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# Part I

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*Abstract*
It is well known that consistent estimators of errors-in-variables models require knowledge of the ratio of error variances. What is not well known is that a Joint Least Squares estimator is robust to a wide misspecification of that ratio. Through a series of Monte Carlo experiments we show that an easy-to-implement estimator produces estimates that are nearly unbiased for a wide range of the ratio of error variances. These MC analyses encompass linear and nonlinear specifications and also a system on nonlinear equations where all the variables are measured with errors.

*Keywords*: Robust Estimators, Ratio of error variances, errors-in-variables, Joint least squares, concentrated joint least squares, Monte Carlo experiments.
*JEL classification*:C130, C200, C310.

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## Robust Estimators of Errors-In-Variables Models – Part I

### 1. Introduction

The errors-in-variables (EIV) problem has a long history and a vast literature. It all began in 1878, when R. J. Adcock presented a one-page paper in *The Analyst* (published in Des Moines, Iowa) entitled *"A Problem in Least Squares."* In that paper, he shows that, when a sample of bivariate information is measured with error, the regression line has an ambiguous meaning depending on which variable is taken as the dependent one. Since then, many distinguished statisticians have tackled the EIV problem and the conditions for consistent estimators are, by now, pretty well understood. Surprisingly, the same cannot be said for maximum likelihood estimators of the EIV model.

Briefly, one of the conditions for a consistent estimator of the EIV model is the knowledge of the ratio between the error variances. Since this knowledge cannot be ascertained from the sample information, consistent estimates of the EIV models are difficult to obtain.

Over the past forty years, the treatment of errors-in-variables models has received a progressively waning consideration in econometric textbooks: From the twenty eight pages of the 1963 edition of Johnson's *Econometric Methods* to the two pages of the 2004 edition of Davidson and MacKinnon's *Econometric Theory and Methods*. In recent years, the EIV problem has been revisited by several econometricians but, in general, there remains the impression that the EIV problem is rather intractable because no generally consistent estimator exists that is also easy to implement.

In this paper we explore an aspect of the EIV problem that seems to have gone unnoticed in the available literature. We argue that a weighted least-squares estimator is robust to misspecifications of the ratio of the error variances. In other words, even though that crucial piece of information is unavailable from the given sample information, a weighted least squares specification that uses misspecified values of the error variances' ratio produces estimates that are rather close to the true values of the parameters. In turn, and in linear models, the use of these parameter estimates produces estimates of the error variances that are close to their true values. All these assessments are deduced from a rather wide spectrum of Monte Carlo (MC) experiments that constitute the main body of this paper. We emphasize that the MC results presented in this paper constitute only a *prima facie* evidence of an empirical regularity that, if confirmed in other contexts, may renew the interest in dealing with EIV models, given the simplicity of the proposed estimators.

In section 2 we give a brief chronological review of the main papers on the EIV problem. In section 3 we discuss the simplest linear model with a single independent variable that is measured with error. In this section we will also review the condition for a consistent estimator. Extensive Monte Carlo evidence of the robustness of the proposed estimator is also presented. In section 4 we extend the linear EIV model to three independent variables and carry out the corresponding Monte Carlo analyses. In section 5 we tackle the nonlinear EIV problem by discussing two nonlinear specifications with two and three independent variables, respectively. By way of Monte Carlo analyses, it is shown again that the misspecified weighted least-squares estimator produces estimates that are close to their true values. In section 6 we deal with a nonlinear system of three equations where all the variables are measured with error. We choose a linear expenditure system (LES) with three commodities as an example of a nonlinear system of equations that produces MC results remarkably close to the true values of the parameters.

In Part II of this study, the linear and nonlinear models analyzed in this Part I will be extended to include a substantially larger number of parameters.

**2. Brief Review of the Literature**

Adcock (1878) appears to be the first scientist to draw attention to the problem within the context of fitting a straight line. He suggested the minimization of the squared sum of orthogonal deviations of the observed points from the line. In a remarkable paper that appeared soon after, Kummell (1879) analyzed the nonlinear EIV problem within the context of a least-squares framework where the errors are weighted by the corresponding variables' precisions. He concluded, for the first time, that the solution of the EIV problem requires the assumption "that the precisions of the $x$ and $y$ which enter the same observation equation are considered to have a constant ratio." Surprisingly, Pearson (1901) followed the approach of Adcock by minimizing the orthogonal distance of the sample points from a line. Gini (1921), who may not have been aware of Kummell's work, recognizes the importance of the ratio of the error variance of the regressor to that of the dependent variable. He gave the quadratic formula for estimating the slope of a line that contains the error variance ratio.

By 1940, a distinction between functional and structural relations had been introduced in the literature. In the EIV problem, a functional relation specifies that the latent variables (associated with the regressor which is measured with error) be considered fixed values. In a structural relation, these latent quantities are assumed to be random variables with a given probability distribution.

In a lengthy paper, Lindley (1947) gives a clear statement of the state of the art for that time about EIV problems. In particular, he discusses the non-existence of the likelihood function for a functional relation because the approach produces the implausible result that the square of the slope of the regression line must be equal to the error variances' ratio. The absence of a finite maximum will be made clearer by Solari (1969) who demonstrated that the likelihood function has a saddle point. Lindley then assumed that the latent variables are distributed normally together with the knowledge of the ratio between the error variances. These two assumptions produce consistent maximum likelihood estimates of the desired parameters except that the assumption of normality of the latent variables will soon be recognized to lead to the non-identification of the model.

The discussion about model identification under EIV began with a paper by Thomson (1916) who noticed that the joint distribution of the sample variates based only upon first and second moments cannot identify the parameters of a linear model. Neyman (1936) demonstrated that a necessary condition for identification is that the latent variables not be normally distributed because the joint distribution of two (or more) normal variates is itself normal and admits different values of the model's parameters. Hence, such a model is not identified. The discussion was concluded by Reiersol (1950) with the demonstration that non-normality is also a sufficient condition for identification. Neyman (1951) and Neyman and Scott (1948) presented significant results involving the non-consistency of maximum likelihood estimators even when a consistent estimator exists.

The quest for consistent estimators of EIV models continued with a paper by Wald (1949) who presented conditions under which maximum likelihood estimates are consistent. These conditions, however, are difficult to verify. His results were re-elaborated by Kiefer and Wolfowitz (1956) in an important paper about mixed distributions that provided another set of conditions for consistent maximum likelihood estimators. Both papers are very theoretical and their existence theorems do not seem to provide operative indications easy to implement as to how consistent maximum likelihood estimates can be obtained.

The current literature on EIV models is vast and growing. The bottom line can be stated as follows. Consistent estimators seem to require one of the following conditions: either knowledge of the ratio of the error variances or the availability of replicate measurements of the latent variables. In general, both of these conditions are difficult to implement. For this reason, we assess the primal-dual estimator proposed in this paper by performing a Monte Carlo analysis using a sample of real data that we assume as the benchmark observations of the latent variables and compare the estimates obtained with the specification of the true ratio of the error variances with those obtained with mis-specified ratios. The interesting conclusion is that the estimator is robust to mis-specification of the true ratio of the error variances.

## 3. A Linear EIV Model With One Independent Variable

In this section we analyze the simplest linear model without intercept and with only one independent variable. This streamlined specification will allow a clear understanding of many issues pertaining to the EIV problem including the process of obtaining a consistent estimate of the model's parameter. The linear model is specified as follows

(1) $$y_i = \beta x_i^* + u_i$$

(2) $$x_i = x_i^* + \varepsilon_i$$

where $X^*$ is considered a latent variable with unknown realizations $x_i^*$, $i = 1,...N$ that are measured with error. These unknown entities can be considered either as fixed values or as draws from a given distribution. In the first case, model (1)-(2) is regarded as a *functional* specification while, in the second case, equations (1) and (2) are regarded as a *structural* model. The observed sample information is given by the $N$ pairs $(y_i, x_i)$. The error terms are assumed to be iid random variables. A more definite specification, utilized also in the Monte Carlo analyses, will regard the error terms to be distributed as normal random variable as $u_i \sim N(0, \sigma_u)$, $\varepsilon_i \sim N(0, \sigma_\varepsilon)$, where $\sigma_u$ and $\sigma_\varepsilon$ are the standard deviations of the corresponding errors. The independence hypothesis implies that $\text{cov}(x_i^*, \varepsilon_i) = 0$, $\text{cov}(y_i, \varepsilon_i) = 0$, $\text{cov}(x_i^*, u_i) = 0$, $i = 1,...,N$. In order to satisfy the identification requirements stated by Riersol (1950), we assume that the latent variable is distributed according to the uniform distribution.

It is known that the likelihood function (based upon normal distributions of the measurement errors in a functional model) of the EIV model specified in equations (1) and (2) is not a feasible estimator because the likelihood function does not possess a finite maximum as a consequence of its saddle function structure (Solari, 1969). For the structural EIV model, Kiefer and Wolfowitz (1956) demonstrated the consistency of the maximum likelihood estimator but the empirical implementation of their results has never appeared in the literature. The difficulty encountered in a feasible specification of the proper likelihood approach has induced a re-visitation of the least-squares estimator. It is, thus, known that a consistent and feasible estimator is the following weighted Joint Least Squares (JLS) estimator, where the two sums of squared residuals are weighted by the corresponding error variances

(3) $$\min_{\beta, x_i^*} JLS = \sum_{i=1}^{N} (y_i - \beta x_i^*)^2 / \sigma_u^2 + \sum_{i=1}^{N} (x_i - x_i^*)^2 / \sigma_\varepsilon^2 .$$

In order to implement the JLS consistent estimator given by equation (3), the ratio of the error variances must be known a priori. Hence, we posit the following parameter $\lambda \overset{\text{def}}{=} \sigma_\varepsilon^2 / \sigma_u^2$ that transforms equation (3) into the following expression

(4) $$\min_{\beta, x_i^*} JLS = \sum_{i=1}^{N} (y_i - \beta x_i^*)^2 + \sum_{i=1}^{N} (x_i - x_i^*)^2 / \lambda$$

where $\lambda > 0$. The first order necessary conditions of problem (4) are as follows

(5) $$\frac{\partial JLS}{\partial \beta} = -2 \sum_{i=1}^{N} (y_i - \beta x_i^*) x_i^* = 0 ,$$

(6) $$\frac{\partial JLS}{\partial x_i^*} = -2(y_i - \beta x_i^*)\beta - 2(x_i - x_i^*)/\lambda = 0 .$$

The second order necessary conditions require that the following Hessian matrix be positive definite:

$$H = \begin{bmatrix} \sum_i (x_i^*)^2 & -(y_i - 2\beta x_i^*) & -(y_N - 2\beta x_N^*) \\ -(y_i - 2\beta x_i^*) & \beta^2 + 1/\lambda & \\ -(y_N - 2\beta x_N^*) & & \beta^2 + 1/\lambda \end{bmatrix} .$$

Notice that it is possible to rewrite equation (4) in concentrated form by exploiting the information of equation (6) which gives

(7) $$x_i^* = \frac{\lambda\beta y_i + x_i}{1+\lambda\beta^2}$$

and by replacing the latent variable $x_i^*$ into equation (4) with the following result

(8) $$\min_{\beta} CJLS = \frac{1}{(1+\lambda\beta^2)} \sum_{i=1}^{N} (y_i - \beta x_i)^2$$

where *CJLS* stands for concentrated joint least squares. The solution of problems (4) and (8), for the same value of the $\lambda$ parameter, gives identical estimates of the parameter $\beta$. The CJLS estimator does not have a closed form solution and, therefore, it requires access to an efficient mathematical programming package such as, for example, GAMS.

*3.1 Consistent Estimates of the $\beta$ parameter*

We now show that, with knowledge of the true value of the $\lambda$ ratio, equation (8) is a consistent estimator of parameter $\beta$. We begin by taking the derivative of equation (8), that is, the first order necessary condition of the minimization problem (8):

(9) $$\frac{\partial CJLS}{\partial\beta} = -2\frac{1}{(1+\lambda\beta^2)} \sum_{i=1}^{N} (y_i - \beta x_i)x_i - 2\frac{\lambda\beta}{(1+\lambda\beta^2)} \sum_{i=1}^{N} (y_i - \beta x_i)^2 = 0$$

which can be simplified as the following quadratic equation in the $\beta$ parameter:

(10) $$-\sum_{i=1}^{N} y_i x_i + \beta\left(\sum_{i=1}^{N} x_i^2 - \lambda \sum_{i=1}^{N} y_i^2\right) + \beta^2 \lambda \sum_{i=1}^{N} y_i x_i = 0$$

with solution

(11) $$\tilde{\beta} = \frac{-(\sum_i x_i^2 - \lambda \sum_i y_i^2) \pm \sqrt{(\sum_i x_i^2 - \lambda \sum_i y_i^2)^2 + 4\lambda(\sum_i y_i x_i)^2}}{2\lambda \sum_i y_i x_i}.$$

Hence, the probability limit of the $\tilde{\beta}$ estimator can be stated as follows:

$$\underset{N\to\infty}{P\lim}\tilde{\beta} = \frac{-[\sum_i (x_i^* + \varepsilon_i)^2 - \lambda \sum_i (\beta x_i^* + u_i)^2] + \sqrt{[\sum_i (x_i^* + \varepsilon_i)^2 - \lambda \sum_i (\beta x_i^* + u_i)^2]^2 + 4\lambda [\sum_i (\beta x_i^* + u_i)(x_i^* + \varepsilon_i)]^2}}{2\lambda \sum_i (\beta x_i^* + u_i)(x_i^* + \varepsilon_i)}$$

$$\underset{N\to\infty}{P\lim}\tilde{\beta} = \frac{-[\sum_i x_i^{*2} + \sigma_\varepsilon^2 - \lambda\beta^2 \sum_i x_i^{*2} - \lambda\sigma_u^2] + \sqrt{[\sum_i x_i^{*2} + \sigma_\varepsilon^2 - \lambda\beta^2 \sum_i x_i^{*2} - \lambda\sigma_u^2]^2 + 4\lambda\beta^2 [\sum_i x_i^{*2}]^2}}{2\lambda\beta \sum_i x_i^{*2}}$$

$$\underset{N\to\infty}{P\lim}\tilde{\beta} = \frac{-[\sum_i x_i^{*2} - \frac{\sigma_\varepsilon^2}{\sigma_u^2}\beta^2 \sum_i x_i^{*2}] + \sqrt{[\sum_i x_i^{*2} + \frac{\sigma_\varepsilon^2}{\sigma_u^2}\beta^2 \sum_i x_i^{*2}]^2}}{2\frac{\sigma_\varepsilon^2}{\sigma_u^2}\beta \sum_i x_i^{*2}}$$

$$\underset{N\to\infty}{P\lim}\tilde{\beta} = \frac{-\sum_i x_i^{*2} + \frac{\sigma_\varepsilon^2}{\sigma_u^2}\beta^2 \sum_i x_i^{*2} + [\sum_i x_i^{*2} + \frac{\sigma_\varepsilon^2}{\sigma_u^2}\beta^2 \sum_i x_i^{*2}]}{2\frac{\sigma_\varepsilon^2}{\sigma_u^2}\beta \sum_i x_i^{*2}} = \frac{2\frac{\sigma_\varepsilon^2}{\sigma_u^2}\beta^2 \sum_i x_i^{*2}}{2\frac{\sigma_\varepsilon^2}{\sigma_u^2}\beta \sum_i x_i^{*2}}$$

(12) $$\underset{N\to\infty}{P\lim}\tilde{\beta} = \beta$$

after using the fact that $\lambda \overset{\text{def}}{=} \sigma_\varepsilon^2 / \sigma_u^2$ and all the distributional assumptions stated above. Knowledge of the true error variances' ratio is, thus, crucial for obtaining consistent estimates of the $\beta$ parameter.

*3.2 Monte Carlo Experiments for Model (1)-(2)*

The empirical results of this section will show that the CJLS estimator, given by equation (8), produces estimates of the $\beta$ parameter that are very close to the true value of $\beta$ even though the ratio of the error variances is grossly misspecified. The definition of "very close" is admittedly subjective but it will be taken to be a discrepancy from the true value of the $\beta$ parameter smaller than a few percentage points. We appeal to Monte Carlo experiments in order to gauge the performance of the CJLS estimator under a variety of error specifications.

In this initial set of Monte Carlo experiments the choice of parameters and distributions is reported in Table 1.

Table 1. Parameter and distribution choice for the MC experiments of model (1)-(2)

| Set | $\beta$ | $x_i^*$ | $\sigma_u$ | $\sigma_\varepsilon$ | True $\lambda$ | $N$ |
|---|---|---|---|---|---|---|
| 1 | 1.30 | *U(2,20)* | 2.0 | 3.0 | 2.25 | 200 |
| 2 | 1.30 | *U(2,20)* | 2.0 | 3.0 | 2.25 | 5000 |
| 3 | 1.30 | *U(2,20)* | 1.0 | 1.5 | 2.25 | 200 |
| 4 | 1.30 | *U(2,20)* | 1.0 | 1.5 | 2.25 | 5000 |
| 5 | 1.30 | *U(2,20)* | 1.0 | 1.0 | 1.0 | 200 |
| 6 | 1.30 | *U(2,20)* | 1.0 | 1.0 | 1.0 | 5000 |
| 7 | 1.30 | *U(2,20)* | 2.0 | 1.0 | 0.25 | 200 |
| 8 | 1.30 | *U(2,20)* | 2.0 | 1.0 | 0.25 | 5000 |
| 9 | 1.30 | *U(2,20)* | 1.5 | 1.0 | 0.4444 | 200 |
| | | | | | | |
| 10 | 1.30 | *U(2,20)* | 2.0 | 1.0 | 0.25 | 200 |
| 11 | 1.30 | *U(2,20)* | 1.5 | 1.0 | 0.4444 | 200 |

The choice of two sample sizes is based on the desire to gauge the discrepancy from the consistency of the EIV estimator with a large sample size ($N$ = 5000) and the conjecture that a more realistic sample size ($N$ = 200), associated with 100 sample drawings, could reveal the extent of discrepancy from an unbiased estimate of the $\beta$ parameter. The results of the Monte Carlo experiments are reported in Tables 2 and 3. In order to allow for a meaningful numerical estimation of the model even with very large sample sizes, all the (latent and error) variables where scaled by the squared root of $N$, the sample size.

