclarity of exactly which irrationals Leibniz intends to accommodate) as sums of infinite series of rationals, he lacks anything resembling a rule for taking such sums. His Principle of Continuity operates at a merely conceptual level, and appears to be intended to elucidate the conceptual coherence of whichever irrational numbers Leibniz wants to subsume under his general account of number. But a rigorous mathematical procedure it is not, and so Leibniz's understanding

of irrationals as homogeneous to unity lacks the transparency -- and the applicability to entire classes of numbers -- that his understanding of rationals possesses. It is clear, on Leibniz's account of rational numbers, how to generate any given rational number from unity; it remains opaque, given Leibniz's Principle of Continuity, how to generate any given *irrational* number from unity.

4. Leibniz on Negative and Complex Numbers

It is difficult to understand Leibniz's position on negative and complex numbers, both in light of his general conception of number and independently of it. First, Leibniz clearly *intends* to reject negative numbers. When he writes about them, he often shifts between talk of quantity and talk of number, but it is clear enough that he wishes to deny the legitimacy of both negative quantities and the numbers that would express them. In one typical text, he writes that “negative quantities, where a greater is to be subtracted from a lesser, often arise in calculation”, indicating that “the question has been conceived badly” (GM VII 70). Presumably, this amounts to an outright rejection of the legitimacy of negative quantities and the numbers that express them “in calculation”. It is easy enough to see why Leibniz wants to reject negative *quantities*. Recall Leibniz's definition of quantity as that which can be understood in terms of a number of parts: a quantity that is less than nothing, if it can be understood at all, certainly cannot be understood as collection of parts. If it could be so understood -- if it could be divided up into a collection of parts in accordance with a unit of measurement -- then it would have to have positive size, and would not be less than nothing after all. As the very notion of a negative quantity

is incoherent, negative quantities cannot be real quantities, and when they appear to arise in calculation, as Leibniz says, they are mere indications that something has gone wrong.

At this point, one might think that Leibniz rejects negative numbers because such numbers purport to express incoherent quantities. But as we have seen, Leibniz defines number independently of quantity: he holds that number is conceptually prior to quantity, and provides a definition of number that does not make the reality of any number dependent on the ability to express a quantity. It is interesting to note, then, that Leibniz's definition of number seems to fail to rule out negative numbers, even if negative *quantities* are incoherent. In fact, the definition seems to commit Leibniz to the *acceptance* of negative numbers. This is best illustrated by example: at a first pass, it seems that the number -3 qualifies as homogeneous to unity, as it can be rendered back into unity by adding 4. The same applies to any negative number, since there is always a way to get back to the number 1 by addition. Negative numbers, as Leibniz would say, can be compared with unity and added to or subtracted from it. Thus, it seems that Leibniz's definition of number *by itself* cannot straightforwardly rule out negative numbers as real, despite Leibniz's clear desire to do so -- indeed, it seems that Leibniz's definition should commit him to the view that negative numbers are real.

Leibniz's treatment of the imaginary root and complex numbers is even more difficult to square with his general definition of number. First, Leibniz often runs together his treatment of the imaginary root in itself, on the one hand, and his treatment of complex numbers that involve a real part. Additionally, as he does for negative numbers, he often shifts between talk of quantity and talk of number. In discussing these "impossible or

imaginary quantities”, whose “nature is amazing” (GM VII 73), he writes: “[F]or although they themselves signify something impossible in itself... but they also, with the assistance of themselves, can be expressed by real quantities” (ibid). A few lines later in the same text, he writes that expressions such as $\sqrt{-1}$ “have this miraculous thing: that in calculation, they involve nothing of the absurd or contradictory, and nevertheless they cannot be exhibited in the nature of things or concretely” (ibid). In another text, adducing examples of the utility of imaginary roots, Leibniz writes:

Even though these are called imaginary, they continue to be useful and even necessary in expressing real magnitudes analytically. For example, it is impossible to express the analytic value of a straight line necessary to trisect a given angle without the aid of imaginaries... Furthermore, imaginary roots likewise have a real foundation. So when I told the late Mr. Huygens that $\sqrt{1+\sqrt{-3}} - \sqrt{1-\sqrt{-3}} = \sqrt{6}$, he found this so remarkable that he replied that there is something incomprehensible to us in the matter. (L 544/GM IV 93)

In these texts, Leibniz is unclear about the status of imaginary roots, complex numbers, and the quantities that may or may not be expressed by them. Indeed, at first glance, he appears to lean toward admitting both the imaginary root and complex numbers as real. He writes that not only are roots of negative numbers indispensable for expressing certain mathematical objects when used in tandem with real numbers, but these roots themselves also have a “real foundation”, as evidenced by the fact that computations involving imaginary roots sometimes yield a real number. Leibniz adduces as an example of the latter phenomenon the operation $\sqrt{1+\sqrt{-3}} - \sqrt{1-\sqrt{-3}}$, which is equal to $\sqrt{6}$. Another example of the “real foundation” of imaginary numbers is found in Leibniz's claim that $\sqrt{-2}$ is equal to “ $\sqrt{2}$ multiplied by $\sqrt{-1}$ ” (GM VII 73); here is a case in which the square root of a negative number is generated by an operation

involving the square root of a positive number. More precisely, this is a case in which a calculation yields the square root of a *negative* number by an operation involving the square root of a *positive* number. Finally, unlike in the cases canvassed in the previous section, Leibniz claims that imaginary roots “involve nothing of the absurd or contradictory”. For the reasons explored above, Leibniz seems to hold that infinite cardinals and infinitesimals both involve absurdities, and are fictitious; but he states the opposite about imaginary roots.

Prima facie, it would be strange for Leibniz to *want* to admit imaginary roots as real given his desire to exclude negative numbers: if imaginary roots are the square roots of negative numbers, and negative numbers are not real numbers, then how could Leibniz consistently maintain that imaginary numbers do not themselves fail to exist? Put more simply, if negative numbers do not exist, then how can their square roots not also fail to exist? However, as was the case with negative numbers, Leibniz's intentions may not line up with the implications of his own definition of number. Recall that Leibniz's desire to reject negative numbers is not straightforwardly supported by his definition of number, and so a rejection of imaginary roots that *follows* from a rejection of negative numbers fails to be supported in the same way. Indeed, if Leibniz's definition of number actually commits him to an *acceptance* of negative numbers as real -- given their apparent homogeneity to unity -- then this definition might also commit him to an acceptance of complex numbers as real. For example, if -2 counts as a number on Leibniz's conception because it can be turned into unity by addition, then why would its square root not also count as a number? Given that Leibniz's definition of number seems to commit him to

the reality of negative numbers, I suggest that it is not implausible to think that the definition might also commit him to the reality of complex numbers.

Furthermore, if Leibniz is committed to the reality of negative numbers *and* to the transitivity of homogeneity -- which I have argued that he is -- then it may be possible to demonstrate rigorously that he is committed to the reality of complex numbers. This can be done via an argument analogous to the argument similar to that which demonstrated his commitment to the reality of algebraic irrationals. The argument proceeds as follows, where x again stands for a positive integer:

- (1) Something is a number if and only if it is homogeneous to unity
- (2) A is homogeneous to B if and only if A can be rendered similar to B by a transformation
- (3) If A is homogeneous to B , and B is homogeneous to C , then A is homogeneous to C
- (4) The n th root of $-x$ can be rendered similar to x by the operation of raising to the n th power
- (5) Therefore, the n th root of $-x$ is homogeneous to $-x$
- (6) $-x$ can be rendered similar to unity by the operation of addition
- (7) Therefore, $-x$ is homogeneous to unity
- (8) Therefore, the n th root of $-x$ is homogeneous to unity Therefore, the n th root of $-x$ is a number

Thus, if Leibniz endorses the transitivity of homogeneity (which I argued earlier that there is good reason to think he does), then he is apparently committed to the reality of the roots

of negative numbers. This is, again, despite his repeated remarks to the effect that such numbers at least cannot be *fully* real, and despite his repeated remarks that negative numbers themselves are completely *unreal*.

5. Leibniz's Account of Number and his Rejection of Infinite Cardinals and Infinitesimals

By way of concluding the main body of this chapter, I now investigate what bearing Leibniz's account of number might have on his views of infinite cardinals and infinitesimals. Leibniz's views on these are well-known: in short, he rejects infinite cardinal numbers and infinitesimals. These purported numbers, for Leibniz, are not numbers at all. But it is worth exploring his rejection of these numbers from the point of view of the general account of number that has been reconstructed on his behalf.

First, as we have seen, Leibniz rejects infinite aggregates and the numbers that purport to express them -- in modern terms, he rejects infinite cardinal numbers -- because these entities violate the part-whole axiom, in the sense that they are as large as at least one of their proper parts¹³. Leibniz's rejection of infinite cardinals is consistent with his definition of number as that which is homogeneous to unity. An infinite cardinal straightforwardly fails to satisfy this definition: an infinite cardinal is not *in any way* intelligible in terms of unity. Whereas an irrational number, in Leibniz's account, is at least *supposed to be* related to unity as the sum of an infinite series of rational numbers (though as we have seen, Leibniz only provides a rudimentary account of how this might

¹³ A selection of texts where Leibniz makes this argument, along with helpful commentary, is found in LOC

actually work), in the case of an infinite cardinal, there is only a series that fails to have a sum. Leibniz's series for $\pi/4$ sums to $\pi/4$; the series $(1+1+1+1+1+1\dots)$, which Leibniz thinks of as representing the first infinite cardinal, merely increases without end. This is the crucial conceptual difference, for Leibniz between an infinite cardinal and an irrational number. An irrational number can be understood in terms of unity for Leibniz because it is generated from unity by an intelligible process whose "terminus" is conceivable to the human mind. This does not appear to be the case for an infinite cardinal, which is not a completed whole because the infinite series of unities composing it has no sum. It is merely the notion of the ceaseless iteration of unities, and fails to qualify as a number on Leibniz's view.

Leibniz's rejection of infinitesimals is also well-known and is also consistent with the definition of number explored in this chapter. The standard interpretation of Leibniz's view is that he denies actual infinitesimals and adopts the language of infinitesimals as a useful fiction for the purpose of problems that require the differential calculus¹⁴. Talk of infinitesimals, for Leibniz, is elliptical for talk of quantities that we can take as small as we like while yielding the same results in calculations. Accordingly, talk of any purported "number" that would express such a quantity is similarly elliptical. Passages such as these are taken to support this interpretation:

It will be sufficient if, when we speak of... infinitely small quantities... it is understood that we mean quantities that are... indefinitely small, i.e... as small as you please, so that the error that any one may assign may be less than a certain assigned quantity. Also, since in general it will appear that, when any small error is assigned, it can be shown that it should be less, it

¹⁴ This interpretation is articulated by a wide variety of authors; see e.g..Bos (1974); Ishiguro (1990); the chapters collected in Goldenbaum and Jesseph (2008); Mancosu (1996); and Knobloch (2002).

follows that the error is absolutely nothing; an almost exactly similar kind of argument is used in different places by Euclid, Theodosius and others... [I]t will be sufficient simply to make use of [infinitely small quantities] as a tool that has advantages for the purpose of calculation, just as the algebraists retain imaginary roots with great profit. (HODC xx)

But at the same time we must consider that these incomparable magnitudes themselves, as commonly understood, are not at all fixed or determined but can be taken to be as small as we wish in our geometrical reasoning and so have the effect of the infinitely small in the rigorous sense. If any opponent tries to contradict this proposition, it follows from our calculus that the error will be less than any possible assignable error, since it is in our power to make this incomparably small magnitude small enough for this purpose, inasmuch as we can always take a magnitude as small as we wish... [E]ven if someone refuses to admit infinite and infinitesimal lines in a rigorous metaphysical sense and as real things, he can still use them with confidence as ideal concepts which shorten his reasoning... (L 543/GP IV 92)

Leibniz thus appears to hold that talk of infinitesimals is simply a convenient way of expressing that we can take a given quantity as small as we wish during the course of a calculation, rather than denoting quantities that are actually infinitely small.

