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Synthesis and Verification of Networked Systems with Applications to Transportation Networks

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Synthesis and Verification of Networked Systems with Applications to Transportation Networks

by

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Professor Murat Arcak, Chair
Professor Andrew Packard
Professor Pravin Varaiya
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Abstract

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Engineering advances have enabled systems that are increasingly complex while, simultaneously, expectations for the closed-loop behavior of these systems have become more demanding. The resulting combination of complexity and desired behavior requires correct-by-design approaches to system analysis and control. In this dissertation, we develop a class of scalable and automated verification and control synthesis techniques for networked systems. To motivate our study and to demonstrate the applicability of our results, we focus on transportation networks.

The formal analysis and synthesis approaches which are the emphasis of much of this dissertation require a discrete representation of the system in the form of a finite abstraction for the system's behavior. A finite abstraction models the states and dynamics of the system with a finite set of properties and transitions which capture all the phenomena of interest. We first present a broad class of systems which are *mixed monotone*, that is, systems for which the state evolution can be decomposed into an increasing and decreasing component. For such systems, we develop an efficient abstraction algorithm to enable correct-by-design control synthesis. We then use this approach to synthesize control strategies for transportation flow networks.

We next turn our attention to qualitative analysis of the dynamics present in traffic flow networks. We propose a macroscopic network flow model in continuous time suitable for analysis as a dynamical system, and we analyze equilibrium flows as well as convergence. We show that the same mixed monotone property that enables efficient abstraction allows us to prove global asymptotic stability by embedding the flow network in a larger system.

These contributions provide the theoretical foundation for correct-by-design control of a large class of networked physical systems. By focusing on transportation networks, we show concretely how these results apply to a domain that is of considerable practical importance.

Contents

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I first wish to express my deep gratitude to Murat Arcak for advising this PhD dissertation. When I arrived at Berkeley in the summer of 2010, I was eager to get moving in some direction, but I truthfully did not know where to start, let alone in which direction to move. Fortunately, Murat has a remarkable ability to provide the perfect balance of direction, freedom, and advice at each stage of the PhD process. Over time, my eagerness turned into greater independence, and I always had the resources and encouragement to explore my interests and hone my research abilities. I thank Murat for being such a tremendous teacher and mentor.

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Chapter 1

Introduction

Engineered systems increasingly incorporate subsystems that couple physical processes with digital computation. This coupling induces global behavior in the networked system not immediately apparent from the isolated behavior of the constituent components. Control theory has long been the standard-bearer for unraveling this coupling and has provided an astounding body of theoretical and practical tools for designing engineered systems and for understanding natural systems.

In this tradition, we study the verification and synthesis of networked, dynamical systems. This dissertation contributes formal techniques for analyzing the behavior of a large class of physically-motivated systems as well as algorithms for efficient and provably correct control of such systems. While these fundamental contributions are general, this dissertation focuses especially on applications to transportation flow networks. In addition to inspiring much of the theoretical contributions of the dissertation, transportation networks are of independent practical and theoretical interest.

1.1 Control of Networked Systems

For well over half a century, feedback control theory, rooted in rigorous mathematics, has provided steady progress towards mastering evermore complex systems which has enabled technologies such as the electricity grid, process control, telephony, and automatic flight control [AK14]. Furthermore, the field of feedback control has quickly incorporated advances in other domains. For example, the advent of digital communication and computing led to a wealth of advances in control theory and had a multiplicative effect on the field's impact. Many formidable problems that once seemed daunting, including optimal, stochastic, and robust control, gradually (or in some cases, abruptly) yielded to continued research and provided results that were elegant, general, and practical.

Despite these advances, a number of important problems have remained stubbornly difficult to solve. For example, in networked systems composed of interconnected subsystems, it is a natural objective to seek control strategies that are similarly decentralized to improve scalability and implementability. Yet the intractability and nonconvexity of many problems in decentralized control have stymied progress. As another example, many control problems incorporate restrictions on the regime of operation or on the available set of control inputs, and furthermore possess dynamics with significant nonlinearities. Model predictive control provides an appealing general approach to solving such control problems and has seen significant practical use. However, theoretical results remain elusive, the conditions which can be accommodated are relatively limited, and factors such as the curse of dimensionality present significant hurdles.

As a final example, classic feedback control theory is not well-suited for studying systems that experience discrete changes in dynamics or allow discrete choices of control inputs. This particular deficiency has received considerable attention, giving rise first to the field of hybrid systems with the ambitious objective of marrying the discrete world of computer science with the continuous world of control theory. Despite more than a decade of research, a plethora of rigorous definitions, and a number of notable advances, challenges and questions abound. It remains remarkably difficult to predict or control the behavior of systems for which the dynamics may experience discrete mode changes, are subject to discrete inputs and outputs, or interact with discrete entities such as software programs. Indeed, it has become apparent that a primary source of difficulty for such systems is the tight integration of discrete digital components such as software with the continuous physical world, and thus these systems are often now called *cyber-physical systems*.

At the same time that these systems have become more complex, the expectation for their behavior has become more demanding. The costs of incorrect configuration, as well as safety and security concerns, require automated and provably correct techniques for verification and synthesis. Furthermore, the expected behavior of such systems often extends well beyond standard objectives such as stability or invariance. For example, an autonomous robotic system may be required to perform a complex choreography of tasks in a provably correct manner. This requirement has further driven the need for bringing ideas from theoretical computer science to control systems. In particular, formal methods that have been developed for understanding and designing software and hardware systems have recently found considerable application to hybrid and cyber-physical systems.

Specifications for the behavior of networked systems often include requiring that the system eventually reach a certain condition (liveness), avoid certain unsafe conditions (invariance), infinitely often exhibit certain behavior (fairness), or exhibit certain behavior only after achieving a prior condition (sequentiality). Such specifications are easily expressed in various temporal logics, and formal methods exist for the verification and synthesis of control strategies that satisfy these temporal logic specifications. While temporal logic allows rich control specifications, the formal analysis and synthesis techniques require a discrete representation of the system in the form of a finite abstraction for the system's behavior. A finite abstraction models the states and dynamics of the system with a finite set of properties and transitions which capture all the phenomena of interest, but difficulties in computing finite abstractions often prevent scalability to new applications.

Much of this thesis focuses on exploiting structural properties in the dynamics and interconnection of networked systems to alleviate the computationally daunting task of creating symbolic models. Such structural properties result from physical limitations of the systems of interest and enable scalable control synthesis techniques. To demonstrate the practicability of these techniques, and as a motivation for much of the developed theory, we focus on applying the tools developed in this thesis to transportation networks.

1.2 Transportation Networks

Transportation is a quintessential cyber-physical system, coupling the physical movement of vehicles along physical infrastructure with digital communication and control among vehicles and the infrastructure. Furthermore, inefficient traffic management leads to congestion, the costs of which have increased five-fold in the past three decades to $120 billion annually and include 5.5 billion hours of additional travel time and 2.9 billion gallons of wasted fuel [LSE12]. Tomorrow's smarter cities will require intelligent transportation systems that mitigate these problems, and the next generation of transportation systems will include connected vehicles, connected infrastructure, and increased automation. In addition, these advances must coexist with legacy technology into the foreseeable future. This complexity makes the goal of improved mobility and safety ever more daunting.

Of particular relevance to the results of this thesis, control of networks of signalized intersections has received considerable attention in recent decades [Pap+03]. Many of the existing strategies are fixed-time and thus cannot accomodate changes in network demand, do not consider coordination of adjacent intersections, or are only applicable when the network demand is low. Similarly, for freeway networks, ramp metering strategies that control the rate of flow of vehicles from onramps onto the freeway are used to achieve objectives such as decreased travel time or decreased congestion. Use of ramp metering is now ubiquitous in many US cities, but, despite such prevalence, many existing control strategies do not consider coordination of multiple onramps, are designed for basic objectives such as set-point tracking, or fail to prevent onramp spillback which can cause significant congestion on adjacent arterial networks.

The theoretical results of this thesis are primarily applied to control of traffic networks to enable correct-by-design control synthesis, for which temporal logic is well-suited. For example, the objective may be to develop a ramp metering and signal coordination strategy that avoids congestion on the freeway but prevents onramp queues from blocking adjacent arterial traffic.

1.3 Preview of the Thesis

The objective of this dissertation is to develop scalable and automated verification and synthesis techniques for networked systems with particular emphasis on applying these techniques to transportation networks. In Chapter 2, we introduce the notion of *mixed mono-*

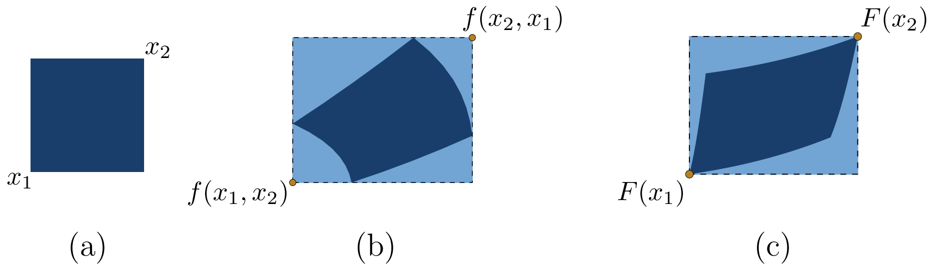


Figure 1.1: Mixed monotonity enables efficient overapproximation of the one-step reachable set from **(a)** a rectangular set defined by x_1 and x_2. In particular, **(b)** the box defined by $f(x_1, x_2)$ and $f(x_2, x_1)$ overapproximates the reachable set, depicted in dark blue. **(c)** For monotone systems, we recover the special case for which this reachable set overapproximated by the box defined by $F(x_1)$ and $F(x_2)$.

tonicity for dynamical systems and study how this property leads to efficient finite state abstraction. In Chapter 3, we leverage this result and propose a methodology for automatically synthesizing traffic control strategies such that the resulting traffic dynamics satisfy a control objective expressed in *linear temporal logic* which allows rich control specifications. In Chapter 4, we qualitatively study the dynamical properties of traffic flow networks, and in Chapter 5 we specifically focus on the mixed monotone properties of traffic flow.

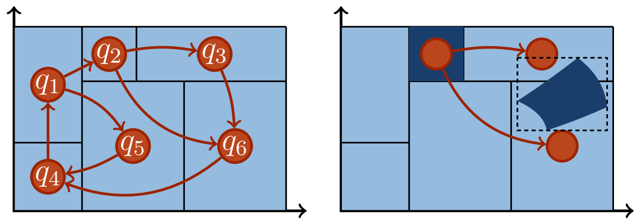
Efficient Finite Abstraction of Mixed Monotone Systems

In Chapter 2, we present an efficient computational procedure for finite abstraction of discrete-time *mixed monotone* systems. The discrete-time dynamical system

$$x^+ = F(x) \tag{1.1}$$

with state x is *monotone* if $x_1 \leq x_2$ implies $F(x_1) \leq F(x_2)$ where $\leq$ is the standard order in Euclidean space, that is, F is nondecreasing with respect to its argument. Such systems have received extensive attention in the controls community [Hir85; Smi95; AS03]. In this chapter, we extend this notion and consider *mixed monotone* systems characterized by the decomposition of the update map F into an increasing and decreasing component. Specifically, (1.1) is *mixed monotone* if there exists a *decomposition function* $f(x, y)$ such that $f(x, x) = F(x)$ and f is nondecreasing in its first argument and nonincreasing in its second argument. Many physical systems are found to be mixed monotone, including traffic flow networks, which is fully explored in Chapters 3 and 5.

For mixed monotone systems, we tightly overapproximate the one-step reachable set from a box of initial conditions by evaluating the decomposition function at only two points, regardless of the dimension of the state space. The approach is illustrated in Fig. 1.1. Consider two points x_1 and x_2 and the rectangular set $\{x \mid x_1 \leq x \leq x_2\}$ as shown in Fig. 1.1(a) for which we wish to overapproximate the one-step reachable set $\{F(x) \mid x_1 \leq x \leq x_2\}$.



(a) (b)

Figure 1.2: Schematic representation of a finite state abstraction for a mixed monotone system. **(a)** A rectangular partition of the state space maps a set of states in the continuous domain to a single state in the abstraction. **(b)** Efficient overapproximation of reachables sets enables efficient computation of the transitions in the abstraction.

The mixed monotone properties of F implies

$$\{F(x) \mid x_1 \leq x \leq x_2\} \subseteq \{x' \mid f(x_1, x_2) \leq x' \leq f(x_2, x_1)\}. \tag{1.2}$$

This result recovers the known special case for monotone systems, for which $\{F(x) \mid x_1 \leq x \leq x_2\} \subseteq \{x' \mid F(x_1) \leq x' \leq F(x_2)\}$. These two cases are illustrated in Fig. 1.1(b) and Fig. 1.1(c).

Efficient one-step reachable set computation enables efficient finite state abstraction of mixed monotone systems. A finite state abstraction, or simply abstraction, is a finite representation of the system's behavior. An abstraction is obtained by partitioning the state-space into a finite number of regions, and then possible transitions between regions are obtained *via* reachability analysis to obtain a finite state transition system. For mixed monotone systems, the reachability approximations described above suggest a rectangular partition of the state space as schematically depicted in Fig. 1.2(a).

From the finite state abstraction, the behavior of the original system can be verified to satisfy desired specifications such as avoidance of unsafe operating conditions using automated techniques. Furthermore, these ideas are naturally generalized to systems with multiple operating modes for which a particular mode may be chosen at each time-step as a controlled input. From the finite state abstraction, a control strategy can then be automatically synthesized to ensure that the closed-loop system satisfies a specification on its behavior. We apply these results to verify the dynamical behavior of a model for insect population dynamics and to synthesize a signaling strategy for a traffic network. The results of Chapter 2 appear in [CA15b].