The Monte Carlo computations were carried out for a range of the $\lambda$ parameter that differs from the true value of the error variances' ratio up to a factor of five. In Table 2 the shaded rows indicate the average estimates (over 100 samples) corresponding to the true value of the error variances' ratio, under the first six sets of distributional specifications as reported in Table 1. It can be seen that, for values of the error variances' ratio that differ considerably from the true value of the $\lambda$ parameter, the average value of the estimates of the $\beta$ parameter do not differ greatly.

Table 2. Average estimates (100 samples) of the $\beta$ parameter for given $\lambda$, model (8)

| Number | λ value | set 1 | set 2 | set 3 | set 4 | set 5 | set 6 |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| 1 | 0.25 | 1.2446 | 1.2525 | 1.2874 | 1.2898 | 1.2969 | 1.2963 |
| 2 | 0.50 | 1.2636 | 1.2679 | 1.2924 | 1.2938 | 1.2995 | 1.2984 |
| 3 | 0.75 | 1.2758 | 1.2779 | 1.2955 | 1.2964 | 1.3013 | 1.2998 |
| 4 | 1.00 | 1.2843 | 1.2848 | 1.2977 | 1.2981 | 1.3025 | 1.3008 |
| 5 | 1.25 | 1.2906 | 1.2898 | 1.2992 | 1.2994 | 1.3035 | 1.3016 |
| 6 | 1.50 | 1.2953 | 1.2937 | 1.3004 | 1.3004 | 1.3042 | 1.3022 |
| 7 | 1.75 | 1.2991 | 1.2968 | 1.3014 | 1.3011 | 1.3048 | 1.3026 |
| 8 | 2.00 | 1.3021 | 1.2992 | 1.3021 | 1.3018 | 1.3052 | 1.3030 |
| 9 | 2.25 | 1.3047 | 1.3013 | 1.3028 | 1.2932 | 1.3056 | 1.3033 |
| 10 | 2.50 | 1.3068 | 1.3030 | 1.3033 | 1.3027 | 1.2998 | 1.2975 |
| 11 | 2.75 | 1.3085 | 1.3044 | 1.2909 | 1.3031 | 1.3063 | 1.3038 |
| 12 | 3.00 | 1.3101 | 1.3057 | 1.3041 | 1.3034 | 1.3065 | 1.3040 |
| 13 | 3.25 | 1.3114 | 1.2937 | 1.3045 | 1.3036 | 1.2938 | 1.3042 |
| 14 | 3.50 | 1.3126 | 1.3077 | 1.3048 | 1.3039 | 1.3069 | 1.3044 |
| 15 | 3.75 | 1.3136 | 1.3085 | 1.3050 | 1.3041 | 1.3071 | 1.3045 |
| 16 | 4.00 | 1.3146 | 1.3093 | 1.3052 | 1.3043 | 1.3072 | 1.3046 |
| 17 | 4.25 | 1.3154 | 1.3100 | 1.3054 | 1.3044 | 1.3074 | 1.3047 |
| 18 | 4.50 | 1.3162 | 1.3106 | 1.3056 | 1.3046 | 1.3075 | 1.3048 |
| 19 | 4.75 | 1.3168 | 1.3111 | 1.3058 | 1.3047 | 1.3076 | 1.3049 |
| 20 | 5.00 | 1.3175 | 1.3116 | 1.3060 | 1.3049 | 1.3077 | 1.3050 |

Table 3 reports the results of the Monte Carlo experiments when the true value of the $\lambda$ parameter is smaller than one. The average estimates of the $\beta$ parameter for the sets of distributional specifications 7, 8 and 9 of Table 1 are somewhat larger than the similar estimates of Table 2. It appears, therefore, that when the true value of the $\lambda$ parameter is smaller than one, a more appropriate specification of the CJLS objective function may be a slightly modified CJLS of equation (8) that now reads

(13) $$\min_{\beta} CJLS = \frac{\lambda}{(1+\lambda\beta^2)} \sum_{i=1}^{N} (y_i - \beta x_i)^2 .$$

Equation (13) corresponds to an original JLS objective function where the $\lambda \overset{\text{def}}{=} \sigma_\varepsilon^2 / \sigma_u^2$ parameter multiplies the first residual sum of squares, as indicated below

(14) $$\min_{\beta, x_i^*} JLS = \lambda \sum_{i=1}^{N} (y_i - \beta x_i^*)^2 + \sum_{i=1}^{N} (x_i - x_i^*)^2 .$$

The results of using equation (13) and the same distributional specifications of sets 7 and 9 of Table 1 are reported in Table 3 under sets 10 and 11, respectively. Hence, with this alternative specification of the CJLS objective function, the average estimate of the $\beta$ parameter is closer to its true value for the entire spectrum of the $\lambda$ parameter.

Table 3. Average estimates of the $\beta$ parameter for given $\lambda$, models (8) and (13)

| Number | $\lambda$ value | set 7 | set 8 | set 9 | | set 10 | set 11 |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| 1 | 0.05 | 1.0056 | 0.7653 | 0.8791 | | 1.2938 | 1.2927 |
| 2 | 0.10 | 1.3113 | 1.3237 | 1.3161 | | 1.2962 | 1.2944 |
| 3 | 0.15 | 1.3134 | 1.3254 | 1.3176 | | 1.2983 | 1.2959 |
| 4 | 0.20 | 1.3153 | 1.3270 | 1.3189 | | 1.3002 | 1.2972 |
| 5 | 0.25 | 1.3170 | 1.3283 | 1.3201 | | 1.3019 | 1.2983 |
| 6 | 0.30 | 1.3185 | 1.3296 | 1.3211 | | 1.3033 | 1.2993 |
| 7 | 0.35 | 1.3198 | 1.3306 | 1.3220 | | 1.3046 | 1.3002 |
| 8 | 0.40 | 1.3210 | 1.3316 | 1.3229 | | 1.3058 | 1.3010 |
| 9 | 0.45 | 1.3221 | 1.3325 | 1.3236 | | 1.3069 | 1.3018 |
| 10 | 0.50 | 1.3231 | 1.3333 | 1.3243 | | 1.3078 | 1.3024 |
| 11 | 0.55 | 1.3240 | 1.3340 | 1.3249 | | 1.3087 | 1.3030 |
| 12 | 0.60 | 1.3248 | 1.3347 | 1.3255 | | 1.3095 | 1.3036 |
| 13 | 0.65 | 1.3255 | 1.3353 | 1.3260 | | 1.3103 | 1.3041 |
| 14 | 0.70 | 1.3262 | 1.3359 | 1.3265 | | 1.3110 | 1.3046 |
| 15 | 0.75 | 1.3269 | 1.3364 | 1.3269 | | 1.3116 | 1.3050 |
| 16 | 0.80 | 1.3275 | 1.3369 | 1.3273 | | 1.3122 | 1.3054 |
| 17 | 0.85 | 1.3280 | 1.3374 | 1.3277 | | 1.3127 | 1.3058 |
| 18 | 0.90 | 1.3285 | 1.3378 | 1.3281 | | 1.3133 | 1.3062 |
| 19 | 0.95 | 1.3290 | 1.3382 | 1.3284 | | 1.3137 | 1.3065 |
| 20 | 1.00 | 1.3295 | 1.3385 | 1.3287 | | 1.3142 | 1.3068 |

*3.3 Estimation of the Error Variances*

Although the principal objective of this paper is the estimation of the $\beta$ parameter, we now show how it is possible to obtain reasonable estimates of the error variances by using the estimate of the $\beta$ parameter itself. In the structural interpretation of model (1)-(2), the latent variable $X^*$ is assumed to have a uniform distribution (recall that for identification purposes its distribution cannot be normal). Under the assumptions of model (1)-(2), therefore, we have the following second moment relations

(15) $$\text{var}(y) = \beta^2\sigma^2_{X^*} + \sigma^2_u$$

(16) $$\text{var}(x) = \sigma^2_{X^*} + \sigma^2_\varepsilon$$

(17) $$\text{cov}(y,x) = \beta\sigma^2_{X^*}.$$

The terms on the left-hand-side of the equality sign can be approximated by the corresponding sample estimates of the variance and covariance quantities. This system of three equations has four unknowns $(\beta,\sigma^2_{X^*},\sigma^2_u,\sigma^2_\varepsilon)$. However, given a "good" estimate of the $\beta$ parameter, the other three unknowns $(\sigma^2_{X^*},\sigma^2_u,\sigma^2_\varepsilon)$ can be determined univocally and with surprising precision. The results of applying this estimation procedure (using the estimates of the $\beta$ parameter reported in Tables 2 and 3) are given in Tables 4, 5 and 6.

Given the chosen limits for the uniform distribution of the latent variable $X^*$, $U(a,b) = U(2,20)$, the variance of $X^*$ is equal to $\sigma^2_{X^*} = (b-a)^2/12 = 27$. The estimated values of the variances of the error and latent variables are reasonably close to their true values especially for $N = 5000$.

Table 4. Estimates of the Error Variances for Model (1)-(2), Sets 1 and 2 of Table 1

| | | set 1 of table 1, N=200 | | | set 2 of table 1, N=5000 | | |
|---|---|---|---|---|---|---|---|
| Number | λ value | $\sigma_u^2 = 4$ | $\sigma_\varepsilon^2 = 9$ | $\sigma_{X^*}^2 = 27$ | $\sigma_u^2 = 4$ | $\sigma_\varepsilon^2 = 9$ | $\sigma_{X^*}^2 = 27$ |
| | | | | | | | |
| 1 | 0.25 | 6.0987 | 7.6677 | 25.4913 | 5.7068 | 7.9787 | 27.7935 |
| 2 | 0.50 | 5.4993 | 8.0487 | 25.1102 | 5.1695 | 8.3173 | 27.4549 |
| 3 | 0.75 | 5.1118 | 8.2891 | 24.8698 | 4.8238 | 8.5308 | 27.2413 |
| 4 | 1.00 | 4.8423 | 8.4536 | 24.7053 | 4.5837 | 8.6772 | 27.0950 |
| 5 | 1.25 | 4.6444 | 8.5730 | 24.5859 | 4.4076 | 8.7835 | 26.9886 |
| 6 | 1.50 | 4.4933 | 8.6635 | 24.4954 | 4.2731 | 8.8642 | 26.9080 |
| 7 | 1.75 | 4.3741 | 8.7344 | 24.4246 | 4.1671 | 8.9275 | 26.8447 |
| 8 | 2.00 | 4.2777 | 8.7914 | 24.3676 | 4.0814 | 8.9784 | 26.7938 |
| 9 | 2.25 | 4.1983 | 8.8382 | 24.3208 | 4.0107 | 9.0202 | 26.7520 |
| 10 | 2.50 | 4.1317 | 8.8773 | 24.2817 | 3.9515 | 9.0552 | 26.7170 |
| 11 | 2.75 | 4.0750 | 8.9105 | 24.2485 | 3.9010 | 9.0849 | 26.6873 |
| 12 | 3.00 | 4.0262 | 8.9389 | 24.22 | 3.8576 | 9.1104 | 26.6618 |
| 13 | 3.25 | 3.9837 | 8.9637 | 24.1953 | 3.8199 | 9.1326 | 26.6396 |
| 14 | 3.50 | 3.9465 | 8.9853 | 24.1736 | 3.7867 | 9.1520 | 26.6202 |
| 15 | 3.75 | 3.9135 | 9.0045 | 24.1545 | 3.7574 | 9.1691 | 26.6030 |
| 16 | 4.00 | 3.8841 | 9.0215 | 24.1375 | 3.7312 | 9.1844 | 26.5878 |
| 17 | 4.25 | 3.8578 | 9.0367 | 24.1222 | 3.7078 | 9.1981 | 26.5741 |
| 18 | 4.50 | 3.8341 | 9.0505 | 24.1085 | 3.6866 | 9.2104 | 26.5618 |
| 19 | 4.75 | 3.8125 | 9.0629 | 24.0961 | 3.6675 | 9.2216 | 26.5506 |
| 20 | 5.00 | 3.7930 | 9.0742 | 24.0848 | 3.6500 | 9.2317 | 26.5404 |

Table 5. Estimates of the Error Variances for Model (1)-(2), Sets 5 and 6 of Table 1

| | | set 5 of table 1, N=200 | | | set 6 of table 1, N=5000 | | |
|---|---|---|---|---|---|---|---|
| Number | λ value | $\sigma_u^2 = 1$ | $\sigma_\varepsilon^2 = 1$ | $\sigma_{X*}^2 = 27$ | $\sigma_u^2 = 1$ | $\sigma_\varepsilon^2 = 1$ | $\sigma_{X*}^2 = 27$ |
| | | | | | | | |
| 1 | 0.25 | 1.2428 | 0.8535 | 24.6163 | 1.1626 | 0.9091 | 26.8578 |
| 2 | 0.50 | 1.1533 | 0.9067 | 24.5632 | 1.0835 | 0.9562 | 26.8108 |
| 3 | 0.75 | 1.0971 | 0.9400 | 24.5299 | 1.0337 | 0.9857 | 26.7813 |
| 4 | 1.00 | 1.0585 | 0.9628 | 24.507 | 0.9996 | 1.0059 | 26.7610 |
| 5 | 1.25 | 1.0303 | 0.9794 | 24.4904 | 0.9747 | 1.0206 | 26.7463 |
| 6 | 1.50 | 1.0089 | 0.9920 | 24.4778 | 0.9557 | 1.0318 | 26.7351 |
| 7 | 1.75 | 0.9920 | 1.0020 | 24.4679 | 0.9408 | 1.0406 | 26.7263 |
| 8 | 2.00 | 0.9784 | 1.0100 | 24.4599 | 0.9288 | 1.0477 | 26.7192 |
| 9 | 2.25 | 0.9672 | 1.0166 | 24.4533 | 0.9189 | 1.0536 | 26.7134 |
| 10 | 2.50 | 0.9579 | 1.0221 | 24.4478 | 0.9105 | 1.0585 | 26.7085 |
| 11 | 2.75 | 0.9499 | 1.0268 | 24.4431 | 0.9035 | 1.0626 | 26.7043 |
| 12 | 3.00 | 0.9430 | 1.0308 | 24.4391 | 0.8974 | 1.0662 | 26.7007 |
| 13 | 3.25 | 0.9370 | 1.0343 | 24.4355 | 0.8921 | 1.0693 | 26.6976 |
| 14 | 3.50 | 0.9318 | 1.0374 | 24.4325 | 0.8875 | 1.0721 | 26.6949 |
| 15 | 3.75 | 0.9271 | 1.0401 | 24.4297 | 0.8834 | 1.0745 | 26.6925 |
| 16 | 4.00 | 0.9230 | 1.0425 | 24.4273 | 0.8797 | 1.0766 | 26.6903 |
| 17 | 4.25 | 0.9193 | 1.0447 | 24.4251 | 0.8764 | 1.0786 | 26.6884 |
| 18 | 4.50 | 0.9160 | 1.0467 | 24.4232 | 0.8734 | 1.0803 | 26.6866 |
| 19 | 4.75 | 0.9129 | 1.0484 | 24.4214 | 0.8708 | 1.0819 | 26.6851 |
| 20 | 5.00 | 0.9102 | 1.0501 | 24.4198 | 0.8683 | 1.0833 | 26.6836 |

Table 6. Estimates of the Error Variances for Model (1)-(2), Sets 7 and 8 of Table 1

| | | set 7 of table 1, N=200 | | | set 8 of table 1, N=5000 | | |
|---|---|---|---|---|---|---|---|
| Number | λ value | $\sigma_u^2 = 4$ | $\sigma_\varepsilon^2 = 1$ | $\sigma_{X*}^2 = 27$ | $\sigma_u^2 = 4$ | $\sigma_\varepsilon^2 = 1$ | $\sigma_{X*}^2 = 27$ |
| | | | | | | | |
| 1 | 0.05 | 4.30787 | 0.82181 | 24.648 | 4.2438 | 0.8729 | 26.8940 |
| 2 | 0.10 | 4.22969 | 0.86842 | 24.6014 | 4.1746 | 0.9141 | 26.8528 |
| 3 | 0.15 | 4.16181 | 0.90876 | 24.5611 | 4.1146 | 0.9498 | 26.8171 |
| 4 | 0.02 | 4.10233 | 0.94399 | 24.5259 | 4.0620 | 0.9810 | 26.7859 |
| 5 | 0.25 | 4.04982 | 0.97501 | 24.4948 | 4.0156 | 1.0085 | 26.7585 |
| 6 | 0.30 | 4.00313 | 1.00253 | 24.4673 | 3.9744 | 1.0328 | 26.7341 |
| 7 | 0.35 | 3.96135 | 1.0271 | 24.4428 | 3.9375 | 1.0546 | 26.7123 |
| 8 | 0.40 | 3.92376 | 1.04916 | 24.4207 | 3.9043 | 1.0742 | 26.6928 |
| 9 | 0.45 | 3.88976 | 1.06909 | 24.4008 | 3.8742 | 1.0918 | 26.6751 |
| 10 | 0.50 | 3.85887 | 1.08716 | 24.3827 | 3.8470 | 1.1078 | 26.6591 |
| 11 | 0.55 | 3.83068 | 1.10363 | 24.3662 | 3.8221 | 1.1224 | 26.6445 |
| 12 | 0.60 | 3.80485 | 1.11871 | 24.3511 | 3.7992 | 1.1358 | 26.6311 |
| 13 | 0.65 | 3.78111 | 1.13255 | 24.3373 | 3.7782 | 1.1481 | 26.6188 |
| 14 | 0.70 | 3.7592 | 1.1453 | 24.3245 | 3.7589 | 1.1594 | 26.6075 |
| 15 | 0.75 | 3.73893 | 1.15709 | 24.3128 | 3.7410 | 1.1699 | 26.5970 |
| 16 | 0.80 | 3.72012 | 1.16802 | 24.3018 | 3.7244 | 1.1796 | 26.5873 |
| 17 | 0.85 | 3.70261 | 1.17819 | 24.2917 | 3.7089 | 1.1886 | 26.5783 |
| 18 | 0.90 | 3.68629 | 1.18766 | 24.2822 | 3.6944 | 1.1971 | 26.5699 |
| 19 | 0.95 | 3.67102 | 1.19651 | 24.2733 | 3.6810 | 1.2049 | 26.5620 |
| 20 | 1.00 | 3.65672 | 1.20479 | 24.2651 | 3.6683 | 1.2123 | 26.5547 |

*3.4 A Linear EIV Model With One Independent Variable and Intercept*

We repeat the Monte Carlo analysis with the addition of an $\alpha$ intercept to the linear model (1)-(2) to read

(18) $$y_i = \alpha + \beta x_i^* + u_i$$

(19) $$x_i = x_i^* + \varepsilon_i .$$

The specifications for the various Monte Carlo experiments associated with model (18)-(19) are stated in Table 7.