Just as for infinite cardinals, it is difficult to see how an infinitely small number could be the sort of thing that is homogeneous to unity. In the case of a number that is *finitely small* -- $3/100,000,000$, say -- we have an aggregate of parts of unity, as Leibniz defines fractions. No matter how small a fraction one proposes, we still have something that is understood in terms of unity and can be rendered back into unity by means of the requisite addition. We have, as Leibniz would once again say, something that is comparable to unity and can be added to or subtracted from it. By contrast, a purported number that is *actually* infinitely small, rather than a number that is taken as small as needed for the purpose of some calculation, seems by definition to fail to be comprehensible in terms of unity. Presumably, the concept of an infinitely small number

includes the key feature that the number is not expressible by means of any positive real number, no matter how small; if it is not so expressible, then it would seem to fail to be comprehensible in terms of unity, as what makes the positive real numbers what they are is their intelligibility in terms of unity. Another way of putting the point is to ask: by what sort of transformation is an actually infinitely small number related to unity, by means of which it might be rendered back into unity? If such a number is smaller than any positive real number, and the common feature of all real numbers is that they are transformable into unity, then it seems unlikely that any satisfactory answer will be forthcoming.

Chapter 5: Leibniz's Mathematical Epistemology

1. Introduction

Until now, this work has not been primarily concerned with epistemology. I have been mostly concerned with the conceptual details of Leibniz's account of number, the ontological implications thereof, and the precise extension of the class of numbers Leibniz might be able to subsume therein. The question of how Leibniz thinks we acquire knowledge of numbers -- or of mathematics and its objects in general -- has not been my focus. The goal of this chapter is to outline an epistemology of mathematics on Leibniz's behalf, with a particular focus on number -- though toward the end of the chapter, I suggest that Leibniz is committed to a deep epistemological contrast between arithmetic and geometry.

This chapter proceeds in the following way. I first treat Leibniz's general epistemology, arguing that it is broadly Platonistic, and that its Platonistic character extends to our knowledge of mathematics in general and number in particular. I then shift my focus to a pervasive misunderstanding in Leibniz scholarship: the view that the imagination has an essential role to play in mathematical -- particularly arithmetical -- cognition. Correcting this misunderstanding requires three steps. First, it is necessary to understand what the imagination *is* according to Leibniz. Second, it is necessary to understand why scholars have thought that Leibniz's epistemology of mathematics gives a primary role to the imagination in mathematical cognition. Finally, with this exposition in place, I will be in a position to argue that this view misrepresents

Leibniz's epistemological views with respect to mathematics, and that Leibniz is not committed to certain ontological claims that he *would* be committed to if this interpretation were accurate. The overarching aim of this chapter is to dispense with this view in favor of an interpretation on which the intellect, and not the imagination, is the primary engine of mathematical (particularly, arithmetical) cognition for Leibniz, and to show that this epistemological thesis squares perfectly with the ontological status that Leibniz ascribes to numbers.

2. Leibniz's Platonistic Epistemology of Mathematics

2.1. Leibniz's General Epistemology. A particularly lucid and concise summary of Leibniz's general epistemological scheme appears a 1702 letter to Queen Sophie Charlotte of Prussia; in this text, Leibniz explicitly puts himself in league with Plato with regard to the structure of human knowledge:

[W]hat the ancient Platonists have remarked is very true, and very worthy of consideration, that the existence of intelligible things... is incomparably more certain than the existence of sensible things, and thus, at bottom, there should only be these intelligible substances, and that sensible things should only be appearances. However, our lack of attention lets us take sensible things for the only true things. It is also worth observing that, if in dreaming I should discover some demonstrative truth, mathematical or otherwise... it would be as certain as if I had been awake. This allows us to see the extent to which intelligible truth is independent of the truth or the existence of sensible and material things outside of us. (AG 189/GP VI 502-503)

What is notable in the first instance about this passage is Leibniz's separation of the world into *intelligible* and *sensible* realms, and his claim that the objects populating the former realm are “incomparably” better known than those of the latter realm. In other words, we

are able to acquire a much greater degree of certainty about intelligible objects than we are about sensible ones. This idea, as Leibniz notes, comes from Plato, who posits a metaphysical and epistemological distinction between what is known by the senses and what is known by the intellect. For Plato, the world divides into two realms -- the intelligible and the sensible -- in the following two ways: first, the objects of the sensible realm are “less real” than the objects of the intelligible realm (this is the *metaphysical* distinction, which Leibniz also seems to endorse here); second, our knowledge of the sensible realm is inferior to our knowledge of the intelligible realm, and these two kinds of knowledge are even acquired by different faculties, one of which (sense perception) is inferior to the other (reason). Accordingly, for Plato and apparently for Leibniz, the acquisition of knowledge about the objects of the intelligible realm, whatever they are, proceeds independently of the senses, as encapsulated by Leibniz's remark that we could in principle acquire certain knowledge of an intelligible object while dreaming, when no veridical sense perception occurs. We should also note that Leibniz explicitly invokes *mathematical discovery* in his dream example, suggesting that mathematical truths are one class of truths discoverable independently of the senses, and suggesting in turn that the objects of which those truths hold exist in the intelligible realm.

Elsewhere in the same text, Leibniz uses a different kind of dream example to illustrate the distinction he is after, a distinction that accords well with the one he attributes to Plato: “*Being* itself and *truth* are not known wholly through the senses. For it would not be impossible for a creature to have long and orderly dreams resembling our *life*, such that everything it believed it perceived by the senses was nothing but mere

appearances. There must therefore be something beyond the senses which distinguishes the true from the apparent” (AG 188/GP VI 502). Here, Leibniz employs the example of a vivid, orderly, lifelike dream to illustrate the idea that the kind of knowledge acquired by the senses is in some sense uncertain or insecure, owing to our inability to distinguish between this kind of dream and the actual sensible world in purely sensory terms. In other words: sense data, and the various inferences they license, are insufficient to distinguish what is real from what is only apparent. Leibniz elaborates on these remarks in the following way:

But the truth of the demonstrative sciences is exempt from these doubts, and must even serve to judge the truth of sensible things. For, as able ancient and modern philosophers have already remarked, even if everything I believed I saw were only a dream, it would always still be true that I (who in dreaming thinks) would be something, and would, in fact, think in many ways, for which there must always be some reason. (AG 188/GP VI 502)

The metaphysical distinction between “the apparent” and “the true” corresponds to an epistemological distinction between two ways of acquiring knowledge -- through sense perception, on the one hand, and through “demonstration”, as exemplified by the “demonstrative sciences”, on the other. Demonstration yields knowledge of what is true independently of the sensible world; it yields knowledge of intelligible objects, which would exist as objects of certain knowledge even if our whole sensible lives turned out to be a vivid and orderly dream. In other words, there are objects that would exist, objects of which we could acquire knowledge and about which many statements would be true (and many false), regardless of any particular way the sensible world happened to be -- regardless, even, of whether any of our sense-perceptions were veridical. For

Leibniz, mathematics is one of the demonstrative sciences, implying that mathematical objects are intelligible and our knowledge of them (through demonstration) certain.

Interestingly, it seems that a fundamental Leibnizian distinction between two kinds of truths lines up exactly with the distinctions under scrutiny thus far. He makes a very suggestive remark elsewhere in the letter to Sophie Charlotte that illustrates this point, in discussing the process of demonstration in the demonstrative sciences: “[T]he force of the demonstrations depends upon intelligible notions and truths, *which alone are capable of allowing us to judge what is necessary*” (AG 189/GP VI 504, emphasis mine). Here, he indicates a deep connection between the notion of *necessity*, or that which would hold independently of any particular way the world is, and the notion of a truth's being discoverable by demonstration. It seems that the demonstrative sciences -- mathematics included -- yield knowledge of necessary truths, which makes sense in light of Leibniz's claim that the objects of the intelligible realm exist independently of the sensible world and any particular way it happens to be. In other words, since intelligible objects are the same regardless of how the sensible world is, the truths that hold of them are necessary and our knowledge of these truths is certain. For Leibniz, “the senses can, in some way, make known what there is, but they cannot make known what must be or what cannot be otherwise” (AG 190/GP VI 504). The truths discoverable by the senses are *contingent* -- the sensible world could easily have been otherwise, and the objects of the senses constantly undergo change -- whereas the truths discoverable by whatever faculty yields knowledge of the sense-independent world are *necessary*, since the intelligible realm could not have been otherwise. Leibniz makes the link between necessity and

demonstrative knowledge more explicit in the preface to the *New Essays Concerning Human Understanding*: “[I]t appears that necessary truths, such as we find in pure mathematics and particularly in arithmetic and geometry, must have principles whose proof does not depend on instances nor, consequently, on the testimony of the senses...” (RB 50).

There is also a corresponding deep connection between necessity and the existence of an intelligible part of the world that is immutable and independent of the sensible part, as Leibniz makes clear in “On the Ultimate Origination of Things”:

Neither... essences nor the so-called eternal truths pertaining to them are fictitious. Rather, they exist in a certain realm of ideas, so to speak, namely in God himself, the source of every essence and of the existence of the rest. The very existence of the actual series of things shows that we have not spoken without grounds. For the reason for things must be sought in metaphysical necessities or in eternal truths, since... it cannot be found in the actual series of things. But existing things cannot derive from anything but existing things... So it is necessary that eternal truths have their existence in a certain absolute or metaphysically necessary subject, that is, in God, through whom those things which would otherwise be imaginary are realized... (AG 151-152/GP VII 304-305)

First, Leibniz here confirms that the necessary truths hold of a certain kind of object that exists in a domain separate from that of the sensible world. The truths about these things are “eternal”, in the sense that these truths could not have been false -- propositions that are true of these things are true of them for all time. Indeed, the existence of immutable abstract things that are described by necessarily true propositions in some sense grounds whatever existence the sensible world *does* have, even if the latter's constituents are ever-changing -- and even if the sensible world could have been radically different from the

way it is. Leibniz says that the very “reason for things”, i.e. the ontological ground for the actually existent series of particulars, “must be sought” in the intelligible realm.

It is important to note that Leibniz’s intelligible realm is God’s mind, and its inhabitants are God’s ideas. Given this, Mondadori argues that they bear some resemblance to Plato’s Forms, to the extent that they are “representation[s] of what would be the case if [they] were exemplified” (1990, 170). Individual concepts, for example, are representations of exactly what an individual would be like if it were created. According to Plato, sensible objects have their properties by imperfectly resembling various abstract things that exist in the intelligible realm. Plato calls these abstracta the “Forms”, and a sensible particular (e.g. a red apple) has its properties (e.g. being red) by “participating in”, or imperfectly instantiating, the Forms (e.g. the “Form of Redness”). For Plato, the series of actually existent particulars is what it is by virtue of instantiating, in a shifting and ephemeral way, different Forms at different times. The Forms, however, do not change, and their immutability provides both the ontological and epistemological ground for the realm of everyday sensory experience. Although knowledge of sensory ephemera is inherently uncertain, the properties we observe in particulars at least give us a imperfect kind of knowledge of the abstracta on which the properties are grounded, and also *trigger* the mind to start reasoning about these properties *by themselves*, as opposed to the particular ways in which sensibles instantiate them. For Leibniz, too, it seems that the immutable intelligible objects ground the very existence of the elements of sensory experience, and also the possibility of anything resembling certain knowledge of the sensible world -- witness his earlier remark that “the truth of the demonstrative sciences...

must even serve to determine the truth of sensible things". We lack certainty about sensible things, but whatever secure knowledge we do have of them comes from knowledge of the objects of the demonstrative sciences -- i.e. the objects of the intelligible realm.

