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Smooth Field Theories and Homotopy Field Theories

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by

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Abstract

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In this thesis we assemble machinery to create a map from the field theories of Stolz and Teichner (see [ST]), which we call *smooth field theories*, to the field theories of Lurie (see [Lur1]), which we term *homotopy field theories*. Finally, we upgrade this map to work on inner-homs. That is, we provide a map from the fibred category of smooth field theories to the Segal space of homotopy field theories. In particular, along the way we present a definition of *symmetric monoidal Segal space*, and use this notion to complete the sketch of the defintion of homotopy bordism category employed in [Lur1] to prove the cobordism hypothesis.

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Contents

List of Figures

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Chapter 1

Introduction

Naively, a field theory is an expected value function on some set of fields. In this case, the term field can refer to things such as a smooth function representing temperature on a chunk of space-time (i.e. bordism), or a vector field or differential form representing an electric or magnetic field. The field theory gives an expected value to each field, which can be thought of as the probability that reality will manifest in the way described by the field; for example temperature will evolve as predicted by the smooth function. Despite the best attempts of mathematicians, it is not well understood yet how to fit field theories that represent physical reality into the usual framework of expected value—namely integration with respect to a probability measure. In the case of field theories this elusive integration value is the so-called path integral. The issue—again naively—is that the spaces of fields are usually 'too big' to have a well-behaved probability measure.

One approach that mathematicians have taken in order to get a rigorously defined notion of field theory is to abstract the properties that the path integral should satisfy, and thus provide a 'top down' definition of field theories. Taking as our fields smooth maps from d-dimensional bordisms (that is, d-dimensional smooth manifolds with boundary) to a fixed target X, a first approximation to such an abstraction results in the following.

Definition A *field theory* is a symmetric monoidal functor from the d-dimensional bordism category to the category of vector spaces.

Here functorialty is a reflection of the fact that integrals are additive over gluing of intervals, and the monoidal property comes from the Fubini theorem.

Broadly speaking, there are two reasons that the above definition is only a first approximation. The first reason is that there is no locality in the definition. If a field theory is meant to formalize something physical, then it must be local in the sense that the results of an experiment can only depend on what what particles might have interacted with the system describing the experiment. The accepted method for formalizing this idea is that the bordism category should be extended all the way down to points. That is, boundaries of arbitrary codimension are allowed, resulting in some flavor of d-category, and then field theories should the corresponding d-functors.

The second feature missing from the definition above is that it does not take into account any continuity properties that we expect from a path integral. In particular, if we change a temperature by a small amount, its expected value should not change very much either.

Research around the area of field theories defined in such top down categorical fashion has formed a very rich subject in mathematics over the last quarter of a century at least. One reason for this is the wide variety of approaches to expressing the above two properties of field theories in categorical languange. In this dissertation, we study two approaches which appear outwardly very different, and we produce a map which connects them.

The first type of field theories, which we will call 'smooth field theories' captures the continuity expected from the path integral by fibering the functors of the above definition over the category of manifolds. Smoothness, and therefore continuity, then follows, since the field theory accepts smooth families of fields over a base manifold, and then outputs a smooth family of results over the same base. In order to reflect the symmetric monoidal property, these fibred categories must posess a point-wise symmetric monoidal structure as well.These field theories are due to Stephan Stolz and Peter Teichner (see [ST], [STHK], [STH]). Smooth field theories are difficult to classify in general, although there are some partial results. In [STHK] it is shown that a smooth (topological) 0-dimensional field theory over X is just a smooth function on X. In [STD] it is shown that a smooth 1-dimensional field theory over X is a smooth vector bundle on X with a connection. Part of the difficulty is that efforts are ongoing to give a suitable definition of the smooth bordism category extended down to points.

One the other hand we have the 'homotopy field theories' of Jacob Lurie. Here continuity of the field theories is expressed by 'extending the bordism categories up'. Namely having higher morphisms in the bordism categories corresponding to k paramter families of diffeomorphisms of the bordisms, and having the field theories act on those as well. Formally, the bordism category here is a symmetric monoidal Segal space. In contrast to smooth field theories, homotopy field theories are easy to classify all at once. In [Lur1], Jacob Lurie establishes the so-called cobordism hypothesis for homotopy field theories, which essentially says that a field theory is completely determing by where it sends points.

In this dissertation, the aim is to connect these two methods of capturing continuity of field theories by producing a map from symmetric monoidally fibred categories to symmetric monoidal Segal spaces. We begin this task by providing precise definitions of both notions, given in Section 2.1 and Section 2.4. Chapter 1 is devoted to motivating the definition of symmetric monoidal Segal space, and also developing the machinery to produce our map, which is assembled in Section 3.4.

We make a point of staying as general as possible, and hence only introduce the smooth and homotopy bordism categories in Chapter 2, where their full definitions are presented. One benefit of this approach is that we can immediately apply the same machinery to get a map from smooth Euclidean field theories, where the bordisms are equipped with Euclidean structure, to homotopy Euclidean field theories. Replacing 'Euclidean' with any other type of geometry is also easily done.

Finally, we upgrade the map to be more 'functorial'. That is, there is a notion of fibred category of smooth field theories, and a notion of Segal space of homotopy field theories. We show that the former can be mapped to the latter in a precise sense in Chapter 3.

We conclude the introduction by explaining some notational conventions we adopt for this paper. For abstract categories, we use calligraphic font capital letters, such as $\mathcal{C}, \mathcal{D}, \mathcal{S}, \cdots$. For specific categories such as the category of simplicial sets, we use boldface, **sSet**, **Cat**, **Kan** etcetera. For abstract categories that are enriched, say over **Cat** or **sSet**, we use boldface letters like $\mathbf{C}, \mathbf{D}, \cdots$, and we also use boldface for fibred categories, such as $\mathbf{E}$. Note that all the categories named above have many enrichment structures. Finally, we use the name of the category itself to denote homsets, like $\mathcal{C}(X,Y)$ for the set of morphisms from X to Y. Variations on this notation will be used for various similar notions, such as hom-objects. One exception to this rule is that we use the notation $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to denote the category of functors and natural transformations of such $\mathcal{C} \to \mathcal{D}$. We will discuss 2-categories extensively, which for us will be categories enriched in **Cat**. In particular, $\circ$ is a functor which on objects is composition of 1-morphisms, and on morphisms is horizontal composition of 2-morphisms. Vertical composition of 2-morphisms we denote by $\bullet$. Diagramatically, 1-morphisms we denote by single arrows $\rightarrow$, and 2-morphisms we denote by double arrows $\Rightarrow$. As an example of this, we will sometimes want to consider natural transformations as 1-morphisms in a functor category, and sometimes as 2-morphisms in **Cat**; they will be labelled with single and double arrows accordingly. Finally, we will use 1 to denote identity morphisms, except when confusion with the number 1 might arise, in which case we use id.

Chapter 2

Symmetric Monoidal $(\infty, 1)$-Categories

The main goal of this chapter is to give a definition symmetric monoidal Segal spaces. In the next chapter we will provide an important example, namely the homotopy bordism category, which is the source of homotopy field theories. Our defintion of symmetric monoidal Segal space requires substantial motivation, including a natural construction for the symmetric monoidal Segal space induced by an ordinary symmetric monoidal category. In order to produce this machinery, and motivate our definition, we employ the Grothendieck construction, which gives a correspondence between (op)fibred categories and pseudofunctors with target **Cat**. Since we want to understand the Grothendieck construction in depth, we present a more elaborate version of it in Section 2.1, namely, the analogue where there are symmetric monoidal structures present. The rest of the chapter is spent in explaining the aforementioned path from symmetric monoidal category to symmetric monoidal Segal space, and uses only the ordinary Grothendieck construction. However, in the following chapter, we will use the full strength of the constructions in Section 2.1.

2.1 Symmetric Monoidal Fibred Categories

The goal of this section is to explain the equivalence between fibred categories $\mathbf{E} \to \mathbf{B}$ where the fibres have a symmetric monoidal structure, and pseudofunctors $\mathbf{B}^{\mathrm{op}} \to \mathbf{SMCat}$. Furthermore, we will explain how the pseudofuctors and fibred categories can be strictified in a natural way. We won't need the symmetric monoidal version of this result until the next chapter, but we will need the standard version of the equivalence between (op)fibred categories and functors $\mathbf{B}^{(\mathrm{op})} \to \mathbf{Cat}$ throughout this chapter, so we will present the more complicated situation here.

Throughout this section we will be studying functors $p : \mathbf{E} \to \mathbf{B}$. We will be considering the image of diagrams in $\mathbf{E}$ under p, and it will be convenient to draw both a diagram in $\mathbf{E}$

and its image at the same time, as in

$$\begin{tikzcd}
e' \arrow[r, "f"] \arrow[d, mapsto] & e \arrow[d, mapsto] \\
p(e') \arrow[r, "p(f)"] & p(e)
\end{tikzcd}$$

Here, we are using the $\mapsto$ arrows to indicate an image under p. To avoid clutter, we only draw these arrows between objects, and it is implicit that the morphisms in $\mathbf{E}$ map to those in p below them.

Definition 2.1.1. We say that a morphism $\phi : e' \to e$ in $\mathbf{E}$ is *cartesian* if given the solid diagram below, the dotted arrow exists and is unique:

$$\begin{tikzcd}
e'' \arrow[rrd, bend left, "f"] \arrow[rd, dotted] \arrow[dd, mapsto] & & \\
& e' \arrow[r, "\phi"] \arrow[dd, mapsto] & e \arrow[dd, mapsto] \\
p(e'') \arrow[rd] \arrow[rrd, bend left] & & \\
& p(e') \arrow[r] & p(e)
\end{tikzcd}$$

In words, any morphism whose image under p factors through $p(\phi)$ has a unique compatible factorization through ϕ.

Clearly cartesian morphisms are closed under composition. They are meant to be a categorical definition of fibre-wise isomorphism.

Remark 2.1.2. In the situation of Definition 2.1.1, if f is itself cartesian, we will call the dotted arrow a *cartesian factorization.* Note that by uniqueness, cartesian factorizations are isomorphisms. Also note that, given a composable sequence of cartesian morphisms, one may either factor through each one in turn, or through the composition at once. The result is the same.

Definition 2.1.3. The functor p is called a *Grothendieck fibration* or simply a *fibration* if every solid diagram below can be filled in with the dotted data such that ϕ is cartesian.

$$\begin{tikzcd}
e' \arrow[r, dotted, "\phi"] \arrow[d, dotted, mapsto] & e \arrow[d, mapsto] \\
b' \arrow[r, "f"] & p(e)
\end{tikzcd}$$

we typically only write e' by abuse of notation, and call it a *pullback* of e along f. A choice of pullback for every diagram

$$\begin{tikzcd} & e \arrow[d, mapsto] \\ b' \arrow[r, "f"] & p(e) \end{tikzcd}$$

is called a *cleavage*, and in the presence of a cleavage, the preferred pullbacks are denoted by f^*e.

In short p is a fibration when pullbacks exist. Note that cleavages of fibrations always exist, provided one assumes the axiom of choice. We will always assume that our cleavages are *normalized* in the sense that the chosen pullback over an identity is an identity.

Example 2.1.4. Let $\mathbf{Vect}_{\mathbf{Man}}$ denote the category of smooth vector bundles over all base manifolds. Let $\pi : \mathbf{Vect}_{\mathbf{Man}} \to \mathbf{Man}$ be the functor that forgets the bundle and keeps the base space. Then π is a fibration. The usual notion of pullback constitutes a cleavage.

It is clear that fibred categories should themselves be the objects in a category, so we must define the morphisms. There is a general notion where the base category $\mathbf{B}$ is allowed to vary, but we will not be concerned with this case here.

Definition 2.1.5. The category $\mathrm{Fib}(\mathbf{B})$ of categories fibred over $\mathbf{B}$ has objects fibrations $p : \mathbf{E} \to \mathbf{B}$ and given another such object $p' : \mathbf{E}' \to \mathbf{B}$ a morphism is a functor f such that

$$\begin{tikzcd} \mathbf{E}' \arrow[rr, "f"] \arrow[dr, "p'"'] & & \mathbf{E} \arrow[dl, "p"] \\ & \mathbf{B} & \end{tikzcd}$$

commutes, and such that f preserves cartesian arrows. The category $\mathrm{opFib}(\mathbf{B})$ of categories *opfibred* over $\mathbf{B}$ has objects fibrations $p : \mathbf{E}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}$, in which case we will call p^{op} an *opfibration*, and say that $\mathbf{E}$ is *opfibred* over $\mathbf{B}$. Also in this case we call the analogue of a pullback a *pushforward.*

Now we need to formalize the notion of a fibre-wise symmetric monoidal structure. The data and conditions will be in exact correspondence with that of a symmetric monoidal category, except that everything is over $\mathbf{B}$. We begin by recalling the definition of a symmetric monoidal category.

Definition 2.1.6. A *symmetric monoidal category* is an ordinary category $\mathcal{M}$ with the following additional data

- ◇ A functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$

- ◇ A functor $\mathbb{1} : \mathbf{1} \to \mathcal{M}$, where $\mathbf{1}$ is the terminal category. This functor, or by abuse of notation the object in its image, is called the *monoidal unit*.
- ◇ A natural isomorphism
$$\alpha : \otimes \circ (\otimes \times 1) \Rightarrow \otimes \circ (1 \times \otimes)$$
of functors $\mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the *associator*.
- ◇ A natural isomorphism
$$\lambda_L : \otimes \circ (\mathbb{1} \times 1) \Rightarrow 1$$
of functors $\mathcal{M} \cong \mathbf{1} \times \mathcal{M} \to \mathcal{M}$, and a natural transformation
$$\lambda_R : \otimes \circ (1 \times \mathbb{1}) \Rightarrow 1.$$
These are called the left and right *unitors*, respectively.
- ◇ A natural isomorphism
$$\gamma : \otimes \Rightarrow \otimes \circ (\pi_2, \pi_1)$$
of functors $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the *braiding*. Here π_k is the projection of a product onto the k^{th} factor.

The data are required to obey the following axioms, where we use notation such as

$$(\cdot_2 \otimes \cdot_1) \otimes \cdot_3 := \otimes \circ (\otimes \times 1) \circ (\pi_2, \pi_1, \pi_3)$$

to make things more readable.

- ◇ The pentagon axiom: The following diagram of functors $\prod_1^4 \mathcal{M} \to \mathcal{M}$ and natural transformations of such commutes.

$$\begin{tikzcd}
& & ((\cdot_1 \otimes \cdot_2) \otimes \cdot_3) \otimes \cdot_4 \arrow[dll, Rightarrow, "\alpha"'] \arrow[drr, Rightarrow, "\alpha\otimes 1"] & & \\
(\cdot_1 \otimes \cdot_2) \otimes (\cdot_3 \otimes \cdot_4) \arrow[dr, Rightarrow, "\alpha"'] & & & & (\cdot_1 \otimes (\cdot_2 \otimes \cdot_3)) \otimes \cdot_4 \arrow[dl, Rightarrow, "\alpha"] \\
& \cdot_1 \otimes (\cdot_2 \otimes (\cdot_3 \otimes \cdot_4)) \arrow[rr, Rightarrow, "1\otimes\alpha"'] & & \cdot_1 \otimes ((\cdot_2 \otimes \cdot_3) \otimes \cdot_4) &
\end{tikzcd}$$

- ◇ The following diagram of functors $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and natural transformations commutes.

$$\begin{tikzcd}
(\cdot_1 \otimes \mathbb{1}) \otimes \cdot_2 \arrow[rr, Rightarrow, "\alpha_{\cdot_1, \mathbb{1}, \cdot_2}"] \arrow[dr, Rightarrow, "\lambda_R \otimes 1"'] & & \cdot_1 \otimes (\mathbb{1} \otimes \cdot_2) \arrow[dl, Rightarrow, "1 \otimes \lambda_L"] \\
& \cdot_1 \otimes \cdot_2 &
\end{tikzcd}$$

We refer to this as the coherence of unitors and the associator.

- ◇ The hexagon axioms: The following diagrams of functors and natural transformations $\mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ commute

$$\begin{tikzcd}
 & (\cdot_1 \otimes \cdot_2) \otimes \cdot_3 \arrow[r, Rightarrow, "\gamma"] \arrow[dl, Rightarrow, "\alpha"'] & \cdot_3 \otimes (\cdot_1 \otimes \cdot_2) & \\
\cdot_1 \otimes (\cdot_2 \otimes \cdot_3) \arrow[dr, Rightarrow, "1\otimes\gamma"'] & & & (\cdot_3 \otimes \cdot_1) \otimes \cdot_2 \arrow[ul, Rightarrow, "\alpha"'] \\
 & \cdot_1 \otimes (\cdot_3 \otimes \cdot_2) \arrow[r, Rightarrow, "\alpha^{-1}"'] & (\cdot_1 \otimes \cdot_3) \otimes \cdot_2 \arrow[ur, Rightarrow, "\gamma\otimes 1"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \cdot_1 \otimes (\cdot_2 \otimes \cdot_3) \arrow[r, Rightarrow, "\gamma"] \arrow[dl, Rightarrow, "\alpha^{-1}"'] & (\cdot_2 \otimes \cdot_3) \otimes \cdot_1 & \\
(\cdot_1 \otimes \cdot_2) \otimes \cdot_3 \arrow[dr, Rightarrow, "\gamma\otimes 1"'] & & & \cdot_2 \otimes (\cdot_3 \otimes \cdot_1) \arrow[ul, Rightarrow, "\alpha^{-1}"'] \\
 & (\cdot_2 \otimes \cdot_1) \otimes \cdot_3 \arrow[r, Rightarrow, "\alpha"'] & \cdot_2 \otimes (\cdot_1 \otimes \cdot_3) \arrow[ur, Rightarrow, "1\otimes\gamma"'] &
\end{tikzcd}$$

- ◇ The braiding is *symmetric*: The following diagram of functors and natural transformations $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ commutes

$$\begin{tikzcd}
\cdot_1 \otimes \cdot_2 \arrow[rr, "\gamma"] \arrow[dr, "1"'] & & \cdot_2 \otimes \cdot_1 \arrow[dl, "\gamma"] \\
 & \cdot_1 \otimes \cdot_1 &
\end{tikzcd}$$

As is usual in these situations, the extra data are typically suppressed in the notation, and we refer to a symmetric monoidal category by one symbol $\mathcal{M}$.

Definition 2.1.7. A *symmetric monoidal fibred category* consists of the data:

(a) A fibration $p : \mathbf{E} \to \mathbf{B}$

(b) A section $\mathbb{1} : \mathbf{B} \to \mathbf{E}$ of p (i.e. $p \circ \mathbb{1} = 1$). This section is required to preserve cartesian arrows in the sense that $\mathbb{1}(f)$ is cartesian for any $f : b' \to b$.

(c) A fibred functor $\otimes : \mathbf{E} \times_{\mathbf{B}} \mathbf{E} \to \mathbf{E}$ of categories fibred over $\mathbf{B}$.

(d) A natural isomorphism $\boldsymbol{\gamma}$ of functors $\mathbf{E} \times_{\mathbf{B}} \mathbf{E} \to \mathbf{E}$ over $\mathbf{B}$ with source $\otimes$ and target $\otimes \circ (\pi_2, \pi_1)$ (the second functor is 'flip and then tensor'). Being over $\mathbf{B}$ means that $\boldsymbol{\gamma}$ of an object over b is a morphism mapped to 1_b by p.

(e) A natural isomorphism $\boldsymbol{\lambda}_L$ of functors over $\mathbf{B}$ with source

$$\mathbb{1} \otimes \cdot = \otimes \circ (\mathbb{1} \times 1) \circ (p, 1_{\mathbf{E}}) : \mathbf{E} \to \mathbf{B} \times_{\mathbf{B}} \mathbf{E} \to \mathbf{E} \times_{\mathbf{B}} \mathbf{E} \to \mathbf{E}$$

and target $1_{\mathbf{E}}$, and a similar natural isomorphism $\boldsymbol{\lambda}_R$ from $\cdot \otimes \mathbb{1}$ to $1_{\mathbf{E}}$.

(f) A natural isomorphism $\boldsymbol{\alpha}$, over $\mathbf{B}$ with source

$$\otimes \circ (\otimes, 1) : \mathbf{E} \times_{\mathbf{B}} \mathbf{E} \times_{\mathbf{B}} \mathbf{E} \to \mathbf{E} \times_{\mathbf{B}} \mathbf{E} \to \mathbf{E}$$

and target $\otimes \circ (1, \otimes)$.

These data are required to satisfy coherence conditions exactly analagous to those of a symmetric monoidal category. The diagrams are identical to those given in Definition 2.1.6, remembering that all objects in the diagrams are over a single object in $\mathbf{B}$.

The only differences to keep in mind between a symmetric monoidal category and a symmetric monoidal fibred category is that what used to be the unit object $\mathbb{1}$ is now a section of the fibration, and that all the structure functors and transformations are over $\mathbf{B}$. In other words, all of the symmetric monoidal structure takes place 'within the fibres'.

Example 2.1.8. Let $\mathbf{Vect_{Man}}$ be as in example 2.1.4. This is a symmetric monoidal fibred category under the fibre-wise tensor product of vector spaces, with unit section $\mathbb{1}$ the trivial line bundle.

We complete the definition by describing the 1-morphisms and 2-morphisms.

Definition 2.1.9. The category $\mathrm{Fib}^{\otimes}(\mathbf{B})$ of categories *symmetric monoidally fibred* over $\mathbf{B}$ has morphisms given by the data

- ◇ A commutative diagram

$$\begin{tikzcd}
\mathbf{E}' \arrow[rr, "f"] \arrow[dr, "p'"'] & & \mathbf{E} \arrow[dl, "p"] \\
& \mathbf{B} &
\end{tikzcd}$$

- ◇ A natural isomorphism $f_{\mathbb{1}} : \mathbb{1}_{\mathbf{E}} \Rightarrow f \circ \mathbb{1}_{\mathbf{E}'}$ over $\mathbf{B}$
- ◇ A natural isomorphism $f_{\otimes} : \otimes_{\mathbf{E}} \circ (f \times f) \Rightarrow f \circ \otimes_{\mathbf{E}'}$ over $\mathbf{B}$

These data are required to satisfy coherence conditions with the symmetric monoidal data from $\mathbf{E}'$ and $\mathbf{E}$. The conditions are identical to those of an ordinary symmetric monoidal functor, with everything over $\mathbf{B}$:

- $$f_{\otimes} \bullet (\boldsymbol{\gamma}_{\mathbf{E}'} \circ (f \times f)) = (f \circ \boldsymbol{\gamma}_{\mathbf{E}}) \bullet f_{\otimes}.$$

  Here $\bullet$ is used to denote the vertical composition of natural transformations of functors $\mathbf{E}' \times_{\mathbf{B}} \mathbf{E}' \to \mathbf{E}$.

- $$f_{\otimes} \bullet (1 \otimes f_{\otimes}) \bullet \boldsymbol{\alpha}_{\mathbf{E}} = \boldsymbol{\alpha}_{\mathbf{E}'} \bullet f_{\otimes} \bullet (f_{\otimes} \otimes 1)$$

- $$\boldsymbol{\lambda}_{\boldsymbol{L}_{\mathbf{E}}} \bullet f_{\otimes} \bullet (1 \otimes f_{\mathbb{1}}) = \boldsymbol{\lambda}_{\boldsymbol{L}_{\mathbf{E}'}}$$

  and a similar identity involving the $\boldsymbol{\lambda}_R$'s

Definition 2.1.10. Given two morphisms $f, g : \mathbf{E}' \to \mathbf{E}$ in $\mathrm{Fib}^{\otimes}(\mathbf{B})$, a *monoidally fibred natural transformation* $\theta : f \Rightarrow g$ is a natural transformation of functors over $\mathbf{B}$, with the extra conditions that

$$g_{\otimes} \bullet \theta \otimes \theta = \theta \bullet f_{\otimes}$$

and

$$\theta \bullet f_{\otimes} = g_{\otimes}.$$

Again, these conditions are identical to those of a monoidal natural transformation, with everything over $\mathbf{B}$.

Remark 2.1.11. We will also be working with ordinary symmetric monoidal functors, and in this case, we will use the notation of the previous definitions for the corresponding structural isomorphisms.

Returning to fibrations, note that given a fibration $p : \mathbf{E} \to \mathbf{B}$ we *almost* get a functor

$$\mathbf{B}^{op} \to \mathbf{Cat},$$

by sending b to the fiber $p^{-1}(b)$ and sending $f : b' \to b$ to a functor which sends an object in $p^{-1}(b)$ to some chosen pullback.

The problem is that functoriality is not satisfied, because choosing a cleavage does not guarantee that pullbacks compose (e.g. Example 2.1.4). A cleavage where the composition of two pullbacks is a pullback is called a *splitting*. Hence, split fibrations have an associated functor $\mathbf{B}^{op} \to \mathbf{Cat}$.

Returning to the example of $\mathbf{Vect}_{\mathbf{Man}}$, note that while the pullback of a pullback is not *equal* to a pullback, the two are isomorphic. In fact they are functorially isomorphic in the sense that there is a natural transformation $(f \circ g)^* \Rightarrow g^* f^*$. This weakening can be incorporated into maps $\mathbf{B}^{op} \to \mathbf{Cat}$ resulting in the notion of pseudofunctors.

Definition 2.1.12. Let $\mathbf{C}$ and $\mathbf{D}$ be strict 2-categories. A *pseudofunctor* $F : \mathbf{C} \to \mathbf{D}$ is

- For each object $x \in \mathbf{C}$ an object $Fx \in \mathbf{D}$
- For each hom-category $\mathbf{C}(x, y)$ of $\mathbf{C}$ a functor $F_{x,y} : \mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$.
- For each $x \in \mathbf{C}$ an invertible 2-morphism $F_{1_x} : 1_{Fx} \Rightarrow F_{x,x}(1_x)$, called a *unitor.*
- For each pair of composable morphisms $f : x \to y, g : y \to z \in \mathbf{C}$ an isomorphism $F_{f,g} : F_{y,z}(g) \circ F_{x,y}(f) \Rightarrow F_{x,z}(g \circ f)$ which is natural in f and g. These are called *compositors.*

Subject to the following diagrams commuting:

- In $\mathbf{C}(x, y)$, given $f : x \to y$,

$$\begin{tikzcd}
F_{x,y}(f) \circ 1_{Fx} \arrow[r, Rightarrow, "1 \circ F_{1_x}"] \arrow[d, Rightarrow, "="'] & F_{x,y}(f) \circ F_{x,x}(1_x) \arrow[d, Rightarrow, "F_{1_x, f}"] \\
F_{x,y}(f) & F_{x,y}(f \circ 1_x) \arrow[l, Rightarrow, "="']
\end{tikzcd}$$

- In $\mathbf{C}(x, y)$ the analagous diagram with 1_y's on the left.
- In $\mathbf{C}(w, z)$, given $f : w \to x, g : x \to y, h : y \to z$,

$$\begin{tikzcd}
(F_{y,z}h \circ F_{x,y}g) \circ F_{w,x}f \arrow[r, Rightarrow, "="] \arrow[d, Rightarrow, "F_{g,h} \circ 1_{F_{w,x}f}"'] & F_{y,z}h \circ (F_{x,y}g \circ F_{w,x}f) \arrow[d, Rightarrow, "1_{F_{y,z}h} \circ F_{f,g}"] \\
F_{x,z}(h \circ g) \circ F_{w,x}f \arrow[d, Rightarrow, "F_{f,h \circ g}"'] & F_{y,z}h \circ F_{w,y}(g \circ f) \arrow[d, Rightarrow, "F_{g \circ f, h}"] \\
F_{w,z}(h \circ (g \circ f)) \arrow[r, Rightarrow, "="'] & F_{w,z}((h \circ g) \circ f)
\end{tikzcd}$$

A *lax functor* is the above definition where the 2-morphisms F_{1_x} and $F_{f,g}$ are not required to be invertible. For Theorem 2.3.1 below, it will be convenient to switch the direction of these to

$$F_{f,g} : F(g \circ f) \Rightarrow F(g) \circ F(f).$$

some sources call these *oplax* functors.

We will drop the subscripts on F as much as possible as long as no confusion is likely to arise. We would like to note two things about this definition. First, the 2-morphisms in the diagrams above which are equalities in strict 2-categories need to be filled in with appropriate associators and unitors to define a pseudofunctor of bicategories. We will not need that generality here. Secondly, we can safely ignore the data of the invertible 2-morphisms

$F_{1_x} : 1_{Fx} \to F_{x,x}(1_x)$ by simply redefining F so that it always sends identities to identities, and "absorbing" these morphisms into the $F_{1,f}$, $F_{f,1}$ 2-isomorphisms. At any rate, the pseudofunctors we are interested always send identities to identities. Note that this is equivalent to cleavages being normalized, and we will therefore call a pseudofunctor with F_{1_x} being identities a *normalized* pseudofunctor.

We should also mention the associated definition of pseudonatural transformation, and of modification.

Definition 2.1.13. Let $F, G : \mathbf{C} \to \mathbf{D}$ be pseudfunctors. Then a *pseudonatural transformation* $\alpha : F \Rightarrow G$ is

- ◇ For each object $x \in \mathbf{C}$ a 1-morphism $\alpha x : Fx \to Gx$.
- ◇ For each 1-morphism $f \in \mathbf{C}(x, y)$, an invertible 2-morphism αf filling in the diagram

$$\begin{tikzcd}
Fx \arrow[r, "\alpha x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\
Fy \arrow[r, "\alpha y"] \arrow[ur, Rightarrow, "\alpha f"] & Gy
\end{tikzcd}$$

It is required that α is functorial on 1-morphisms in the sense that, given the $f : x \to y$ in $\mathbf{C}$ and $g : y \to z$ in $\mathbf{C}$, the 2-morphism filling the diagram

$$\begin{tikzcd}
Fx \arrow[r, "\alpha x"] \arrow[d, "Ff"'] \arrow[dd, bend right=80, "F(g\circ f)"', ""{name=A}] & Gx \arrow[d, "Gf"] \arrow[dd, bend left=80, "G(g\circ f)", ""'{name=B}] \\
Fy \arrow[r, "\alpha y"'] \arrow[d, "Fg"'] \arrow[ur, Rightarrow, "\alpha f"] \arrow[to=A, Rightarrow, "F_{f,g}"] & Gy \arrow[d, "Gg"] \arrow[to=B, Rightarrow, "G_{f,g}"'] \\
Fz \arrow[r, "\alpha z"'] \arrow[ur, Rightarrow, "\alpha g"'] & Gz
\end{tikzcd}$$

is equal to $\alpha(g \circ f)$; and that for each $x \in \mathbf{C}$, the 2-morphism filling the diagram

$$\begin{tikzcd}
Fx \arrow[r, "\alpha x"] \arrow[d, "F(1_x)", ""'{name=C}] \arrow[d, bend right=80, "1_{Fx}"', ""{name=A}] & Gx \arrow[d, "G(1_x)"', ""{name=D}] \arrow[d, bend left=80, "1_{Gx}", ""'{name=B}] \\
Fx \arrow[r, "\alpha x"'] \arrow[ur, Rightarrow, "\alpha(1_x)"] & Gx
\arrow[from=A, to=C, Rightarrow, "F_{1_x}"]
\arrow[from=B, to=D, Rightarrow, "G_{1_x}"']
\end{tikzcd}$$

is equal to $1_{\alpha x}$. Note that when F and G are normalized, so that the unitors are equalities, this reduces to $\alpha 1_x = 1_{\alpha x}$. Finally, α is required to be natural with respect to 2-morphisms, in the sense that if $\phi : f \Rightarrow g$ is a 2-morphism between 1-morphisms $f, g : x \to y$, we have an equality

$$\begin{tikzcd}
Fx \arrow[r, "\alpha x"] \arrow[d, "Ff", ""'{name=B}] \arrow[d, bend right=80, "Fg"', ""{name=A}] & Gx \arrow[d, "Gf"] \\
Fy \arrow[r, "\alpha y"'] \arrow[ur, Rightarrow, "\alpha f"] & Gx
\arrow[Rightarrow, from=A, to=B, "F\phi"]
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, "\alpha x"] \arrow[d, "Fg"] & Gx \arrow[d, "Gg"', ""{name=C}] \arrow[d, bend left=80, "Gf", ""'{name=D}] \\
Fy \arrow[r, "\alpha y"'] \arrow[ur, Rightarrow, "\alpha g"] & Gx
\arrow[Rightarrow, from=C, to=D, "G\phi"]
\end{tikzcd}$$

A *pseudonatural isomorphism* is a pseudonatural transformation where all the above images under α are invertible. Also, a *lax transformation* is the above where we drop the condition that the 2-morphisms $\alpha(f)$ are invertible.

Definition 2.1.14. Let $F, G : \mathbf{C} \to \mathbf{D}$ be pseudofunctors, and $\alpha, \beta : F \Rightarrow G$ be pseudonatural transformations. A *modification* m from α to β is an assignment $x \in \mathbf{C} \mapsto mx : \alpha x \Rightarrow \beta x$ of objects to 2-morphisms, such that for any 1-morphism $f : x \to y$ in $\mathbf{C}$, the following diagram commutes

$$\begin{tikzcd}
\alpha y \circ Ff \arrow[r, Rightarrow, "my \circ 1"] \arrow[d, Rightarrow, "\alpha f"'] & \beta y \circ Ff \arrow[d, Rightarrow, "\beta f"] \\
Gf \circ \alpha x \arrow[r, Rightarrow, "1 \circ mx"'] & Gf \circ \beta x
\end{tikzcd}$$

We can now speak of the 2-category of pseudofunctors $\mathbf{B}^{\mathrm{op}} \to \mathbf{SMCat}$, pseudonatural transformations, and modifications of such.

Returning to the situation of fibred categories, one can actually construct a pseudofunctor $\tilde{\mathbf{E}} : \mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ given a fibred category $p : \mathbf{E} \to \mathbf{B}$. This process is invertible in some weak sense, and the inverse is typically called the Grothendieck construction (perhaps 'a Grothendieck construction' would be more appropriate, as Grothendieck constructed many things). In [Jo] part B chapter 1 this is explained in detail. Here we would like to spell out the case where there are symmetric monoidal structures present. As far as the author knows, this is not present in the literature.

Definition 2.1.15. Let $F : \mathbf{B}^{\mathrm{op}} \to \mathbf{SMCat}$ be a pseudofunctor. We define a category $\mathcal{G}(F)$. The objects are pairs (b, x), where $a \in \mathbf{B}$ and $x \in F(b)$. Given another such object (b', x'), the morphisms from (b', x') to (b, x) are the set of pairs

$$\{(f, g) | f : b' \to b, g : x' \to Ff(x)\}.$$

The composition of $(f, g) : (b', x') \to (b, x)$ with $(f', g') : (b'', x'') \to (b', x')$ is the pair $f \circ f'$ and

$$x'' \xrightarrow{g'} Ff'(x') \xrightarrow{Ff'(g)} (Ff' \circ Ff)(x) \xrightarrow{F_{f',f}(x)} F(f \circ f')(x)$$

The coherence of the natural transformations $F_{\cdot,\cdot}$ with triples of morphisms guarantees that this composition is associative.

It is not hard to see that $\mathcal{G}(F)$ is fibred over $\mathbf{B}$, and in fact comes with a canonical cleavage. See part (a) of Theorem 2.1.16 below for a description of this cleavage, and we observe that it constitutes a splitting precisely when the natural transformations $F_{f',f}$ are the identity, i.e. when F is strict. So far this is identical to the Grothendieck construction in the ordinary fibred case. It is more difficult to check that this construction results in a symmetric monoidally fibred category when the target is **SMCat** in place of **Cat**.

Theorem 2.1.16. The category $\mathcal{G}(F)$ is symmetric monoidally fibred over $\mathbf{B}$.

Proof. We will go through the data and conditions of Definition 2.1.7

(a) The projection functor p simply sends objects (b, x) to b and morphisms (f, g) to f. Cartesian morphisms are of the form (f, ϕ), where ϕ is an isomorphism, and $(f, 1_{Ffx})$ for all f constitutes a cleavage. Hence, $p : \mathcal{G}(F) \to \mathbf{B}$ is a fibration.

(b) The section $\mathbb{1} : \mathbf{B} \to \mathcal{G}(F)$ on objects is $b \mapsto (b, \mathbb{1}_{Fb})$ and on morphisms it is $f \mapsto (f, (Ff)_{\mathbb{1}})$. This is clearly a section. To check functoriality, take $f' : b'' \to b'$ and $f : b' \to b$. Then, under $\mathbb{1}$,

$$f \circ f' \mapsto (f \circ f', F(f \circ f')_{\mathbb{1}}).$$

And composing after applying $\mathbb{1}$, we obtain $(f \circ f', h)$, where h is the composite

$$\mathbb{1}_{b''} \xrightarrow{Ff'_{\mathbb{1}}} Ff'(\mathbb{1}_{b'}) \xrightarrow{Ff'(Ff_{\mathbb{1}})} (Ff' \circ Ff)(\mathbb{1}_b) \xrightarrow{F_{f',f}(\mathbb{1}_b)} F(f \circ f')(\mathbb{1}_b)$$

The composition of the first two is, by definition, $(Ff' \circ Ff)_{\mathbb{1}}$, and the composition of all three is $F(f \circ f')_{\mathbb{1}}$, since $F_{f',f}$ is a 2-morphism in **SMCat**, i.e. a *symmetric monoidal* natural transformation, so in particular it preserves the monoidal unit morphism datum of monoidal functors.

(c) The functor $\otimes : \mathcal{G}(F) \times_{\mathbf{B}} \mathcal{G}(F) \to \mathcal{G}(F)$ on objects is simply $(b, x, y) \mapsto (b, x \otimes y)$. On morphisms, for $(f, g, h) : (b', x', y') \to (b, x, y)$ it is

$$(f, g, h) \mapsto (f, (Ff)_{\otimes}(x, y) \circ (g \otimes h)) \, .$$

Here note $g \otimes h : x' \otimes y' \to Ff(x) \otimes Ff(y)$. To check functoriality, suppose we have the data above together with $(f', g', h') : (b'', x'', y'') \to (b', x', y')$. Then $\otimes$ sends $(f, g, h) \circ (f', g', h')$ to

$$(f \circ f', F(f \circ f')_{\otimes}(x, y) \circ ((F_{f',f}x \circ Ffg \circ g') \otimes (F_{f',f}y \circ Ffh \circ h'))) \, ,$$

which, after using the interchange

$$(F_{f',f}x \circ Ffg \circ g') \otimes (F_{f',f}y \circ Ffh \circ h') = (F_{f',f}x \otimes F_{f',f}y) \circ (Ffg \otimes Ffh) \circ (g' \otimes h'),$$

which follows from functoriality of $\otimes$, is the top path followed by the rightmost vertical morphism in the following commutative diagram

$$\begin{tikzcd}
 & & & Ff'Ffx \otimes Ff'Ffy \arrow[r, "F_{f',f}x \otimes F_{f',f}y"] \arrow[dl, "Ff'_{\otimes}"'] \arrow[dd, "(Ff'Ff)_{\otimes}"] & F(f \circ f')x \otimes F(f \circ f')y \arrow[dd, "F(f \circ f')_{\otimes}"] \\
 & Ff'x' \otimes Ff'y \arrow[urr, "Ff'g \otimes Ff'h"] \arrow[d, "Ff'_{\otimes}"] & Ff'(Ffx \otimes Ffy) \arrow[dr, "Ff'(Ff_{\otimes})"] & & \\
x'' \otimes y'' \arrow[ur, "g' \otimes h'"] \arrow[r, "Ff'_{\otimes} \circ (g' \otimes h')"'] & Ff'(x' \otimes y') \arrow[ur, "Ff'(g \otimes h)"] \arrow[rr, "Ff'(Ff_{\otimes} \circ (g \otimes h))"'] & & Ff'Ff(x \otimes y) \arrow[r, "F_{f',f}(x \otimes y)"'] & F(f \circ f')(x \otimes y)
\end{tikzcd}$$

Note: we have abused the notation of natural transformations such as $Ff_{\otimes}$ in this diagram by not explicitly writing what object they have been applied to. The commutativity of the rightmost square follows from the fact that $F_{f',f}$ is a monoidal natural transformation. The other parts of the diagram commute by the definition of how the structure isomorphisms of a symmetric monoidal functor compose. The bottom path is the result of applying $\otimes$ and then composing, so we obtain functoriality. Finally, this functor preserves cartesian morphisms since the monoidal product of isomorphisms is an isomorphism by functoriality of $\otimes$, and since we have assumed the structure morphisms of the symmetric monoidal functors in **SMCat**, such as $Ff_{\otimes}$, are isomorphisms.

(d) The natural isomorphism $\boldsymbol{\gamma}$ sends an object (b, x, y) of $\mathbf{E} \times_{\mathbf{B}} \mathbf{E}$ to the morphism $(1_b, \gamma_{Fb})$, where $\gamma_{Fb}(x, y) : x \otimes y \to y \otimes x$ is the braiding isomorphism of the symmetric monoidal category Fb. To check naturality, suppose $(f, g, h) : (b', x', y') \to (b, x, y)$ is a morphism of $\mathbf{E} \times_{\mathbf{B}} \mathbf{E}$. Then $\boldsymbol{\gamma}$ maps (f, g, h) to the outer rectangle in the commutative diagram

$$\begin{tikzcd}
x' \otimes y' \arrow[r, "\gamma_{Fb'}(x', y')"] \arrow[d, "(g \otimes h)"'] & y' \otimes x' \arrow[d, "(h \otimes g)"'] \\
Ffx \otimes Ffy \arrow[r, "\gamma_{Fb'}(Ffx, Ffy)"] \arrow[d, "Ff_{\otimes}(x,y)"'] & Ffy \otimes Ffx \arrow[d, "Ff_{\otimes}(Ffy, Ffx)"] \\
Ff(x \otimes y) \arrow[r, "Ff(\gamma_{Fb}(x,y))"] & Ff(y \otimes x)
\end{tikzcd}$$

The top square commutes because $\gamma_{Fb'}$ is a natural transformation, and the bottom commutes because Ff is a symmetric monoidal functor. It is clear that $\boldsymbol{\gamma}$ is over $\mathbf{B}$ by definition.

(e) The natural isomorphism $\boldsymbol{\lambda}_L$ sends an object (b, x) to the morphism $(1_b, \lambda_{L_{Fb}})$. It sends a morphism $(f, g) : (b', x') \to (b, x)$ to the commutative diagram

$$\begin{tikzcd}
\mathbb{1}_{Fb'} \otimes x' \arrow[r, "\lambda_{L_{Fb}} x'"] \arrow[d, "1\otimes g"'] & x' \arrow[d, "g"] \\
\mathbb{1}_{Fb'} \otimes Ffx \arrow[r, "\lambda_{L_{Fb}} Ffx"] & Ffx
\end{tikzcd}$$

which is commutative by naturality of $\lambda_{L_{Fb}}$. We define $\boldsymbol{\lambda}_R$ similarly.