Table 7. Parameter and distribution choice for the MC experiments of model (18)-(19)

| Set | $\alpha$ | $\beta$ | $x_i^*$ | $\sigma_u$ | $\sigma_\varepsilon$ | True $\lambda$ | $N$ |
|---|---|---|---|---|---|---|---|
| 12 | 1.0 | 0.80 | *U(2,10)* | 1.0 | 1.5 | 2.25 | 200 |
| 13 | 1.0 | 0.80 | *U(2,10)* | 1.0 | 1.5 | 2.25 | 5000 |
| 14 | 1.0 | 0.80 | *U(2,20)* | 1.0 | 1.5 | 2.25 | 200 |
| 15 | 1.0 | 0.80 | *U(2,30)* | 1.0 | 1.5 | 2.25 | 200 |

As the regression line goes through the centroid of model (18)-(19), that is $\bar{y} = \alpha + \beta\bar{x}$, where $\bar{y}$ and $\bar{x}$ are the sample averages of the corresponding variables and where $\bar{x} = \bar{x}^*$, it is convenient to transform model (18)-(19) in deviations from the mean

(20) $$\tilde{y}_i = \beta\tilde{x}_i^* + u_i$$

(21) $$\tilde{x}_i = \tilde{x}_i^* + \varepsilon_i$$

where $\tilde{y}_i = y_i - \bar{y}$ and $\tilde{x}_i = x_i - \bar{x}$. Then, model (20)-(21) can be estimated with the same CJLS estimator given by equation (8), using the additional step for $\tilde{\alpha} = \bar{y} - \tilde{\beta}\bar{x}$. The results of the Monte Carlo experiments associated with Table 7 are reported in Tables 8, 9, 10, and 11.

The information produced by this set of Monte Carlo experiments confirms the robustness of the CJLS estimator for Tables 10 and 11, while for Tables 8 and 9 the estimates exhibit a rate of departure from the true values that is higher than in Tables 2-6, 10 and 11. As the distribution of the latent variable widens, the CJLS estimator regains its original level of robustness. It appears, therefore, that in these MC experiments the introduction of an intercept perturbs the robustness of the CJLS estimator with one independent variable to an unexpected degree. In a subsequent linear model with three independent variables, the introduction of an intercept has less significant consequences.

The reason why the introduction of an intercept term disrupts the robustness of the CJLS estimator (for linear models) may have to do with the non identifiability of the intercept parameter: "If a parameter is not identifiable, no consistent (near consistent, in our case) estimate of the parameter will exist" (Reiersol, 1950). Notice, however, that in a translog model analyzed in section 5.2, the introduction of an intercept does not disrupts the robustness of the corresponding weighted least squares estimator.

Table 8. Average estimates of $\alpha, \beta, \sigma_u^2, \sigma_\varepsilon^2, \sigma_{X^*}^2$ parameters for given $\lambda$, model (20)-(21)

| | | set 12 of table 7, X*~U(2,10), N = 200 | | | | |
|---|---|---|---|---|---|---|
| Number | λ value | β slope | $\sigma_u^2 = 1$ | $\sigma_\varepsilon^2 = 2.25$ | $\sigma_{X^*}^2 = 5.33$ | α intercept |
| | | | | | | |
| 1 | 0.25 | 0.5896 | 1.8290 | 0.4573 | 6.6257 | 2.1468 |
| 2 | 0.50 | 0.6265 | 1.6863 | 0.8431 | 6.2399 | 1.9468 |
| 3 | 0.75 | 0.6609 | 1.5530 | 1.1648 | 5.9182 | 1.7600 |
| 4 | 1.00 | 0.6925 | 1.4312 | 1.4312 | 5.6518 | 1.5891 |
| 5 | 1.25 | 0.7209 | 1.3214 | 1.6518 | 5.4312 | 1.4349 |
| 6 | 1.50 | 0.7337 | 1.2034 | 1.8051 | 5.1271 | 1.2553 |
| 7 | 1.75 | 0.7689 | 1.1360 | 1.9880 | 5.0950 | 1.1744 |
| 8 | 2.00 | 0.7890 | 1.0584 | 2.1168 | 4.9662 | 1.0653 |
| 9 | 2.25 | 0.8069 | 0.9894 | 2.2260 | 4.8570 | 0.9683 |
| 10 | 2.50 | 0.8229 | 0.9278 | 2.3195 | 4.7635 | 0.8818 |
| 11 | 2.75 | 0.8302 | 0.8649 | 2.3785 | 4.6248 | 0.7864 |
| 12 | 3.00 | 0.8499 | 0.8234 | 2.4703 | 4.6127 | 0.7351 |
| 13 | 3.25 | 0.8614 | 0.7790 | 2.5317 | 4.5513 | 0.6727 |
| 14 | 3.50 | 0.8719 | 0.7388 | 2.5858 | 4.4972 | 0.6162 |
| 15 | 3.75 | 0.8813 | 0.7024 | 2.6338 | 4.4492 | 0.5650 |
| 16 | 4.00 | 0.8899 | 0.6692 | 2.6767 | 4.4063 | 0.5183 |
| 17 | 4.25 | 0.8978 | 0.6388 | 2.7151 | 4.3679 | 0.4757 |
| 18 | 4.50 | 0.9050 | 0.6111 | 2.7498 | 4.3332 | 0.4366 |
| 19 | 4.75 | 0.9116 | 0.5855 | 2.7812 | 4.3018 | 0.4007 |
| 20 | 5.00 | 0.9177 | 0.5619 | 2.8097 | 4.2733 | 0.3676 |

Table 9. Average estimates of $\alpha, \beta, \sigma_u^2, \sigma_\varepsilon^2, \sigma_{X^*}^2$ parameters, given $\lambda$, model (20)-(21)

| | | set 13 of table 7, X*~U(2,10),N=5000 | | | | |
|---|---|---|---|---|---|---|
| Number | λ value | β parameter | $\sigma_u^2 = 1$ | $\sigma_\varepsilon^2 = 2.25$ | $\sigma_{X^*}^2 = 5.33$ | α intercept |
| | | | | | | |
| 1 | 0.25 | 0.5989 | 1.8649 | 0.4662 | 7.0800 | 2.2082 |
| 2 | 0.50 | 0.6341 | 1.7163 | 0.8582 | 6.6881 | 1.9985 |
| 3 | 0.75 | 0.6666 | 1.5789 | 1.1841 | 6.3621 | 1.8044 |
| 4 | 1.00 | 0.6961 | 1.4539 | 1.4539 | 6.0923 | 1.6281 |
| 5 | 1.25 | 0.7226 | 1.3418 | 1.6773 | 5.8690 | 1.4698 |
| 6 | 1.50 | 0.7462 | 1.2419 | 1.8629 | 5.6834 | 1.3288 |
| 7 | 1.75 | 0.7672 | 1.1532 | 2.0182 | 5.5281 | 1.2036 |
| 8 | 2.00 | 0.7858 | 1.0745 | 2.1490 | 5.3972 | 1.0925 |
| 9 | 2.25 | 0.8023 | 1.0046 | 2.2602 | 5.2860 | 0.9938 |
| 10 | 2.50 | 0.8170 | 0.9422 | 2.3555 | 5.1907 | 0.9058 |
| 11 | 2.75 | 0.8302 | 0.8865 | 2.4379 | 5.1084 | 0.8271 |
| 12 | 3.00 | 0.8420 | 0.8365 | 2.5095 | 5.0367 | 0.7566 |
| 13 | 3.25 | 0.8527 | 0.7915 | 2.5724 | 4.9739 | 0.6930 |
| 14 | 3.50 | 0.8623 | 0.7508 | 2.6278 | 4.9184 | 0.6356 |
| 15 | 3.75 | 0.8710 | 0.7139 | 2.6771 | 4.8692 | 0.5835 |
| 16 | 4.00 | 0.8789 | 0.6803 | 2.7211 | 4.8252 | 0.5360 |
| 17 | 4.25 | 0.8862 | 0.6496 | 2.7606 | 4.7856 | 0.4927 |
| 18 | 4.50 | 0.8929 | 0.6214 | 2.7963 | 4.7500 | 0.4529 |
| 19 | 4.75 | 0.8811 | 0.5838 | 2.7732 | 4.6236 | 0.4078 |
| 20 | 5.00 | 0.9046 | 0.5716 | 2.8581 | 4.6882 | 0.3827 |

Table 10. Average estimates of $\alpha, \beta, \sigma_u^2, \sigma_\varepsilon^2, \sigma_{X^*}^2$ parameters, given $\lambda$, model (20)-(21)

| | | set 14 of table 7, X*~U(2,20), N = 200 | | | | |
|---|---|---|---|---|---|---|
| Number | λ value | β slope | $\sigma_u^2 = 1$ | $\sigma_\varepsilon^2 = 2.25$ | $\sigma_{X^*}^2 = 27$ | α intercept |
| | | | | | | |
| 1 | 0.25 | 0.7486 | 2.0202 | 0.5051 | 26.3029 | 1.4959 |
| 2 | 0.50 | 0.7600 | 1.7963 | 0.8981 | 25.9098 | 1.3854 |
| 3 | 0.75 | 0.7693 | 1.6136 | 1.2102 | 25.5977 | 1.2953 |
| 4 | 1.00 | 0.7770 | 1.4626 | 1.4626 | 25.3453 | 1.2208 |
| 5 | 1.25 | 0.7835 | 1.3362 | 1.6703 | 25.1377 | 1.1584 |
| 6 | 1.50 | 0.7889 | 1.2291 | 1.8436 | 24.9643 | 1.1055 |
| 7 | 1.75 | 0.7936 | 1.1373 | 1.9902 | 24.8177 | 1.0603 |
| 8 | 2.00 | 0.7976 | 1.0579 | 2.1157 | 24.6922 | 1.0211 |
| 9 | 2.25 | 0.8011 | 0.9886 | 2.2243 | 24.5837 | 0.9869 |
| 10 | 2.50 | 0.8042 | 0.9276 | 2.3190 | 24.4890 | 0.9568 |
| 11 | 2.75 | 0.8070 | 0.8736 | 2.4023 | 24.4057 | 0.9301 |
| 12 | 3.00 | 0.8094 | 0.8254 | 2.4761 | 24.3318 | 0.9064 |
| 13 | 3.25 | 0.8116 | 0.7821 | 2.5420 | 24.2660 | 0.8850 |
| 14 | 3.50 | 0.8136 | 0.7432 | 2.6011 | 24.2069 | 0.8658 |
| 15 | 3.75 | 0.8154 | 0.7078 | 2.6544 | 24.1536 | 0.8484 |
| 16 | 4.00 | 0.8171 | 0.6757 | 2.7027 | 24.1053 | 0.8325 |
| 17 | 4.25 | 0.8186 | 0.6463 | 2.7467 | 24.0613 | 0.8180 |
| 18 | 4.50 | 0.8199 | 0.6193 | 2.7869 | 24.0210 | 0.8047 |
| 19 | 4.75 | 0.8212 | 0.5945 | 2.8238 | 23.9841 | 0.7924 |
| 20 | 5.00 | 0.8224 | 0.5716 | 2.8578 | 23.9501 | 0.7811 |

Table 11. Average estimates of $\alpha, \beta, \sigma_u^2, \sigma_\varepsilon^2, \sigma_{X^*}^2$ parameters, given $\lambda$, model (20)-(21)

| | | set 15 of table 7, X*~U(2,30), N = 200 | | | | |
|---|---|---|---|---|---|---|
| Number | λ value | β slope | $\sigma_u^2 = 1$ | $\sigma_\varepsilon^2 = 2.25$ | $\sigma_{X^*}^2 = 65.33$ | α intercept |
| | | | | | | |
| 1 | 0.25 | 0.7781 | 2.0547 | 0.5137 | 61.2148 | 1.3040 |
| 2 | 0.50 | 0.7831 | 1.8147 | 0.9073 | 60.8211 | 1.2335 |
| 3 | 0.75 | 0.7872 | 1.6233 | 1.2175 | 60.5110 | 1.1774 |
| 4 | 1.00 | 0.7904 | 1.4676 | 1.4676 | 60.2608 | 1.1317 |
| 5 | 1.25 | 0.7931 | 1.3387 | 1.6733 | 60.0551 | 1.0938 |
| 6 | 1.50 | 0.7954 | 1.2302 | 1.8453 | 59.8832 | 1.0620 |
| 7 | 1.75 | 0.7974 | 1.1377 | 1.9910 | 59.7374 | 1.0348 |
| 8 | 2.00 | 0.7990 | 1.0580 | 2.1161 | 59.6124 | 1.0115 |
| 9 | 2.25 | 0.8005 | 0.9887 | 2.2245 | 59.5039 | 0.9911 |
| 10 | 2.50 | 0.8018 | 0.9278 | 2.3194 | 59.4090 | 0.9732 |
| 11 | 2.75 | 0.8029 | 0.8739 | 2.4032 | 59.3253 | 0.9574 |
| 12 | 3.00 | 0.8039 | 0.8259 | 2.4776 | 59.2509 | 0.9433 |
| 13 | 3.25 | 0.8048 | 0.7828 | 2.5441 | 59.1843 | 0.9307 |
| 14 | 3.50 | 0.8056 | 0.7440 | 2.6040 | 59.1245 | 0.9193 |
| 15 | 3.75 | 0.8064 | 0.7088 | 2.6581 | 59.0703 | 0.9090 |
| 16 | 4.00 | 0.8070 | 0.6768 | 2.7073 | 59.0211 | 0.8996 |
| 17 | 4.25 | 0.8077 | 0.6476 | 2.7522 | 58.9762 | 0.8910 |
| 18 | 4.50 | 0.8082 | 0.6207 | 2.7934 | 58.9351 | 0.8831 |
| 19 | 4.75 | 0.8087 | 0.5960 | 2.8312 | 58.8973 | 0.8758 |
| 20 | 5.00 | 0.8092 | 0.5732 | 2.8661 | 58.8624 | 0.8691 |

An indirect way to assess the quality of the CJLS estimates of model (20 )-(21) is to consider the ordinary least squares estimates resulting from the use of the same sample information. The results are reported in Table 12 and they indicate that the OLS estimates of the intercept and of the slope are considerably more biased than the corresponding estimates of the CJLS estimator under a wide spectrum of the λ parameter's values.

Table 12. OLS estimates using the sample information of sets 12 and 15 of Table 7

| Set | $\alpha$=1.0 | $\beta$= 0.80 | $x_i^*$ | $\sigma_u$ | $\sigma_\varepsilon$ | $N$ |
|---|---|---|---|---|---|---|
| 12 | 2.3517 | 0.55033 | *U(2,10)* | 1.0 | 1.5 | 200 |
| 15 | 1.3946 | 0.77161 | *U(2,30)* | 1.0 | 1.5 | 200 |

## 4. A Linear EIV Model With Three Independent Variables

Next, we analyze a linear model with three independent variables exhibiting the following structure

(22) $$y_i = \beta_1 x_{1i}^* + \beta_2 x_{2i}^* + \beta_3 x_{3i}^* + u_i$$

(23) $$x_{ji} = x_{ji}^* + \varepsilon_{ji}$$

$j = 1,...,3$. As in section 3, we assume that the error terms are iid variables. More specifically, $u_i \sim N(0,\sigma_u)$, $\varepsilon_{ji} \sim N(0,\sigma_{j\varepsilon})$, where $\sigma_u$ and $\sigma_{j\varepsilon}$ are the standard deviations of the corresponding errors, $j = 1,...,3$. The independence hypothesis implies that $\text{cov}(x_{ji}^*,\varepsilon_{ji}) = 0$, $\text{cov}(y_i,\varepsilon_{ji}) = 0$, $\text{cov}(x_{ji}^*,u_i) = 0$, $i = 1,...,N$, $j = 1,...,3$. Furthermore, the latent variables $X_j^*$ are assumed to be identically, independently, and uniformly distributed. Initially, model (22)-(23) has no intercept. An intercept will be introduced later in this section.

Under these specifications, the CJLS objective function assumes the following structure:

(24) $$\min_{\beta_1,\beta_2,\beta_3} CJLS = \frac{1}{(1+\sum_j \lambda_j \beta_j^2)} \sum_{i=1}^{N} (y_i - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i})^2$$

where $\lambda_j \overset{\text{def}}{=} \sigma_{\varepsilon_j}^2 / \sigma_u^2$, $j = 1,2,3$.

The second moment relationships among the variables of model (22)-(23) can be stated explicitly as

(25) $$\text{var}(y) = \beta_1^2 \sigma_{X_1^*}^2 + \beta_2^2 \sigma_{X_2^*}^2 + \beta_3^2 \sigma_{X_3^*}^2 + 2\beta_1\beta_2 \sigma_{X_1^* X_2^*} + 2\beta_1\beta_3 \sigma_{X_1^* X_3^*} 2\beta_2\beta_3 \sigma_{X_2^* X_3^*} + \sigma_u^2$$

(26) $$\text{var}(x_1) = \sigma_{X_1^*}^2 + \sigma_{\varepsilon_1}^2$$

(27) $$\text{var}(x_1) = \sigma_{X_1^*}^2 + \sigma_{\varepsilon_1}^2$$

(28) $\operatorname{var}(x_3) = \sigma^2_{X_3^*} + \sigma^2_{\varepsilon_3}$

(29) $\operatorname{cov}(x_1, x_1) = \sigma_{X_1^* X_2^*} + \sigma_{\varepsilon_1 \varepsilon_2}$

(30) $\operatorname{cov}(x_1, x_3) = \sigma_{X_1^* X_3^*} + \sigma_{\varepsilon_1 \varepsilon_3}$

(31) $\operatorname{cov}(x_2, x_3) = \sigma_{X_2^* X_3^*} + \sigma_{\varepsilon_2 \varepsilon_3}$

(32) $\operatorname{cov}(y, x_1) = \beta_1 \sigma^2_{X_1^*} + \beta_2 \sigma_{X_1^* X_2^*} + \beta_3 \sigma_{X_1^* X_3^*} + \sigma_{u \varepsilon_1}$

(33) $\operatorname{cov}(y, x_2) = \beta_1 \sigma_{X_1^* X_2^*} + \beta_2 \sigma^2_{X_2^*} + \beta_3 \sigma_{X_2^* X_3^*} + \sigma_{u \varepsilon_2}$

(34) $\operatorname{cov}(y, x_3) = \beta_1 \sigma_{X_1^* X_3^*} + \beta_2 \sigma_{X_2^* X_3^*} + \beta_3 \sigma^2_{X_3^*} + \sigma_{u \varepsilon_3}$.