Leibniz's distinctions among our faculties of knowledge acquisition also become pertinent here. Broadly speaking, Leibniz seems to think that the faculty by which we acquire knowledge of intelligible things is innate, and that it is only by virtue of this innate ability that we are able to get beyond the senses in the first place:

[T]here is an *inborn light within us*. For since the senses and induction can never teach us truths that are fully universal, nor what is absolutely necessary, but only what is, and what is found in particular examples, and since, nonetheless, we know some universal and necessary truths in the sciences... it follows that we have derived these truths, in part, from what is within us. Thus one can lead a child to them in the way that Socrates did, by simple questions, without telling him anything, and without having him experiment at all about the truth of that which is asked of him. And this can very easily be carried out with numbers and other similar matters. (AG 191/GP VI 505-506, emphasis in original)

Here, it is abundantly clear that the senses cannot get us beyond the particulars that they allow us to perceive; the senses only license inductive inferences, which never guarantee certainty according to Leibniz. Induction can never help us discover necessary truths; this kind of discovery requires some other kind of reasoning procedure, which Leibniz seems to think of as a kind of deduction. Since the senses give us all of our external inputs, the kind of reasoning process that yields necessary truths, licensing conclusions of a universal and necessary character, must arise "from what is within us". Leibniz acknowledges that "in the present state, the senses are necessary for our thinking, and that if we did not have any, we would not think"; in other words, the senses provide a necessary cognitive stimulus for reasoning toward necessary truths that hold of abstracta. But, at the same

time, “that which is necessary for something does not, for all that, constitute its essence... The senses provide us material for reasoning... but reasoning requires something else besides that which is sensible”. The idea that the senses “trigger” reasoning about abstract things is, as previously noted, a doctrine whose roots are found in Plato's philosophy.

Additionally, in the above passage, Leibniz obliquely registers approval of the procedure that Plato carries out in the *Meno*; in that dialogue, the character Socrates leads a young slave through a demonstration of some properties of squares, purportedly without telling him anything that would give away the conclusion the boy is supposed to draw. Plato intends this example to illustrate the idea that we possess a faculty of reasoning that can in principle proceed independently of the senses, even if the senses often serve as its trigger. It is precisely this faculty that Leibniz calls the “natural light”, and as far as necessary truths are concerned, “it is generally true that we know them only by [it], and not at all by the experiences of the senses” (AG 189/GP VI 504). Leibniz paints a similar picture of his epistemology in other texts, perhaps most notably in the *New Essays on Human Understanding*, in which he intends to establish his epistemological scheme against that of Locke and empiricism in general. Leibniz again draws inspiration from Plato in arguing against Locke and the latter's supposedly Aristotelian influences:

There is the question whether the soul in itself is completely blank like a writing tablet upon which nothing has as yet been written -- a *tabula rasa* -- as Aristotle and [Locke] maintain, and whether everything which is inscribed there comes solely from the senses and experience; or whether the soul inherently contains the sources of various notions and doctrines, which external objects merely rouse up on suitable occasions, as I believe and as do Plato and even the Schoolmen... (RB 48)

Here, Leibniz adds a doctrine of innate ideas to his earlier declarations in favor of a doctrine of an innate faculty to reason about abstract things. In other words, the mind comes equipped both with a robust reasoning faculty and with a set of categories -- “various notions and doctrines” -- that allow reason do its work. Leibniz elaborates:

The Stoics call these sources... fundamental assumptions, or things taken for granted in advance. Mathematicians call them common notions... Modern philosophers give them other fine names and Julius Scaliger, in particular, used to call them 'seeds of eternity' and also *zopyra* -- meaning living fires or flashes of light hidden inside us but made visible by the stimulation of the senses, as sparks can be struck from a steel. And we have reason to believe that these flashes reveal something divine and eternal: this appears especially in the case of necessary truths. (RB 49)

The mind has access to the objects of the intelligible realm by possessing a combination of an innate *faculty* to reason about abstract things and a set of innate *ideas* about their nature. On this picture, the basic innate categories reveal something to us about the general character of the intelligible realm's constituents -- this is what Leibniz means by characterizing innate ideas as “flashes of light” that “reveal something eternal” -- while the innate faculty of reason allows us to draw conclusions from these revelations. This last point is what Leibniz must mean when he says that what the innate categories reveal is something that “appears especially in necessary truths”, i.e. in the truths arrived at by employing the reasoning faculty. In fact, Leibniz is explicit about the contents of innate ideas, claiming that “we include Being, Unity, Substance, Duration, Change, Action, Perception, Pleasure, and hosts of other objects of our intellectual ideas” (RB 51).

Leibniz also affirms in the *New Essays* the “triggering” doctrine discussed earlier, both in the passage from the preface to that work quoted in the last paragraph, and in the first few pages of the main text, declaring that “one should in my opinion say that there

are ideas and principles which do not reach us through the senses, and which we find in ourselves without having formed them, *though the senses bring them to our awareness*” (RB 74, emphasis mine). As has been noted, this idea comes directly from Plato, and Leibniz at times explicitly situates himself so closely to Plato that one might think he actually adopts something like the latter's theory of recollection, which is roughly the idea that innate principles were implanted in the eternal soul at a time prior to its present bodily existence. In discussing “the origin of necessary truths, whose source is in the understanding” (RB 75), Leibniz suggests that “teaching from outside merely brings to life what was already in us” (RB 76), which is what Socrates' examination of the slave in the *Meno* is supposed to illustrate. In defending the coherence of the doctrine of innate ideas triggered in some way by sense-perception, Leibniz appeals to the intuitively plausible thesis that “we know an infinity of things which we are not aware of all the time, even when we need them; it is the function of memory to store them, and of recollection to put them before us again... [But] recollection needs some assistance. Something must make us revive one rather than another of the multitude of items of knowledge” (RB 77). In other words, there is a sense in which we are *remembering something we already knew* when we derive necessary truths from our innate conceptual apparatus. Leibniz makes this explicit when he “grant[s] the point... as applied to necessary truths or truths of reason” that “all truths... are already imprinted on the soul” (RB 77).

2.2. General Implications for Mathematics. Leibniz takes pains to indicate the implications of these theses for mathematics: “On this view, the whole of arithmetic and

of geometry should be regarded as innate, and contained within us in an implicit way, so that we can find them within ourselves by attending carefully and methodically to what is already in our minds..." (RB 77). He takes Plato to have "showed this, in a dialogue where he had Socrates leading a child to abstruse truths just by asking questions and without teaching him anything" (RB 77). It should be noted that Leibniz does not agree with the full-fledged Platonic theory of recollection -- if our present innate ideas were implanted in the soul at an earlier time, then "it is obvious that if there was an earlier state, however far back, it too must have involved some innate knowledge, just as our present state does: such knowledge must then either have come from a still earlier state or else have been innate and created with the soul..." (RB 79). Leibniz opts for the latter option on pain of infinite regress. But the point is just that Leibniz seems to regard his general epistemological scheme as deriving directly from but improving upon that of Plato, and the implications for abstract things, such as the objects of mathematics, are at least that Leibniz cannot be the kind of nominalist that Mates has labeled him.

To reinforce the last claim, it is worth emphasizing the many references to mathematics in these passages about the mind's innate abilities, references of a kind found repeatedly in investigating Leibniz's epistemological remarks. First, recall the claim that one kind of object to which the "natural light" -- the innate faculty of reason -- applies is numbers. Also, the example from the *Meno* that he takes to illustrate the concept of the natural light is explicitly mathematical, having to do with the properties of diagonals in quadrilaterals. Earlier in the letter to Sophie, he claims that "it is also by this *natural light* that the *axioms* of mathematics are recognized, for example that if we take away the same

quantity from two equal things, the things remaining are equal” (AG 189/GP VI 503, emphasis in original). Furthermore, the various branches of mathematics, and of the demonstrative sciences in general, rest on these kinds of basic axioms: “[i]t is on such foundations that we establish arithmetic, geometry, mechanics, and other demonstrative sciences...” (AG 189/GP VI 503). In the *New Essays* as well, as we saw earlier, the “necessary truths, such as we find in pure mathematics and particularly in arithmetic and geometry, must have principles whose proof does not depend on... the testimony of the senses” (RB 77). These “principles”, it should be clear by now, are grasped as innate ideas and employed by the natural light to discover mathematical truths, which are a species of necessary truth.

Leibniz sums this up concisely, with a focus on geometry, in another passage from the *New Essays*: “These ideas which are said to come from more than one sense -- such as those of space, figure, motion, rest -- come rather from the common sense, that is, from the mind itself; for they are ideas of the pure understanding... and so they admit of definitions and of demonstrations” (RB 128). Elsewhere in that work, Leibniz explicitly states that “the ideas of numbers are intellectual ones” (RB 81), and once again approvingly cites the *Meno* and declares that all of arithmetic and geometry are discoverable independently of experience. In general, for Leibniz, mathematics always serves as an exemplar for distinguishing the intelligible realm from the sensible realm -- metaphysically, epistemologically, and modally (i.e. in terms of the kinds of truths that hold of either realm's objects). The abstract entities of mathematics, in other words, exemplify the kind of thing that exists in the intelligible realm, are grasped by

innate ideas and sense-independent reasoning, and are correctly described by necessarily true statements.

3. The Problem of the Imagination

3.1. The Problem Described. In several texts, Leibniz seems to suggest that the imagination plays a key role in mathematical thinking -- indeed, that mathematics is actually *the study of* whatever falls under the faculty of imagination.¹⁵ On the other hand, as detailed in an earlier section of this chapter, Leibniz is clear that the *intellect* plays a primary and decisive role in mathematical thinking. The precise relationship in his thought between the imagination and the intellect is not generally well-understood, and some prominent secondary literature advances seriously flawed accounts of this material.

My task in this section is to provide a general reconstruction of Leibniz's view of the role of the imagination in *arithmetical* thought . Ultimately, where numbers are concerned, the role of the imagination is nothing more than incidental -- as a “trigger” for the intellect to consider certain clear and distinct innate ideas -- and its invocation by Leibniz has no bearing on his ontology of number and very little bearing on his epistemology of number. Much of what has been said in previous chapters should already

¹⁵ For example, at A.VI.4.511, he says that “mathematics is the science of imaginable things”. Many of Leibniz’s declarations to this effect occur within texts that concern “mathesis universalis”, a lifelong unfinished project involving the development of a general science of quantity and quality, from which “special mathematics is excluded, concerning numbers, situation, [and] motion” (A.VI.4.513-514). These texts are often cryptic and in one instance possibly contradictory: in the “Elementa Nova”, he characterizes mathesis universalis as the general science of quantity and quality (A.VI.4.514; see also A.VI.4.362), while in another text, he characterizes it as the general science of quantity only, making no mention of quality (GM VII 53). It would be a mistake, then, to think that there is a clearly identifiable textual tendency in Leibniz’s corpus to assign an essential role to the imagination in “special” mathematics, with one of whose branches -- arithmetic -- I am concerned here.

suggest this: Leibniz's ontology of number is generally Platonist, and his epistemology of number is founded on the intellect's access to the intelligible realm of ideas. Whatever bearing the imagination has on Leibniz's epistemology of number is consistent with claims made in an earlier chapter on Leibniz's purported nominalism: Leibniz endorses something like Plato's thesis that the senses "trigger" the intellect to initiate cognition that then proceeds independently of the senses and involves only intellectual representations. The imagination plays the role of a mediator in the triggering process. Interestingly, many of the clearest examples that Leibniz adduces in discussing the role of the imagination in mathematical thought are geometrical, and it is in these examples that Leibniz appears to give a more substantive role to the imagination. This feature of his argumentation hints at a possible difference between the imagination's roles in geometrical and arithmetical thought. This contrast actually constitutes further evidence that where arithmetic is concerned, the imagination has little to no essential role even in concept formation. Another goal of this section is to highlight this possible contrast.