(f) As with all the above data, $\boldsymbol{\alpha}$ is defined on objects over b simply via the associator of $Fb \in \mathbf{SMCat}$. That is,

$$\boldsymbol{\alpha}(b, x, y, z) = (1_b, \alpha_{Fb}).$$

Given a morphism $(f, g, h, i) : (b', x', y', z') \to (b, x, y, z)$, we obtain the commutative diagram

$$\begin{tikzcd}
(x' \otimes y') \otimes z' \arrow[r, "\alpha_{Fb'}(x',y',z')"] \arrow[d, "(g\otimes h)\otimes i"'] & x' \otimes (y' \otimes z') \arrow[d, "g\otimes(h\otimes i)"] \\
(Ffx \otimes Ffy) \otimes Ffz \arrow[r, "\alpha_{Fb'}(Ffx,Ffy,Ffz)"] \arrow[d, "Ff_{\otimes}(x,y)\otimes 1"'] & Ffx \otimes (Ffy \otimes Ffz) \arrow[d, "1\otimes Ff_{\otimes}(y,z)"] \\
Ff(x \otimes y) \otimes Ffz \arrow[d, "Ff_{\otimes}(x\otimes y,z)"'] & Ffx \otimes (Ffy \otimes Ffz) \arrow[d, "Ff_{\otimes}(x,y\otimes z)"] \\
Ff((x \otimes y) \otimes z) \arrow[r, "Ff(\alpha_{Fb}(x,y,z))"] & Ff(x \otimes (y \otimes z))
\end{tikzcd}$$

The upper square commutes because $\alpha_{Fb'}$ is natural, the lower rectangle commutes because Ff is a monoidal functor, and the commutativity of the outer rectangle gives naturality of $\boldsymbol{\alpha}$.

As stated in Definition 2.1.7, the data

$$\mathbb{1}, \boldsymbol{\gamma}, \boldsymbol{\lambda}_L, \boldsymbol{\lambda}_R, \boldsymbol{\alpha}$$

are required to satisfy coherence conditions identical to those for a symmetric monoidal category. These conditions solely involve the natural transformations applied to objects, which means that they corresponding diagrams are contained in a single symmetric monoidal category Fb and furthermore that the monoidal functor data from the Ff's are not involved. The upshot is that all the coherence conditions follow from the coherence conditions following from the pointwise definitions of the data (e.g $\boldsymbol{\alpha}$ acts on objects over b by α_{Fb}, which satisfies the usual associator coherence conditions). This completes the proof. □

Next we describe the construction that produces a pseudofunctor from a symmetric monoidal fibred category, and show that these constructions are inverse in a higher categorical sense.

Theorem 2.1.17. Let $p : \mathbf{E} \to \mathbf{B}$ be a symmetric monoidal fibration with a chosen cleavage. Then there is a pseudofunctor $\tilde{\mathbf{E}} : \mathbf{B}^{op} \to \mathbf{SMCat}$, and moreover $\mathcal{G}(\tilde{\mathbf{E}})$ is isomorphic in $\mathrm{Fib}^{\otimes}(\mathbf{B})$ to $\mathbf{E}$. Finally, for a pseudofunctor F, $\widetilde{\mathcal{G}(F)}$ is naturally isomorphic to F.

Proof. On objects, $\tilde{\mathbf{E}}(b) = p^{-1}(b)$, objects over b and morphisms over 1_b. The data making $\mathbf{E}$ symmetric monoidal restricted over b gives this the structure of a symmetric monoidal category. On morphisms $f : b' \to b$, $\tilde{E}(f)$ sends objects x over b to f^*x. Let $g : x \to y$ be a morphism over 1_b. Then by the factorization property of cartesian morphisms, there is a unique morphism f^*g making the diagram

$$\begin{tikzcd}
f^*x \arrow[rr] \arrow[dr, dotted, "f^*g"] \arrow[dd, mapsto] & & x \arrow[d, "g"] \\
& f^*y \arrow[r] \arrow[d, mapsto] & y \arrow[d, mapsto] \\
b' \arrow[r, "1_b"] & b' \arrow[r, "f"] & b
\end{tikzcd}$$

commute. Define $\tilde{E}(f)g = f^*g$, and note that functoriality in g follows from uniqueness of this factorization. For the symmetric monoidal structure of this functor, we have the diagrams

$$\begin{tikzcd}
\mathbb{1}(b') \arrow[d, dotted] \arrow[dr, "\mathbb{1}(f)"] & \\
f^*\mathbb{1}(b) \arrow[r] \arrow[d, mapsto] & \mathbb{1}(b) \arrow[d, mapsto] \\
b' \arrow[r] & b
\end{tikzcd}$$

with the dotted arrow defining $\tilde{E}(f)_{\mathbb{1}}$, and the diagram

$$\begin{tikzcd}
f^*x \otimes f^*y \arrow[d, dotted] \arrow[dr] & \\
f^*(x \otimes y) \arrow[r] \arrow[d, mapsto] & x \otimes y \arrow[d, mapsto] \\
b' \arrow[r] & b
\end{tikzcd}$$

defining $\tilde{E}(f)_{\otimes}$. The diagonal morphism above is the monoidal product

$$\otimes : \mathbf{E} \times_{\mathbf{B}} \mathbf{E} \to \mathbf{E}$$

applied to the cartesian morphisms $f^*x \to x$ and $f^*y \to y$. Note that both morphisms are over f, and both are cartesian; since $\otimes$ preserves (pairs of) cartesian morphisms, the diagonal morphism is cartesian. Hence the above factorizations were of cartesian morphisms, and are therefore invertible, again by uniqueness of factorizations. Similarly, the coherence conditions for these data to make $\tilde{E}(f)$ a symmetric monoidal functor follow since all paths in the coherence diagrams are factorizations through various cartesian morphisms, and so must agree. Finally, given $f' : b'' \to b'$ and f as before, we can factor the composition $f'^* f^* x \to f^* x \to x$ of cartesian morphisms through $(f \circ f')^* x \to x$ to produce the natural isomorphism

$$\tilde{E}_{f',f} : \tilde{E}(f') \circ \tilde{E}(f) \Rightarrow \tilde{E}(f \circ f')$$

The pseudofunctor coherency conditions of $\tilde{E}$ again follow from uniqueness of cartesian factorizations, as does the property of $\tilde{E}_{f',f}$ being a *symmetric monoidal* natural transformation of symmetric monoidal functors.

Next, there is an isomorphism $\mathcal{G}(\tilde{E}) \to \mathbf{E}$ which sends an object (b, x) to x and a morphism (f, g) to the composite

$$\begin{tikzcd} x' \arrow[r, "g"] & \tilde{E}(f)x := f^*x \arrow[r] & x \end{tikzcd}$$

The inverse sends x to $(p(x), x)$ and $g : x' \to x$ to $(p(g), h)$, where $h : x' \to p(g)^* x$ is again a cartesian factorization.

Finally, there is a natural isomorphism $\widetilde{\mathcal{G}(F)} \to F$ which is essentially an equality, provided that one uses the canonical cleavage $(f, 1_{Ffx})$ for $\mathcal{G}(F)$. □

We recall a standard theorem from the the literature on 2-categories ([SP],[Jo])

Theorem 2.1.18 (Whitehead's Theorem for 2-categories)**.** A pseudofunctor $F : \mathbf{C} \to \mathbf{D}$ is an equivalence of bicategories if and only if the following conditions are satisfied.

(i) F is essentially surjective on objects in the sense that for any object $d \in \mathbf{D}$, there is an object $c \in \mathbf{C}$ such that $F(c) \cong d$.

(ii) $F_{x,y} : \mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$ is essentially surjective for $x, y \in \mathbf{C}$.

(iii) $F_{x,y} : \mathbf{C}(x, y) \to \mathbf{D}(Fx, Fy)$ is fully-faithful for $x, y \in \mathbf{C}$.

With this in mind, we see that we are most of the way toward proving the following theorem.

Theorem 2.1.19. The (strict) 2-category of pseudofunctors $\mathbf{B}^{op} \to \mathbf{SMCat}$, pseudonatural transformations, and modifications, is equivalent to the 2-category $\mathrm{Fib}^{\otimes}(\mathbf{B})$, of symmetric monoidally fibred categories, symmetric monoidally fibred functors, and monoidal natural transformations over $\mathbf{B}$.

Proof. To complete this proof, we need to promote $\mathcal{G}$ to a pseudofunctor, and show that the conditions of Whitehead's Theorem for 2-categories are satisfied. To wit, given $\alpha : F \Rightarrow G$, a pseudonatural transformation of pseudofunctors $F, G : \mathbf{B}^{op} \to \mathbf{SMCat}$, we define $\mathcal{G}_{F,G}(\alpha) : \mathcal{G}(F) \Rightarrow \mathcal{G}(G)$ on objects by

$$\mathcal{G}(\alpha)(b, x) := (b, \alpha b(x))$$

and on morphisms $(f, g) : (b', x') \to (b, x)$ by

$$\mathcal{G}(\alpha)(f, g) := (f, (\alpha f(x)) \circ (\alpha b'(g)))\,.$$

Note that

$$(\alpha f(x)) \circ (\alpha b'(g)) : \alpha b'(x') \to \alpha b'(Ffx) \to Gf(\alpha b(x)),$$

as required. Recalling that morphisms (f, g) are cartesian precisely when g is an isomorphism, we see that $\mathcal{G}(\alpha)$ preserves cartesian morphisms. Indeed, αf is invertible by the definition of pseudonatural transformation, and $\alpha b'(g)$ is invertible since $\alpha b'$ is a functor, and g is invertible. Functoriality of $\mathcal{G}(\alpha)$ follows directly from pseudonaturality of α and pseudofunctoriality of F, G. We will not give the details since they are identical to the non-monoidal case.

Now we need to produce the extra data associated to a symmetric monoidally fibered functor, namely $\mathcal{G}(\alpha)_{\mathbb{1}}$ and $\mathcal{G}(\alpha)_{\otimes}$. For $b \in \mathbf{B}$, define

$$\mathcal{G}(\alpha)_{\mathbb{1}}(b) = (1_b, \alpha b_{\mathbb{1}}),$$

which is natural in b since the αf are monoidal natural transformations. Similarly, let $(b, x, y) \in \mathcal{G}(F) \times_{\mathbf{B}} \mathcal{G}(F)$ and define

$$\mathcal{G}(\alpha)_{\otimes}(b, x, y) = (1_b, (\alpha b)_{\otimes}).$$

By definition, it is clear that $\mathcal{G}(\alpha)_{\mathbb{1}}$ and $\mathcal{G}(\alpha)_{\otimes}$ are over $\mathbf{B}$. To check naturality, let $(f, g, h) : (b', x', y') \to (b, x, y)$, and consider the following diagram in Gb', where the left and right vertical paths are the functors $\otimes_{\mathcal{G}(G)} \circ (\mathcal{G}(\alpha) \times \mathcal{G}(\alpha))$ and $\mathcal{G}(\alpha) \circ \otimes_{\mathcal{G}(F)}$ applied to (f, g, h), respectively.

$$\begin{tikzcd}
\alpha b' x' \otimes \alpha b' y' \arrow[r, "\alpha b'_{\otimes}(x',y')"] \arrow[d, "\alpha b'(g) \otimes \alpha b'(h)"'] & \alpha b'(x' \otimes y') \arrow[d, "\alpha b'(g \otimes h)"] \\
\alpha b' F f x \otimes \alpha b' F f y \arrow[r, "\alpha b'_{\otimes}(Ffx, Ffy)"] \arrow[d, "\alpha f(Ffx) \otimes \alpha f(Ffy)"'] \arrow[dr, "(\alpha b' F f)_{\otimes}(x,y)"] & \alpha b'(Ffx \otimes Ffy) \arrow[d, "\alpha b'(Ff_{\otimes}(x,y))"] \\
Gf\alpha b x \otimes Gf\alpha b y \arrow[d, "Gf_{\otimes}(\alpha b x, \alpha b y)"'] \arrow[dr, "(Gf\alpha b)_{\otimes}(x,y)"] & \alpha b' F f(x \otimes y) \arrow[d, "af(Ff(x \otimes y))"] \\
Gf(\alpha b x \otimes \alpha b y) \arrow[r, "Gf(\alpha b_{\otimes}(x,y))"'] & Gf\alpha b(x \otimes y)
\end{tikzcd}$$

The top square commutes since $\alpha b'$ is a monoidal functor. The triangles commute by the definition of composition of monoidal functors. Finally, the parallelogram commutes because αf is a monoidal natural transformation.

Now, we must check all the coherency diagrams involving the data of this symmetric monoidally fibred natural transformation $\mathcal{G}(\alpha)$ and the data making $\mathcal{G}(F)$ and $\mathcal{G}(G)$ into symmetrically monoidally fibred categories. However, these coherency diagrams only involve evaluating these data, which are natural transformations, on objects. The definitions on objects are all given by corresponding structures on things in **SMCat**, and therefore satisfy all needed diagrams.

Lastly, we must define the action of $\mathcal{G}$ on modifications. So take a modification m between $\alpha, \beta : F \Rightarrow G$. Note that a modification here sends an object $b \in \mathbf{B}$ to a monoidal natural transformation of functors $mb : \alpha b \Rightarrow \beta b$. Define $\mathcal{G}(m)$ by

$$\mathcal{G}(m)(b, x) := (1_b, mb(x)).$$

For a morphism $(f, g) : (b', x') \to (b, x)$, we obtain the diagram

$$\begin{tikzcd}
\alpha b' x' \arrow[r, "mb'x'"] \arrow[d, "\alpha b' g"'] & \beta b' x' \arrow[d, "\beta b' g"] \\
\alpha b'(Ffx) \arrow[r, "mb'(Ffx)"] \arrow[d, "\alpha fx"'] & \beta b'(Ffx) \arrow[d, "\beta fx"'] \\
Gf\alpha bx \arrow[r, "Gfmbx"'] & Gf\beta bx
\end{tikzcd}$$

The bottom square commutes because m is a modification, and the top square commutes because mb' is a natural transformation. As in previous cases, the condition required for $\mathcal{G}(m)$ to be monoidal follows from mb being monoidal, since this condition only requires checking pointwise.

Checking functoriality is identical here to the non-monoidal case, so we need only check the conditions of Whitehead's Theorem 2.1.18 to show that we have an equivalence. Essentially surjective on objects was shown in theorem 2.1.17. For essentially full, suppose we have a monoidally fibred functor $\Gamma : \mathcal{G}(F) \Rightarrow \mathcal{G}(G)$. Define a pseudonatural transformation $\hat{\Gamma} : F \Rightarrow G$ as follows. Given an object $b \in \mathbf{B}$, $\hat{\Gamma} b = \Gamma|_{p^{-1}(b)}$, with p the canonical projection functor. By this restriction, we mean 'ignore that Γ is the identity on the first entry of tuples (b, x)'. This is clearly symmetric monoidal, since a Γ is fibrewise symmetric monoidal. For a morphism $f : b' \to b$, and for $x \in Fb$, we have the diagram

$$\begin{tikzcd}
(b', \Gamma F f x) \arrow[d, dotted, leftrightarrow] \arrow[dr, "\Gamma(f,1_x)"] & \\
(b', G f \Gamma x) \arrow[r, "(f,1_{\Gamma x})"] \arrow[d, mapsto] & (b, \Gamma x) \arrow[d, mapsto] \\
b' \arrow[r, "f"] & b
\end{tikzcd}$$

Where the dotted arrow is a (unique) cartesian factorization, and therefore invertible (see Remark 2.1.2). This morphism we define to be the image of $\hat{\Gamma}(f)(x)$. Now, if we consider $\mathcal{G}(\hat{\Gamma})$, on objects this does the same thing as Γ. On morphisms, it does the same thing as well, provided one uses the canonical cleavage on $\mathcal{G}(F)$ and $\mathcal{G}(G)$ to produce the factorization. Hence we produced an inverse to the map $\alpha \mapsto \mathcal{G}(\alpha)$. Finally, to check fully-faithfulness, let η be a monoidal natural transformation over $\mathbf{B}$ and define a modification via $mb = \eta|_{p^{-1}(b)}$. This is clearly the inverse of $m \mapsto \mathcal{G}(m)$. $\square$

Next we turn our attention to strictifying the pseudofunctors $\tilde{\mathbf{E}}$ in a canonical way.

A fibration whose associated pseudofunctor factors through **Set** is called a discrete fibration. Note that any pseudofunctor factoring through **Set** must in fact be a functor, and so any discrete fibration is split.

Remark 2.1.20. Given any object $b \in \mathbf{B}$ we can form a discrete fibration via the Yoneda embedding. The fibre over $b' \in \mathbf{B}$ is $\mathbf{B}(b', b)$, and the pullback of an object over b, i.e. a morphism $g : b' \to b$ along a morphism $f : b'' \to b'$ is given by precomposition. The associated functor $\mathbf{B}^{\mathrm{op}} \to \mathbf{Set}$ in this case is the Yoneda embedding itself. We will use the notation $\underline{b}$ to denote this category (discretely) fibred over $\mathbf{B}$. Note that this is a split fibration, and may just as well be considered as a functor $\mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$.

Definition 2.1.21. We will denote the hom categories of $\mathrm{Fib}(\mathbf{B})$ by $\mathrm{Fun}_{\mathbf{B}}$. The objects are fibred functors and the morphisms are natural transformations over $\mathbf{B}$. Given $\mathbf{E} \in \mathrm{Fib}(\mathbf{B})$, we can get a (strict) functor $\mathbf{B}^{\mathrm{op}} \to \mathbf{Cat}$ defined by $X \mapsto \mathrm{Fun}_{\mathbf{B}}(\underline{X}, \mathbf{E})$. If $\mathbf{E}$ was symmetric monoidally fibred, we can similarly get a functor $\mathbf{B}^{op} \to \mathbf{SMCat}$, denoted $\mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{X}, \mathbf{E})$ by taking the monoidal product point-wise.

We can go further and define an inner-hom in $\mathrm{Fib}(\mathbf{B})$. Its fibres are given by

$$\underline{\mathrm{Fun}_{\mathbf{B}}}(\mathbf{E}', \mathbf{E})_b = \mathrm{Fun}_{\mathbf{B}}(\mathbf{E}' \times_{\mathbf{B}} \underline{b}, \mathbf{E})$$

We finally promote this inner-hom to $\mathrm{Fib}^{\otimes}(\mathbf{B})$ by the formula

$$\underline{\mathrm{Fun}^{\otimes}_{\mathbf{B}}}(\mathbf{E}', \mathbf{E})_b = \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\mathbf{E}', \underline{\mathrm{Fun}_{\mathbf{B}}}(\underline{b}, \mathbf{E})),$$

where as above the monoidal structure on $\underline{\mathrm{Fun}_{\mathbf{B}}}(\underline{b}, \mathbf{E})$ is induced point-wise.

Let us describe the relationship between these constructions and the Grothendieck constructions from above.

Theorem 2.1.22. The functor $\mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{\cdot}, \mathbf{E})$ is pseudonaturally isomorphic to the pseudofunctor $\tilde{E}$.

Proof. We define a pseudonatural transformation

$$\alpha : \tilde{E} \Rightarrow \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{\cdot}, \mathbf{E})$$

Given $b \in \mathbf{B}$, we must give a symmetric monoidal functor from the fibre of $\mathbf{E}$ over b to $\mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b}, \mathbf{E})$. On objects x, we define it by $x \mapsto ((f : b' \to b) \mapsto f^*x)$. Send morphisms $g : x \to y$ to the natural transformations $(f : b' \to b) \mapsto f^*g$ which are functorial in g and therefore natural. In short, we define

$$(\alpha b)(f) := \tilde{E}(f)$$

These are therefore symmetric monoidal with the same structure maps as $\tilde{E}(f)$. These αb are invertible up to isomorphism, with inverse given by sending functors and natural transformations $\phi \in \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b}, \mathbf{E})$ to $\phi(1_b)$. For $f : b' \to b$, we have the diagram

$$\begin{tikzcd}
\tilde{E}(b) \arrow[r, "\alpha b"] \arrow[d, "f^*"'] & \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b}, \mathbf{E}) \arrow[d] \\
\tilde{E}(b') \arrow[r, "\alpha b'"] & \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b'}, \mathbf{E})
\end{tikzcd}$$

The functor going along the top first sends x to the functor $(f' : b'' \to b') \mapsto (f \circ f')^*x$. Down and right results in $(f' : b'' \to b') \mapsto f'^*f^*x$. Hence we may define

$$(\alpha f)(f') := \tilde{E}_{f',f}$$

Which is monoidal since $\tilde{E}_{f',f}$ is a monoidal natural transformation. The coherence of $\tilde{E}_{\cdot,\cdot}$ on triples is equivalent to the functoriality of α on morphisms, as can be seen in the diagram

$$\begin{tikzcd}
\tilde{\mathbf{E}}(b) \arrow[rr, "\alpha b"] \arrow[dd, "\tilde{\mathbf{E}}(f \circ f')"'] \arrow[dr, "f^*"] & & \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b}, \mathbf{E}) \arrow[d, "f\circ"] \\
& \tilde{\mathbf{E}}(b') \arrow[r, "\alpha b'"] \arrow[dl, "f'^*"] \arrow[l, Rightarrow, "\tilde{\mathbf{E}}_{f',f}"'] \arrow[ur, Rightarrow, "\tilde{\mathbf{E}}_{\cdot,f}"] & \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b'}, \mathbf{E}) \arrow[d, "f'\circ"] \\
\tilde{\mathbf{E}}(b'') \arrow[rr, "\alpha b''"'] \arrow[urr, Rightarrow, shorten <=8em, "\tilde{\mathbf{E}}_{\cdot,f'}"] & & \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b''}, \mathbf{E})
\end{tikzcd}$$

The fact that the composition of the two trapezoidal 2-cells is equal to the triangle 2-cell with the 2-cell $\alpha(f \circ f') := \tilde{\mathbf{E}}_{\cdot, f \circ f'}$ which fills the whole diagram is precisely the coherence of $\tilde{E}_{\cdot,\cdot}$ with triples. Recall finally that these $\tilde{E}_{\cdot,\cdot}$ are isomorphisms, completing the proof. $\square$

Therefore we have shown that pseudofunctors with target **SMCat** can be strictified in a natural way. It is clear that pseudofunctors with target **Cat** can be strictified in the same way, by the same proof without symmetric monoidal considerations.

Definition 2.1.23. In analogy with the notation $\underline{b}$, for the discrete fibred category arising from the Yoneda embedding, we define

$$\underline{\mathbf{E}} := \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{\cdot}, \mathbf{E}).$$

We conclude by extending this construction to fibred functors.

Definition 2.1.24. Let $\mathbf{E}', \mathbf{E} \in \mathrm{Fib}^{\otimes}(\mathbf{B})$, and let $F : \mathbf{E}' \to \mathbf{E}$ be a symmetric monoidally fibred functor. Define $\underline{F}$ by postcomposition. Explicitly, given $b \in \mathbf{B}$, and a fibred functor $G \in \mathrm{Fun}_{\mathbf{B}}(\underline{b}, \mathbf{E})$, define $\underline{F} : \underline{E'} \to \underline{E}$ by

$$\underline{F}(b)(G) = (F \circ G),$$

and the definition on fibred natural transformations is given by 'whiskering': $\underline{F}(b)(\alpha) := (1_F \bullet \alpha)(b)$.

Lemma 2.1.25. From the definition above, $\underline{F}$ is a monoidal (strictly) natural transformation. Indeed, given $f : b' \to b$, we obtain the diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b}, \mathbf{E}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b}, \mathbf{E}') \arrow[d] \\
\mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b'}, \mathbf{E}) \arrow[r] & \mathrm{Fun}^{\otimes}_{\mathbf{B}}(\underline{b'}, \mathbf{E}')
\end{tikzcd}$$

Again taking $G \in \mathrm{Fun}_{\mathbf{B}}(\underline{b}, \mathbf{E})$, and a morphism $f' : b'' \to b'$ both paths in the diagram result in the functor

$$b' \mapsto (F \circ G)(f \circ f')$$

2.2 Symmetric Monoidal Categories and Opfibrations

In this section, we describe an alternate model of the category **SMCat**. This will serve the dual purpose of giving a motivation for the definition of symmetric monoidal $(\infty, 1)$-catgeory that we will propose and also provide a step in the map of field theories that is the main goal of this work.

We begin with a characterization of symmetric monoidal categories as a full subcategory of certain opfibrations. The ideas here come from [Lur] chapter 2. We remark that we will repeatedly use the fact that in a symmetric monoidal category $\mathcal{M}$, when taking the monoidal product of a tuple of objects ordered and associated in two different ways, the various isomorphisms between these elements generated by the data

$$\alpha, \gamma, \lambda_L, \lambda_R$$

are all *equal*. This is sometimes calle MacLane's coherence theorem.

Definition 2.2.1. We define the category $\mathbf{\Gamma}$ as follows. The objects of $\mathbf{\Gamma}$ are pointed sets $\{*, 1, \cdots, n\}$, denoted by $\langle n \rangle$, for $n = 0, 1, 2, \ldots$. The morphisms of $\mathbf{\Gamma}$ are morphisms of pointed sets, i.e. set maps $\alpha : \langle n \rangle \to \langle m \rangle$ such that $\alpha(*) = *$.

Now, given a symmetric monoidal category $\mathcal{M}$, we define a new category $\mathcal{M}^{\otimes}$ which is opfibred over $\mathbf{\Gamma}$.

Definition 2.2.2. Let $\mathcal{M}$ be a symmetric monoidal category. The objects of $\mathcal{M}^{\otimes}$ are (possibly empty) ordered tuples of objects of $\mathcal{M}$: $(x_1, \cdots, x_l)$. The morphisms of $\mathcal{M}^{\otimes}$ are pairs $(\alpha, f_i) : (x'_1, \cdots, x'_k) \to (x_1, \cdots, x_l)$, where $\alpha : \langle k \rangle \to \langle l \rangle$ is a morphism in $\mathbf{\Gamma}$, and f_i, for $i = 1, \cdots, l$ is a morphism in $\mathcal{M}$:

$$f_i : \bigotimes_{j \in (\alpha^{-1}(i) \setminus \{*\})} x'_j \to x_i.$$

In order to define the products over $\alpha^{-1}(i) \setminus \{*\}$ uniquely, we make the convention that the empty product is the monoidal unit $\mathbb{1}$, that the ordering on the product is induced by the ordering of $\alpha^{-1}(i)$ as a subset of $\{*, 1, \cdots, k\}$, and finally that the products are grouped 'all the way to the left', as in

$$(\cdots(((\bullet \otimes \bullet) \otimes \bullet) \otimes \bullet) \cdots \otimes \bullet).$$

The composition of (α, f_i) with (β, g_j) is the pair $(\beta \circ \alpha, g_j \circ \otimes f_i \circ \theta_j)$, where θ_j is the canonical isomorphism

$$\bigotimes_{n \in (\beta\circ\alpha)^{-1}(j) \setminus \{*\})} x''_n \to \bigotimes_{m \in (\beta^{-1}(n) \setminus \{*\})} \left(\bigotimes_{n \in (\alpha^{-1}(j) \setminus \{*\})} x''_n \right)$$

in $\mathcal{M}$ which permutes the ordering and grouping of the first product to that of the second. The uniqueness of this isomorphism follows from the axioms of a symmetric monoidal category, which imply that any combination of the structure maps inducing this reordering and regrouping is equal to any other (Maclane's coherence). Furthermore, associativity of this composition follows by naturality of the structure isomorphism (to commute the θ's with the f's and g's) combined again with coherence (to combine θ's).

There is a forgetful functor $p : \mathcal{M}^{\otimes} \to \mathbf{\Gamma}$ which sends an element $(m_1, \cdots, m_l)$ to $\langle l \rangle$, and sends a morphism (α, f_i) to α.

Lemma 2.2.3. The functor $p : \mathcal{M}^{\otimes} \to \mathbf{\Gamma}$ is an opfibration of categories.

Proof. The co-cartesian morphisms are precisely those pairs (α, f_i), where the f_i are isomorphisms. A cleavage is then given by the assignment

$$((x_1, \cdots, x_k), \alpha) \mapsto (\alpha, 1_{\bigotimes_{j\in(\alpha^{-1}(i)\setminus\{*\})} x_j})$$

□

We will use the notation $\alpha_!$ for the pushforward morphisms in this cleavage. Note that, by the definition of composition in $\mathcal{M}^{\otimes}$, this cleavage does not split.

For example, let $t : \langle 2\rangle \to \langle 2\rangle$ be the isomorphism that switches 1 and 2, and let $m : \langle 2\rangle \to \langle 1\rangle$ be the morphism that sends 1 and 2 to 1. Note that $m \circ t = m$. The pushforward of (x_1, x_2) over t is (x_2, x_1), and the further pushforward over m is $x_2 \otimes x_1$. In the composition of these morphisms, $(m \circ t, f_1)$, f_1 is, by definition, an isomorphism $x_1 \otimes x_2 \to x_2 \otimes x_1$, i.e. the braiding of $\mathcal{M}$. In particular, it is not the identity, and so the cleavage is not a splitting.

Let us make some further observations on the category $\mathcal{M}^{\otimes}$.

Remark 2.2.4. For numbers n, i, let $\pi^i_{\langle n\rangle} : \langle n\rangle \to \langle 1\rangle$ be the morphism defined by

$$\pi^i_{\langle n\rangle}(j) = \begin{cases} 1 & \text{if } j = i \\ * & \text{otherwise} \end{cases}$$

Let $\mathcal{M}^{\otimes}_{\langle n\rangle}$ denote the fibre over $\langle n\rangle$. Then the functor

$$\phi_n := \big((\pi^1_{\langle n\rangle})_!, \cdots, (\pi^n_{\langle n\rangle})_!\big) : \mathcal{M}^{\otimes}_{\langle n\rangle} \to \prod_1^n \mathcal{M}^{\otimes}_{\langle 1\rangle}$$

is an isomorphism of categories. Furthermore, if $\sigma : \langle n\rangle \to \langle n\rangle$, is an isomorphism, then the following diagram commutes

$$\begin{tikzcd}
\mathcal{M}^{\otimes}_{\langle n\rangle} \arrow[r, "\phi_n"] \arrow[d, "\sigma_!"'] & \prod_1^n \mathcal{M}^{\otimes}_{\langle 1\rangle} \arrow[d, "(\pi_{\sigma(1)}, \cdots, \pi_{\sigma(n)})"] \\
\mathcal{M}^{\otimes}_{\langle n\rangle} \arrow[r, "\phi_n"] & \prod_1^n \mathcal{M}^{\otimes}_{\langle 1\rangle}
\end{tikzcd}$$

In other words, the ϕ_n is equivariant with respect to the obvious action of the symmetric group Σ_n on source and target.

Noting that $\pi^i_{\langle n\rangle} \circ \sigma = \pi^{\sigma(i)}_{\langle n\rangle}$, we can rephrase this by saying that pushforward commutes with composition of $\pi^i_{\langle n\rangle}$ maps and permutations. We make the following definition.

Definition 2.2.5. A morphism $\alpha : \langle n\rangle \to \langle m\rangle$ in $\mathbf{\Gamma}$ is called *static* if, for each $i = 1, \cdots, m$, $\alpha^{-1}(i)$ contains at most one element.

In words, a morphism is static if it is an injection away from $*$. A composition of static morphisms is static, so we may define the subcategory $\mathbf{\Gamma}_s$ of $\mathbf{\Gamma}$ consisting of all objects and only the static morphisms. The following follows directly from the definition of composition in $\mathcal{M}^{\otimes}$.

Lemma 2.2.6. In the pullback diagram

$$\begin{tikzcd}
\mathcal{M}_s^{\otimes} \arrow[r] \arrow[d, "p_s"'] & \mathcal{M}^{\otimes} \arrow[d, "p"] \\
\mathbf{\Gamma}_s \arrow[r, hook] & \mathbf{\Gamma}
\end{tikzcd}$$

The canonical cleavage on $p : \mathcal{M}^{\otimes} \to \mathbf{\Gamma}$ is sent to a splitting of the opfibration $p_s : \mathcal{M}_s^{\otimes} \to \mathbf{\Gamma}_s$.

The point is that the canonical cleavage only fails to be a splitting when there is some choice made in the ordering and grouping of the tensor products appearing in the definition of morphisms (α, f_i). Note that the condition of this lemma implies, among other things, the Σ_n equivariance noted in remark 2.2.4. We now make a definition abstracting these observations.

Definition 2.2.7. Define the category $\text{opFib}(\mathbf{\Gamma})_s$ to be the full subcategory of $\text{opFib}(\mathbf{\Gamma})$ with objects opfibrations $\mathbf{D} \to \mathbf{\Gamma}$ with a cleavage such that

◇ The functors

$$\phi_n := \left((\pi^1_{\langle n \rangle})_!, \cdots, (\pi^n_{\langle n \rangle})_!\right) : \mathbf{D}_{\langle n \rangle} \to \prod_1^n \mathbf{D}_{\langle 1 \rangle}$$

are *isomorphisms* of categories.

◇ In the pullback diagram

$$\begin{tikzcd}
\mathbf{D}_s \arrow[r] \arrow[d, "p_s"'] & \mathbf{D} \arrow[d, "p"] \\
\mathbf{\Gamma}_s \arrow[r, hook] & \mathbf{\Gamma},
\end{tikzcd}$$

the cleavage induced on the opfibration $\mathbf{D}_s \to \mathbf{\Gamma}_s$ splits.

Next we show that these opfibrations have the structure to induce a symmetric monoidal structure on $\mathbf{D}_{\langle 1 \rangle}$.

Theorem 2.2.8. Let $\mathbf{D} \in \text{opFib}(\mathbf{\Gamma})_s$. Then $\mathbf{D}_{\langle 1 \rangle}$ has the structure of a symmetric monoidal category.

Proof. Throughtout the proof, we will use the notation $\alpha_!$ for the pushforwards in a fixed cleavage for $\mathbf{D}$. Recall from the previous section that these pushforwards can be interpreted as functors on the fibres. The monoidal product of $\mathbf{D}_{\langle 1\rangle}$ is the composition of functors

$$m_! \circ \phi_2^{-1} : \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \to \mathbf{D}_{\langle 1\rangle}$$

Note that ϕ_0 gives an isomorphism of $\mathbf{D}_{\langle 0\rangle}$ with the empty product, i.e. the terminal category of one object and its identity. Let $\iota : \langle 0\rangle \to \langle 1\rangle$ be the unique morphism of $\mathbf{\Gamma}$. Then the monoidal unit of $\mathbf{D}$ is the image under $\iota_!$ of the unique object of $\mathbf{D}_{\langle 0\rangle}$. Recall again from the previous section that, given α', α, two composable morphisms in $\mathbf{\Gamma}$, there is a natural isomorphism of functors

$$\tilde{\mathbf{D}}_{\alpha',\alpha} : \alpha_! \circ \alpha'_! \Rightarrow (\alpha \circ \alpha')_!$$

Here we have used the notation $\tilde{\mathbf{D}}$ for the pseudofunctor $\mathbf{\Gamma} \to \mathbf{Cat}$ associated to the opfibration $\mathbf{D} \to \mathbf{\Gamma}$, in analogy to Theorem 2.1.17. Furthermore, given a third composable morphism α'', we have the commutative diagram

$$\begin{tikzcd}
\alpha_! \circ \alpha'_! \circ \alpha''_! \arrow[r, Rightarrow, "\tilde{\mathbf{D}}_{\alpha',\alpha} \circ 1_{\alpha''_!}"] \arrow[d, Rightarrow, "1_{\alpha_!} \circ \tilde{\mathbf{D}}_{\alpha'',\alpha'}"'] & (\alpha \circ \alpha')_! \circ \alpha''_! \arrow[d, Rightarrow, "\tilde{\mathbf{D}}_{\alpha'',\alpha\circ\alpha'}"] \\
\alpha_! \circ (\alpha' \circ \alpha'')_! \arrow[r, Rightarrow, "\tilde{\mathbf{D}}_{\alpha'\circ\alpha'',\alpha}"'] & (\alpha \circ \alpha' \circ \alpha'')_!
\end{tikzcd}$$

arising from the pseudofunctoriality of $\tilde{\mathbf{D}}$.

Consider the diagram

$$\begin{tikzcd}
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"] \arrow[d, "{(\pi_2,\pi_1)}"'] & \mathbf{D}_{\langle 2\rangle} \arrow[rr, "m_!=(m\circ t)_!"] \arrow[dr, "t_!"'] & & \mathbf{D}_{\langle 1\rangle} \\
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[rr, "\phi_2^{-1}"] & & \mathbf{D}_{\langle 2\rangle} \arrow[ur, "m_!"'] \arrow[u, Rightarrow, "\tilde{\mathbf{D}}_{t,m}"] &
\end{tikzcd}$$

The path along the top is the morphism the monoidal product. The path along the bottom is a 'flip' followed by the monoidal product. The trapezoid commutes, as a special case of the equivariance explained in remark 2.2.4. The natural isomorphism filling in the diagram is the braiding isomorphism γ. To see that this braiding is symmteric, consider the diagram

$$\begin{tikzcd}
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"] \arrow[d, "{(\pi_2,\pi_1)}"'] & \mathbf{D}_{\langle 2\rangle} \arrow[r, "m_!"] \arrow[d, "t_!"'] & \mathbf{D}_{\langle 1\rangle} \arrow[d, "="] \\
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"] \arrow[d, "{(\pi_2,\pi_1)}"'] & \mathbf{D}_{\langle 2\rangle} \arrow[r, "m!"] \arrow[d, "t_!"'] \arrow[ur, Rightarrow, "\tilde{\mathbf{D}}_{t,m}"] & \mathbf{D}_{\langle 1\rangle} \arrow[d, "="] \\
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"] & \mathbf{D}_{\langle 2\rangle} \arrow[r, "m!"] \arrow[ur, Rightarrow, "\tilde{\mathbf{D}}_{t,m}"] & \mathbf{D}_{\langle 1\rangle}
\end{tikzcd}$$

The natural isomorphism filling the whole diagram is $\gamma \bullet (\gamma \circ 1_{\text{flip}})$, which we need to show is the identity transformation. To wit, applying the pseudofunctoriality of $\tilde{\mathbf{D}}$ to the triple t, t, m, we find that the natural transformation filling the right rectangle is equal to

$$\tilde{\mathbf{D}}_{t \circ t, m} \bullet (1_{m_!} \circ \tilde{\mathbf{D}}_{t,t}).$$

However, t is static, so $\tilde{\mathbf{D}}_{t,t} = 1$, and $t \circ t = 1$, so

$$\tilde{\mathbf{D}}_{t \circ t, m} = \tilde{\mathbf{D}}_{1,m} = 1$$

since we assume the cleavage is normalized. Finally, the rectangle on the left commutes (i.e. is filled with the identity transformation), so the braiding is symmetric.

For $n \geq 2$ and $1 \geq i \leq n-1$, define $\mu^i_{\langle n \rangle} : \langle n \rangle \to \langle n-1 \rangle$, a morphism in $\mathbf{\Gamma}$ by

$$\mu^i_{\langle n \rangle}(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \\ * & \text{if } j = * \end{cases}$$

Furthermore, for $n \geq 1$, define $m_{\langle n \rangle} : \langle n \rangle \to \langle 1 \rangle$ by the formula

$$m_{\langle n \rangle}(j) = \begin{cases} 1 & \text{if } j \neq * \\ * & \text{if } j = * \end{cases}$$

For example, $\mu^1_{\langle 2 \rangle} = m_{\langle 2 \rangle} = m$. Note also that

$$m \circ \mu^1_{\langle 3 \rangle} = m_{\langle 3 \rangle} = m \circ \mu^2_{\langle 3 \rangle}.$$

From the first equality, we obtain the rightmost natural transformation in the following diagram, which lies over the obvious (strictly) commutative diagram in free products of objects of $\mathbf{\Gamma}$.

$$\begin{tikzcd}
\mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r, "\phi_2^{-1} \times 1"] & \mathbf{D}_{\langle 2 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r] \arrow[d, "m_! \times 1"'] & \mathbf{D}_{\langle 3 \rangle} \arrow[dl] \arrow[d, "(\mu^1_{\langle 3 \rangle})_!"] \arrow[dd, bend left=60, "(m_3)_!"] \\
& \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r, "\phi_2^{-1}"'] & \mathbf{D}_{\langle 2 \rangle} \arrow[d, "m_!"] \arrow[r, Rightarrow] & {} \\
& & \mathbf{D}_{\langle 1 \rangle}
\end{tikzcd}$$

The square is the 'product' of the squares

$$\begin{tikzcd}
\mathbf{D}_{\langle 3 \rangle} \arrow[r] \arrow[d, "(\mu^1_{\langle 3 \rangle})_!"'] & \mathbf{D}_{\langle 2 \rangle} \arrow[d, "(m_!)"] \\
\mathbf{D}_{\langle 2 \rangle} \arrow[r, "\pi^1_{\langle 2 \rangle}"'] & \mathbf{D}_{\langle 1 \rangle}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{D}_{\langle 3\rangle} \arrow[r, "\pi^1_{\langle 3\rangle}"] \arrow[d, "(\mu^1_{\langle 3\rangle})_!"'] & \mathbf{D}_{\langle 1\rangle} \arrow[d, "1"] \\
\mathbf{D}_{\langle 1\rangle} \arrow[r, "1"'] & \mathbf{D}_{\langle 1\rangle}
\end{tikzcd}$$

both of which are over commutative diagrams in $\mathbf{\Gamma}$, and can therefore be filled in with natural isomorphisms of functors (in fact the second square commutes in $\mathbf{D}$ as well). We use these natural isomorphisms as the coordinates of a natural transformation filling the square in the larger diagram.

In particular, there is a natural isomorphism of functors filling the diagram

$$\begin{tikzcd}
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}\times 1"] & \mathbf{D}_{\langle 2\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r] \arrow[d, "m_!\times 1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 3\rangle} \arrow[d, "(\mu^1_{\langle 3\rangle})_!"] \arrow[dd, bend left=60, "(m_3)_!"] & \\
& \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"'] & \mathbf{D}_{\langle 2\rangle} \arrow[d, "m_!"] \arrow[r, Rightarrow] & {} \\
& & \mathbf{D}_{\langle 1\rangle} &
\end{tikzcd}$$

The bottom path is by definition the functor which on objects is

$$(A,B,C) \mapsto (A\otimes B)\otimes C.$$

Noting that the the composition of the top two functors is *equal* to ϕ_3^{-1}, by the splitting of our cleavage over static morphisms, we see that the top path is $(m_3)_! \circ \phi_3^{-1}$. This should be considered as a 'triple monoidal product', which we will denote $\otimes_3$. We may produce a similar diagram where the bottom path is the other association of monoidal product of three objects:

$$(A,B,C) \mapsto A\otimes (B\otimes C).$$

Namely,

$$\begin{tikzcd}
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "1\times\phi_2^{-1}"'] \arrow[rr, bend left=15, "\phi_3^{-1}"] & \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 2\rangle} \arrow[r] \arrow[d, "1\times m_!"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 3\rangle} \arrow[d, "(\mu^2_{\langle 3\rangle})_!"] \arrow[dd, bend left=60, "(m_3)_!"] & \\
& \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"'] & \mathbf{D}_{\langle 2\rangle} \arrow[d, "m_!"] \arrow[r, Rightarrow] & {} \\
& & \mathbf{D}_{\langle 1\rangle} &
\end{tikzcd}$$

The associator for the monoidal product, α, is the composition of the first natural transformation with the inverse of the second.