Using the initial assumptions of this model, which imply $\sigma_{\varepsilon_1 \varepsilon_2} = \sigma_{\varepsilon_1 \varepsilon_3} = \sigma_{\varepsilon_2 \varepsilon_3} = 0$ and $\sigma_{u \varepsilon_2} = \sigma_{u \varepsilon_2} = \sigma_{u \varepsilon_3} = 0$, the system of ten equations (25)-(34) can be solved for the remaining ten unknown variances and covariances, given good estimates of the three $\beta$ parameters. This procedure is an extension of a similar procedure developed in section 3.

Table 13 gives the specification of the Monte Carlo experiments associated with model (22)-(23). The choice of the $\beta$ parameter values intends to represent positive and negative coefficients with magnitude greater and smaller than unity. The choice of the variances' level intends to create an unfavorable environment for the CJLS estimator using a wider distribution of the measurement errors of the independent variables relative to that of the dependent variable. We chose the same variance for the errors of the independent variables in order to simplify slightly the reporting of results.

Table 13. Parameter specification for the Monte Carlo experiments of model (22)-(23)

| Set | $\beta_1$ | $\beta_2$ | $\beta_3$ | $x_i^*$ | $\sigma_u$ | $\sigma_{\varepsilon_j}$ $j = 1,2,3$ | True $\lambda$ | $N$ |
|---|---|---|---|---|---|---|---|---|
| 16 | 1.30 | -1.0 | 0.70 | *U(2,20)* | 2.0 | 3.0 | 2.25 | 200 |
| 17 | 1.30 | -1.0 | 0.70 | *U(2,20)* | 2.0 | 3.0 | 2.25 | 5000 |
| 18 | 1.30 | -1.0 | 0.70 | *U(10,20)* | 2.0 | 3.0 | 2.25 | 200 |
| 19 | 1.30 | -1.0 | 0.70 | *U(2,30)* | 2.0 | 3.0 | 2.25 | 200 |
| 20 | 1.30 | -1.0 | 0.70 | *U(2,20)* | 1.0 | 1.0 | 1.00 | 200 |
| 21 | 1.30 | -1.0 | 0.70 | *U(2,20)* | 2.0 | 1.0 | 0.25 | 200 |
| 22 | Same as set 16 with an intercept | | | | | | | |

The results of this section are presented in Table 14 through Table 20. This Monte Carlo evidence confirms that the CJLS estimator is robust with respect to the $\lambda$ parameter also in the case of a linear model with three independent variables. The average estimates (over 100 samples) of each of the three $\beta$ parameters are rather close to their true values for a wide spectrum of the misspecified ratio of the error variances. Although the main objective of this paper was a robust estimation of the $\beta$ parameters, we carried out also an exploratory study of estimating the error variances. Using the estimates of the $\beta$ parameters, the variances of the various measurement errors were computed with less accurate results: it is worth to point out, however, that the values of the estimated variances for the measurement errors of the independent variables are within a relatively small range of the true values; the estimated variance values of the measurement error of the dependent variable are much less accurate. This is undoubtedly due to the complexity of relation (25) where the estimate of the error variance $\sigma_u^2$ depends on several other parameter estimates.

The introduction of an intercept into model (22)-(23) produces the results reported in Table 20. They do not differ greatly from those already commented above.

Table 14. Results of Set 16, $N = 200$, of Table 13.

| Number | λ ratio | $\beta_1 = 1.30$ | $\beta_2 = -1.0$ | $\beta_3 = 0.70$ | $\sigma_u^2 = 4$ | $\sigma_{\varepsilon_1}^2 = 9$ | $\sigma_{\varepsilon_2}^2 = 9$ | $\sigma_{\varepsilon_3}^2 = 9$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 1.1629 | -0.8241 | 0.6519 | 15.1664 | 5.5609 | 2.2553 | 6.8551 |
| 2 | 0.50 | 1.2139 | -0.8899 | 0.6719 | 11.0466 | 6.8522 | 4.9187 | 7.6488 |
| 3 | 0.75 | 1.2434 | -0.9283 | 0.6835 | 8.6552 | 7.5528 | 6.2953 | 8.0867 |
| 4 | 1.00 | 1.2623 | -0.9529 | 0.6909 | 7.1182 | 7.9857 | 7.1229 | 8.3598 |
| 5 | 1.25 | 1.2755 | -0.9700 | 0.6960 | 6.0536 | 8.2779 | 7.6719 | 8.5454 |
| 6 | 1.50 | 1.2850 | -0.9825 | 0.6997 | 5.2749 | 8.4878 | 8.0616 | 8.6793 |
| 7 | 1.75 | 1.2923 | -0.9921 | 0.7026 | 4.6814 | 8.6456 | 8.3522 | 8.7803 |
| 8 | 2.00 | 1.2981 | -0.9996 | 0.7048 | 4.2144 | 8.7686 | 8.5770 | 8.8591 |
| 9 | 2.25 | 1.3027 | -1.0057 | 0.7066 | 3.8376 | 8.8670 | 8.7559 | 8.9224 |
| 10 | 2.50 | 1.3065 | -1.0107 | 0.7081 | 3.5273 | 8.9475 | 8.9017 | 8.9742 |
| 11 | 2.75 | 1.3097 | -1.0148 | 0.7094 | 3.2673 | 9.0146 | 9.0228 | 9.0174 |
| 12 | 3.00 | 1.3125 | -1.0184 | 0.7104 | 3.0464 | 9.0713 | 9.1249 | 9.0540 |
| 13 | 3.25 | 1.3148 | -1.0215 | 0.7113 | 2.8563 | 9.1199 | 9.2122 | 9.0855 |
| 14 | 3.50 | 1.3168 | -1.0241 | 0.7121 | 2.6912 | 9.1621 | 9.2877 | 9.1127 |
| 15 | 3.75 | 1.3186 | -1.0265 | 0.7128 | 2.5462 | 9.1989 | 9.3535 | 9.1365 |
| 16 | 4.00 | 1.3202 | -1.0285 | 0.7134 | 2.4181 | 9.2314 | 9.4116 | 9.1575 |
| 17 | 4.25 | 1.3216 | -1.0304 | 0.7140 | 2.3040 | 9.2603 | 9.4630 | 9.1762 |
| 18 | 4.50 | 1.3228 | -1.0320 | 0.7145 | 2.2017 | 9.2861 | 9.5090 | 9.1930 |
| 19 | 4.75 | 1.3240 | -1.0335 | 0.7149 | 2.1095 | 9.3093 | 9.5503 | 9.2080 |
| 20 | 5.00 | 1.3250 | -1.0348 | 0.7153 | 2.0260 | 9.3303 | 9.5876 | 9.2217 |

Table 15. Results of Set 17, $N = 5000$, of Table 13.

| Number | λ ratio | $\beta_1 = 1.30$ | $\beta_2 = -1.0$ | $\beta_3 = 0.70$ | $\sigma_u^2 = 4$ | $\sigma_{\varepsilon_1}^2 = 9$ | $\sigma_{\varepsilon_2}^2 = 9$ | $\sigma_{\varepsilon_3}^2 = 9$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 1.1531 | -0.8059 | 0.6394 | 15.5244 | 5.5291 | 2.4048 | 6.5731 |
| 2 | 0.50 | 1.2065 | -0.8758 | 0.6611 | 11.3641 | 6.8727 | 5.0982 | 7.4955 |
| 3 | 0.75 | 1.2378 | -0.9168 | 0.6737 | 8.9258 | 7.6048 | 6.4898 | 8.0067 |
| 4 | 1.00 | 1.2579 | -0.9434 | 0.6818 | 7.3520 | 8.0575 | 7.3249 | 8.3259 |
| 5 | 1.25 | 1.2719 | -0.9618 | 0.6875 | 6.2596 | 8.3630 | 7.8780 | 8.5428 |
| 6 | 1.50 | 1.2821 | -0.9754 | 0.6916 | 5.4595 | 8.5825 | 8.2701 | 8.6993 |
| 7 | 1.75 | 1.2899 | -0.9857 | 0.6948 | 4.8493 | 8.7475 | 8.5620 | 8.8173 |
| 8 | 2.00 | 1.2961 | -0.9938 | 0.6972 | 4.3690 | 8.8759 | 8.7877 | 8.9094 |
| 9 | 2.25 | 1.3010 | -1.0003 | 0.6992 | 3.9813 | 8.9787 | 8.9672 | 8.9833 |
| 10 | 2.50 | 1.3051 | -1.0057 | 0.7009 | 3.6618 | 9.0627 | 9.1134 | 9.0438 |
| 11 | 2.75 | 1.3085 | -1.0103 | 0.7022 | 3.3942 | 9.1328 | 9.2346 | 9.0943 |
| 12 | 3.00 | 1.3114 | -1.0141 | 0.7034 | 3.1667 | 9.1920 | 9.3369 | 9.1370 |
| 13 | 3.25 | 1.3139 | -1.0174 | 0.7044 | 2.9710 | 9.2427 | 9.4243 | 9.1737 |
| 14 | 3.50 | 1.3161 | -1.0203 | 0.7053 | 2.8009 | 9.2866 | 9.4997 | 9.2054 |
| 15 | 3.75 | 1.3180 | -1.0228 | 0.7061 | 2.6516 | 9.3251 | 9.5656 | 9.2332 |
| 16 | 4.00 | 1.3197 | -1.0251 | 0.7068 | 2.5196 | 9.3590 | 9.6236 | 9.2578 |
| 17 | 4.25 | 1.3212 | -1.0271 | 0.7074 | 2.4021 | 9.3891 | 9.6751 | 9.2796 |
| 18 | 4.50 | 1.3225 | -1.0288 | 0.7079 | 2.2967 | 9.4160 | 9.7210 | 9.2991 |
| 19 | 4.75 | 1.3238 | -1.0305 | 0.7084 | 2.2017 | 9.4402 | 9.7623 | 9.3166 |
| 20 | 5.00 | 1.3249 | -1.0319 | 0.7088 | 2.1157 | 9.4621 | 9.7996 | 9.3325 |

Table 16. Results of Set 18, $N = 200$, of Table 13.

| Number | λ ratio | $\beta_1 = 1.30$ | $\beta_2 = -1.0$ | $\beta_3 = 0.70$ | $\sigma_u^2 = 4$ | $\sigma_{\varepsilon_1}^2 = 9$ | $\sigma_{\varepsilon_2}^2 = 9$ | $\sigma_{\varepsilon_3}^2 = 9$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.9460 | -0.5211 | 0.5705 | 13.0238 | 5.6452 | 0.1453 | 7.0822 |
| 2 | 0.50 | 1.0524 | -0.6662 | 0.6122 | 10.3087 | 6.8080 | 4.0103 | 7.6969 |
| 3 | 0.75 | 1.1283 | -0.7702 | 0.6420 | 8.3687 | 7.5060 | 5.8807 | 8.0891 |
| 4 | 1.00 | 1.1819 | -0.8439 | 0.6631 | 6.9964 | 7.9459 | 6.9254 | 8.3457 |
| 5 | 1.25 | 1.2208 | -0.8975 | 0.6784 | 5.9996 | 8.2412 | 7.5766 | 8.5218 |
| 6 | 1.50 | 1.2500 | -0.9378 | 0.6899 | 5.2515 | 8.4507 | 8.0163 | 8.6486 |
| 7 | 1.75 | 1.2725 | -0.9690 | 0.6988 | 4.6727 | 8.6061 | 8.3312 | 8.7435 |
| 8 | 2.00 | 1.2904 | -0.9937 | 0.7059 | 4.2131 | 8.7255 | 8.5672 | 8.8170 |
| 9 | 2.25 | 1.3049 | -1.0138 | 0.7117 | 3.8401 | 8.8200 | 8.7502 | 8.8755 |
| 10 | 2.50 | 1.3169 | -1.0304 | 0.7165 | 3.5316 | 8.8965 | 8.8961 | 8.9229 |
| 11 | 2.75 | 1.3269 | -1.0444 | 0.7205 | 3.2724 | 8.9597 | 9.0151 | 8.9622 |
| 12 | 3.00 | 1.3355 | -1.0563 | 0.7239 | 3.0518 | 9.0127 | 9.1139 | 8.9953 |
| 13 | 3.25 | 1.3429 | -1.0665 | 0.7269 | 2.8617 | 9.0578 | 9.1972 | 9.0235 |
| 14 | 3.05 | 1.3493 | -1.0755 | 0.7294 | 2.6964 | 9.0966 | 9.2684 | 9.0477 |
| 15 | 3.75 | 1.3549 | -1.0833 | 0.7317 | 2.5512 | 9.1304 | 9.3299 | 9.0689 |
| 16 | 4.00 | 1.3599 | -1.0902 | 0.7337 | 2.4228 | 9.1600 | 9.3835 | 9.0874 |
| 17 | 4.25 | 1.3643 | -1.0964 | 0.7355 | 2.3084 | 9.1862 | 9.4308 | 9.1038 |
| 18 | 4.50 | 1.3683 | -1.1019 | 0.7370 | 2.2058 | 9.2095 | 9.4727 | 9.1185 |
| 19 | 4.75 | 1.3719 | -1.1069 | 0.7385 | 2.1134 | 9.2305 | 9.5101 | 9.1316 |
| 20 | 5.00 | 1.3752 | -1.1114 | 0.7398 | 2.0296 | 9.2493 | 9.5437 | 9.1434 |

Table 17. Results of Set 19, $N = 200$, of Table 13.

| Number | λ ratio | $\beta_1 = 1.30$ | $\beta_2 = -1.0$ | $\beta_3 = 0.70$ | $\sigma_u^2 = 4$ | $\sigma_{\varepsilon_1}^2 = 9$ | $\sigma_{\varepsilon_2}^2 = 9$ | $\sigma_{\varepsilon_3}^2 = 9$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 1.2408 | -0.9242 | 0.6801 | 15.7796 | 5.5998 | 2.6470 | 6.8493 |
| 2 | 0.50 | 1.2642 | -0.9543 | 0.6892 | 11.2178 | 6.8925 | 5.0874 | 7.6721 |
| 3 | 0.75 | 1.2770 | -0.9708 | 0.6941 | 8.7086 | 7.5831 | 6.3669 | 8.1149 |
| 4 | 1.00 | 1.2850 | -0.9812 | 0.6973 | 7.1312 | 8.0102 | 7.1498 | 8.3898 |
| 5 | 1.25 | 1.2906 | -0.9883 | 0.6994 | 6.0503 | 8.2998 | 7.6769 | 8.5766 |
| 6 | 1.50 | 1.2946 | -0.9935 | 0.7010 | 5.2643 | 8.5087 | 8.0556 | 8.7117 |
| 7 | 1.75 | 1.2976 | -0.9975 | 0.7021 | 4.6673 | 8.6666 | 8.3406 | 8.8139 |
| 8 | 2.00 | 1.3000 | -1.0006 | 0.7031 | 4.1987 | 8.7900 | 8.5628 | 8.8938 |
| 9 | 2.25 | 1.3019 | -1.0031 | 0.7038 | 3.8210 | 8.8891 | 8.7409 | 8.9581 |
| 10 | 2.50 | 1.3035 | -1.0051 | 0.7044 | 3.5103 | 8.9704 | 8.8868 | 9.0109 |
| 11 | 2.75 | 1.3048 | -1.0068 | 0.7049 | 3.2501 | 9.0383 | 9.0085 | 9.0550 |
| 12 | 3.00 | 1.3060 | -1.0083 | 0.7054 | 3.0292 | 9.0959 | 9.1116 | 9.0924 |
| 13 | 3.25 | 1.3069 | -1.0096 | 0.7058 | 2.8391 | 9.1453 | 9.1999 | 9.1245 |
| 14 | 3.50 | 1.3078 | -1.0106 | 0.7061 | 2.6740 | 9.1882 | 9.2766 | 9.1524 |
| 15 | 3.75 | 1.3085 | -1.0116 | 0.7064 | 2.5292 | 9.2258 | 9.3436 | 9.1768 |
| 16 | 4.00 | 1.3092 | -1.0125 | 0.7066 | 2.4011 | 9.2590 | 9.4029 | 9.1984 |
| 17 | 4.25 | 1.3097 | -1.0132 | 0.7068 | 2.2871 | 9.2886 | 9.4555 | 9.2176 |
| 18 | 4.50 | 1.3103 | -1.0139 | 0.7071 | 2.1849 | 9.3150 | 9.5026 | 9.2348 |
| 19 | 4.75 | 1.3107 | -1.0145 | 0.7072 | 2.0928 | 9.3388 | 9.5450 | 9.2503 |
| 20 | 5.00 | 1.3112 | -1.0150 | 0.7074 | 2.0094 | 9.3604 | 9.5834 | 9.2644 |

Table 18. Results of Set 20, $N = 200$, of Table 13.