Although many of the texts examined in this section may make Leibniz appear at first to be a kind of empiricist about mathematics -- such that the "objects" of mathematics are certain imaginative abstractions from sensory inputs -- I emphasize that this is *not* his position, at least for arithmetic. Ultimately, while he does think that the imagination is *prompted* by sensory inputs to have certain abstract representations that may in turn trigger the intellect, he certainly does not think that the imagination has any role in securing *distinct ideas* of numbers, or *knowledge* of the properties of numbers and the number system. Both of the latter come from the intellect; the imagination, whose

representations are based on sensory inputs, cannot supply them. In fact, his considered view seems to be that the imagination has no *essential* role even in our initial conceptual access to numbers.¹⁶ A passage from the *Discourse on Metaphysics* encapsulates his view: “One can also say that we receive knowledge from outside by way of the senses, because some external things contain or express more particularly the reasons that determine our soul to certain thoughts. But... it is important to recognize the extent and independence of our soul, which goes infinitely further than is commonly thought...” (AG 59/GP IV 452).

3.2. What is the Imagination? The current reconstruction must begin with an account of how Leibniz characterizes the imagination in general. One well-known locus for his thoughts on the matter is the letter to Queen Sophie Charlotte of Prussia quoted in an earlier section. In this letter, Leibniz spells out his conception of the relationship between the senses, the imagination, and the intellect, as well as the relationship between the *objects* of these faculties. Leibniz first claims that the senses provide us with *clear* but *confused* notions; these are notions of certain sensible qualities, like colors, whose instances we can recognize, but for which we are unable to give a definition -- a definition that would “make another person understand what the thing is” (AG 59/GP IV 452). If we could give such a definition, these notions would become *clear* and *distinct* notions, and it turns out that the senses do furnish us with these notions as well. However, these

¹⁶ There is some evidence that Leibniz *does* take the imagination to play an essential role in *geometrical* cognition, in addition to triggering the intellect to consider its clear and distinct ideas. As mentioned in the previous paragraph, I will touch on this point, but I will not address it in detail, as my focus in this work is on arithmetic. For a comprehensive treatment of Leibniz's philosophy of geometry, containing a case for the imagination's role in geometrical cognition, see De Risi (2007)

notions do not come from any one particular sense, but from what Leibniz calls the “common sense”. He explains:

Yet we must do justice to the senses by acknowledging that... they allow us to recognize other, more manifest, qualities which furnish us with more distinct notions. These are the notions we attribute to the *common sense* because there is no external sense to which they are particularly attached and belong. It is here that definitions of the terms or words we use can be given. Such is the idea of *number*, which is found equally in sounds, colors, and tactile qualities. It is in this way that we also perceive *shapes* which are common to colors and tactile qualities, but which we do not observe in sounds. (ibid, emphasis in original)

So, in addition to the clear but confused notions which come from the individual external senses -- as the notions of colors come from sight -- there are certain clear and *distinct* notions that come equally from multiple external senses. It is suggestive that Leibniz adduces the ideas of numbers and shapes as prime examples of such notions, but I will come to this aspect of his account later. What is important initially is the key difference between notions furnished by individual senses and notions furnished equally by several senses.

Leibniz provides a more detailed explanation of this basic distinction between notions furnished by the senses in an earlier text, the “Meditations on Knowledge, Truth, and Ideas”. It is worth quoting the key passages at some length; of particular interest is Leibniz's remark that even though *we* cannot give definitions of sensible qualities like colors, the qualities *themselves* (and their notions) still have objectively correct “resolutions” into their fundamental components:

[K]nowledge is *clear* when I have the means for recognizing the thing represented. Clear knowledge, again, is either confused or distinct. It is confused when I cannot enumerate one by one marks sufficient for differentiating a thing from others, even though the thing does indeed have

such marks and requisites into which its notion can be resolved. And so we recognize colors, smells, tastes, and other particular objects of the senses clearly enough, and we distinguish them from one another, but only through the simple testimony of the senses, not by way of explicit marks. Thus we cannot explain what red is to a blind man, nor can we make such things clear to others except by leading them into the presence of the thing and making them see, smell, or taste the same thing we do, or, at the very least, by reminding them of some past perception that is similar. This is so even though it is certain that the notions of these qualities are composite and can be resolved because, of course, they do have causes. (AG 24/GP IV 422, emphasis in original)

The first kind of notion given by the senses, then, is the kind that allows us to recognize its instances and distinguish them from each other only by “the simple testimony” of the senses themselves -- that is, only in terms of the brute impressions that their instances make on the senses. So, for example, Leibniz thinks that we are able to distinguish instances of red from instances of blue, or instances of different *shades* of red from one another, only in particular cases and only with the aid of some sensory apparatus. We cannot distinguish them in general; doing so would require us to be able to give a list of marks which distinguish *any* instance of red from *any* instance of blue or any other color. Consequently, we also cannot provide a general explanation of redness that would make anyone understand its notion or even be able to conjure up an image of a red thing -- the only way to introduce someone to the color red is to “lead them into the presence” of it. As Leibniz puts it in the “General Inquiries about the Analysis of Concepts”, colors and the like are things “which we perceive clearly but cannot explain distinctly or define by other concepts” (P 51). To do any of this, we would have to be able to give a definition of the notion of red, which would involve breaking that notion up into its more primitive components. And as has been noted, Leibniz thinks that the notion of red is in fact

composite, being built up out of simpler notions; it is just that we are unable to enumerate these components. The color red, Leibniz thinks, “has causes”, and giving a definition of the notion of red would involve enumerating these causes.

Immediately after the passage just quoted, Leibniz contrasts the clear but confused notions provided by the individual senses with the clear and *distinct* notions provided equally by several senses:

But a *distinct notion* is like the notion an assayer has of gold, that is, a notion connected with marks and tests sufficient to distinguish a thing from all other similar bodies. Notions common to several senses, like notions of number, magnitude, shape are usually of such a kind, as are those pertaining to many states of mind, such as hope or fear, in a word, those that pertain to everything for which we have a *nominal definition* (which is nothing but an enumeration of sufficient marks). (AG 24/GP IV 423, emphasis in original)

Note the way in which he seems to take *mathematical* notions to differ from those furnished by individual senses. The notions of shapes are perhaps the most perspicuous here. We receive such notions, to the extent that they are furnished by the senses, from the sense of sight and the sense of touch. And unlike the case of color notions, for example, we are actually able to give definitions of shapes by enumerating the components of their notions. We are able to consider shapes at a level of generality beyond that at which we can consider colors; for example, we can easily enumerate the components of the notion of a triangle, rendering it distinct. Accordingly, we can also distinguish one kind of shape from any other kind of shape at a general level, by simply enumerating the components of the relevant notions. Finally, we can explain to others what the different shapes are, without recourse to any sensory aids or ostension.

With these last points in mind, consider again the letter to Sophie. In that text, after making the distinction between the two kinds of notions furnished by the senses, Leibniz suggests that there must be some mental faculty that allows us to do things with these notions even when their instances are not currently being perceived externally:

Therefore, since our soul compares the numbers and shapes that are in color, for example, with the numbers and shapes that are in tactile qualities, there must be an *internal sense* in which the perceptions of these different external senses are found united. This is called *imagination*, which contains both the *notions of the particular senses*, which are *clear but confused*, and the *notions of the common sense*, which are *clear and distinct*. (AG 187/GP VI 501, emphasis in original)

The argument is simple: we do in fact manipulate, in various ways, the notions furnished by individual senses and those furnished by the common sense, and we can manipulate them even when their instances are not present to any of the external senses. We can compare shades of red, for example, even when instances of those shades of red are not actually present to the eye. And we can compare different shapes, or different numbers, even when examples of such shapes or numbers aren't present to any of the external senses. In sum, we can do things with sensible notions *internally*, without the aid of concurrent external sensory perception. So Leibniz concludes that there must be an independent faculty allowing us to do this, since *ex hypothesi* the faculty will not be identical to any one of the external senses or with any multiple external senses taken together. The faculty will be an *internal sense* that allows us to compare notions furnished by the senses, and this is what Leibniz calls the imagination.

In other texts, Leibniz elaborates on this basic conception of the imagination as a kind of internal sense. One way to approach the imagination, he thinks, is by way of a

direct analogy with perception: perception has the same relation to certain *external* things that the imagination has to certain *internal* things. In particular, while the objects of perception are external things present to the various senses, the objects of *imagination* are *images* that are present internally, in the “mind's eye”. In one text, pursuing this line of reasoning, Leibniz writes that “an image is the continuation of a passion in the organ although the action of the object has stopped”, and given this definition of an image, “imagination is the perception of an image” (A.VI.4.1394). Images, Leibniz thinks, are furnished by perception, and thereafter can be conjured up in the imagination even after the “action of the object has stopped”. In perceiving a red thing, we get an image of red, which we can recall in the imagination long after the red thing is gone. In perceiving an instance of a particular shape, we get an image of that shape, which then becomes subject to the imagination. It is interesting to note that in the letter to Sophie, Leibniz speaks of *notions* being furnished by the senses and then taken over by the imagination, while in this text, he speaks of *images* in that way. This need not pose an interpretative problem, because Leibniz seems to think that perception furnishes both actual *images* of sensible qualities and the general *notions* of those qualities. For Leibniz, our formation of the *ideas* of sensible qualities seems to go hand in hand with the presentation of those qualities to the senses, whether we are talking about that which is present to individual senses or to several senses at once. The senses furnish *notions* of sensible qualities by first giving us *images* of them -- even though ideas, for Leibniz, are not themselves images. In another text, Leibniz even declares that “he who is furnished with more images... is, other things being equal, furnished with more truths”, because at least some images “contain

something of a distinct concept” (A.VI.4.802). And the imagination can manipulate the general notions of sensible qualities just as much as it can recall the actual images of their instances.

Admittedly, Leibniz writes very little about the imagination in his corpus. There is not much direct exposition of this faculty other than in the texts examined thus far. The idea that the imagination is an internal sense that considers and manipulates notions furnished by the senses, together with the lack of elaboration on the precise nature of the imagination, raises the question whether the imagination and the common sense are really distinct. In other words, Leibniz may think that the imagination and the common sense are really the same faculty doing two different tasks: we have a particular mental faculty that, on the one hand, notices that which is common to sensory inputs of different kinds -- for example, number or shape -- and on the other hand, is able to abstract general notions from those common features of different sensory modalities. In addition, this mental faculty is able to conjure up images of sensory qualities furnished by individual senses, and is able to abstract incomplete notions of such qualities from their sensory presentation. To be clear, then, I do not wish to claim that Leibniz decisively separates the imagination and the common sense -- especially because the question whether he *does* distinguish them sharply is of no particular importance for the rest of my analysis.

3.3. The Imagination and the “Objects of the Mathematical Sciences,” In the last subsection, I said I would leave for later an exploration of Leibniz's use of numbers and shapes as examples of clear and distinct notions furnished by several senses, and subject to the imagination. I am now in a position to carry out such an investigation, and I shall

do so in order to understand the relationship between the imagination and mathematics. In the letter to Sophie, Leibniz says the following about the clear and distinct notions furnished by the “common sense”:

And these clear and distinct ideas, subject to imagination, are the objects of the *mathematical sciences*, namely arithmetic and geometry... We also see that particular sensible qualities are capable of being explained and reasoned about only insofar as they contain what is common to the objects of the several external senses, and belong to the internal sense. For those who attempt to explain sensible qualities distinctly always have recourse to the ideas of mathematics, and these ideas always contain *magnitude*, or multitude of parts. (AG 187-188/GP VI 501) , emphasis in original)

A cursory reading of this passage may suggest that Leibniz believes the ideas furnished by the common sense and subject to the imagination -- e.g. the notions of numbers and shapes -- are the subject matter of mathematics. As noted earlier, passages like this one, and passages quoted in the previous section, make Leibniz appear to be a kind of empiricist about mathematics: the thesis that mathematics studies clear and distinct sensory abstractions seems to commit him to the claim that there is nothing to mathematical objects over and above the sensory abstractions that constitute them. But this cannot be Leibniz’s view. In fact, as noted previously, it is the clear and distinct ideas *themselves* that are the subject matter of mathematics -- *not the ideas insofar as they are subject to imagination*. In particular, it is the clear and distinct ideas of numbers that constitute the basic subject matter of arithmetic, not the ideas of numbers insofar as they are capable of imaginative representation. Leibniz’s remark that these ideas are “subject to imagination” is misleading: it is true that we can produce an imaginative representation of the number two, for example, by imagining two objects. And Leibniz does hold that such imaginative representations, abstracted from sensory inputs, may have a role in

triggering the intellect to consider the corresponding ideas of numbers. But given the general philosophy of mathematics outlined in the first section of this chapter, this can be the only role of the imagination, and it is incidental to the cognitive machinery and physical situation of humans, as the texts previously quoted from the *New Essays* make clear. In principle, mathematical discovery could occur without any sensory input and without the corresponding imaginative abstraction.