To show that the pentagon holds, note that there is a natural isomorphism from any association of a quadruple monoidal product to the 'quadruple monoidal product' $(m_4)_! \circ \phi_4^{-1}$. For example, the functor

$$(A, B, C, D) \mapsto ((A \otimes B) \otimes C) \otimes D$$

is the bottom path in the diagram

$$\begin{tikzcd}
\prod_1^4 \mathbf{D}_{\langle 1 \rangle} \arrow[r] \arrow[rrr, bend left, "\phi_4^{-1}"] & \mathbf{D}_{\langle 2 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r] \arrow[d, "(m)_! \times 1 \times 1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 3 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r] \arrow[d, "(\mu^1_{\langle 3 \rangle})_! \times 1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 4 \rangle} \arrow[d, "(\mu^1_{\langle 4 \rangle})_!"] \arrow[ddd, bend left=60, "(m_4)_!"] & \\
 & \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r] & \mathbf{D}_{\langle 2 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r] \arrow[d, "m_! \times 1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 3 \rangle} \arrow[r, Rightarrow] \arrow[d, "(\mu^1_{\langle 3 \rangle})_!"] & {} \\
 & & \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r, "\phi_2^{-1}"'] & \mathbf{D}_{\langle 2 \rangle} \arrow[d, "m_!"] & \\
 & & & \mathbf{D}_{\langle 1 \rangle} &
\end{tikzcd}$$

and we fill in the squares with natural isomorphisms in the same manner as the triple product case. The natural isomorphism on the right is the (unique) natural isomorphism

$$m_! \circ (\mu^1_{\langle 3 \rangle})_! \circ (\mu^1_{\langle 4 \rangle})_! \Rightarrow (m \circ \mu^1_{\langle 3 \rangle} \circ \mu^1_{\langle 4 \rangle})_! = (m_4)_!$$

There is a natural isomorphism from the functor

$$(A, B, C, D) \mapsto (A \otimes (B \otimes C)) \otimes D$$

to the quadruple product functor which comes from the diagram

$$\begin{tikzcd}
\prod_1^4 \mathbf{D}_{\langle 1 \rangle} \arrow[r] \arrow[rrr, bend left, "\phi_4^{-1}"] & \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 2 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r] \arrow[d, "1 \times (m)_! \times 1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 3 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r] \arrow[d, "(\mu^2_{\langle 3 \rangle})_! \times 1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 4 \rangle} \arrow[d, "(\mu^2_{\langle 4 \rangle})_!"] \arrow[ddd, bend left=60, "(m_4)_!"] & \\
 & \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r] & \mathbf{D}_{\langle 2 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r] \arrow[d, "m_! \times 1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 3 \rangle} \arrow[r, Rightarrow] \arrow[d, "(\mu^1_{\langle 3 \rangle})_!"] & {} \\
 & & \mathbf{D}_{\langle 1 \rangle} \times \mathbf{D}_{\langle 1 \rangle} \arrow[r, "\phi_2^{-1}"'] & \mathbf{D}_{\langle 2 \rangle} \arrow[d, "m_!"] & \\
 & & & \mathbf{D}_{\langle 1 \rangle} &
\end{tikzcd}$$

We now have two natural isomorphisms

$$((A \otimes B) \otimes C) \otimes D \Rightarrow (A \otimes (B \otimes C)) \otimes D$$

The first is $\alpha_{A,B,C} \times 1_D$, and the second is the natural isomorphism going to the quadruple product and back. However, examining the above diagrams component-wise, we see that all natural isomorphisms were produced by factorization of various co-cartesian morphisms through other co-cartesian morphisms. Since these factorizations are unique by the definition of an opfibration, the resulting natural isomorphisms must be equal. Let

$$\otimes_{\langle 4\rangle} := (m_4)_! \circ \phi_4^{-1} : \mathbf{D}_{\langle 4\rangle} \to \mathbf{D}_{\langle 1\rangle}.$$

We can produce analogous diagrams an use the same argument to show that the following diagram of natural transformations commutes

$$\begin{tikzcd}
& & ((\cdot_1 \otimes \cdot_2) \otimes \cdot_3) \otimes \cdot_4 \arrow[lld, Rightarrow, "\alpha"'] \arrow[d, Rightarrow] \arrow[rrd, Rightarrow, "\alpha\otimes 1"] & & \\
(\cdot_1 \otimes \cdot_2) \otimes (\cdot_3 \otimes \cdot_4) \arrow[rr, Rightarrow] \arrow[rd, Rightarrow, "\alpha"'] & & \otimes_{\langle 4\rangle} & & (\cdot_1 \otimes (\cdot_2 \otimes \cdot_3)) \otimes \cdot_4 \arrow[ll, Rightarrow] \arrow[ld, Rightarrow, "\alpha"] \\
& \cdot_1 \otimes (\cdot_2 \otimes (\cdot_3 \otimes \cdot_4)) \arrow[ru, Rightarrow] \arrow[rr, Rightarrow, "1\otimes\alpha"'] & & \cdot_1 \otimes ((\cdot_2 \otimes \cdot_3) \otimes \cdot_4) \arrow[lu, Rightarrow] &
\end{tikzcd}$$

The outer pentagon is the one from the pentagon axiom.

For the left unitor, consider the following diagram, which is over the obvious commutative diagram in products of objects in $\mathbf{\Gamma}$.

$$\begin{tikzcd}
& \mathbf{D}_{\langle 1\rangle} \arrow[ld, "(\iota_!{,}1)"'] \arrow[d] \arrow[dd, bend left=60, "1"] & \\
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"] \arrow[rd, "\otimes"'] & \mathbf{D}_{\langle 2\rangle} \arrow[r, Rightarrow] \arrow[d, "m_!"] & {} \\
& \mathbf{D}_{\langle 1\rangle} &
\end{tikzcd}$$

The natural isomorphism is induced by a factorization of pushforwards, the upper triangle commutes by the property of static morphisms, and the lower triangle commutes by definition. This natural isomorphism is the left unitor λ_L. The right unitor comes from the same diagram with $\iota_!$ and the identity switched in the upper diagonal functor. To see that the left an right unitors are coherent with the associator, consider the diagram (over the obvious commutative diagram)

$$\begin{tikzcd}
& \mathbf{D}_{\langle 1\rangle}\times\mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"] \arrow[dl, "1\times\iota\times 1"'] \arrow[d] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 2\rangle} \arrow[d] \arrow[ddd, bend left=80, "m_!"] & \\
\mathbf{D}_{\langle 1\rangle}\times\mathbf{D}_{\langle 1\rangle}\times\mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}\times 1"] & \mathbf{D}_{\langle 2\rangle}\times\mathbf{D}_{\langle 1\rangle} \arrow[r] \arrow[d, "m_!\times 1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 3\rangle} \arrow[d, "(\mu^1_{\langle 3\rangle})_!"'] \arrow[r, Rightarrow] \arrow[dd, bend left=60, "(m_3)_!"] & {} \\
& \mathbf{D}_{\langle 1\rangle}\times\mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"'] & \mathbf{D}_{\langle 2\rangle} \arrow[d, "m_!"] \arrow[r, Rightarrow] & {} \\
& & \mathbf{D}_{\langle 1\rangle} &
\end{tikzcd}$$

The path beginning with the upper-left $\mathbf{D}_{\langle 2\rangle}\times\mathbf{D}_{\langle 2\rangle}$ and going down around the outside of the diagram is the functor

$$(A,B)\mapsto(A\otimes\mathbb{1})\otimes B$$

and so the diagram exhibits a natural isomorphism from this functor to $m_!\circ\phi_2^{-1}=\otimes$. Restricting to factors, the natural transformations are all induced by co-cartesian factorizations through co-cartesian morphisms, as is the natural transformation $\lambda_R\times 1$. Therefore they are equal by uniqueness.

Taking the natural isomorphism resulting from removing the right-most 2-cell, and composing with the inverse of natural isomorphism filling the diagram

$$\begin{tikzcd}
& \mathbf{D}_{\langle 1\rangle}\times\mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"] \arrow[dl, "1\times\iota\times 1"'] \arrow[d] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 2\rangle} \arrow[d] & \\
\mathbf{D}_{\langle 1\rangle}\times\mathbf{D}_{\langle 1\rangle}\times\mathbf{D}_{\langle 1\rangle} \arrow[r, "1\times\phi_2^{-1}"] & \mathbf{D}_{\langle 1\rangle}\times\mathbf{D}_{\langle 2\rangle} \arrow[r] \arrow[d, "m_!\times 1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 3\rangle} \arrow[d, "(\mu^2_{\langle 3\rangle})_!"'] \arrow[dd, bend left=60, "(m_3)_!"] & \\
& \mathbf{D}_{\langle 1\rangle}\times\mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"'] & \mathbf{D}_{\langle 2\rangle} \arrow[d, "m_!"] \arrow[r, Rightarrow] & {} \\
& & \mathbf{D}_{\langle 1\rangle} &
\end{tikzcd}$$

we obtain, by definition, the natural transformation $\alpha_{\cdot,\mathbb{1},\cdot}$. Now composing this again with the natural isomorphism

$$\begin{tikzcd}
 & \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"] \arrow[dl, "1\times\iota\times1"'] \arrow[d] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 2\rangle} \arrow[d] \arrow[ddd, bend left=60, "m_!"] & \\
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "1\times\phi_2^{-1}"] & \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 2\rangle} \arrow[r] \arrow[d, "m_!\times1"'] \arrow[dr, Rightarrow] & \mathbf{D}_{\langle 3\rangle} \arrow[r, Rightarrow] \arrow[d, "(\mu^2_{\langle 3\rangle})_!"'] & {} \\
 & \mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"'] & \mathbf{D}_{\langle 2\rangle} \arrow[d, "m_!"] & \\
 & & \mathbf{D}_{\langle 1\rangle} &
\end{tikzcd}$$

which is, by the same argument as above, equal to $1 \times \lambda_L$, we obtain a natural isomorphism, equal to the composition

$$(1 \times \lambda_L) \circ \alpha_{\cdot,\mathbb{1},\cdot}$$

which is created entirely by co-cartesian factorizations. Hence, it must also be equal to $\lambda_R \times 1$, which proves the coherence of unitors and the associator.

In order to prove the hexagon axioms hold, we use the same strategy as in proving the pentagon axiom. For example, to obtain the first hexagon axiom. we produce a diagram

$$\begin{tikzcd}
 & (\cdot_1 \otimes \cdot_2) \otimes \cdot_3 \arrow[rr, Rightarrow, "\gamma"] \arrow[dl, Rightarrow, "\alpha"'] \arrow[dr, Rightarrow] & & \cdot_3 \otimes (\cdot_1 \otimes \cdot_2) \arrow[dl, Rightarrow] & \\
\cdot_1 \otimes (\cdot_2 \otimes \cdot_3) \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "1\otimes\gamma"'] & & \otimes_3 & & (\cdot_3 \otimes \cdot_1) \otimes \cdot_2 \arrow[ll, Rightarrow] \arrow[ul, Rightarrow, "\alpha"'] \\
 & \cdot_1 \otimes (\cdot_3 \otimes \cdot_2) \arrow[rr, Rightarrow, "\alpha^{-1}"'] \arrow[ur, Rightarrow] & & (\cdot_1 \otimes \cdot_3) \otimes \cdot_2 \arrow[ul, Rightarrow] \arrow[ur, Rightarrow, "\gamma\otimes1"'] &
\end{tikzcd}$$

which commutes since all natural isomorphisms were produced via co-cartesian factorizations. We remark also that the triangles with outer edge α commute by definition of α. The other hexagon is similar. ☐

Remark 2.2.9. In the proof of Theorem 2.2.8 we repeatedly used the fact that co-cartesian factorizations are unique (see Remark 2.1.2). This should be considered as a reflection of the fact that given two different choices of orderings and groupings of monoidal products of the same objects in a symmetric monoidal category, say for example

$$((A \otimes C) \otimes B) \otimes (E \otimes D)$$

and

$$((E \otimes B) \otimes (C \otimes D)) \otimes A$$

the space of isomorphisms between them generated by the data

$$\gamma, \lambda_L, \lambda_R, \alpha$$

forms a contractible groupoid (Maclane coherence). In particular, there can be no similar characterizan of braided monoidal categories as opfibrations, because in the absence of symmetry, one can form distinct isomorphisms from the data. The easiest example is of course

$$\gamma \cdot (\gamma \circ 1_{\text{flip}}) \neq 1.$$

Uniqueness of co-cartesian factorizations means that an opfibration cannot capture this.

Theorem 2.2.10. The category **SMCat** of symmetric monoidal categories and symmetric monoidal functors is equivalent to the category $\text{opFib}(\mathbf{\Gamma})_s$.

Proof. On objects, the functor $\mathbf{SMCat} \to \text{opFib}(\mathbf{\Gamma})_s$ was defined above by $\mathcal{M} \mapsto \mathcal{M}^{\otimes}$. On symmetric monoidal functors $F : \mathcal{M}' \to \mathcal{M}$ we define $F^{\otimes}$ on tuples by

$$F^{\otimes}(x_1, \cdots, x_k) = (Fx_1, \cdots, Fx_k)$$

and on morphisms by

$$F(\alpha, f_i) = (\alpha, Ff_i \circ \Psi_i)$$

Where Ψ_i is the (unique) isomorphism

$$\otimes_{j \in (\alpha^{-1}(i) \setminus \{*\})} F x'_j \to F \otimes_{j \in (\alpha^{-1}(i) \setminus \{*\})} x'_j$$

induced by the datum $F_{\otimes} : \otimes \circ (F \times F) \Rightarrow F \circ \otimes$ making F monoidal, and which is unique since F is symmetric monoidal. The functoriality of $F^{\otimes}(\cdot)$ is a consequence of the naturality of the structure isomorphisms and coherence. The functoriality of $F \mapsto F^{\otimes}$ on objects is clear, and on morphisms again follows from naturality and coherence.

The functor going in the opposite direction $\text{opFib}(\mathbf{\Gamma})_s$ on objects is restriction to the fibre $\mathbf{D} \to \mathbf{\Gamma} \mapsto \mathbf{D}_{\langle 1 \rangle}$, which has a symmetric monoidal structure as shown in Theorem 2.2.8.

Suppose now that $\mathbf{C}, \mathbf{D} \in \text{opFib}(\mathbf{\Gamma})_s$, and let $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ be a fibred functor. We want to show that the restriction $\mathcal{F}_{\langle 1 \rangle}$ has the structure of a symmetric monoidal functor. Indeed, let $*_{\mathbf{C}} \in \mathbf{C}_{\langle 0 \rangle}$ denote the unique object, so that $\iota_!(*_{\mathbf{C}})$ is the monoidal unit of $\mathbf{C}_{\langle 1 \rangle}$. Similarly, let $*_{\mathbf{D}} \in_{\langle 1 \rangle}$ denote the unique object. Then (see Remark 2.1.2) there is a unique isomorphism

$$(\mathcal{F}_{\langle 1 \rangle})_{\otimes} : \mathcal{F}\iota_!(*_{\mathbf{C}}) \to \iota_!(\mathcal{F}*_{\mathbf{C}}) = \iota_!(*_{\mathbf{D}})$$

induced as usual by cocartesian factorization. We have abused notation in the above slightly, by giving pushforwards in $\mathbf{C}$ and $\mathbf{D}$ the same names, and note also we needed the fact that

$\mathcal{F}$ sends cocartesian morphisms to cocartesian morphisms. Now, the following diagram of functors is over the obvious commutative diagram of products of objects of $\mathbf{\Gamma}$

$$\begin{tikzcd}
\mathbf{C}_{\langle 1\rangle} \times \mathbf{C}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"] \arrow[d, "\mathcal{F}_{\langle 1\rangle} \times \mathcal{F}_{\langle 1\rangle}"'] & \mathbf{C}_{\langle 2\rangle} \arrow[r, "m_!"] \arrow[d, "\mathcal{F}_{\langle 2\rangle}"] & \mathbf{C}_{\langle 1\rangle} \arrow[d, "\mathcal{F}_{\langle 1\rangle}"] \\
\mathbf{D}_{\langle 1\rangle} \times \mathbf{D}_{\langle 1\rangle} \arrow[r, "\phi_2^{-1}"'] & \mathbf{D}_{\langle 2\rangle} \arrow[r, "m_!"'] & \mathbf{D}_{\langle 1\rangle}
\end{tikzcd}$$

Therefore, it can be filled in with a natural isomorphism of functors

$$(\mathcal{F}_{\langle 1\rangle})_{\otimes} : \mathcal{F}_{\langle 1\rangle} \circ m_! \circ \phi_2^{-1} \Rightarrow m_! \circ \phi_2^{-1} \circ (\mathcal{F}_{\langle 1\rangle} \times \mathcal{F}_{\langle 1\rangle}).$$

We have produced the data required for $\mathcal{F}_{\langle 1\rangle}$ to be symmetric monoidal, and as in the proof of Theorem 2.2.8, the paths in the diagrams required are all various combinations of co-cartesian factorizations, and therefore must all agree.

The composition

$$\mathcal{M} \mapsto \mathcal{M}^{\otimes} \mapsto (\mathcal{M}^{\otimes})_{\langle 1\rangle}$$

is clearly the identity. Also, the isomorphisms ϕ_n given as part of the data of an object $\mathbf{D} \in \mathrm{opFib}(\mathbf{\Gamma})_s$ give a fibred equivalence $\mathbf{D}^{\otimes}_{\langle 1\rangle} \Rightarrow \mathbf{D}$, and these equivalences assemble to form a natural isomorphism to the identity functor. This completes the proof. □

Corollary 2.2.11. Applying Theorem 2.1.19 in the non-symmetric monoidal case (and switching to the opfibration case) to the result of the previous theorem we obtain an equivalence of categories between **SMCat** and the full 2-subcategory of pseudofunctors, pseudonatural transformations, and modifications $\mathbf{\Gamma} \to \mathbf{Cat}$ whose objects F have the properties

- ◇ For each n,

  $$(F\pi^1_{\langle n\rangle}, \cdots, F\pi^n_{\langle n\rangle}) : F(\langle n\rangle) \to \prod_1^n F(\langle 1\rangle)$$

  is an isomorphism of categories.

- ◇ $F|_{\mathbf{\Gamma}_s}$ is functorial.

Remark 2.2.12. In [Lur], it is claimed in the discussion beginning chapter 2, with a proof sketched, that the second condition is unnecessary, and that isomorphism in the first condition can be weakened to equivalence. In the sequel we will therefore adopt these changes.

This brings us to the equivalence

Theorem 2.2.13. There is an equivalence of 2-categories between **SMCat** and the full category of pseudofunctors $\mathbf{\Gamma} \to \mathbf{Cat}$, pseudonatural transformations, and modifications, whose objects F have the property that

$$(F\pi^1_{\langle n\rangle}, \cdots, F\pi^n_{\langle n\rangle}) : F(\langle n\rangle) \to \prod_1^n F(\langle 1\rangle)$$

is an equivalence of categories for each n.

2.3 Pseudofunctors and Enriched Functors

There is one remaining step, namely to replace the pseudofunctors with functors enriched in categories, or in other words, strictify the pseudofunctors. An analogous construction to the one below appears in [GJ]. As always, the notation $[n]$ will refer to the linearly ordered set

$$0 < 1 < \cdots < n.$$

In this section we will treat these as categories.

Also for this section we temporarily use the notation $s, t : [0] \to [1]$ for the inclusion of the source and target object of the single nonidentity morphism in $[1]$.

We next remark that the commutative diagram

$$\begin{tikzcd}
{[0]} \arrow[r, "s"] \arrow[d, "t"] & {[1]} \arrow[d] \\
{[1]} \arrow[r] & {[2]}
\end{tikzcd}$$

is a pushout in **Cat**. The upshot is that giving a pair of composable morphisms in some category $\mathcal{C}$, i.e. a pair of functors $f, g : [1] \to \mathcal{C}$ with $g \circ s = f \circ t$, is equivalent to giving a functor $[2] \to \mathcal{C}$. Thus $[2]$ should be thought of as a category modeling '2 composable morphisms', and similarly, $[n]$ is 'n composable morphisms'.

Given a category $\mathcal{J}$, we construct a category $\lhd\mathcal{J}$ enriched in categories as follows. The objects of $\lhd\mathcal{J}$ are the objects of $\mathcal{J}$. Given $j', j \in \mathcal{J}$, the category

$$\lhd\mathcal{J}(j', j)$$

has as objects functors $c : [n] \to \mathcal{J}$, with the property that $c(0) = j'$ and $c(n) = j$. In other words, the objects are chains of composable morphisms whose composition is a morphism

$j' \to j$. The morphisms of $\lhd\mathcal{J}(j', j)$ consist of commutative diagrams

$$\begin{tikzcd}
{[n']} \arrow[dd, "\alpha"'] \arrow[dr, "c'"] & \\
& J \\
{[n]} \arrow[ur, "c"'] &
\end{tikzcd}$$

where $\alpha : [n'] \to [n]$ has the property that $\alpha(0) = 0$ and $\alpha(n') = n$, that is, α is endpoint-preserving. We will sometimes refer to such an α as a *path homotopy.*

Now, given two composable chains of composable morphisms,

$$c' : [n'] \to \mathcal{J} \in \lhd\mathcal{J}(j'', j') \text{ and } c : [n] \to \mathcal{J} \in \lhd\mathcal{J}(j', j)$$

we define the composition functor

$$\lhd\mathcal{J}(j'', j') \times \lhd\mathcal{J}(j', j) \to \lhd\mathcal{J}(j'', j)$$

to be concatenation, that is, the functor defined on objects by

$$c' \circ c(k) \begin{cases} c'(k) & \text{if } k \leq n' \\ c(k - n') & \text{if } k \geq n' \end{cases}$$

Note that $c'(n') = c(0) = j'$ by hypothesis. We define composition on morphisms also to be (horizontal) concatenation of the α's. This composition functor has two-sided inverses given by the 'object' functors $[0] \to \mathcal{J}$. In particular, observe that these are distinct from functors $[1] \to \mathcal{J}$ that hit identities.

By definition of the morphisms of $\lhd\mathcal{J}(j', j)$, there are no morphisms between chains with distinct compositions. Let $\lhd\mathcal{J}(j', j)_f$ denote the subcategory of chains that compose to $f \in \mathcal{J}(j', j)$. Note that $[1]$ is initial in the category with objects $[n]$ and morphisms endpoint preserving functors, so that $\lhd\mathcal{J}(j', j)_f$ has initial object the functor $[1] \to \mathcal{J}$ with image f. We conclude that there is an equivalence of categories

$$\lhd\mathcal{J}(j', j)_f \cong \{f, 1_f\}, \text{ and therefore } \lhd \mathcal{J}(j', j) \cong \mathcal{J}(j', j)$$

the latter a set considered as a discrete category. To put this another way, it is clear how to use this construction to make a functor $\lhd : \mathbf{Cat} \to \mathbf{Cat}_{\mathbf{Cat}}$, and we have shown that $(\pi_0)_* \circ \lhd = 1_{\mathbf{Cat}}$ (see Remark A.1.3).

The point of this construction is the following theorem.

Theorem 2.3.1. Let $\mathbf{C} \in \mathbf{Cat}_{\mathbf{Cat}}$ be a strict 2-category, and let $\mathcal{J}$ be a small category. The following 2-categories are isomorphic.

- Lax functors $\mathcal{J} \to \mathbf{C}$, lax transformations, and modifications.

  and

- Enriched functors $\triangleleft\mathcal{J} \to \mathbf{C}$ (i.e. strict 2-functors), lax transformations, and modifications.

Proof. Let $F : \mathcal{J} \to \mathbf{C}$ be a lax functor. We define a functor $\tilde{F} : \triangleleft\mathcal{J} \to \mathbf{C}$, which does the same thing on objects as F. Given an object in some morphism category

$$c : [n] \to \mathcal{J} \in \triangleleft\mathcal{J}(j', j),$$

we set

$$\tilde{F}(c) = F(c(f_n)) \circ \cdots \circ F(c(f_1)),$$

where f_k is the unique morphism $k - 1 \to k$ in $[n]$. The composition on the right is taken to be the identity if the chain is empty, i.e. if $n = 0$. This is required, since empty chains are the identities, and $\tilde{F}$ is meant to be strict. Finally, we define the action of $\tilde{F}$ on morphisms of chains by defining it on generators. It is well known that a functor $\alpha : [n'] \to [n]$, i.e. a morphism of $\mathbf{\Delta}$, can be decomposed into a sequence of cofaces $d_i^{[m]} : [m-1] \to [m]$ defined by

$$d^i_{[m]}(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$

where $0 \leq i \leq m$, and codegeneracies $s^i_{[m]} : [m+1] \to [m]$:

$$s^i_{[m]}(k) = \begin{cases} k & \text{if } k < i \\ k-1 & \text{if } k \geq i \end{cases}$$

where $0 \leq i \leq m$. It is not hard to see that the endpoint-preserving functors α are generated by all codegeneracies and only the cofaces $d^i_{[n]}$ with $0 < i < n$. So, given a commutative diagram D:

$$\begin{tikzcd}
{[n-1]} \arrow[dr, "c'"] \arrow[dd, "d^i_{[n]}"'] & \\
& J \\
{[n]} \arrow[ur, "c"'] &
\end{tikzcd}$$

noting that commutativity here implies

$$c'(f_i) = c(f_{i+1}) \circ c(f_i)$$

we define

$$\tilde{F}(D) = 1_{F(c(f_1))} \circ \cdots \circ F_{c(f_i), c(f_{i+1})} \circ \cdots \circ 1_{F(c(f_n))}$$

Similarly, given a diagram D'

$$\begin{tikzcd}
{[n+1]} \arrow[dd, "s^j_{[n]}"'] \arrow[dr, "c'"] & \\
& J \\
{[n]} \arrow[ur, "c"'] &
\end{tikzcd}$$

define

$$\tilde{F}(D') = 1_{F(c(f_1))} \circ \cdots \circ F_{1_{c(i)}} \circ \cdots 1_{F(c(f_n))}$$

We need to check that the cosimplicial identities hold for this definition, in the sense that if a diagram D like those above can be produced with the vertical morphism being decomposed in two different ways into generators, the definitions of $\tilde{F}$ resulting from these decompositions agree. We recall these cosimlpicial identities (see [JT], for example):

$$\begin{array}{lll}
s^k_{[n-1]} d^k_{[n]} = 1 = s^k_{[n-1]} d^{k+1}_{[n]} & & \\
d^k_{[n+1]} d^i_{[n]} = d^i_{[n+1]} d^{k-1}_{[n]} & & \text{if } i < k \\
s^k_{[n]} d^i_{[n-1]} = d^i_{[n]} s^{k-1}_{[n-1]} & & \text{if } i < k \\
s^k_{[n]} d^i_{[n-1]} = d^{i-1}_{[n]} s^k_{[n-1]} & & \text{if } i > k+1 \\
s^k_{[n]} s^i_{[n+1]} = s^i_{[n+1]} s^{k+1}_{[n]} & & \text{if } i \leq k
\end{array}$$

The first pair of identities hold because of the compatibility of $F_{f,1_y}$ with F_{1_y} and the compatibility of $F_{1_x,f}$ with F_{1_x}, respectively. The second identity holds for $i = k - 1$ by the compatibility of $F_{\cdot,\cdot}$ with triples, and holds for other values of i by the interchange law of horizontal and vertical composition of 2-morphisms. The other three identities all hold by interchange as well. This establishes that $\tilde{F}$ is well-defined as a functor on the hom-categories of $\triangleleft J$. It is clear that $\tilde{F}$ is compatible with concatenation of strings by definition, so it is an enriched functor.

Since the construction $F \mapsto \tilde{F}$ above used all of the data which established F as a pseudofunctor, it is injective. Now, given an enriched functor $G : \triangleleft J \to \mathbf{C}$, we produce a lax functor that does the same thing on objects, and sends a morphism to G applied to the diagram $[1] \to \triangleleft J$ representing that morphism. Diagrams

$$\begin{tikzcd}
{[1]} \arrow[dd, "d^1_{[2]}"'] \arrow[dr, "c'"] & \\
& J \\
{[2]} \arrow[ur, "c"'] &
\end{tikzcd}$$

give the action on pairs of composable morphisms, and diagrams

$$\begin{tikzcd}
{[1]} \arrow[dd, "s^0_{[0]}"'] \arrow[dr, "c'"] & \\
& J \\
{[0]} \arrow[ur, "c"'] &
\end{tikzcd}$$

give the unitors. The identities

$$\begin{aligned}
s^1_{[1]} d^1_{[2]} &= 1 \\
s^0_{[1]} d^1_{[2]} &= 1 \\
d^2_{[3]} d^1_{[2]} &= d^1_{[3]} d^1_{[2]},
\end{aligned}$$

give the compatibility conditions of a lax functor (in the same order as presented in Definition 2.1.12). This is clearly the inverse of the construction $F \mapsto \tilde{F}$, so we have a bijection.

Now, if $\alpha : F \Rightarrow G$ is a lax transformation we define $\tilde{\alpha}$ to act the same on objects. If

$$c : [n] \to \mathcal{J} \in \triangleleft\mathcal{J}(j', j),$$

is a (nonempty) composable chain, we set

$$\tilde{\alpha}(c) := \alpha(c(f_n)) \bullet_{\alpha(c(n-1))} \cdots \bullet_{\alpha(c(n))} \alpha(c(f_1))$$

where the notations means we are vertically composing along 1-morphisms, as in the following diagram for composable morphisms $g' : j'' \to j'$ $g : j' \to j$.

$$\begin{tikzcd}
\tilde{F}(j'') \arrow[r, "\alpha(j'')"] \arrow[d, "\tilde{F}(g')"'] & \tilde{G}(j'') \arrow[d, "\tilde{G}(g')"] \\
\tilde{F}(j') \arrow[r, "\alpha(j')"'] \arrow[ur, Rightarrow, "\alpha(g')"] \arrow[d, "\tilde{F}(g)"'] & \tilde{G}(j') \arrow[d, "\tilde{G}(g)"] \\
\tilde{F}(j) \arrow[r, "\alpha(j)"'] \arrow[ur, Rightarrow, "\alpha(g)"'] & \tilde{G}(j)
\end{tikzcd}$$

For an empty chain $\mathrm{id}_j : [0] \to \mathcal{J}$ hitting an object j (recall these are the identities w.r.t. concatenation), we set $\tilde{\alpha}(\mathrm{id}_j) = 1_{\alpha j}$. Note that this is a requirement, since $\tilde{F}$ and $\tilde{G}$ are strict.

The above assignments are clearly compatible with concatenation of chains, and $\tilde{F}$ and $\tilde{G}$ are strictly functorial, so $\tilde{\alpha}$ is functorial on 1-morphisms automatically. Finally, naturality of $\tilde{\alpha}$ with respect to 2-morphisms is equivalent to functoriality of α on 1-morphisms.

Since we utilized all the data provided by α to construct $\tilde{\alpha}$, this assignment is injective. Furthermore, the only condition on α we 'did not' need to produce $\tilde{\alpha}$ is its naturality on 2-morphisms, but this condition is vacuous, since $\mathcal{J}$ has only trivial 2-morphisms. So this assignment is surjective as well. A more explicit argument for surjectivity can be established by constructing an inverse to $\alpha \mapsto \tilde{\alpha}$.

Finally, if $m : \alpha \to \beta$ is some modification we make $\tilde{m}$ do the same thing to objects. On chains of length zero, naturality is clear, and on chains of length one, naturality follows from from that of m. Now, if

$$c : [n] \to \mathcal{J} \in \lhd\mathcal{J}(j', j),$$

is a general chain, then

$$\tilde{\alpha}(c(n)) \circ \tilde{F}(c) = \alpha(c(n)) \circ F(c(f_n)) \circ \cdots \circ F(c(f_1)).$$

And from similar formulas, one sees that the outer rectangle of the diagram

$$\begin{tikzcd}
\alpha(c(n)) \circ F(c(f_n)) \circ \cdots \circ F(c(f_1)) \arrow[r, Rightarrow, "m(c(n))\circ 1"] \arrow[d, Rightarrow, "\alpha(c(f_n))\circ 1"'] & \beta(c(n)) \circ F(c(f_n)) \circ \cdots \circ F(c(f_1)) \arrow[d, Rightarrow, "\beta(c(f_n))\circ 1"] \\
G(c(f_n)) \circ \alpha(c(n-1)) \circ F(c(f_{n-1})) \circ \cdots \circ F(c(f_1)) \arrow[r, Rightarrow, "1\circ m(c(n-1))\circ 1"'] \arrow[d, Rightarrow, "1\circ\alpha(c(f_{n-1}))\circ 1"'] & G(c(f_n)) \circ \beta(c(n-1)) \circ F(c(f_{n-1})) \circ \cdots \circ F(c(f_1)) \arrow[d, Rightarrow, "1\circ\beta(c(f_{n-1}))\circ 1"] \\
\vdots \arrow[d, Rightarrow, "1\circ\alpha(c(f_2))\circ 1"'] & \vdots \arrow[d, Rightarrow, "1\circ\beta(c(f_2))\circ 1"] \\
G(c(f_n)) \circ \cdots \circ G(c(f_2)) \circ \alpha(c(1)) \circ F(c(f_1)) \arrow[r, Rightarrow, "1\circ m(c(2))\circ 1"] \arrow[d, Rightarrow, "1\circ\alpha(c(f_1))"'] & G(c(f_n)) \circ \cdots \circ G(c(f_2)) \circ \beta(c(1)) \circ F(c(f_1)) \arrow[d, Rightarrow, "1\circ\beta(c(f_1))"] \\
G(c(f_n)) \circ \cdots \circ G(c(f_1)) \circ \alpha(c(0)) \arrow[r, Rightarrow, "1\circ m(c(0))"'] & G(c(f_n)) \circ \cdots \circ G(c(f_1)) \circ \beta(c(0))
\end{tikzcd}$$

is precisely the diagram

$$\begin{tikzcd}
\tilde{\alpha}(j) \circ \tilde{F}(c) \arrow[r, Rightarrow, "\tilde{m}(j)\circ 1"] \arrow[d, Rightarrow, "\tilde{\alpha}(c)"'] & \tilde{\beta}(c) \circ \tilde{F}(c) \arrow[d, Rightarrow, "\tilde{\beta}(c)"] \\
\tilde{G}(c) \circ \tilde{\alpha}(j') \arrow[r, Rightarrow, "1\circ\tilde{m}(j')"'] & \tilde{G}(c) \circ \beta(j')
\end{tikzcd}$$

whose commutativity verifies the naturality of $\tilde{m}$. Each square above commutes by repeated application of the naturality of m. Again, we used all data and conditions in the construction $m \mapsto \tilde{m}$, so this is a bijection.

This completes the proof. □

In words, the construction $\mathcal{J} \mapsto \lhd\mathcal{J}$ gives us a way to strictify lax functors, at least when the domain is an ordinary category, and leave lax transformations and modifications untouched.

Remark 2.3.2. If one prefers to work with normalized lax functors (corresponding to the normalized opfibrations over $\mathbf{\Gamma}$ representing symmetric monoidal categories), one can restrict the morphisms in $\triangleleft\mathcal{J}$ to be only *semi-simplicial* α, i.e. strictly order preserving. In this case the proof of Theorem 2.3.1 goes through unchanged, with the cosimplicial identities involving the $s^i_{[j]}$ deleted. We will refer to this 2-category as $\triangleleft_s\mathcal{J}$.

When applied to the situation with symmetric monoidal categories, we have now shown that there is an embedded subcategory of lax functors, lax transformations, and modifications $\triangleleft\mathbf{\Gamma} \to \mathbf{Cat}$ which is equivalent to **SMCat**. There is now the extra condition on objects (lax functors) and 1-morphisms (lax transformations) that they happen to be pseudo—i.e. the coherence 2-morphisms they hit are all invertible. We note that this can be enforced by only considering the invertible symmetric monoidal natural transformations **SMCat** to begin with.

2.4 Symmetric Monoidal Segal Spaces

In this section we arrive at the main goals of this chapter: a definition of symmetric monoidal $(\infty, 1)$-category, and the symmetric monoidal $(\infty, 1)$-category induced by an ordinary symmetric monoidal category. The model of $(\infty, 1)$-categories that these will be built on is that of (complete) Segal spaces. See Appendix A.5.1. We will use the language of enriched categories extensively in this section. See Appendix A.1 for a review, and for explanation of our notation.

Definition 2.4.1. We define an S-category $\mathbf{A}$ to be a *simplicially enriched category* which is tensored and cotensored over **sSet**. We denote the hom-objects for $A, B \in \mathbf{A}$ by

$$\mathbf{A}_{\mathbf{sSet}}(A, B),$$

and for $S_\bullet \in \mathbf{sSet}$, the cotensor and tensor structure by

$$\mathbf{sSet}^{\mathbf{A}}(S_\bullet, A), \qquad S_\bullet \otimes A,$$

respectively. In other words, S-categories are the objects of a full subcategory of $\mathbf{Cat}_{\mathbf{sSet}}$. We will call enriched functors between S-categories simplicial functors.

We will use **SSp** to denote the full subcategory of **ssSet** whose objects are simplicial Segal spaces. Recall our notation N_* for hom-wise nerve (Remark A.1.3).

Definition 2.4.2. A *symmetric monoidal simplicial Segal space* $\boldsymbol{S}^{\otimes}$ is a simplicial functor

$$\boldsymbol{S}^{\otimes} : \mathrm{N}_*(\triangleleft\mathbf{\Gamma}) \to \mathbf{SSp}$$

Such that

$$\left(\boldsymbol{S}^{\otimes}\pi^1_{\langle n\rangle}, \cdots, \boldsymbol{S}^{\otimes}\pi^n_{\langle n\rangle}\right) : \boldsymbol{S}^{\otimes}(\langle n\rangle) \to \prod_1^n \boldsymbol{S}^{\otimes}(\langle 1\rangle)$$

is a Reedy weak equivalence. Here, by $\pi^k_{\langle n\rangle}$, we mean the 0-simplex in

$$\mathrm{N}\left(\lhd\boldsymbol{\Gamma}(\langle n\rangle, \langle 1\rangle)\right)$$

represented by the functor $[1] \to \boldsymbol{\Gamma}$ whose image is $\pi^k_{\langle n\rangle}$.

Remark 2.4.3. In Theorem 2.3.1 we showed that enriched functors with source $\lhd\mathcal{J}$ corresdpond to lax functors with source $\mathcal{J}$. What we really want to generalize is pseudofunctors. However, as we show in Remark 2.4.10, **SSp** is locally Kan, and therefore the 2-morphisms and higher are automatically invertible. In this case lax and pseudo amount to the same thing.

As motivation for this definition, we explain how to make a symmetric monoidal simplicial Segal space out of an ordinary symmetric monoidal category. To wit, let $\mathcal{C} \in \mathbf{SMCat}$. Using the constructions in the previous sections we get an enriched functor

$$\widetilde{\mathcal{C}^{\otimes}} : \lhd\boldsymbol{\Gamma} \to \mathbf{Cat}$$

now we apply hom-wise-nerve N_* and post-compose with N_S from Appendix A.5.1 to obtain a symmetric monoidal simplicial Segal space defined by the composite

$$\mathrm{N}_*(\lhd\boldsymbol{\Gamma}) \xrightarrow{\mathrm{N}_*\widetilde{\mathcal{C}^{\otimes}}} \mathrm{N}_*\,\mathbf{Cat} \xrightarrow{\mathrm{N}_S} \mathbf{SSp}$$

From A.5.12, the equivalence condition satisfied by $\widetilde{\mathcal{C}^{\otimes}}$ translates into the Reedy equivalence condition we are after. Actually, the result lands in complete simplicial Segal spaces, and so should perhaps be called a symmetric monoidal complete simplicial Segal space.

We now approach the question of what morphisms should be allowed between symmetric monoidal simplicial Segal spaces.

In Appendix A.3, we show that the standard formula expressing natural transformations as ends can be upgraded to expressing lax transformations and modifications as a lax end:

$$\mathbf{Lax}(F, G) \cong \oint_{c\in\mathbf{C}} \mathbf{D}_{\mathbf{Cat}}(Fc, Gc).$$

Our next goal in this section is to define a simplicial analogue to lax end. That is, where the categories in question are enriched in **sSet** in a way compatible with their categorical structure. The notion we are after, called a simplicially coherent end, is introduced and explored in detal in [CP].

Let $\mathbf{A}$ be a small S-category. For $A, B \in \mathbf{A}$, form the bisimiplicial set $X(A, B)$ with n^{th} space

$$X(a, b)_n := \coprod_{A_0, \cdots, A_n \in \mathbf{A}} \mathbf{A}_{\mathbf{sSet}}(A, A_0) \times \mathbf{A}_{\mathbf{sSet}}(A_0, A_1) \times \cdots \times \mathbf{A}_{\mathbf{sSet}}(A_n, B).$$

Define the face maps

$$d_j^{[n]} : X(A, B)_n \to X(A, B)_{n-1}$$

via composition

$$\mathbf{A}_{\mathbf{sSet}}(A_{j-1}, A_j) \times \mathbf{A}_{\mathbf{sSet}}(A_j, A_{j+1}) \to \mathbf{A}_{\mathbf{sSet}}(A_{j-1}, A_{j+1}),$$

and the degeneracy maps $s_i^{[n]}$ via insertion of identity

$$1_{A_i} : \Delta(0) \to \mathbf{A}_{\mathbf{sSet}}(A_i, A_i).$$

Now define $\hat{\mathbf{A}}(A, B) := \operatorname{diag} X(A, B)$.

Example 2.4.4. Let $\mathbf{A}$ be a small category, and consider it as an S-category with discrete simplicial hom-objects. Then $\hat{\mathbf{A}}(A, B)$ is the nerve of the category of objects under A and over B.

Example 2.4.5. For simplicial categories of the form $\mathrm{N}_*(\lhd\mathcal{J})$, $X(j', j)_{m,n}$ is the set of n composable chains of composable chains of morphisms (of arbitrary length) in $\mathcal{J}$, each of which have m path homotopies between them. After concatenating the path homotopies, and the chains, we get a simplicial set map

$$\widehat{\mathrm{N}_*(\lhd\mathcal{J})}(j', j) \to \mathrm{N}(\lhd\mathcal{J}(j', j)).$$

Note that there is a map

$$\mathrm{N}(\lhd\mathcal{J}(j', j)) \to \widehat{\mathrm{N}_*(\lhd\mathcal{J})}(j', j),$$

Which sends a composable chain of path homotopies of chains to the same thing in $\widehat{\mathrm{N}_*(\lhd\mathcal{J})}(j', j)$ with identities $[0] \to \mathcal{J}$ hitting j concatenated to get the same number as there are path homotopies. The comopsition

$$\mathrm{N}(\lhd\mathcal{J}(j', j)) \to \widehat{\mathrm{N}_*(\lhd\mathcal{J})}(j', j) \to \mathrm{N}(\lhd\mathcal{J}(j', j))$$

is the identity. For the other direction, the map

$$\widehat{\mathrm{N}_*(\lhd\mathcal{J})}(j', j) \to \mathrm{N}(\lhd\mathcal{J}(j', j)) \to \widehat{\mathrm{N}_*(\lhd\mathcal{J})}(j', j)$$

sends a chain of composable chains with path homotopies to the same thing composed into the first variable, with identities on the end. We construct a homotopy

$$\widehat{\mathrm{N}_*(\lhd\mathcal{J})}(j', j) \times \Delta(1) \to \widehat{\mathrm{N}_*(\lhd\mathcal{J})}(j', j)$$

which 'pushes a chain off the last factor' by concatenating the first two chains and then appending an identity chain. This specifies where to send the simplicies of the form $(x, 0)$, $(x, 1)$, and we send simplicies of the form (x, f) (f the 1-simplex of $\Delta(1)$) to to x with an identity concatenated at the end. Since the face maps are composition (in the horizontal direction and the path homotopy direction) the last face map composes an identity which sends the simplex back to x, and the others compose in the middle resulting in a chain with an identity on the end, as required. We then construct similar homotopies which push a chain of each factor until it is only in the first factor. This establishes that there is a weak equivalence

$$\widehat{\mathrm{N}_*(\lhd\mathcal{J})}(j', j) \simeq \mathrm{N}(\lhd\mathcal{J}(j', j)).$$

In fact, it is a deformation retract onto the target.