Control of Traffic Networks from Linear Temporal Logic Specifications

In Chapter 3, we propose a framework for generating a signal control policy for a traffic network of signalized intersections to accomplish complex control objectives expressible using

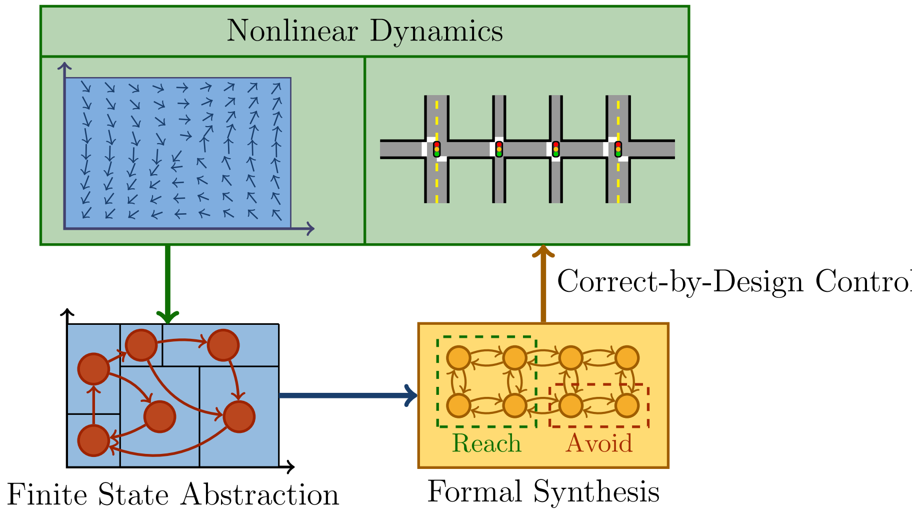


Figure 1.3: Scalable, correct-by-design control of networked transportation systems. A formal methods approach is capable of, *e.g.*, addressing the complexity of coordinating ramp metering with traffic signals to increase mobility and safety on the freeway without causing gridlock on adjacent arterials. The inherent mixed monotonicity of traffic flow dynamics enables efficient computation.

linear temporal logic, an extension of Boolean logic that incorporates temporal modalities. We first model a traffic network as a set of *links* $\mathcal{L}$ interconnected via a set of signalized intersections or *nodes* $\mathcal{V}$. The state of link $\ell \in \mathcal{L}$ at discrete time t is denoted by

$$x_\ell[t] \in [0, x_\ell^{\text{cap}}] \quad \forall \ell \in \mathcal{L} \tag{1.3}$$

and represents the number of vehicles occupying link ℓ where x_ℓ^{cap} is the maximum number of vehicles accommodated by link ℓ. Traffic flow from link to link is restricted by the *demand* of vehicles to flow along a link as well as the *supply* of road capacity downstream. Each link $\ell \in \mathcal{L}$ possesses a *demand* function $\Phi_\ell^{\text{out}}(\cdot)$ and a *supply* function $\Phi_\ell^{\text{in}}(\cdot)$ given by:

$$\Phi_\ell^{\text{out}}(x_\ell[t]) = \min\{x_\ell[t], q_\ell^{\max}\} \qquad \Phi_\ell^{\text{in}}(x_\ell[t]) = (x_\ell^{\text{cap}} - x_\ell[t]) \tag{1.4}$$

for constants $q_\ell^{\max} > 0$.

To accomodate junctions with multiple outputs, we introduce fixed *turn ratios* which dictate how the flow out of an incoming link is divided among the outgoing links. Similarly, to accomodate junctions with multiple inputs, fixed *supply ratios* determine the fraction of supply of an outgoing link available to each incoming link. Specifically,

- $\beta_{\ell k}$ is the fraction of the total flow exiting link ℓ that is routed to link k, and

- $\alpha_{\ell k}$ is the fraction of supply of link k available to link ℓ.

Then, the flow of vehicles exiting a link is the minimum of demand and available supply:

$$f_\ell^{\text{out}}(x[t]) \triangleq s_\ell[t] \cdot \min \left\{ \Phi_\ell^{\text{out}}(x_\ell[t]), \min_{\substack{k \text{ s.t.} \\ \beta_{\ell k} \neq 0}} \left\{ \frac{\alpha_{\ell k}}{\beta_{\ell k}} \Phi_k^{\text{in}}(x_k[t]) \right\} \right\} \qquad \forall \ell \in \mathcal{L} \tag{1.5}$$

where $s_\ell[t] \in \{0,1\}$ is a binary variable indicating whether or not link ℓ is *actuated*, that is, whether flow from link ℓ is allowed. For signalized networks, $s_\ell[t]$ is the control input determined by traffic light signals. Mass conservation completes the model:

$$x_\ell[t+1] = x_\ell[t] + d_\ell[t] - f_\ell^{\text{out}}(x[t]) + \sum_{j \in \mathcal{L}} \beta_{j\ell} f_j^{\text{out}}(x[t]) \tag{1.6}$$

where $d_\ell[t]$ is the exogenous flow of vehicles onto link ℓ and the summation term is the flow of vehicles onto link ℓ from upstream links.

Using this model, we develop a control synthesis approach for determining the traffic signaling scheme such that the resulting closed loop network satisfies rich control objectives expressed using *linear temporal logic (LTL)* [Pnu77; CGP99; BK08]. LTL formulae are defined inductively over a set of observations or *propositions* and the standard logical connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\implies$ (implication), and temporal operators such as $\bigcirc$ (next), U (until), $\Box$ (always), $\Diamond$ (eventually). LTL formulae are interpreted over infinite sequences of propositions.

For example, consider the specification "Eventually, link ℓ will have less than C vehicles and this will remain true for all time thereafter." This objective is written in LTL syntax as follows:

$$\varphi_1 = \Diamond \Box (x_\ell \leq C). \tag{1.7}$$

By applying techniques from model checking and formal methods, we obtain a correct-by-design controller from the finite state abstraction. We then refine this controller to the original traffic network so that the traffic dynamics are guaranteed to satisfy the given LTL specification. The results of this chapter appear in [Coo+15b].

A Compartmental Model for Traffic Flow Networks

In Chapter 4, we propose a macroscopic traffic network flow model in continuous time suitable for analysis as a dynamical system. This model builds on the queueing model presented in Chapter 3 and is rooted in the Cell Transmission Model (CTM) for freeway traffic flow. Few works have investigated the qualitative properties of the CTM as a dynamical system, despite the prevalence of the CTM as a simulation and modeling tool. The results of this chapter address this deficiency, and we qualitatively analyze equilibrium flows as well as convergence of our general network flow model.

As in Chapter 3, flows at a junction are determined by downstream *supply* of capacity, now only assumed to be a decreasing function of density, as well as upstream *demand* of traffic wishing to flow through the junction, an increasing function of density. Furthermore, we pose the *proportional priority, first-in-first-out* (PP/FIFO) rule for resolving the flows through multi-input and multi-output junctions. This rule is a modification of the rule proposed in Chapter 3 and is well suited for freeway traffic. According to the PP/FIFO rule, the incoming flow to any downstream link at a junction is the largest possible such that downstream capacity is not exceeded, upstream demand is not exceeded, and the collection of outgoing flows of upstream links is proportional to the collection of demands for these links.

The PP/FIFO rule imbues the traffic dynamics with certain structural properties that allow us to analyze stability and convergence. A particularly important property is that the PP/FIFO rules precludes the network from being monotone in general; this is in contrast to many other types of dynamic flow networks which often do exhibit monotone structure such as water flowing through pipes. We show that the lack of monotonicity is in fact a useful feature that allows traffic control methods, such as ramp metering, to be effective, and we develop a linear program for optimal ramp metering.

Using contraction theory [LS98; Son10], we further show that, for certain network topologies, a norm-based Lyapunov function exists. In particular, we show that the weighted one-norm of the vector field serves as a Lyapunov function for the traffic network. The one-norm plays an important role in the analysis of mass conserving systems such as traffic networks. Inspired by these results, in Section 4.B, we present an independent result on the use of norm-based Lyapunov functions *via* contraction analysis. The results of Chapter 3 appear in [CA15a].

Mixed Monotonicity in Traffic Flow Dynamics

Chapter 5 continues analysis of the macroscopic traffic network flow model proposed in Chapter 4, focusing on the mixed monotone properties of the dynamics. The continuous-time analogue of mixed monotonicity as defined in Chapter 2 is presented, and it is shown that, under a mild restriction, the traffic flow model studied in Chapter 4 is mixed monotone.

A continuous-time system $\dot{x} = G(x)$ with G differentiable is mixed monotone if there exists a decomposition function $g(x, y)$ such that $g(x, x) = G(x)$, $\partial g_i/\partial x_j \geq 0$ for all $i \neq j$, and $\partial g_i/\partial y_j \leq 0$ for all i, j. Given a mixed monotone system, we then study the symmetric system

$$\dot{x} = g(x, y) \tag{1.8}$$

$$\dot{y} = g(y, x), \tag{1.9}$$

which is seen to be monotone with respect to the *southeast* orthant order. Furthermore, the dynamical behavior of the original system $\dot{x} = G(x)$ is preserved along the invariant subspace $\{(x, y) \mid x = y\}$ for which we have $\dot{x} = \dot{y} = g(x, x) = g(y, y) = G(x) = G(y)$. Monotonicity of this symmetric system allows us to apply results from monotone system theory which, coupled with the structure induced by the symmetry in (1.8)–(1.9), are sufficient for proving global asymptotic stability.

The primary connecting thread through these chapters is the notion that networked, physical systems such as traffic networks possess an abundance of intrinsic structure which this thesis uncovers and uses for efficient analysis and design. Finding and exploiting such structure in systems of physical relevance is key for obtaining scalable and tractable solutions to many of the challenges in control of networked systems.

Chapter 2

Efficient Finite Abstraction of Mixed Monotone Systems

Networked systems often possess intrinsic structure that significantly simplifies analysis and control. An important class of systems exhibiting such structure is *monotone systems* for which trajectories maintain a partial ordering on states [Hir85; Smi95]. The notion of monotonicity is applicable to both continuous-time systems [Smi95] and discrete-time systems [HS05b], and has been extended to control systems with inputs in [AS03].

References [GH94; Smi06; Smi08; ESS06] have observed that dynamics which are not monotone may nonetheless be decomposable into increasing and decreasing components. Such systems are called *mixed monotone* and significantly generalize the class of monotone systems. Unlike the references above which exploit mixed monotonicity for stability analysis, here we demonstrate that mixed monotonicity enables efficient finite state abstraction.

Increased interest in verification and synthesis of *cyber-physical* systems has motivated symbolic models that abstract the underlying system into a finite set of symbols and transitions between symbols which reflect the dynamics [Alu+00; TK02; Tab09; LS11]. The main reason for obtaining finite state abstractions is to allow formal verification and synthesis for specifications given in, *e.g.*, temporal logic [WTM12; Liu+13; CGP99; BK08].

In rare cases, exact symbolic models exactly capture the underlying dynamics [Hen+98; TP06]. In other cases, exact symbolic models are either impossible to obtain or computationally prohibitive, however it is still useful to obtain an abstraction which approximately captures the underlying dynamics [Liu+13; GP07; GPT10; Rei11]. For example, in [Yor+12; ADA13], the authors consider piecewise affine (PWA) systems and construct a finite state abstraction using polyhedral computations.

In this work, we compute finite state abstractions of mixed monotone, discrete-time systems by considering a rectangular partition of the state space. In particular, we show that the reachable set from a box of initial conditions is efficiently overapproximated by evaluating a decomposition function, obtained from the mixed monotone system, at only two points. We accommodate disturbance inputs in the dynamics by suitably generalizing the definition of a mixed monotone system in [Smi06]. Furthermore, we characterize a special class of mixed monotone systems in which the dynamics are componentwise monotone and show that our overapproximation is tight in a particular sense to be made precise. Additionally, we suggest an efficient algorithm for identifying a class of spurious trajectories from the abstraction.

The importance of monotonicity for reachability computation and abstraction has been noted in [MR02; GR06; RMC10]. In particular, the authors of [MR02] study discrete-time systems that are monotone with respect to the positive orthant in Euclidean space and show that the reachable set from a box of initial conditions is overapproximated by propagating only the least and greatest points within this box. This chapter studies the much broader class of mixed monotone systems and recovers [MR02] as a special case.

In Section 2.1, we introduce the notation. In Section 2.2, we pose the general problem statement and introduce mixed monotone systems. In Section 2.3, we present an algorithm for efficiently constructing finite state abstractions of mixed monotone systems. In the case studies of Section 2.4, we analyze a model for insect population dynamics and synthesize a signal controller for a traffic network.

2.1 Preliminaries

For $x \in \mathbb{R}^n$, we use superscripts to index the elements of x, *i.e.*, x^i is the ith component of x and $x = (x^1, \dots, x^n)$, except in the case studies of Section 2.4 where we use subscripts for clarity. Let $\mathbb{R}_{\geq 0} = \{x \mid x \geq 0\}$ and $\mathbb{R}^n_{\geq 0} = (\mathbb{R}_{\geq 0})^n$. For a set $\mathcal{Z} \subset \mathbb{R}^n$, $\text{int}(\mathcal{Z})$ denotes the interior of $\mathcal{Z}$.

Consider a set $\mathcal{X} \subset \mathbb{R}^n$ along with a *positive cone* $\mathcal{Y}_+ \subset \mathbb{R}^n$ satisfying $\alpha\mathcal{Y}_+ \subset \mathcal{Y}_+$ for all $\alpha \in \mathbb{R}_{\geq 0}$, $\mathcal{Y}_+ + \mathcal{Y}_+ \subset \mathcal{Y}_+$, and $\mathcal{Y}_+ \cap (-\mathcal{Y}_+) = 0$. The positive cone $\mathcal{Y}_+$ induces an order relation $\leq$ on $\mathcal{X}$ defined by: $x \leq y$ if and only if $y - x \in \mathcal{Y}_+$ for $x, y \in \mathcal{X}$. Given $x, y \in \mathcal{X}$ with $x \leq y$, we define the *interval*

$$[x, y] \triangleq \{z \in \mathcal{X} \mid x \leq z \leq y\}. \tag{2.1}$$

For $\mathcal{Y}_+ = \mathbb{R}^n_{\geq 0}$, $\leq$ denotes coordinate-wise inequality; we distinguish this partial order by $\leq_+$ and generalize it to arbitrary orthants in the following way: Let $\nu = (\nu_1, \dots, \nu_n)$ with $\nu_i \in \{0, 1\}$ for all i, and define $K_\nu = \{x \in \mathbb{R}^n \mid (-1)^{\nu_i} x^i \geq 0 \ \forall i\}$. K_ν is a cone corresponding to an orthant of $\mathbb{R}^n$, and we denote the induced *orthant order* by $\leq_{K_\nu}$.

For a matrix $M \in \mathbb{R}^{n \times p}$, we additionally interpet $0 \leq_+ M$ to mean M is elementwise nonnegative.

A set $\mathcal{Z} \subset \mathbb{R}^n$ is said to be a *box* if it is the Cartesian product of closed intervals of $\mathbb{R}$, that is, if there exists $a_i, b_i \in \mathbb{R}$ for $i = 1, \dots n$ such that $a_i \leq b_i$ and $\mathcal{Z} = \prod_{i=1}^n [a_i, b_i]_{\mathbb{R}_{\geq 0}}$ where $[\cdot, \cdot]_{\mathbb{R}_{\geq 0}}$ denotes the usual interval on $\mathbb{R}$.

We let $x^+ = F(x, d)$ describe a discrete-time dynamical system where the state x^+ at the next time step is a function of the current state x and a disturbance input d. We denote the ith coordinate mapping of F by F^i, that is, $(x^i)^+ = F^i(x, d)$ and $F(x, d) = (F^1(x, d), \dots, F^n(x, d))$.

2.2 Mixed Monotone Systems

2.2.1 Problem Statement

We first consider discrete-time dynamical systems of the form

$$x^+ = F(x, d) \tag{2.2}$$

with state $x \in \mathcal{X} \subset \mathbb{R}^n$, disturbance input $d \in \mathcal{D} \subset \mathbb{R}^p$, and a continuous map $F : \mathcal{X} \times \mathcal{D} \to \mathcal{X}$. We present a technique for efficiently computing a finite state abstraction of (2.2) when F is *mixed monotone* as defined below. The resulting symbolic model is amenable to standard formal methods techniques to verify desirable properties, as demonstrated in the case study in Section 2.4.1.