Leibniz's considered view is suggested later in the letter to Sophie Charlotte:

However, it is true that in order to conceive numbers, and even shapes, distinctly, and to build sciences from them, we must have recourse to something which the senses cannot provide and which the understanding adds to the senses... It is true that the mathematical sciences would not be demonstrative and would consist only in simple induction or observation... if something higher, something that intelligence alone can provide, did not come to the aid of the *imagination* and *senses*. (AG 187-188/GP VI 501, emphasis in original)

This line of thought is of a piece with a passage in another text:

[O]ne has distinct knowledge of an indefinable notion, since it is *primitive*, or its own mark, that is, since it is irresolvable and is understood only through itself and therefore lacks requisites. But in composite notions, since, again, the individual marks composing them are sometimes understood clearly but confusedly, like heaviness, color, solubility in *aqua fortis*, and others, which are among the marks of gold, such knowledge of gold may be distinct, yet *inadequate*. When everything that enters into a distinct notion is, again, distinctly known, or when analysis has been carried to completion, then knowledge is *adequate* (I don't know whether humans can provide a perfect example of this, although the knowledge of numbers certainly approaches it). (AG 24/GP IV 423, emphasis in original)

Mathematical notions, like those of numbers and shapes, are *subject* to the imagination insofar as their instances can be detected by the senses and represented in the imagination; but they are also subject to the *intellect* insofar as they are clear and distinct -- indeed, the

intellect is required if they are to be rendered distinct. To use the example of the notions of numbers: such notions are clear and distinct, since their marks can be enumerated in such a way that any number can be distinguished from any other at a general level, without external sensory input. The intellect, of course, is the faculty that performs this task, and not the imagination.

3.4. The Imagination and the Status of Mathematical Objects. One might now wonder how Leibniz's purported realism about numbers might square with his *apparent* thesis that mathematics is the logic of the imagination, and that the ultimate objects of study in mathematics are imaginable things. In his (1995), Robert McRae draws what one might think is the obvious conclusion from Leibniz's *prima facie* claim that mathematics is the logic of the imagination: namely, that mathematical objects are imaginary things without any ultimate mind-independent reality. I now investigate this line of thought with a view toward two goals: first, refuting McRae's reading, but more generally, rendering a clear verdict on the question of the role the imagination plays in arithmetical thought.

McRae's claim relies on a mistaken inference from the claim that mathematical objects are imaginable to the claim that they are *merely imaginary*. The latter claim is much stronger than the former, and the inference is unwarranted in the absence of evidence that Leibniz thinks that anything imaginable is also merely imaginary. McRae's line of thought starts with Leibniz's claim that number is like extension and time in being incomplete or abstract with regard to the created world: just as numbers presuppose numbered things, extension presupposes extended things and time presupposes temporally related things. McRae then reasons in the following way:

Extension is always the extension or diffusion of something. In the case of space, its subject is the diffusion of place... Leibniz calls space, or diffusion of place, the primary subject of extension. By virtue of it we are able to speak of physical bodies as being situated in space. But what then is place? The concept of place is an ideal thing, a being of the imagination formed by abstraction... The concept of time, or the temporal continuum, is formed in a way analogous to the formation of the concept of space, that is, with times like places being formed by abstraction... *Number*, the subject of arithmetic, has the same mental or imaginary status as space and time. It is an abstraction from numbered things... Numbers are in the same case as space and the surfaces, lines, and points that are conceived in it, that is, they are nothing but relations or order and have no ultimate components. (1995, 183-184)

McRae makes two distinct claims here about the ontological status of numbers for Leibniz. The first is that numbers are imaginary because they are conceived in the imagination by a process of abstraction from particular numbered things. The second is that numbers are imaginary because “they are nothing but relations or order and have no ultimate components”.

With regard to the first claim, Leibniz does initially *appear* to hold that number concepts are delivered to the mind through the imagination: as has been noted, a cursory reading of the letter to Sophie and associated texts might indicate that the imagination initially detaches the abstract concept of the number two, for example, from the simultaneous sensory presentation of two objects -- and also that Leibniz believes this kind of process is all there is to mathematical concept-formation. But the investigations of this chapter have indicated that Leibniz thinks the intellect drives mathematical thinking, and arithmetical thinking in particular. The notions of numbers are innate -- or at least, the notions required for generating the notions of numbers are innate -- and the intellect is the faculty that gives us knowledge of numbers insofar as it allows us to render

our notions of them distinct and allows us to discover the way they are systematically and necessarily connected to one another. It may be the case that sensory inputs *prompt* the imagination to have certain representations, for example of a discrete quantity that represents a positive integer, but the *distinct ideas of numbers* provided by the intellect are *not* abstractions from sensory inputs. When Leibniz characterizes mathematics as the science of imaginable things, he does *not* mean that it is nothing more than the study of imaginative abstractions from the deliverances of the common sense. He means something quite different: that mathematics is the study of the clear and distinct ideas, provided by the intellect, to which certain imaginative representations happen to *correspond*, or which certain imaginative representations express in a limited way. These imaginative representations may act as stimuli for the intellect to begin investigating and rendering distinct the ideas to which they correspond, but the representations themselves are not the subject matter of mathematics.

Returning, now, to McRae's first claim -- the inference from Leibniz's account of mathematical concept-formation to an account of mathematical ontology -- Leibniz does think that in the created world, the number two will not be found apart from any two particular concrete things. But he also thinks that numbers ultimately have a robust mind-independent ontological status as relations *in abstracto*, or what comes to the same thing, *possibilities for the way things might be* -- possibilities known to the divine mind and present to it at all times. So McRae's inference from purported imaginative numerical concept-formation to the ultimate status of numbers would be illegitimate even if numerical concept-formation did proceed primarily by imaginative abstraction -- which

it does not. From the fact that *we* initially form the concepts of the various positive integers by a process of imaginative abstraction (if we did in fact do so), it would not follow that Leibniz thinks that numbers are ultimately *no more than* imaginative abstractions from sensory inputs. With regard to McRae's related second claim -- that numbers are imaginary because they are relations -- not much more need be said than has already been said in previous sections. That numbers are just a particular kind of relation *in abstracto* detracts in no way from their mind-independence or objectivity -- quite the opposite, in fact. Though my focus here is not on geometry, I think something similar can be said for the “surfaces, lines, and points” about which McRae also attributes to Leibniz the claim of ultimate unreality. Just as the objects of arithmetic are a kind of abstract relation, the objects of geometry can be understood in Leibnizian terms as possibilities for the way things might be with regard to each other in space, without regard to the resolution of the quantities of those things by means of numbers. And just as with numbers, it is true that abstractly conceived shapes will not be found floating around in the created world; but they will be found in the divine mind, insofar as God knows about all the possible relations of situation that created things might bear to one another.

3.5. A Possible Contrast Between Arithmetic and Geometry. It was noted in the introduction to this section that the geometrical examples Leibniz uses to illustrate the operations of the imagination might indicate a possible contrast in his thought between the role the imagination plays in geometrical thought, on the one hand, and arithmetical thought, on the other. I have not discussed this point yet because it was necessary to lay out the main arguments of the section first. Additionally, my focus in this work is on

arithmetic. However, Leibniz does use the example of geometrical shapes repeatedly in his discussion of the imagination, and it is worth addressing what his use of these examples, together with other considerations, might entail for his account of the role of the imagination in geometrical cognition in contrast to its role in arithmetical cognition.

Two quotations from Wallis' *Mathesis Universalis* in a previous chapter are worth repeating here for the sake of setting the stage:

I say that there are two pure mathematical disciplines: arithmetic and geometry, of which the one is about discrete quantity, or number; and the other about continuous quantity, or magnitude. And indeed of these the one is more, the other less pure: for the subject of arithmetic is more pure and more abstract than the subject of geometry; therefore it has more universal speculations, which are equally applicable to geometrical things and to other things. (1695, 18)

If someone asserts that a line of three feet added to a line of two feet makes a line five feet long, he asserts this because the numbers two and three added together make five... for the assertion of the equality of the number five with the numbers two and three taken together is a general assertion, applicable to other kinds of things... no less than to geometrical objects. For also two angels and three angels make five angels. (1695, 56 as in Jesseph 1999, 38-39)

A certain view of the scope of arithmetic can be seen in these passages: namely, that arithmetic is more “pure” than geometry in the sense that it is more abstract and has a wider scope of application. Arithmetic, Wallis seems to assert, applies to everything countable, while geometry applies only to continuous magnitude. Presumably, the domain of the countable is much larger than the domain of continuous magnitude: the latter domain, Wallis says, is part of the former. And so arithmetic, we might say, is more *general* than geometry.

To tie this line of thought to Leibniz, the following passage, quoted previously, indicates that he also views arithmetic as more general than geometry, since arithmetic applies more universally:

There is an old saying according to which *God* created everything according to weight, measure, and number. But there are things which cannot be weighed, those namely which have no force or power. There are also things which have no parts and hence admit of no measure. But there is nothing which is not subordinate to number. Number is thus a basic metaphysical figure, as it were, and arithmetic is a kind of statics of the universe by which the powers of things are discovered. (L 221/GP VII 184, emphasis in original)

In saying that “there is nothing which is not subordinate to number”, Leibniz maintains the universal applicability of arithmetic. In other words, he maintains that everything is countable. By contrast, geometry is merely “the science of extension”; it is “subordinated to arithmetic, since... there is repetition or multitude in extension...” (AG 251-252/GM VI 100). There is a suggestive relationship between these views and the role of the imagination in geometrical and arithmetical thought. Leibniz says that “qualities mediated by corporeal organs are either sensible or common to many organs. The latter are *number*, which is perceived by all the external senses and is the basis of *arithmetic*, and *extension* with its various modes, which are perceived by sight and touch only and are the basis of *geometry*” (L 89-90, emphasis in original). The fact that number is “perceived” by all the external senses, whereas extension is perceived by only sight and touch, may be related to the much greater generality of arithmetic, and the subordination of the content of geometry to the content of arithmetic. In particular, Leibniz's view seems to be that while arithmetic ranges over everything countable, geometry ranges only over everything visible or touchable.

To explore this line of thought further, it is an important corollary of this view that geometrical thought is tied to the senses, and so to the imagination, more closely than arithmetic is. Number, as explained earlier, is “subject” to the imagination in that the imagination can abstract the idea of number from the common deliverances of all six senses. But such abstraction is not an essential part of arithmetical cognition because the innate ideas of numbers, as we can now see, range over everything that can possibly be counted. Innate number concepts are so general that we require no assistance from the imagination in coming to know about them. By contrast, geometrical concepts, though innate, are not nearly as general. The imagination can only detect instances of, and “abstract”, geometrical concepts through sight and touch. If specific senses, rather than no particular senses, are required to generate imaginative representations of shapes, thereby “triggering” the intellect to begin reasoning about geometrical concepts, then it seems reasonable to suggest that Leibniz thinks the senses, and so the imagination, have a more substantive role to play in generating the cognitive processes that yield geometrical knowledge. One way to read Leibniz, in other words, is as claiming that the senses of sight and touch are indispensable for generating the imaginative representations that “trigger” geometrical thought, whereas the imaginative representations that “trigger” arithmetical thought require no particular sense, and the content of arithmetical concepts is so general -- ranging over everything countable -- that it is unlikely that imaginative triggering is even necessary for arithmetical thought.