Definition 2.4.6. Let $\mathbf{A}, \mathbf{C}$ be S-categories, with $\mathbf{A}$ small. Let

$$F : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{C}$$

be a simplicial functor. Define the *simplicially coherent end* of F to be the object

$$\oint_{A \in \mathbf{A}} F(A, A) := \int_{(A'', A') \in \mathbf{A}^{\mathrm{op}} \times \mathbf{A}} \mathbf{sSet}^{\mathbf{A}}(\hat{\mathbf{A}}(A'', A'), F(A'', A')),$$

of $\mathbf{C}$, provided it exists. We will usually simply say *coherent end.*

Note that we are abusing notation here, in using the same notation for a coherent end as a lax end. Motivation for this abuse is the following fact. In [CP], through detailed investigation of the low-dimensional simplicies, they show (see discussion following Proposition 1.2 of [CP]):

Proposition 2.4.7. Let $\mathbf{A}, \mathbf{C} \in \mathbf{Cat}_{\mathbf{Cat}}$, and let $F : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{C}$ be an enriched functor. Then there is an isomorphism

$$\mathrm{N}_* \oint_{A \in \mathbf{A}} F(A, A) \cong \oint_{A' \in \mathrm{N}_* \mathbf{A}} \mathrm{N}_* F(A', A'),$$

of objects of $\mathrm{N}_* \mathbf{C}$. The left-hand side is a lax end, the right-hand side is a coherent end.

This, together with the formulas expressing natural and lax transformations as ends and lax ends, suggests a definition of coherent transformation. Again, the following construction is studied in detail in [CP].

Definition 2.4.8. Let $\mathbf{A}, \mathbf{C}$ be S-categories, and let $F, G : \mathbf{A} \to \mathbf{C}$ be simplicial functors. We define the simplicial set of *coherent transformations* from F to G to be

$$\mathrm{Coh}(F, G) := \oint_{A \in \mathbf{A}} \mathbf{C}_{\mathbf{sSet}}(FA, GA).$$

As usual, the right hand side indicates the coherent end of the composition

$$\mathbf{C}_{\mathbf{sSet}} \circ (F^{\mathrm{op}} \times G).$$

Let us prove Proposition 2.4.7 in the special case where $\mathbf{A} = \lhd\mathcal{J}$, and where

$$H = \underline{\mathrm{Fun}} \circ (F^{\mathrm{op}} \times G) : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Cat},$$

so here $\mathbf{C} = \mathbf{Cat} \in \mathbf{Cat}_{\mathbf{Cat}}$. Logically, this is the only case we need to motivate 2.4.9 below.

Proof. Here we need the fact that fundamental category $\pi_1 : \mathbf{sSet} \to \mathbf{Cat}$, is left-adjoint to nerve (see [JT]), so that (in what follows, we are using hom-*sets*, and $\mathbf{Cat}$ and $\mathrm{N}_* \mathbf{Cat}$ have the same hom-sets).

$$\begin{aligned}\mathrm{N}_* \mathbf{Cat}(X, \mathbf{sSet}^{\mathrm{N}_* \mathbf{Cat}}(S_\bullet, Y)) \cong& \mathbf{sSet}(S_\bullet, \mathrm{N}\, \underline{\mathbf{Cat}}(X, Y)) \\ \cong& \,\mathrm{N}_* \mathbf{Cat}(\pi_1 S_\bullet, \underline{\mathbf{Cat}}(X, Y)).\end{aligned}$$

Setting $X = [0]$ gives the result

$$\mathbf{sSet}^{\mathrm{N}_* \mathbf{Cat}}(S_\bullet, Y) = \mathrm{N}_* \mathbf{Cat}(\pi_1 S_\bullet, Y).$$

So, in this case the universal wedge on the right hand side of the formula in 2.4.7 is of the form ($X \in \mathbf{Cat}$ since the objects of $\mathbf{Cat}$ and $\mathrm{N}_* \mathbf{Cat}$ are the same):

$$\begin{aligned}\omega(j', j) : X \Rightarrow& \mathbf{sSet}^{\mathrm{N}_* \mathbf{Cat}}(\mathrm{N} \lhd\mathcal{J}(j', j), \underline{\mathbf{Cat}}(Fj', Gj)) \\ \cong& \underline{\mathbf{Cat}}(\pi_1 \mathrm{N} \lhd\mathcal{J}(j', j), \underline{\mathbf{Cat}}(Fj', Gj)) \\ \cong& \underline{\mathrm{Fun}}(Fj' \times \lhd\mathcal{J}(j', j), Gj).\end{aligned}$$

In the first step we have already used the result from Example 2.4.5, and in the last step we used Lemma 4.3.4, which implies that the counit of the adjunction π_1, N is the identity. Formally, $\omega(j', j)$ is a morphism in $\mathrm{N}_* \mathbf{Cat}$, but again, this is just a functor. We construct from this a lax wedge $\tilde{\omega} : X \Rightarrow H$. For $f : x' \to x$ in X, and $g : j' \to j$ in $\mathcal{J}$, define

$$\begin{aligned}\tilde{\omega}(j)(x) &= \omega(j, j)(x \times j) \\ \tilde{\omega}(j)(f) &= \omega(j, j)(f \times 1_j) \\ \tilde{\omega}(g)(x) &= \omega(j', j)(1_x \times g).\end{aligned}$$

Functoriality of $\tilde{\omega}(j)$ follows from functoriality of $\omega(j, j)$. Naturality of $\tilde{\omega}(g)$ follow from the identity

$$(f \times 1_j) \circ (1_x \times g) = (1_y \times g) \circ (f \times 1_{j'})$$

and combined with the coherence of $\omega(\cdot, \cdot)$ with morphisms. Finally, condition 1 of Definition A.3.1 follows from functoriality of $\omega(j', j)$ in the second variable (compatibility with chains), and condition 2 follows similarly. Universality of this lax wedge follows directly from universality of ω, so the proof is complete. ☐

Definition 2.4.9. Let $\boldsymbol{S}^\otimes, \boldsymbol{T}^\otimes$ be symmetric monoidal Segal spaces. A *symmetric monoidal functor* $\mathcal{F} : \boldsymbol{S}^\otimes \to \boldsymbol{T}^\otimes$ is a 0-simplex of $\mathrm{Coh}(\boldsymbol{S}^\otimes, \boldsymbol{T}^\otimes)$.

Remark 2.4.10. Note that coherent ends are a generalization of lax ends, for which we had the formula (Theorem A.3.3)

$$\mathbf{Lax}(F, G) \cong \oint_{c\in\mathbf{C}} \mathbf{D}_{\mathbf{Cat}}(Fc, Gc).$$

In particular, the left hand side is *lax* transformations, and not pseudonatural transformations. So, it might seem that the Definition 2.4.9 is actually of *lax* symmetric monoidal functors. However, by Remark A.4.17,

$$\mathbf{ssSet}_{\mathbf{sSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet)$$

is Kan, provided that $\boldsymbol{S}_\bullet$ is Reedy cofibrant (an empty condition) and $\boldsymbol{T}_\bullet$ is Reedy fibrant (true of a simplicial Segal space $\boldsymbol{T}_\bullet$). So, we have

$$\mathrm{Coh}(\boldsymbol{S}^\otimes, \boldsymbol{T}^\otimes) \cong \int_{(\langle n'\rangle, \langle n\rangle)} \underline{\mathbf{sSet}}\big(\mathrm{N}\left(\lhd\boldsymbol{\Gamma}(\langle n'\rangle, \langle n\rangle)\right), \mathbf{ssSet}_{\mathbf{sSet}}\left(\boldsymbol{S}^\otimes(\langle n'\rangle), \boldsymbol{T}^\otimes(\langle n\rangle)\right)\big)$$

we see that the 1-simplicies of

$$\widehat{\mathrm{N}_*(\lhd\boldsymbol{\Gamma})}(\langle n'\rangle, \langle n\rangle) \cong \mathrm{N}\left(\lhd\boldsymbol{\Gamma}(\langle n'\rangle, \langle n\rangle)\right),$$

which correspond to the morphisms of $\boldsymbol{\Gamma}$, are mapped into a Kan complex, and so hit invertible morphisms. This is analagous to the situation of a lax end of a functor to a target that happens to be enriched in groupoids—they are automatically iso-lax. Another way to think of this is that since **SSp** is locally Kan, it is an $(\infty, 1)$-category (see Appendix A.5.2). In particular there are no non-invertible 2-morphisms to hit.

Remark 2.4.11. One defect of the above definition is that there is no canonical way to compose symmetric monoidal functors. However, in [CP] §4, a *choice* of composition is explained, which is associative up to homotopy. The discussion is reminiscent of quasi-categories, and we make the conjecture that it is possible to construct a quasi-category of symmetric monoidal simplicial Segal spaces. We will not need that construction here, since we will be interested in field theories, which have a fixed target, and so will not need to be composed.

Remark 2.4.12. As further motivation for our definition, note that in [Lur], Lurie gives a definition of symmetric monoidal quasi-category that is analogous to the opfibration characterization of a symmetric monoidal category, in Definition 2.0.0.7. In some sense, the entire book is devoted to showing this definition is reasonable. Also, in [Lur2] (Theorem 3.2.0.1), Lurie proves an ∞ analogue to the Grothendieck isomorphism discussed in Section 2.1. If one applies this to the ∞-opfibration definition, it results in a defintion which looks very much like ours, although with **SSp** replaced by the category of quasi-categories as a model for the "category of $(\infty,1)$-categories". The difference is that the domain is $\mathfrak{C}\,\mathrm{N}\,\mathbf{\Gamma}$ instead of our $\mathrm{N}_* \lhd \mathbf{\Gamma}$. To complete the association, one can show that $\mathrm{N}_* \lhd_s \mathbf{\Gamma}$ is DK-equivalent to a simplicial resolution to a certain comonad on **Cat**, and that this in turn is isomorphic to $\mathfrak{C}\,\mathrm{N}\,\mathbf{\Gamma}$ ([Rie], and see [Lur2] for an explanation of the notation $\mathfrak{C}$). We prefer our construction, since $\mathrm{N}_* \lhd \mathbf{\Gamma}$ is locally a quasi-category, and so behaves like a kind of $(\infty,2)$-category, whereas $\mathfrak{C}\,\mathrm{N}\,\mathbf{\Gamma}$ is not (again see [Rie]). We also remark that the full subcategory of **sSet** whose objects are quasi-categories is *not* locally Kan. So in order to make an analogue of Definition 2.4.9 using quasi-categories in place of Segal spaces, one must discard non-invertible 1-morphisms in the morphism spaces.

Next, note that there is an obvious candidate for the Segal space of symmetric monoidal functors between two symmetric monoidal simplicial Segal spaces. Indeed, take $\boldsymbol{S}^{\otimes}, \boldsymbol{T}^{\otimes}$ to be symmetric monoidal simplicial Segal spaces, and define

$$\underline{\mathrm{Fun}}^{\otimes}(\boldsymbol{S}^{\otimes}, \boldsymbol{T}^{\otimes}) := \oint_{\langle n\rangle} \underline{\mathbf{ssSet}}(\boldsymbol{S}^{\otimes}(\langle n\rangle), \boldsymbol{T}^{\otimes}(\langle n\rangle)).$$

Now, for particular $\langle n'\rangle, \langle n\rangle$, the integrand is

$$\underline{\mathbf{ssSet}}\left(\mathrm{disc}\,\widehat{\mathbf{ssSet}}(\boldsymbol{S}^{\otimes}(\langle n'\rangle), \boldsymbol{T}^{\otimes}(\langle n\rangle)), \underline{\mathbf{ssSet}}(\boldsymbol{S}^{\otimes}(\langle n'\rangle), \boldsymbol{T}^{\otimes}(\langle n\rangle))\right).$$

Since the category of simplicial Segal spaces is cartesian closed (see [Rez]), this is pointwise a Segal space, and therefore the Segal map is pointwise a weak equivalence. Note that the k^{th} space of the coherent end is given by

$$\underline{\mathbf{ssSet}}\left(\mathrm{disc}\,\Delta(k), \oint_{\langle n\rangle} \underline{\mathbf{ssSet}}(\boldsymbol{S}^{\otimes}(\langle n\rangle), \boldsymbol{T}^{\otimes}(\langle n\rangle))\right).$$

In particular it is a right adjoint, and so commutes with coherent end, which is a particular end. The fibred products appearing in the target of the Segal map also commute with coherent ends. We may then invoke the fact that a coherent end of a pointwise weak equivalence is a weak equivalence, which is Corollary 2.4 of [CP], to obtain that this definition actually yields a simplicial Segal space of functors.

Finally, we note that it is easily to formulate the definitions of

- ◇ Symmetric monoidal complete simplicial Segal spaces.
- ◇ Symmetric monoidal Segal spaces.
- ◇ Symmetric monoidal complete Segal spaces.

by replacing **SSp** in Definition 2.4.2 with the appropriate target. Recall our notation **Sp** for the category of Segal spaces (Appendix A.5.1). We also define symmetric monoidal functors of such in the same way as Definition 2.4.9. First, we note that to consider segal Spaces as a simplicial category, we use Lemma 4.1.2 to enrich them over **Top**, and then apply Sing_* to enrich them over **sSet**. Also note that the Segal space of monoidal functors can be defined similarly as above.

Chapter 3

Field Theories

The goal of this chapter is to give complete definitions of smooth and homotopy field theories. A vital ingredient to each of these definitions is the corresponding bordism category. We will also call these the smooth and homotopy bordism categories, respectively. After defining the field theories, we devote two sections (4 and 5), to explaining the machinery that produces a homotopy field theory from a smooth one in full. Finally, we analyze this map in low dimensions, where both smooth and homotopy field theories have been classified.

3.1 The Smooth Bordism Category

The goal of this section is to present the definition of the Stolz-Teichner bordism category. Here we choose the version presented in [STH], which is a symmetric monoidal fibred category over the category **Man** of smooth manifolds and smooth maps. There is a slightly more complicated version presented in [ST], which we will discuss in item 2 of Section 4.6.

We will call this the *smooth bordism category* because it is fibred over **Man**, the category of smooth manifolds and functions, and also because functors fibred over **Man** should be thought of as smooth functors in some sense.

We will begin by describing the bordism category where the fibred aspect is absent for the sake of clarity. In other words, we begin with the description of the objects over the point **pt** ∈ **Man**, and morphisms over $1_{\mathbf{pt}}$. We will denote this category by d-$\mathbf{B}_{\mathbf{pt}}$. We remark that the [STH] gives the definition of the Riemannian bordism category; other than that difference, we will use the same notation so that the reader has less difficulty going there for more details.

Definition 3.1.1. The objects of the category d-$\mathbf{B}_{\mathbf{pt}}$ are quadruples $(Y, Y^c, Y^{\pm})$, where Y is a smooth d-manifold without boundary, although not necessarily compact, as in the figure below, and $Y^c \subset Y$ is a compact codimension one submanifold, called the *core* of Y. The data $Y^{\pm}$ are a decomposition of the complement of the core, i.e.

$$Y \setminus Y^c = Y^+ \coprod Y^-.$$

With $Y^{\pm} \subset Y$ are open such that their closures contain Y^c.

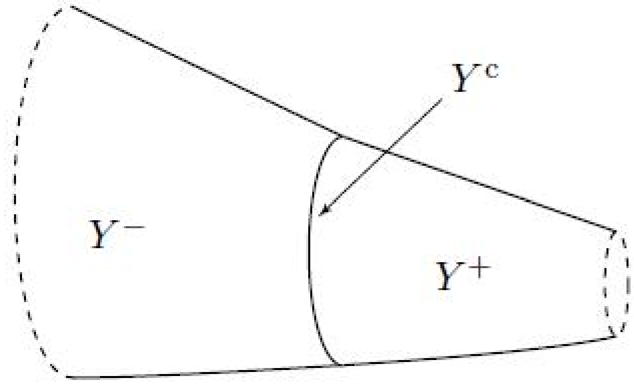

Figure 3.1: An object $(Y, Y^c, Y^{\pm})$ of 2-$\mathbf{B}$

An *isomorphism* in d-$\mathbf{B_{pt}}$ from $(Y_0, Y_0^c, Y_0^{\pm})$ to $(Y_1, Y_1^c, Y_1^{\pm})$ is the germ of a diffeomorphism $\phi : W_0 \to W_1$, where $W_j \subset Y_j$ are open neighborhoods of the cores Y_j^c. The diffeomorphism is required to preserve the cores, i.e. $\phi(Y_0^c) = Y_1^c$ and also preserve the decompositions $W_j^{\pm} := W_j \cap Y_j^{\pm}$ of the complements of the cores in these restricted neighborhoods. When we use the terminology *germ*, we mean that, as usual, two diffeomorphisms that agree on a smaller open neighborhood of Y_0^c are identified. Under disjoint union, the above forms a symmetric monoidal groupoid. We will sometimes refer to an object $(Y, Y^c, Y^{\pm})$ simply by Y.

Note that for the above definition to work, we are considering the empty set as a smooth manifold of every dimension, and this is the monoidal unit under disjoint union. We next introduce more general morphisms of d-$\mathbf{B_{pt}}$.

Definition 3.1.2. A *bordism* from $Y_0 = (Y_0, Y_0^c, Y_0^{\pm})$ to $(Y_1, Y_1^c, Y_1^{\pm})$ is a triple (Σ, i_0, i_1), where Σ is a smooth d-manifold, and i_j are smooth maps $i_j : W_j \to \Sigma$. As above, $W_j \subset Y_j$ are open neighborhoods of the cores. Define $W_j^{\pm} := W_j \cap Y_j^{\pm}$. The data are subject to the following conditions.

1. The restrictions i_j^+ of i_j to W_j^+ are embeddings into $\Sigma \setminus i_1(W_1^- \cup Y_1^c)$,
2. The *core* $\Sigma^c := \Sigma \setminus \big(i_0(W_0^+) \cup i_1(W_1^-)\big)$ is compact.

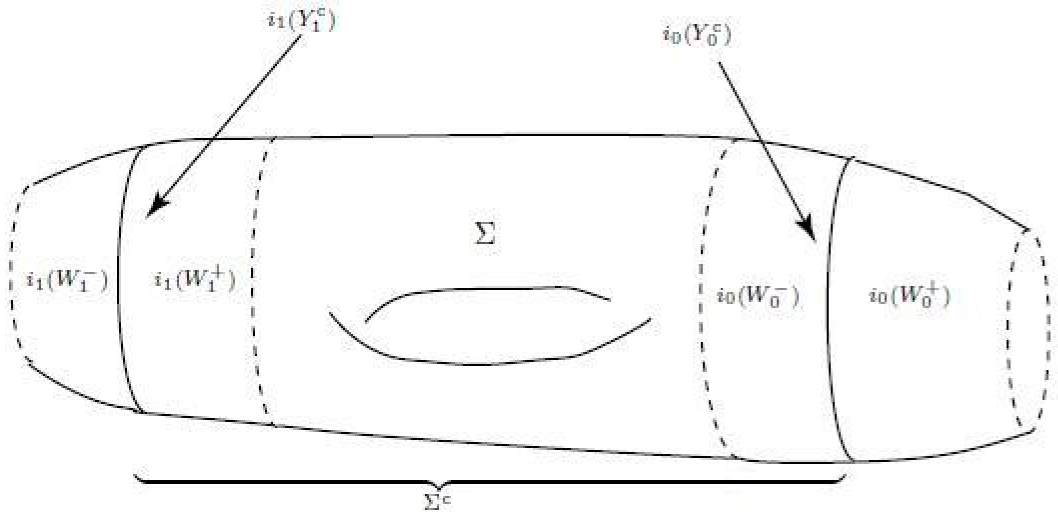

Figure 3.2: A bordisim (Σ, i_0, i_1) in 2-**B**

Now given a bordism $(\Sigma, i_0, i_1) : Y_0 \to Y_1$ and a bordism $(\Sigma', i_1', i_2') : Y_1 \to Y_2$, the composition is defined as follows. As part of the data for the two bordisms Σ, Σ', we have the smooth maps $i_1 : W_1 \to \Sigma$ and $i_1' : W_1' \to \Sigma'$, where $W_1, W_1' \subset Y_1$ are open neighborhoods of the core Y_1^c. Set $W_1'' := W_1 \cap W_1'$. The first condition above guarantees that i_1 and i_1' restrict to embeddings of $(W_1'')^+ := W_1'' \cap Y_1^+$. Using these embeddings, we glue Σ and Σ':

$$\Sigma' \circ \Sigma := \left(\Sigma' \setminus i_1'((W_1')^+ \setminus (W_1'')^+)\right) \coprod_{(W_1'')^+} \left(\Sigma \setminus i_1(W_1^- \cup Y_1^c)\right).$$

The maps $i_0 : W_0 \to \Sigma$ and $i_2 : W_2 \to \Sigma'$ are restricted to smaller open neighborhoods, say U_0 and U_2, to obtain smooth maps to $\Sigma' \circ \Sigma$. The embedding and compactness conditions above are satisfied, and furthermore $\Sigma' \circ \Sigma$ is a manifold by Remark 53 and Lemma 54 of [STH].

An *isomorphism* between Riemannian bordisms $\Sigma, \Sigma' : Y_0 \to Y_1$ is a germ of a triple of diffeomorphisms

$$F : X \to X', \quad f_0 : V_0 \to V_0', \text{ and } f_1 : V_1 \to V_1',$$

where $X \subset \Sigma$ is an open neighborhood of Σ^c, V_j are open neighborhoods of Y_j^c contained in $W_j \cap i_j^{-1}(X)$, and similarly for X', V_0', V_1'. The maps f_j are required to induce isomorphisms of d-$\mathbf{B_{pt}}$ as in Definition 3.1.1, and the following diagram is required to commute.

$$\begin{tikzcd}
V_1 \arrow[r, "i_1"] \arrow[d, "f_1"'] & X \arrow[d, "F"] & V_0 \arrow[l, "i_0"'] \arrow[d, "f_0"] \\
V_1' \arrow[r, "i_1'"'] & X' & V_0' \arrow[l, "i_0'"]
\end{tikzcd}$$

Again, germ means that two such triples (F, f_0, f_1) and (G, g_0, g_1) are identified if the maps agree on smaller open domains that still contain the various cores.

Isomorphisms of the form $(F, \mathrm{id}_{V_0}, \mathrm{id}_{V_1})$ are said to be isomorphisms *rel boundary.*

We remark that the isometries of objects described in Definition 3.1.1 are recovered as a special case of the bordisms defined above: given objects Y_0, Y_1 and an isomorphism represented by $f : W_0 \to W_1$, we can define a bordism $\Sigma = W_1$, $i_1 = \mathrm{id}_{W_1}$, and $i_0 = f$.

Definition 3.1.3. The category d-$\mathbf{B_{pt}}$ has objects as defined in Definition 3.1.1, and morphisms rel boundary isomorphism classes of bordisms as in Definition 3.1.2.

We now turn our attention to the full definition of d-$\mathbf{B}$ as a category fibred over **Man**.

Definition 3.1.4. An object in d-$\mathbf{B}$ consists of the data $(S, Y, p, Y^c, Y^{\pm})$, where S is a smooth manifold (of any dimension), $p : Y \to S$ is a fibre bundle with fibres d-dimensional smooth manifolds without boundary, and Y^c is a sub-bundle of Y which is fibre-wise codimension 1. We also impose the condition that $Y^c \to S$ is proper, so that, in particular, each fibre is compact. As before, $Y^{\pm}$ is a decomposition of $Y \setminus Y^c$ into two open subsets that contain Y^c in their closure. The projection functor $\pi : d\text{-}\mathbf{B} \to \mathbf{Man}$ sends an object $(S, Y, p, Y^c, Y^{\pm})$ to S.

Consider a smooth map $f : S \to S'$ of manifolds, and objects (S, Y) and (S', Y'), of d-$\mathbf{B}$ as defined above. We can get another object over S by pulling back all of the data over S' associated to (S', Y'), that is, the object $(S, f^*(Y'))$. We denote this object by (S, Y_1). Then a *family of bordisms* from (S, Y_0) to (S', Y') consists of the quadruple $(\Sigma, p_\Sigma, i_0, i_1)$ where

- ◇ $p_\Sigma : \Sigma \to S$ is a smooth fibre bundle of d-dimensional manifolds.
- ◇ For $j = 0, 1$, $i_j : W_j \to \Sigma$ are smooth maps over S, where W_j are open sub-bundles of Y_j which are fibrewise neighborhoods of the cores Y^c.

We also impose two conditions, which are the bundle versions of those for bordisms over **pt**.

1. Let $i_j^{\pm}$ be the restrictions of i_j to $W_j^{\pm} := W_j \cap Y_j^{\pm}$. Then i_j^+ is a smooth embedding of W_j^+ into $\Sigma \setminus i_1(W_1^- \cup Y_1^c)$.

   and

2. The restriction of p_Σ to the *core* $\Sigma^c := \Sigma \setminus \big(i_0(W_0^+) \cup i_1(W_1^-)\big)$ is proper.

We define isomorphisms of objects and morphisms (rel boundary) exactly as in Definitions 3.1.1 and 3.1.2 with the (germs of) diffeomorphisms being over S, and with the bordism families lying over the same $f : S \to S'$. Finally, a morphism of d-$\mathbf{B}$ is an isomorphism class rel boundary of families of bordisms as defined above. The action of the projection functor $\pi : d\text{-}\mathbf{B} \to \mathbf{Man}$ is clearly built in to the definition of family of bordisms, as $f : S \to S'$ was part of the data, and we specified that isomorphism families lie over the same base, so this functor is well-defined.

Generalizing the notation d-$\mathbf{B}_{\mathbf{pt}}$ for the fibre over the point, we write d-$\mathbf{B}_S$ for the fibre over a fixed base S, in particular with morphisms over id_S. The operation of fibre-wise disjoint union of objects and bordism families induces the structure of a symmetric monoidal fibred category on d-$\mathbf{B} \to \mathbf{Man}$. The monoidal unit section is the empty bundle. We will call d-$\mathbf{B}$ the *smooth d-dimensional bordism category.*

Remark 3.1.5. We note that in Definition 3.1.4, d-$\mathbf{B}$ has been defined *explicitly* as something in the image of the Grothendieck construction of 2.1.15. Namely, the morphisms in d-$\mathbf{B}$ lying over non-identity morphisms f of $\mathbf{Man}$ are pairs (f, Σ), where Σ is a morphism from the source to the pullback along f of the target, so that Σ essentially lies over an identity.

Definition 3.1.6. For $X \in \mathbf{Man}$, we define the *smooth d-dimensional bordism category over X*, denoted d-$\mathbf{B}(X)$, in the same manner as in Definition 3.1.4, except that objects (S, Y) have the additional datum of a smooth map $Y \to X$, bordism families (f, Σ) have the datum $\Sigma \to X$ as well, and all data involving maps must all be over X. Note in particular that in defining bordism families over nonidentity maps f, we pull back the map to X, and furthermore we must define isomorphisms of bordism families rel boundary *and over X* to define the morphisms of d-$\mathbf{B}(X)$.

Remark 3.1.7. The reason for including the data involving germs of collars in the definition of d-$\mathbf{B}$ is to provide something to glue along when composing bordisms. The astute reader may point out that collars are not really necessary in d-$\mathbf{B}$—when working up to diffeomorphism rel boundary, glueing smoothly along boundary components of manifolds is well-defined. We choose to retain the extra complexity for two reasons. First, it is clear how to define different smooth bordism categories where the bordisms are given extra structure such as a Riemannian metric. In this case collars *are* necessary to obtain a well-defined glueing, so the definition with collars is easier to generalize. Second, the eventual goal motivated by the ideas in this work is to extend down to glue along arbitrary codimension. In this case, collar-like constructions are needed as well. For the case of codimension 2, see [SP].

3.2 Smooth Field Theories

In the most general setting, a *smooth field theory* (over X) will be a symmetric monoidal fibred functor d-$\mathbf{B}(X) \to \mathbf{E}$, where $\mathbf{E}$ is an arbitrary symmetric monoidal fibred category over $\mathbf{Man}$. The reason for this flexibility is the cobordism hypothesis 3.6.1; we are comparing these smooth field theories with the 'homotopy field theories' of Lurie, which can be classified easily for general target. When we turn to calculation in Section 3.6 however, we will want use the standard target $\mathbf{Vect}_{\mathbf{Man}}$, which is a symmetric monoidal fibred category over $\mathbf{Man}$ (see Remark 2.1.8). We will agree to fix this definition for our purposes here.

Definition 3.2.1. A *smooth field theory over X of dimension d* is a symmetric monoidal fibred functor $\mathcal{F}$

$$\begin{tikzcd}
d\text{-}\mathbf{B} \arrow[rr, "\mathcal{F}"] \arrow[dr] & & \mathbf{Vect}_{\mathbf{Man}} \arrow[dl] \\
& \mathbf{Man} &
\end{tikzcd}$$

We note that field theories are the objects in a symmetric monoidal category, with the morphisms given by symmetric monoidally fibred natural transformations, and the monoidal structure given by pointwise monoidal product.

Remark 3.2.2. Eventually, a more elaborate definition of smooth field theory will be given, which will 'extend down to points' instead of stopping at codimension 1 submanifolds. The definition above should then be called a *nonextended* field theory. We will briefly discuss that situation in Remark 3, but unless explicitly mentioned otherwise, our smooth field theories will only extend down to codimension 1.

As discussed at the end of Section 2.1, there is an inner-hom in $\mathrm{Fib}^{\otimes}(\mathbf{Man})$. Applying this observation to the situation of smooth field theories, one obtains a symmetric monoidal fibred category of smooth field theory. We conclude this brief section with notation for this gadget.

Definition 3.2.3. The symmetric monoidal fibred category of d-dimensonal smooth field theories over X is defined as follows.

$$d\text{-}\mathbf{TFT}(X) := \underline{\mathrm{Fun}}^{\otimes}_{\mathbf{Man}}(d\text{-}\mathbf{B}(X), \mathbf{Vect}_{\mathbf{Man}})$$

Recall that a fibre over $S \in \mathbf{Man}$ is given by

$$d\text{-}\mathbf{TFT}(X)_S = \mathrm{Fun}^{\otimes}_{\mathbf{Man}}(d\text{-}\mathbf{B}(X), \underline{\mathrm{Fun}}_{\mathbf{Man}}(\underline{S}, \mathbf{Vect}_{\mathbf{Man}})),$$

where the monoidal structure on the target is defined pointwise. As an example, $d\text{-}\mathbf{TFT}(X)_{\mathbf{pt}}$ is exactly the symmetric monoidal category discussed at the end of Definition 3.2.1.

To complete the unwinding of definitions, note that

$$\underline{\mathrm{Fun}}_{\mathbf{Man}}(\underline{S}, \mathbf{Vect}_{\mathbf{Man}})_T := \mathrm{Fun}_{\mathbf{Man}}(\underline{S \times T}, \mathbf{Vect}_{\mathbf{Man}}).$$

This, by pulling back along the map to $S \times T$, is equivalent to the category of vector bundles over $S \times T$. In this way we see that smooth field theories over some base S 'stick in' an extra factor of S on the output vector bundles.

3.3 The Homotopy Bordism Category

The goal of this section is to present the (nonextended) version of the bordism category utilized in [Lur2]. In the paper, it is treated as a Segal space, and in particular its symmetric monoidal structure is not formalized. Following the premise of [Rez], that the category of bisimplicial sets with the complete Segal space model structure (see Appendix A.5.1) is a model for the 'homotopy theory of homotopy theory', we will call the bordism category presented in this section the *homotopy bordism category.*

Definition 3.3.1. Let V be a finite dimensional (real) vector space. Let n be a non-negative integer. We define $d\text{-}\mathbf{Cob}_n^V$ as a set to be the collection of pairs

$$(M, \{t_0 \leq \cdots \leq t_n\})$$

where M is a d dimensional submanifold of $V \times \mathbb{R}$ (without boundary, possibly noncompact), and $t_j \in \mathbb{R}$ for $j = 0, 1, \cdots, n$. Let π_M be the restriction to M of the projection $\pi_{\mathbb{R}} : V \times \mathbb{R} \to \mathbb{R}$. These pairs are additionally subject to the following conditions.

- ◇ The map π_M is proper.
- ◇ The critical values of π_M are disjoint from the set $\{t_0 \leq \cdots \leq t_n\}$.

This set can be given a topology induced by the topology of the space of embeddings $M \to V \times \mathbb{R}$. We refer the reader to [Lur2] and the appendix of [Gal] for more details on this topology.

When thinking of the points of $d\text{-}\mathbf{Cob}_n^V$, the reader should imagine a sequence of n composed bordisms. The second condition says that $\pi_M^{-1}\{t_j\}$ are codimension 1 submanifolds around which the projection π_M is a trivial fibre bundle. The properness condition says that in particular, inverse images $\pi_M^{-1}[t_j, t_{j+1}]$ are the usual classical compact bordisms with boundary, such as the ubiquitous 'pair of pants'. The vector space V serves only to induce a topology on $d\text{-}\mathbf{Cob}_n^V$ (and to keep the collection $(M, \{t_0 \leq \cdots \leq t_n\})$ small enough to be a set). We now remove dependence on V by taking a direct limit.

Definition 3.3.2. We define the space $d\text{-}\mathbf{Cob}_n$ to be the direct limit of $d\text{-}\mathbf{Cob}_n^V$ over all finitie dimensional vector spaces. Note that $d\text{-}\mathbf{Cob}_\bullet$ is a simplicial space.

Remark 3.3.3. We recall the well-known fact that if M is a compact manifold, then the direct limit of spaces of embeddings of M into finite vector spaces V taken over all V serves as a model of EDiff(M). In particular, it has a free action of Diff(M) on it (apply a diffeomorphism and then the embedding), and quotienting out by this action results in a space BDiff(M) which classifies bundles with fibre M. One can alternatively remove the action of Diff(M) by taking the spaces of *embedded* manifolds i.e. submanifolds. With this in mind we see that $d\text{-}\mathbf{Cob}_n$ is supposed to be a classifying space of fibre bundles of n 'glued' bordisms.

Remark 3.3.4. By pushing the nontrivial parts of the manifolds in d-$\mathbf{Cob}_n$ that are outside of $[t_0, t_n]$ away to infinity, it is not hard to believe that the Segal condition is satisfied, so that d-$\mathbf{Cob}_\bullet$ forms a Segal space (see Definition A.5.6). It is however, not complete (see Warning 2.2.8 of [Lur2]). We note that we have departed somewhat from the notation of [Lur2], in that the simplicial space we have called d-**Cob** is, in his language $\mathbf{PreCob}(d)$ or sometimes $\mathbf{PreCob}(d)^{un}$ to emphasize that the manifolds are unoriented. It is not explicilty mentioned, but d-**Cob** would be, in Lurie's notation the completion of this Segal space. We choose to remove this subtlety, since we will always be mapping d-**Cob** into complete Segal spaces, and the results are identical with or without completion.

As with d-**B**, more structure on the manifolds M can easily be incorporated into the definition, such as geometry, orientations, or maps to some other fixed manifold X. We briefly discuss the two of the most common structures that [Lur2] uses and give them notation.

Definition 3.3.5. Let $M \in \mathbf{Man}$, and let $E \to M$ be a k-dimensional vector bundle over M. The *frame bundle* of E is the $\mathrm{GL}_k(\mathbb{R})$-principal bundle of *frames*, or ordered bases, of the fibres. A *framing* of M is a section of the frame bundle of the tangent bundle of M, or in other words a trivialization.

Definition 3.3.6. The Segal space d-$\mathbf{Cob}^{\mathrm{fr}}$ is defined in the same way as d-**Cob**, except that the manifolds are all equipped with framings.

Definition 3.3.7. Let $M \in \mathbf{Man}$, let X be a space, and let $\zeta \to X$ be a n-dimensional vector bundle on X. Assume that $\dim M = m \leq n$. An (X, ζ)-*structure* on M is the data of

- A continuous map $f : M \to X$
- An isomorphism of vector bundles

$$TM \oplus \underline{\mathbb{R}}^{n-m} \cong f^*\zeta$$

over M. Here $\underline{\mathbb{R}}^k$ denotes the trivial bundle of dimension k.

Definition 3.3.8. Let X be a space and let ζ be a vector bundle of dimention at least d over X. The Segal space d-$\mathbf{Cob}(X, \zeta)$ is defined in the same way as d-**Cob**, except that the manifolds are equipped with (X, ζ) structures.

We now turn our attention to the construction of d-$\mathbf{Cob}^{\otimes}$. Let V be an n-dimensional framed vector space, and consider the space of framing preserving smooth embeddings

$$\coprod_{1}^{k} V \to V.$$

Let us call this space E_k^V. By contracting the images of the embedded copies of V along framing preserving homotopies, one sees that this space has the homotopy type of $F_k(V; G)$, the space of k ordered points in V, each labelled with an element of the group of framing preserving diffeomorphisms $V \to V$. The group of diffeomorphisms $V \to V$ is homotopy equivalent to $\mathrm{Gl}(n)$, via the map that sends an isomorphism to its derivative at 0. Under this map, framing preserving diffeomorphisms are sent to diagonal matrices with positive entries. As this space is contractible, we conclude that G is contractible, so that $F_k(V; G)$ is homotopy equivalent to $F_k(V)$, the space of k ordered points in V.

Now, observe that there is a fibration $F_k(V) \to F_{k-1}(V)$ which is induced by 'forgetting where the last point is'. The fibre is all possible places to put the last point, which is V with $k-1$ points removed. This is homotopy equivalent to a wedge of $k-1$ spheres of dimension $n-1$ so we have the fibration

$$\begin{tikzcd}
\bigvee_{k-1} S^{n-1} \arrow[r] & F_k(V) \arrow[d] \\
 & F_{k-1}(V)
\end{tikzcd}$$

Now, $F_1(V) \simeq V \simeq *$, and for large n, S^{n-1} is highly connected. Looking at the long exact sequence of homotopy groups for a fibration, and inducting on k, we conclude that E^n_k becomes highly connected as $n \to \infty$. If $V \subseteq V'$ is an inclusion of framed vector spaces, there is an obvious inlcusion $E^V_k \to E^{V'}_k$, so it makes sense to take the direct limit over all finite dimensional vector spaces to produce a contractible space E^∞_k. These spaces form an operad via composition; in fact they are an infinite version of the little cubes operad of Boardman and Vogt [BV]. The full structure of an operad will not be important for us, but we note that E^∞_k, as an operad, comes with composition maps

$$E^\infty_k \times \left(E^\infty_{j_1} \times \cdots \times E^\infty_{j_k}\right) \to E^\infty_{j_1+\cdots+j_k}$$

Since the spaces E^∞_k are contractible, we may extend solid diagrams

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & E^\infty_k \\
\Delta^n \arrow[ur, dotted] &
\end{tikzcd}$$

to the diagonal dotted arrow. We use this repeatedly to produce the following data, in which we use the notation $(k) = [k] \setminus \{0\}$.

1. A point p_k in each space E^∞_k. We choose p_1 to correspond to the identity.

2. The points p_k, together with the composition maps of E^∞_k produce many more points of E^∞_k. The first iteration can be indexed as follows. Let $f : (k) \to (k')$ be a surjective set map. Then we have an induced operad composition map

$$E^\infty_{k'} \times \left(E^\infty_{|f^{-1}(1)|} \times \cdots \times E^\infty_{|f^{-1}(k')|}\right) \to E^\infty_k .$$

Plugging the points chosen in 1 into this map results in a point $p_{f,k'}$ in E_k^∞. Similarly, given a composable chain of surjective set maps

$$(k) \xrightarrow{f_1} (k_1) \xrightarrow{f_2} \cdots \xrightarrow{f_n} (k_n),$$

one obtains a point $p_{f_1,\cdots,f_n,k_n}$ in E_k^∞ by feeding the original chosen points from 1 through n compositions of the operad $E_\bullet^\infty$. We will abuse notation, by denoting the unique surjective set map $(k) \to (1)$ by k. In this way, we have, for each composable chain of surjective set maps whose first source is (k) and whose ultimate target is (1), a point $p_{\dots} \in E_k^\infty$.

3. Consider a commutative diagram

$$\begin{tikzcd}
{[s']} \arrow[dd, "\alpha"'] \arrow[dr] & \\
& \mathbf{Set} \\
{[s]} \arrow[ur] &
\end{tikzcd}$$

with α endpoint preserving, i.e. a path homotopy, and with the initial source and final target of $[s']$ and $[s]$ being (k) and (1), respectively. In other words, this is a morphism in

$$\triangleleft\mathbf{Set}((k),(1)).$$

From 2, there are points in E_k^∞ induced by the functors from $[s]$ and $[s']$ to **Set**, which we will denote by $p_{[s]}$ and $p_{[s']}$. For each α as above, we choose a path from $p_{[s']}$ to $p_{[s]}$, which we denote by Δ_α. Now given a composable pair $\beta \circ \alpha$ with α, β path homotopies as above, we have the map

$$\partial\Delta^2 \to E_k^\infty$$

induced by the paths $\Delta_\alpha, \Delta_\beta$, and $\Delta_{\beta\circ\alpha}$. We choose a 2-simplex $\Delta_{\alpha,\beta}$ filling in this map. Continuing inductively, we choose n-simplicies $\Delta_{\alpha_1,\cdots,\alpha_n}$ in $\mathrm{Sing}(E_k^\infty)$ for each composable chain of path homotopies.

Remark 3.3.9. Let V be a finite dimensional vector space, and let $\mathrm{Emb}(V)$ denote the space of smooth submanifolds M of V. Then $\mathrm{Emb}(V)$ is an algebra over the operad $E_\bullet^V$ via composition. That is, there are continuous maps

$$\prod_1^k \mathrm{Emb}(V) \times E_k^V \to \mathrm{Emb}(V).$$

They are compatible with inclusions $V \to V'$ of framed vector spaces, and therefore pass to the direct limit. In a similar manner $\mathrm{Emb}(V \times \mathbb{R})$ is an algebra over $E_\bullet^V$ where the action is trivial on the $\mathbb{R}$ coordinate. Finally, we remark that this action induces continuous maps

$$\prod_1^k d\text{-}\mathbf{Cob}_j^V \times E_k^V \to d\text{-}\mathbf{Cob}_j^V,$$

by forgetting the data of the points t_j, applying the action on $\mathrm{Emb}(V \times \mathbb{R})$, and then remembering the points t_j, whose $\mathbb{R}$ factor is fixed under the E_k^V action, so that the points have not been moved. Again, passing to direct limits, we have that $d\text{-}\mathbf{Cob}_j$ is an algebra over $E_\bullet^\infty$.