Next, we consider the problem of controlling the switched discrete-time dynamical system

$$x^+ = F_m(x, d) \tag{2.3}$$

for $m \in \mathcal{M}$ where $\mathcal{M}$ is a finite set of modes and each $F_m : \mathcal{X} \times \mathcal{D} \to \mathcal{X}$ is continuous. For switched systems of the form (2.3), the control input is the mode m at each time step. When each F_m satisfies a mixed monotonicity property, we propose an efficient algorithm for obtaining a finite state abstraction. As demonstrated in the case study of Section 2.4.2, this abstraction is amenable to synthesis algorithms to meet complex control objectives expressible in, *e.g.*, *Linear Temporal Logic* (LTL).

2.2.2 Basic Definitions and Results

For systems of the form (2.2), we let $\leq_\mathcal{X}$ and $\leq_\mathcal{D}$ denote order relations on $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^p$, respectively, induced by positive cones. The notation $[\cdot,\cdot]_\mathcal{X}$ (resp. $[\cdot,\cdot]_\mathcal{D}$) denotes an interval with respect to $\leq_\mathcal{X}$ (resp. $\leq_\mathcal{D}$). For systems of the form (2.3), we wish to allow potentially different order relations on $\mathcal{X}$, and thus consider a set $\{\leq_m\}_{m\in\mathcal{M}}$ of order relations on $\mathcal{X}$ and $\leq_\mathcal{D}$, a fixed order relation on $\mathcal{D}$. The notation $[\cdot,\cdot]_m$ denotes an interval with respect to $\leq_m$. For notational convenience, we assume that the same partial order on $\mathcal{D}$ holds for all modes, however different partial orders on $\mathcal{D}$ for each mode are possible with suitable alterations to the development below.

We begin with the well-known class of monotone dynamical systems:

Definition 2.2.1 (Monotonicity)**.** *The system* (2.2) *is* monotone with respect to $\leq_\mathcal{X}$ and $\leq_\mathcal{D}$*, or simply* monotone*, if*

$$x_1 \leq_\mathcal{X} x_2 \text{ and } d_1 \leq_\mathcal{D} d_2 \implies F(x_1, d_1) \leq_\mathcal{X} F(x_2, d_2). \tag{2.4}$$

We say that the switched system (2.3) *is* monotone with respect to $\{\leq_m\}_{m\in\mathcal{M}}$ and $\leq_\mathcal{D}$*, or simply* monotone*, if each mode m is monotone with respect to $\leq_m$ and $\leq_\mathcal{D}$.*

We next provide a significant generalization of Definition 2.2.1:

Definition 2.2.2 (Mixed monotonicity)**.** *The system* (2.2) *is said to be* mixed monotone with respect to $\leq_{\mathcal{X}}$ and $\leq_{\mathcal{D}}$*, or simply* mixed monotone *[Smi06], if there exists a function* $f : \mathcal{X} \times \mathcal{D} \times \mathcal{X} \times \mathcal{D} \to \mathcal{X}$ *satisfying:*

C1) $\forall x \in \mathcal{X},\ \forall d \in \mathcal{D}$*:* $F(x, d) = f(x, d, x, d)$

C2) $\forall x_1, x_2, y \in \mathcal{X},\ \forall d_1, d_2, e \in \mathcal{D}$*:* $x_1 \leq_{\mathcal{X}} x_2$ *and* $d_1 \leq_{\mathcal{D}} d_2$ *implies* $f(x_1, d_1, y, e) \leq_{\mathcal{X}} f(x_2, d_2, y, e)$

C3) $\forall x, y_1, y_2 \in \mathcal{X},\ \forall d, e_1, e_2 \in \mathcal{D}$*:* $y_1 \leq_{\mathcal{X}} y_2$ *and* $e_1 \leq_{\mathcal{D}} e_2$ *implies* $f(x, d, y_2, e_2) \leq_{\mathcal{X}} f(x, d, y_1, e_1)$*.*

We say that the switched system (2.3) *is* mixed monotone with respect to $\{\leq_m\}_{m\in\mathcal{M}}$ and $\leq_{\mathcal{D}}$*, or simply* mixed monotone*, if each mode* $x^+ = F_m(x, d)$ *is mixed monotone with respect to* $\leq_m$*.*

The function f is nondecreasing in the first pair of variables and nonincreasing in the second pair of variables, and is henceforth called a *decomposition function*:

Definition 2.2.3 (Decomposition function)**.** *A function* f *satisfying C1)–C3) above is a* decomposition function *for* $F(x, d)$*.*

Clearly every monotone system is mixed monotone with $f(x, d, y, e) \triangleq F(x, d)$. In the case of a switched system (2.3), we denote by f_m a corresponding decomposition function for each mode $m \in \mathcal{M}$.

Example 2.2.1. *Consider the system*

$$x^+ = G(x, d) - H(x, d) \tag{2.5}$$

for $x \in \mathcal{X} \subset \mathbb{R}^n$*,* $d \in \mathcal{D} \subset \mathbb{R}^p$*, and* $G, H : \mathcal{X} \times \mathcal{D} \to \mathcal{X}$ *such that* $x^+ = G(x, d)$ *and* $x^+ = H(x, d)$ *are monotone systems for* $\leq_{\mathcal{X}}=\leq_+$ *and* $\leq_{\mathcal{D}}=\leq_+$*. Then* (2.5) *is mixed monotone for* $\leq_{\mathcal{X}}=\leq_+$ *and* $\leq_{\mathcal{D}}=\leq_+$ *and* $f(x, d, y, e) \triangleq G(x, d) - H(y, e)$ *is a decomposition function.*

Example 2.2.2. *Consider the system*

$$x^+ = A(x, d)x + B(x, d)d =: F(x, d) \tag{2.6}$$

for $x \in \mathcal{X} \subset \mathbb{R}^n_{\geq 0}$*,* $d \in \mathcal{D} \subset \mathbb{R}^p_{\geq 0}$*, such that:*

- $0 \leq_+ A(x, d)$ *and* $0 \leq_+ B(x, d)$ *for all* $x \in \mathcal{X}$ *for all* $d \in \mathcal{D}$*,*
- $x_1 \leq_+ x_2$ *and* $d_1 \leq_+ d_2 \implies A(x_2, d_2) \leq_+ A(x_1, d_1)$ *and* $B(x_2, d_2) \leq_+ B(x_1, d_1)$*.*

Equations of the form (2.6) *arise in the study of population dynamics, [Cus98]. Taking* $f(x, d, y, e) = A(y, e)x + B(y, e)d$*, system* (2.6) *is mixed monotone for* $\leq_{\mathcal{X}}=\leq_+$ *and* $\leq_{\mathcal{D}}=\leq_+$*.*

We now characterize a special class of mixed monotone systems in terms of the sign of the entries in $\partial F/\partial x$ and $\partial F/\partial d$, the Jacobians of F with respect to x and d.

Proposition 2.2.1. *Consider the system (2.2) where $x \in \mathcal{X} \subset \mathbb{R}^n$, $d \in \mathcal{D} \subset \mathbb{R}^p$, $\mathcal{X}$ and $\mathcal{D}$ are boxes, and F is continuously differentiable. If for all $i \in \{1, \ldots, n\}$,*

$$\forall j \in \{1, \ldots, n\}\ \exists s_j \in \{0,1\} : (-1)^{s_j} \frac{\partial F^i}{\partial x^j}(x,d) \geq 0\ \forall x, d \tag{2.7}$$

and

$$\forall j \in \{1, \ldots, p\}\ \exists \sigma_j \in \{0,1\} : (-1)^{\sigma_j} \frac{\partial F^i}{\partial d^j}(x,d) \geq 0\ \forall x, d \tag{2.8}$$

then (2.2) is mixed monotone with respect to any orthant order on $\mathcal{X}$ and $\mathcal{D}$.

Proof. Let $\nu \in \{0,1\}^n$ and $\mu \in \{0,1\}^p$ characterize arbitrary orthant orders $\leq_{K_\nu}$ and $\leq_{K_\mu}$ on $\mathcal{X}$ and $\mathcal{D}$, respectively. Define

$$f^i(x,d,y,e) \triangleq F^i(z^i, w^i) \tag{2.9}$$

where $z^i = (z^{i,1}, \ldots, z^{i,n})$, $w^i = (w^{i,1}, \ldots, w^{i,p})$, and

$$z^{i,j} \triangleq \begin{cases} x^j & \text{if } (-1)^{\nu_i+\nu_j} \partial F^i/\partial x^j \geq 0\ \forall x \in \mathcal{X}, d \in \mathcal{D} \\ y^j & \text{if } (-1)^{\nu_i+\nu_j} \partial F^i/\partial x^j \leq 0\ \forall x \in \mathcal{X}, d \in \mathcal{D} \end{cases} \tag{2.10}$$

$$w^{i,j} \triangleq \begin{cases} d^j & \text{if } (-1)^{\nu_i+\mu_j} \partial F^i/\partial d^j \geq 0\ \forall x \in \mathcal{X}, d \in \mathcal{D} \\ e^j & \text{if } (-1)^{\nu_i+\mu_j} \partial F^i/\partial d^j \leq 0\ \forall x \in \mathcal{X}, d \in \mathcal{D}. \end{cases} \tag{2.11}$$

If $\partial F^i/\partial x^j = 0\ \forall x, d$ for some i, j, then the assignment to $z^{i,j}$ is arbitrary, likewise for $w^{i,j}$ if $\partial F^i/\partial d^j = 0\ \forall x, d$ for some i, j. Let $f(x,d,y,e) = (f^1(x,d,y,e), \ldots, f^n(x,d,y,e))$. Clearly $f(x,d,x,d) = F(x,d)$, and a modification of the *Kamke* conditions for monotonicity [Smi95, Section 3.1] proves that f satisfies the remaining conditions of Definition 2.2.2. In particular, it follows that:

$$\frac{\partial f^i}{\partial x^j}(x,d,y,e) = \begin{cases} \frac{\partial F^i}{\partial x^j}(z^i, w^i) & \text{if } z^{i,j} = x^j \\ 0 & \text{else} \end{cases} \quad \forall i, j \tag{2.12}$$

$$\frac{\partial f^i}{\partial y^j}(x,d,y,e) = \begin{cases} \frac{\partial F^i}{\partial x^j}(z^i, w^i) & \text{if } z^{i,j} = y^j \\ 0 & \text{else} \end{cases} \quad \forall i, j \tag{2.13}$$

$$\frac{\partial f^i}{\partial d^j}(x,d,y,e) = \begin{cases} \frac{\partial F^i}{\partial d^j}(z^i, w^i) & \text{if } w^{i,j} = d^j \\ 0 & \text{else} \end{cases} \quad \forall i, j \tag{2.14}$$

$$\frac{\partial f^i}{\partial e^j}(x,d,y,e) = \begin{cases} \frac{\partial F^i}{\partial d^j}(z^i, w^i) & \text{if } w^{i,j} = e^j \\ 0 & \text{else.} \end{cases} \quad \forall i, j \tag{2.15}$$

Let $f(x,d,y,e) = \begin{bmatrix} f^1(x,d,y,e) & \dots & f^n(x,d,y,e) \end{bmatrix}^T$. Trivially, $f(x,d,x,d) = F(x,d)$. Consider $x_1 \leq_{K_\nu} x_2$ and $d_1 \leq_{K_\mu} d_2$. We first must show

$$(-1)^{\nu_i}\left(f^i(x_2,d_2,y,e) - f^i(x_1,d_1,y,e)\right) \geq 0 \tag{2.16}$$

for all y, e. By the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
&(-1)^{\nu_i}\left(f^i(x_2,d_1,y,e) - f^i(x_1,d_1,y,e)\right) && (2.17)\\
&= (-1)^{\nu_i}\cdot && (2.18)\\
&\left(\int_0^1 \sum_{j=1}^{n} \frac{\partial f^i}{\partial x^j}(x_1 + r(x_2 - x_1), d_1, y, e)(x_2^j - x_1^j)dr\right) && (2.19)\\
&= (-1)^{\nu_i}\left(\int_0^1 \sum_{j=1}^{n} (-1)^{\nu_j}\frac{\partial f^i}{\partial x^j}(x_1 + r(x_2 - x_1), d_1, y, e)\cdot\right.\\
&\qquad\qquad \left.(-1)^{\nu_j}(x_2^j - x_1^j)dr\right) && (2.20)\\
&\geq 0 && (2.21)
\end{aligned}$$

where (2.21) follows because $(-1)^{\nu_j}(x_2^j - x_1^j) \geq 0$ and

$$(-1)^{\nu_i+\nu_j}\partial f^i/\partial x^j \geq 0 \tag{2.22}$$

by (2.10) and (2.12). Similarly,

$$\begin{aligned}
&(-1)^{\nu_i}\left(f^i(x_2,d_2,y,e) - f^i(x_2,d_1,y,e)\right) && (2.23)\\
&= (-1)^{\nu_i}\cdot && (2.24)\\
&\left(\int_0^1 \sum_{j=1}^{p} \frac{\partial f^i}{\partial d^j}(x_2, d_1 + r(d_2 - d_1), y, e)(d_2^j - d_1^j)dr\right) && (2.25)\\
&\geq 0, && (2.26)
\end{aligned}$$

thus (2.23) holds. A symmetric analysis shows for all i,

$$(-1)^{\nu_i}\left(f^i(x,d,y_2,e_2) - f^i(x,d,y_1,e_1)\right) \leq 0 \tag{2.27}$$

for $y_1 \leq_{K_\nu} y_2$, $e_1 \leq_{K_\mu} e_2$ for all x, d. As ν and μ were arbitrary binary vectors, the proof is concluded.

□

Proposition 2.2.1 states that if the partial derivatives of F are *sign stable* over $\mathcal{X}\times\mathcal{D}$, then (2.2) is mixed monotone with respect to any orthant order on $\mathcal{X}$ and $\mathcal{D}$. The special class characterized in Proposition 2.2.1 plays an important role in the case study of Section 2.4.2; see [KM06] for a similar characterization that excludes disturbance inputs.

Example 2.2.3. *Let* $\mathcal{X} = \mathbb{R}^2_{\geq 0}$*,* $\mathcal{D} = \mathbb{R}^2_{\geq 0}$*, and consider the system*

$$x^+ = F(x, d) = (F^1(x, d), F^2(x, d)) \tag{2.28}$$
$$= (5x_1 - x_2^3 + 5d_1^2, x_1^2 + 3x_2x_1 - 6d_1d_2) \tag{2.29}$$

where $x = (x_1, x_2) \in \mathbb{R}^2_{\geq 0}$ *and* $d = (d_1, d_2) \in \mathbb{R}^2_{\geq 0}$ *(we momentarily abandon our superscript convention for notational convenience). For all* $x \in \mathcal{X}$*,* $d \in \mathcal{D}$*,*

$$\partial F^1/\partial x_1 = 5 \geq 0 \qquad \partial F^1/\partial x_2 = -3x_2^2 \leq 0 \tag{2.30}$$
$$\partial F^2/\partial x_1 = 2x_1 + 3x_2 \geq 0 \qquad \partial F^2/\partial x_2 = 3x_1 \geq 0 \tag{2.31}$$
$$\partial F^1/\partial d_1 = 10d_1 \geq 0 \qquad \partial F^1/\partial d_2 = 0 \tag{2.32}$$
$$\partial F^2/\partial d_1 = -6d_2 \leq 0 \qquad \partial F^2/\partial d_2 = -6d_1 \leq 0. \tag{2.33}$$

Thus, the system is mixed monotone by Proposition 2.2.1. Taking $\leq_\mathcal{X} = \leq_+$ *and* $\leq_\mathcal{D} = \leq_+$*, we have that*

$$f(x, d, y, e) = (5x_1 - y_2^3 + 5d_1^2, x_1^2 + 3x_2x_1 - 6e_1e_2) \tag{2.34}$$

is a decomposition function where $y = (y_1, y_2)$*,* $e = (e_1, e_2)$*.*

Remark 1. *We remark that, while Proposition 2.2.1 assumed that* F *is continuously differentiable, the results in fact hold if* F *is continuous and* piecewise differentiable*, and thus nondifferentiable on a set of measure zero as in the case study of Section 2.4.2.*

2.2.3 Reachable Set Computation

In this section, we show that an overapproximation of the reachable set from a box of initial states is efficiently computed by evaluating the decomposition function at only two points, regardless of the state space dimension. In the next section, we use this result to obtain finite state abstractions of mixed monotone systems.