| Number | λ ratio | $\beta_1 = 1.30$ | $\beta_2 = -1.0$ | $\beta_3 = 0.70$ | $\sigma_u^2 = 1$ | $\sigma_{\varepsilon_1}^2 = 1$ | $\sigma_{\varepsilon_2}^2 = 1$ | $\sigma_{\varepsilon_3}^2 = 1$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 1.2869 | -0.9831 | 0.6966 | 2.0986 | 0.6985 | 0.3958 | 0.8294 |
| 2 | 0.50 | 1.2944 | -0.9930 | 0.6995 | 1.4820 | 0.8626 | 0.6950 | 0.9355 |
| 3 | 0.75 | 1.2985 | -0.9983 | 0.7010 | 1.1522 | 0.9495 | 0.8526 | 0.9918 |
| 4 | 1.00 | 1.3010 | -1.0015 | 0.7020 | 0.9473 | 1.0033 | 0.9497 | 1.0267 |
| 5 | 1.25 | 1.3027 | -1.0038 | 0.7027 | 0.8077 | 1.0398 | 1.0155 | 1.0504 |
| 6 | 1.50 | 1.3040 | -1.0054 | 0.7032 | 0.7065 | 1.0662 | 1.0631 | 1.0676 |
| 7 | 1.75 | 1.3049 | -1.0066 | 0.7035 | 0.6298 | 1.0861 | 1.0990 | 1.0805 |
| 8 | 2.00 | 1.3056 | -1.0075 | 0.7038 | 0.5697 | 1.1018 | 1.1271 | 1.0907 |
| 9 | 2.25 | 1.3062 | -1.0084 | 0.7040 | 0.5213 | 1.1143 | 1.1497 | 1.0989 |
| 10 | 2.50 | 1.3067 | -1.0090 | 0.7042 | 0.4815 | 1.1247 | 1.1682 | 1.1056 |
| 11 | 2.75 | 1.3071 | -1.0095 | 0.7044 | 0.4482 | 1.1333 | 1.1837 | 1.1112 |
| 12 | 3.00 | 1.3075 | -1.0100 | 0.7045 | 0.4199 | 1.1406 | 1.1969 | 1.1160 |
| 13 | 3.25 | 1.3078 | -1.0104 | 0.7046 | 0.3956 | 1.1469 | 1.2081 | 1.1201 |
| 14 | 3.50 | 1.3080 | -1.0107 | 0.7047 | 0.3745 | 1.1524 | 1.2180 | 1.1237 |
| 15 | 3.75 | 1.3083 | -1.0110 | 0.7048 | 0.3560 | 1.1572 | 1.2265 | 1.1268 |
| 16 | 4.00 | 1.3085 | -1.0113 | 0.7049 | 0.3396 | 1.1615 | 1.2341 | 1.1295 |
| 17 | 4.25 | 1.3086 | -1.0115 | 0.7050 | 0.3250 | 1.1652 | 1.2409 | 1.1320 |
| 18 | 4.50 | 1.3088 | -1.0117 | 0.7050 | 0.3119 | 1.1686 | 1.2470 | 1.1342 |
| 19 | 4.75 | 1.3090 | -1.0119 | 0.7051 | 0.3002 | 1.1716 | 1.2524 | 1.1362 |
| 20 | 5.00 | 1.3091 | -1.0121 | 0.7051 | 0.2895 | 1.1744 | 1.2573 | 1.1380 |

Table 19. Results of Set 21, $N = 200$, of Table 13.

| Number | λ ratio | $\beta_1 = 1.30$ | $\beta_2 = -1.0$ | $\beta_3 = 0.70$ | $\sigma_u^2 = 4$ | $\sigma_{\varepsilon_1}^2 = 1$ | $\sigma_{\varepsilon_2}^2 = 1$ | $\sigma_{\varepsilon_3}^2 = 1$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.05 | 0.2778 | -0.9720 | 0.6941 | 5.7234 | 0.5397 | -0.0154 | 0.7754 |
| 2 | 0.10 | 0.2856 | -0.9821 | 0.6971 | 5.0921 | 0.7096 | 0.2983 | 0.8847 |
| 3 | 0.15 | 0.2917 | -0.9902 | 0.6995 | 4.5889 | 0.8435 | 0.5439 | 0.9711 |
| 4 | 0.20 | 0.2968 | -0.9968 | 0.7015 | 4.1794 | 0.9516 | 0.7409 | 1.0410 |
| 5 | 0.25 | 0.3009 | -1.0022 | 0.7031 | 3.8402 | 1.0404 | 0.9021 | 1.0986 |
| 6 | 0.30 | 0.3044 | -1.0068 | 0.7044 | 3.5551 | 1.1147 | 1.0364 | 1.1468 |
| 7 | 0.35 | 0.3074 | -1.0107 | 0.7056 | 3.3121 | 1.1777 | 1.1499 | 1.1877 |
| 8 | 0.40 | 0.3100 | -1.0140 | 0.7066 | 3.1028 | 1.2317 | 1.2469 | 1.2228 |
| 9 | 0.45 | 0.3122 | -1.0169 | 0.7074 | 2.9207 | 1.2785 | 1.3309 | 1.2533 |
| 10 | 0.50 | 0.3142 | -1.0195 | 0.7082 | 2.7608 | 1.3195 | 1.4042 | 1.2800 |
| 11 | 0.55 | 0.3159 | -1.0218 | 0.7089 | 2.6194 | 1.3556 | 1.4688 | 1.3036 |
| 12 | 0.60 | 0.3174 | -1.0238 | 0.7094 | 2.4934 | 1.3878 | 1.5260 | 1.3245 |
| 13 | 0.65 | 0.3188 | -1.0256 | 0.7100 | 2.3805 | 1.4165 | 1.5772 | 1.3433 |
| 14 | 0.70 | 0.3201 | -1.0272 | 0.7105 | 2.2787 | 1.4423 | 1.6231 | 1.3601 |
| 15 | 0.75 | 0.3212 | -1.0287 | 0.7109 | 2.1866 | 1.4657 | 1.6646 | 1.3754 |
| 16 | 0.80 | 0.3222 | -1.0301 | 0.7113 | 2.1027 | 1.4869 | 1.7023 | 1.3893 |
| 17 | 0.85 | 0.3232 | -1.0313 | 0.7117 | 2.0261 | 1.5062 | 1.7366 | 1.4019 |
| 18 | 0.90 | 0.3240 | -1.0324 | 0.7120 | 1.9558 | 1.5240 | 1.7680 | 1.4135 |
| 19 | 0.95 | 0.3248 | -1.0335 | 0.7123 | 1.8910 | 1.5403 | 1.7969 | 1.4242 |
| 20 | 1.00 | 0.3256 | -1.0344 | 0.7126 | 1.8313 | 1.5553 | 1.8235 | 1.4340 |

Table 20. Results of Set 22, $N = 200$, of Table 13, with intercept.

| Number | λ ratio | $\alpha_1 = 1.0$ | $\beta_1 = 1.30$ | $\beta_2 = -1.0$ | $\beta_3 = 0.70$ | $\sigma_u^2 = 4$ | $\sigma_{\varepsilon_1}^2 = 9$ | $\sigma_{\varepsilon_2}^2 = 9$ | $\sigma_{\varepsilon_3}^2 = 9$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 2.930 | 1.108 | -0.875 | 0.590 | 16.556 | 4.139 | 4.139 | 4.139 |
| 2 | 0.50 | 2.271 | 1.177 | -0.924 | 0.631 | 11.968 | 5.984 | 5.984 | 5.984 |
| 3 | 0.75 | 1.878 | 1.218 | -0.952 | 0.655 | 9.289 | 6.967 | 6.967 | 6.967 |
| 4 | 1.00 | 1.623 | 1.245 | -0.970 | 0.671 | 7.565 | 7.565 | 7.565 | 7.565 |
| 5 | 1.25 | 1.445 | 1.263 | -0.982 | 0.682 | 6.373 | 7.966 | 7.966 | 7.966 |
| 6 | 1.50 | 1.314 | 1.276 | -0.991 | 0.690 | 5.501 | 8.251 | 8.251 | 8.251 |
| 7 | 1.75 | 1.214 | 1.286 | -0.998 | 0.696 | 4.837 | 8.465 | 8.465 | 8.465 |
| 8 | 2.00 | 1.135 | 1.294 | -1.003 | 0.700 | 4.315 | 8.630 | 8.630 | 8.630 |
| 9 | 2.25 | 1.071 | 1.301 | -1.008 | 0.704 | 3.894 | 8.762 | 8.762 | 8.762 |
| 10 | 2.50 | 1.018 | 1.306 | -1.011 | 0.708 | 3.548 | 8.870 | 8.870 | 8.870 |
| 11 | 2.75 | 0.974 | 1.310 | -1.014 | 0.710 | 3.258 | 8.960 | 8.960 | 8.960 |
| 12 | 3.00 | 0.937 | 1.314 | -1.017 | 0.713 | 3.012 | 9.035 | 9.035 | 9.035 |
| 13 | 3.25 | 0.904 | 1.317 | -1.019 | 0.714 | 2.800 | 9.100 | 9.100 | 9.100 |
| 14 | 3.50 | 0.876 | 1.320 | -1.021 | 0.716 | 2.616 | 9.156 | 9.156 | 9.156 |
| 15 | 3.75 | 0.852 | 1.323 | -1.022 | 0.718 | 2.455 | 9.205 | 9.205 | 9.205 |
| 16 | 4.00 | 0.830 | 1.325 | -1.024 | 0.719 | 2.312 | 9.248 | 9.248 | 9.248 |
| 17 | 4.25 | 0.810 | 1.327 | -1.025 | 0.720 | 2.185 | 9.286 | 9.286 | 9.286 |
| 18 | 4.50 | 0.793 | 1.329 | -1.026 | 0.721 | 2.071 | 9.320 | 9.320 | 9.320 |
| 19 | 4.75 | 0.777 | 1.330 | -1.027 | 0.722 | 1.969 | 9.351 | 9.351 | 9.351 |
| 20 | 5.00 | 0.763 | 1.332 | -1.028 | 0.723 | 1.876 | 9.379 | 9.379 | 9.379 |

## 5. Nonlinear EIV Models

When independent variables are measured with error in a nonlinear model, no easy concentration of the joint least-squares objective function is, in general, available. Furthermore, the set of nonlinear functions that may be admissible for representing regression models is large.

### *5.1 A Cobb-Douglas Model*

For this initial analysis, therefore, we choose a Cobb-Douglas functional form. Hence the nonlinear regression model with measurement errors has the following structure

(35) $$y_i = A\prod_{j=1}^{3}(x_{ji}^{*})^{\beta_j} + u_i$$

(36) $$x_{ji} = x_{ji}^{*} + \varepsilon_{ji}$$

where $j = 1,...,3$. As in section 4, we assume that the error terms are iid variables. More specifically, $u_i \sim N(0,\sigma_u)$, $\varepsilon_{ji} \sim N(0,\sigma_{j\varepsilon})$, where $\sigma_u$ and $\sigma_{j\varepsilon}$ are the standard deviations of the corresponding errors, $j = 1,...,3$. The independence hypothesis implies that $\mathrm{cov}(x_{ji}^{*},\varepsilon_{ji}) = 0$, $\mathrm{cov}(y_i,\varepsilon_{ji}) = 0$, $\mathrm{cov}(x_{ji}^{*},u_i) = 0$, $i = 1,...,N$, $j = 1,...,3$. Furthermore, the latent variables $X_j^{*}$ are assumed to be identically, independently, and uniformly distributed.

The nonlinear joint least-squares (NJLS) EIV estimator utilized in this section for estimating model (35)-(36) has the following structure:

(37) $$NJLS = \min_{A,\beta_j,x_{ji}^{*},u_i,\varepsilon_{ji}} \left\{ \sum_{i=1}^{N}[y_i - A\prod_{j=1}^{3}(x_{ji}^{*})^{\beta_j}]^2 + \sum_{j=1}^{3}\sum_{i=1}^{N}(x_i - x_{ji}^{*})^2/\lambda_j \right\}$$

where $\lambda_j > 0$, $j = 1,2,3$.

As before, the objective is to assess the performance of the NJLS EIV estimator by a sequence of Monte Carlo experiments. In this nonlinear case, the second moment relationships for estimating the error variances are not easily obtained. Hence, the empirical results will concentrate on the estimation of the $A$ and $\beta$ parameters of model (35)-(36). The choice of parameter values and error variances is given in Table 21. All the draws were scaled by a factor equal to $\sqrt{N}$.

Table 21. Specification for the Monte Carlo experiments of the C-D model (35)-(36)

| Set | $A$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $x_i^*$ | $\sigma_u$ | $\sigma_{\varepsilon_j}$ $j=1,2,3$ | True $\lambda$ | $N$ |
|---|---|---|---|---|---|---|---|---|---|
| 23 | 2.0 | 0.3 | 0.5 | 0.70 | *U(10,20)* | 2.0 | 3.0 | 2.25 | 200 |
| 24 | 2.0 | 0.3 | 0.5 | 0.70 | *U(10,20)* | 2.0 | 3.0 | 2.25 | 5000 |
| 25 | 2.0 | 0.3 | 0.5 | 0.70 | *U(10,30)* | 2.0 | 3.0 | 2.25 | 200 |
| 26 | 2.0 | 0.3 | 0.5 | 0.70 | *U(10,20)* | 1.0 | 2.0 | 4.00 | 200 |
| 27 | 2.0 | 0.3 | 0.5 | 0.70 | *U(10,20)* | 1.0 | 1.0 | 1.00 | 200 |
| 28 | 2.0 | 0.3 | 0.5 | 0.70 | *U(10,20)* | 2.0 | 1.0 | 0.25 | 200 |

The results in all the Tables (22)-(27) show a remarkable precision of the estimates (including the $A$ parameter) for almost all the values of the error variances' ratio. An alternative MC run for specification set 26 with a scale factor equal to 10 (not reported here) gave equally accurate results.

Table 22. Results of Set 23, $N = 200$, of Table 21.

| Number | λ ratio | $A = 2.0$ | $\beta_1 = 0.3$ | $\beta_2 = 0.5$ | $\beta_3 = 0.7$ |
|---|---|---|---|---|---|
| 1 | 0.25 | 2.0033 | 0.1652 | 0.2911 | 0.3848 |
| 2 | 0.50 | 2.0737 | 0.2113 | 0.3642 | 0.4884 |
| 3 | 0.75 | 2.0654 | 0.2385 | 0.4043 | 0.5477 |
| 4 | 1.00 | 2.0600 | 0.2575 | 0.4312 | 0.5884 |
| 5 | 1.25 | 2.0562 | 0.2711 | 0.4503 | 0.6175 |
| 6 | 1.50 | 2.0535 | 0.2814 | 0.4644 | 0.6391 |
| 7 | 1.75 | 2.0515 | 0.2893 | 0.4752 | 0.6558 |
| 8 | 2.00 | 2.0499 | 0.2955 | 0.4837 | 0.6690 |
| 9 | 2.25 | 2.0486 | 0.3006 | 0.4906 | 0.6797 |
| 10 | 2.50 | 2.0476 | 0.3048 | 0.4963 | 0.6886 |
| 11 | 2.75 | 2.0467 | 0.3084 | 0.5011 | 0.6960 |
| 12 | 3.00 | 2.0460 | 0.3114 | 0.5051 | 0.7023 |
| 13 | 3.25 | 2.0453 | 0.3140 | 0.5086 | 0.7077 |
| 14 | 3.50 | 2.0448 | 0.3163 | 0.5117 | 0.7125 |
| 15 | 3.75 | 2.0443 | 0.3183 | 0.5143 | 0.7166 |
| 16 | 4.00 | 2.0439 | 0.3201 | 0.5167 | 0.7203 |
| 17 | 4.25 | 2.0435 | 0.3216 | 0.5188 | 0.7235 |
| 18 | 4.50 | 2.0432 | 0.3231 | 0.5206 | 0.7265 |
| 19 | 4.75 | 2.0429 | 0.3243 | 0.5222 | 0.7291 |
| 20 | 5.00 | 2.0426 | 0.3255 | 0.5238 | 0.7315 |

Table 23. Results of Set 24, $N = 5000$, of Table 21.

| Number | λ ratio | $A = 2.0$ | $\beta_1 = 0.3$ | $\beta_2 = 0.5$ | $\beta_3 = 0.7$ |
|---|---|---|---|---|---|
| 1 | 0.25 | 2.5240 | 0.1859 | 0.3186 | 0.4503 |
| 2 | 0.50 | 2.3440 | 0.2249 | 0.3810 | 0.5375 |
| 3 | 0.75 | 2.2453 | 0.2481 | 0.4174 | 0.5880 |
| 4 | 1.00 | 2.1860 | 0.2628 | 0.4400 | 0.6195 |
| 5 | 1.25 | 2.1471 | 0.2727 | 0.4552 | 0.6407 |
| 6 | 1.50 | 2.1197 | 0.2799 | 0.4661 | 0.6559 |
| 7 | 1.75 | 2.0993 | 0.2852 | 0.4743 | 0.6672 |
| 8 | 2.00 | 2.0837 | 0.2894 | 0.4806 | 0.6760 |
| 9 | 2.25 | 2.0714 | 0.2928 | 0.4857 | 0.6831 |
| 10 | 2.50 | 2.0613 | 0.2955 | 0.4898 | 0.6888 |
| 11 | 2.75 | 2.0530 | 0.2978 | 0.4933 | 0.6936 |
| 12 | 3.00 | 2.0460 | 0.2997 | 0.4962 | 0.6976 |
| 13 | 3.25 | 2.0400 | 0.3014 | 0.4987 | 0.7010 |
| 14 | 3.50 | 2.0349 | 0.3028 | 0.5008 | 0.7040 |
| 15 | 3.75 | 2.0304 | 0.3041 | 0.5027 | 0.7066 |
| 16 | 4.00 | 2.0264 | 0.3052 | 0.5043 | 0.7089 |
| 17 | 4.25 | 2.0229 | 0.3062 | 0.5058 | 0.7110 |
| 18 | 4.50 | 2.0198 | 0.3070 | 0.5071 | 0.7128 |
| 19 | 4.75 | 2.0170 | 0.3078 | 0.5083 | 0.7144 |
| 20 | 5.00 | 2.0145 | 0.3085 | 0.5094 | 0.7159 |

Table 24. Results of Set 25, $N = 200$, of Table 21.