Definition 3.3.10. Using the data chosen above, we construct a symmetric monoidal Segal space

$$\mathrm{N}_* \lhd \mathbf{\Gamma} \xrightarrow{d\text{-}\mathbf{Cob}^\otimes} \mathbf{Sp}.$$

For S a finite set, define $X_{S,j}$ to be the space of pairs

$$(M, \{t_0, \cdots, t_j\}),$$

where M is a manifold embedded in

$$S \times \mathbb{R}^\infty \times \mathbb{R},$$

and $t_0 \leq \cdots \leq t_j \in \mathbb{R}$ are numbers disjoint from the set of critical values of the induced projection $M \to \mathbb{R}$. So in partuclar, if S has one element, then

$$X_{S,j} \cong d\text{-}\mathbf{Cob}_j.$$

With this notation, on objects $\langle n \rangle \in \mathbf{\Gamma}$, we set

$$d\text{-}\mathbf{Cob}^\otimes(\langle n \rangle)_j := X_{(n),j}.$$

For 0-simplicies of the hom-simplicial sets of $\mathrm{N}_* \lhd \mathbf{\Gamma}$ of the form $[1] \to \mathbf{\Gamma}$, or in other words morphisms $a : \langle n' \rangle \to \langle n \rangle$ of gamma, we must produce continuous maps

$$d\text{-}\mathbf{Cob}^\otimes(\langle n' \rangle)_j \to d\text{-}\mathbf{Cob}^\otimes(\langle n \rangle)_j$$

which are compatible with the face and degeneracy maps in the j coordinate. To do this, note that there are continuous maps

$$X_{S_1 \cup S_2, j} \to X_{S_1, j}$$

induced by 'forgetting the piece of M embedded in the S_2 part'. We can then form the composite

$$\begin{aligned} d\text{-}\mathbf{Cob}^{\otimes}(\langle n'\rangle)_j := X_{(n'),j} &\to \prod_{m=1}^{n'} X_{\{m\},j} \\ &\to \prod_{i=1}^{n}\left[\left(\prod_{m\in a^{-1}(i)} X_{\{m\},j}\right)\times\{p_{|a^{-1}(i)|}\}\right] \\ &\hookrightarrow \prod_{i=1}^{n}\left[\left(\prod_{m=1}^{|a^{-1}(i)|} d\text{-}\mathbf{Cob}_j\right)\times E^{\infty}_{|a^{-1}(i)|}\right] \\ &\to \prod_{1}^{n} d\text{-}\mathbf{Cob}_j \cong \prod_{i=1}^{n} X_{\{i\},j} \end{aligned}$$

Here we used the points p_k chosen above and the action map from Remark 3.3.9. Since this action does not affect the $\mathbb{R}$ of the ambient space the above composition factors through the map

$$d\text{-}\mathbf{Cob}^{\otimes}(\langle n\rangle)_j := X_{\{1,\cdots,n\},j} \to \prod_{\ell=1}^{n} X_{\{\ell\},j};$$

in words, the points in the product all have the same points $t_0 \leq \cdots \leq t_j$ marked on the $\mathbb{R}$ factor. Again, since the $\mathbb{R}$ factor is untouched by the map $d\text{-}\mathbf{Cob}^{\otimes}(a)_j$ as defined above, it commutes with simplicial maps in the j coordinate and hance defines a map of Segal spaces.

For more general 0-simplicies of hom-simplicial sets of $\mathrm{N}_* \lhd \mathbf{\Gamma}$, of the form $c : [s] \to \mathbf{\Gamma}$, say with final target $\langle n\rangle \in \mathbf{\Gamma}$, we are forced, by functoriality, to send these to Segal space map which is induced, in a similar manner to that above, by a sequence of actions of the operad $E^{\infty}_{\bullet}$ on the algebra $d\text{-}\mathbf{Cob}_j$. Since algebra actions by an operad are associative, we get the same result if we perform the appropriate compositions in the operad first and then use the action. To make this precise, for $i = 1, \cdots, n$, let $a_{s,i}$ denote the restriction of $c(f(s))$ to $c(f(s))^{-1}(i)$. Similarly, let $a_{s-1,i}$ denote the restriction of $c(f(s-1))$ to the inverse image of $a_{s,i}$ on its domain. Continuing in this manner, for each i, we obtain a chain of composable surjective set maps

$$a_{s,i} \circ \cdots \circ a_{1,i} : f^{-1}(i) \to \{i\} \cong (1),$$

where f is the composition of all the morphisms in the image of c. Suppose that $f : \langle n'\rangle \to \langle n\rangle$.

Then the map d-$\mathbf{Cob}^{\otimes}(c)_j$ is the composite

$$\begin{aligned}
d\text{-}\mathbf{Cob}^{\otimes}(\langle n'\rangle)_j := X_{(n'),j} &\to \prod_{m=1}^{n'} X_{\{m\},j} \\
&\to \prod_{i=1}^{n}\left[\left(\prod_{m\in f^{-1}(i)} X_{\{m\},j}\right)\times\{p_{a_{1,i},\cdots,a_{s,i}}\}\right] \\
&\hookrightarrow \prod_{i=1}^{n}\left[\left(\prod_{m=1}^{|f^{-1}(i)|} d\text{-}\mathbf{Cob}_j\right)\times E^{\infty}_{|f^{-1}(i)|}\right] \\
&\to \prod_{1}^{n} d\text{-}\mathbf{Cob}_j \cong \prod_{i=1}^{n} X_{\{i\},j}
\end{aligned}$$

Where the $p_{\dots}$ are the points chosen above in the spaces $E^{\infty}_{\bullet}$. Again this is compatible with simplicial maps in j, because the $\mathbb{R}$ coordinate is left alone.

Finally, given some composable chain of path homotopies

$$\alpha_k \circ \cdots \circ \alpha_1 : [s'] \to [s]$$

Suppose that the common initial source and final target of $[s]$ and $[s']$ are $\langle n'\rangle$ and $\langle n\rangle$, respectively. Then we must produce an m-simplex of maps

$$d\text{-}\mathbf{Cob}^{\otimes}(\langle n'\rangle)_j \times \Delta^k \to d\text{-}\mathbf{Cob}^{\otimes}(\langle n\rangle)_j$$

which is compatible with simplicial maps in the coordinate j. Let f denote the common composition of all the chains c appearing as source and target of the α's. We begin by parsing each chain $c : [s] \to \mathbf{\Gamma}$ as we did in the paragraph above, to produce n chains of surjective set maps $f^{-1}(i) \to \{i\}$ for each $i = 1, \cdots, n$. Next we parse the α's in the same way, forming chains

$$\alpha_{k,i} \circ \cdots \alpha_{k,i}$$

for $i = 1, \cdots, n$, of path homotopies of chains of surjective set maps $f^{-1}(i) \to \{i\}$. These, when ‘stacked’ along i, are the original α’s away from $*$. We can now form the composite

$$\begin{aligned} d\text{-}\mathbf{Cob}^{\otimes}(\langle n'\rangle)_j \times \Delta^k &\to \prod_{m=1}^{n'} X_{\{m\},j} \times \Delta^k \\ &\to \prod_{i=1}^{n}\left[\left(\prod_{m\in f^{-1}(i)} X_{\{m\},j}\right) \times \Delta_{\alpha_{1,i},\cdots,\alpha_{k,i}}\right] \\ &\to \prod_{i=1}^{n}\left[\left(\prod_{m=1}^{|f^{-1}(i)|} d\text{-}\mathbf{Cob}_j\right) \times E^{\infty}_{|f^{-1}(i)|}\right] \\ &\to \prod_{1}^{n} d\text{-}\mathbf{Cob}_j \cong \prod_{i=1}^{n} X_{\{i\},j} \end{aligned}$$

where the second arrow is induced by the diagonal map $\Delta^k \to \prod_1^n \Delta^k$. As always, this map commutes with simplicial maps in the j coordinate. Moreover, it is functorial with respect to composition, because the hom-simplicial sets of $\mathbf{Sp}$ are of the form

$$\int_j \mathrm{Sing}(\underline{\mathbf{Top}}(X_j, Y_j)) = \mathrm{Sing}\left(\int_j \underline{\mathbf{Top}}(X_j, Y_j)\right)$$

and the composition map for $\mathrm{Sing}_* \,\mathbf{Top}$

$$\mathrm{Sing}\,\underline{\mathbf{Top}}(X,Y) \times \mathrm{Sing}\,\underline{\mathbf{Top}}(Y,Z) \to \mathrm{Sing}\,\underline{\mathbf{Top}}(X,Z)$$

is induced by the diagonal map $\Delta^k \to \Delta^k \times \Delta^k$, as well. Above we used the fact that Sing commutes with ends, being a right adjoint (see Chapter 3).

Lastly, we must check the condition that the morphism

$$d\text{-}\mathbf{Cob}^{\otimes}(\langle n\rangle) \to \prod_{1}^{n} d\text{-}\mathbf{Cob}^{\otimes}(\langle 1\rangle)$$

induced by the morphisms $\pi^j_{\langle n\rangle}$ of $\mathbf{\Gamma}$ is a weak equivalence of Segal spaces. Since we chose the points p_1 to correspond to the identity above, we see that the difference between the two sides is that the points of $d\text{-}\mathbf{Cob}^{\otimes}(\langle n\rangle)_j$ correspond to points of $d\text{-}\mathbf{Cob}^{\otimes}(\langle 1\rangle)_j$ which happen to have the same marked $t_0 \le \cdots \le t_j$. However, the embedded manifolds and points can be pushed around continuously so that the points t_j are at $0 < 1 < \cdots < j$ for non-degenerate bordisms, and with appropriate doubling for degenerate bordisms. This gives a homotopy inverse to the map above, so it is a weak equivalence of Segal spaces, as required.

Remark 3.3.11. While Definition 3.3.10 is quite involved, the intuition is straightforward. The bordisms in the homotopy bordism category are embedded in ambient space, and so to formalize the notion that 'disjoint union is the monoidal product', we must choose *how* to take the disjoint union of manifolds within the ambient space. To do this, we simply took the disjoint union of ambient spaces themselves and embedded the whole result into another copy of the ambient space. We chose a way to do this for any k-fold disjoint union (these were the points p_k). However, with this data, we have many possible ways to form disjoint unions of triples etcetera (the points $p_{\dots}$). The higher simplices $\Delta_{\dots}$ express the fact that there are homotopies between the various ways, and all these homotopies form a 'homotopy coherent' system. Finally Definition 3.3.10 takes this coherent system of monoidal structure on the ambient space, and uses it to induce the disjoint union monoidal structure on d-**Cob**, resulting in a symmetric monoidal Segal space.

Definition 3.3.12. We define $d\text{-}\mathbf{Cob}^{\otimes}(X,\zeta)$ in the same manner as $d\text{-}\mathbf{Cob}^{\otimes}$ was defined in Definition 3.3.10, except that the manifolds have (X,ζ) structures.

We conclude by defining the homotopy field theories.

Definition 3.3.13. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal Segal space. A *homotopy field theory* with target $\mathcal{C}^{\otimes}$ and structure (X,ζ) is a 0-simplex in

$$\mathrm{Coh}(d\text{-}\mathbf{Cob}^{\otimes}(X,\zeta),\mathcal{C}^{\otimes}),$$

or in other words (see Definition 2.4.9), a symmetric monoidal functor.

3.4 The Machinery

The goal of this section is to explain how to obtain a symmetric monoidal Segal space from a category symmetric monoidally fibred over **Man**. We will then apply this construction to the smooth bordism category, and compare the result to a certain homotopy bordism category. The construction will be similar to that presented at the end of chapter 1, where we described the symmetric monoidal simplicial Segal space induced by an ordinary monoidal category.

Consider the 2-category $\mathrm{Fib}^{\otimes}(\mathbf{Man})$ described in Section 2.1. In Definitions 2.1.23 and 2.1.24 we constructed a functor from this 2-category to the 2-category with objects strict functors

$$\mathbf{Man}^{\mathrm{op}} \to \mathbf{SMCat}$$

and morphism categories consisting of strict natural transformations and modifications of such, which was denoted by underline: $\mathbf{E} \mapsto \underline{\mathbf{E}}$.

Next, we demonstrated that the 2-category **SMCat** is equivalent to the 2-category with objects pseudofunctors

$$F : \mathbf{\Gamma} \to \mathbf{Cat}$$

with the property that

$$(F\pi^1_{\langle n\rangle},\cdots,F\pi^n_{\langle n\rangle}) : F(\langle n\rangle) \to \prod_1^n F(\langle 1\rangle)$$

is an equivalence of categories for each n.

Combining the above two constructions, we have a functor from the 2-category $\mathrm{Fib}^{\otimes}(\mathbf{Man})$ to the 2-category with objects strict functors

$$\mathbf{Man}^{\mathrm{op}} \to (\mathbf{\Gamma} \to \mathbf{Cat}),$$

where the target consists of pseudofunctors, pseudonatural transformations, and modifications.

Remark 3.4.1. In **Man**, there is the subcategory with objects *extended simplicies*

$$\Delta^k_{\mathrm{ext}} = \left\{ (t_0, \cdots, t_k) \in \mathbb{R}^{k+1} | \sum t_i = 1 \right\},$$

and morphisms generated by face maps $\Delta^k \to \Delta^{k-1}$:

$$(t_0, \cdots, t_i, t_{i+1}, \cdots, t_k) \mapsto (t_0, \cdots, t_i + t_{i+1}, \cdots, t_k)$$

and degeneracy maps $\Delta^k_{\mathrm{ext}} \to \Delta^{k+1}_{\mathrm{ext}}$:

$$(t_0, \cdots, t_i, t_{i+1}, \cdots, t_k) \mapsto (t_0, \cdots, t_i, 0, t_{i+1}, \cdots, t_k).$$

This category is clearly isomorphic to $\mathbf{\Delta}$. So we have an embedding $\mathbf{\Delta} \to \mathbf{Man}$.

We restrict the domain $\mathbf{Man}^{\mathrm{op}}$ via this embedding to get to the 2-category with objects strict functors

$$\mathbf{\Delta}^{\mathrm{op}} \to (\mathbf{\Gamma} \to \mathbf{Cat}).$$

Now, by currying, we get an equivalence of this 2-category to the 2-category

$$\mathbf{\Gamma} \to (\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}).$$

We define the currying procedure explicitly on the objects. If $F : \mathbf{\Delta}^{\mathrm{op}} \to (\mathbf{\Gamma} \to \mathbf{Cat})$, which pseudo in the $\mathbf{\Gamma}$ coordinate, we define

$$\hat{F} : \mathbf{\Gamma} \to (\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}),$$

$$\begin{aligned} \hat{F}(\langle n \rangle)([m]) &:= F([m])(\langle n \rangle) \\ \hat{F}(a : \langle n' \rangle \to \langle n \rangle)([m]) &:= F([m])(a) \\ \hat{F}_{a,b}([m]) &:= (F([m]))_{a,b} \\ &\cdots \end{aligned}$$

and so forth. The point is that being functorial or pseudofunctorial is a property in each variable separately.

In Section 2.3, we showed that pseudofunctors can be replaced by strict functors via the $\lhd$ construction on the domain. Combining this with the above, we have a functor from the 2-category $\mathrm{Fib}^{\otimes}(\mathbf{Man})$ to the 2-category with objects strict functors

$$\lhd\mathbf{\Gamma} \to (\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat})$$

where the target 2-category is all strict (i.e. strict functors, natural transformations, and modifications). We now apply the functor N_* to this diagram, which sends the **Cat**-enriched functors to simplicial functors, and the pseudonatural transformations of such to homotopy coherent transformations, as explained in Section 2.4. Now, since nerve commutes with products and inner-hom of cat (see Section 4.3) it can be regarded as a simplicial functor

$$\mathrm{N} : \mathrm{N}_*(\mathbf{Cat}) \to \mathbf{sSet}.$$

Applying this observation to the above we now end up with simplicial functors

$$\mathrm{N}_* \lhd\mathbf{\Gamma} \to (\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}).$$

At this point we recall that the outer $\mathbf{\Delta}^{\mathrm{op}}$ coordinate (the one separate from **sSet**) is the restriction of the $\mathbf{Man}^{\mathrm{op}}$ coordinate coming from the original fibration, and the inner $\mathbf{\Delta}^{\mathrm{op}}$ factor (of **sSet**) is induced by the nerve functor. We switch these two $\mathbf{\Delta}^{\mathrm{op}}$ factors. The intuition behind this switch is that in a simplicial Segal space (see Appendix A.5.1), the outer $\mathbf{\Delta}^{\mathrm{op}}$ coordinate keeps track of how many composable morphisms there are, which is exactly what the nerve coordinate counts, and the inner $\mathbf{\Delta}^{\mathrm{op}}$ coordinate 'feels out' the topology on the morphism spaces, which is what the smooth families of bordisms are doing. Let us check how close the bisimplicial sets that are hit are to being simplicial Segal spaces. Let Φ denote the map constructed so far; then taking $\mathbf{E} \in \mathrm{Fib}^{\otimes}(\mathbf{Man})$, the k^{th} simplicial set $\Phi(\mathbf{E})(\langle n\rangle)_k$ is $\mathrm{N}(\prod_1^n \underline{\mathbf{E}}_{\Delta^{\bullet}_{\mathrm{ext}}})_k$, where $\underline{\mathbf{E}}_{\Delta^{\bullet}_{\mathrm{ext}}}$ denotes the restriction of $\underline{\mathbf{E}}$ to the subcategory of extended simplicies.

In [Rez] the following characterization of Reedy fibrant is given. For $\boldsymbol{X}_{\bullet} : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$, consider the simplicial set maps

$$\mathbf{ssSet}_{\mathbf{sSet}}(\mathrm{disc}\,\Delta(k), \boldsymbol{X}_{\bullet}) \to \mathbf{ssSet}_{\mathbf{sSet}}(\mathrm{disc}\,\partial\Delta(k), \boldsymbol{X}_{\bullet})$$

induced by the inclusion $\partial\Delta(k) \hookrightarrow \Delta(k)$. For an explanation of this notation, refer to the beginning of Section 4.3. Then $\boldsymbol{X}_{\bullet}$ is Reedy fibrant if and only if these maps are fibrations. We recall that being a (Kan) fibration (Definition A.5.2) means that given a solid diagram

$$\begin{tikzcd}
\Lambda(n,j) \arrow[r] \arrow[d] & \mathbf{ssSet}_{\mathbf{sSet}}(\mathrm{disc}\,\Delta(k), \boldsymbol{X}_{\bullet}) \arrow[d] \\
\Delta(n) \arrow[r] \arrow[ur, dotted] & \mathbf{ssSet}_{\mathbf{sSet}}(\mathrm{disc}\,\partial\Delta(k), \boldsymbol{X}_{\bullet})
\end{tikzcd}$$

the dotted arrow exists. Combining this with the above characterization of what bisimplicial sets are hit, we see that Reedy fibrant amounts to various special cases of the statement: given a diagram in $\mathbf{E}$ over a manifold S that commutes when pulled back to some submanifold of codimension 1, then it also commutes over all of S. This does not seem to be true in general, or in particular in the case $\mathbf{E} = d\text{-}\mathbf{B}(X)$. However, in the special case where $k = 0$, the statement is quite different. It becomes: the simplicial set of k simplicies is Kan. This is the case, since an object over an inner horn can be pulled back to an object over the the filled in simplex. We will use this fact in Section 4.4.

The Segal maps (see Appendix A.5.1)

$$\Phi(\mathbf{E})(\langle n\rangle)_k \to \underbrace{\Phi(\mathbf{E})(\langle n\rangle)_1 \times_{\Phi(\mathbf{E})(\langle n\rangle)_0} \cdots \times_{\Phi(\mathbf{E})(\langle n\rangle)_0} \Phi(\mathbf{E})(\langle n\rangle)_1}_{k \text{ times}}$$

on the other hand, are isomorphisms, since k simplicies in nerves are precisely composable chains of morphisms, which is the right hand side.

The last step in the map is to go from bisimplicial sets to simplicial spaces. This is accomplished by postcomposing with geometric realization

$$\Delta^{\mathrm{op}} \to \mathbf{sSet} \xrightarrow{|\cdot|} \mathbf{Top}.$$

We will denote the resulting functor $G : \mathbf{ssSet} \to \mathbf{sTop}$; G is a simplicial functor, since $|\cdot|$ commutes with finite limits, and in particular products (see [JT]). Let $\phi(\boldsymbol{X}_\bullet)_k$ denote the k^{th} Segal map of a bisimplicial set $\boldsymbol{X}_\bullet$, and let $\hat{\phi}(Y_\bullet)_k$ denote the k^{th} Segal map of a simplicial space $Y_\bullet$. Then we have the following diagram.

$$\begin{tikzcd}
& {|\boldsymbol{X}_1| \times_{|\boldsymbol{X}_0|} \cdots \times_{|\boldsymbol{X}_0|} |\boldsymbol{X}_1|} \\
G(\boldsymbol{X}_\bullet)_k = |\boldsymbol{X}_k| \arrow[ur, "\hat{\phi}(G(\boldsymbol{X}_\bullet))_k"] \arrow[dr, "|\phi(\boldsymbol{X}_\bullet)_k|"'] & \\
& {|\boldsymbol{X}_1 \times_{\boldsymbol{X}_0} \cdots \times_{\boldsymbol{X}_0} \boldsymbol{X}_1|} \arrow[uu, "\cong"']
\end{tikzcd}$$

The diagram commutes since it commutes on each factor $|\boldsymbol{X}_1|$. If it happened that the Segal morphisms $\phi(\boldsymbol{X}_\bullet)_k$ were isomorphisms, $|\phi(\boldsymbol{X}_\bullet)_k|$ are isomorphisms, and therefore so are $\hat{\phi}(G(\boldsymbol{X}_\bullet))_k$. Since the Segal morphisms for $\Phi(\mathbf{E})(\langle n\rangle)$ were isomorphisms, we conclude that $G(\Phi(\mathbf{E})(\langle n\rangle))$ satisfies the Segal condition as well. Hence sending a symmetric monoidally fibred category over **Man** through the above machinery, we get a symmetric monoidal Segal space. Note that the Reedy fibrant condition is dropped for Segal spaces.

We summarize the machinery developed in this section with the following notation. Given $\mathbf{E} \in \mathrm{Fib}^{\otimes}(\mathbf{Man})$, we denote the associated symmetric monoidal Segal space by $\mathfrak{M}(\mathbf{E})$. This symmetric monoidal Segal space was constructed by the following steps.

1. Apply the 'strictified Grothendieck construction'
$$\mathbf{E} \mapsto (\underline{\mathbf{E}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{SMCat}).$$
2. Replace **SMCat** with the equivalent 2-category with objects pseudofunctors $\mathbf{\Gamma} \to \mathbf{Cat}$.
3. Restrict $\mathbf{Man}^{\mathrm{op}}$ to extended simplicies, a subcategory of $\mathbf{Man}^{\mathrm{op}}$ equivalent to $\mathbf{\Delta}^{\mathrm{op}}$.
4. Switch the $\mathbf{\Gamma}$ factor (which is pseudofunctorial) with the $\mathbf{\Delta}^{\mathrm{op}}$ factor (which is strictly functorial), to obtain pseudofunctors, pseudonatural transformations, and modifications
$$\mathbf{\Gamma} \to (\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}).$$
5. Strictify the functors in the $\mathbf{\Gamma}$ factor via the 'hammock construction' $\mathbf{\Gamma} \rightsquigarrow \triangleleft\mathbf{\Gamma}$. This brings us to the category of **Cat**-enriched functors
$$\triangleleft\mathbf{\Gamma} \to (\mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{Cat}),$$
and pseudonatural transformations and modifications of such.
6. Apply the nerve pushforward N_* to get to simplicial (**sSet**-enriched) functors
$$\mathrm{N}_* \triangleleft\mathbf{\Gamma} \to \mathrm{N}_*(\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}) \cong (\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{N}_* \mathbf{Cat}).$$
The pseudonatural transformations can be regarded as objects of a certain lax end (see Appendix A.3).
7. Apply the nerve functor, regarded as a simplicial functor $\mathrm{N} : \mathrm{N}_*(\mathbf{Cat}) \to \mathbf{sSet}$, to get to simplicial functors
$$\mathrm{N}_* \triangleleft\mathbf{\Gamma} \to \mathbf{ssSet}$$
The lax ends are sent to coherent ends, and in particular the pseudonatural transformations are sent to coherent transformations.
8. Switch the two $\mathbf{\Delta}^{\mathrm{op}}$ factors of **ssSet**.
9. Apply geometric realization, considered as a simplicial functor $G : \mathbf{ssSet} \to \mathbf{sTop}$ to obtain simplicial functors
$$\mathrm{N}_* \triangleleft\mathbf{\Gamma} \to \mathbf{sTop}$$
and coherent transformations. Since the Segal condition is satisfied, the simplicial functors produced factor through **Sp**; in particular they are symmetric monoidal Segal spaces.

Given $\mathbf{E}', \mathbf{E} \in \mathrm{Fib}^{\otimes}(\mathbf{Man})$, and a symmetric monoidally fibred functor $F : \mathbf{E}' \to \mathbf{E}$, we denote the associated symmetric monoidal functor of symmetric monoidal Segal spaces (Definition 2.4.9) by

$$\mathfrak{M}(F) : \mathfrak{M}(\mathbf{E}') \to \mathfrak{M}(\mathbf{E}).$$

Again, note that the machinery above does produce such a symmetric monoidal functor, since each step involved enriched functors of the appropriate type, or involved a change of the enrichment. For example, when we applied N_*, what was a lax transformation (which happened to be iso-lax), was sent to a coherent transformation (see Proposition 2.4.7).

3.5 The Map on Field Theories

The goal of this section is to construct a map from smooth field theories to homotopy field theories. We construct this map by applying the machinery of Section 3.4 to the source of smooth field theories, namely the smooth bordism category, and their target, vector bundles. This results in a map from smooth field theores to coherent transformations

$$\mathfrak{M}(d\text{-}\mathbf{B}(X)) \to \mathfrak{M}(\mathbf{Vect}_{\mathbf{Man}}),$$

which we will then precompose with a fixed morphism from a homotopy bordism category to get a certain kind of homotopy field theories.

We first need to figure out which homotopy bordism category is appropriate. In other words, we want to find which pair (Y, ζ), where Y is a space, and ζ is a vector bundle over Y, is such that (Y, ζ) structures on a manifold M correspond to maps to X. Recall that the maifolds of d-$\mathbf{B}(X)$ are equipped with smooth maps to $X \in \mathbf{Man}$. We first treat the case where $X = \mathbf{pt}$. In this case there is no additional datum beyond the bordisms. This is mirrored by the homotopy bordism category

$$d\text{-}\mathbf{Cob}^{\otimes}(\mathrm{BO}(d), \gamma_{\mathrm{BO}(d)}),$$

with $\gamma_{\mathrm{BO}(d)}$ the canonical bundle over $\mathrm{BO}(d)$. The manifolds in this homotopy bordism category are equipped with a map to $\mathrm{BO}(d)$ and an isomorphism of their tangent bundle with the pullback of the canonical bundle. However, their tangent bundles, equipped with some choice of Riemannian metric, being d-dimensional, are classified by such a map (well-defined up to homotopy) and such an isomorphism. Hence, there are no additional data (up to contractible space of choices). One could avoid the choice of Riemannian metric and instead work with

$$d\text{-}\mathbf{Cob}^{\otimes}(\mathrm{GL}(d), \gamma_{\mathrm{GL}(d)}),$$

but we choose to follow the convention of [Lur1], which consistently chooses metrics everywhere.

To add back in a map to X, we simply map into the product $\mathrm{BO}(d) \times X$, and take the bundle over this space which over a point (p, x), is the subspace of $\mathbb{R}^\infty$ defined by p, i.e. the canonical bundle extended trivially in the X direction, which we can write as a pullback along the projection: $\pi^*_{\mathrm{BO}(d)}\gamma_{\mathrm{BO}(d)}$, with $\pi_{\mathrm{BO}(d)} : \mathrm{BO}(d) \times X \to \mathrm{BO}(d)$. It is clear then from the preceding discussion that the manifolds of the homotopy bordism category

$$d\text{-}\mathbf{Cob}^{\otimes}(\mathrm{BO}(d) \times X, \pi^*_{\mathrm{BO}(d)}\gamma_{\mathrm{BO}(d)})$$

have the data of a map to X.

Remark 3.5.1. The reason that we specifically chose to define homotopy bordism categories of the form $d\text{-}\mathbf{Cob}^{\otimes}(Y, \zeta)$ is Theorem 2.4.18 of [Lur1], which classifies (extended) field theories out of such bordism categories. We will take advantage of this classification in the next section, in the cases where extended and non-extended amount to the same thing, i.e. $d = 0, 1$.

Now we apply the machinery of Section 3.4 to $d\text{-}\mathbf{B}(X)$. There is one subtlety, which is that $\mathfrak{M}$ calls for taking nerves of categories of the form

$$\prod_1^n \underline{d\text{-}\mathbf{B}(X)}_{\Delta^k_{\mathrm{ext}}} \cong \prod_1^n \underline{d\text{-}\mathbf{B}(X)}_{\mathbb{R}^k}$$

However, the categories

$$\underline{d\text{-}\mathbf{B}(X)}_{\mathbb{R}^k} := \mathrm{Fun}_{\mathbf{Man}}(\underline{\mathbb{R}^k}, d\text{-}\mathbf{B}(X))$$

are large. In particular, there is a distinct fibred functor $\underline{\mathbb{R}^k} \to d\text{-}\mathbf{B}(X)$ for each assignment

$$(\mathrm{id} : \mathbb{R}^k \to \mathbb{R}^k) \mapsto (Y \to \mathbb{R}^k),$$

with $Y = (Y, Y^c, Y^{\pm})$ any bordism over $\mathbb{R}^k$.

There are at least two solutions to this problem. The first, which seems to be the option of choice when developing abstract theory (see [Rez],[Lur2]) is to use the formalism of *Grothendieck universes.* Namely, posit that the sets underlying the manifolds such as Y are elements of some universe U, and that all such manifolds do in fact form a set in some larger universe U'.

The other solution, which we will use here, is to replace the fibred categories $d\text{-}\mathbf{B}(X)$ with equivalent fibre-wise small categories. This obviously does not work in general, for not every category is equivalent to a small one. However, in our case it is possible, and the method will serve the dual purpose of clarifying the relationship to homotopy bordism categories.

The fibre-wise small replacement of $d\text{-}\mathbf{B}(X)$ is simply the same definition of $d\text{-}\mathbf{B}(X)$ except that all object and morphism data involving manifolds are submanifolds of $\mathbb{R}^\infty$, in the same direct limit sense of Section 3.3. We will denote this fibred category $d\text{-}\mathbf{B}(X)^{\mathrm{emb}}$.

Remark 3.5.2. It is important to note that when we are at the stage of pseudofunctors

$$\mathbf{\Gamma} \to (\Delta^{\mathrm{op}} \to \mathbf{Cat})$$

the monoidal structure is encoded in the $\mathbf{\Gamma}$ factor. Here, functors that are strict correspond to strict symmetric monoidal categories, where the structure isomorphisms are all identities, which one can think of as 0-coherent structure. For more general pseudofunctors, corresponding to more general symmetric monoidal categories, Maclane coherence says that the structure is 1-coherent. Unfortunately, $d\text{-}\mathbf{B}(X)^{\mathrm{emb}}$ is then *not* symmetric monoidally fibred. This is the case because disjoint union cannot be realized as a 1-coherent structure when the manifolds are embedded in some ambient space. The best one can do, as explained in Section 3.3 is use disjoint union to induce a symmetric monoidal structure up to an ∞-coherent system of homotopies. We use the same procedure of Definition 3.3.10 to make at this stage, a symmetric monoidal Segal space

$$N_* \lhd \mathbf{\Gamma} \to \mathbf{sTop},$$

which we will denote by $\mathfrak{M}(d\text{-}\mathbf{B}(X))$ (despite the fact that its construction contained the additional step of passing to equivalent small categories), with

$$\mathfrak{M}(d\text{-}\mathbf{B}(X))(\langle n\rangle)_k := \left|\prod_1^n \mathrm{N}\left(\underline{d\text{-}\mathbf{B}(X)^{\mathrm{emb}}}_{\Delta^{\bullet}_{\mathrm{ext}}}\right)_k\right|,$$

and with the images of simplicies in the $\langle n\rangle$ coordinate induced by the same ambient space disjoint union embeddings chosen in the discussion preceding Defintion 3.3.10.

With this construction, we have the following result.

Theorem 3.5.3. There is a natural transformation of simplicial functors

$$\mathfrak{T} : d\text{-}\mathbf{Cob}^{\otimes}\left(\mathrm{BO}(d)\times X, \pi^*_{\mathrm{BO}(d)}\gamma_{\mathrm{BO}(d)}\right) \Rightarrow \mathfrak{M}(d\text{-}\mathbf{B}(X)),$$

which is pointwise a weak equivalence.

Proof. By definition

$$\mathfrak{T}(\langle n\rangle)_k$$

must be a continuous map from the space of manifolds embedded in

$$(n)\times\mathbb{R}^{\infty}\times\mathbb{R}$$

with points $t_0 \le \cdots \le t_k \in \mathbb{R}$, such that the critical values of the projection of the manifolds to $\mathbb{R}$ are disjoint from t_i, to the geometric realization of

$$\prod_1^n \mathrm{N}\left(\underline{d\text{-}\mathbf{B}(X)^{\mathrm{emb}}}_{\Delta^{\bullet}_{\mathrm{ext}}}\right)_k.$$

Let us denote the former space by $Y(\langle n\rangle)_k$.

Unfortunately, it is in general difficult to map *into* a geometric realization; for example it is a *left* adjoint, not a right adjoint. We make the following construction. First, replace the geometric realization with the *extended geometric realization*, which is the coend

$$|S_\bullet|_{\mathrm{ext}} := \int^{[k]} \Delta^k_{\mathrm{ext}} \times \operatorname{disc} S_k.$$

By the same abstract nonsense as used for the usual geometric realization (i.e. Proposition 3.1.5 of [Hov]), one obtains that it is left adjoint to

$$(\mathrm{Sing}_{\mathrm{ext}}\, X)_k = \mathbf{Top}(\Delta^k_{\mathrm{ext}}, X).$$

Since extended simplicies deformation retract onto the standard simplicies, it is easy to see that $|S_\bullet|_{\mathrm{ext}}$ and $|S_\bullet|$ are homotopy equivalent. Now, we do have a simplicial map

$$\mathrm{Sing}_{\mathrm{ext}}(Y(\langle n\rangle)_k) \to \prod_1^n \mathrm{N}\left(\underline{d\text{-}\mathbf{B}(X)^{\mathrm{emb}}}_{\Delta^\bullet_{\mathrm{ext}}}\right)_k.$$

Let us describe this map when $n = 1$. An m simplex on the left hand side is a map

$$\Delta^m_{\mathrm{ext}} \to Y(\langle n\rangle)_k.$$

The points of the target are embedded manifolds with various extra data, and it is clear that the target has a universal bundle of such fibres over it. Hence, by pulling back this bundle, we get a bundle over Δ^m_{ext} with fibres given by manifolds M embedded in $\mathbb{R}^\infty \times \mathbb{R}$ with points $t_0 \leq \cdots \leq t_k \in \mathbb{R}$, such that the critical values of the projection of the manifolds to $\mathbb{R}$ are disjoint from t_i. Let π_M denote the restriction of the projection of the total space of this bundle to $\mathbb{R}$ fibrewise to M. Note that π_M is proper.

We send this bundle to a the following k composable families bordisms on the right hand side. The j^{th} bordism is a morphism from the object

$$(Y_{j-1}, Y^c_{j-1}, Y^+_{j-1}, Y^-_{j-1}) := \left(\pi_M^{-1}(\mathbb{R}), \pi_M^{-1}(\{t_{j-1}\}), \pi_M^{-1}(t_{j-1}, \infty), \pi_M^{-1}(-\infty, t_{j-1})\right).$$

to the object Y_j (defined in the same way). The bordism is the rel boundary isomorphism class represented by the triple

$$(\Sigma, i_0, i_1) := (\pi_M^{-1}(\mathbb{R}), \mathrm{id}_{\pi_M^{-1}(\mathbb{R})}, \mathrm{id}_{\pi_M^{-1}(\mathbb{R})}).$$

In particular, the core family is $\Sigma^c = \pi_M^{-1}[t_{j-1}, t_j]$. The bordisms were already equipped with maps to the product $\mathrm{BO}(d) \times X$, so we simply retain the map to X, and also forget the vector bundle isomorphism data. All of the requirements for objects and bordisms are clearly satisfied, and furthermore everything is embedded in $\mathbb{R}^\infty$, so we obtain an m simplex on the right hand side. Since simplicial maps in the m coordinate are defined identically on both sides, these maps over varying m induce a map of bisimplicial sets. When $n > 1$, we do the same procedure separately on each component embedded in $\{l\} \times \mathbb{R}^\infty \times \mathbb{R}$. Note that the above maps are functorial in the k coordinate. Furthermore, up to homotopy, these maps are the Segal maps of the bisimplicial set $\mathrm{Sing}_{\mathrm{ext}}(Y(\langle n\rangle)_\bullet)$, which are weak equivalences since $Y(\langle n\rangle)$ is a Segal space, and $\mathrm{Sing}_{\mathrm{ext}}$ is a right Quillen functor.

Now, applying extended geometric realization in the m coordinate, we obtain a map of simplicial spaces

$$|\,\mathrm{Sing}_{\mathrm{ext}}(Y(\langle n\rangle)_k)|_{\mathrm{ext}} \to \left|\prod_1^n \mathrm{N}\left(\underline{d\text{-}\mathbf{B}(X)^{\mathrm{emb}}}_{\Delta^\bullet_{\mathrm{ext}}}\right)_k\right|_{\mathrm{ext}} \to \mathfrak{M}(d\text{-}\mathbf{B}(X))(\langle n\rangle)_k.$$

The second arrow is induced by the inclusions $\Delta^k \hookrightarrow \Delta^k_{\mathrm{ext}}$, which are homotopy equivalences. Note that the first map is also a weak equivalence, since geometric realization is a left Quillen functor, and all simplicial sets are cofibrant (apply Ken Brown's Lemma).

Next, we observe that the counit (or evaluation) map of the adjunction $|\cdot|_{\mathrm{ext}} \dashv \mathrm{Sing}_{\mathrm{ext}}$ provides a map

$$|\,\mathrm{Sing}_{\mathrm{ext}}(Y(\langle n\rangle)_k)|_{\mathrm{ext}} \to (Y(\langle n\rangle)_k)$$

which is a weak equivalence.

Now, in [GMTW], the homotopy type of the cobordism category d-**Cob** is calculated, and it is in particular homotopy equivalent to a CW complex. This counit map $|\,\mathrm{Sing}_{\mathrm{ext}} Z|_{\mathrm{ext}} \to Z$ can be inverted up to homotopy provided that Z is equivalent to a CW complex, so we obtain maps

$$Y(\langle n\rangle)_k \to |\,\mathrm{Sing}_{\mathrm{ext}}(Y(\langle n\rangle)_k)|_{\mathrm{ext}} \to \mathfrak{M}(d\text{-}\mathbf{B}(X))(\langle n\rangle)_k.$$

It is clear that these can be arranged to be functorial in the k coordinate. Finally, since we used the same coherence data for disjoint union on both sides, the construction is functorial in $\langle n\rangle$ as well; we therefore conclude that there is a natural transformation of simplicial functors

$$\mathfrak{T} : d\text{-}\mathbf{Cob}^{\otimes}\left(\mathrm{BO}(d) \times X, \pi^*_{\mathrm{BO}(d)}\gamma_{\mathrm{BO}(d)}\right) \Rightarrow \mathfrak{M}(d\text{-}\mathbf{B}(X)).$$

The fact that it is a pointwise weak equivalence since it is a pointwise composition of weak equivalences. □

In order to complete our map on field theories, we need to explain how to use the natural transformation from Theorem 3.5.3 to 'precompose' coherent transformations in the image of $\mathfrak{M}$ and obtain homotopy field theories. The following definition and result, taken from [CP] shows how this is done.

Definition 3.5.4. Let $\mathbf{A}$ be an S-category, and $\mathbf{B}$ be a locally Kan complete S-category (such as $\mathbf{Sp}$). Let $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{A} \to \mathbf{sSet}$ be S functors. Define the *coherent mean cotensor* of F and G by

$$\overline{\mathbf{B}^{\mathbf{A}}}(G, F) := \oint_A \mathbf{B}^{\mathbf{sSet}}(GA, FA).$$

This induces a new functor $\overline{F}$ defined by

$$\overline{F}(A) := \overline{\mathbf{B}^{\mathbf{A}}}(\mathbf{A}_{\mathbf{sSet}}(A, \bullet), F(\bullet)).$$

The importance of $\overline{F}$ is that it "absorbs coherence". The following is Proposition 3.3.i of [CP].

Proposition 3.5.5. Let $\mathbf{A}, \mathbf{B}$ be S-categories, with $\mathbf{B}$ complete. For simplicial functors $F, G : \mathbf{A} \to \mathbf{B}$, there is a natural isomorphism

$$\mathrm{Coh}(F, G) \cong \mathrm{Nat}(F, \overline{G}).$$

Armed with this result, we can now precompose coherent transformations with natural ones, and so we get a simplicial map

$$\mathrm{Coh}(\mathfrak{M}(d\text{-}\mathbf{B}(X)), \mathfrak{M}\mathbf{Vect}_{\mathbf{Man}}) \to \mathrm{Coh}\left(d\text{-}\mathbf{Cob}^{\otimes}\left(\mathrm{BO}(d) \times X, \pi^*_{\mathrm{BO}(d)}\gamma_{\mathrm{BO}(d)}\right), \mathfrak{M}\mathbf{Vect}_{\mathbf{Man}}\right).$$

The above map applied to 0-simplicies gives our desired map of field theories.

3.6 The Cases $d = 0, 1$

The goal of this section is to investigate the action of the map on field theories we have described, in cases where both smooth and homotopy field theories can be calculated.

For smooth field theories, $d = 0$ is treated in the paper [STHK]. In fact much more elaborate field theories of dimension 0 are classified in this paper, but along the way the authors establish that the category of smooth field theories (see Definition 3.2.1) of dimension 0 over X is equivalent to the (discrete) category of smooth functions on X, i.e. $\mathbf{Man}(X, \mathbb{R})$. In fact, this equivalence is a bijection on objects. There is only one object in $0\text{-}\boldsymbol{B}(X)$, namely the empty manifold. Bordisms over the point are finite collections of points with a map to X. For any singleton point mapped into X, we get a linear map of the monoidal unit of $\mathbf{Vect}_{\mathbf{pt}}$, which is $\mathbb{R}$, and a linear endomorphism of $\mathbb{R}$ can be identified with a number. In this way we get a *set* map $X \to \mathbb{R}$, and the fibering over **Man** enforces that it be smooth. On the other hand, any smooth field theory can be entirely reconstructed from this map to $\mathbb{R}$.