We begin with the following key theorem:

Theorem 2.2.1. *Let (2.2) be a mixed monotone system with decomposition function* $f(x, d, y, e)$*. Given* $x_1, x_2 \in \mathcal{X}$ *and* $d_1, d_2 \in \mathcal{D}$ *with* $x_1 \leq_\mathcal{X} x_2$ *and* $d_1 \leq_\mathcal{D} d_2$*,*

$$f(x_1, d_1, x_2, d_2) \leq_\mathcal{X} F(x, d) \leq_\mathcal{X} f(x_2, d_2, x_1, d_1)$$
$$\forall\, x \in [x_1, x_2]_\mathcal{X}\ \forall\, d \in [d_1, d_2]_\mathcal{D}. \tag{2.35}$$

Proof. Consider x, d, y, e satisfying

$$x_1 \leq_\mathcal{X} x \text{ and } d_1 \leq_\mathcal{X} d, \text{ and} \tag{2.36}$$
$$y \leq_\mathcal{X} x_2 \text{ and } e \leq_\mathcal{X} d_2. \tag{2.37}$$

It follows that

$$f(x_1, d_1, x_2, d_2) \leq_{\mathcal{X}} f(x, d, y, e), \qquad \textit{and} \tag{2.38}$$
$$f(y, e, x, d) \leq_{\mathcal{X}} f(x_2, d_2, x_1, d_1). \tag{2.39}$$

Restricting to the set $\{(x, d, y, e) \mid x = y \text{ and } d = e\}$, we obtain

$$\begin{aligned} f(x_1, d_1, x_2, d_2) &\leq_{\mathcal{X}} f(x, d, x, d) = F(x, d) \\ &\leq_{\mathcal{X}} f(x_2, d_2, x_1, d_1). \end{aligned} \tag{2.40}$$

□

The analogous result for monotone systems is:

Corollary 2.2.1. *Given $x_1, x_2 \in \mathcal{X}$ and $d_1, d_2 \in \mathcal{D}$ with $x_1 \leq_{\mathcal{X}} x_2$ and $d_1 \leq_{\mathcal{D}} d_2$. If system (2.2) is monotone, then*

$$\begin{aligned} &F(x_1, d_1) \leq_{\mathcal{X}} F(x, d) \leq_{\mathcal{X}} F(x_2, d_2) \\ &\quad \forall\, x \in [x_1, x_2]_{\mathcal{X}}\ \forall\, d \in [d_1, d_2]_{\mathcal{D}}. \end{aligned} \tag{2.41}$$

The result in [MR02] is a special case of Corollary 2.2.1 restricted to systems with no disturbance input and $\leq_{\mathcal{X}} = \leq_+$.

For $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{D}' \subseteq \mathcal{D}$, we define the shorthand notation

$$F(\mathcal{X}', \mathcal{D}') \triangleq \{F(x, d) \mid x \in \mathcal{X}' \text{ and } d \in \mathcal{D}'\}. \tag{2.42}$$

Then we respectively write (2.35) and (2.41) as

$$F([x_1, x_2]_{\mathcal{X}}, [d_1, d_2]_{\mathcal{D}}) \subseteq [f(x_1, d_1, x_2, d_2), f(x_2, d_2, x_1, d_1)]_{\mathcal{X}} \tag{2.43}$$

and

$$F([x_1, x_2]_{\mathcal{X}}, [d_1, d_2]_{\mathcal{D}}) \subseteq [F(x_1, d_1), F(x_2, d_2)]_{\mathcal{X}}. \tag{2.44}$$

Definition 2.2.4. *The set $F(\mathcal{X}', \mathcal{D}')$ given in (2.42) is the* one-step reachable set from $\mathcal{X}'$ and $\mathcal{D}'$*.*

Example 2.2.4. *Consider again Example 2.2.3 and let $\underline{x} = (0.6, 0.3)$, $\bar{x} = (1, 1)$, $\underline{d} = (0, 0)$, $\bar{d} = (0.3, 0.3)$. From Theorem 2.2.1, it follows that*

$$\begin{aligned} F([\underline{x}, \bar{x}]_{\mathcal{X}}, [\underline{d}, \bar{d}]_{\mathcal{D}}) &\subset [f(\underline{x}, \underline{d}, \bar{x}, \bar{d}), f(\bar{x}, \bar{d}, \underline{x}, \underline{d})]_{\mathcal{X}} \\ &= [(2, 0.36), (5.423, 4)]_{\mathcal{X}}. \end{aligned} \tag{2.45}$$

Figure 2.1 shows the set $\{F(x, d) \mid x \in [\underline{x}, \bar{x}]_{\mathcal{X}}, d \in [\underline{d}, \bar{d}]_{\mathcal{D}}\}$ as a shaded region, and plots $f(\underline{x}, \underline{d}, \bar{x}, \bar{d})$ $f(\bar{x}, \bar{d}, \underline{x}, \underline{d})$ as two corners of a box that bounds this set.

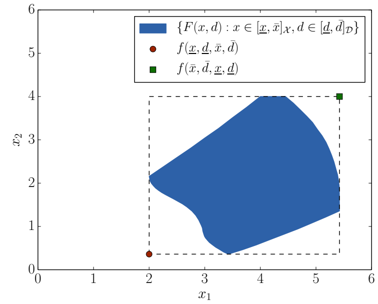


Figure 2.1: The mixed monotone system in Examples 2.2.3 and 2.2.4. This system satisfies the conditions of Theorem 2.2.1, thus we bound $F(x, d)$ when x and d are confined to lie within a given rectangle by evaluating the decomposition function at two points, and the bounding is tight. This example readily generalizes to higher dimensions.

For monotone systems, Corollary 2.2.1 provides tight bounds since the upper and lower bounds are achieved. For mixed monotone systems satisfying (2.7)–(2.8) of Proposition 2.2.1, the bounds given in Theorem 2.2.1 are also tight as suggested in Figure 2.1 for Example 4. We make this precise in the following proposition, which follows immediately from the definition in (2.9):

Proposition 2.2.2. *Suppose* $\leq_{\mathcal{X}} = \leq_{K_\nu}$ *and* $\leq_{\mathcal{D}} = \leq_{K_\mu}$ *for some orthants* K_ν *and* K_μ*. If* (2.2) *is mixed monotone by* (2.7)*–*(2.8) *of Proposition 2.2.1, and* f *is the decomposition function as defined in* (2.9)*–*(2.11)*, then for all* $i \in \{1, \dots, n\}$ *there exists* $\underline{z}^i, \bar{z}^i \in [x_1, x_2]_{\mathcal{X}}$ *and* $\underline{w}^i, \bar{w}^i \in [d_1, d_2]_{\mathcal{D}}$ *such that*

$$f^i(x_1, d_1, x_2, d_2) = F^i(\underline{z}^i, \underline{w}^i), \quad \textit{and} \tag{2.46}$$

$$f^i(x_2, d_2, x_1, d_1) = F^i(\bar{z}^i, \bar{w}^i). \tag{2.47}$$

In particular, $\underline{z}^i$ as in (2.10) with $x = x_1$ and $y = x_2$, and $\underline{w}^i$ as in (2.11) with $d = d_1$ and $e = d_2$ satisfies (2.46). A symmetric results holds for (2.47) after interchanging x_1, x_2 and d_1, d_2.

2.3 Abstraction of Mixed Monotone Systems

We have seen that for mixed monotone systems, an overapproximation of the one-step reachable set from the set $[x_1, x_2]_{\mathcal{X}}$ under a disturbance input from the set $[d_1, d_2]_{\mathcal{D}}$ can be computed by evaluating the decomposition function f at only two particular points. We now exploit Theorem 2.2.1 and Corollary 2.2.1 and present an efficient algorithm for computing a *finite state abstraction* of a mixed monotone system. For systems of the form (2.2), we wish to *verify* that a certain property, usually given in a temporal logic, holds under all possible disturbance inputs. For switched systems of the form (2.3), we wish to *synthesize* a feedback controller that, at each time step, selects a mode $m \in \mathcal{M}$ such that the resulting system satisfies a given property.

2.3.1 Finite State Abstraction

Now we introduce a partition of the domain $\mathcal{X}$ by intervals and construct a finite state abstraction from the partition. We discuss systems of the form (2.3), since (2.2) is a special case.

Assume system (2.3) is mixed monotone with respect to $\{\leq_m\}_{m\in\mathcal{M}}$ and $\leq_\mathcal{D}$. Furthermore, assume $\mathcal{D}$ is representable as the union of intervals:

$$\mathcal{D} = \bigcup_{\ell=1}^{L} \mathcal{D}^\ell \tag{2.48}$$

where $\mathcal{D}^\ell \triangleq [d_1^\ell, d_2^\ell]_\mathcal{D}$ for $d_1^\ell \leq_\mathcal{D} d_2^\ell$.

Definition 2.3.1 (Interval Partition)**.** *The collection $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$ for finite set $\mathcal{Q}$ with $\mathcal{I}_q \subseteq \mathcal{X}$ for all $q \in \mathcal{Q}$ is an* interval partition *of $\mathcal{X}$ if:*

1. *For all $m \in \mathcal{M}$ and for all $q \in \mathcal{Q}$, there exists $x_1^{q,m}, x_2^{q,m} \in \mathcal{X}$ satisfying $x_1^{q,m} \leq_m x_2^{q,m}$ and $\mathcal{I}_q = [x_1^{q,m}, x_2^{q,m}]_m$,*
2. *$\bigcup_{q\in\mathcal{Q}} \mathcal{I}_q = \mathcal{X}$,*
3. *$int(\mathcal{I}_q) \cap int(\mathcal{I}_{q'}) = \emptyset$ for all $q, q' \in \mathcal{Q}, q \neq q'$.*

In other words, $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$ is an *interval partition* of $\mathcal{X}$ if the sets $\mathcal{I}_q$, $q \in \mathcal{Q}$ partition $\mathcal{X}$ and each $\mathcal{I}_q$ is representable as an interval of $\mathcal{X}$ with respect to each order $\leq_m$. In defining a partition, we ignore the set of measure zero where intervals overlap for notational convenience, as is done in, *e.g.*, [Yor+12]. However, as noted in [Yor+12] and [ADA13], if the dynamics are such that trajectories remain within the boundaries after a certain time, one should account for such sets.

For example, if each $\leq_m$ is an orthant order, then a partition $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$ with each $\mathcal{I}_q$ a box constitutes an interval partition of $\mathcal{X} \subset \mathbb{R}^n$. For this special case, we further call the partition a *gridded partition* if for each $i \in \{1, \dots, n\}$ there exists $N_i > 0$ and $\{\xi_{i,1}, \dots, \xi_{i,N_i+1}\}$ such that $\mathcal{Q} = \prod_{i=1}^n \{1, \dots, N_i\}$ and for each $q = (\iota_1, \dots, \iota_n) \in \mathcal{Q}$, we have $\mathcal{I}_q = \prod_{i=1}^n [\xi_{i,\iota_i}, \xi_{i,\iota_i+1}]_{\mathbb{R}_{\geq 0}}$. Figure 2.2 shows schematic depictions of two interval partitions, one of which is a gridded partition.

When clear from context, we refer to the index set $\mathcal{Q}$ itself as an interval partition with the associated notation as above. From such a partition, we readily construct a finite state abstraction of the resulting dynamics.

Consider a map $\delta : \mathcal{Q} \times \mathcal{M} \to 2^\mathcal{Q}$ that satisfies the following property:

$$\begin{aligned} &\text{If } \exists x \in \mathcal{I}_q,\ \exists d \in \mathcal{D} \ \text{ such that } F_m(x, d) \in \mathcal{I}_{q'} \\ &\text{Then } q' \in \delta(q, m). \end{aligned} \tag{2.49}$$

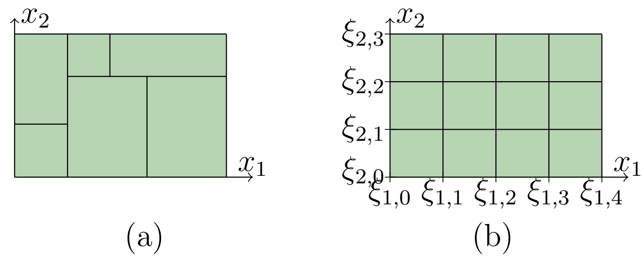


Figure 2.2: Stylized depiction of (a) an interval partition, and (b) a gridded partition.

The map δ includes a transition from q to q' whenever it is possible for the state x to transition from the interval $\mathcal{I}_q$ to $\mathcal{I}_{q'}$ (although δ may also include additional transitions).

Definition 2.3.2 (Interval finite state abstraction)**.** *An* interval finite state abstraction *or simply* abstraction *of system* (2.3) *is a tuple* $\mathcal{T} = (\mathcal{Q}, \mathcal{M}, \delta)$ *where* $\mathcal{Q}$ *is an interval partition of* $\mathcal{X}$ *and* δ *satisfies* (2.49). *We call* δ *a* transition function *and say* $q' \in \mathcal{Q}$ *is a* successor *of* q *in mode* m *if* $q' \in \delta(q, m)$.

$\mathcal{T}$ is a nondeterministic transition system, *i.e.*, $\delta(q, m)$ is, in general, not a singleton set. The nondeterminism arises because $\mathcal{T}$ abstracts an entire set of states into one state or *symbol*, and the transitions account for all possible states in the symbol as well as the disturbance. Nonetheless, $\mathcal{T}$ is a transition system that overapproximates the dynamics (2.3), that is, for every trajectory $x[t]$ satisfying $x[t+1] = F_{m[t]}(x[t], d[t])$ such that $m[t] \in \mathcal{M}$ and $d[t] \in \mathcal{D}$ for all t, there exists $q[t]$ such that $x[t] \in \mathcal{I}_{q[t]}$ and $q[t+1] \in \delta(q[t], m[t])$ for all t.

Computing a transition function δ that is useful in practice is a serious difficulty for standard abstraction approaches. Many existing results apply only to linear or piecewise linear systems, and even in this case, scale poorly with the state space. For example, the polytope-based computations suggested in [Yor+12] require computing F_m at a number of points that scales exponentially with the dimension of the state space and disturbance space. By exploiting the mixed monotonicity properties of system (2.3), we propose an efficient method for computing an abstraction that requires evaluating f_m at only two points for each $q \in Q$ and $m \in \mathcal{M}$.