| Number | λ ratio | $A = 2.0$ | $\beta_1 = 0.3$ | $\beta_2 = 0.5$ | $\beta_3 = 0.7$ |
|---|---|---|---|---|---|
| 1 | 0.25 | 2.1381 | 0.2614 | 0.4461 | 0.6167 |
| 2 | 0.50 | 2.0967 | 0.2745 | 0.4647 | 0.6476 |
| 3 | 0.75 | 2.0767 | 0.2811 | 0.4738 | 0.6630 |
| 4 | 1.00 | 2.0649 | 0.2850 | 0.4792 | 0.6723 |
| 5 | 1.25 | 2.0572 | 0.2876 | 0.4828 | 0.6784 |
| 6 | 1.50 | 2.0517 | 0.2895 | 0.4853 | 0.6827 |
| 7 | 1.75 | 2.0476 | 0.2909 | 0.4872 | 0.6860 |
| 8 | 2.00 | 2.0445 | 0.2920 | 0.4887 | 0.6885 |
| 9 | 2.25 | 2.0419 | 0.2928 | 0.4899 | 0.6906 |
| 10 | 2.50 | 2.0399 | 0.2936 | 0.4908 | 0.6922 |
| 11 | 2.75 | 2.0382 | 0.2942 | 0.4916 | 0.6936 |
| 12 | 3.00 | 2.0367 | 0.2947 | 0.4923 | 0.6947 |
| 13 | 3.25 | 2.0355 | 0.2951 | 0.4929 | 0.6957 |
| 14 | 3.50 | 2.0344 | 0.2954 | 0.4934 | 0.6966 |
| 15 | 3.75 | 2.0335 | 0.2958 | 0.4938 | 0.6973 |
| 16 | 4.00 | 2.0327 | 0.2961 | 0.4942 | 0.6980 |
| 17 | 4.25 | 2.0320 | 0.2963 | 0.4946 | 0.6986 |
| 18 | 4.50 | 2.0313 | 0.2965 | 0.4949 | 0.6991 |
| 19 | 4.75 | 2.0307 | 0.2967 | 0.4951 | 0.6996 |
| 20 | 5.00 | 2.0302 | 0.2969 | 0.4954 | 0.7000 |

Table 25. Results of Set 26, $N = 200$, of Table 21.

| Number | λ ratio | $A = 2.0$ | $\beta_1 = 0.3$ | $\beta_2 = 0.5$ | $\beta_3 = 0.7$ |
|---|---|---|---|---|---|
| 1 | 1.25 | 2.0260 | 0.2780 | 0.4658 | 0.6468 |
| 2 | 1.50 | 2.0246 | 0.2822 | 0.4717 | 0.6564 |
| 3 | 1.75 | 2.0235 | 0.2854 | 0.4763 | 0.6637 |
| 4 | 2.00 | 2.0226 | 0.2879 | 0.4799 | 0.6695 |
| 5 | 2.25 | 2.0219 | 0.2899 | 0.4827 | 0.6741 |
| 6 | 2.50 | 2.0213 | 0.2916 | 0.4851 | 0.6779 |
| 7 | 2.75 | 2.0209 | 0.2930 | 0.4871 | 0.6811 |
| 8 | 3.00 | 2.0205 | 0.2942 | 0.4888 | 0.6839 |
| 9 | 3.25 | 2.0201 | 0.2953 | 0.4902 | 0.6862 |
| 10 | 3.50 | 2.0198 | 0.2962 | 0.4915 | 0.6883 |
| 11 | 3.75 | 2.0195 | 0.2970 | 0.4926 | 0.6900 |
| 12 | 4.00 | 2.0193 | 0.2977 | 0.4936 | 0.6916 |
| 13 | 4.25 | 2.0191 | 0.2983 | 0.4945 | 0.6930 |
| 14 | 4.50 | 2.0189 | 0.2988 | 0.4953 | 0.6943 |
| 15 | 4.75 | 2.0188 | 0.2994 | 0.4960 | 0.6954 |
| 16 | 5.00 | 2.0186 | 0.2998 | 0.4966 | 0.6965 |
| 17 | 5.25 | 2.0185 | 0.3002 | 0.4972 | 0.6974 |
| 18 | 5.50 | 2.0183 | 0.3006 | 0.4977 | 0.6983 |
| 19 | 5.75 | 2.0182 | 0.3010 | 0.4982 | 0.6991 |
| 20 | 6.00 | 2.0181 | 0.3013 | 0.4986 | 0.6998 |

Table 26. Results of Set 27, $N = 200$, of Table 21.

| Number | λ ratio | $A = 2.0$ | $\beta_1 = 0.3$ | $\beta_2 = 0.5$ | $\beta_3 = 0.7$ |
|---|---|---|---|---|---|
| 1 | 0.25 | 2.0113 | 0.2831 | 0.4746 | 0.6622 |
| 2 | 0.50 | 2.0080 | 0.2909 | 0.4862 | 0.6811 |
| 3 | 0.75 | 2.0063 | 0.2952 | 0.4923 | 0.6912 |
| 4 | 1.00 | 2.0053 | 0.2978 | 0.4962 | 0.6975 |
| 5 | 1.25 | 2.0046 | 0.2996 | 0.4988 | 0.7017 |
| 6 | 1.50 | 2.0041 | 0.3009 | 0.5006 | 0.7048 |
| 7 | 1.75 | 2.0037 | 0.3019 | 0.5021 | 0.7071 |
| 8 | 2.00 | 2.0034 | 0.3027 | 0.5032 | 0.7090 |
| 9 | 2.25 | 2.0031 | 0.3033 | 0.5041 | 0.7104 |
| 10 | 2.50 | 2.0029 | 0.3038 | 0.5048 | 0.7116 |
| 11 | 2.75 | 2.0028 | 0.3042 | 0.5054 | 0.7126 |
| 12 | 3.00 | 2.0026 | 0.3046 | 0.5059 | 0.7135 |
| 13 | 3.25 | 2.0025 | 0.3049 | 0.5064 | 0.7142 |
| 14 | 3.50 | 2.0024 | 0.3052 | 0.5068 | 0.7148 |
| 15 | 3.75 | 2.0023 | 0.3054 | 0.5071 | 0.7154 |
| 16 | 4.00 | 2.0022 | 0.3056 | 0.5074 | 0.7159 |
| 17 | 4.25 | 2.0022 | 0.3058 | 0.5077 | 0.7163 |
| 18 | 4.50 | 2.0021 | 0.3060 | 0.5079 | 0.7167 |
| 19 | 4.75 | 2.0020 | 0.3062 | 0.5081 | 0.7171 |
| 20 | 5.00 | 2.0020 | 0.3063 | 0.5083 | 0.7174 |

Table 27. Results of Set 28, $N = 200$, of Table 21.

| Number | λ ratio | $A = 2.0$ | $\beta_1 = 0.3$ | $\beta_2 = 0.5$ | $\beta_3 = 0.7$ |
|---|---|---|---|---|---|
| 1 | 0.05 | 2.0162 | 0.2724 | 0.4589 | 0.6388 |
| 2 | 0.10 | 2.0128 | 0.2802 | 0.4706 | 0.6579 |
| 3 | 0.15 | 2.0100 | 0.2866 | 0.4801 | 0.6734 |
| 4 | 0.20 | 2.0077 | 0.2919 | 0.4878 | 0.6862 |
| 5 | 0.25 | 2.0058 | 0.2962 | 0.4942 | 0.6968 |
| 6 | 0.30 | 2.0043 | 0.3000 | 0.4996 | 0.7057 |
| 7 | 0.35 | 2.0030 | 0.3031 | 0.5042 | 0.7133 |
| 8 | 0.40 | 2.0018 | 0.3059 | 0.5082 | 0.7199 |
| 9 | 0.45 | 2.0009 | 0.3083 | 0.5116 | 0.7256 |
| 10 | 0.50 | 2.0000 | 0.3104 | 0.5146 | 0.7306 |
| 11 | 0.55 | 1.9993 | 0.3122 | 0.5173 | 0.7350 |
| 12 | 0.60 | 1.9986 | 0.3139 | 0.5197 | 0.7390 |
| 13 | 0.65 | 1.9980 | 0.3154 | 0.5218 | 0.7425 |
| 14 | 0.70 | 1.9975 | 0.3167 | 0.5237 | 0.7456 |
| 15 | 0.75 | 1.9970 | 0.3179 | 0.5254 | 0.7485 |
| 16 | 0.80 | 1.9966 | 0.3190 | 0.5270 | 0.7511 |
| 17 | 0.85 | 1.9962 | 0.3201 | 0.5284 | 0.7535 |
| 18 | 0.90 | 1.9958 | 0.3210 | 0.5297 | 0.7556 |
| 19 | 0.95 | 1.9955 | 0.3218 | 0.5309 | 0.7576 |
| 20 | 1.00 | 1.9952 | 0.3226 | 0.5320 | 0.7595 |

*5.2 A Translog Model*

The next model takes the form of a translog specification with two independent variables

(38) $$\log(y_i) = \alpha + \sum_{j=1}^{2}\beta_j \log(x_{ji}^*) + \sum_{j=1}^{2}\sum_{k=1}^{2}\gamma_{jk} \log(x_{ji}^*)\log(x_{ki}^*) + u_i$$

(39) $$x_{ji} = x_{ji}^* + \varepsilon_{ji}$$

with symmetry of the $\gamma$ parameters, that is, $\gamma_{jk} = \gamma_{kj}$, $j,k = 1,2$, and with the same assumptions on the error terms and latent variables as stated above. We notice that the above translog model is nonlinear in the parameters because the latent variables are unknown. This specification clearly increases the difficulty of estimating a nonlinear EIV model and, thus, puts the *NJLS* estimator to a serious scrutiny.

The parameter specifications for the translog model are given in Table 28 while the results are reported in Table 29 through Table 34. The choice of parameter values mimics the structure of a generic function and of a production function. All the draws were scaled by a factor equal to 20.

Table 28. Specification for the Monte Carlo experiments of the translog model (38)-(39)

| Set | $\alpha$ | $\beta_1$ | $\beta_2$ | $\gamma_{11}$ | $\gamma_{12}$ | $\gamma_{22}$ | $x_i^*$ | $\sigma_u$ | $\sigma_{\varepsilon_j}$ | True $\lambda$ | $N$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 29 | -0.5 | 0.4 | 0.5 | 1.0 | -0.2 | 0.8 | *U(10,20)* | 0.316 | 0.5 | 2.25 | 200 |
| 30 | -0.5 | 0.4 | 0.5 | 1.0 | -0.2 | 0.8 | *U(10,20)* | 0.5 | 0.5 | 1.00 | 200 |
| 31 | -0.5 | 0.4 | 0.5 | 1.0 | -0.2 | 0.8 | *U(10,20)* | 1.0 | 0.5 | 0.25 | 200 |
| 32 | 1.5 | 0.6 | 0.3 | -1.0 | -0.4 | -0.7 | *U(10,20)* | 0.316 | 0.5 | 2.25 | 200 |
| 33 | 1.5 | 0.6 | 0.3 | -1.0 | -0.4 | -0.7 | *U(10,20)* | 0.5 | 0.5 | 1.00 | 200 |
| 34 | 1.5 | 0.6 | 0.3 | -1.0 | -0.4 | -0.7 | *U(10,20)* | 1.0 | 0.5 | 0.25 | 200 |

Notice that in all the translog models of Table 28, the introduction of an intercept term does not disrupt the robustness of the *NJLS* estimator.

Table 29. Results of Set 29, $N = 200$, of Table 28

| Number | λ ratio | $\alpha = -0.5$ | $\beta_1 = 0.4$ | $\beta_2 = 0.5$ | $\gamma_{11} = 1.0$ | $\gamma_{12} = -0.2$ | $\gamma_{22} = 0.8$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | -0.4967 | 0.3950 | 0.4935 | 0.9169 | -0.1884 | 0.7591 |
| 2 | 0.50 | -0.4978 | 0.3964 | 0.4952 | 0.9323 | -0.1903 | 0.7699 |
| 3 | 0.75 | -0.4987 | 0.3976 | 0.4966 | 0.9444 | -0.1920 | 0.7780 |
| 4 | 1.00 | -0.4993 | 0.3986 | 0.4978 | 0.9542 | -0.1935 | 0.7843 |
| 5 | 1.25 | -0.4999 | 0.3995 | 0.4988 | 0.9623 | -0.1950 | 0.7892 |
| 6 | 1.50 | -0.5011 | 0.3960 | 0.5036 | 0.9808 | -0.1892 | 0.7802 |
| 7 | 1.75 | -0.5006 | 0.4010 | 0.5004 | 0.9748 | -0.1976 | 0.7964 |
| 8 | 2.00 | -0.5010 | 0.4016 | 0.5011 | 0.9798 | -0.1988 | 0.7989 |
| 9 | 2.25 | -0.5012 | 0.4021 | 0.5017 | 0.9841 | -0.1999 | 0.8010 |
| 10 | 2.50 | -0.5014 | 0.4027 | 0.5022 | 0.9879 | -0.2010 | 0.8028 |
| 11 | 2.75 | -0.5016 | 0.4031 | 0.5027 | 0.9913 | -0.2020 | 0.8041 |
| 12 | 3.00 | -0.5018 | 0.4035 | 0.5032 | 0.9944 | -0.2029 | 0.8052 |
| 13 | 3.25 | -0.5019 | 0.4039 | 0.5036 | 0.9971 | -0.2039 | 0.8061 |
| 14 | 3.50 | -0.5021 | 0.4043 | 0.5040 | 0.9996 | -0.2048 | 0.8068 |
| 15 | 3.75 | -0.5022 | 0.4046 | 0.5044 | 1.0019 | -0.2057 | 0.8074 |
| 16 | 4.00 | -0.5023 | 0.4050 | 0.5048 | 1.0039 | -0.2065 | 0.8079 |
| 17 | 4.25 | -0.5024 | 0.4053 | 0.5051 | 1.0058 | -0.2073 | 0.8082 |
| 18 | 4.50 | -0.5024 | 0.4055 | 0.5054 | 1.0075 | -0.2081 | 0.8084 |
| 19 | 4.75 | -0.5025 | 0.4058 | 0.5057 | 1.0091 | -0.2089 | 0.8086 |
| 20 | 5.00 | -0.5026 | 0.4061 | 0.5060 | 1.0106 | -0.2096 | 0.8086 |

Table 30. Results of Set 30, $N = 200$, of Table 28

| Number | λ ratio | $\alpha = -0.5$ | $\beta_1 = 0.4$ | $\beta_2 = 0.5$ | $\gamma_{11} = 1.0$ | $\gamma_{12} = -0.2$ | $\gamma_{22} = 0.8$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | -0.4975 | 0.3972 | 0.4943 | 0.9265 | -0.1912 | 0.7664 |
| 2 | 0.50 | -0.4992 | 0.3997 | 0.4972 | 0.9507 | -0.1948 | 0.7823 |
| 3 | 0.75 | -0.5005 | 0.4018 | 0.4996 | 0.9702 | -0.1983 | 0.7944 |
| 4 | 1.00 | -0.5016 | 0.4037 | 0.5016 | 0.9861 | -0.2015 | 0.8036 |
| 5 | 1.25 | -0.5024 | 0.4053 | 0.5034 | 0.9995 | -0.2045 | 0.8108 |
| 6 | 1.50 | -0.5035 | 0.3991 | 0.5086 | 1.0100 | -0.2045 | 0.8135 |
| 7 | 1.75 | -0.5037 | 0.4080 | 0.5062 | 1.0206 | -0.2102 | 0.8210 |
| 8 | 2.00 | -0.5042 | 0.4092 | 0.5075 | 1.0291 | -0.2128 | 0.8247 |
| 9 | 2.25 | -0.5046 | 0.4102 | 0.5086 | 1.0372 | -0.2153 | 0.8267 |
| 10 | 2.50 | -0.5050 | 0.4112 | 0.5096 | 1.0443 | -0.2177 | 0.8284 |
| 11 | 2.75 | -0.5053 | 0.4121 | 0.5106 | 1.0507 | -0.2200 | 0.8296 |
| 12 | 3.00 | -0.5056 | 0.4126 | 0.5115 | 1.0563 | -0.2225 | 0.8287 |
| 13 | 3.25 | -0.5058 | 0.4137 | 0.5124 | 1.0619 | -0.2245 | 0.8308 |
| 14 | 3.50 | -0.5060 | 0.4144 | 0.5132 | 1.0667 | -0.2266 | 0.8310 |
| 15 | 3.75 | -0.5062 | 0.4151 | 0.5140 | 1.0715 | -0.2285 | 0.8303 |
| 16 | 4.00 | -0.5064 | 0.4156 | 0.5147 | 1.0774 | -0.2301 | 0.8279 |
| 17 | 4.25 | -0.5065 | 0.4163 | 0.5154 | 1.0800 | -0.2323 | 0.8288 |
| 18 | 4.50 | -0.5066 | 0.4169 | 0.5161 | 1.0836 | -0.2341 | 0.8281 |
| 19 | 4.75 | -0.5067 | 0.4174 | 0.5167 | 1.0881 | -0.2356 | 0.8261 |
| 20 | 5.00 | -0.5069 | 0.4180 | 0.5172 | 1.0886 | -0.2378 | 0.8284 |

Table 31. Results of Set 31, $N = 200$, of Table 28

| Number | λ ratio | $\alpha = -0.5$ | $\beta_1 = 0.4$ | $\beta_2 = 0.5$ | $\gamma_{11} = 1.0$ | $\gamma_{12} = -0.2$ | $\gamma_{22} = 0.8$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.05 | -0.4974 | 0.3965 | 0.4926 | 0.9358 | -0.2015 | 0.7624 |
| 2 | 0.10 | -0.4987 | 0.3984 | 0.4948 | 0.9548 | -0.2042 | 0.7743 |
| 3 | 0.15 | -0.5000 | 0.4003 | 0.4969 | 0.9736 | -0.2069 | 0.7858 |
| 4 | 0.20 | -0.5012 | 0.4022 | 0.4990 | 0.9920 | -0.2096 | 0.7969 |
| 5 | 0.25 | -0.5025 | 0.4040 | 0.5009 | 1.0102 | -0.2123 | 0.8075 |
| 6 | 0.30 | -0.5036 | 0.4057 | 0.5028 | 1.0279 | -0.2150 | 0.8176 |
| 7 | 0.35 | -0.5048 | 0.4074 | 0.5045 | 1.0454 | -0.2178 | 0.8274 |
| 8 | 0.40 | -0.5059 | 0.4090 | 0.5062 | 1.0625 | -0.2205 | 0.8367 |
| 9 | 0.45 | -0.5069 | 0.4106 | 0.5078 | 1.0792 | -0.2233 | 0.8457 |
| 10 | 0.50 | -0.5080 | 0.4121 | 0.5094 | 1.0957 | -0.2260 | 0.8543 |
| 11 | 0.55 | -0.5090 | 0.4136 | 0.5109 | 1.1120 | -0.2288 | 0.8626 |
| 12 | 0.60 | -0.5100 | 0.4151 | 0.5123 | 1.1280 | -0.2316 | 0.8706 |
| 13 | 0.65 | -0.5110 | 0.4165 | 0.5137 | 1.1437 | -0.2344 | 0.8782 |
| 14 | 0.70 | -0.5119 | 0.4179 | 0.5151 | 1.1592 | -0.2372 | 0.8856 |
| 15 | 0.75 | -0.5128 | 0.4193 | 0.5164 | 1.1744 | -0.2400 | 0.8928 |
| 16 | 0.80 | -0.5137 | 0.4206 | 0.5177 | 1.1892 | -0.2428 | 0.8997 |
| 17 | 0.85 | -0.5146 | 0.4220 | 0.5189 | 1.2037 | -0.2456 | 0.9064 |
| 18 | 0.90 | -0.5155 | 0.4233 | 0.5201 | 1.2178 | -0.2484 | 0.9129 |
| 19 | 0.95 | -0.5163 | 0.4246 | 0.5213 | 1.2318 | -0.2512 | 0.9191 |
| 20 | 1.00 | -0.5171 | 0.4258 | 0.5225 | 1.2456 | -0.2539 | 0.9248 |