Also for smooth field theories, $d = 1$ is treated in the paper [STD]. In this case, it is shown that there is a bijection between (unoriented) 1-dimensional smooth field theories over X and smooth vector bundles over X with connection and fibre-wise pairing. This bijection is seen as follows. Connected objects of $1\text{-}\boldsymbol{B}(X)_{\mathbf{pt}}$ are collections of (collared) points with maps to X, which get sent to vector spaces under a smooth field theory. These vector spaces assemble to give a smooth vector bundle on X; again smoothness follows from things being fibred over **Man**. Connected bordisms over a point are paths mapping into X, which get sent to linear maps from the fibre over the start of the path to that of the end of the path.

These data assemble to give a parallel transport of the bundle on X, which are the same data as that of a connection. Again, it can be shown that an entire smooth field theory can be reconstructed just from the vector bundle and the connection.

As for homotopy field theories, the *extended* case has been solved completely in [Lur1]. Since we will be interested in the (Y, ζ) structure variety here, we present the general result for them, which is Theorem 2.4.18 in [Lur1].

Theorem 3.6.1. Let $\mathcal{C}$ be a symmetric monoidal (∞, d)-category with duals with $d > 0$, let Y be a CW complex, let ζ be a d-dimensional vector bundle over X equipped with an inner product, and let $\tilde{Y} \to X$ be the associated principal $\mathrm{O}(d)$ bundle of frames of ζ. Then there is an equivalence of $(\infty, 0)$-categories

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_d^{(Y,\zeta)}, \mathcal{C}) \simeq \underline{\mathbf{Top}}_{\mathrm{O}(d)}(\tilde{Y}, \mathcal{C}_{\mathbf{0}}).$$

We need to explain some of the notation above. First, 'with duals' refers to a higher categorical abstraction of the fact that finitie dimensional vector spaces have duals. In particular, $\mathfrak{M}\mathbf{Vect}_{\mathbf{Man}}$ has duals. Second, $\mathbf{Bord}_d^{(Y,\zeta)}$ is the analogue of d-$\mathbf{Cob}(Y, \zeta)$ extended to points. In particular, in the cases $d = 0, 1$, they agree. Next, an equivalence of $(\infty, 0)$ categories means a homotopy equivalence of spaces. In the notation of the right hand side

$$\underline{\mathbf{Top}}_{\mathrm{O}(d)}(\tilde{Y}, \mathcal{C}_{\mathbf{0}})$$

we are referring to $\mathrm{O}(d)$-equivariant maps of spaces, and the right hand side, zeroth space of $\mathcal{C}_{\mathbf{0}}$ should be thought of as the $(\infty, 0)$-category obtained from $\mathcal{C}$ by discarding non-invertible morphisms. See Appendix A.5.1 for an explanation of this. Finally, an explanation of why such a space obtained from a symmetric monoidal (∞, d)-category with duals, see Corollary 2.4.10 and Remark 2.4.11 in [Lur1]. For our purposes here, we only need to understand the action for $d = 0, 1$. In the case $d = 0$, $\mathrm{O}(d)$ is trivial, so there is no data. In the case $d = 1$, on objects the action of $\mathrm{O}(d) = \mathbb{Z}/2\mathbb{Z}$ is given by sending an object to its dual, and in the case of vector spaces, on morphisms the action sends an invertible linear map to the inverse of its adjoint (see Example 2.4.12 of [Lur1]).

We should also explain how the above equivelence arises. Given a point $\tilde{y} \in \tilde{Y}$, $\tilde{x}$ has an obvious (Y, ζ) structure, and therefore can be plugged into a homotopy field theory, yielding an object of $\mathcal{C}$. Again, the entire field theory can be reconstructed from this datum.

In this case $d = 0$, it is important to remember that points actually correspond to 1-morphisms. Indeed, in this case, the unique empty bordism must be sent to the space of monoidal units of $\mathcal{C}$, that is, the image of $\iota : \langle 0 \rangle \to \langle 1 \rangle$ of $\mathbf{\Gamma}$, which we will denote by $\mathcal{C}_{\mathbb{1}}$. Hence the right hand side should be replaced with

$$\underline{\mathbf{Top}}(X, \Omega\mathcal{C}_{\mathbb{1}}),$$

where we dropped the equivariance since $\mathrm{O}(0)$ is trivial. Here the Ω denotes the space of endomorphisms. The target is a loopspace of a contractible space, and is therefore contractible. So up to homotopy, such homotopy field theories are trivial.

Given a map $f \in \mathbf{Man}(X, \mathbb{R})$, we may construct a smooth field theory which sends a point $x \in X$ to the linear map defined by $f(x)$, considered as a morphism in $\mathbf{Vect_{pt}}$. This datum is essentially unchanged through the construction $\mathfrak{M}$; the point $x \in X$ corresponds to a point in

$$\mathfrak{M}(0\text{-}B(X))(\langle 1\rangle)_1 = \left|\mathrm{Mor}\left(0\text{-}B(X)_{\Delta^\bullet_{\mathrm{ext}}}\right)\right|$$

which is over a point, and is to the point in

$$\mathfrak{M}\left(\mathbf{Vect_{Man}}\right)(\langle 1\rangle)_1 = \left|\mathrm{Mor}\left(\left(\mathbf{Vect_{Man}}\right)_{\Delta^\bullet_{\mathrm{ext}}}\right)\right|$$

which is also over a point, corresponding to $f(x)$. If we arrange that the homotopy inverse used in Theorem 3.5.3 sends points of X to points of X, then we can identify the above data with a (continuous) map from X to $\mathbb{R}$, which is f, the one we started with. In this way, our construction in the case $d = 0$ can be interpreted as 'forget that the map was smooth'.

Next we calculate the right hand side of the formula in Theorem 3.6.1 in the remaining pertinent case. As explained in the previous section, if we start with X as the target for our smooth bordisms, the analogous structure for the homotopy bordisms is the pair

$$\left(\mathrm{BO}(d) \times X, \pi^*_{\mathrm{BO}(d)}\gamma_{\mathrm{BO}(d)}\right)$$

Plugging in $d = 1$, and passing to the principal O(1) bundle induced by the line bundle, we get

$$\mathrm{EO}(1) \times X \xrightarrow{p\times \mathrm{id}_X} \mathrm{BO}(1) \times X.$$

A model of EO(1) is given by S^∞, and in this case the $\mathrm{O}(1) = \mathbb{Z}/2\mathbb{Z}$ action is by the antipodal map. Hence 1-dimensional field theories with structure given by the above pair are equivalent to the space

$$\underline{\mathbf{Top}}_{\mathrm{O}(1)}\left(S^\infty \times X, \left|\mathrm{Obj}\left(\left(\mathbf{Vect_{Man}}\right)_{\Delta^\bullet_{\mathrm{ext}}}\right)\right|\right).$$

Now, the target is homotopy equivalent to $\coprod_k \mathrm{BO}(k)$, by homotopy equivalence of the counit, and the fact that this space is the classifying space for vector bundles of arbitrary dimension. So we have

$$\underline{\mathbf{Top}}_{\mathrm{O}(1)}(S^\infty \times X, \coprod_k \mathrm{BO}(k)) = \underline{\mathbf{Top}}(X, \underline{\mathbf{Top}}_{\mathrm{O}(1)}(S^\infty, \coprod_k \mathrm{BO}(k)) \simeq \underline{\mathbf{Top}}\left(X, \left(\coprod_k \mathrm{BO}(k)\right)^{h\mathbb{Z}/2\mathbb{Z}}\right),$$

where the target denotes the space of homotopy fixed points of the action described above (again, see 2.4.12 of [Lur1]). One model of the target is the space of subspaces of $\mathbb{R}^\infty$ with a chosen inner product, so that they are canonically self-dual (see Example 2.4.28 in [Lur1]). Hence, the map on field theories in this case amounts to forgetting the bundle is smooth, and forgetting the connection. Again, we can think of this as 'forgetting smoothness'.

Chapter 4

Categories of Field Theories

The goal of this chapter is to upgrade the map from smooth field theories to homotopy field theories, produced in chapters 1 and 2, to a map from the fibred category of smooth field theories to the Segal space of homotopy field theories. This means in particular that we need maps

$$\mathfrak{M}\underline{\mathrm{Fun}}^{\otimes}_{\mathbf{Man}}(d\text{-}\mathbf{B}(X), \mathbf{Vect}_{\mathbf{Man}}) \to \underline{\mathrm{Fun}}^{\otimes}(\mathfrak{M}d\text{-}\mathbf{B}(X), \mathfrak{M}\mathbf{Vect}_{\mathbf{Man}})$$

We proceed through the construction of this $\mathfrak{M}$, (from Section 3.4) and produce maps which commute each step with the intermediate inner-hom as we go.

4.1 Functor Categories and Hom-Objects

The goal of this section is two technical Lemmas, the first of which we will use throughout this chapter, and the second of which we need for Section 4.3.

We will use power notation to denote functor category in this section. We will also use the notation $\overline{c}$ to denote a constant functor sending everything to a fixed object c. We avoid using the usual Δ for this 'diagonal functor' to avoid confusion with the simplicial stuff ubiquitous in this work. For a discussion of ends, refer to Appendix A.2.

Remark 4.1.1. Let $\mathcal{C}$ be a cartesian closed category. Then inner-hom induces a functor

$$\underline{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}.$$

This is a consequence of a pointwise-adjunction (inner-hom) inducing an adjuntion as explained in [Mac].

The next result shows that hom-objects in a functor category are a special case of ends.

Lemma 4.1.2. Suppose that $\mathcal{C}$ is a cartesian closed category and that $\mathcal{J}$ is a small category. Suppose also that $\mathcal{C}$ is small-complete indexed over $\mathcal{J}$. Then $\mathcal{C}^{\mathcal{J}}$ is enriched over $\mathcal{C}$ with the cartesian monoidal structure, and the hom-objects satisfy the following adjunction property

$$\mathcal{C}(c, \mathcal{C}^{\mathcal{J}}_{\mathcal{C}}(F, G)) \cong \mathcal{C}^{\mathcal{J}}(F \times \overline{c}, G).$$

Here, the notation $\mathcal{C}^{\mathcal{J}}_{\mathcal{C}}$ indicates the $\mathcal{C}$ hom-objects of $\mathcal{C}^{\mathcal{J}}$.

Proof. We need to define the hom-objects. As stated above, they are ends. Let $F, G : \mathcal{J} \to \mathcal{C}$, and form the functor

$$\underline{\mathcal{C}} \circ (F^{op} \times G) : \mathcal{J}^{op} \times \mathcal{J} \to \mathcal{C}$$

which has an end by completeness, denoted

$$\int_j \underline{\mathcal{C}}(F(j), G(j))$$

and these, for given F, G, are the hom-objects. The identity and composition properties follow from those of the inner-hom and the 'Fubini theorem' for ends. As for for the adjunction property we have

$$\begin{aligned} \mathcal{C}\left(c, \int_j \underline{\mathcal{C}}(F(j), G(j))\right) &\cong \int_j \mathcal{C}\left(c, \underline{\mathcal{C}}(F(j), G(j))\right) && (4.1)\\ &\cong \int_j \mathcal{C}(c \times F(j), G(j)) && (4.2)\\ &\cong \mathcal{C}^{\mathcal{J}}(\overline{c} \times F, G). && (4.3)\end{aligned}$$

In the first step, we used that an end is a limit, so commutes with hom. In the last step, we used the fact that hom-sets in functor categories are natural transformations, and applied Lemma A.2.4. ☐

We note here that the adjunction property says that the hom-objects represent a certain functor. In particular, they are automatically unique up to isomorphism, no matter how we 'produce' them.

Finally, we arrive at the result that will be the key to proving the main theorem of Section 4.3.

Lemma 4.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed, small-complete categories, and let $\mathcal{J}$ be a small category. Suppose $\mathrm{N} : \mathcal{C} \to \mathcal{D}$ is a right adjoint, and suppose also that N is compatible with inner-hom, in the sense that

$$\mathrm{N}\, \underline{\mathcal{C}}(X, Y) \cong \underline{\mathcal{D}}(\mathrm{N}\, X, \mathrm{N}\, Y),$$

naturally for $X, Y \in \mathcal{C}$. Then for $F, G \in \mathcal{C}^{\mathcal{J}}$, we have

$$\mathrm{N}\, \mathcal{C}^{\mathcal{J}}_{\mathcal{C}}(F, G) = \mathcal{D}^{\mathcal{J}}_{\mathcal{D}}(\mathrm{N}\, F, \mathrm{N}\, G).$$

Proof. In 4.1.2, we showed that the hom-objects in functor categories are ends. We therefore compute

$$\begin{aligned} \mathrm{N}\,\mathcal{C}^{\mathcal{J}}_{\mathcal{C}}(F,G) &\cong \mathrm{N}\int_j \underline{\mathcal{C}}(F(j),G(j)) \\ &\cong \int_j \mathrm{N}\,\underline{\mathcal{C}}(F(j),G(j)) \\ &\cong \int_j \underline{\mathcal{D}}(\mathrm{N}\,F(j),\mathrm{N}\,G(j)) \\ &\cong \mathcal{D}^{\mathcal{J}}_{\mathcal{D}}(\mathrm{N}\,F,\mathrm{N}\,G), \end{aligned}$$

where in the second step, we used that N is a right adjoint, and therefore preserves ends. In the third step, we used that limits of naturally isomorphic diagrams are isomorphic. ☐

4.2 Strictification and Restriction

The first step in our construction is to send categories symmetrically fibred over **Man** to strict functors $\mathbf{Man}^{\mathrm{op}} \to \mathbf{SMCat}$ via the construction $\mathbf{E} \mapsto \underline{\mathbf{E}}$ and send fibred functors to strictly natural transformations via $F \mapsto \underline{F}$. These constructions were introduced at the end of Section 2.1. Our goal is to show that this step commutes with inner-hom in the following sense.

Theorem 4.2.1. Let $\mathbf{E}', \mathbf{E} \in \mathrm{Fib}^{\otimes}(\mathcal{S})$, for some base category $\mathcal{S}$. Then there is a natural isomorphism of functors

$$\underline{\underline{\mathrm{Fun}^{\otimes}_{\mathcal{S}}(\mathbf{E},\mathcal{F})}} \cong \underline{\mathbf{SMCat}^{\mathcal{S}^{\mathrm{op}}}}(\underline{\mathbf{E}'},\underline{\mathbf{E}})$$

First, we explaine what the right hand side means.

Remark 4.2.2. There is an inner-hom in $\mathbf{SMCat}^{\mathcal{S}^{\mathrm{op}}}$. Given two functors $\mathcal{F}', \mathcal{F} : \mathcal{S}^{\mathrm{op}} \to \mathbf{SMCat}$, we have

$$\underline{\mathbf{SMCat}^{\mathcal{S}^{\mathrm{op}}}}(\mathcal{F}',\mathcal{F})(S) = \mathbf{SMCat}^{\mathcal{S}^{\mathrm{op}}}_{\mathbf{SMCat}}(\mathcal{F}' \times \underline{S}, \mathcal{F}).$$

Here the enrichment is given by Lemma 4.1.2, and $\underline{S}$ is the Yoneda construction of Remark 2.1.20.

Now, we are ready for the proof.

Proof. We calculate the value of each side at $S \in \mathcal{S}$. The left hand side is

$$\begin{aligned} \underline{\underline{\mathrm{Fun}^{\otimes}_{\mathcal{S}}(\mathbf{E},\mathcal{F})}}(S) &= \mathrm{Fun}^{\otimes}_{\mathcal{S}}(\underline{S}, \underline{\mathrm{Fun}^{\otimes}_{\mathcal{S}}}(\mathbf{E}',\mathbf{E})) \\ &\cong \mathrm{Fun}^{\otimes}_{\mathcal{S}}(\mathbf{E}' \times \underline{S}, \mathbf{E}) \end{aligned}$$

The monoidal structures here are induced pointwise, as the domains $\underline{S}$ do not have a monoidal structure. The right hand side, in view of the remark above, is

$$\begin{aligned}\underline{\mathbf{SMCat}^{\mathcal{S}^{\mathrm{op}}}}(\underline{\mathbf{E}}', \underline{\mathbf{E}}) &\cong \int_{S'} \mathrm{Fun}^{\otimes}(\underline{\mathbf{E}}'(S') \times \underline{S}(S'), \underline{\mathbf{E}}(S')) \\ &= \int_{S'} \mathrm{Fun}^{\otimes}\left(\mathrm{Fun}^{\otimes}_{\mathcal{S}}(\underline{S'}, \mathbf{E}' \times \underline{S}), \mathrm{Fun}^{\otimes}_{\mathcal{S}}(\underline{S'}, \mathbf{E})\right).\end{aligned}$$

Now there is an obvious wedge

$$\mathrm{Fun}^{\otimes}_{\mathcal{S}}(\mathbf{E}' \times \underline{S}, \mathbf{E}) \Rightarrow \mathrm{Fun}^{\otimes}\left(\mathrm{Fun}^{\otimes}_{\mathcal{S}}(\underline{\bullet}, \mathbf{E}' \times \underline{S}), \mathrm{Fun}^{\otimes}_{\mathcal{S}}(\underline{\bullet}, \mathbf{E})\right)$$

given by post composition. The wedge condition follows since composition is associative. It is clear that this wedge is initial, and finally, it is natural in S' also because composition is associative. This completes the proof. □

The next step in the construction of $\mathfrak{M}$ was to replace the target **SMCat** with an equivalent full subcategory of $\mathbf{\Gamma} \to \mathbf{Cat}$. In this step, we already studied what happens to the category of symmetric monoidal functors between two symmetric monoidal categories; they correspond to pseudonatural transformations, which in turn correspond to lax ends (see Appendix A.3). Hence, at the point in the construction where our objects are strict functors

$$\mathbf{Man}^{\mathrm{op}} \to (\mathbf{\Gamma} \to \mathbf{Cat}),$$

where the target 2-category consists of pseudofunctors, pseudonatural transformations, and modifications, the version of inner-hom is

$$\mathcal{F}', \mathcal{F} \mapsto \oint_{\langle n \rangle} \mathrm{Fun}\left(\mathcal{F}'(\bullet)(\langle n \rangle), \mathcal{F}(\bullet)(\langle n \rangle)\right),$$

considered as a functor $\mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$.

Next, we restricted the domain from $\mathbf{Man}^{\mathrm{op}}$ to the embedded copy of $\mathbf{\Delta}^{\mathrm{op}}$ defined by the extended simplicies. It is easiest to demonstrate the commutator map for inner-hom if we perform this restriction initially with the symmetrical monoidally fibred categories. Changing the order to do the restriction first clearly does not change $\mathfrak{M}$.

Lemma 4.2.3. Let $\mathbf{E}', \mathbf{E} \in \mathrm{Fib}^{\otimes}(\mathbf{Man})$, and let $\mathbf{E}_{\mathbf{\Delta}}$ denote the restriction to $\mathbf{\Delta} \hookrightarrow \mathbf{Man}$ given by the extended simplicies; which is the pullback along this embedding. Then there is a fibred functor

$$\underline{\mathrm{Fun}}^{\otimes}_{\mathbf{Man}}(\mathbf{E}', \mathbf{E})_{\mathbf{\Delta}} \to \underline{\mathrm{Fun}}^{\otimes}_{\mathbf{\Delta}}(\mathbf{E}'_{\mathbf{\Delta}}, \mathbf{E}_{\mathbf{\Delta}})$$

Proof. We have

$$\left(\underline{\mathrm{Fun}}^{\otimes}_{\mathbf{Man}}(\mathbf{E}', \mathbf{E})_{\mathbf{\Delta}}\right)_{[k]} = \mathrm{Fun}^{\otimes}_{\mathbf{Man}}(\mathbf{E}' \times \underline{\Delta^k_{\mathrm{ext}}}, \mathbf{E})$$

Now, in the above, $\underline{\Delta^k_{\text{ext}}}$ is considered as fibred over **Man**. In particular, its value at another extended simplex consists of the set of *all* smooth maps with target Δ^k_{ext}. We remind of this fact with a superscript **Man**. We may forget what these fibred functors do over the fibres that are not extended simplices, and continue with the following.

$$\begin{aligned}
\mathrm{Fun}^{\otimes}_{\mathbf{Man}}(\mathbf{E}' \times \underline{\Delta^k_{\text{ext}}}^{\mathbf{Man}}, \mathbf{E}_{\mathbf{\Delta}}) &\to \mathrm{Fun}^{\otimes}_{\mathbf{\Delta}}(\mathbf{E}'_{\mathbf{\Delta}} \times \underline{\Delta^k_{\text{ext}}}_{\mathbf{\Delta}}^{\mathbf{Man}}, \mathbf{E}_{\mathbf{\Delta}}) \\
&\to \mathrm{Fun}^{\otimes}_{\mathbf{\Delta}}(\mathbf{E}'_{\mathbf{\Delta}} \times \underline{[k]}, \mathbf{E}_{\mathbf{\Delta}}) \\
&= \mathrm{Fun}^{\otimes}_{\mathbf{\Delta}}(\mathbf{E}'_{\mathbf{\Delta}}, \mathbf{E}_{\mathbf{\Delta}})
\end{aligned}$$

The second map is induced by the inclusion of categories fibred over $\mathbf{\Delta}$

$$\underline{[k]} \hookrightarrow \underline{\Delta^k_{\text{ext}}}_{\mathbf{\Delta}}^{\mathbf{Man}}$$

given by inclusion of the simplicial maps into all smooth maps. □

The next step involves switching factors of the domain. We make the following remark.

Remark 4.2.4. Since inner-homs are defined to be the right adjoint of certain functors, it is clear that they are compatible with equivalences of categories, in the sense that if $\phi : \mathcal{C} \to \mathcal{D}$ is an equivalence of cartesian closed categories, then

$$\phi\underline{\mathcal{C}}(X, Y) \cong \underline{\mathcal{D}}(\phi X, \phi Y)$$

This remark also applies to the step which replaces pseudofunctors with domain $\mathbf{\Gamma}$ with strict functors with domain $\triangleleft\mathbf{\Gamma}$.

The next step in the construction is to apply the hom-wise nerve functor N_*. Proposition 2.4.7 says exactly that this functor commutes with inner-hom.

Finally, we apply nerve to the inner-most target **Cat**, switch the two $\mathbf{\Delta}^{\mathrm{op}}$ factors (an equivalence, in fact an isomorphism) which commutes with inner-hom, and then take geometric realization along the inner $\mathbf{\Delta}^{\mathrm{op}}$ coordinate. Finding the commutator maps of inner-hom for the nerve and realization steps will take up the next few sections.

4.3 Simplicial Categories and Bisimplicial Sets

Let **C** and **D** be functors $\mathbf{\Delta}^{op} \to \mathbf{Cat}$. We will call the category of such functors *simplicial categories*, denoted **sCat**. Let N denote the nerve functor, and note that we can form a simplicial-simplicial set by taking pointwise nerve:

$$(\mathrm{N}\,\mathbf{C})[k] = \mathrm{N}\,\mathbf{C}([k]).$$

This is just a composition of functors.

The notation **sCat** is a special case of putting an **s** in front of the name of a category to indicate functors from $\mathbf{\Delta}^{op}$ into it. So **ssSet** is the category of bisimplicial sets. It is convenient in the following to think of **ssSet** as $\mathbf{sSet}^{\mathbf{\Delta}^{op}}$. As usual, we will use bar to denote the constant functor.

Remark 4.3.1. Note that **ssSet** is properly tensored (see Appendix A.1) over **sSet**, with

$$\mathbf{ssSet}_{\mathbf{sSet}}(\boldsymbol{X}_\bullet, \boldsymbol{Y}_\bullet)_k := \mathbf{ssSet}(\boldsymbol{X}_\bullet \times \overline{\Delta(k)}, \boldsymbol{Y}_\bullet)$$

Furthermore, **ssSet** is cartesian closed, with the inner-hom being defined in terms of the **sSet**-homs.

$$\underline{\mathbf{ssSet}}(\boldsymbol{X}_\bullet, \boldsymbol{Y}_\bullet)_k := \mathbf{ssSet}_{\mathbf{sSet}}(\boldsymbol{X}_\bullet \times \operatorname{disc}(\Delta(k)), \boldsymbol{Y}_\bullet)$$

Here we are using the functor **sSet** → **ssSet** that is constant in the other simplicial direction. In other words $\operatorname{disc} S_\bullet : \boldsymbol{\Delta}^{op} \to \mathbf{sSet}$ sends $[k]$ to the constant simplicial set S_k.

We note that we can enrich **ssSet** over **sSet** in either coordinate. We make the convention that when talking about Segal spaces $\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet$ (see Appendix A.5.1) we enrich in the coordinate so that

$$\underline{\mathbf{ssSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet)$$

is a Kan complex. Note that this is required for the Segal space model structure to be simplicial.

Remark 4.3.2. **sCat** has a version of inner-hom. It is defined by

$$\underline{\mathbf{sCat}}(\mathcal{F}, \mathcal{G})[j] = \mathbf{sCat}(\mathcal{F} \times \underline{[j]}, \mathcal{G}),$$

where the right hand side is considered as a category. Also, by $\underline{[j]}$, we mean the simplicial category with discrete fibers $\boldsymbol{\Delta}([k], [j])$.

We now have the machinery necessary to state the main theorem of the section.

Theorem 4.3.3. Let $\mathbf{C}, \mathbf{D} \in \mathbf{sCat}$. Then there is an isomorphism of bisimplicial sets

$$\mathrm{N}\, \underline{\mathbf{sCat}}(\mathbf{C}, \mathbf{D}) \cong \underline{\mathbf{ssSet}}(\mathrm{N}\,\mathbf{C}, \mathrm{N}\,\mathbf{D}).$$

For the proof we need a couple of more intermediate results. First, a well-known fact, whose proof we include for completeness:

Lemma 4.3.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then there is an isomorphism of simplicial sets

$$\mathrm{N}\left(\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})\right) \cong \underline{\mathbf{sSet}}(\mathrm{N}\,\mathcal{C}, \mathrm{N}\,\mathcal{D})$$

Proof. We recall one definition of nerve:

$$\mathrm{N}\,\mathcal{C}_k := \mathrm{Fun}([k], \mathcal{C}),$$

where the ordinal $[k]$ is considered as a category via its poset structure, and the right hand side is the set of functors (as opposed to category).

Now, we check that there is a bijection of *sets*:

$$\mathrm{N}\left(\mathrm{Fun}(\mathcal{C},\mathcal{D})\right)_0 \cong \mathbf{sSet}(\mathrm{N}\,\mathcal{C},\mathrm{N}\,\mathcal{D})$$

The left hand side is the set of functors $\mathcal{C} \to \mathcal{D}$. A functor clearly induces a simplicial map on nerves, using the definition above. The right hand side is the simplicial maps $\mathrm{N}\,\mathcal{C} \to \mathrm{N}\,\mathcal{D}$, and the simplicial identities in dimensions 0,1,2 for a simplicial set map $\mathbf{sSet}(\mathrm{N}\,\mathcal{C},\mathrm{N}\,\mathcal{D})$ are exactly the conditions a functor must satisfy.

Next, we have

$$\begin{aligned}\mathrm{N}\left(\mathrm{Fun}(\mathcal{C},\mathcal{D})\right)_k &= \mathrm{Fun}([k],\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D}))\\ &\cong \mathrm{Fun}([k]\times\mathcal{C},\mathcal{D})\\ &\cong \mathbf{sSet}(\mathrm{N}[k]\times\mathrm{N}\,\mathcal{C},\mathrm{N}\,\mathcal{D})\\ &\cong \mathbf{sSet}(\Delta(k)\times\mathrm{N}\,\mathcal{C},\mathrm{N}\,\mathcal{D})\\ &\cong \underline{\mathbf{sSet}}(\mathrm{N}\,\mathcal{C},\mathrm{N}\,\mathcal{D})_k.\end{aligned}$$

In the second-to-last step we used the observation that

$$\mathrm{N}[k]_j = \mathrm{Fun}([j],[k]) \cong \mathbf{\Delta}([j],[k]) =: \Delta(k).$$

since a functor on ordinals considered as posets is precisely a nondecreasing map. The steps are clearly natural in all variables, so we get the desired result.

□

Lemma 4.3.5. Let $\mathcal{F}$ be a simplicial category. Then,

$$\mathrm{N}\,\mathcal{F}\times\mathrm{disc}\,\Delta(k) \simeq \mathrm{N}\left(\mathcal{F}\times\underline{[k]}\right).$$

And the isormorphism is natural in $\mathcal{F}$ and k.

Proof. We compute

$$\begin{aligned}\left(\mathrm{N}\,\mathcal{F}\times\mathrm{disc}\,\Delta(k)\right)_j &= \mathrm{N}\,\mathcal{F}[j]\times\overline{\mathbf{\Delta}([j],[k])}\\ &\cong \mathrm{N}\,\mathcal{F}[j]\times\mathrm{N}\,\mathbf{\Delta}([j],[k])\\ &= \mathrm{N}\,\mathcal{F}[j]\times\mathrm{N}\,\underline{[k]}_j\\ &\cong \mathrm{N}\left(\mathcal{F}\times\underline{[k]}\right)_j\end{aligned}$$

In its first appearance, $\overline{\mathbf{\Delta}([j],[k])}$, was a constant simplicial set (i.e. the zero-simplicies are the only non-degenerate ones). Next, we considered $\mathbf{\Delta}([j],[k])$ as a discrete category. We also used that nerve has a left adjoint, and therefore preserves products. All the steps above clearly commute with the face and degeneracy maps in the coordinate j. □

We are now ready for the proof of the main result of this section:

Proof of 4.3.3. We are showing

$$\mathrm{N}\,\underline{\mathbf{sCat}}(\mathcal{F},\mathcal{G}) \cong \underline{\mathbf{ssSet}}(\mathrm{N}\,\mathcal{F},\mathrm{N}\,\mathcal{G}),$$

where $\mathcal{F},\mathcal{G} : \mathbf{\Delta}^{op} \to \mathbf{Cat}$. As usual, we look at each side's value point-wise on $\mathbf{\Delta}^{op}$.

$$\mathrm{N}\,\underline{\mathbf{sCat}}(\mathcal{F},\mathcal{G})[k] = \mathrm{N}\,\mathbf{sCat}(\mathcal{F}\times\underline{[k]},\mathcal{G})$$

And similarly,

$$\begin{aligned}\underline{\mathbf{ssSet}}(\mathrm{N}\,\mathcal{F},\mathrm{N}\,\mathcal{G})[k] &\simeq \mathbf{ssSet}_{\mathbf{sSet}}(\mathrm{N}\,\mathcal{F}\times\operatorname{disc}\Delta(k),\mathrm{N}\,\mathcal{G})\\ &\simeq \mathbf{ssSet}_{\mathbf{sSet}}\left(\mathrm{N}\left(\mathcal{F}\times\underline{[k]}\right),\mathrm{N}\,\mathcal{G}\right),\end{aligned}$$

where we used 4.3.5 in the second step. Since everything is natural, we have reduced the problem to

$$\mathrm{N}\,\mathbf{sCat}(\mathcal{F},\mathcal{G}) \simeq \mathbf{ssSet}_{\mathbf{sSet}}(\mathrm{N}\,\mathcal{F},\mathrm{N}\,\mathcal{G}),$$

as simplicial sets.

Now we can apply 4.1.3 with **Cat** and **sSet** in place of **A** and **B**, $\mathbf{\Delta}$ in place of $\mathcal{J}$, and finally nerve in place of N. Nerve preserves inner-hom by 4.3.4 and the fact that it has a left adjoint, called fundamental category, is proved in [JT]. □

4.4 Bisimplicial Sets and Geometric Realization

For this section, we will use the notation **Top** for the category of compactly generated Hausdorff spaces, and **sTop** for the functor category $\mathbf{Top}^{\mathbf{\Delta}^{op}}$. Recall the geometric realization functor

$$|\cdot| : \mathbf{sSet} \to \mathbf{Top}$$

Which can be defined as the coend

$$|S_\bullet| := \int^n S_n \times \Delta^n,$$

with S_n given the discrete topology, and $\Delta^n = \{(t_0,\cdots,t_n)\in\mathbb{R}^{n+1} | \sum t_i = 1\}$ the standard n simplex (see [JT]). Also recall the singular simplices functor

$$\operatorname{Sing} : \mathbf{Top} \to \mathbf{sSet}$$

defined by $\operatorname{Sing}(X)_k = \mathbf{Top}(\Delta^k, X)$. It is well-known that Sing is right-adjoint to $|\cdot|$ ([JT]).

The goal of this section is to provide a map analogous to the isomorphism in Theorem 4.3.3 for the simplicial realization functor. In this case, we only get a map, which we conjecture to be a weak equivalence on the full subcategory of simplicial Segal spaces.

Theorem 4.4.1. Let $\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet \in \mathbf{ssSet}$. Then there is a natural transformation

$$|\underline{\mathbf{ssSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet)| \to \underline{\mathbf{sTop}}(|\boldsymbol{S}_\bullet|, |\boldsymbol{T}_\bullet|)$$

that is, natural in $\boldsymbol{S}_\bullet$ and $\boldsymbol{T}_\bullet$.

Lemma 4.4.2. Let $F, G : \mathbf{sSet}^{op} \times \mathbf{sSet} \to \mathbf{Top}$ denote the given by taking inner-hom, then realizing, and realizing and then taking inner-hom, respectively. That is,

$$F(S_\bullet, T_\bullet) = |\underline{\mathbf{sSet}}(S_\bullet, T_\bullet)|\,,$$

and

$$G(S_\bullet, T_\bullet) = \underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|).$$

Then there is a natural transformation $\phi : F \Rightarrow G$.

Proof. We must construct morphisms

$$\phi(S_\bullet, T_\bullet) \in \mathbf{Top}\left(|\underline{\mathbf{sSet}}(S_\bullet, T_\bullet)|\,, \underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|)\right)$$

Natural in $S_\bullet$ and $T_\bullet$. Note that

$$\mathbf{Top}\left(|\underline{\mathbf{sSet}}(S_\bullet, T_\bullet)|\,, \underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|)\right) \cong \mathbf{sSet}\left(\underline{\mathbf{sSet}}(S_\bullet, T_\bullet), \operatorname{Sing}\underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|)\right)$$

And furthermore

$$\begin{aligned} \operatorname{Sing}\underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|)_{[k]} &= \mathbf{Top}\left(\Delta^k, \underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|)\right) \\ &\simeq \mathbf{Top}\left(|S_\bullet| \times \Delta^k, |T_\bullet|\right) \end{aligned}$$

Recall that $|\Delta(k)|$ is the coend

$$|\Delta(k)| = \int^n \boldsymbol{\Delta}(n, k) \times \Delta^n.$$

For all n, we have maps $\boldsymbol{\Delta}(n, k) \times \Delta^n \to \Delta^k$, induced by the standard embedding $\boldsymbol{\Delta} \to \mathbf{Top}$ followed by an evaluation map. These maps assemble to form a wedge

$$\boldsymbol{\Delta}(\cdot, k) \times \Delta^{\cdot} \Rightarrow \Delta^k$$

and therefore by the universal property of coends, a map $\Delta^k \to |\Delta(k)|$, *which is natural in* k. Hence we obtain a map

$$\begin{aligned} \mathbf{Top}\left(|S_\bullet| \times \Delta^k, |T_\bullet|\right) &\leftarrow \mathbf{Top}\left(|S_\bullet| \times |\Delta(k)|, |T_\bullet|\right) \\ &\cong \mathbf{Top}\left(|S_\bullet \times \Delta(k)|, |T_\bullet|\right) \\ &\cong \mathbf{sSet}\left(S_\bullet \times \Delta(k), \operatorname{Sing}|T_\bullet|\right). \end{aligned}$$

where in the second step we used that simplicial realization commutes with finite limits (see [JT]).

The resulting maps

$$\mathbf{sSet}(S_\bullet \times \Delta(k), \operatorname{Sing}|T_\bullet|) \to \operatorname{Sing}\underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|)_{[k]}$$

created by composing the above maps and equivalences are natural in $S_\bullet$ and $T_\bullet$ since the adjunctions involved are, and natural in $[k]$ as well, since evaluation is natural. Naturality in $[k]$ implies that they assemble to form a map of simplicial sets:

$$\underline{\mathbf{sSet}}(S_\bullet, \operatorname{Sing}|T_\bullet|) \to \operatorname{Sing}\underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|)$$

Note that the simplicial sets above induce the usual simplicial model category structure on **sSet** and **Top**. Since $|\bullet| : \mathbf{sSet} \rightleftharpoons \mathbf{Top} : \operatorname{Sing}$ is an equivalence of *cartesian* model categories ([JT]), we have the weak equvalence

$$\mathbf{sSet}_{\mathbf{sSet}}(S_\bullet, \operatorname{Sing} X) \cong \mathbf{Top}_{\mathbf{sSet}}(|S_\bullet|, X),$$

The map above is therefore a weak equivalence, being the special case where $X = |T_\bullet|$.

We also have a map of simplicial sets

$$\mathbf{sSet}(S_\bullet, T_\bullet) \to \mathbf{sSet}(S_\bullet, \operatorname{Sing}|T_\bullet|)$$

induced by the unit of the adjunction $|\bullet| \dashv \operatorname{Sing}$.

Composing the morphism $\mathbf{sSet}(S_\bullet, T_\bullet) \to \mathbf{sSet}(S_\bullet, \operatorname{Sing}|T_\bullet|)$ and the weak equivalence $\underline{\mathbf{sSet}}(S_\bullet, \operatorname{Sing}|T_\bullet|) \to \operatorname{Sing}\underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|)$, we obtain

$$\hat{\phi}(S_\bullet, T_\bullet) : \underline{\mathbf{sSet}}(S_\bullet, T_\bullet) \to \operatorname{Sing}\underline{\mathbf{Top}}(|S_\bullet|, |T_\bullet|)$$

The adjoint of these are the $\phi(S_\bullet, T_\bullet)$ we are after. Again, this construction consisted of natural transformations at every step, so the resulting ϕ's form a natural transformation of functors $\phi : F \Rightarrow G$. □

The following example shows why we cannot expect 4.4.2 to produce an isomorphism or even a weak equivalence in general.

Example 4.4.3. Let $\mathcal{C}$ be the category $\bullet \rightrightarrows \bullet$. Then $B\mathcal{C}$ is homotopy equivalent to a circle. Hence, $\underline{\mathbf{Top}}(B\mathcal{C}, B\mathcal{C})$ has the homotopy type of $\mathbb{Z} \times S^1$. On the other hand, there are only finitely many functors $\mathcal{C} \to \mathcal{C}$ (in fact there are exactly four). So $B\underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{C})$ only has finitely many connected components, and therefore is not homotopy equivalent to $\underline{\mathbf{Top}}(B\mathcal{C}, B\mathcal{C})$.

The next lemma is key in showing that the morphism produced in 4.4.2 *is* a weak equivalence provided that $T_\bullet$ is a fibrant, i.e. Kan, simplicial set.

Lemma 4.4.4. Let $S_\bullet \in \mathbf{sSet}$. Then the functor $F : \mathbf{sSet} \to \mathbf{sSet}$ defined by $F(T_\bullet) = \underline{\mathbf{sSet}}(S_\bullet, T_\bullet)$ sends weak equivalences of fibrant objects to weak equivalences.

Proof. We use a standard fact from model category theory (see Appendix A.4) that a weak equivalence of fibrant objects is actually a (left or right) homotopy equivalence. We also need the fact that $\underline{\mathbf{sSet}}(\Delta(1), X_\bullet)$ is a path object for any $X_\bullet$, with 'source' and 'target' projections induced by the inclusions of the zero simplices $\Delta(0) \hookrightarrow \Delta(1)$ (see [JT]). Let $f : T'_\bullet \to T_\bullet$ be a weak equivalence of fibrant objects, which therefore has a right homotopy inverse g.

We claim that the induced morphisms $F(f) : \underline{\mathbf{sSet}}(S_\bullet, T'_\bullet) \to \underline{\mathbf{sSet}}(S_\bullet, T_\bullet)$ and g_* constitute a (right) homotopy equivalence $\underline{\mathbf{sSet}}(S_\bullet, T'_\bullet) \sim \underline{\mathbf{sSet}}(S_\bullet, T_\bullet)$, and hence $F(f)$ is a weak equivalence

Let $h : T_\bullet \to \underline{\mathbf{sSet}}(\Delta(1), T_\bullet)$ be a right homotopy from fg to $1_{T_\bullet}$. Then h induces a morphism

$$F(h) : \underline{\mathbf{sSet}}(S_\bullet, T_\bullet) \to \underline{\mathbf{sSet}}(S_\bullet, \underline{\mathbf{sSet}}(\Delta(1), T_\bullet))$$

And the target of this morphism is a path object for $\underline{\mathbf{sSet}}(S_\bullet, T_\bullet)$, indicated by Yoneda's lemma applied to this bijection of hom-sets:

$$\begin{aligned}\mathbf{sSet}(X_\bullet, \underline{\mathbf{sSet}}(S_\bullet, \underline{\mathbf{sSet}}(\Delta(1), T_\bullet))) &\cong \mathbf{sSet}((X_\bullet \times S_\bullet) \times \Delta(1), T_\bullet) \\ &\cong \mathbf{sSet}((X_\bullet \times \Delta(1)) \times S_\bullet, T_\bullet) \\ &\cong \mathbf{sSet}(X_\bullet, \underline{\mathbf{sSet}}(\Delta(1), \underline{\mathbf{sSet}}(S_\bullet, T_\bullet)))\end{aligned}$$

The steps above are natural in the $\Delta(1)$ entry, and F is natural, being a functor, so we have

$$F(h)|_{t=0} = F(h|_{t=0}) = F(fg) = F(f) \circ F(g)$$

and similarly $F(h)|_{t=1} = F(1_{T_\bullet}) = 1$. ☐

Note that one can alternatively use Ken Brown's Lemma (A.4.8) directly.

Theorem 4.4.5. With F and G as defined in Lemma 4.4.2, and restricting their domains to $\mathbf{sSet}^{op} \times \mathbf{Kan}$, there is a natural transformation $F \Rightarrow G$ which is a point-wise weak equivalence.

Proof. Looking back through the proof of 4.4.2 and applying 4.4.4 we can see that $\hat{\phi}(S_\bullet, T_\bullet)$ is a weak equivalence provided that $T_\bullet$ is Kan. (Sing $|T_\bullet|$ is Kan, since Sing lands in the subcategory of Kan complexes.) Thus, $\phi(S_\bullet, T_\bullet)$ is a weak equivalence; the hom adjunction takes weak equivalences to weak equivalences, being part of a Quillen equivalence. ☐

Now, as in section 4.3, we need to add in the extra simplicial factor to ramp this up to a natural transformation

$$|\underline{\mathbf{ssSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet)| \simeq \underline{\mathbf{sTop}}(|\boldsymbol{S}_\bullet|, |\boldsymbol{T}_\bullet|).$$

Here $\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet$ are bisimplicial sets, where $\boldsymbol{S}_k$ is the k^{th} simplicial set of $\boldsymbol{S}_\bullet$. Note in particular that we are only realizing along one of the simplicial coordinates.