As my focus here is solely on arithmetic, I will not explore in detail the role that the imagination might play in geometrical thought. But geometry does turn out to be a

useful contrast class for arithmetic, and it is worth clarifying exactly how this is so. Leibniz's claim about the extreme generality of number concepts, in contrast to the more restricted domain of shape concepts, is of a piece with his claim that no particular senses are required to generate imaginative representations of number concepts, whereas either sight or touch is required to do this for shape concepts. Where number concepts are concerned, the imagination's role appears to be incidental for Leibniz, as claimed at the beginning of this section. If Leibniz holds that number concepts are so general in their range of application that the imagination can detect their presence in representations from *any* sense whatsoever, then he likely also holds that arithmetical thought could begin in the human intellect without any sensory triggering at all. Leibniz holds that the human intellect is *in fact* triggered to begin reasoning about its innate arithmetical concepts by the generation of imaginative representations, but he is not committed to the view that such triggering is necessary for the inception of arithmetical thought. Indeed, given Leibniz's view that there is nothing that fails to be subject to number concepts in some way or other, he is committed to the view that the domain of such concepts far outstrips the sensory realm. So it would be puzzling for him to maintain that reasoning about such concepts is dependent upon the senses and the imagination.

By contrast, it is more difficult to maintain that Leibniz has the same view of geometrical thought: if he holds that geometrical concepts only range over the domain of the extended, and only sight and touch can take up features of the extended for the generation of imaginative representations, then it is unlikely that Leibniz is as optimistic about the prospects for geometrical thought in the absence of a sensory apparatus that

includes either sight or touch. Indeed, it seems likely that Leibniz has a view of geometrical thought as tied “intermediately” to the senses and imagination: geometrical truths are necessarily true, independent of discovery by the human intellect or any other intellect, and apply to any possible extended domain, but no intellect -- human or otherwise -- without the appropriate sensory apparatus would ever begin to think in a way that would allow it to discover geometrical truths. So there may ultimately be a degree of truth to what McRae maintains about Leibniz's epistemology of mathematics when it comes to geometry. This is emphatically not the case for arithmetic, however.

At this point, I have gone as far as I reasonably can within the scope of this work into Leibniz's view of geometrical cognition; the primary utility of this line of investigation has been to illustrate further the independence of arithmetical thought from the deliverances of the senses and the representations of the imagination. However, this is not to say that Leibniz's epistemology of geometry is a subject unworthy of investigation, and it is worth repeating here that De Risi (2007) contains a detailed treatment of that subject and its relationship to other issues in Leibniz's philosophy of geometry specifically and his philosophy of mathematics in general. One issue in particular is worth highlighting: the idea that geometrical truths are necessary, but their discovery depends on certain features of sensibility, has a distinctly Kantian flavor.

Chapter 6: Leibniz and the Logicians

1. Introduction

The main body of this work has been concerned with aspects of Leibniz's philosophy of mathematics that scholars have either paid little attention to or have misunderstood: foremost, his definition and ontology of number, and the extent to which these distinguish him from his predecessors and contemporaries in mathematics and the philosophy of mathematics. Chapter 4 also treated epistemological questions: foremost, Leibniz's conception of arithmetic as epistemically independent of the senses, such that its body of universally applicable concepts and truths can be grasped and known without any aid from sensory intuition or the imagination. The material has all been treated either specifically as it pertains to Leibniz's own work or in a backward-looking fashion, with respect to how various Leibnizian theses improve on the work of his predecessors -- with the notable exception of my argument that Leibniz's way of understanding irrational numbers has a modern flavor insofar as it anticipates the way contemporary mathematics employs infinite series to understand such numbers. The purpose of this final chapter is to treat some of the conclusions reached about Leibniz's conception of number in a forward-looking fashion, with respect to how aspects of that philosophy anticipate later developments in the philosophy of mathematics, particularly the core tenets of the logicist program.

In one sense, Leibniz's general anticipation of logicism, such that he is considered the first logicist, is one of the best known aspects of his philosophy: it is "almost

universally recognized” that Leibniz is the first to advocate the general logicist thesis that mathematics, or some part of it, follows from logical principles (Godwyn and Irvine 2003, 174). It is not the goal of this chapter to recapitulate this general point about Leibniz's broader philosophy. Instead, the goal is to examine how specific aspects of Leibniz's philosophy of arithmetic anticipate specific aspects of the way logicism treats numbers and arithmetical truths. I investigate two points: first, the extent to which Leibniz's epistemological thesis about the knowability of arithmetic mirrors similar theses found in Frege, Russell, and Dedekind; and second, the extent to which Leibniz's definition of number mirrors in a fundamental way the definitions of number proposed by Frege and Russell. I split this chapter into two main parts: the first concerning epistemological issues and the second concerning the definition of number. Each part first extracts the relevant logicist theses from the work of Dedekind, Frege, and Russell, and then proceeds to exhibit the similarities they bear to the relevant Leibnizian claims.

One of the main claims of this work is that Leibniz is the first philosopher of mathematics to combine a general definition of number with background views that liberate the study of non- integral numbers from geometrical methods, placing all positive real numbers on the same definitional, ontological, and epistemic footing. One might think that the features of Leibniz's philosophy of mathematics that allow him to do this are precisely those features that most closely resemble aspects of the logicist program: for example, his purely conceptual definition of number and his conception of numbers as mind-independent abstract relations whose natures can be grasped through pure thought alone. The logicists, as I will review, certainly exhibit the former tendency,

defining numbers in a way that is philosophically similar to the way Leibniz does. They also share Leibniz's epistemological view about arithmetic. Indeed, Leibniz's name appears repeatedly in their work, especially Frege's *Foundations of Arithmetic* (1980). But it is also the case that the founding logicians disagree on matters of ontology; in fact, their ontological views are the subject of ongoing scholarly debate. So the analogy between Leibniz's views and those of the logicians cannot be pushed quite as far as one might like. Nevertheless, their views about the definition of number and our epistemic access to arithmetic are so similar as to merit an investigation like this one.

2. Leibniz and the Logicians on Arithmetical Knowledge

Perhaps the most general way to put the most basic tenet of logicism is that mathematics is part of, or reducible to, logic, in some sense to be further specified. This means different things to different people: for example, one might claim merely that all the theorems of mathematics are provable from merely logical premises, or one might claim that all mathematical truths -- theorems or not -- are provable from merely logical premises. Logicism has a storied history stretching from the present day back to the late nineteenth century, and there are significant differences of detail between the so-called neologicists and the original, founding members of the logicist school, usually taken to be Dedekind, Frege, and Russell, in chronological order. Furthermore, there are significant differences between the logicist programs outlined by the three founding members. Nonetheless, there are key similarities in the central philosophical claims about mathematics advanced by the three founding members and their successors in the later

twentieth and twenty-first centuries. It would be orthogonal to the purpose of this chapter to rehearse that history, especially since it has been so well documented. Instead, it will be helpful here to extract two central logicist claims: one claim about the place of mathematics in the structure of human knowledge, and another about the nature of numbers.¹⁷

For the first claim, we would do well to begin with some remarks due to Dedekind:

In science nothing capable of proof ought to be accepted without proof. Though this demand seems so reasonable yet I cannot regard it as having been met even in the most recent methods of laying the foundations of the simplest science; viz., that part of logic which deals with the theory of numbers. In speaking of arithmetic (algebra, analysis) as a part of logic I meant to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate consequence of the laws of thought. (1963, 31)

Dedekind's remarks here encapsulate the central logicist claim about the place of mathematics, specifically arithmetic, in our epistemological scheme: the concept of number, and so our eventual knowledge of arithmetic, is "an immediate consequence of the laws of thought", i.e. the laws of logic. The acquisition of number concepts and knowledge of arithmetical truths are attainable by attending to such laws alone, without appeal to the "notions or intuitions of space and time" -- in direct opposition to the views of Kant, who is widely regarded as one of the last important philosophers of mathematics

¹⁷ Throughout this chapter, when I treat Russell's views, I deliberately ignore the differences between his early and late logicism, where the transition between the two is characterized by the addition of type theory in order to avoid paradoxes such as that arising from the class of all classes that do not contain themselves, named after Russell himself. The differences between the two involve the technical apparatus used and perhaps the ontological commitments entailed; the philosophical views with which I am concerned here remain the same through the transition. For a sustained account of the two phases of Russell's logicism, see Godwyn and Irvine (2003).

before the inception of logicism, despite having advanced his views some one hundred years prior.¹⁸ For Kant, the acquisition of arithmetical knowledge requires something beyond an appeal to the laws of thought; in fact, arithmetical truths are not even analytic, not discoverable by conceptual analysis alone:

To be sure, one might initially think that the proposition ' $7+5=12$ ' is a merely analytic proposition that follows from the concept of a sum of seven and five in accordance with the principle of contradiction. Yet if one considers it more closely, one finds that the concept of the sum 7 and 5 contains nothing more than the unification of both numbers in a single one, through which it is not at all thought what this single number is which comprehends the two of them. The concept of twelve is by no means already thought merely by my thinking of that unification of seven and five, and no matter how long I analyze my concept of such a possible sum I will still not find twelve in it. (CPR B15)

The poverty of number concepts, says Kant, implies that an appeal to a non-conceptual faculty of the mind is required if we are to acquire knowledge of the relations among numbers:

One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say... and one after another add the units of the five given in the intuition to the concept of seven. For I take first the number 7, and, as I take the fingers on my hand as an intuition for assistance with the concept of 5, to that image of mine I now add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise. (CPR B15-16)

The logicist project is deeply opposed to such a view of arithmetic -- the view that number concepts are too impoverished to yield arithmetical knowledge by their analysis, and the

¹⁸ The other, arguably, is Mill, who advances an empiricist view of mathematical objects and knowledge. Frege explicitly frames his views in the opening sections of *Foundations of Arithmetic* as reactions to both Kant's view that arithmetical knowledge is partially grounded in perceptual intuition and Mill's view that all of mathematics consists of mere empirical generalizations

resulting claim that an appeal to a faculty of perceptual intuition is a crucial in explaining our knowledge of arithmetical truths.

Frege's remarks in his *Foundations of Arithmetic* mirror Dedekind's in their essentials: Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one... The laws of number... are not really applicable to external things; they are not laws of nature. They are, however, applicable to judgments holding good of things in the external world: they are laws of the laws of nature. They assert not connections between phenomena, but connections between judgments; and among judgments are included the laws of nature. (1980, 99)

For Frege, as for Dedekind, the truths of arithmetic are just so many more truths of logic, and so whatever faculty allows us access to the truths of logic, and to their attendant concepts, allows us access to the truths of arithmetic and *their* attendant concepts. Frege also points out the high level of generality of arithmetical truths: their application includes, but is not limited to, the laws of nature. Interestingly, earlier in the work, Frege signals his alignment with what he takes to be Leibniz's view about what is subject to the laws of arithmetic, declaring that "Leibniz long ago rebutted the view of the schoolmen that number results from the mere division of the continuum and cannot be applied to immaterial things" (52). There is a deep connection between the nature of arithmetical knowledge and the generality of arithmetical truths: logic consists of the most general laws there are -- laws that range over the widest domain possible -- and so if arithmetic is part of logic, then it must consist of similarly general laws. Additionally, since logic consists of such laws, it is implausible to maintain that logical knowledge is attained through anything like spatiotemporal intuition or the deliverances of the senses, and so arithmetic, as part of logic, must similarly be grasped independently of the senses via pure

thought, or as Leibniz might say, the intellect. Frege puts this aspect of the logicist enterprise eloquently in *Foundations*:

The basis of arithmetic lies deeper, it seems, than that of any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable. *This is the widest domain of them all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought?* (1980, 21, emphasis mine)

In the famous opening sentences of *The Principles of Mathematics*, his first logicist treatise, Russell is even more explicit than Frege about collapsing the domain of arithmetic into that of logic:

Pure Mathematics is the class of all propositions of the form ‘p implies q’, where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. (1996, 3)

Russell later elaborates:

The distinction of mathematics from logic is very arbitrary, but if a distinction is desired... Logic consists of the premisses of mathematics, together with all other propositions which are concerned exclusively with logical constants and with variables but do not fulfil the above definition of mathematics. Mathematics consists of all the consequences of the above premisses which assert formal implications containing variables, together with such of the premisses themselves that have these marks. (ibid, 9)

Once again, mathematics is part of logic, implying that mathematical (and so arithmetical) knowledge is a species of logical knowledge, and so is not attained through any faculty involving the senses, but rather through pure thought alone. Before moving forward, it is important to note that none of the three founding logicists, or for that matter any

contemporary neologist, is an empiricist about logic itself. The view that logical concepts and eventual logical knowledge are somehow abstracted or generalized from empirical phenomena, together with the view that mathematics is part of logic, would entail that mathematical concepts and knowledge are also abstractions from empirical phenomena. This is emphatically not the view of logic that the logicist adopts; indeed, one of the purposes of reducing mathematics to logic, as far as the logicist is concerned, is to secure for mathematics the same “a priori certainty” (Godwyn and Irvine 2003, 177) that logic enjoys. Thus, the whole enterprise would be futile if the logicist were an empiricist about logic, since presumably empirical generalizations do not carry the kind of certainty that the logicist seeks for mathematics.