Theorem 2.3.1. *Consider the mixed monotone system* (2.3) *with interval partition* $\mathcal{Q}$. *Let* $\delta : \mathcal{Q} \times \mathcal{M} \to 2^{\mathcal{Q}}$ *be given by* $q' \in \delta(q, m)$ *if and only if*

$$\begin{aligned} &\exists \ell : [f_m(x_1^{q,m}, d_1^\ell, x_2^{q,m}, d_2^\ell), f_m(x_2^{q,m}, d_2^\ell, x_1^{q,m}, d_1^\ell)]_{\mathcal{X}} \\ &\cap [x_1^{q',m}, x_2^{q',m}]_{\mathcal{X}} \neq \emptyset. \end{aligned} \tag{2.50}$$

Then $\mathcal{T} = (\mathcal{Q}, \mathcal{M}, \delta)$ *is a finite state abstraction of* (2.3).

function FINITESTATEABTRACTION(*system*, $\mathcal{D}$, $\mathcal{Q}$) **returns** $\mathcal{T}$
  inputs: *system*, a mixed monotone system (2.3) with domain $\mathcal{X}$, modes $\mathcal{M}$ and decomposition functions $\{f_m\}_{m\in\mathcal{M}}$
  $\mathcal{D}$, the disturbance set $\mathcal{D} = \cup_{\ell=1}^{L}\mathcal{D}^\ell$ with $\mathcal{D}^\ell \triangleq [d_1^\ell, d_2^\ell]_{\mathcal{D}}$ for $d_1^\ell \leq_{\mathcal{D}} d_2^\ell$
  $\mathcal{Q}$, an interval partition $\mathcal{X}$
  for each $m \in \mathcal{M}$ **do**
    for each $q \in \mathcal{Q}$ **do**
      $\delta(q,m) := \emptyset$
      for $\ell := 1$ **to** L **do**
        $y_1 := f_m(x_1^{q,m}, d_1^\ell, x_2^{q,m}, d_2^\ell)$
        $y_2 := f_m(x_2^{q,m}, d_2^\ell, x_1^{q,m}, d_1^\ell)$
        $\mathcal{Q}' :=$ COMPUTESUCCESSORS$(y_1, y_2, \mathcal{Q})$
        $\delta(q,m) := \delta(q,m) \cup \mathcal{Q}'$
      end for
    end for
  end for
  return $\mathcal{T} := (\mathcal{Q}, \mathcal{M}, \delta)$ ▷ abstraction of (2.3)
end function

Algorithm 2.3.1: Algorithm for computing an interval finite state abstraction of (2.3).

Proof. Consider $x \in \mathcal{I}_q$ and $d \in \mathcal{D}$ such that $x' = F_m(x,d) \in \mathcal{I}_{q'} = [x_1^{q',m}, x_2^{q',m}]_m$. Let $\ell \in \{1,\dots,L\}$ be such that $d \in \mathcal{D}^\ell$. From Theorem 2.2.1, it holds that also

$$x' \in [f_m(x_1^{q,m}, d_1^\ell, x_2^{q,m}, d_2^\ell), f_m(x_2^{q,m}, d_2^\ell, x_1^{q,m}, d_1^\ell)]_m,$$

which implies $q' \in \delta(q,m)$, thus δ satisfies (2.49). □

Corollary 2.3.1. *Consider monotone system* (2.3) *with interval partition* $\mathcal{Q}$. *Let* $\delta : \mathcal{Q} \times \mathcal{M} \to 2^{\mathcal{Q}}$ *be given by* $q' \in \delta(q,m)$ *if and only if*

$$\exists \ell : [F_m(x_1^{q,m}, d_1^\ell), F_m(x_2^{q,m}, d_2^\ell)] \cap [x_1^{q',m}, x_2^{q',m}] \neq \emptyset. \tag{2.51}$$

Then $\mathcal{T} = (\mathcal{Q}, \mathcal{M}, \delta)$ *is a finite state abstraction of* (2.3).

We summarize the algorithm implied by Theorem 2.3.1 in Algorithm 2.3.1. For systems of the form (2.2), we interpret $\mathcal{M}$ as a singleton and proceed as above. We then notationally omit $\mathcal{M}$ and instead write $\mathcal{T} = (\mathcal{Q}, \delta)$, and $\delta(q) \subset \mathcal{Q}$.

2.3.2 Computing Successor States

Theorem 2.3.1 and Corollary 2.3.1 provided a method for overapproximating the one-step reachable set of an interval. How do we identify the successor states from this overapproximation? Lemma 2.3.1 below provides an efficient method for determining if two intervals

function ComputeSuccessors(y_1, y_2, $\mathcal{Q}$) **returns** $\mathcal{Q}'$
  inputs: y_1 and y_2, points in domain $\mathcal{X} \subset \mathbb{R}^n$
    $\mathcal{Q}$, an interval partition of $\mathcal{X}$
  initialize: $\mathcal{Q}' = \emptyset$
  for each $q' \in \mathcal{Q}$ **do**
    if $(y_1 \leq_m x_2^{q',m}) \wedge (x_1^{q',m} \leq_m y_2)$ **then**
      $\mathcal{Q}' := \mathcal{Q}' \cup \{q'\}$
    end if
  end for
  return $\mathcal{Q}'$ ▷ successor states from $[y_1, y_2]_m$
end function

Algorithm 2.3.2: A universal algorithm for overapproximating successor states.

overlap. With this lemma, we establish a universal algorithm for computing the set of one-step reachable intervals in Figure 2.3.2.

Lemma 2.3.1. *Consider* $[\alpha_1, \beta_1]_\mathcal{X}$ *and* $[\alpha_2, \beta_2]_\mathcal{X}$ *for* $\alpha_1, \beta_1 \in \mathcal{X}$ *and* $\alpha_2, \beta_2 \in \mathcal{X}$*. Then* $[\alpha_1, \beta_1]_\mathcal{X} \cap [\alpha_2, \beta_2]_\mathcal{X} \neq \emptyset$ *implies* $\alpha_1 \leq_\mathcal{X} \beta_2$ *and* $\alpha_2 \leq_\mathcal{X} \beta_1$*.*

Proof. Choosing $x \in [\alpha_1, \beta_1]_\mathcal{X} \cap [\alpha_2, \beta_2]_\mathcal{X}$, the lemma follows from transitivity of $\leq_\mathcal{X}$. □

For special types of partitions, however, more efficient methods exist for computing the successor states. In particular, when each $\leq_m$ is an orthant order and $\mathcal{Q}$ is a gridded partition, computing successor states is accomplished by considering each coordinate separately, as in Algorithm 2.3.3. This algorithm scales linearly with $\sum_{i=1}^{n} N_i$. When all N_i are approximately the same, the algorithm scales approximately linearly with n, the dimension of $\mathcal{X}$.

2.3.3 Spurious Self-Loops

An abstraction may produce *spurious* trajectories that do not correspond to any trajectories of (2.3). While such spurious trajectories are often unavoidable, we can identify and ameliorate the effect of a particular type of spurious trajectory that are generated from "self-loops" of the finite state abstraction. $\mathcal{T}$ contains a *self-loop* at state $q^* \in \mathcal{Q}$ for modes $\mathcal{M}' \subseteq \mathcal{M}$ if $q^* \in \delta(q^*, m)$ for all $m \in \mathcal{M}'$. A self-loop implies that under any control action satisfying $\sigma[t] \in \mathcal{M}'$ for all t, the trajectory $q[t] = q^*$ for all t is possible in $\mathcal{T}$. If there is no corresponding trajectory in the original system (2.3) for any such choice of $\sigma[t]$, the state and input set pair $(q^*, \mathcal{M}')$ is said to be *stuttering*. A similar definition of stuttering inputs is given [Yor+12]. For systems of the form (2.2), we instead say q^* is *stuttering* if the above holds with $\mathcal{M}'$ interpreted to be the singleton set corresponding to F.

In verification problems where the dynamics have the form (2.2), it is sometimes possible to simply remove stuttering transitions. In particular, this is possible if the condition to be verified belongs to the fragment of LTL without the "next" operator [PW97]. In other cases,

function ComputeSuccessors(y_1, y_2, $\mathcal{Q}$) **returns** $\mathcal{Q}'$
  inputs: y_1 and y_2, points in domain $\mathcal{X} \subset \mathbb{R}^n$
    where $y_j = (y_j^1, \ldots, y_j^n)$ for $j = 1, 2$
   $\mathcal{Q}$, a grid interval partition of $\mathcal{X}$, *i.e.*,
    $\mathcal{Q} = \prod_{i=1}^n \{1, \ldots, N_i\}$ and
    $\mathcal{I}_q = \prod_{i=1}^n [\xi_{i,\iota_i}, \xi_{i,\iota_i+1}]_{\mathbb{R}_{\geq 0}}$ for each
     $q = (\iota_1, \ldots, \iota_n) \in \mathcal{Q}$
  for $i := 1$ **to** n **do**
    if $\min\{y_1^i, y_2^i\} \leq \xi_{i,1}$ **then** $\underline{\iota}_i := 1$ **else**
      $\underline{\iota}_i := \arg\max_{\iota \in \{1,\ldots,N_i\}}$ s.t. $\xi_{i,\iota} \leq \min\{y_1^i, y_2^i\}$
    if $\xi_{i,N_i+1} \leq \max\{y_1^i, y_2^i\}$ **then** $\bar{\iota}_i := N_i$ **else**
      $\bar{\iota}_i := \arg\min_{\iota \in \{1,\ldots,N_i\}}$ s.t. $\xi_{i,\iota+1} \geq \max\{y_1^i, y_2^i\}$
  end for
  return $\mathcal{Q}' := \{(\iota_1, \ldots, \iota_n) \mid \iota_i \in \{\underline{\iota}_i, \ldots, \bar{\iota}_i\} \; \forall i\}$
end function

Algorithm 2.3.3: An algorithm to identify successor states when $\mathcal{Q}$ is a gridded partition of $\mathbb{R}^n$.

knowledge of stuttering inputs leads to less conservative control strategies; see [Yor+12] for a detailed discussion.

A sufficient condition for determining if $(q, \mathcal{M}')$ is stuttering is to compute a sequence of one-step reachable sets that eventually do not intersect $\mathcal{I}_q$. As an illustration, consider system (2.2) with the standard order $\leq_+$ on $\mathcal{X} \subset \mathbb{R}^2$. Figure 2.3 shows that the overapproximation of $F(\mathcal{I}_q, \mathcal{D})$ intersects $\mathcal{I}_q$, and thus $q \in \delta(q)$. We then overapproximate the one-step reachable set from $\mathcal{I}_q \cap F(\mathcal{I}_q, \mathcal{D})$, which no longer intersects $\mathcal{I}_q$, and thus we conclude that q is stuttering because no trajectory of (2.2) can remain within $\mathcal{I}_q$ for all time.

We generalize this idea and provide Algorithm 2.3.4 for determining if $(q, \mathcal{M}')$ is stuttering. The algorithm requires a function getNewInt which returns a set of points $\{\zeta_1^m, \zeta_2^m\}_{m \in \mathcal{M}'}$ such that for each $m \in \mathcal{M}'$,

$$[\zeta_1^m, \zeta_2^m]_m \supseteq [y_1^{m',\ell}, y_2^{m',\ell}]_{m'} \cap \mathcal{I}_q \quad \forall m' \in \mathcal{M}' \quad \forall \ell \in \{1, \ldots, L\}. \tag{2.52}$$

These points are used in the next iteration when computing the one-step reachable set. In Figure 2.3, these points correspond to the points defining the darkly shaded interval. As suggested by this example, implementing getNewInt for Euclidean spaces with orthant orders can be done coordinate-wise.

2.3.4 Computational Requirements

We now address the computational requirements of the proposed algorithms. Determining $\delta(q, m)$ requires first evaluating the decomposition function f_m at $2L$ points where L is

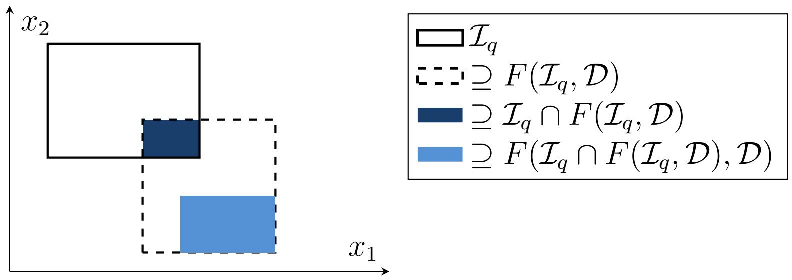


Figure 2.3: Finding stuttering inputs. The solid outline denotes $\mathcal{I}_q$, and the dashed outline denotes the overapproximation of the one-step reachable set from $\mathcal{I}_q$. By overapproximating the one-step reachable set (lightly shaded region) of the intersection (darkly shaded region), we determine that q is stuttering because this region no longer intersects $\mathcal{I}_q$.

the number of boxes constituting the disturbance set $\mathcal{D}$. For each $\ell = 1, \dots L$, the corresponding pair of evaluations of f_m is then used to determine successor states representing an overapproximation of the reachable set from q. In Algorithm 2.3.2, this requires $2|\mathcal{Q}|$ order comparisons of vectors in $\mathbb{R}^n$, and each comparison scales linearly with n. For gridded partitions, determining successor states requires $\sum_{i=1}^{n} N_i$ scalar order comparisons as seen in Algorithm 2.3.3.

Thus, computing δ scales linearly with $|\mathcal{M}|$ and linearly with L. Using Algorithm 2.3.2, the computation further scales quadratically with $|\mathcal{Q}|$ and linearly with n, and using Algorithm 2.3.3, it scales linearly with $|\mathcal{Q}|$ and linearly with $\sum_{i=1}^{n} N_i$. In contrast, computing successor states from a polyhedral region as in, *e.g.*, [Yor+12] requires polyhedral computations that scale exponentially in both n and p [KV10b]. Above, we have assumed that f_m requires constant computation time. This is reasonable in some cases, such as the case study in Section 2.4.2 where intrinsic sparsity of traffic networks implies that the required computation time of f_m does not scale with n or p. However, in other cases, the complexity of evaluating f_m must be taken into account.

We further remark that $|\mathcal{Q}|$ typically increases exponentially with n. However, this dependence can be mitigated *via* various techniques such as interval partitions that incorporate domain specific knowledge. For example, the authors of [MR02] consider monotone systems that converge to a low-dimensional manifold, and suggest a methodology for abstracting the low dimensional manifold while retaining the intrinsic high dimensional dynamics. Future research will investigate related techniques for mixed monotone systems.