We proceed in the same manner as in Section 4.3, moving from simplicial sets to the **sSet**-homs of **ssSet**, and finally to inner-hom in **ssSet**. More care must be taken at certain steps, since weak equivalences do not always enjoy the same properties as isomorphisms. For example, the limit of a diagram of weak equivalences is not necessarily a weak equivalence. Here again we are thinkin of **ssSet** as $\mathbf{sSet}^{\boldsymbol{\Delta}^{op}}$, and this 'outer' simplicial coordinate will be the only one tampered with.

Lemma 4.4.6. Let $\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet \in \mathbf{ssSet}$ be Segal spaces. Then there is a map

$$|\mathbf{ssSet}_{\mathbf{sSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet)| \to \mathbf{sTop}_{\mathbf{Top}}(|\boldsymbol{S}_\bullet|, |\boldsymbol{T}_\bullet|),$$

which is natural in $\boldsymbol{S}_\bullet$ and $\boldsymbol{T}_\bullet$.

Proof. We look for such a map in the set

$$\mathbf{Top}\left(|\mathbf{ssSet}_{\mathbf{sSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet)|\,, \mathbf{sTop}_{\mathbf{Top}}(|\boldsymbol{S}_\bullet|, |\boldsymbol{T}_\bullet|)\right),$$

or, equivalently one in

$$\mathbf{sSet}\left(\mathbf{ssSet}_{\mathbf{sSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet), \operatorname{Sing} \mathbf{sTop}_{\mathbf{Top}}(|\boldsymbol{S}_\bullet|, |\boldsymbol{T}_\bullet|)\right).$$

Recalling lemma 4.1.2, these hom-objects are ends, which commute with the right-adjoint Sing. Hence we are looking in the set

$$\mathbf{sSet}\left(\int \underline{\mathbf{sSet}}(\boldsymbol{S}_k, \boldsymbol{T}_k), \int \operatorname{Sing} \underline{\mathbf{Top}}(|\boldsymbol{S}_k|, |\boldsymbol{T}_k|)\right)$$

The morphism we are after is

$$\int \hat{\phi}(\boldsymbol{S}_k, \boldsymbol{T}_k) : \int \underline{\mathbf{sSet}}(\boldsymbol{S}_k, \boldsymbol{T}_k) \to \int \operatorname{Sing} \underline{\mathbf{Top}}(|\boldsymbol{S}_k|, |\boldsymbol{T}_k|)$$

with $\hat{\phi}$ from the proof of lemma 4.4.2. Since $\boldsymbol{T}_k$ was assumed to be Kan (a condition of $\boldsymbol{T}_\bullet$ being a Segal space, this is an end of a point-wise weak equivalence with domain $\boldsymbol{\Delta} \times \boldsymbol{\Delta}^{op}$. □

Remark 4.4.7. We conjecture that the map produced in Lemma 4.4.6 is a weak equivalence. This would follow from some statement of the form: ends are left-Quillen bifunctors.

As before, in the last step, we move from the hom-objects like $\mathbf{ssSet}_{\mathbf{sSet}}$ to the full-blown inner-homs $\underline{\mathbf{ssSet}}$. Recall their relationship, given in remark 4.3.1:

$$\underline{\mathbf{ssSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet)_k = \mathbf{ssSet}_{\mathbf{sSet}}(\boldsymbol{S}_\bullet \times \operatorname{disc}(\Delta(k)), \boldsymbol{T}_\bullet)$$

We also recall the following facts about simplicial Segal spaces, which are explained in section 2 of [Rez].

Lemma 4.4.8. A bisimplicial set in the image of disc : **ssSet** → **sSet** is a simplicial Segal space. Furthermore, simplicial Segal spaces are closed under products.

The lemma clearly implies that the domain $\boldsymbol{S_\bullet} \times \mathrm{disc}(\Delta(k))$ is a simplicial Segal space, provided that $\boldsymbol{S_\bullet}$ is.

We remark that the inner-hom in **sTop** is defined similarly in terms of its **Top** hom-objects:

$$\underline{\mathbf{sTop}}(X,Y)_k = \mathbf{sTop}_{\mathbf{Top}}(X \times \mathrm{disc}(\Delta(k)), Y),$$

where now by abuse of notation disc means discrete topological space. Lemma 4.4.8 remains true with **sSet** replaced by **Top**. We remark that the only property of Segal spaces we need is that they are point-wise Kan, in order to invoke Theorem thm:ssetwe.

We are now equipped to prove the desired result of the section

Proof of Theorem 4.4.1. We need to check point-wise, as stated in the theorem. We have

$$|\underline{\mathbf{ssSet}}(\boldsymbol{S_\bullet}, \boldsymbol{T_\bullet})|_k = |\mathbf{ssSet}_{\mathbf{sSet}}(\boldsymbol{S_\bullet} \times \mathrm{disc}(\Delta(k)), \boldsymbol{T_\bullet})|$$

by the comments following lemma 4.4.8, the domain here is a Segal space, so we can apply 4.4.6 to obtain a weak equivalence

$$|\mathbf{ssSet}_{\mathbf{sSet}}(\boldsymbol{S_\bullet} \times \mathrm{disc}(\Delta(k)), \boldsymbol{T_\bullet})| \to \mathbf{sTop}_{\mathbf{Top}}(|\boldsymbol{S_\bullet} \times \mathrm{disc}(\Delta(k))|, |\boldsymbol{T_\bullet}|)$$

We now use that geometric realization commutes with products, that the realization of a discrete simplicial set is a discrete space up to weak equivalence, and finally that a weak equivalence of cofibrant objects (i.e. any object) in the first coordinate of enriched-hom induces a weak equivalence, to obtain a map

$$\mathbf{sTop}_{\mathbf{Top}}(|\boldsymbol{S_\bullet} \times \mathrm{disc}(\Delta(k))|, |\boldsymbol{T_\bullet}|) \to \mathbf{sTop}_{\mathbf{Top}}(|\boldsymbol{S_\bullet}| \times \mathrm{disc}(\Delta(k)), |\boldsymbol{T_\bullet}|),$$

which will be a weak equivalence, provided the conjecture of Remark 4.4.7 holds true. Finally, note that the target above is the definition of the k^{th} space of $\underline{\mathbf{sTop}}(|\boldsymbol{S_\bullet}|, |\boldsymbol{T_\bullet}|)$.

So composing the two maps above, we have the desired map

$$|\underline{\mathbf{ssSet}}(\boldsymbol{S_\bullet}, \boldsymbol{T_\bullet})| \to \underline{\mathbf{sTop}}(|\boldsymbol{S_\bullet}|, |\boldsymbol{T_\bullet}|).$$

□

Remark 4.4.9. We make the conjecture that the map obtained in Theorem 4.4.1 is actually a point-wise weak equivalence provided that $\boldsymbol{S_\bullet}$ and $\boldsymbol{T_\bullet}$ are simplicial Segal spaces. That is, a weak equivalence in the Reedy model structure on **sTop**. As noted, this would follow from the conjecture in Remark 4.4.7. It should hold, for example, if the categories of Segal spaces and simplicial Segal spaces are Quillen equivalent via geometric realization.

4.5 The Monoidal Coordinate

In this section, we ramp up the maps from the previous two one final time to add in the monoidal coordinate $\mathbf{\Gamma}$. We begin with the nerve step. According to Theorem 4.3.3, we have an isomorphism of bisimplicial sets

$$\mathrm{N}\,\underline{\mathbf{sCat}}(\mathbf{C},\mathbf{D}) \cong \underline{\mathbf{ssSet}}(\mathrm{N}\,\mathbf{C},\mathrm{N}\,\mathbf{D}).$$

We need two lemmas to add in the simplicial coordinate.

Lemma 4.5.1. The cotensoring of **sCat** over **sSet** is described by the following formula

$$\mathbf{sSet}^{\mathbf{sCat}}(S_\bullet,\mathbf{D}) = \underline{\mathbf{sCat}}(\overline{\pi_1 S_\bullet},\mathbf{D}),$$

where π_1 is the left adjoint to nerve (fundamental category), and bar indicates constant in the simplicial factor.

Proof. We calculate

$$\begin{aligned}
\mathbf{sCat}(\mathbf{C},\underline{\mathbf{sCat}}(\overline{\pi_1 S_\bullet},\mathbf{D})) &\cong \mathbf{sCat}(\overline{\pi_1 S_\bullet},\underline{\mathbf{sCat}}(\mathbf{C},\mathbf{D}))\\
&\cong \mathbf{ssSet}(\overline{S_\bullet},\mathrm{N}\,\underline{\mathbf{sCat}}(\mathbf{C},\mathbf{D}))\\
&\cong \mathbf{sSet}(S_\bullet,\mathrm{N}\,\mathbf{sCat}_{\mathbf{Cat}}(\mathbf{C},\mathbf{D})) \quad = \mathbf{sSet}(S_\bullet,\mathbf{sCat}_{\mathbf{sSet}}(\mathbf{C},\mathbf{D}))
\end{aligned}$$

The final line is by definition, since the **sSet** enrichment of **sCat** is induced by nerve. Since this is the adjunction formula that the cotensoring $\mathbf{sSet}^{\mathbf{sCat}}$ must satisfy, the proof is complete. ☐

Lemma 4.5.2. Let $\mathbf{C},\mathbf{D} \in \mathbf{sCat}$. Then

$$\mathrm{N}\,\pi_1\widehat{\mathbf{sCat}}(\mathbf{C},\mathbf{D}) \cong \widehat{\mathbf{sCat}}(\mathbf{C},\mathbf{D}).$$

Proof. We prove the result first for ordinary categories. Let $\mathcal{C},\mathcal{D} \in \mathbf{Cat}$. Refer to Section 2.4 for the definitions of the following notations. We have

$$\begin{aligned}
\mathrm{N}\,\pi_1\widehat{\mathbf{sCat}}(\mathcal{C},\mathcal{D})_j := \mathrm{N}\,\pi_1\,\mathrm{diag}\,X(\mathcal{C},\mathcal{D})_j &= \mathrm{N}\,\pi_1\int^k \Delta(k)\times X(\mathcal{C},\mathcal{D})_{k,j}\\
&\cong \mathrm{N}\int^k \pi_1\Delta(k)\times\prod \mathrm{N}\,\mathrm{Fun}(\mathcal{C},\mathcal{C}_0)_j\times\cdots\times\mathrm{N}\,\mathrm{Fun}(\mathcal{C}_k,\mathcal{D})_j\\
&\cong \mathrm{N}\int^k \pi_1\,\mathrm{N}([k]\times\prod \mathrm{Fun}(\mathcal{C},\mathcal{C}_0)\times\cdots\times\mathrm{Fun}(\mathcal{C}_k,\mathcal{D}))_j\\
&\cong \mathrm{N}\int^k [k]\times\prod \mathrm{Fun}(\mathcal{C},\mathcal{C}_0)\times\cdots\times\mathrm{Fun}(\mathcal{C}_k,\mathcal{D}))
\end{aligned}$$

Here we used that diag is a coend and therefore a colimit, and π_1 is a left adjoint, and so commutes with coends. Then we used that nerve commutes with products, and finally that $\pi_1 \mathrm{N} = \mathrm{id}_{\mathbf{Cat}}$. It is not hard to produce cowedges to show that the above is isomorphic to

$$\widehat{\mathbf{sCat}}(\mathcal{C}, \mathcal{D}),$$

as required. Finally, in $\widehat{\mathbf{sCat}}(\mathbf{C}, \mathbf{D})$, the enrichment over **sSet** is given by

$$\mathbf{sCat}_{\mathbf{sSet}}(\mathbf{C}, \mathbf{D}) = \int_k \mathrm{N}\,\mathrm{Fun}(\mathbf{C}_k, \mathbf{D}_k).$$

These extra ends commute with the nerves, making the above argument work in this case as well. ☐

We can use these lemma in combination with Theorem 4.3.3 to obtain the following result.

Theorem 4.5.3. Let $\mathbf{C}^{\otimes}, \mathbf{D}^{\otimes}$ be functors $\mathrm{N}_* \lhd \mathbf{\Gamma} \to \mathbf{sCat}$. Then there is an isomorphism

$$\mathrm{N} \oint_{\langle n \rangle} \underline{\mathbf{sCat}}(\mathbf{C}(\langle n \rangle), \mathbf{D}(\langle n \rangle)) \cong \oint_{\langle n \rangle} \underline{\mathbf{ssSet}}(\mathrm{N}\,\mathbf{C}(\langle n \rangle), \mathbf{D}(\langle n \rangle))$$

Proof. We calculate

$$\begin{aligned}
&\mathrm{N} \oint_{\langle n \rangle} \underline{\mathbf{sCat}}(\mathbf{C}\left\langle n \right\rangle, \mathbf{D}\left\langle n \right\rangle) \\
&:= \mathrm{N} \int_{\langle n' \rangle, \langle n \rangle} \mathbf{sSet}^{\mathbf{sCat}} \left(\widehat{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \mathbf{D}\left\langle n \right\rangle), \underline{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \left\langle n \right\rangle)\right) \\
&\cong \mathrm{N} \int_{\langle n' \rangle, \langle n \rangle} \underline{\mathbf{sCat}} \left(\overline{\pi_1 \widehat{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \mathbf{D}\left\langle n \right\rangle)}, \underline{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \left\langle n \right\rangle)\right) \\
&\cong \int_{\langle n' \rangle, \langle n \rangle} \mathrm{N}\, \underline{\mathbf{sCat}} \left(\overline{\pi_1 \widehat{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \mathbf{D}\left\langle n \right\rangle)}, \underline{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \mathbf{D}\left\langle n \right\rangle)\right) \\
&\cong \int_{\langle n' \rangle, \langle n \rangle} \underline{\mathbf{ssSet}} \left(\mathrm{N}\, \overline{\pi_1 \widehat{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \mathbf{D}\left\langle n \right\rangle)}, \mathrm{N}\, \underline{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \mathbf{D}\left\langle n \right\rangle)\right) \\
&\cong \int_{\langle n' \rangle, \langle n \rangle} \underline{\mathbf{ssSet}} \left(\overline{\widehat{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \mathbf{D}\left\langle n \right\rangle)}, \underline{\mathbf{ssSet}}(\mathrm{N}\,\mathbf{C}\left\langle n' \right\rangle, \mathrm{N}\,\mathbf{D}\left\langle n \right\rangle)\right) \\
&\cong \int_{\langle n' \rangle, \langle n \rangle} \mathbf{sSet}^{\mathbf{ssSet}} \left(\overline{\widehat{\mathbf{sCat}}(\mathbf{C}\left\langle n' \right\rangle, \mathbf{D}\left\langle n \right\rangle)}, \underline{\mathbf{ssSet}}(\mathrm{N}\,\mathbf{C}\left\langle n' \right\rangle, \mathrm{N}\,\mathbf{D}\left\langle n \right\rangle)\right) \\
&= \oint_{\langle n \rangle} \underline{\mathbf{ssSet}}(\mathrm{N}\,\mathbf{C}(\langle n \rangle), \mathbf{D}(\langle n \rangle))
\end{aligned}$$

In the second step, we used Lemma 4.5.1. In the fifth step, we used Lemma 4.5.2. We used Theorem 4.3.3 in steps four and five. ☐

Next we approach the geometric realization step. In Theorem 4.4.1, we got a natural transformation

$$\phi_{\boldsymbol{S}_\bullet,\boldsymbol{T}_\bullet} : |\underline{\mathbf{ssSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet)| \to \underline{\mathbf{sTop}}(|\boldsymbol{S}_\bullet|, |\boldsymbol{T}_\bullet|).$$

Consider the adjoint

$$\hat{\phi}_{\boldsymbol{S}_\bullet,\boldsymbol{T}_\bullet} : \underline{\mathbf{ssSet}}(\boldsymbol{S}_\bullet, \boldsymbol{T}_\bullet) \to \mathrm{Sing}\, \underline{\mathbf{sTop}}(|\boldsymbol{S}_\bullet|, |\boldsymbol{T}_\bullet|).$$

For $\boldsymbol{S}^\otimes, \boldsymbol{T}^\otimes : \mathrm{N}_* \triangleleft \mathbf{\Gamma} \to \mathbf{ssSet}$, we take the coherent end of $\hat{\phi}$ to obtain

$$\oint_{\langle n \rangle} \hat{\phi} : \oint_{\langle n \rangle} \underline{\mathbf{ssSet}}(\boldsymbol{S}^\otimes \langle n \rangle, \boldsymbol{T}^\otimes \langle n \rangle) \to \oint \mathrm{Sing}\, \underline{\mathbf{sTop}}(|\boldsymbol{S}^\otimes \langle n \rangle_.\,|, |\boldsymbol{T}^\otimes \langle n \rangle_.\,|).$$

Since Sing is a right adjoint, it commutes with coherent ends, and then taking the adjoint of this result, we get a map

$$\left| \oint_{\langle n \rangle} \underline{\mathbf{ssSet}}(\boldsymbol{S}^\otimes \langle n \rangle_.\,, \boldsymbol{T}^\otimes \langle n \rangle_.) \right| \to \underline{\mathbf{sTop}}(|\boldsymbol{S}^\otimes \langle n \rangle_.\,|, |\boldsymbol{T}^\otimes \langle n \rangle_.\,|),$$

which is the last map required to produce a morphism of Segal spaces

$$\mathfrak{M}\underline{\mathrm{Fun}}^\otimes_{\mathbf{Man}}(\mathbf{E}', \mathbf{E}) \to \underline{\mathrm{Fun}}^\otimes(\mathfrak{M}\mathbf{E}', \mathfrak{M}\mathbf{E}),$$

for any $\mathbf{E}', \mathbf{E} \in \mathrm{Fib}^\otimes(\mathbf{Man})$. Of course, we are most interested in the pair d-$\mathbf{B}(X), \mathbf{Vect}_{\mathbf{Man}}$.

Finally, to get a map to homotopy field theories, we again refer to Proposition 3.5.5 to precompose this with the natural transformation of Theorem 3.5.3, and we obtain our upgraded map of field theories

$$\mathfrak{M}\underline{\mathrm{Fun}}^\otimes_{\mathbf{Man}}(d\text{-}\mathbf{B}(X), \mathbf{Vect}_{\mathbf{Man}}) \to \mathrm{Fun}^\otimes\left(d\text{-}\mathbf{Cob}\left(\mathrm{BO}(d) \times X, \pi^*_{\mathrm{BO}(d)}\gamma_{\mathrm{BO}(d)}\right), \mathfrak{M}\mathbf{Vect}_{\mathbf{Man}}\right)$$

4.6 Further Work

In this section we explain some possibilities for further work on the material presented in this thesis.

1. We conjecture that the morphism of Segal spaces

$$\begin{aligned}&\mathfrak{M}\underline{\mathrm{Fun}}^\otimes_{\mathbf{Man}}(d\text{-}\mathbf{B}(X), \mathbf{Vect}_{\mathbf{Man}})\\ &\to \mathrm{Fun}^\otimes\left(d\text{-}\mathbf{Cob}\left(\mathrm{BO}(d) \times X, \pi^*_{\mathrm{BO}(d)}\gamma_{\mathrm{BO}(d)}\right), \mathfrak{M}\mathbf{Vect}_{\mathbf{Man}}\right)\end{aligned}$$

we have constructed is a weak equivalence. Aside from technical considerations such as the behavior of weak equivalences of Segal spaces under coherent ends, there are two facts missing in obtaining this result. First, the restriction to $\mathbf{\Delta} \hookrightarrow \mathbf{Man}$ of

Fib$^{\otimes}$(**Man**) should commute with inner-hom up to isomorphism; that is, the fibred functor produced in Lemma 4.2.3 needs to be an isomorphism. There is no reason to expect this to hold true in general, but perhaps it is true for symmetric monoidal *stacks*. Both d-$\mathbf{B}(X)$ and $\mathbf{Vect}_{\mathbf{Man}}$ are symmetric monoidal stacks (see the Appendix of [SP] for a definition). Finally, the conjecture of Remark 4.4.9 must be established.

2. In the paper [ST], Stolz and Teichner use a more complicated version of smooth bordism category than we have used here. Instead of being an object of Fib$^{\otimes}$(**Man**), it is a category *internal* to Fib$^{\otimes}$(**Man**)$^{\sim}$ (the $\sim$ indicates groupoids). One could attempt to upgrade our map from smooth field theories to homotopy field theories to smooth field theories of this type which has been 'extended up'. The internal categories are certain diagrams in Fib$^{\otimes}$(**Man**)$^{\sim}$, which we could map to diagrams of symmetric monoidal Segal spaces via $\mathfrak{M}$. The missing ingredient would be a map from such 'internal' symmetric monoidal Segal spaces to ordinary symmetric monoidal Segal spaces. Morally, such a construction should be possible; a category internal to the 2-category **Grpd** can be thought of as a type of $(2,1)$-category, and Segal spaces are already 'extended up' infinitely.

3. In order to connect smooth field theories in a more coherent manner to homotopy field theories, the smooth bordism category should be extended down to points. The work of defining a fully extended smooth bordism category is quite technical; refer to [SP] for an extension of 2-**B** to points.

Appendix A

A.1 Enriched Categories

In this section, we give the definition of enriched categories and their enriched functors. Of particular interest will be categories enriched in **Cat**, the category of small categories, and also those enriched in **sSet**, the category of simplicial sets.

Definition A.1.1. A category $\mathbf{C}$ *enriched* over a monoidal category $(\mathcal{M}, \otimes)$ is a collection of objects with the following additional data.

1. (hom-objects) For each pair of objects $X, Y \in \mathbf{C}$, there is an object $\mathbf{C}_{\mathcal{M}}(X, Y)$ in $\mathcal{M}$.
2. (identities) For each object $X \in \mathcal{C}$ there is a morphism $\mathbb{1}_{\mathcal{M}} \to \mathbf{C}_{\mathcal{M}}(X, X)$, denoted 1_X.
3. (composition) For each triple of objects $X, Y, Z \in \mathbf{C}$, there is a morphism

$$\mathbf{C}_{\mathcal{M}}\,(Y, Z) \otimes \mathbf{C}_{\mathcal{M}}\,(X, Y) \to \mathbf{C}_{\mathcal{M}}\,(X, Z).$$

   denoted $\circ$.

subject to

1. (associativity) For $W, X, Y, Z \in \mathbf{C}$, the following diagram (in $\mathcal{M}$) commutes.

$$\begin{tikzcd}
(\mathbf{C}_{\mathcal{M}}(Y,Z) \otimes \mathbf{C}_{\mathcal{M}}(X,Y)) \otimes \mathbf{C}_{\mathcal{M}}(W,X) \arrow[rr] \arrow[d, "\circ\otimes 1"'] & & \mathbf{C}_{\mathcal{M}}(Y,Z) \otimes (\mathbf{C}_{\mathcal{M}}(X,Y) \otimes \mathbf{C}_{\mathcal{M}}(W,X)) \arrow[d, "1\otimes\circ"] \\
\mathbf{C}_{\mathcal{M}}(X,Z) \otimes \mathbf{C}_{\mathcal{M}}(W,X) \arrow[dr, "\circ"'] & & \mathbf{C}_{\mathcal{M}}(Y,Z) \otimes \mathbf{C}_{\mathcal{M}}(W,Y) \arrow[dl, "\circ"] \\
& \mathbf{C}_{\mathcal{M}}(W,Z) &
\end{tikzcd}$$

   The top horizontal arrow is the associator for $(\mathcal{M}, \otimes)$.

2. (right-identity) For $X, Y \in \mathbf{C}$, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{1} \otimes \mathbf{C}_{\mathcal{M}}(X,Y) \arrow[r, "1_Y \otimes 1"] \arrow[dr] & \mathbf{C}_{\mathcal{M}}(Y,Y) \otimes \mathbf{C}_{\mathcal{M}}(X,Y) \arrow[d, "\circ"] \\
 & \mathbf{C}_{\mathcal{M}}(X,Y)
\end{tikzcd}$$

The diagonal arrow is the left-identity property for the monoidal unit $\mathbb{1}$ of $\mathcal{M}$.

3. (right-identity) Same as above, with the identity on the right.

Note that if $\mathcal{M}$ is symmetric monoidal, the last two conditions are redundant. Also note that if $\mathcal{M}$ is symmetric monoidal, the composition morphisms can be written

$$\mathbf{C}_{\mathcal{M}}(X,Y) \otimes \mathbf{C}_{\mathcal{M}}(Y,Z) \to \mathbf{C}_{\mathcal{M}}(X,Z).$$

A category enriched over $\mathcal{M}$ will sometimes be called an $\mathcal{M}$-category.

These $\mathcal{M}$-categories can be arranged into a category $\mathbf{Cat}_{\mathcal{M}}$, whose morphisms we now define.

Definition A.1.2. Let $\mathbf{C}, \mathbf{D}$ be $\mathcal{M}$-categories. An *enriched functor* $F : \mathbf{C} \to \mathbf{D}$ consists of the data

- For each object $X \in \mathbf{C}$, an object $FX \in \mathbf{D}$.
- For each hom-object $\mathbf{C}_{\mathcal{M}}(X,Y) \in \mathcal{M}$, a morphism in $\mathcal{M}$:

$$F_{X,Y} : \mathbf{C}_{\mathcal{M}}(X,Y) \to \mathbf{D}_{\mathcal{M}}(FX,FY)$$

subject to the conditions

1. For $X, Y, Z \in \mathbf{C}$, the following diagram in $\mathcal{M}$ commutes.

$$\begin{tikzcd}
\mathbf{C}_{\mathcal{M}}(Y,Z) \otimes \mathbf{C}_{\mathcal{M}}(X,Y) \arrow[r, "F_{Y,Z} \otimes F_{X,Y}"] \arrow[d, "\circ_{\mathbf{C}}"'] & \mathbf{D}_{\mathcal{M}}(FY,FZ) \otimes \mathbf{D}_{\mathcal{M}}(FX,FY) \arrow[d, "\circ_{\mathbf{D}}"] \\
\mathbf{C}_{\mathcal{M}}(X,Z) \arrow[r, "F_{X,Z}"] & \mathbf{D}_{\mathcal{M}}(FX,FZ)
\end{tikzcd}$$

2. For $X \in \mathbf{C}$, the following diagram in $\mathcal{M}$ commutes.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "1_X"'] \arrow[dr, "1_{FX}"'] & \mathbf{C}_{\mathcal{M}}(X,X) \arrow[d, "F_{X,X}"] \\
 & \mathbf{D}_{\mathcal{M}}(FX,FX)
\end{tikzcd}$$

We have an additional structure on $\mathbf{Cat}_{\mathcal{M}}$ when $\mathcal{M}$ is allowed to vary, given by the following 'push-forward' construction.

Remark A.1.3. Let $\mathcal{M}', \mathcal{M}$ be monoidal categories, and let $\mathbf{Cat}_{\mathcal{M}'}$ and $\mathbf{Cat}_{\mathcal{M}}$ denote the categories of $\mathcal{M}$-enriched, and $\mathcal{M}'$-enriched categories and enriched functors, respectively. Let $F : \mathcal{M}' \to \mathcal{M}$ be a monoidal functor. Then there is a functor

$$F_* : \mathbf{Cat}_{\mathcal{M}'} \to \mathbf{Cat}_{\mathcal{M}}$$

defined by applying F hom-object-wise.

Also note that there is an obvious 'opposite' construction on $\mathbf{Cat}_{\mathcal{M}}$ just as with ordinary categories, defined by

$$\mathbf{C}^{\mathrm{op}}_{\mathcal{M}}(X, Y) = \mathbf{C}_{\mathcal{M}}(Y, X).$$

The following definition provides another common example of enrichment.

Definition A.1.4. A monoidal category $\mathcal{M}$ is called *closed* if for each object m of $\mathcal{M}$, there is a chosen functor denoted

$$m' \mapsto \underline{\mathcal{M}}(m, m')$$

which is right adjoint to the functor given by left-tensoring with m:

$$m' \mapsto m \otimes m'.$$

Proposition A.1.5. Every closed monoidal category is enriched over itself.

Proof. Let $\mathcal{M}$ be a closed monoidal category. Define

$$\mathcal{M}_{\mathcal{M}}(m', m) := \underline{\mathcal{M}}(m', m),$$

and set 1_m to be the image of the morphism

$$\lambda_L : m \otimes \mathbb{1} \to m$$

under the closure adjunction of $\mathcal{M}$. The definition of composition, and the details of the conditions are straightforward adjunction manipulations, and are spelled out for example in [Kel]. ☐

Now we turn our attention to other ways in which an enriched category and its base category can interact.

Definition A.1.6. Let $\mathcal{C}$ be an $\mathcal{M}$-category, where $(\mathcal{M}, \otimes)$ is a monoidal category which is enriched over itself (for example $\mathcal{M}$ could be closed). If there is a functor $\mathcal{M} \times \mathcal{C} \to \mathcal{C}$, also denoted $\otimes$ which is associative, in the sense that

$$(m \otimes n) \otimes X \cong m \otimes (n \otimes X),$$

and natural in $m, n \in \mathcal{M}$ and $X \in \mathcal{C}$, and finally it satisfies the adjunction property

$$\underline{\mathcal{M}}(m, \mathcal{C}_{\mathcal{M}}(X, Y)) \cong \mathcal{C}_{\mathcal{M}}(m \otimes X, Y),$$

naturally in $m \in \mathcal{M}$, and $X, Y \in \mathcal{C}$, we say that $\mathcal{C}$ is *tensored* over $\mathcal{M}$.

Note that the tensor structure induces a category structure on $\mathbf{C}$ via $\mathbf{C}(X, Y) := \mathcal{M}(\mathbb{1}, \mathbf{C}_{\mathcal{M}}(X, Y))$ and furthermore, the adjunction property guarantees that the category structure we assume in our definition agrees with this up to isomorphism of categories:

$$\mathcal{M}(\mathbb{1}, \mathbf{C}_{\mathcal{M}}(X, Y)) \cong \mathbf{C}(\mathbb{1} \otimes X, Y) \cong \mathbf{C}(X, Y).$$

There is an obvious dual notion:

Definition A.1.7. An enriched category $\mathbf{C} \in \mathbf{Cat}_{\mathcal{M}}$, with $\mathcal{M}$ enriched over itself is *cotensored* over $\mathcal{M}$ if, for $m \in \mathcal{M}$ and $Y \in \mathbf{C}$, there is an object $\mathcal{M}^{\mathbf{C}}(m, Y) \in \mathbf{C}$ which satisfies the adjunction formula

$$\underline{\mathcal{M}}(m, \mathbf{C}_{\mathcal{M}}(X, Y)) \cong \mathbf{C}_{\mathcal{M}}(X, \mathcal{M}^{\mathbf{C}}(m, Y))$$

for any $X \in \mathbf{C}$. Furthermore, the cotensoring is required to be associative, in the sense that there is an isomorphism

$$\mathcal{M}^{\mathbf{C}}(m', \mathcal{M}^{\mathbf{C}}(m, Y)) \cong \mathcal{M}^{\mathbf{C}}(m' \otimes m, Y),$$

natural in m', m. Note that if $\mathbf{C}$ is also considered as a category via

$$\mathcal{M}(\mathbb{1}, \mathbf{C}_{\mathcal{M}}(X, Y)) \cong \mathbf{C}(\mathbb{1} \otimes X, Y) \cong \mathbf{C}(X, Y),$$

we can rephrase the adjunction formula by saying that the functor

$$\mathbf{C}_{\mathcal{M}}(\cdot, Y) : \mathbf{C}^{\mathrm{op}} \to \mathcal{M}$$

has a left adjoint, denoted

$$\mathcal{M}^{\mathbf{C}}(m, \cdot) : \mathcal{M} \to \mathbf{C}^{\mathrm{op}}.$$

We remark that the notions of tensoring and cotensoring can be extended to the case where $\mathcal{M}$ is not self-enriched by writing the same formulas above but dropping the underlines on the $\mathcal{M}$'s and the subscript $\mathcal{M}$'s on the $\mathcal{C}$'s. In other words use ordinary hom-sets. For example, in the case of tensoring, a natural isomorphism of hom-sets

$$\mathcal{M}(m, \mathcal{C}_{\mathcal{M}}(X, Y)) \cong \mathcal{C}(m \otimes X, Y),$$

enforces the natural isomorphism of $\mathcal{M}$ hom-objects of Definition A.1.6. To see this, map a test $m' \in \mathcal{M}$ into both sides, use the usual adjunction manipulations, and apply the Yoneda lemma to the result. Indeed, we have

$$\begin{aligned}\mathcal{M}(m', \underline{\mathcal{M}}(m, \mathcal{C}_{\mathcal{M}}(X,Y)) &\cong M(m' \otimes m, \mathcal{C}_{\mathcal{M}}(X,Y)) \\ &\cong \mathcal{C}((m' \otimes m) \otimes X, Y) \\ &\cong \mathcal{C}(m' \otimes (m \otimes X), Y) \\ &\cong \mathcal{M}(m', \mathcal{C}_{\mathcal{M}}(m \otimes X, Y)),\end{aligned}$$

where the third isomorphism is induced by the associator of $\mathcal{M}$.

We return to tensoring, and note that it is natural to ask for more if the monoidal structure on $\mathcal{M}$ is induced by product.

Proposition A.1.8. Consider the special case where the monoidal structure on $\mathcal{M}$ is induced by product, and the tensor structure $\mathcal{M} \times \mathbf{C} \to \mathbf{C}$ is induced by a functor $\mathcal{M} \to \mathbf{C}$, followed by the product. We will use the notation $m \mapsto \overline{m}$, so that $m \otimes X$ is written $\overline{m} \times X$.

In this situation, the only additional data needed for $\mathbf{C}$ to be tensored over $\mathcal{M}$ are the hom-objects, and the only condition needed is the adjunction property of hom-objects.

Proof. Suppose we have hom-objects, and the tensor functor $\mathcal{M} \otimes \mathbf{C} \to \mathbf{C}$ satisfies the property

$$\mathcal{M}(m, \mathbf{C}_{\mathcal{M}}(X,Y)) \cong \mathbf{C}(\overline{m} \otimes X, Y),$$

First, we will produce the identity arrows. Note that the monoidal unit under cartesian product is a terminal object. Yoneda's Lemma says that natural transformations hom-functors $\mathrm{Hom}(-,X) \Rightarrow \mathrm{Hom}(-,Y)$ are in bijection with arrows $\mathrm{Hom}(X,Y)$. Studying the diagram

$$\begin{tikzcd} \mathcal{M}(m, \mathbb{1}) \arrow[r] \arrow[dr] & \mathcal{M}(\overline{m} \times X, X) \arrow[d] \\ & \mathcal{M}\left(m, \mathbf{C}_{\mathcal{M}}(X,X)\right) \end{tikzcd}$$

We see that the upper-left hom-set is simply a point, and there is a canonical choice for the horizontal map: send the point to the projection map. Composing these two maps (which are natural in m), yields a natural transformation and by Yoneda we obtain 1_X.

Next we can obtain the composition maps following this diagram counter-clockwise:

$$\begin{tikzcd}
\mathcal{M}\left(m, \mathbf{C}_{\mathcal{M}}(Y,Z) \times \mathbf{C}_{\mathcal{M}}(X,Y)\right) \arrow[r, "\circ_*"] \arrow[d, "\sim"'] & \mathcal{M}\left(m, \mathbf{C}_{\mathcal{M}}(X,Z)\right) \\
\mathcal{M}\left(m, \mathbf{C}_{\mathcal{M}}(Y,Z)\right) \times \mathcal{M}\left(m, \mathbf{C}_{\mathcal{M}}(X,Y)\right) \arrow[d, "\sim"'] & \\
\mathbf{C}\left(\overline{m} \times Y, Z\right) \times \mathbf{C}\left(\overline{m} \times X, Y\right) \arrow[d] & \\
\mathbf{C}\left(\overline{m} \times Y, Z\right) \times \mathbf{C}\left(\overline{m} \times X, \overline{m} \times Y\right) \arrow[r, "\circ_{\mathbf{C}}"] & \mathbf{C}\left(\overline{m} \times X, Z\right) \arrow[uuu, "\sim"']
\end{tikzcd}$$

The point is that this situation creates a natural compatibility between the compositions of $\mathbf{C}$ as an ordinary category and $\mathbf{C}$ as an enriched category.

Let us also check the left-identity requirement. Consider:

$$\begin{tikzcd}
\mathcal{M}\left(m, \mathbb{1} \times \mathbf{C}_{\mathcal{M}}(X,Y)\right) \arrow[r] \arrow[dr] & \mathcal{M}\left(m, \mathbf{C}_{\mathcal{M}}(X,Y) \times \mathbf{C}_{\mathcal{M}}(X,X)\right) \arrow[d] \\
& \mathcal{M}(m, \mathbf{C}_{\mathcal{M}}(X,Y))
\end{tikzcd}$$

Using our various isomorphisms of hom-sets, we can obtain a diagram:

$$\begin{tikzcd}
\mathcal{M}(m, \mathbb{1}) \times \mathbf{C}(\overline{m} \times X, Y) \arrow[r] \arrow[dr] & \mathbf{C}(\overline{m} \times X, Y) \times \mathbf{C}(\overline{m} \times X, X) \arrow[d] \\
& \mathbf{C}(\overline{m} \times X, Y)
\end{tikzcd}$$

Where the arrows are defined by the above diagram, i.e. this diagram commutes if and only if the above does.

The diagonal map does nothing. The horizontal arrow takes a map $f \in \mathbf{C}(\overline{m} \times X, Y)$ to the pair (f, π_X). The vertical map makes π_X into a map to $\overline{m} \times X$ by having it be the projection on the $\overline{m}$ factor, then it composes f with the result. However, the map $\mathbf{C}(\overline{m} \times X, \overline{m} \times X)$ identified with $(\pi_{\overline{m}}, \pi_X)$ is the identity, so the diagram commutes.

Right-identity follows, since cartesian product is a symmetric monoidal structure.

A similar argument where we map an arbitrary $m \in \mathcal{M}$ into everything and use all hom-set isomorphisms shows that the compositions is also associative, since composition in $\mathbf{C}$ is associative. □

Definition A.1.9. We will say in the situation of Proposition A.1.8 that $\mathbf{C}$ is *properly tensored* over $(\mathcal{M}, \times)$.

Examples A.1.10. 1. All ordinary categories $\mathcal{C}$ are enriched over **Set**, and those with products and a 'free' functor **Set** → $\mathcal{C}$ are properly tensored over **Set**.

2. A simplicial model category is tensored over **sSet**. In many examples, it is properly tensored (**Top**, **ssSet**, etc.)
3. A closed monoidal category is tensored over itself. A cartesian closed category (closed model category under the cartesian monoidal structure) is properly tensored over itself.

A.2 Ends

In this section we introduce ends and prove some useful properties of them. Ends and their generalizations are used extensively throughout this paper.

Definition A.2.1. Let $F : \mathcal{J}^{op} \times \mathcal{J} \to \mathcal{C}$ be a functor. A *wedge* from an object $c \in \mathcal{C}$ to F is an map ω from objects in $\mathcal{J}$ to morphisms $\omega(j) : c \to F(j, j)$ in $\mathcal{C}$ such that the following diagram commutes for all $j, j' \in \mathcal{J}$ and $f : j \to j'$:

$$\begin{tikzcd}
c \arrow[r, "\omega(j)"] \arrow[d, "\omega(j')"'] & F(j,j) \arrow[d, "F(1,f)"] \\
F(j',j') \arrow[r, "F(f,1)"'] & F(j,j')
\end{tikzcd}$$

We will use the notation $\omega : c \Rightarrow F$

Definition A.2.2. Let $F : \mathcal{J}^{op} \times \mathcal{J} \to \mathcal{C}$ be a functor. An *end* of F is an object $e \in \mathcal{C}$ and a wedge $\omega : e \Rightarrow F$ that is universal in the sense that given any other wedge $\gamma : d \Rightarrow F$, there is a unique morphism $g : d \to e$ such that $\omega(j)g = \gamma(j)$ for all $j \in \mathcal{J}$. The object d is clearly unique up to unique isomorphism. By abuse of notation, we usually only refer to the object e, and write it as

$$\int_j F(j, j),$$

or sometimes simply $\int F$ if no confusion arises.

Note that cowedges can also be defined $F \Rightarrow d$ by reversing all arrows, and the corresponding universal cowedges are called *coends*.

The next few results on ends are taken from [Mac].

Remark A.2.3. The universal property of an end:

$$\mathrm{Wedge}(e, F) \cong \mathrm{Hom}(e, \int F),$$

looks very much like that of a limit:

$$\mathrm{Cone}(e, F) \cong \mathrm{Hom}(e, \lim F),$$

which we recall can be rewritten as an adjunction. To do so, we note that a cone from e to a functor $F : \mathcal{J} \to \mathcal{C}$ is simply a natural transformation from the constant functor hitting the object e, which we will denote $\overline{e} : \mathcal{J} \to \mathcal{C}$, to F. Going further, a natural transformation is a morphism in the functor category $\mathcal{C}^{\mathcal{J}}$. Hence we have

$$\mathcal{C}^{\mathcal{J}}(\overline{e}, F) \cong \mathcal{C}(e, \lim F).$$

Here we are using the name of a category to denote its hom functor.

If there is a framework in which wedges become cones, then we will have shown that ends are special cases of limits, and therefore right adjoints.

Such a framework is given by the *subdivision category*. The subdivision category $\mathcal{J}'$ of a category $\mathcal{J}$ is a category with objects

$$\{\text{objects of } \mathcal{J}\} \bigcup \{\text{morphisms of } \mathcal{J}\}$$

and nonidentity morphisms $s \to f \leftarrow t$ if f is an arrow and s and t are its source and target. Note that there are no meaningful compositions in this category.

Now given a functor $F : \mathcal{J}^{op} \times \mathcal{J} \to \mathcal{C}$, there is an induced functor $F' : \mathcal{J}' \to \mathcal{C}$, given by $F'(j) = F(j,j)$, $F'(f : s \to t) = F(s,t)$, $F'(s \to f) = F(1_s, f)$, and $F'(f \leftarrow t) = F(f, 1_t)$.

Given a cone, i.e. a natural transformation $\omega' : \overline{e} \Rightarrow F'$, we get the following diagram by applying ω' to the diagram $s \to f \leftarrow t$ in $\mathcal{J}'$:

$$\begin{tikzcd}
c \arrow[r, "\omega'(s)"] \arrow[d, "1_c"'] & F(s,s) \arrow[d, "{F(1,f)}"] \\
c \arrow[r, "\omega'(f)"'] & F(s,t) \\
c \arrow[u, "1_c"] \arrow[r, "\omega'(t)"'] & F(t,t) \arrow[u, "{F(f,1)}"']
\end{tikzcd}$$

This is identical to the diagram for a wedge, where $\omega'(f)$ is the diagonal composition. A similar observation shows that a wedge determines a cone.

We have converted wedges to cones, and so have shown that ends are limits, In particular they enjoy the same properties as limits (e.g. they commute with right adjoints).

We now provide some examples of ends.