Table 32. Results of Set 32, $N = 200$, of Table 28

| Number | λ ratio | $\alpha = 1.5$ | $\beta_1 = 0.6$ | $\beta_2 = 0.3$ | $\gamma_{11} = -1.0$ | $\gamma_{12} = -0.4$ | $\gamma_{22} = -0.7$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 1.4958 | 0.5837 | 0.2866 | -0.8982 | -0.3888 | -0.6281 |
| 2 | 0.50 | 1.4981 | 0.5885 | 0.2897 | -0.9415 | -0.3926 | -0.6464 |
| 3 | 0.75 | 1.4993 | 0.5916 | 0.2918 | -0.9641 | -0.3949 | -0.6589 |
| 4 | 1.00 | 1.5001 | 0.5937 | 0.2933 | -0.9779 | -0.3965 | -0.6686 |
| 5 | 1.25 | 1.5007 | 0.5953 | 0.2945 | -0.9872 | -0.3978 | -0.6767 |
| 6 | 1.50 | 1.5012 | 0.5967 | 0.2957 | -0.9944 | -0.3996 | -0.6839 |
| 7 | 1.75 | 1.5015 | 0.5978 | 0.2964 | -0.9989 | -0.3998 | -0.6898 |
| 8 | 2.00 | 1.5017 | 0.5983 | 0.2970 | -1.0007 | -0.3997 | -0.6958 |
| 9 | 2.25 | 1.5020 | 0.5996 | 0.2979 | -1.0062 | -0.4012 | -0.7006 |
| 10 | 2.50 | 1.5024 | 0.6000 | 0.2986 | -1.0065 | -0.4023 | -0.7080 |
| 11 | 2.75 | 1.5024 | 0.6011 | 0.2991 | -1.0114 | -0.4024 | -0.7099 |
| 12 | 3.00 | 1.5026 | 0.6021 | 0.2998 | -1.0141 | -0.4032 | -0.7151 |
| 13 | 3.25 | 1.5027 | 0.6024 | 0.3001 | -1.0154 | -0.4035 | -0.7184 |
| 14 | 3.50 | 1.5028 | 0.6028 | 0.3003 | -1.0192 | -0.4017 | -0.7220 |
| 15 | 3.75 | 1.5025 | 0.6034 | 0.3007 | -1.0182 | -0.4045 | -0.7240 |
| 16 | 4.00 | 1.5028 | 0.6042 | 0.3010 | -1.0169 | -0.4038 | -0.7222 |
| 17 | 4.25 | 1.5029 | 0.6053 | 0.3016 | -1.0155 | -0.4057 | -0.7304 |
| 18 | 4.50 | 1.5034 | 0.6051 | 0.3024 | -1.0229 | -0.4055 | -0.7371 |
| 19 | 4.75 | 1.5036 | 0.6056 | 0.3028 | -1.0241 | -0.4059 | -0.7406 |
| 20 | 5.00 | 1.5037 | 0.6060 | 0.3032 | -1.0253 | -0.4062 | -0.7441 |

Table 33. Results of Set 33, $N = 200$, of Table 28

| Number | λ ratio | $\alpha = 1.5$ | $\beta_1 = 0.6$ | $\beta_2 = 0.3$ | $\gamma_{11} = -1.0$ | $\gamma_{12} = -0.4$ | $\gamma_{22} = -0.7$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 1.4970 | 0.5870 | 0.2878 | -0.9173 | -0.3922 | -0.6404 |
| 2 | 0.50 | 1.4998 | 0.5933 | 0.2922 | -0.9678 | -0.3976 | -0.6683 |
| 3 | 0.75 | 1.5015 | 0.5975 | 0.2953 | -0.9944 | -0.4011 | -0.6892 |
| 4 | 1.00 | 1.5026 | 0.6006 | 0.2977 | -1.0110 | -0.4038 | -0.7066 |
| 5 | 1.25 | 1.5034 | 0.6032 | 0.2997 | -1.0225 | -0.4060 | -0.7218 |
| 6 | 1.50 | 1.5041 | 0.6053 | 0.3015 | -1.0313 | -0.4079 | -0.7355 |
| 7 | 1.75 | 1.5047 | 0.6072 | 0.3030 | -1.0384 | -0.4095 | -0.7483 |
| 8 | 2.00 | 1.5052 | 0.6090 | 0.3044 | -1.0443 | -0.4110 | -0.7604 |
| 9 | 2.25 | 1.5056 | 0.6105 | 0.3057 | -1.0486 | -0.4118 | -0.7699 |
| 10 | 2.50 | 1.5063 | 0.6121 | 0.3064 | -1.0550 | -0.4116 | -0.7808 |
| 11 | 2.75 | 1.5064 | 0.6138 | 0.3084 | -1.0584 | -0.4163 | -0.7919 |
| 12 | 3.00 | 1.5071 | 0.6146 | 0.3096 | -1.0608 | -0.4170 | -0.8059 |
| 13 | 3.25 | 1.5075 | 0.6162 | 0.3106 | -1.0668 | -0.4173 | -0.8161 |
| 14 | 3.50 | 1.5077 | 0.6176 | 0.3115 | -1.0696 | -0.4149 | -0.8258 |
| 15 | 3.75 | 1.5078 | 0.6196 | 0.3138 | -1.0727 | -0.4235 | -0.8380 |
| 16 | 4.00 | 1.5086 | 0.6196 | 0.3148 | -1.0723 | -0.4212 | -0.8506 |
| 17 | 4.25 | 1.5083 | 0.6209 | 0.3144 | -1.0744 | -0.4198 | -0.8496 |
| 18 | 4.50 | 1.5095 | 0.6227 | 0.3164 | -1.0861 | -0.4234 | -0.8704 |
| 19 | 4.75 | 1.5099 | 0.6240 | 0.3176 | -1.0898 | -0.4246 | -0.8812 |
| 20 | 5.00 | 1.5103 | 0.6253 | 0.3188 | -1.0932 | -0.4258 | -0.8919 |

Table 34. Results of Set 34, $N = 200$, of Table 28

| Number | λ ratio | $\alpha = 1.5$ | $\beta_1 = 0.6$ | $\beta_2 = 0.3$ | $\gamma_{11} = -1.0$ | $\gamma_{12} = -0.4$ | $\gamma_{22} = -0.7$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.05 | 1.4935 | 0.5835 | 0.2822 | -0.8568 | -0.3884 | -0.6175 |
| 2 | 0.10 | 1.4966 | 0.5889 | 0.2861 | -0.9090 | -0.3946 | -0.6434 |
| 3 | 0.15 | 1.4991 | 0.5936 | 0.2895 | -0.9507 | -0.3993 | -0.6663 |
| 4 | 0.20 | 1.5011 | 0.5977 | 0.2926 | -0.9847 | -0.4033 | -0.6875 |
| 5 | 0.25 | 1.5029 | 0.6013 | 0.2955 | -1.0128 | -0.4068 | -0.7074 |
| 6 | 0.30 | 1.5045 | 0.6046 | 0.2981 | -1.0365 | -0.4100 | -0.7264 |
| 7 | 0.35 | 1.5059 | 0.6077 | 0.3005 | -1.0568 | -0.4129 | -0.7448 |
| 8 | 0.40 | 1.5071 | 0.6105 | 0.3029 | -1.0746 | -0.4157 | -0.7626 |
| 9 | 0.45 | 1.5083 | 0.6131 | 0.3050 | -1.0903 | -0.4183 | -0.7799 |
| 10 | 0.50 | 1.5094 | 0.6155 | 0.3071 | -1.1045 | -0.4208 | -0.7969 |
| 11 | 0.55 | 1.5104 | 0.6178 | 0.3091 | -1.1173 | -0.4232 | -0.8135 |
| 12 | 0.60 | 1.5113 | 0.6200 | 0.3110 | -1.1292 | -0.4255 | -0.8298 |
| 13 | 0.65 | 1.5123 | 0.6221 | 0.3129 | -1.1402 | -0.4277 | -0.8459 |
| 14 | 0.70 | 1.5131 | 0.6242 | 0.3146 | -1.1506 | -0.4299 | -0.8619 |
| 15 | 0.75 | 1.5140 | 0.6261 | 0.3164 | -1.1605 | -0.4320 | -0.8777 |
| 16 | 0.80 | 1.5148 | 0.6280 | 0.3181 | -1.1699 | -0.4341 | -0.8935 |
| 17 | 0.85 | 1.5156 | 0.6299 | 0.3197 | -1.1790 | -0.4361 | -0.9092 |
| 18 | 0.90 | 1.5164 | 0.6317 | 0.3213 | -1.1877 | -0.4381 | -0.9249 |
| 19 | 0.95 | 1.5172 | 0.6334 | 0.3229 | -1.1963 | -0.4400 | -0.9407 |
| 20 | 1.00 | 1.5181 | 0.6346 | 0.3244 | -1.2014 | -0.4398 | -0.9578 |

## 6. A Nonlinear EIV System of Equations

The next model has the structure of a simultaneous system of three nonlinear equations with all the variables measured with error. In general, the formulation of a Monte Carlo specification for a simultaneous nonlinear equations system is rather difficult because the choice of parameter values and random draws of the sample variables may rarely fit the desired relations. Fortunately, there exists a nonlinear system of equations that can serve as the framework for a Monte Carlo experiment. Such a system is the Linear Expenditure System (LES) of demand analysis. We define a LES model with three commodities

(40) $$x_{ji}^{*} = \gamma_j + \beta_j (y_i^{*} - \sum_{k=1}^{3} \gamma_k p_{ki}^{*}) / p_{ji}^{*}$$

(41) $$y_i^{*} = \sum_{j=1}^{3} p_{ji}^{*} x_{ji}^{*}$$

(42) $$y_i = y_i^{*} + u_i$$

(43) $$x_{ji} = x_{ji}^{*} + \varepsilon_{ji}$$

(44) $$p_{ji} = p_{ji}^{*} + v_{ji}$$

where $j = 1,2,3$ and $\sum_j \beta_j = 1$. The $x_{ji}^{*}$ variables are interpreted as commodity quantities while $p_{ji}^{*}$ variables are the corresponding prices. Hence, $y_i^{*} = \sum_j p_{ji}^{*} x_{ji}^{*}$ is disposable income. The $\gamma_j$ parameters are interpreted as subsistence levels of the corresponding commodities with $\gamma_j \leq \min_i x_{ji}^{*}$. We assume that quantities, prices and income are iid uniformly random variables. The measurement errors are iid normal random variables.

To generate the Monte Carlo sample information we first draw the prices $p_{ji}^{*}$ and income $y_i^{*}$ from the uniform distribution and define the quantities $x_{ji}^{*}$ using relations (40). The structure of the LES model guarantees that the adding up condition on disposable income $y_i^* = \sum_j p_{ji}^* x_{ji}^*$ is satisfied. Then, errors $\varepsilon_{ij}$ and $\nu_{ij}$ are drawn from the normal distribution according to the desired specification. The "observed" quantities, prices and disposable income are, thus, defined as $x_{ij} = x_{ij}^* + \varepsilon_{ij}$, $p_{ij} = p_{ij}^* + \nu_{ij}$, and $y_i = \sum_j x_{ij} p_{ij}$.

The empirical solution of the nonlinear model (40)-(44) requires access to a very efficient mathematical programming code. In all the computations reported in this paper we used the commercial package GAMS. We notice that the Monte Carlo experiments associated with model (40)-(44) necessitated the definition of upper and lower bounds on all the unknowns $(\beta_j, \gamma_j, x_{ij}^*, p_{ij}^*, y_i^*)$ without which the optimal solution is difficult to obtain. The solution of each problem required 1500 iterations, on average. In these Monte Carlo experiments we limited the sample repetitions to 10.

The specification of the parameter and variance values is given in Table 35. All the draws were measured in the original units.

Table 35. Specification of the parameter values for the LES Model (40)-(44)

| Parameter | Set 1 | Set 2 | Set 3 | Set 4 | Set 5 | Set 6 |
|---|---|---|---|---|---|---|
| $\beta_1$ | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 |
| $\beta_2$ | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| $\beta_3$ | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 |
| $\gamma_1$ | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 |
| $\gamma_2$ | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 |
| $\gamma_3$ | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| $\sigma_{\varepsilon_1}$ | 0.3 | 0.4 | 0.5 | 0.6 | 0.4 | 0.6 |
| $\sigma_{\nu_1}$ | 0.3 | 0.4 | 0.5 | 0.6 | 0.6 | 0.4 |
| Latent $Y^*$ | U(10,20) | U(10,20) | U(10,20) | U(10,20) | U(10,20) | U(10,20) |
| Latent $P_j^*, j = 1,2,3$ | U(1,4) | U(1,4) | U(1,4) | U(1,4) | U(1,4) | U(1,4) |
| Sample size | N = 200 | N = 200 | N = 200 | N = 200 | N = 200 | N = 200 |

The choice of the beta parameters values is dictated by the necessity that these parameters be positive and add to unity. The choice of the gamma parameters tries to approximate the theoretical restriction $\gamma_j \leq \min_i x^*_{ji}$, although it may not be achievable for all the sample values. The choice of the standard deviations of the error terms attempts to strike a balance between large errors and the level of the observed variables. In order to judge the magnitude of the errors in relation to the level of the various variables we computed the ratio between the average largest absolute value of the errors and the average largest value of the corresponding variable. Hence, in the sample the relative errors can be even larger for some observations. These computations are reported in Table 35B. It shows that the choice of the standard deviations of the error terms reported in Table 35 produces errors that can be as large as the ratio values given in Table 35B. Thirty and forty percent errors are large error and, therefore, the estimator is put to a serious test.

Table 35B. Ratios of errors to levels of the variables

| Table 35 Set | $\frac{\max\mid\varepsilon_1\mid}{\max x_1}$ | $\frac{\max\mid\varepsilon_2\mid}{\max x_2}$ | $\frac{\max\mid\varepsilon_3\mid}{\max x_3}$ | $\frac{\max\mid\nu_1\mid}{\max p_1}$ | $\frac{\max\mid\nu_2\mid}{\max p_2}$ | $\frac{\max\mid\nu_3\mid}{\max p_3}$ |
|---|---|---|---|---|---|---|
| 1 | 0.160 | 0.119 | 0.284 | 0.200 | 0.253 | 0.166 |
| 2 | 0.213 | 0.158 | 0.359 | 0.251 | 0.326 | 0.212 |
| 3 | 0.261 | 0.197 | 0.427 | 0.297 | 0.393 | 0.253 |
| 4 | 0.306 | 0.236 | 0.489 | 0.338 | 0.455 | 0.300 |
| 5 | 0.213 | 0.158 | 0.359 | 0.338 | 0.455 | 0.291 |
| 6 | 0.306 | 0.236 | 0.489 | 0.251 | 0.326 | 0.212 |

The Monte Carlo results are reported in Tables 36 through 41. The suggested EIV estimator produces very accurate estimates of the $\beta$ parameters regardless of the error variances' ratio and of the distribution of the error terms. On the contrary, the $\gamma$

parameters are estimated with a lesser level of accuracy as the distribution of the error term widens. Also, when the error of the quantities is relatively small and the error of the prices is relatively large (set 5 of Table 35), the estimates of the $\gamma$ parameters overshoot the true values up to 30 percent. The opposite is true when the relative magnitude of the errors is reversed (set 6 of Table 35). We would like to underline again that the error distributions selected in these Monte Carlo experiments have a significant dispersion as reported in Table 35B.