Thus, we summarize by emphasizing that the computational complexity of the proposed approach effectively does not depend directly on the state-space dimension n; this contrasts with many existing abstraction approaches for which n is a significant bottleneck. However, the complexity still depends crucially on $|\mathcal{Q}|$, the number of partitions in the abstraction.

function STUTTERING(*system*, $\mathcal{D}$, $\mathcal{T}$, $(q, \mathcal{M}')$) **returns** `isStuttering`
  inputs: *system*, a mixed monotone system (2.3)
    $\mathcal{D}$, the disturbance set $\mathcal{D} = \cup_{\ell=1}^{L} \mathcal{D}^\ell$
    $\mathcal{T} = (\mathcal{Q}, \mathcal{M}, \delta)$, abstraction
    $(q, \mathcal{M}')$, a stuttering pair candidate
  initialize: `isStuttering` $:=$ `Null`
    `iter` $:= 1$
    $\zeta_1^m := x_1^{q,m}$, $\zeta_2^m := x_2^{q,m}$ **for all** $m \in \mathcal{M}'$
  while `iter` $\leq N_{\max}$ **do**
    for $\ell := 1$ **to** L **do**
      for each $m \in \mathcal{M}'$ **do**
        $y_1^{m,\ell} := f_m(\zeta_1^m, d_1^\ell, \zeta_2^m, d_2^\ell)$
        $y_2^{m,\ell} := f_m(\zeta_2^m, d_2^\ell, \zeta_1^m, d_1^\ell)$
      end for
    end for
    if $\exists \ell \in \{0, \ldots, L\}\ \exists m \in \mathcal{M}'$ s.t. $(y_1^{m,\ell} \leq_m x_2^{q,m}) \wedge (x_1^{q,m} \leq_m y_2^{m,\ell})$ **then**
      $\{\zeta_1^m, \zeta_2^m\}_{m \in \mathcal{M}'} :=$ GETNEWINT$(\{y_1^{m,\ell}, y_2^{m,\ell}\}_{m \in \mathcal{M}'}, \mathcal{I}_q)$
      `iter` $:=$ `iter` $+ 1$
    else
      `isStuttering` $:=$ `True`
      break
    end if
  end while
  return `isStuttering`
end function

Algorithm 2.3.4: An algorithm to determine if $(q, \mathcal{M}')$ is stuttering. The parameter $N_{\max}$ determines how many time steps should be considered. The function GETNEWINT returns a set of points $\{\zeta_1^m, \zeta_2^m\}_{m \in \mathcal{M}}$ such that for each $m \in \mathcal{M}'$, $[\zeta_1^m, \zeta_2^m]_m \supseteq [y_1^{m',\ell}, y_2^{m',\ell}]_{m'} \cap \mathcal{I}_q$ for all $m' \in \mathcal{M}'$ and $\ell = 1, \ldots, L$.

2.4 Case Studies

2.4.1 Verifying Oscillations in Insect Population Dynamics

We consider the following model from [Cos+95] for the population dynamics of the flour beetle *Tribolium castaneum*:

$$x^+ = A(x)x, \qquad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{2.53}$$

$$A(x) = \begin{pmatrix} 0 & 0 & b\exp(-c_{e\ell}x_1 - c_{ea}x_3) \\ p & 0 & 0 \\ 0 & \exp(-c_{pa}x_3) & q \end{pmatrix}, \tag{2.54}$$

where x_1, x_2, and x_3 represent populations of the insect at various stages of life (larvae, pupae, and adults, respectively), and $p, q \in (0, 1]$ are probabilities of survival. The exponential nonlinearities are the result of cannibalism of eggs and pupae. The dynamics are mixed monotone with $f(x, d, y, e) = A(y)x$ where $\leq_{\mathcal{X}}=\leq_{+}$.

Using parameters from [Cos+95], we let $b = 7.88$, $c_{ea} = 0.011$, $c_{e\ell} = 0.014$, $p = 0.839$, $q = 0.5$, and $c_{pa} = 0.0047$, with a time step of 2 weeks. We first note that the domain

$$\mathcal{X} = [0, (265, 225, 450)]_{+} \tag{2.55}$$

is invariant. This follows because $bx_3 \exp(-c_{ea}x_3) \leq 265$ for all $x_3 \geq 0$ and, thus, $x_1 \leq 265$ is invariant, from which $x_2 \leq p \cdot 265 \leq 225$. Since $x_3^+ \leq x_2 + qx_3$, we conclude that $x_3 \leq 225/(1 - q) = 450$ is invariant.

For certain sets of parameters, the dynamics (2.53)–(2.54) induce oscillations in the number of larvae—a phenomenon documented in controlled laboratory experiments [Cos+95]. We wish to verify the following LTL formula which is a consequence of this oscillatory behavior:

$$\square\Big(\big((x_1 \leq 10) \wedge (x_3 \geq 40)\big) \rightarrow \lozenge(x_1 \geq 150)\Big). \tag{2.56}$$

In words, "if the larvae population (x_1) reduces to a small number or zero and the adult population (x_3) is not too small, then the larvae population will eventually reach a large population size in the future."

We partition the state space into 2,376 intervals using a gridded partition. Computing the finite state abstraction takes less than one second on a standard personal computer. By applying Algorithm 2.3.4, we remove 14 self transitions that are stuttering. Checking the model with SPIN [Hol97] took 103 seconds, and we verify that (2.56) is satisfied. Figure 2.4 shows a sample trajectory of the population dynamics initialized at $(x_1, x_2, x_3) = (0, 0, 300)$. We see that the larvae population does not reach the desired population 150 immediately, but it does so eventually around week 26.

2.4.2 Synthesizing Control Laws for Traffic Networks

We next synthesize a traffic signal control policy for a network of signalized intersections, providing a preview of the results developed in Chapter 3. We consider a discrete-time model of traffic flow where each road *link* contains a queue of vehicles waiting to proceed through an intersection. Each intersection signal actuates a subset of its queues at a given time step, and the vehicles in actuated queues are allowed to flow to downstream links if there is available space.

We consider a network of $\mathcal{L}$ *links* and a set $\mathcal{V}$ of *signalized intersections*. We assume each link $\ell \in \mathcal{L}$ has a queue of size $x_\ell \in [0, x_\ell^{\text{crit}}]$ representing the number of vehicles on the link

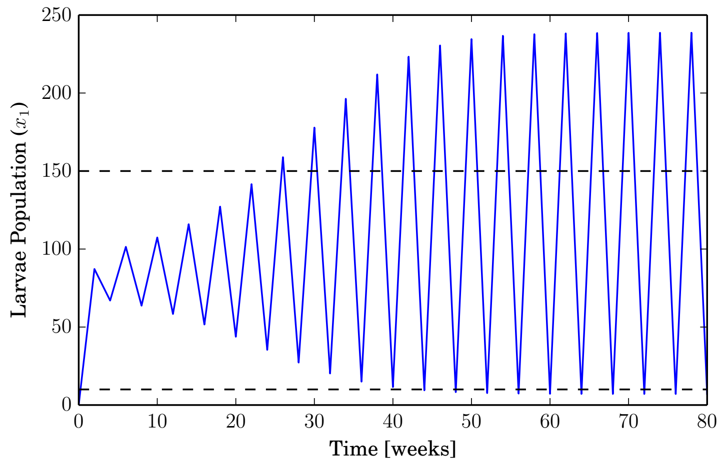


Figure 2.4: Sample trajectory of the insect population model (2.53)–(2.54), plotting x_1 over time when the system is initialized at $(x_1, x_2, x_3) = (0, 0, 300)$. The trajectory satisfies (2.56).

where $x_\ell^{\text{crit}} > 0$ is the capacity of link $\ell \in \mathcal{L}$. By allowing x_ℓ to be continuous, we adopt a *fluid* model of traffic flow.

For $\ell \in \mathcal{L}$, let $\eta(\ell) \in \mathcal{V}$ denote the head node of link ℓ and let $\tau(\ell) \in \mathcal{V} \cup \emptyset$ denote the tail node. A link ℓ with $\tau(\ell) = \emptyset$ serves as an entry-point into the network, and we assume $\eta(\ell) \neq \tau(\ell)$ for all $\ell \in \mathcal{L}$ (*i.e.*, no self-loops). Link $k \neq \ell$ is *upstream* of link ℓ if $\eta(k) = \tau(\ell)$, *downstream* of link ℓ if $\tau(k) = \eta(\ell)$, and *adjacent to* link ℓ if $\tau(k) = \tau(\ell)$. Roads exiting the traffic network are not modeled explicitly. For each $v \in \mathcal{V}$, define

$$\mathcal{L}_v^{\text{in}} = \{\ell \mid \eta(\ell) = v\}, \qquad \mathcal{L}_v^{\text{out}} = \{\ell \mid \tau(\ell) = v\}. \tag{2.57}$$

For simplicity of notation, we assume each intersection $v \in \mathcal{V}$ has two possible states actuating either "East-West" (EW) incoming links or "North-South" (NS) incoming links. Thus, we have the partition $\mathcal{L} = \mathcal{L}^{\text{EW}} \cup \mathcal{L}^{\text{NS}}$, $\mathcal{L}^{\text{EW}} \cap \mathcal{L}^{\text{NS}} = \emptyset$. At each junction $v \in \mathcal{V}$, we define the signal variable $m_v \in \{0, 1\}$ as follows:

$$m_v = \begin{cases} 1 & \text{if links } \mathcal{L}_v^{\text{in}} \cap \mathcal{L}^{\text{EW}} \text{ are actuated} \\ 0 & \text{if links } \mathcal{L}_v^{\text{in}} \cap \mathcal{L}^{\text{NS}} \text{ are actuated.} \end{cases} \tag{2.58}$$

Let $m = \{m_v\}_{v \in \mathcal{V}}$ so that $\mathcal{M} = \{0, 1\}^{\mathcal{V}}$. When a link ℓ is actuated, the *turn ratio* $\beta_{\ell k}$ denotes the fraction of vehicles exiting link ℓ that is routed to link k. It follows that $\beta_{\ell k} \neq 0$ only if $\eta(\ell) = \tau(k)$ and

$$\sum_{k \in \mathcal{L}_{\eta(\ell)}^{\text{out}}} \beta_{\ell k} \leq 1. \tag{2.59}$$

Strict inequality in (2.59) implies that a fraction of vehicles on link ℓ are routed off the network via unmodeled roads.

Each link $\ell \in \mathcal{L}$ possesses a *demand* function $\Phi_\ell^{\text{out}} : [0, x_\ell^{\text{crit}}] \to \mathbb{R}$ that gives the number of vehicles wishing to flow downstream in one time step and a *supply* function $\Phi_\ell^{\text{in}} : [0, x_\ell^{\text{crit}}] \to \mathbb{R}$ that gives the available road space for incoming upstream vehicles in one time step. Thus, Φ_ℓ^{out} is an increasing function and Φ_ℓ^{in} is a decreasing function of queue length. In this example, we let

$$\Phi_\ell^{\text{out}}(x_\ell) = c_\ell(1 - \exp(-x_\ell/c_\ell)) \tag{2.60}$$

$$\Phi_\ell^{\text{in}}(x_\ell) = w_\ell(x_\ell^{\text{crit}} - x_\ell) \tag{2.61}$$

where $c_\ell > 0$ is a *saturation rate* and $0 < w_\ell < 1$ scales the available queue capacity to account for, *e.g.*, vehicles still traveling on the link and not enqueue. This demand-supply approach to vehicular traffic flow is rooted in the Cell Transmission Model [Dag95].

Movement of vehicles among link queues is governed by mass-conservation laws and the state of the signalized intersections. When a link is *actuated*, a maximum of $\Phi_\ell^{\text{out}}(x_\ell)$ vehicles are allowed to flow from link ℓ to links $\mathcal{L}_{\eta(\ell)}^{\text{out}}$ per time step. We let $\alpha_{\ell k}$ denote the fraction of link k's supply available to link ℓ. Since only incoming EW or NS links are actuated in each time step, we have

$$\sum_{\ell \in \mathcal{L}_{\tau(k)}^{\text{in}} \cap \mathcal{L}^{\text{EW}}} \alpha_{\ell k} = \sum_{\ell \in \mathcal{L}_{\tau(k)}^{\text{in}} \cap \mathcal{L}^{\text{NS}}} \alpha_{\ell k} = 1 \tag{2.62}$$

for all $k \in \mathcal{L}$. It then follows that the dynamics on link ℓ are given by

$$x_\ell^+ = F_m^\ell(x, d) \tag{2.63}$$

$$\triangleq x_\ell - f_\ell^{\text{out}}(x, m) + \sum_{j \in \mathcal{L}_{\tau(\ell)}^{\text{in}}} \beta_{j\ell} f_j^{\text{out}}(x, m) + d_\ell \tag{2.64}$$

where

$$f_\ell^{\text{out}}(x, m) = s_\ell(m) \cdot \min \left\{ \Phi_\ell^{\text{out}}(x_\ell), \min_{\substack{k \text{ s.t.} \\ \beta_{\ell k} \neq 0}} \frac{\alpha_{\ell k}}{\beta_{\ell k}} \Phi_k^{\text{in}}(x_k) \right\} \tag{2.65}$$

$$s_\ell(m) = \begin{cases} m_{\eta(\ell)} & \text{if } \ell \in \mathcal{L}^{\text{EW}} \\ 1 - m_{\eta(\ell)} & \text{if } \ell \in \mathcal{L}^{\text{NS}}. \end{cases} \tag{2.66}$$

Assumption 2.4.1. *For all $\ell \in \mathcal{L}$ and all k upstream of ℓ,*

$$\exp\left(\frac{-1}{c_\ell} \left(x_\ell^{crit} - \frac{\beta_{k\ell}}{w_\ell \alpha_{k\ell}} c_k \right) \right) \leq 1 - w_\ell. \tag{2.67}$$

Assumption 2.4.1 ensures that an increase in x_i does not lead to a decrease in x_i^+. This assumption is mild because (2.67) is satisfied for small enough c_ℓ and c_k, and these parameters decrease for shorter time steps; indeed, violation of the assumption would indicate that the chosen time step is to large to accurately capture the queue dynamics.

Lemma 2.4.1. *Assumption 2.4.1 ensures that* $\frac{\partial F_m^\ell}{\partial x_\ell}(x,d) \geq 0$ *for all* m *whenever the partial derivative exists.*

Proof. We have

$$\frac{\partial F_m^\ell}{\partial x_\ell}(x,d) = 1 - \frac{\partial f_\ell^{\text{out}}}{\partial x_\ell}(x_\ell, m) + \sum_{j \in \mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta_{j\ell} \frac{\partial f_j^{\text{out}}}{\partial x_\ell}(x, m). \tag{2.68}$$

Note that $\frac{\partial f_\ell^{\text{out}}}{\partial x_\ell}(x_\ell, m) \leq 1$. Furthermore, $\frac{\partial f_j^{\text{out}}}{\partial x_\ell}(x,m) \neq 0$ only if $s_j(m) = 1$ and $\frac{\alpha_{j\ell}}{\beta_{j\ell}}\Phi_\ell^{\text{in}}(x_\ell)$ is the minimizer in (2.65). As $\Phi_j^{\text{out}}(x_j) \leq c_j$, the latter condition can only occur if

$$c_j \geq \frac{\alpha_{j\ell}}{\beta_{j\ell}} w_\ell (x_\ell^{\text{crit}} - x_\ell) \iff x_\ell \geq x_\ell^{\text{crit}} - \frac{\beta_{j\ell}}{w_\ell \alpha_{j\ell}} c_j. \tag{2.69}$$

It then follows that $\sum_{j \in \mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta_{j\ell} \frac{\partial f_j^{\text{out}}}{\partial x_\ell}(x,m) < 0$ only if there exists $j \in \mathcal{L}^{\text{in}}_{\tau(\ell)}$ such that the inequalities in (2.69) hold. But this implies $\frac{\partial f_\ell^{\text{out}}}{\partial x_\ell}(x_\ell, m) \leq 1 - w_\ell$ by Assumption 2.4.1 and the fact that $\exp(-\frac{1}{c_\ell} x_\ell)$ decreases in x_ℓ. Furthermore, $\sum_{j \in \mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta_{j\ell} \frac{\partial f_j^{\text{out}}}{\partial x_\ell}(x,m) \geq -w_\ell$, and we thus conclude that $\frac{\partial F_m^\ell}{\partial x_\ell}(x,d) \geq 0$. □

Proposition 2.4.1. *The traffic dynamics* (2.63)–(2.64) *are mixed monotone.*

In Chapter 3, we provide a proof of 2.4.1 for a modified case where the supply and demand functions are piecewise linear in Theorem 3.4.1. The proof of mixed monotonicity for the present model is nearly the same, except Assumption 2.4.1 replaces Assumption 3.4.1.