Lemma A.2.4. Suppose $F, G : \mathcal{C} \to \mathcal{D}$ are functors. Then

$$\mathrm{Nat}(F,G) \cong \int_c \mathcal{D}(Fc, Gc)$$

where the right hand side is an end of the hom-functor composition

$$\mathcal{D} \circ (F^{\mathrm{op}} \times G) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}.$$

Proof. Suppose that $S \in \mathbf{Set}$. If we have a wedge $\omega : S \Rightarrow \mathcal{D} \circ (F^{op} \times G)$, then for each $c \in C$ we get a set map $\omega(c) : S \to \mathcal{D}(Fc, Gc)$, or in other words, for each pair $c, p \in Ob(\mathcal{C}) \times S$, a morphism $\omega(c, p) : Fc \to Gc$. Looking at the wedge diagram, we see that it really says that

$$\begin{tikzcd}
Fc \arrow[r, "\omega(c)"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\
Fc' \arrow[r, "\omega(c')"] & Gc'
\end{tikzcd}$$

commutes. In other words, that $\omega(\cdot, p)$ is a natural transformation. In particular, we have the wedge $c \mapsto \alpha(c) \in \mathcal{D}(Fc, Gc)$ for $\alpha \in \mathrm{Nat}(F, G)$. And the map that sends $p \in S$ to the natural transformation $\omega(\cdot, p)$ witnesses the fact that this wedge is universal. □

Remark A.2.5. The above proof goes through essentially unchanged if $\mathcal{C}$ and $\mathcal{D}$ are replaced by strict 2-categories, and we have the hom-category functor: Suppose $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}_{\mathbf{Cat}}$ are categories enriched over **Cat** (i.e. strict 2-categories). Let $F, G : \mathbf{C} \to \mathbf{D}$ be enriched functors (i.e. strict 2-functors). Then there is an *isomorphism* of categories

$$\mathrm{Nat}(F, G) \cong \int_c \mathbf{D}_{\mathbf{Cat}}(Fc, Gc)$$

Where the left hand side is the category of natural transformations and modifications, i.e. $\mathbf{D}^{\mathbf{C}}(F, G)$. We will weaken ends suitably in the next section to produce pseudonatural transformations.

Example A.2.6. The geometric realization functor $\mathbf{sSet} \to \mathbf{Top}$ can be written as a coend $\boldsymbol{\Delta} \times \boldsymbol{\Delta}^{op} \to \mathbf{Top}$:

$$|S_\bullet| = \int^{[k]} \Delta^k \times \operatorname{disc} S_k,$$

where Δ^k is the standard k simplex

$$\{(t_0, \cdots, t_k) \in \mathbb{R}^{k+1} | \sum t_i = 1\}.$$

Hence, we get that geometric realization is a left adjoint immediately.

Example A.2.7. The diagonal functor $\mathrm{diag} : \mathbf{ssSet} \to \mathbf{sSet}$ is the coend

$$(\operatorname{diag} \boldsymbol{S})_\bullet = \int^{[k]} \Delta(k) \times \boldsymbol{S}_{k,\bullet}$$

A.3 Lax Ends

In this section we introduce lax ends, and prove analogues to some of the material in the previous section. We will stick to strict functors and strict 2-categories, or in other words enriched functors of **Cat**-enriched categories, but one may define lax ends in the more general setting of bicategories and pseudofunctors. According to other sources, this material is covered in [Boz], presumably in the more general case of bicategories (see e.g. [Boz1]). Unfortunately, the author was was unable to locate a copy, but the definitions and proofs below are fairly straightforward, so we will produce them from scratch.

Definition A.3.1. Let $\mathbf{A}, \mathbf{C} \in \mathbf{Cat}_{\mathbf{Cat}}$, let $F : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{C}$ be an enriched functor, and let $c \in \mathbf{C}$. We define a *lax wedge* $\omega : c \Rightarrow F$ to be an assignment of objects $a \in \mathbf{A}$ to 1-morphisms $\omega(a) : c \to F(a,a)$, and an assigmnet from 1-morphisms $f : a' \to a$ in $\mathbf{A}$ to 2-morphisms

$$\begin{tikzcd}
c \arrow[r, "\omega(a)"] \arrow[d, "\omega(a')"'] & F(a,a) \arrow[d, "F(f{,}1)"] \\
F(a',a') \arrow[r, "F(1{,}f)"'] \arrow[ur, Rightarrow, shorten=4mm, "\omega(f)"] & F(a',a)
\end{tikzcd}$$

Subject to the following conditions:

1. For $f' : a'' \to a$, $f : a' \to a$, the 2-morphism filling the diagram

$$\begin{tikzcd}
c \arrow[rr, "\omega(a)"] \arrow[dr, "\omega(a')"] \arrow[dd, "\omega(a'')"'] & & F(a,a) \arrow[d, "F(f{,}1)"] \\
& F(a',a') \arrow[r, "F(1{,}f)"'] \arrow[d, "F(f'{,}1)"] \arrow[ur, Rightarrow, shorten=4mm, "\omega(f)"] & F(a',a) \arrow[d, "F(f'{,}1)"] \\
F(a'',a'') \arrow[r, "F(1{,}f')"'] \arrow[ur, Rightarrow, shorten=4mm, "\omega(f')"] & F(a'',a') \arrow[r, "F(1{,}f)"'] \arrow[ur, equal, shorten=4mm] & F(a'',a)
\end{tikzcd}$$

is equal to $\omega(f \circ f')$.

2. For a 2-morphism $\phi : g \to f$, $f, g : a' \to a$, the 2-morphism filling the diagram

$$\begin{tikzcd}
c \arrow[r, "\omega(a)"] \arrow[d, "\omega(a')"'] & F(a,a) \arrow[d, "F(f{,}1)"', ""{name=A}] \arrow[d, bend left=60, "F(g{,}1)", ""'{name=B}] \\
F(a',a') \arrow[r, "F(1{,}f)", ""'{name=C}] \arrow[r, bend right=60, "F(1{,}g)"', ""{name=D}] \arrow[ur, Rightarrow, "\omega(f)"] & F(a',a)
\arrow[Rightarrow, from=A, to=B, "F(\phi{,}1)"]
\arrow[Rightarrow, from=D, to=C, "F(1{,}\phi)"]
\end{tikzcd}$$

is equal to $\omega(g)$.

If in addition we require the images $\omega(f)$ to be invertible 1-morphisms, we call ω an *iso-lax wedge.*

Definition A.3.2. Let $\mathbf{A}, \mathbf{C} \in \mathbf{Cat}_{\mathbf{Cat}}$, and let $F : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{C}$. We define a *lax end* of F to be an object $e \in \mathbf{C}$, together with a lax wedge $\omega : e \Rightarrow F$ which is final in the sense that, given another wedge $\gamma : d \Rightarrow F$, there is a 1-morphism $g : d \to e$, and unique invertible 2-morphisms

$$\omega(a) \circ g \Rightarrow \omega(a)$$

such that, for $f : a' \to a$, the 2-morphism filling the diagram

$$\begin{tikzcd}
d \arrow[dr, "g"] \arrow[drr, bend left, "\gamma(a)", ""'{name=G}] \arrow[ddr, bend right, "\gamma(a')"', ""{name=H}] & & \\
& e \arrow[r, "\omega(a)"'] \arrow[d, "\omega(a')"] \arrow[Rightarrow, to=G] \arrow[Rightarrow, to=H] & F(a,a) \arrow[d, "F(f{,}1)"] \\
& F(a',a') \arrow[r, "F(1{,}f)"'] \arrow[ur, Rightarrow, "\omega(f)"] & F(a',a)
\end{tikzcd}$$

is equal to $\gamma(f)$. The object e is unique up to equivalence, and we usually denote the lax end by

$$\oint_{a \in \mathbf{A}} F(a,a)$$

or by other obvious variations on $a \in \mathbf{A}$. An *iso-lax end* will be the same definition with a final iso-lax wedge.

Theorem A.3.3. Let $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}_{\mathbf{Cat}}$, and let $F, G : \mathbf{C} \to \mathbf{D}$ be enriched functors. Then there is an equivalence of categories

$$\mathbf{Lax}(F,G) \cong \oint_{c \in \mathbf{C}} \mathbf{D}_{\mathbf{Cat}}(Fc, Gc),$$

where the left hand side is the category of lax transformations and modifications.

Proof. We produce a lax wedge

$$\omega : \mathbf{Lax}(F, G) \Rightarrow \mathbf{D}_{\mathbf{Cat}} \circ (F^{\mathrm{op}} \times G),$$

and show that it is final among such lax wedges. Indeed, let $f : c' \to c \in \mathbf{C}$, let $\alpha', \alpha : F \Rightarrow G$ be a lax transformations, and let $m : \alpha' \to \alpha$ be a modification. We make the following definitions

$$\begin{aligned} \omega(c)(\alpha) &:= \alpha(c) \\ \omega(c)(m) &:= m(c) \\ \omega(f)(\alpha) &:= \alpha(f). \end{aligned}$$

The functoriality of $\omega(c)$ is clear, and the naturality of $\omega(f)$ on modifications follows from the naturality condition for modifications. Condition 1 of Definition A.3.1 follows from the functoriality of α on 1-morphisms. Condition 2 follows from the naturality of α on 1-morphisms. The universal property of this lax wedge follows from the observation that the conditions for it to be a lax wedge were equivalent to the conditions satisfied by lax transformations and modifications. We have therefore established the equivalence, since lax ends are well-defined up to equivalence. ☐

Remark A.3.4. The details in the proof above go through unchanged to give an equivalence

$$\mathbf{pNat}(F, G) \cong \oint_{c \in \mathbf{C}}^{\sim} \mathbf{D}_{\mathbf{Cat}}(Fc, Gc)$$

where the left hand side is now the category of pseudonatural transformations and modifications, and the right hand side is the iso-lax end.

A.4 Model Categories

In this appendix we give the definition of a model category, and also that of a simplicial model category. These turn out to be an appropriate framework for forming various models of $(\infty, 1)$ categories, which will be discussed in the next section. Good introductions on model categories is given in [Hov], and [GJ], which also discusses simplicial model categories. They were first introduced by Quillen, which explains the terminology 'Quillen equivalence'.

We begin with some preliminary terminology.

Definition A.4.1. 1. Let $\mathcal{C}$ be a category, and let $f : A \to B$, $g : C \to D$ be morphisms of $\mathcal{C}$. We say that f is a *retract* of g if there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "1"'] & D \arrow[r] & B
\end{tikzcd}$$

Note in particular the existence of the diagram implies that A is a retract of C and B is a retract of B in the sense of objects.

2. A *functorial factorization* is an ordered pair of functors (α_1, α_2) from the category of morphisms and commutative squares of $\mathcal{C}$ to itself with the property that

$$\alpha_2(f) \circ \alpha_1(f) = f$$

for every morphism f of $\mathcal{C}$. In particular, this composition is assumed to exist.

Definition A.4.2. Let $\mathcal{C}$ be a category. A *model structure* on $\mathcal{C}$ consists of the data

- ◇ Three collections of morphisms of $\mathcal{C}$, called *fibrations*, *cofibrations*, and *weak equivalences.*
- ◇ Two functorial factorizations of $\mathcal{C}$: (α_1, α_2), and (β_1, β_2).

These data are subject to the following conditions.

1. (2-out-of-3). If f and g are composable morphisms of $\mathcal{C}$ such that two of the morphisms f, g, and $g \circ f$ are weak equivalences, then so is the third.
2. (retract) The three collections of fibrations, cofibrations and weak equivalences are closed under retracts.
3. (lifting) We say a morphism is a *trivial fibration* if it is both a fibration and a weak equivalence, and similarly define a *trivial cofibration.* Given the solid commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

the dotted arrow exists, making the diagram commute, provided that i is a trivial cofibration and p is a fibration, or that i is a cofibration and p is a trivial fibration. We describe this situation by saying that i has the *left lifting property* with respect to p and that p has the *right lifting property* with respect to i.

4. (factorization) The factorization (α_1, α_2) factors morphisms into a cofibration followed by a trivial fibration, and the factorization (β_1, β_2) factors morphisms into a trivial cofibration followed by a fibration.

Definition A.4.3. A *model category* $\mathcal{C}$ is a category with all small limits and small colimits together with a model structure.

We note the following characterization of cofibrations and fibrations, which (in the 'if' direction) follows from the retract axiom (see 1.1.9-1.1.10 in [Hov]) .

Lemma A.4.4. Let $\mathcal{C}$ be a model category. Then a morphism is a cofibration (resp. trivial cofibration) if and only if it satisfies the left lifting property with respect to all trivial fibrations (resp. fibrations). A morphism is a fibration (resp. trivial fibration) if it has the right lifting property with respect to all trivial cofibrations (resp. cofibrations).

The 'only if' direction is satisfied by the lifting axiom. There is a nice corollary.

Corollary A.4.5. Cofibrations and trivial cofibrations are closed under pushout, in the sense that if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

is a pushout square, then if f is a cofibration or trivial cofibration, so is g. Similarly, fibrations and trivial fibrations are closed under pullback.

Proof. Pushout preserves the left lifting property, and pullback preserves the right lifting property. $\square$

In most sources, the factorizations in the fourth axiom are merely assumed to exist, rather than constitute a functorial factorization. However, they can always be made to be functorial, and it makes various constructions using them natural, such as the following.

Definition A.4.6.

1. In a model category $\mathcal{C}$, an object X is called *fibrant* if the canonical morphism $X \to *$, with $*$ the final object, is a fibration. We dually define *cofibrant*. The *fibrant replacement* of an object X, is the object $\tilde{X}$ in the diagram

$$\begin{tikzcd} X \arrow[rr, "1"] \arrow[dr, "\alpha_1(1)"'] & & X \\ & \tilde{X} \arrow[ur, "\alpha_2(1)"'] & \end{tikzcd}$$

   arising from the factorization of 1_X into a cofibration followed by a trivial fibration. We similarly define cofibrant replacement via (β_1, β_2) applied to 1_X.

2. Similarly to 1. we can factor the fold map $X \coprod X \to X$ into a cofibration $X \coprod X \to CX$ followed by a trivial fibration $CX \to X$; CX is called a *cylinder object* for X. We similarly apply the other factorization to the diagonal map $X \to X \times X$ to define a *path object* PX with a trivial cofibration $X \to PX$.

Cylinder and path objects are used to define the notion of homotopy in model categories.

Definition A.4.7. Let $f, g : X \to Y$ be two morphisms in a model category $\mathcal{C}$.

1. A *left homotopy* from f to g is a morphism $h : CX \to Y$ such that with the inclusions $i_1, i_2 : X \to X \coprod X$ we have $h \circ i_1 = f$ and $h \circ i_2 = g$.
2. A *right homotopy* from f to g is a morphism $h : X \to PY$ such that with the projections $\pi_1, \pi_2 : X \times X \to X$, we have $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.
3. We say that f, g are *homotopic* if there is both a left and right homotopy from f to g. We write this as $f \sim g$.
4. We say that f is a *homotopy equivalence* if there is a morphism $f' : Y \to X$ such that $f' \circ f \sim 1_X$ and $f \circ f' \sim 1_Y$.

We next recall a very useful lemma, whose proof we include for completeness.

Lemma A.4.8 (Ken Brown's Lemma)**.** Suppose $\mathcal{C}$ is a model category and $\mathcal{D}$ is a category with a subset of morphisms called weak equivalences satisfying the 2-out-of-3 rule. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor that takes trivial fibrations of fibrant objects to trivial fibrations. Then F takes all weak equivalences of fibrant objects to weak equivalences.

Proof. Let $f : X \to Y$ be a weak equivalence of fibrant objects. Since X and Y are fibrant, the right and bottom morphisms in the diagram

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & *
\end{tikzcd}$$

are fibrations. Hence π_X, π_Y are fibrations as well–fibrations are closed under pull-back. Compositions of fibrations are fibrations (solve the lifting problem on the inner factor and then the outer) and therefore $X \times Y$ is fibrant. Factor the map $(1_X, f) : X \to X \times Y$ into a trivial cofibration $i : X \to C$ followed by a fibration $p : C \to X \times Y$. Again, since compositions of fibrations are fibrant, C is fibrant. Note that $\pi_X \circ p \circ i = 1_X$, so $\pi_X \circ p$ is a weak equivalence (of fibrant objects) by 2-out-of-3. Similarly, $\pi_Y \circ p \circ i = f$ is a weak equivalence, so $\pi_Y \circ p$ is a weak equivalence. Both $\pi_X \circ p$ and $\pi_Y \circ p$ are fibrations as well, being compositions of such, and hence they are trivial fibrations (of fibrant objects). By hypothesis, this means $F(\pi_X \circ p)$ and $F(\pi_Y \circ p)$ are weak equivalences. By 2-out-of-3 (in $\mathcal{D}$), the former fact, and the observation $F(\pi_X \circ p \circ i) = F(1_X)$ is a weak equivalence implies that $F(i)$ is a weak equivalence. Again by 2-out-of-3, $F(\pi_Y \circ p) \circ F(i) = F(f)$ is a weak equivalence, proving the result. □

Remark A.4.9. There is a duality in the theory of model categories. Namely, if $\mathcal{C}$ is a model category, then $\mathcal{C}^{op}$ has a canonical model structure with the fibrations of $\mathcal{C}^{op}$ corresponding to the cofibrations of $\mathcal{C}$, and vice-versa. When this duality is applied to Ken Brown's Lemma, for example, we have the statement: if $F : \mathcal{C} \to \mathcal{D}$ is a functor taking trivial cofibrations of cofibrant objects to trivial cofibrations, then it takes all weak equivalences of cofibrant objects to weak equivalences.

By judicial application of Ken Brown's lemma, together with the axioms of a model category, one obtains the following important facts. For detailed proofs, see Chapter 1 of [Hov].

Theorem A.4.10. Let X, Y be objects of a model category $\mathcal{C}$ that is both fibrant and cofibrant.

1. The relation $\sim$ on $\mathcal{C}(X, Y)$ is an equivalence relation which is compatible with composition. Furthermore this relation is identical to that of right homotopy and that of left homotopy (alone).

2. The homotopy equivalences between X and Y correspond exactly to the weak equivalences.

The *homotopy category* of a model category $\mathcal{C}$, denoted $\mathrm{h}\mathcal{C}$ is (informally) the category obtained by formally inverting all of the weak equivalences of $\mathcal{C}$. The above facts motivate an equivalence between this category and the category with objects the fibrant, cofibrant objects of $\mathcal{C}$, and morphisms given by equivalence classes of homotopic morphisms.

We next define an appropriate notion of equivalence of model categories, which subsumes the obvious candidate of an 'equivalence of categories preserving all relevant structure'.

Definition A.4.11. A *Quillen adjunction* between model categories $\mathcal{C}, \mathcal{D}$ is an adjunction $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ which satisfies the following two equivalent conditions.

1. F preseves cofibrations and trivial cofibrations.

2. G preserves fibrations and trivial fibrations.

That the two conditions are equivalent is Lemma 1.3.4 in [Hov].

Definition A.4.12. A Quillen adjunction $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ is called a *Quillen equivalence* if, for $X \in \mathcal{C}$ cofibrant amd $Y \in \mathcal{D}$ fibrant, the bijection

$$\mathcal{C}(X, GY) \overset{\sim}{\to} \mathcal{D}(FX, Y)$$

induced by the adjunction preserves weak equivalences.

The importance of Quillen equivalences is that they are precisely the Quillen adjunctions which induce an equivalence on homotopy categories (see Proposition 1.3.13 in [Hov]).

Next we head toward the notion of a simplicial model category.

Definition A.4.13. The category of *properly simplicially enriched categories* has as objects categories properly tensored and cotensored over **sSet** with the product monoidal structure, and as morphisms enriched functors.

We use this clunky term simply to leave the term 'simplicial category' available. Enriched categories and the notion of properly tensored are defined in A.1.

Remark A.4.14. The properness of the tensor structure of a properly simplicially enriched category $\mathbf{C}$ implies, among other things, that

$$\mathbf{C}(X,Y) \cong \mathbf{sSet}(*, \mathbf{C}_{\mathbf{sSet}}(X,Y)) \cong \mathbf{C}_{\mathbf{sSet}}(X,Y)_0$$

In other words, the morphisms of $\mathbf{C}$ are precisely the zero simplicies of the 'space of morphisms'.

Definition A.4.15. The *standard model structure* on the category $\mathbf{sSet}$ of simplicial sets has inclusions as cofibrations, Kan fibrations as fibrations (see A.5), and as weak equivalences the morphisms that induce isomorphisms on homotopy groups.

For a detailed explanation of this model structure (and a proof of the model axioms), see [Hov] or [GJ].

Definition A.4.16. A properly simplicially enriched category $\mathbf{C}$ with a model structure is a *simplicial model category* if the following additional property holds. Suppose $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration. Then

$$\mathbf{C}_{\mathbf{sSet}}(B,X) \overset{(i^*,p_*)}{\longrightarrow} \mathbf{C}_{\mathbf{sSet}}(A,X) \times_{\mathbf{C}_{\mathbf{sSet}}(A,Y)} \mathbf{C}_{\mathbf{sSet}}(B,Y)$$

is a fibration in the standard model structure of $\mathbf{sSet}$, which is a weak equivalence if i or p is.

The section II.3 in [GJ] contains the following useful remarks.

Remark A.4.17. There are three observations to be made about the above definition. The first is that, by setting A to be the initial object in the above, we get the statement that if B is cofibrant, and $p : X \to Y$ is a fibration, then

$$p_* : \mathbf{C}_{\mathbf{sSet}}(B,X) \to \mathbf{C}_{\mathbf{sSet}}(B,Y)$$

is a fibration. In particular, simplicial set of maps into a fibrant object from a cofibrant object is a Kan complex. Setting Y to be the final object yields a cofibration

$$i^* : \mathbf{C}_{\mathbf{sSet}}(B,X) \to \mathbf{C}_{\mathbf{sSet}}(A,X)$$

provided X is fibrant. The third statement is that the extra axiom is much stronger than the lifting axiom of a model category. Indeed, the solid diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

is precisely a zero simplex in

$$\mathbf{C}_{\mathbf{sSet}}(A,X) \times_{\mathbf{C}_{\mathbf{sSet}}(A,Y)} \mathbf{C}_{\mathbf{sSet}}(B,Y).$$

Which is surjected on by $\mathbf{C}_{\mathbf{sSet}(B,Y)}$, since trivial fibrations (of simplicial sets) are surjective. The dotted arrow can be chosen to be any preimage of the zero simplex corresponding to the diagram. Moreover, since trivial fibrations are closed under pullback, by pulling back the trivial fibration

$$\mathbf{C}_{\mathbf{sSet}}(B,X) \xrightarrow{(i^*,p_*)} \mathbf{C}_{\mathbf{sSet}}(A,X) \times_{\mathbf{C}_{\mathbf{sSet}}(A,Y)} \mathbf{C}_{\mathbf{sSet}}(B,Y)$$

over the zero simplex specified by the solid diagram one sees that there is a contractible space of solutions to the mapping problem.

The last observation in particular shows that a simplicial model category is something more than just an enriched category with compatible model structure, which one might think of as the first two observations.

A.5 $(\infty, 1)$ Categories

The goal of this section is to describe four models for the theory of $(\infty, 1)$ categories, which will ultimately be combined with the previous section to provide the model for symmetric monoidal $(\infty, 1)$ categories we choose to work with.

Recall the abstract notion of an $(\infty, 1)$ category: it is a gadget posessing objects, 1-morphisms, and various higher morphisms. The morphisms above dimension 1 are required to be invertible. Furthermore, the usual axioms of identity and associativity for 1-morphisms in category theory are replaced by various coherence conditions. For example, composition is associative only up to (invertible) 2-morphisms called the associator, and the usual pentagon formed from associating four morphisms in the various ways is in turn filled with a 'pentagonator', which is in turn subject to higher coherence conditions.

One quickly sees the impossibility of specifying all this data individually; so various models have been introduced to model these data all at once. We refer the reader to the survey [Ber], and also to the first chapter of [Lur] for a discussion of the ideas behind these models.

These models for $(\infty, 1)$ categories all share the property that they can be expressed as the full subcategory of fibrant objects of some model category (whose underlying category is fairly simple, but whose model structure can be quite complicated). In [Ber] references are given to various works showing that all these model categories are Quillen equivalent, as one might expect.

We conclude each discussion of a model for $(\infty, 1)$-categories with a description of the $(\infty, 1)$-category induced by an ordinary category. Clearly any reasonable model of $(\infty, 1)$-category should admit such a construction.

A.5.1 Complete Segal Spaces

The first two models of $(\infty, 1)$ categories we will discuss are built off of models of $(\infty, 0)$ categories, i.e. ∞-groupoids. One natural example of something that behaves like an ∞ groupoid is a Kan complex. We recall their definition, which requires some preliminary notation.

Definition A.5.1. Let $\mathbf{sSet} := \mathbf{Set}^{\mathbf{\Delta}^{op}}$ denote the category of simplicial sets. Let $[n]$ denote the totally ordered set

$$0 < 1 < \cdots < n,$$

i.e. a typical object of $\mathbf{\Delta}$. Define the *standard n-simplex* $\Delta(n) \in \mathbf{sSet}$ by the formula

$$\Delta(n)_k = \mathbf{\Delta}([k], [n]).$$

Finally define the k^{th} *horn* of the standard n-simplex, denoted $\Lambda(n, k)$ to be the largest sub-simplicial set of $\Delta(n)$ obtained by removing the face opposite the k^{th} vertex.

Note that $\Delta(n)$ represents the n^{th} simplicies functor:

$$\mathbf{sSet}(\Delta(n), S_\bullet) \cong S_n$$

Definition A.5.2. A *Kan fibration* is a morphism of simplicial sets $p : S_\bullet \to T_\bullet$ such that given the solid diagram, a simplex indicated by the dotted line exists

$$\begin{tikzcd}
\Lambda(n,k) \arrow[r] \arrow[d] & S_\bullet \arrow[d] \\
\Delta(n) \arrow[r] \arrow[ur, dotted] & T_\bullet
\end{tikzcd}$$

Here n is anything, $0 \leq k \leq n$, and the vertical map is the obvious inclusion of the horn. A simplicial set is *Kan* if the unique morphism $S_\bullet \to \Delta(0)$ is a Kan fibration. In this case, we can simply ignore the T in the mapping problem diagram above.

We will take the position that this is a definition. Namely that an ∞ groupoid *is* a Kan complex. Note that Kan complexes are the fibrant objects in the usual model structure on simplicial sets.

Now, given some $(\infty, 1)$ category X, we should be able to recover an ∞ groupoid (i.e. Kan complex) by discarding all the non-invertible 1-morphisms. This can be thought of as considering the terminal category $[0]$ as an $(\infty, 1)$ category with only identities as higher morphisms, taking the $(\infty, 1)$ category $\text{Fun}([0], X)$ and then taking the largest Kan complex inside. We denote this operation of discarding noninvertible morphisms by superscript $\circ$

We could just as well replace the test category $[0]$, by the categories $[n]$ for larger n. That is, we can produce Kan complexes

$$X_n := \text{Fun}([n], X)^\circ.$$

This is similar to the nerve construction, and since nerve induces a fully faithful embedding of the category of small categories into **sSet**, we produce the category of Segal spaces, denoted **SSp** as a full subcategory of the category of bisimplicial sets, **ssSet**, which are in particular pointwise Kan. We need a bit more terminology to state the other condition.

Definition A.5.3. For each n, and $i = 0, \cdots, n-1$, Let $\iota^i_{[n]} : [1] \to [n]$ be the functor (i.e. morphism in $\mathbf{\Delta}$) determined by $\iota^i_{[n]}(0) = i$ and $\iota^i_{[n]}(1) = i+1$. Let $\iota^{[n]}_i$ denote the corresponding morphism in $\mathbf{\Delta}^{op}$.

Let $\boldsymbol{X}_\bullet : \mathbf{\Delta}^{op} \to \mathbf{sSet}$ be a bisimplicial set. Note that the map

$$(\iota^{[n]}_0, \cdots, \iota^{[n]}_{n-1}) : \boldsymbol{X}_n \to \prod_1^n \boldsymbol{X}_1$$

passes to the fibred product $\boldsymbol{X}_1 \times_{\boldsymbol{X}_0} \cdots \times_{\boldsymbol{X}_0} \boldsymbol{X}_1$. We let ψ_n denote the resulting map, sometimes called the *Segal map.*

We now have the following definition, due to Rezk ([Rez]).

Definition A.5.4. A *simplicial Segal space* is a bisimplicial set $\boldsymbol{X}_\bullet : \mathbf{\Delta}^{op} \to \mathbf{sSet}$ which is Reedy fibrant and such that the Segal maps

$$\psi_n : \boldsymbol{X}_n \to \underbrace{\boldsymbol{X}_1 \times_{bmX_0} \cdots \times_{\boldsymbol{X}_0} \boldsymbol{X}_1}_{n \text{ times}}$$

are weak equivalences of simplicial sets.

Note that the Reedy fibrant assumption implies that the above simplicial sets are Kan, so that the Segal map is actually a homotopy equivalence (see appendix A.4). The above definition seems standard in most of the literature on $(\infty, 1)$ categories. Unfortunately, in [Lur1], uses the same term for something slightly different, which we will call a Segal space. We will need some terminology before we give the definition.

Definition A.5.5. Let $f : X \to Z$, and $g : Y \to Z$ be continuous maps in **Top**. Then the *homotopy fibre product* $X \times^{\mathrm{h}}_Z Y$ is the space

$$X \times_Z Z^{[0,1]} \times_Z Y.$$

Given a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

in **Top**, we say that it is a *homotopy pullback square* if the induced map

$$W \to X \times_Z Y \to X \times^{\mathrm{h}}_Z Y$$

is a weak equivalence.

Definition A.5.6. A *Segal space* is a simplicial space $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$ such that, for each pair of natural numbers n, m, the diagram

$$\begin{tikzcd}
X_{m+n} \arrow[r] \arrow[d] & X_m \arrow[d] \\
X_n \arrow[r] & X_0
\end{tikzcd}$$

is a homotopy pullback square. We denote the full subcategory of **sTop** whose objects are Segal spaces by **Sp**.

Remark A.5.7. Notice in particular in Definition A.5.6, the Reedy fibrancy condition has been dropped. We remark that if this condition is retained, then the Segal map condition is analogous to that of simplicial Segal spaces. That is, if $X_\bullet$ is a Segal space $X_\bullet : \mathbf{\Delta}^{op} \to \mathbf{Top}$ which is Reedy fibrant in the Reedy model structure on **sTop**, then the Segal maps

$$\psi_n : X_n \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$$

are weak equivalences of (compactly generated, weakly Hausdorff) topological spaces (see [Lur2] §2.1).

We can make sense of 'objects' of a (simplicial) Segal space; namely the zero simplicies of X_0 (i.e. $X_{0,0}$). We can also make sense of the mapping space between any two objects x, y: it is the fiber of the obvious map $X_1 \to X_0 \times X_0$ over (x, y). We will say that two zero simplicies in the mapping space between x and y (i.e. two maps) are homotopic if they lie in the same path component. One can make sense of composition, by taking a homotopy inverse to the Segal map ψ_2, and the higher Segal maps ensure that this composition is homotopy coherent, which motivates the following definition.

Definition A.5.8. Let $X_\bullet$ be a (simplicial) Segal space. We define the *homotopy category*, $\mathrm{h}\, X$ to have objects $X_{0,0}$ and hom-sets given by the connected components of the mapping spaces of $X_\bullet$ (i.e. π_0).

That the result is indeed a category in the case of simplicial Segal spaces is shown in §5 of [Rez].

This gives enough to make sense of the notion of homotopy equivalence, which obviously subsumes 'identity maps' in the image of the degeneracy map $s_0 : X_0 \to X_1$. We arrive at the following definition, again from [Rez].

Definition A.5.9. Let X_{eq} denote the space of homotopy equivalences of a Segal space X_bullet. We say that $X_\bullet$ is a *complete simplicial Segal space* if the diagonal map in

$$\begin{tikzcd}
X_0 \arrow[r, "s_0"] \arrow[dr] & X_1 \\
 & X_{eq} \arrow[u, hook]
\end{tikzcd}$$

is a weak equivalence of simplicial sets.

We make an anlagous defintion for *complete Segal space.*

We next consider a particular type of morphism of simplicial Segal spaces, named for Dwyer and Kan.

Definition A.5.10. Let $\boldsymbol{X}_\bullet, \boldsymbol{Y}_\bullet \in \mathbf{ssSet}$, and $f : \boldsymbol{X}_\bullet \to \boldsymbol{Y}_\bullet$. We say that f is a *DK equivalence* if

- ◇ For any pair of objects in $\boldsymbol{X}_{0,0}$, the map induced by f on mapping spaces is a weak equivalence of simplicial sets.
- ◇ The functor
$$\mathrm{h}\, f : \mathrm{h}\, \boldsymbol{X} \to \mathrm{h}\, \boldsymbol{Y}$$
is an equivalence of categories.

We can summarize the results of [Rez], §7, with the following theorem

Theorem A.5.11. 1. There is a simplicial model structure on **ssSet** such that

  - ◇ The fibrant objects are exactly the Segal spaces, and all objects are cofibrant.
  - ◇ The trivial fibrations (i.e. weak equivalences of Segal spaces) are precisely the Reedy weak equivalence.

2. There is a simplicial model structure on **ssSet** such that

  - ◇ The fibrant objects are exactly the complete Segal spaces, and all objects are cofibrant.
  - ◇ The trivial fibrations are precisely the DK-equivalences.

3. The complete Segal space model structure is obtained from the Segal space model structure by Bousfield localization at the DK equivalences.

The reason that we care about *complete* (simplicial) Segal spaces is that it is **ssSet** with the complete simplicial Segal space model structure which is Quillen equivalent to the other types of $(\infty, 1)$ categories below. Fortunately, although the examples of $(\infty, 1)$ categories discussed in this paper (and which arise 'from nature') are neither Reedy fibrant nor complete in general, fibrant replacement gives a uniform way to rectify that. In general, if the completeness condition is dropped, then there are many (simplicial) Segal spaces with the same homotopy category. For example both $\mathrm{N}\,\mathcal{C}$ and $\mathrm{N}_S\,\mathcal{C}$ have $\mathcal{C}$ as their homotopy category, where N_S is defined as follows.

Let **cSSp** denote the full subcategory of **ssSet** consisting of complete Segal spaces. Let $I[m]$ denote the groupoid with $m+1$ objects and m composable isomorphisms between them. Let $\mathcal{C} \in \mathbf{Cat}$ be a small category. Define the *Segal nerve* of $\mathcal{C}$ to be the bisimplicial set

$$(\mathrm{N}_S\,\mathcal{C})_{n,m} = \mathrm{Fun}([n] \times I[m], \mathcal{C})$$

This construction is obviously functorial, and Lemmas 3.6-3.9 in [Rez] give the following result.

Theorem A.5.12. The functor $N_s : \mathbf{Cat} \to \mathbf{ssSet}$ is a full embedding of categories, and ends up in the subcategory of complete Segal spaces. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if $\mathrm{N}_S\, F$ is a weak equivalence in the Reedy model structure, and finally we have natural isomorphisms

$$\mathrm{N}_S(\mathcal{C} \times \mathcal{D}) \cong \mathrm{N}_S\,\mathcal{C} \times \mathrm{N}_S\,\mathcal{D} \text{ and } \mathrm{N}_S\left(\mathcal{D}^{\mathcal{C}}\right) \cong \underline{\mathbf{ssSet}}(\mathrm{N}_S\,\mathcal{C}, \mathrm{N}_S\,\mathcal{D}).$$

Remark A.5.13. In view of the second isomorphism, it makes sense to view N_S as an enriched functor $\mathrm{N}_S : \mathrm{N}_*\,\mathbf{Cat} \to \mathbf{ssSet}$, by setting

$$\mathrm{N}_S\left(\mathrm{N}\left(\mathcal{D}^{\mathcal{C}}\right)\right) = \mathbf{ssSet}_{\mathbf{sSet}}(\mathrm{N}_S\,\mathcal{C}, \mathrm{N}_S\,\mathcal{D}).$$

A.5.2 Locally Kan Categories

The second model is based on the idea, which bears out in the case of complete Segal spaces, that one should replace a set of morphisms between any two objects, as in ordinary categories, with a space of morphisms. Another way to view this is that a (strict) 2-category can be thought of as a category enriched in small categories; therefore, an $(\infty, 1)$ category should be enriched in ∞-groupoids. As discussed in the previous section, a reasonable model for ∞-groupoids is provided by the category of Kan complexes. So the idea is that categories enriched over the category **Kan** of Kan complexes is a type of $(\infty, 1)$ category. As it turns out, these are the fibrant objects of a certain model structure on the category of simplicially enriched categories and enriched functors.

Definition A.5.14. A *simplicially enriched category* is a category enriched over **sSet**, considered as a monoidal category with the product monoidal structure. We use the notation $\mathbf{Cat}_{\mathbf{sSet}}$ to denote the category of such objects together with enriched functors, which in this case we will call *simplicial functors*.

A *locally Kan category* is a simplicially enriched category $\mathbf{C}$ such that, for $X, Y \in \mathbf{C}$, the simplicial set

$$\mathbf{C}_{\mathbf{sSet}}(X, Y)$$

is Kan.

For enriched categories, recall that we have the following construction of Remark A.1.3.

As an example, we have the 'path component' functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ which commutes with products, and is therefore monoidal with respect to the product monoidal structures on source and target. Given $\mathbf{C} \in \mathbf{Cat}_{\mathbf{sSet}}$, $(\pi_0)_*\mathbf{C}$ should be considered the 'homotopy category' of $\mathbf{C}$:it is a category enriched in $\mathbf{Set}$, i.e. an ordinary category. We now describe the weak equivalences of the model structure having locally Kan categories as the fibrant objects. These are also called DK-equivalences.

Definition A.5.15. Let $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}_{\mathbf{sSet}}$. A *DK-equivalence* $F : \mathbf{C} \to \mathbf{D}$ is a simplicial functor such that

- For $x, y \in \mathbf{C}$, the simplicial map
$$F_{x,y} : \mathbf{C}_{\mathbf{sSet}}(x, y) \to \mathbf{D}_{\mathbf{sSet}}(Fx, Fy)$$
is a weak equivalence of simplicial sets.
- The functor
$$(\pi_0)_*F : (\pi_0)_*\mathbf{C} \to (\pi_0)_*\mathbf{D}$$
is an equivalence of categories.

We need one more piece of terminology to describe the model structure on $\mathbf{Cat}_{\mathbf{sSet}}$ whose fibrant objects are locally Kan categories.

Definition A.5.16. Let $\mathbf{C} \in \mathbf{Cat}_{\mathbf{sSet}}$. We say that a 'morphism'
$$f \in \mathbf{C}_{\mathbf{sSet}}(x, y)_0$$
is a *homotopy equivalence* if the image of f in $(\pi_0)_*\mathbf{C}$ is an isomorphism.

The main result of [Ber1] is the following.

Theorem A.5.17. There is a model category structure on $\mathbf{Cat}_{\mathbf{sSet}}$ with the following weak equivalences and fibrations.

1. The weak equivalences are the DK-equivalences.
2. The fibrations $F : \mathbf{C} \to \mathbf{D}$ are the simplicial functors such that
    - For $x, y \in \mathbf{C}$, the simplicial map
    $$F_{x,y} : \mathbf{C}_{\mathbf{sSet}}(x, y) \to \mathbf{D}_{\mathbf{sSet}}(Fx, Fy)$$
    is a Kan fibration.

- ◇ For objects $x_1 \in \mathbf{C}$ and $y \in \mathbf{D}$, and a homotopy equivalence $f : Fx_1 \to y$, there is an object $x_2 \in \mathbf{C}$ and a homotopy equivalence $e : x_1 \to x_2$ in $\mathbf{C}$ such that $Fe = f$.

Furthermore the fibrant objects are precisely the locally Kan categories.

Note that we did not specify the cofibrations. This is not necessary by Lemma A.4.4: the cofibrations can be defined to be the morphisms with the left lifting property with respect to the trivial fibrations.

In this case, the locally Kan category induced by an ordinary category is clear: simply consider an ordinary category as enriched over **sSet** with discrete simplicial hom-objects.

A.5.3 Quasi-categories

Quasi-categories were introduced by Boardman and Vogt in [BV], and their theory was developed extensively by Joyal ([Joy]), and Lurie ([Lur2]). They are also sometimes called weak Kan complexes, and ∞-categories (in [Lur2]).

The germ of the idea behind quasi-categories is the following standard exercise.

Lemma A.5.18. A simplicial set $C_\bullet \in \mathbf{sSet}$ is in the essential image of the nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ if and only if the following condition holds: Given the solid diagram where $n \geq 2$ and $0 < k < n$,

$$\begin{tikzcd}
\Lambda(n,k) \arrow[r] \arrow[d, hook] & C_\bullet \\
\Delta(n) \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

The dotted arrow exists and is unique.

Indeed, such a lift is found in the case $n = 2$, $k = 1$ by the composition of morphisms. A horn $\Lambda(n,k)$ is called an *inner horn* when $0 < k < n$, and so we have an alternative definition of an ordinary (small) category: it is a simplicial set where every inner horn can be filled uniquely.

This leads to the weakening:

Definition A.5.19. A simplicial set $C_\bullet \in \mathbf{sSet}$ is called a *quasi-category* if every inner horn $\Lambda(n,k) \to C_\bullet$ can be filled.

There is the following pleasant theorem of Joyal ([Joy]).

Theorem A.5.20. Let $C_\bullet \in \mathbf{sSet}$. Then $C_\bullet$ is a quasi-category if and only if the simplical set map

$$\underline{\mathbf{sSet}}(\Delta(2), C_\bullet) \to \underline{\mathbf{sSet}}(\Lambda(2,1), C_\bullet)$$

on inner-homs induced by the inclusion of the horn is a trivial Kan fibration.

So, similarly to the simplicial model category case, the space of solutions to the mapping problem

$$\begin{tikzcd}
\Lambda(2,1) \arrow[r] \arrow[d, hook] & C_\bullet \\
\Delta(2) \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

i.e., 'the space of compositions', is contractible.

Let $\pi_1 : \mathbf{sSet} \to \mathbf{Cat}$ denote the left adjoint to the nerve functor, i.e. 'fundamental category' (see [JT]). Now we can describe the model structure whose fibrant objects are the quasi-categories.

Theorem A.5.21. There is a model structure on **sSet**, called the *Joyal model structure* (compare the standard model structure), with

1. Weak equivalences are simplicial set maps $f : S_\bullet \to T_\bullet$ such for any quasi-category $C_\bullet$, the functor
$$\pi_1(f^*) : \pi_1\underline{\mathbf{sSet}}(T_\bullet, C_\bullet) \to \pi_1\underline{\mathbf{sSet}}(S_\bullet, C_\bullet)$$
is an equivalence of categories.
2. The fibrations are simplicial set maps $f : S_\bullet \to T_\bullet$ that have the right lifting propert with respect to inner horns: given a solid commutative diagram
$$\begin{tikzcd}
\Lambda(n,k) \arrow[r] \arrow[d] & S_\bullet \arrow[d, "f"] \\
\Delta(n) \arrow[r] \arrow[ur, dotted] & T_\bullet
\end{tikzcd}$$
with $0 < k < n$, a simplex indicated by the dotted arrow exists.
3. Cofibrations are injections of simplicial sets.

Given a category $\mathcal{C}$, taking the nerve $\mathrm{N}\,\mathcal{C}$ produces a quasi-category. In this case the horn-fillers are unique.

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