In sum, for the logicist, some or all of mathematical knowledge is a species of logical knowledge, such that whatever faculty of the mind secures logical knowledge also secures knowledge of the requisite part of mathematics. However, there are important differences between the founding logicists on the subject of just how much of mathematics is at stake here: Russell, from the beginning of his logicist career, holds that *all* of mathematics, including such far-flung branches as analysis, projective geometry, and even (in his early work *The Principles of Mathematics*) a sort of *a priori* dynamics, which he thinks can be “considered as a branch of pure mathematics” (1996, xx). As Grattan-Guinness puts it, for Russell “mathematical logic (with relations) alone could *subsume* all mathematical notions, objects as well as methods of reasoning”; between logic and mathematics -- *all* of mathematics -- there is “no dividing line” (2003, 58). By contrast, Frege’s logicism is much more limited in scope, encompassing only arithmetic

and analysis; he explicitly excludes geometry from the part of mathematics that is knowable through logical means alone and in fact agrees to a large extent with Kant about the grounding of geometrical knowledge in spatial intuition. In the early pages of *Foundations of Arithmetic*, Frege makes this clear:

We shall do well not to overestimate the extent to which arithmetic is akin to geometry... One geometrical point, considered by itself, cannot be distinguished in any way from any other; the same applies to lines and planes. Only when several points, or lines or planes, are included together in a single intuition, do we distinguish them. In geometry, therefore, it is quite intelligible that general propositions should be derived from intuition... [T]he truths of geometry govern all that is spatially intuitable. (1980, 19-20)

So one of the founding logicians actually agrees with Kant about the epistemic status and range of application of geometry, where Kant's view on this subject is akin to his view on arithmetic: to acquire knowledge of geometrical truths, it is insufficient to subject geometrical concepts to logical analysis, and "help must here be gotten from intuition, by means of which alone" (CPR B16) it is possible to come to know about geometry. But it is a core tenet of logicism that *at least* arithmetical knowledge is a species of logical knowledge.

Now, recalling the results of the second chapter, some striking similarities between the relevant Leibnizian and logicist views emerge. The first similarity concerns the ultimate source of arithmetical knowledge. Both Leibniz and the logicist intend to establish that neither our conceptual access to arithmetical concepts, such as number concepts, nor our eventual knowledge of the properties of and relations among the objects picked out by these concepts, arises from any faculty of the mind involving the deliverances of the senses or abstractions therefrom. The logicists, most explicitly Frege,

frame much of their foundational work as a reaction against earlier views, and Kant's views loom the largest. Kant holds that arithmetical knowledge ultimately cannot come from pure thought alone -- or what Leibniz would call the intellect -- and that the certainty of basic propositions such as $7+5=12$ ultimately derives from our ability to "exhibit" them in a concrete, singular representation, for which the faculty of intuition is required. The logicist holds that our knowledge of such propositions derives from pure thought alone and requires no such exhibition in singular perceptual representations.

It is no surprise, then, that Frege and Russell both explicitly invoke Leibniz as a pre-Kantian inspiration for their own programs. Both Frege and Russell represent Leibniz as holding a nascent form of logicism, either about mathematics in general or arithmetic specifically. Russell, for his part, declares that "the general doctrine that all mathematics is deduction by logical principles from logical principles was strongly advocated by Leibniz, who urged constantly that axioms ought to be proved and that all except a few fundamental notions ought to be defined" (1996, 5). Frege says that "statements in Leibniz can only be taken to mean that the laws of number are analytic" (1980, 21). Beyond these general declarations, Frege seems to recognize some of the specifics of Leibniz's arithmetical epistemology. He expresses the outline of my own view about Leibniz's arithmetical epistemology in *Foundations of Arithmetic*: "Leibniz holds... that the necessary truths, such as are found in arithmetic, must have principles whose proof does not depend on examples and therefore not on the evidence of the senses, though doubtless without the senses it would have occurred to no one to think of them". (1980, 17). He also attributes to Leibniz an "inclination] to regard number as an adequate idea,

meaning one that which is so clear that every element contained in it is also clear, or at least as an almost adequate one” (ibid, 27). It is debatable whether Leibniz thinks humans ever acquire any truly adequate ideas, but he at least thinks our ideas of numbers are clear and distinct, as was discussed in Chapter Two. And he does say in a passage quoted in that chapter that the ideas of numbers are perhaps the closest that human beings can come to attaining adequate ideas.

For Leibniz, though in the actual physical and cognitive situation of human beings the senses in fact trigger the intellect to reason arithmetically, our access to and knowledge of the “principles” of arithmetic, such as the properties of numbers and the relations between numbers, depends in no essential way on the incidental role of the senses. The concepts and truths of arithmetic range over the most general domain one can think of: everything is countable, including things not actually or possibly sensible. It is the intellect that secures access to and knowledge of these concepts, and it is the intellect that reasons about the relations among the objects corresponding to them. Though it is certainly the case that Leibniz had nothing like the post-Fregean conception of logic -- which includes set theory -- at his disposal, the similarity between Leibniz’s view of the source of arithmetical knowledge and of the corresponding generality of the range of arithmetic, is undeniable. The reason why the early logicians find such inspiration in Leibniz’s philosophy of mathematics, even though most of the Leibnizian texts I have cited were unknown to them, is that this philosophy is informed by the same fundamental intuitions about the nature of arithmetic -- what it is about and where our knowledge of it comes from -- as theirs is. Should the issue of what counts as logic or logical knowledge

be brought up at this point, it must be stressed that this issue is orthogonal to the basic similarities illustrated here, which concern the ultimate source of arithmetical knowledge and the range of arithmetical concepts and truths. In the final analysis, Leibniz opposes essentially the same view as that against which the logicians frame much of their philosophical work about mathematics.

3. Leibniz and the Logicians on the Analysis of Number

Another central tenet of logicism, part and parcel of the project of reducing mathematics to logic, is that mathematical concepts must be definable through logical means alone. As Shapiro (2000) puts it, “the idea is that the concepts and objects of mathematics, such as ‘number’, can be defined from logical terminology; and with these definitions, the theorems of mathematics can be derived from principles of logic” (108). So for a logicist about arithmetic, logic is capable of furnishing the definitions of numbers, for example, in a way that allows for purely logical derivations of the truths of arithmetic. Different logicians go about defining numbers in different ways, but the definitions share the property of having been thought to be purely logical by those formulating them. Frege, for example, defines the positive integers in terms of the content of number statements of the form “the number of Fs is x”, for example, “the number of moons of Mars is two”. The content of such a statement, he argues, is “an assertion about a concept” (1980, 67): the statement “the number of moons of Mars is two” is an abbreviation of the statement “the number of the concept ‘being a moon of Mars’ is two”. His famous definition in these terms -- that “the number which belongs to the concept F is the extension of the concept ‘equal to the concept F’” (ibid, 79-80) -- only makes use

of vocabulary that Frege takes to be logical, such as “concept” and “extension”, and also the concept of one-to-one correspondence: the equality invoked in his definition, he says, “must be defined in terms of one-one correlation” (ibid, 74). For Frege, concepts and their extensions are part of the province of logic, so this definition satisfies the logicist desideratum about the definitions of mathematical concepts.

Russell’s definition of number in terms of classes has much the same place in his own program. Russell defines “the number of a class” as “the class of all classes similar to the given class” (1996, 115). The notion of similarity, for Russell, falls within the purview of logic, for similarity is a relation. Russell is well known for having done more than perhaps anyone else to extend the logic available at the turn of the twentieth century to accommodate relations in addition to one-place predicates, and his particular brand of logicism is rife with the use of relations to delineate various mathematical notions. His *Introduction to Mathematical Philosophy* explains with particular clarity what it is for one class to be similar to another: “One class is said to be ‘similar’ to another when there is a one-one relation of which the one class is the domain, while the other is the converse domain” (1993, 16). The domain of a relation is simply “the class of those terms that have a relation to something” (ibid); the converse domain is simply the class of terms picked out by the “something”. So Russell, not unlike Frege, defines number in a way that makes essential use of one-to-one correspondence: a Russellian number is a class of classes that can be put into one-to-one correspondence.

Finally, Dedekind’s definition of number should be noted insofar as it differs from Frege’s and Russell’s: Dedekind spells out the content of the concept of number ordinally,

rather than cardinally. In his *Nature and Meaning of Numbers*, he offers the following definition, which appears to characterize what is most naturally thought of as an ordinal number or a system thereof:

If in the the consideration of a simply infinite system N set in order by a transformation ϕ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ , then are these elements called *positive integers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N ... The relations or laws which... are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements, form the first object of the *science of numbers* or *arithmetic*. (1963, 68)

The definition is easy enough to understand without entering into a discussion of the specific meanings for Dedekind of the technical terms such as “transformation”: the positive integers are those elements of any appropriately ordered structure, regardless of the identity of the elements. This definition takes the notion of ordinal number as primary, as the positive integers here do not correspond to the sizes of collections, but rather to the elements, up to isomorphism, of a particular set with the appropriate relation defined on it. Unsurprisingly, this passage is understood by philosophers of mathematics as the first elaboration of a contemporary view known as *structuralism*, which takes mathematical objects such as numbers to be nothing more than elements, up to isomorphism, of a given structure, or a set with relations defined on it. On this view, mathematical objects are individual things, but they are defined entirely in terms of the relations they bear to other things in the appropriate structure, without any non-relational essential properties. So Dedekind is usually taken to be the first logicist, but his work also inspires a rather

different school of thought.¹⁹ Like Frege and Russell, Dedekind takes himself to be defining number in purely logical terms: and also like Frege and Russell, what is taken as purely logical differs significantly from what is taken as such today, in particular the set theoretic notions that all three of the founding logicians employ.²⁰

In sum, the logicist attempts to define number in a way that makes use only of concepts in the province of logic. This part of the logicist project is of a piece with the logicist's general enterprise: if some branch of mathematics is part of logic, then the concepts and objects of the branch of mathematics must be knowable -- and so definable -- by the same means that yield knowledge of logic. Presumably, then, the paraphernalia of the given branch of mathematics must be definable, and their properties and relations knowable, through means that involve only pure thought, devoid of reference to the deliverances of the senses or any kind of intuitive faculty. In particular, the constituents of arithmetic and their relations must be definable and knowable using the same means that yield definitions of logical concepts and knowledge of logic. Interestingly, Frege takes this particular aspect of his project to be inspired by Leibniz just as the general spirit of his project is so inspired. He is aware of at least one text in the Erdmann edition

¹⁹ An elaboration of the Dedekindian roots of structuralism -- as well as an extremely detailed treatment of one form of the view itself, and of its purported applications, can be found in Shapiro (1997). Descriptions of other forms of the view can be found in Hellman (1989) and Chihara (2004)

²⁰ There are other notable tensions between the views of Dedekind and the other founding logicians. Godwyn and Irvine (2003, 178-180) note that Frege criticizes Dedekind for claiming that "numbers are free creations of the human mind" in the previously quoted section from Dedekind's *Nature and Meaning of Numbers*, and also for employing the notions he does, such as the notion of a "system" and its constituents, which Frege apparently takes to be non-logical. The authors also note that the latter criticism is odd, especially considering that Frege uses the analogous notions of concept and extension. Finally, they note that Dedekind's apparent claim that numbers are psychological entities may not have the import that Frege thinks it does. The details of this debate are beyond the scope of the present work.

available at the time where Leibniz “defines number as 1 and 1 and 1 or as units” (1980, 48), and describes himself as “trying to complete the Leibnizian definitions of the individual numbers by giving the definitions of 0 and of 1” (ibid, 67). He does so in a way that allows for the definition of each positive integer in terms of the one that precedes it. Zero is “the number which belongs to the concept ‘not identical with itself’” (ibid, 87); this works because the extension of the concept “equal to the concept ‘not identical with itself’” is empty. The number one, then, is “the number which belongs to the concept ‘identical with 0’” (ibid, 90). Again, exactly one thing falls under the concept ‘identical with 0’: namely, 0. The number two will be the number belonging to the concept “‘identical with 0 or identical with 1’”, and so on.