Consider the traffic network show in Figure 2.5 consisting of two signalized intersections and eight links. We have $\mathcal{L}^{\text{EW}} = \{1,2,3,4\}$ and $\mathcal{L}^{\text{NS}} = \{5,6,7,8\}$. The leftmost signal actuates the EW links 1 and 3 simultaneously, or the NS links 5 and 6 simultaneously, and similarly for the rightmost signal. We take the time step to be 15 seconds and assume $c_1 = c_2 = c_3 = c_4 = 20$, $c_5 = c_6 = c_7 = c_8 = 5$, $x_1^{\text{crit}} = x_4^{\text{crit}} = 50$, $x_2^{\text{crit}} = x_3^{\text{crit}} = 60$, $x_5^{\text{crit}} = x_6^{\text{crit}} = x_7^{\text{crit}} = x_8^{\text{crit}} = 40$, $w_\ell = 0.75$ for all ℓ, $\beta_{12} = \beta_{43} = \beta_{52} = \beta_{62} = \beta_{73} = \beta_{83} = 0.5$, $\alpha_{52} = \alpha_{62} = \alpha_{73} = \alpha_{83} = 0.5$, $\alpha_{12} = 1$, and $\alpha_{43} = 1$. For the disturbance input, we assume that at each time step, up to 7 vehicles join each of the queues on links 1 and 3, or up to 8 vehicles join each of the queues on links 5 and 6, or up to 8 vehicles join each of the queues on links 7 and 8.

We partition the domain of the traffic network, representing the state of all queues, into 3,600 boxes using a gridded partition. Using the mixed monotonicity properties of the dynamics, we obtain a finite state abstraction of the dynamics in 43.8 seconds.

Next, we wish to find a controller that satisfies the specification:

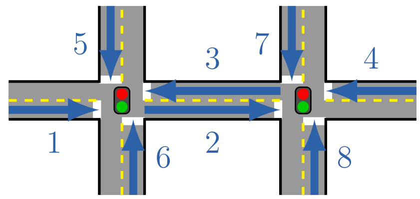


Figure 2.5: A traffic network with two signalized intersections and 8 links. The blue links represent queues of vehicles. The leftmost signal actuates links 1 and 3 simultaneously, or links 5 and 6 simultaneously. Likewise, the rightmost signal actuates links 2 and 4 or 7 and 8 simultaneously.

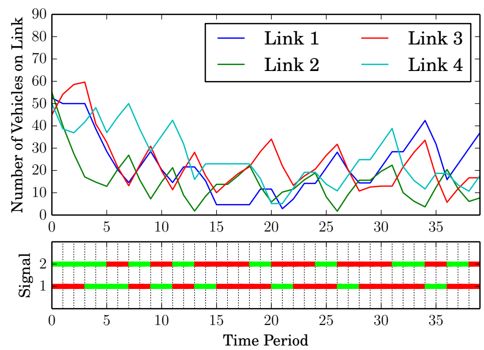


Figure 2.6: An example trajectory of the traffic network for links 1, 2, 3, and 4. Signal 1 (resp. 2) is the leftmost (resp. rightmost) signal in Figure 2.5. The trajectory satisfies the given specification. In the lower plot, green (resp. red) indicates that EW (resp. NS) links are actuated.

> "Infinitely often, the cross streets on links 5 and 6 are actuated, AND infinitely often, the cross streets on links 7 and 8 are actuated, AND eventually, the queue lengths on links 2 and 3 are each less than 40 vehicles and remain so for all future time, AND whenever the queue on link 1 exceeds 40 vehicles, it eventually is less than 30 vehicles, AND whenever the queue on link 4 exceeds 40 vehicles, it eventually is less than 30 vehicles."

The above specification can be expressed in linear temporal logic and encoded in a deterministic *Rabin automaton* [BK08] with 46 states. We then construct a controller that is guaranteed to satisfy the specification by considering a game in a which an adversary seeks to violate the specification and the control inputs seek to prevent this violation. The game is formulated using the finite state abstraction and the Rabin automaton that encodes the specification and may be solved using algorithms as in [Hor05; PP06]. In Figure 2.6, we plot an example trajectory of the system where we assume the maximum number of allowed vehicles enters the network in each time step. We see in the figure that the trajectory satisfies the above specification.

2.5 Discussion

We have efficiently computed finite state abstractions for mixed monotone discrete-time systems. Mixed monotonicity is a general property encompassing many practical systems and provides a powerful tool for analysis and control. The primary feature that permits efficient abstraction is overapproximation of reachable sets by evaluating a decomposition function at two points.

Chapter 3

Control of Traffic Networks from Linear Temporal Logic Specifications

State-of-the-art approaches to coordinated control of signalized intersections often focus on limited objectives such as maximizing throughput [Won+12] or maintaining stability of network queues [Var13b; Var13a]; see [Pap+03] for a review of the literature. However, traffic networks are a natural domain for a much richer class of control objectives that are expressible using *linear temporal logic (LTL)* [CGP99; BK08]. LTL *formulae* allow control objectives such as "actuate traffic flows such that throughput is always greater than C_1" where C_1 is a threshold throughput, or such that "traffic link queues are always less than C_2" where C_2 is a threshold queue length. LTL formulae also allow more complex objectives such as "infinitely often, the queue length on road ℓ should reach 0," "anytime link ℓ becomes congested, it eventually becomes uncongested," or any combination of these conditions. As these examples suggest, many objectives that are difficult or impossible to address using standard control theoretic techniques are easily expressed in LTL.

In this chapter, we leverage the results of Chapter 2 and propose a technique for synthesizing a signal control policy for a traffic network such that the network satisfies a given control objective expressed using LTL. The synthesized policy is a finite-memory, state feedback controller that is provably correct, that is, guaranteed to result in a closed loop system that satisfies the control objective.

Recent approaches to control synthesis from LTL specifications such as [TP06; Tab08; Fai+09; KFP09; KB10; ADD11; WTM12; Yor+12; GLB14; JW12; Top+12; Liu+13; ALB13; PKV13; CA14] allow automatic development of correct-by-construction control laws; however, despite these promising developments, scalability concerns prevent direct application of existing results to large traffic networks.

To overcome these scalability limitations, we identify and exploit the *mixed monotonicity* property introduced in Chapter 2. As alluded to in the previous chapter, mixed monotonicity is inherent in flow networks such as traffic networks. Mixed monotonicity allows efficient computation of bounds on the one-step reachable set from a rectangular box of initial conditions, which in turn allows efficient computation of a finite state abstraction of the dynamics, thereby mitigating a crucial bottleneck in the control synthesis process. A related approach to abstractions of monotone systems is suggested in [MR02], however the componentwise monotonicity properties exploited in this work are much more general and encompass monotone systems as a special case.

This chapter is organized as follows: Section 3.1 gives necessary preliminaries. Section 3.2 presents the model for signalized networks, and Section 3.3 establishes the problem formulation. Section 3.4 identifies componentwise monotonicity properties of the traffic networks, and Section 3.5 presents scalable algorithms that rely on these properties to construct a finite state representation of the traffic network. Section 3.6 describes the controller synthesis approach, and discusses the computation requirements of our method. We present a case study in Section 3.7 and conclude our work in Section 3.8.

3.1 Preliminaries

The set $\mathcal{I} \subseteq \mathbb{R}^n$ is a *box* if it is the cartesian product of intervals, or equivalently, $\mathcal{I}$ is a box if there exists $x, y \in \mathbb{R}^n$ such that $\mathcal{I} = \prod_{i=1}^n \{z \in \mathbb{R} \mid x_i \prec_i^1 z \prec_i^2 y_i\}$ where $\prec_i^1, \prec_i^2 \in \{<, \leq\}$ and x_i, y_i denote the ith coordinate of x and y, respectively. Defining $\prec^1 \triangleq \{\prec_i^1\}_{i=1}^n$ and $\prec^2 \triangleq \{\prec_i^2\}_{i=1}^n$, we may write $\mathcal{I} = \{z \in \mathbb{R}^n \mid x \prec^1 z \prec^2 y\}$. The vector x is the *lower corner* of $\mathcal{I}$, and likewise y is the *upper corner*.

When applied to vectors, $<$, $\leq$, $>$, and $\geq$ are interpreted elementwise. The notation $\mathbf{0}$ denotes the all-zeros vector where the dimension is clear from context. We denote closure of a set Y by $\mathbf{cl}(Y)$. Given an index set $\mathcal{L}$ and a set of values $x_\ell \in \mathbb{R}$ for $\ell \in \mathcal{L}$, $\{x_\ell\}_{\ell \in \mathcal{L}}$ denotes the collection of x_ℓ, $\ell \in \mathcal{L}$, but we also interpret $x = \{x_\ell\}_{\ell \in \mathcal{L}}$ as an element of $\mathbb{R}^{|\mathcal{L}|}$.

A *transition system* is a tuple $\mathcal{T} = (Q, S, \rightarrow)$ where $\mathcal{Q}$ is a finite set of states, $\mathcal{S}$ is a finite set of actions, and $\rightarrow \subset \mathcal{Q} \times \mathcal{S} \times \mathcal{Q}$ is a transition relation. We write $q \xrightarrow{s} q'$ instead of $(q, s, q') \in \rightarrow$. Note that all transition systems in this chapter are *finite* [BK08]. The evolution of a transition system is described by $\rightarrow$. That is, a transition system is initialized in some state $q_0 \in \mathcal{Q}$, and, given an action $s \in \mathcal{S}$, the next state of the transition system is chosen nondeterministically from $\{q' \mid q \xrightarrow{s} q'\}$.

3.2 Signalized Network Traffic Model

A signalized traffic network consists of a set $\mathcal{L}$ of *links* and a set $\mathcal{V}$ of *signalized intersections*. For $\ell \in \mathcal{L}$, let $\eta(\ell) \in \mathcal{V}$ denote the downstream intersection of link ℓ and let $\tau(\ell) \in \mathcal{V} \cup \emptyset$ denote the upstream intersection of link ℓ. A link ℓ with $\tau(\ell) = \emptyset$ serves as an entry-point into the network, and we assume $\eta(\ell) \neq \tau(\ell)$ for all $\ell \in \mathcal{L}$ (*i.e.*, no self-loops). Link $k \neq \ell$ is *upstream* of link ℓ if $\eta(k) = \tau(\ell)$, *downstream* of link ℓ if $\tau(k) = \eta(\ell)$, and *adjacent to* link ℓ if $\tau(k) = \tau(\ell)$. Roads exiting the traffic network are not modeled explicitly. For each $v \in \mathcal{V}$,

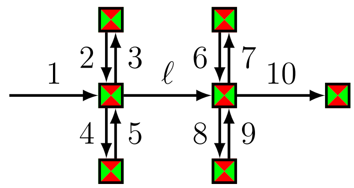


Figure 3.1: A typical traffic network with 11 links and 7 signalized intersections. In the figure, $\mathcal{L}_\ell^{\text{down}} = \{\ell, 7, 8, 10\}$, $\mathcal{L}_\ell^{\text{up}} = \{1, 2, 5\}$, and $\mathcal{L}_\ell^{\text{adj}} = \{3, 4\}$. At each time step, a signal actuates a subset of upstream links.

define $\mathcal{L}_v^{\text{in}} = \{\ell \mid \eta(\ell) = v\}$, $\mathcal{L}_v^{\text{out}} = \{\ell \mid \tau(\ell) = v\}$ and for each $\ell \in \mathcal{L}$, define

$$\mathcal{L}_\ell^{\text{up}} = \{k \in \mathcal{L} \mid \eta(k) = \tau(\ell)\} \tag{3.1}$$

$$\mathcal{L}_\ell^{\text{down}} = \{k \in \mathcal{L} \mid \tau(k) = \eta(\ell)\} \cup \{\ell\} \tag{3.2}$$

$$\mathcal{L}_\ell^{\text{adj}} = \{k \in \mathcal{L} \mid \tau(k) = \tau(\ell)\} \backslash \{\ell\} \tag{3.3}$$

so that $\mathcal{L}_\ell^{\text{down}}$ includes link ℓ and the links downstream of link ℓ, and $\mathcal{L}_\ell^{\text{up}}$ and $\mathcal{L}_\ell^{\text{adj}}$ are the links upstream and adjacent to ℓ, respectively, see Fig. 3.1. We have $\mathcal{L}_\ell^{\text{down}} \cap \mathcal{L}_\ell^{\text{adj}} = \emptyset$ and $\mathcal{L}_\ell^{\text{up}} \cap \mathcal{L}_\ell^{\text{adj}} = \emptyset$, but note that it is possible for $\mathcal{L}_\ell^{\text{down}} \cap \mathcal{L}_\ell^{\text{up}} \neq \emptyset$, in particular, if there is a cycle of length two in the network. Let $\mathcal{L}_\ell^{\text{loc}} = \mathcal{L}_\ell^{\text{down}} \cup \mathcal{L}_\ell^{\text{up}} \cup \mathcal{L}_\ell^{\text{adj}}$ be links "local" to link ℓ.

Each link $\ell \in \mathcal{L}$ possesses a queue $x_\ell[t] \in [0, x_\ell^{\text{cap}}]$ representing the number of vehicles on link ℓ at time step $t \in \mathbb{N} \triangleq \{0, 1, 2, \ldots\}$ where x_ℓ^{cap} is the capacity of link ℓ. We allow x_ℓ to be a continuous quantity, thus adopting a fluid-like model of traffic flow evolving in slotted time as in [Won+12; Var13a; Var13b].

Movement of vehicles among link queues is governed by mass-conservation laws and the state of the signalized intersections. A link is said to be *actuated* if outgoing flow from link ℓ is allowed as determined by the state of the traffic signal at intersection $\eta(\ell)$. At each intersection v,

$$\mathcal{S}_v \subseteq 2^{\mathcal{L}_v^{\text{in}}} \tag{3.4}$$

denotes the set of available signal *phases*, that is, each $s_v \in \mathcal{S}_v$, $s_v \subseteq \mathcal{L}_v^{\text{in}}$ denotes a set of incoming links at intersection v that may be actuated simultaneously. We define

$$\mathcal{S} = \{\cup_{v \in \mathcal{V}} s_v \mid s_v \in \mathcal{S}_v \ \forall v \in \mathcal{V}\} \subseteq 2^{\mathcal{L}} \tag{3.5}$$

so that each $\boldsymbol{s} \in \mathcal{S}$, $\boldsymbol{s} \subseteq \mathcal{L}$ denotes a set of links in the network that may be actuated simultaneously. We identify $\boldsymbol{s} \in \mathcal{S}$ with its constituent phases so that $\boldsymbol{s} = \{s_v\}_{v \in \mathcal{V}}$, and we interpret $\mathcal{S}$ as the set of allowed inputs to the traffic network.