Table 36. Monte Carlo results, LES model, 3 equations, Set 1, $N = 200$, of Table 35

| Number | λ ratio | $\beta_1 = 0.30$ | $\beta_2 = 0.50$ | $\beta_3 = 0.20$ | $\gamma_1 = 1.20$ | $\gamma_2 = 0.90$ | $\gamma_3 = 0.50$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.2997 | 0.4996 | 0.2007 | 1.1893 | 0.9092 | 0.4860 |
| 2 | 0.50 | 0.3005 | 0.4989 | 0.2006 | 1.1878 | 0.9098 | 0.4884 |
| 3 | 0.75 | 0.3006 | 0.4988 | 0.2007 | 1.1873 | 0.9096 | 0.4879 |
| 4 | 1.00 | 0.3006 | 0.4987 | 0.2007 | 1.1871 | 0.9095 | 0.4876 |
| 5 | 1.25 | 0.3006 | 0.4987 | 0.2007 | 1.1869 | 0.9095 | 0.4874 |
| 6 | 1.50 | 0.3006 | 0.4986 | 0.2008 | 1.1868 | 0.9094 | 0.4873 |
| 7 | 1.75 | 0.3006 | 0.4986 | 0.2008 | 1.1867 | 0.9094 | 0.4872 |
| 8 | 2.00 | 0.3006 | 0.4986 | 0.2008 | 1.1867 | 0.9094 | 0.4872 |
| 9 | 2.25 | 0.3006 | 0.4986 | 0.2008 | 1.1866 | 0.9093 | 0.4871 |
| 10 | 2.50 | 0.3006 | 0.4986 | 0.2008 | 1.1866 | 0.9093 | 0.4871 |
| 11 | 2.75 | 0.3006 | 0.4986 | 0.2008 | 1.1866 | 0.9093 | 0.4871 |
| 12 | 3.00 | 0.3006 | 0.4986 | 0.2008 | 1.1866 | 0.9093 | 0.4870 |
| 13 | 3.25 | 0.3006 | 0.4986 | 0.2008 | 1.1865 | 0.9093 | 0.4870 |
| 14 | 3.50 | 0.3006 | 0.4986 | 0.2008 | 1.1865 | 0.9093 | 0.4870 |
| 15 | 3.75 | 0.3006 | 0.4986 | 0.2008 | 1.1865 | 0.9093 | 0.4870 |
| 16 | 4.00 | 0.2994 | 0.4981 | 0.2026 | 1.1930 | 0.9144 | 0.4839 |
| 17 | 4.25 | 0.3006 | 0.4986 | 0.2008 | 1.1865 | 0.9093 | 0.4869 |
| 18 | 4.50 | 0.3006 | 0.4985 | 0.2008 | 1.1865 | 0.9093 | 0.4869 |
| 19 | 4.75 | 0.3006 | 0.4985 | 0.2008 | 1.1865 | 0.9093 | 0.4869 |
| 20 | 5.00 | 0.3006 | 0.4985 | 0.2008 | 1.1864 | 0.9093 | 0.4869 |

Table 37. Monte Carlo results, LES model, 3 equations, Set 2, $N = 200$, of Table 35

| Number | λ ratio | $\beta_1 = 0.30$ | $\beta_2 = 0.50$ | $\beta_3 = 0.20$ | $\gamma_1 = 1.20$ | $\gamma_2 = 0.90$ | $\gamma_3 = 0.50$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.3017 | 0.4973 | 0.2011 | 1.1751 | 0.9126 | 0.4787 |
| 2 | 0.50 | 0.3018 | 0.4967 | 0.2014 | 1.1728 | 0.9115 | 0.4763 |
| 3 | 0.75 | 0.3019 | 0.4966 | 0.2016 | 1.1720 | 0.9111 | 0.4754 |
| 4 | 1.00 | 0.3019 | 0.4965 | 0.2016 | 1.1715 | 0.9108 | 0.4749 |
| 5 | 1.25 | 0.3019 | 0.4964 | 0.2017 | 1.1712 | 0.9107 | 0.4747 |
| 6 | 1.50 | 0.3019 | 0.4964 | 0.2017 | 1.1710 | 0.9106 | 0.4745 |
| 7 | 1.75 | 0.3019 | 0.4963 | 0.2017 | 1.1709 | 0.9105 | 0.4744 |
| 8 | 2.00 | 0.3020 | 0.4963 | 0.2017 | 1.1708 | 0.9104 | 0.4743 |
| 9 | 2.25 | 0.3020 | 0.4963 | 0.2017 | 1.1707 | 0.9104 | 0.4742 |
| 10 | 2.50 | 0.3020 | 0.4963 | 0.2017 | 1.1706 | 0.9103 | 0.4742 |
| 11 | 2.75 | 0.3020 | 0.4963 | 0.2017 | 1.1706 | 0.9103 | 0.4741 |
| 12 | 3.00 | 0.3020 | 0.4963 | 0.2017 | 1.1705 | 0.9103 | 0.4741 |
| 13 | 3.25 | 0.3020 | 0.4963 | 0.2018 | 1.1705 | 0.9103 | 0.4740 |
| 14 | 3.50 | 0.3020 | 0.4963 | 0.2018 | 1.1704 | 0.9102 | 0.4740 |
| 15 | 3.75 | 0.3020 | 0.4963 | 0.2018 | 1.1704 | 0.9102 | 0.4740 |
| 16 | 4.00 | 0.3042 | 0.4959 | 0.2000 | 1.1648 | 0.9114 | 0.4800 |
| 17 | 4.25 | 0.3020 | 0.4963 | 0.2018 | 1.1704 | 0.9102 | 0.4739 |
| 18 | 4.50 | 0.3020 | 0.4962 | 0.2018 | 1.1703 | 0.9102 | 0.4739 |
| 19 | 4.75 | 0.3020 | 0.4962 | 0.2018 | 1.1703 | 0.9102 | 0.4739 |
| 20 | 5.00 | 0.3020 | 0.4962 | 0.2018 | 1.1703 | 0.9102 | 0.4739 |

Table 38. Monte Carlo results, LES model, 3 equations, Set 3, $N = 200$, of Table 35

| Number | λ ratio | $\beta_1 = 0.30$ | $\beta_2 = 0.50$ | $\beta_3 = 0.20$ | $\gamma_1 = 1.20$ | $\gamma_2 = 0.90$ | $\gamma_3 = 0.50$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.3049 | 0.4966 | 0.1985 | 1.1492 | 0.9064 | 0.4782 |
| 2 | 0.50 | 0.3055 | 0.4961 | 0.1984 | 1.1436 | 0.9028 | 0.4771 |
| 3 | 0.75 | 0.3058 | 0.4959 | 0.1983 | 1.1408 | 0.9013 | 0.4768 |
| 4 | 1.00 | 0.3060 | 0.4958 | 0.1982 | 1.1392 | 0.9003 | 0.4765 |
| 5 | 1.25 | 0.3061 | 0.4957 | 0.1982 | 1.1381 | 0.8997 | 0.4764 |
| 6 | 1.50 | 0.3062 | 0.4957 | 0.1981 | 1.1373 | 0.8992 | 0.4762 |
| 7 | 1.75 | 0.3063 | 0.4956 | 0.1981 | 1.1367 | 0.8988 | 0.4761 |
| 8 | 2.00 | 0.3063 | 0.4956 | 0.1981 | 1.1362 | 0.8985 | 0.4760 |
| 9 | 2.25 | 0.3064 | 0.4956 | 0.1981 | 1.1358 | 0.8982 | 0.4760 |
| 10 | 2.50 | 0.3064 | 0.4956 | 0.1981 | 1.1355 | 0.8980 | 0.4759 |
| 11 | 2.75 | 0.3064 | 0.4955 | 0.1980 | 1.1352 | 0.8978 | 0.4758 |
| 12 | 3.00 | 0.3064 | 0.4955 | 0.1980 | 1.1350 | 0.8976 | 0.4758 |
| 13 | 3.25 | 0.3065 | 0.4955 | 0.1980 | 1.1348 | 0.8975 | 0.4758 |
| 14 | 3.50 | 0.3065 | 0.4955 | 0.1980 | 1.1346 | 0.8974 | 0.4757 |
| 15 | 3.75 | 0.3065 | 0.4955 | 0.1980 | 1.1344 | 0.8972 | 0.4757 |
| 16 | 4.00 | 0.3065 | 0.4955 | 0.1980 | 1.1343 | 0.8971 | 0.4756 |
| 17 | 4.25 | 0.3065 | 0.4955 | 0.1980 | 1.1342 | 0.8970 | 0.4756 |
| 18 | 4.50 | 0.3065 | 0.4955 | 0.1980 | 1.1340 | 0.8969 | 0.4756 |
| 19 | 4.75 | 0.3065 | 0.4955 | 0.1980 | 1.1339 | 0.8969 | 0.4756 |
| 20 | 5.00 | 0.3065 | 0.4955 | 0.1980 | 1.1338 | 0.8968 | 0.4755 |

Table 39. Monte Carlo results, LES model, 3 equations, Set 4, $N = 200$, of Table 35

| Number | λ ratio | $\beta_1 = 0.30$ | $\beta_2 = 0.50$ | $\beta_3 = 0.20$ | $\gamma_1 = 1.20$ | $\gamma_2 = 0.90$ | $\gamma_3 = 0.50$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.3098 | 0.4963 | 0.1938 | 1.1089 | 0.8882 | 0.4807 |
| 2 | 0.50 | 0.3113 | 0.4957 | 0.1930 | 1.0946 | 0.8780 | 0.4795 |
| 3 | 0.75 | 0.3121 | 0.4953 | 0.1926 | 1.0875 | 0.8735 | 0.4791 |
| 4 | 1.00 | 0.3126 | 0.4951 | 0.1923 | 1.0832 | 0.8710 | 0.4790 |
| 5 | 1.25 | 0.3129 | 0.4949 | 0.1922 | 1.0802 | 0.8696 | 0.4789 |
| 6 | 1.50 | 0.3132 | 0.4948 | 0.1920 | 1.0781 | 0.8688 | 0.4789 |
| 7 | 1.75 | 0.3134 | 0.4946 | 0.1919 | 1.0766 | 0.8684 | 0.4790 |
| 8 | 2.00 | 0.3136 | 0.4946 | 0.1919 | 1.0754 | 0.8681 | 0.4791 |
| 9 | 2.25 | 0.3137 | 0.4945 | 0.1918 | 1.0744 | 0.8678 | 0.4791 |
| 10 | 2.50 | 0.3139 | 0.4944 | 0.1917 | 1.0736 | 0.8676 | 0.4792 |
| 11 | 2.75 | 0.3140 | 0.4944 | 0.1917 | 1.0729 | 0.8676 | 0.4793 |
| 12 | 3.00 | 0.3140 | 0.4943 | 0.1916 | 1.0724 | 0.8676 | 0.4794 |
| 13 | 3.25 | 0.3141 | 0.4943 | 0.1916 | 1.0720 | 0.8676 | 0.4795 |
| 14 | 3.50 | 0.3142 | 0.4942 | 0.1916 | 1.0717 | 0.8677 | 0.4796 |
| 15 | 3.75 | 0.3142 | 0.4942 | 0.1916 | 1.0714 | 0.8678 | 0.4798 |
| 16 | 4.00 | 0.3143 | 0.4942 | 0.1915 | 1.0712 | 0.8679 | 0.4799 |
| 17 | 4.25 | 0.3143 | 0.4942 | 0.1915 | 1.0710 | 0.8681 | 0.4800 |
| 18 | 4.50 | 0.3144 | 0.4941 | 0.1915 | 1.0709 | 0.8682 | 0.4802 |
| 19 | 4.75 | 0.3144 | 0.4941 | 0.1915 | 1.0708 | 0.8684 | 0.4803 |
| 20 | 5.00 | 0.3145 | 0.4941 | 0.1914 | 1.0707 | 0.8686 | 0.4805 |

Table 40. Monte Carlo results, LES model, 3 equations, Set 5, $N = 200$, of Table 35

| Number | λ ratio | $\beta_1 = 0.30$ | $\beta_2 = 0.50$ | $\beta_3 = 0.20$ | $\gamma_1 = 1.20$ | $\gamma_2 = 0.90$ | $\gamma_3 = 0.50$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.3053 | 0.5108 | 0.1840 | 1.3397 | 1.1554 | 0.6433 |
| 2 | 0.50 | 0.3027 | 0.5102 | 0.1871 | 1.3395 | 1.1524 | 0.6257 |
| 3 | 0.75 | 0.3027 | 0.5099 | 0.1874 | 1.3374 | 1.1498 | 0.6232 |
| 4 | 1.00 | 0.3027 | 0.5097 | 0.1876 | 1.3362 | 1.1481 | 0.6215 |
| 5 | 1.25 | 0.3026 | 0.5096 | 0.1878 | 1.3354 | 1.1469 | 0.6204 |
| 6 | 1.50 | 0.3026 | 0.5096 | 0.1878 | 1.3347 | 1.1459 | 0.6198 |
| 7 | 1.75 | 0.3026 | 0.5096 | 0.1878 | 1.3342 | 1.1452 | 0.6193 |
| 8 | 2.00 | 0.3026 | 0.5095 | 0.1879 | 1.3338 | 1.1446 | 0.6189 |
| 9 | 2.25 | 0.3026 | 0.5095 | 0.1879 | 1.3335 | 1.1442 | 0.6186 |
| 10 | 2.50 | 0.3026 | 0.5095 | 0.1879 | 1.3333 | 1.1439 | 0.6184 |
| 11 | 2.75 | 0.3026 | 0.5095 | 0.1879 | 1.3331 | 1.1436 | 0.6183 |
| 12 | 3.00 | 0.3026 | 0.5095 | 0.1879 | 1.3330 | 1.1435 | 0.6182 |
| 13 | 3.25 | 0.3057 | 0.5093 | 0.1850 | 1.3342 | 1.1552 | 0.6339 |
| 14 | 3.50 | 0.3026 | 0.5095 | 0.1879 | 1.3328 | 1.1433 | 0.6181 |
| 15 | 3.75 | 0.3026 | 0.5095 | 0.1879 | 1.3328 | 1.1433 | 0.6181 |
| 16 | 4.00 | 0.3026 | 0.5095 | 0.1879 | 1.3328 | 1.1433 | 0.6181 |
| 17 | 4.25 | 0.3026 | 0.5095 | 0.1879 | 1.3327 | 1.1433 | 0.6181 |
| 18 | 4.50 | 0.3026 | 0.5095 | 0.1879 | 1.3327 | 1.1433 | 0.6181 |
| 19 | 4.75 | 0.3026 | 0.5095 | 0.1879 | 1.3327 | 1.1434 | 0.6181 |
| 20 | 5.00 | 0.3043 | 0.5078 | 0.1879 | 1.3121 | 1.1249 | 0.6118 |

Table 41. Monte Carlo results, LES model, 3 equations, Set 6, $N = 200$, of Table 35

| Number | λ ratio | $\beta_1 = 0.30$ | $\beta_2 = 0.50$ | $\beta_3 = 0.20$ | $\gamma_1 = 1.20$ | $\gamma_2 = 0.90$ | $\gamma_3 = 0.50$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.25 | 0.3079 | 0.4847 | 0.2074 | 0.8902 | 0.5729 | 0.2916 |
| 2 | 0.50 | 0.3093 | 0.4845 | 0.2062 | 0.8787 | 0.5668 | 0.2953 |
| 3 | 0.75 | 0.3101 | 0.4844 | 0.2055 | 0.8733 | 0.5649 | 0.2978 |
| 4 | 1.00 | 0.3107 | 0.4842 | 0.2051 | 0.8699 | 0.5644 | 0.2992 |
| 5 | 1.25 | 0.3110 | 0.4840 | 0.2049 | 0.8675 | 0.5644 | 0.3000 |
| 6 | 1.50 | 0.3107 | 0.4842 | 0.2051 | 0.8551 | 0.5445 | 0.2943 |
| 7 | 1.75 | 0.3116 | 0.4838 | 0.2046 | 0.8644 | 0.5646 | 0.3012 |
| 8 | 2.00 | 0.3117 | 0.4837 | 0.2045 | 0.8633 | 0.5647 | 0.3016 |
| 9 | 2.25 | 0.3119 | 0.4837 | 0.2045 | 0.8625 | 0.5649 | 0.3019 |
| 10 | 2.50 | 0.3120 | 0.4836 | 0.2044 | 0.8618 | 0.5650 | 0.3022 |
| 11 | 2.75 | 0.3121 | 0.4835 | 0.2043 | 0.8612 | 0.5651 | 0.3024 |
| 12 | 3.00 | 0.3122 | 0.4835 | 0.2043 | 0.8606 | 0.5652 | 0.3026 |
| 13 | 3.25 | 0.3123 | 0.4835 | 0.2042 | 0.8602 | 0.5653 | 0.3028 |
| 14 | 3.50 | 0.3124 | 0.4834 | 0.2042 | 0.8598 | 0.5653 | 0.3029 |
| 15 | 3.75 | 0.3124 | 0.4834 | 0.2042 | 0.8595 | 0.5654 | 0.3031 |
| 16 | 4.00 | 0.3125 | 0.4834 | 0.2042 | 0.8592 | 0.5655 | 0.3032 |
| 17 | 4.25 | 0.3125 | 0.4833 | 0.2041 | 0.8589 | 0.5656 | 0.3033 |
| 18 | 4.50 | 0.3126 | 0.4833 | 0.2041 | 0.8586 | 0.5656 | 0.3034 |
| 19 | 4.75 | 0.3126 | 0.4833 | 0.2041 | 0.8584 | 0.5657 | 0.3035 |
| 20 | 5.00 | 0.3127 | 0.4833 | 0.2041 | 0.8582 | 0.5657 | 0.3036 |

**References**

Adcock, R.J., 1878. A Problem in Least Squares. *The Analyst*, 5, 53-54.

Brooke, A., Kendrick, D. Meeraus, A., 1988. *GAMS, User Guide*, Boyd and Fraser Publishing Company. Danvers, MA.

Gini, C., 1921. Sull'interpolazione di una retta quandi i valori della variabile indipendente sono affetti da errori accidentali. *Metron*, 1, 63-82.

Kiefer, J. and J. Wolfowitz, 1956. Consistency of the Maximum Likelihood Estimator in the Presence of Infinitely Many Incidental Parameters. *The Annals of Mathematical Statistics*, 27, 887-906.

Kummell, C.H., 1879. Reduction of Observation Equations Which Contain More Than One Observed Quantity. *The Analyst*, 6, 97-105.

Lindley, D.V., 1947. Regression Lines and the Linear Functional Relationship. *Journal of the Royal Statistical Society*, Series B, 9, 218-244.

Neyman, J., 1937. Remarks on a Paper by E.C. Rhodes. *Journal of the Royal Statistical Society*. 100, 50-57.

Neyman, J and E.L. Scott, 1948. Consistent Estimates Based on Partially Consistent Observations. *Econometrica*, 16, 1-32.

Neyman, J., 1951. Existence of Consistent Estimates of the Directional Parameter in a Linear Structural Relation Between Two Variables, *The Annals of Mathematical Statistics*, 22, 497-512.

Pearson, K., 1901. On Lines and Planes of Closest Fit to systems of Points in Space, *Philosophical Magazine and Journal of Science*, S. 6. Vol. 2, N. 10, 559-572.

Reiersol, O., 1950. Identifiability of a linear relation between Variables Which Are

Subject to Error. *Econometrica*, 18, 375-389.

Solari, M.E., 1969. The "Maximum Likelihood Solution" of the Problem of Estimating a Linear Functional Relationship. *Journal of the Royal Statistical Society, Series B,* 31, 372-375.

Thompson, G.H., 1916. A Hierarchy Without a General Factor. *British Journal of Psychology*, 8, 271-281.

Wald, A., 1949. Note on the Consistency of the Maximum Likelihood Estimate. *The Annals of Mathematical Statistics*, 20, 595-601.