I have omitted many fascinating details of Frege’s definitions in order to illustrate a particular property of them: Frege’s general definition of number yields a specific characterization of the positive integer system that is *recursive*. In addition to defining “the number of F” in a way that he takes to be purely logical, he is able to produce a recursive definition of any given positive integer. Now, recall the analysis in Chapter Three of Leibniz’s definition of “number in general” and the way his definition of the positive integers is a case of it: number in general is a relation to unity, expressing the homogeneity to unity of a given quantity or collection, where homogeneity is the capacity to be rendered similar by a transformation. The positive integers are those relations to unity that express the wholeness and size of aggregates of individual unities, such that these aggregates can be rendered similar to unity by the removal of their constituents until the aggregate is indistinguishable from unity. I pointed out in Chapter Three that this

analysis of number yields a recursive definition of the positive integers: the number one is the degenerate relation to unity, expressing a collection that is already indistinguishable from unity. The number two is then defined in terms of the number one: two is the relation that expresses that an aggregate needs *one* constituent removed in order to be indistinguishable from unity. The number three is the relation that expresses that a collection needs *two* constituents removed, and so on. So it is with good reason that Frege takes himself to be completing a project begun by Leibniz, even if Frege did not realize just how close Leibniz's characterization of the positive integer system is to that desired by the logicist.

Beyond the technical similarities between Frege's and Leibniz's definitions, a more general philosophical similarity also becomes apparent between Leibniz's definition of number and the general spirit of the logicist's attempt to define number: the logicist defines number in a way that she takes to be purely logical, where one implication of the procedure is that the definition employs only concepts and principles on the same epistemic plane as the concepts and principles of logic. This means that no concepts or principles abstracted from either the deliverances of the senses or the general features of sensibility are allowed. This, as I have explained in earlier chapters and recapitulated in this one, is very similar to Leibniz's view that the arithmetical concepts and truths fall within the purview of pure thought alone, in the sense that the imagination and the senses play merely incidental roles for arithmetical thought. Additionally, it can now be noted that Leibniz's definition of number reflects this view in the same way that any logicistically acceptable definition of number reflects the logicist's view of our access to

the concepts and truths of logic, and our corresponding access to the concepts and truths of arithmetic. The concepts that Leibniz employs in his own definition mirror those that the logicist employs in their generality and their accessibility to the intellect or pure thought: where the logicist might employ classes, Leibniz employs aggregates; where the logicist might employ equality or one-one correspondence, Leibniz employs homogeneity.

4. Conclusion

This chapter has been forward-looking, in contrast to the main body of the work. I have attempted here to delineate the similarities between Leibniz's epistemological views about arithmetic and his definition of number, and the corresponding logicist views about the epistemic status of arithmetic and related efforts to define number. In concluding this line of investigation, it is worth noting the absence of ontological issues here. I have deliberately avoided the question whether Leibniz's ontology of number is similar to that adopted by the logicist, simply because logicism does not commit one to the adoption of any particular ontology of number. This is apparent from the work of the early logicists, particularly Frege and Russell. In *Foundations*, Frege makes claims that *prima facie* seem to commit him to a Platonist view, at least for arithmetic: the subtitle of Chapter IV, for example, is "every individual number is a self-subsistent object" (1980, 67), and Frege's view that the extensions of concepts are objects seems to commit him to the view that numbers are objects, since his famous definition of number has it that numbers are extensions of certain concepts. Russell, by contrast, appears to reject Platonism, saying that classes are "logical fictions" which are not "part of the ultimate

furniture of the world” (1993, 182). But his precise view of their ontological status is unclear.²¹ he also says that “if we can find any way of dealing with [classes] as symbolic fictions... we avoid the need of assuming that there are classes without being compelled to make the opposite assumption that there are no classes. We merely abstain from both assumptions” (ibid, 184).

In sum, logicism by itself does not imply a commitment to any particular ontology of number. But as mentioned in the introduction, this matters very little for the purpose of this chapter, which has been to show that the general logicist philosophy of arithmetic and that of Leibniz are similar in two key respects: in their epistemological view about arithmetical concepts and arithmetical knowledge, and in their effort to define number in a way that reflects the thesis that arithmetic is accessible to pure conceptual thought alone, without recourse to perception, intuition, or any other non-conceptual faculty of the mind. For Leibniz and for the logicist, these claims are of a piece. Indeed, the two claims lie at the very heart of logicism, to the extent that one can be a logicist about a given branch of mathematics only if one adopts these claims. The ontology that a logicist adopts for a given branch of mathematics only needs to be consistent with these claims; it only needs to ensure that the claims about definition and epistemic access hold for that branch of mathematics. As is clear from the work of the founders, more than one ontology will fulfill that purpose. Frege and Russell may adopt deeply opposed ontological views of number, but they are nonetheless both logicists. To the extent that Leibniz endorses the

²¹ As before, see Godwyn and Irvine (2003) for a useful summary of this issue in Russell interpretation.

requisite fundamental claims, I argue that he also counts as a logicist, or at least as having anticipated logicism in a striking way.

General Conclusion

The first chapter of this work surveyed several prominent attempts, in antiquity and the early modern period, to carry out two tasks: (1) to define “number”, thereby delineating the class of legitimate numbers; and (2) to give a philosophical account of the nature of number. There, I noted that during the Renaissance, a trend emerges of accepting non-integral solutions to algebraic problems without explicitly philosophizing about the status of non-integral numbers. Such numbers are accepted implicitly, for the purposes of mathematical practice, but little is said explicitly about their ontological standing or their legitimacy.

Following the Renaissance, mathematicians begin to offer explicit theses about these numbers. These two tasks sometimes inform one another in deep ways: for example, Barrow’s definition of number as the sign of magnitude is intimately related to his philosophical account of number -- and of all mathematics -- as fully reducible to geometry. Numbers, for Barrow, are merely signs of magnitude because there is nothing more to mathematics than the study of magnitude. Furthermore, the reality of geometrical magnitude is grounded in the material constitution of the universe. By contrast, for Wallis, number is separable from geometrical magnitude -- it is a kind of abstract object -- but the only genuine numbers are the positive integers. Arguably, Wallis has so much trouble accepting non-integral numbers because unlike Barrow, he lacks the right

combination of a sufficiently general definition of number and a philosophy of mathematics to support it by metaphysically grounding the different kinds of numbers.

This work has outlined an interpretation of Leibniz's definition of "number in general" and united it with his ontological claim that numbers are relations. It has become clear that Leibniz's approach to tasks (1) and (2) both differs from and represents a significant advance over that of his predecessors. Leibniz has both a fully general definition of number *and* a philosophy of mathematics that supports it to a significant extent. Barrow has both of these as well, but as I have argued, Barrow's mathematics is distinctly non-modern in reducing all of mathematics to geometry and even geometry to the material world. By contrast, Leibniz's conceptual and ontological characterization of number is intended to include all the positive real numbers as equally legitimate *and* to give them robust ontological standing in his philosophical system as relations. While Barrow's approach rules out the study of numbers in any way that is not fully geometrical, Leibniz's approach is intended to allow for the study of numbers without reference to geometrical magnitude. The study of number, on Leibniz's account, is independent of the study of quantity. In fact, as we have seen, Leibniz holds that arithmetical knowledge is *prior* to geometrical knowledge -- prior to knowledge of quantity -- insofar as number is required for the distinct apprehension of magnitude. Number theory, as the study of the properties of the positive integers in a non-geometrical fashion, had existed from antiquity. Thus, it can be said that one dimension of the importance of Leibniz's account of number lies in its opening up the possibility of a general study of what we now call the real numbers, including the irrational numbers, without reference to quantity.

We have also had occasion to observe the deep internal complexity of the account of number Leibniz proposes. His twofold treatment of number holds that numbers are those entities that fall under the conceptual category of “that which is homogeneous to unity”, and, equivalently, that they are relations that express the homogeneity to unity borne by possible aggregates or -- in the case of irrationals -- by possible things that cannot be understood as aggregates. Ultimately, however, Leibniz's account of number only allows him to accommodate irrationals at a purely conceptual level, leaving us without a mathematically rigorous procedure for generating any given irrational number by means of an infinite series. Additionally, Leibniz's account of number involves him in extremely thorny philosophical difficulties concerning the status of negative and complex numbers, to the extent that he is logically committed to their existence, though he adamantly denies any such commitment in his writings. Finally, I have argued that Leibniz's epistemology of mathematics -- and of number in particular -- lines up well with his definitional and ontological views on number, and that his definitional and epistemological efforts to understand the nature of number anticipate those of the founding logicists along several dimensions.

In the final analysis, I wish the take-home point of this work to be the following. Despite Leibniz's inability, on his account of number, to demonstrate exactly how irrationals can be understood as conceptually coherent, existent entities, what he offers us is an account that represents a significant, original advance over those of his predecessors. He threads the eye of the needle insofar as he avoids reducing number to quantity, and so trivializing the very concept of number (as Barrow does), while simultaneously making a substantial effort to accommodate irrationals. While

Wallis -- who also holds that number is conceptually independent of magnitude -- is unsure even whether fractions exist, Leibniz formulates an account that straightforwardly accommodates fractions and at least provides a way forward toward understanding the nature and status of irrational numbers.

In closing, I will mirror a remark I made in my general introduction. Leibniz is not primarily known, except in certain restricted domains (i.e. insofar as he has well-known views on the fundamental concepts involved in the calculus) as a philosopher of mathematics. He is certainly, at any rate, not known as a philosopher with rich or complex views on the nature of number -- on what numbers are, on what numbers count as conceptually legitimate, on what ontological status numbers have, or on how we acquire knowledge of numbers. If I have established anything conclusive in this work, it should be that at the very least, Leibniz should be recognized as a philosopher who has such views, and as a philosopher who does not produce them in a vacuum, but rather in response to the intellectually chaotic climate in seventeenth-century philosophy of mathematics, and in such a way as to bring together the most promising aspects of the competing views to which he responds. The result is a philosophical account of number that improves upon those which came before it, and which anticipates the modern mathematical conception of real number as well as a movement in the philosophy of mathematics -- logicism -- that is still popular today in various updated forms. Leibniz is often called the last universal genius, and it has been the purpose of this work to expose and interpret an aspect of his genius that has gone unnoticed by scholars of all stripes. It is my hope, in closing this work, that at the very least, this aspect of his genius can now see the light of day.

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