When a link is actuated, a maximum of c_ℓ vehicles are allowed to flow from link ℓ to links $\mathcal{L}_{\eta(\ell)}^{\text{out}}$ per time step where c_ℓ is the known *saturation flow* for link ℓ, [Pap+03]. The *turn* *ratio* $\beta_{\ell k}$ denotes the fraction of vehicles exiting link ℓ that are routed to link k, [Var13b]. Then $\beta_{\ell k} \neq 0$ only if $\eta(\ell) = \tau(k)$, and

$$\sum_{k \in \mathcal{L}^{\text{out}}_{\eta(\ell)}} \beta_{\ell k} \leq 1. \tag{3.6}$$

Strict inequality in (3.6) implies that a fraction of vehicles on link ℓ are routed off the network via unmodeled roads that exit the network. Traffic flow can occur only if there is available capacity downstream. To this end, the supply ratio $\alpha^{s_v}_{\ell k}$ denotes the fraction of link k's capacity available to link ℓ during phase $s_v \in \mathcal{S}_{\tau(k)}$. That is, link ℓ may only send $\alpha^{s_v}_{\ell k}(x^{\text{cap}}_k - x_k[t])$ vehicles to link k in time period t under input s_v. As the supply is only divided among actuated incoming links, it follows that for each $k \in \mathcal{L}$

$$\sum_{\ell \in s_v} \alpha^{s_v}_{\ell k} = 1 \quad \forall s_v \in \mathcal{S}_{\tau(k)}, s_v \neq \emptyset. \tag{3.7}$$

Constant turn and supply ratios are a common modeling assumption justified by empirical observations; see [Leb05] for further discussion.

We are now in a position to define the dynamics of the link queues. As we will see subsequently, the flow of vehicles out of link ℓ is only a function of the state of links in $\mathcal{L}^{\text{down}}_\ell$, and the update of link ℓ's state is only a function of links in $\mathcal{L}^{\text{loc}}_\ell$.

Let $\boldsymbol{x}[t] = \{x_\ell[t]\}_{\ell \in \mathcal{L}}$, $\boldsymbol{x}^{\text{down}}_\ell[t] = \{x_k[t]\}_{k \in \mathcal{L}^{\text{down}}_\ell}$, and $\boldsymbol{x}^{\text{loc}}_\ell[t] = \{x_k[t]\}_{k \in \mathcal{L}^{\text{loc}}_\ell}$. The outflow of link $\ell \in \mathcal{L}$ is as follows:

$$f^{\text{out}}_\ell(\boldsymbol{x}^{\text{down}}_\ell, s_{\eta(\ell)}) = \begin{cases} \min\left\{x_\ell[t], c_\ell, \min_{\substack{k \text{ s.t.} \\ \beta_{\ell k} \neq 0}} \left\{\frac{\alpha^{s_{\eta(\ell)}}_{\ell k}}{\beta_{\ell k}}(x^{\text{cap}}_k - x_k[t])\right\}\right\} & \text{if } \ell \in s_{\eta(\ell)} \\ 0 & \text{else.} \end{cases} \tag{3.8}$$

The interpretation of (3.8) is that the flow of vehicles exiting a link ℓ when actuated is the minimum of the link's queue length, its saturation flow, and the downstream *supply* of capacity, weighted appropriately by turn and supply ratios. This modeling approach is based on the *cell transmission model* of traffic flow [Dag94] which restricts flow if there is inadequate capacity downstream. A consequence of (3.8) is that inadequate capacity on one downstream link at an intersection causes congestion that blocks incoming flow to other downstream links. This phenomenon, sometimes called the *first-in-first-out* property, has been widely studied in the transportation literature and occurs even in multilane settings [MD02][1]. The number of vehicles in each link's queue then evolves according to the mass

[1]Even if a turn pocket exists at an intersection, it is often too short to fully mitigate this blocking property. Nonetheless, if the road geometry is such that a sufficient number of dedicated lanes exist for a turning movement, these lanes may be modeled with a separate link.

conservation equation

$$x_\ell[t+1] = F_\ell(\boldsymbol{x}_\ell^{\mathrm{loc}}[t], \boldsymbol{s}_\ell^{\mathrm{loc}}[t], d_\ell[t]) \tag{3.9}$$

$$\triangleq \min\Big\{x_\ell^{\mathrm{cap}}, x_\ell[t] - f_\ell^{\mathrm{out}}(\boldsymbol{x}_\ell^{\mathrm{down}}[t], s_{\eta(\ell)}) + \sum_{j\in\mathcal{L}_\ell^{\mathrm{up}}} \beta_{j\ell} f_j^{\mathrm{out}}(\boldsymbol{x}_j^{\mathrm{down}}[t], s_{\eta(j)}) + d_\ell[t]\Big\} \tag{3.10}$$

where $d_\ell[t]$ is the number of vehicles that exogenously enters the queue on link ℓ in time step t, $\bar{d} = \{d_\ell[t]\}_{\ell\in\mathcal{L}}$, and $\boldsymbol{s}_\ell^{\mathrm{loc}} = \{s_{\eta(\ell)}, s_{\tau(\ell)}\}$ if $\tau(\ell) \neq \emptyset$, $\boldsymbol{s}_\ell^{\mathrm{loc}} = \{s_{\eta(\ell)}\}$ otherwise; that is, $\boldsymbol{s}_\ell^{\mathrm{loc}}$ is the state of the signals that are "local" to link ℓ. The minimization in (3.10) is only needed in case the exogenous input $d_\ell[t]$ would cause the state of link ℓ to exceed x_ℓ^{cap} and ensures that the network dynamics maps

$$\mathcal{X} = \prod_{\ell\in\mathcal{L}} [0, x_\ell^{\mathrm{cap}}] \tag{3.11}$$

to itself. We interpet this as refusal of vehicles attempting to exogenously enter the network when the link is full. Note in particular that the supply/demand formulation prevents upstream inflow from exceeding supply and thus for links with no exogenous input, x_ℓ^{cap} is never the unique minimizer in (3.10).

Remark 2. *An alternative to the above approach is to define an auxiliary sink state* `Out` *in the transition systems* $\mathcal{T}$ *defined in Section 3.5 which captures any trajectories that exit the domain* $\mathcal{X}$*. The temporal logic specification can then incorporate the requirement that the system never enters this* `Out` *state.*

Assumption 3.2.1. *We assume there exists* $\mathcal{D} \subset \mathbb{R}^{\mathcal{L}}$ *such that*

$$\bar{d}[t] \in \mathcal{D} \quad \forall t \tag{3.12}$$

and $\mathcal{D}$ *satisfies* $\mathcal{D} \subset \cup_{i=1}^{n_\mathcal{D}} \mathcal{D}_i$ *where each* $\mathcal{D}_i$ *is given by*

$$\mathcal{D}_i = \{\bar{d} \mid \underline{\bar{d}}^i \leq \bar{d} \leq \overline{\bar{d}}^i\} \tag{3.13}$$

for some $\underline{\bar{d}}^i = \{\underline{d}_\ell^i\}_{\ell\in\mathcal{L}}$, $\overline{\bar{d}}^i = \{\overline{d}_\ell^i\}_{\ell\in\mathcal{L}}$.

In other words, we assume the disturbance is contained within a union of boxes given by (3.13). This assumption is not particularly restrictive, as any compact subset of $\mathbb{R}^{\mathcal{L}}$ can be approximated with boxes to arbitrary precision [KJW02], however the number of boxes $n_\mathcal{D}$ affects the computation time as detailed in Section 3.6.2.

We let $F(\boldsymbol{x}, \boldsymbol{s}, \bar{d}) = \{F_\ell(\boldsymbol{x}_\ell^{\mathrm{loc}}, \boldsymbol{s}_\ell^{\mathrm{loc}}, d_\ell)\}_{\ell\in\mathcal{L}} : \mathcal{X} \times \mathcal{S} \times \mathcal{D} \to \mathcal{X}$ so that

$$\boldsymbol{x}[t+1] = F(\boldsymbol{x}[t], \boldsymbol{s}[t], \bar{d}[t]). \tag{3.14}$$

The set of states of system (3.14) that are reachable from a set $Y \subset \mathcal{X}$ under the control signal $\boldsymbol{s} \in \mathcal{S}$ in one timestep is denoted by the `Post` operator and given by

$$\texttt{Post}(Y, \boldsymbol{s}) = \{\boldsymbol{x}' = F(\boldsymbol{x}, \boldsymbol{s}, \bar{d}) \mid \boldsymbol{x} \in Y, \bar{d} \in \mathcal{D}\}. \tag{3.15}$$

We call $\texttt{Post}(Y, \boldsymbol{s})$ the *one step reachable set from Y under $\boldsymbol{s}$*. The main features of the queue-based modeling approach proposed above such as finite saturation rates, finite queue capacity, a set of available signaling phases, and fixed turn ratios are standard in many modeling and simulation approaches such as [Var13a; Var13b], see also [Pap95; Pap+03] and references therein for discussions of queue-based modeling of traffic networks.

3.3 Problem Formulation and Approach

We now define and motivate the need for control objectives expressible in LTL for traffic networks, and we outline a control synthesis approach which relies on a finite state representation of the traffic dynamics to meet these objectives.

LTL *formulae* are generated inductively using the Boolean operators $\vee$ (disjunction), $\wedge$ (conjunction), $\neg$ (negation), and the temporal operators $\bigcirc$ (next) and U (until). From these, we obtain a suite of derived logical and temporal operators such as $\rightarrow$ (implication), $\Box$ (always), $\Diamond$ (eventually), $\Box\Diamond$ (infinitely often), finite deadlines with repeated $\bigcirc$, and many others, see [BK08; CGP99].

Formally, such formulae are expressed over a set of atomic propositions, which we restrict to be indicator expressions over subsets of $\mathcal{X}$ or predicates over the signaling state. For example, the atomic proposition $x_\ell \leq 10$ is true for all $\boldsymbol{x} \in \mathcal{X}$ that satisfies the condition $x_\ell \leq 10$ (which constitutes a box subset of $\mathcal{X}$), and the atomic proposition $\ell \in \boldsymbol{s}$ is true for all signals that actuate link ℓ. We will see in Section 3.4 and Section 3.5 that restricting to atomic propositions corresponding to box subsets of $\mathcal{X}$ offers significant computational advantages.

Semantically, LTL formulae are interpreted over a trajectory $\boldsymbol{x}[t]$ and the corresponding input sequence $\boldsymbol{s}[t]$ for $t = 0, 1, \ldots$. For example, the state/input sequence $(\boldsymbol{x}[t], \boldsymbol{s}[t])$ *satisfies* the LTL formula $\varphi = \Box(x_\ell \leq 10) \wedge \Box\Diamond(\ell \in \boldsymbol{s})$ if and only if $x_\ell[t] \leq 10$ for all t and $\ell \in \boldsymbol{s}[t]$ infinitely often (*i.e.*, for infinitely many t). Thus a trajectory satisfies a LTL formula if and only if the formula holds for the corresponding trace of atomic propositions that are valid at each time step. A formal definition of the semantics of LTL over traces is readily available in the literature, *e.g.*, [BK08; CGP99], and is a natural interpretation of the above Boolean and temporal operators. For example, a trace satisfies $\Diamond\varphi$ if and only if there exists a suffix of the trace satisfying φ.

Examples of LTL formulae representing desired control objectives relevant to traffic networks include those from the Introduction, as well as:

- $\varphi_1 = \Diamond\Box(x_\ell \leq C)$ for some C
  "Eventually, link ℓ will have less than C vehicles and this will remain true for all time"

- $\varphi_2 = \square\lozenge(\ell \in \boldsymbol{s})$
  "Infinitely often, link ℓ is actuated"

- $\varphi_3 = \square((\ell \in s_{v_1}) \to \bigcirc(k \in s_{v_2}))$
  "Whenever signal v_1 actuates link ℓ, signal v_2 must actuate link k in the next time step"

- $\varphi_4 = \square(x_\ell \geq C_1 \to \lozenge(x_\ell \leq C_2))$
  "Whenever the number of vehicles on link ℓ exceeds C_1, it is eventually the case that the number of vehicles on link ℓ decreases below C_2."

The main problem considered in this chapter is as follows:

Control Synthesis Problem. *Given a traffic network and an LTL formula φ over a set of atomic propositions as described above, find a control strategy that, at each time step, chooses a signaling input such that all trajectories of the traffic network satisfy φ from any initial condition.*

To solve the control synthesis problem, we propose computing a finite state abstraction that simulates (in a manner to be formalized below) the traffic network dynamics. As we discuss in Section 3.6.1, the result is a full-state feedback controller which requires finite memory. We rely on dynamical properties of the traffic network to compute the abstraction, and then apply tools from automata theory and formal methods to synthesize a finite-memory, state feedback control strategy solving the control synthesis problem.

3.4 Mixed Monotonicity of Traffic Networks

To generate control strategies for the traffic network that guarantee satisfaction of a LTL formula, we first construct a finite state representation, or *abstraction*, of the model defined in Section 3.2. As suggested in Section 2.4.2, traffic networks are mixed monotone, which simplifies the abstraction process.

To prove that the traffic network dynamics developed in Section 3.2 are mixed monotone, we first require a technical assumption:

Assumption 3.4.1. *For all $\ell \in \mathcal{L}$,*

$$c_\ell \leq x_\ell^{cap} - \frac{\beta_{k\ell}}{\alpha_{k\ell}} c_k \quad \forall k \in \mathcal{L}_\ell^{up}. \tag{3.16}$$

Assumption 3.4.1 is a sufficient condition for ensuring that if a link has inadequate capacity and blocks upstream flow, then this link's queue will not empty in one time step. This effectively is an assumption that the time step is sufficiently small to appropriately capture the queuing phenomenon. Specifically, the saturation flow rate c_ℓ of link ℓ is in units of vehicles per time step and, thus, is implicitly a function of the chosen time step. Physically, c_ℓ is required to decrease with decreased time step and thus Assumption 3.4.1 is satisfied when a sufficiently small time step is used for the model.

Theorem 3.4.1. *The traffic network model is mixed monotone for any signaling input $\boldsymbol{s} \in \mathcal{S}$. In particular, F_ℓ is increasing in x_k for k downstream or upstream of link ℓ or equal to ℓ, and decreasing in x_k for k adjacent to link ℓ.*

Proof. For fixed $\boldsymbol{s} \in \mathcal{S}$, we show that $F(\boldsymbol{x}, \boldsymbol{s}, \bar{d})$ satisfies conditions (2.7) and (2.8) of Proposition 2.2.1 with $\boldsymbol{x}, \bar{d}$ replacing x, d. Observe that F is continuous and piecewise differentiable by (3.8)–(3.10) and (3.14), thus it is Lipschitz continuous [Sch12]. The minimum function in (3.8) implies that F is differentiable almost everywhere. We first have $\frac{\partial F_\ell}{\partial d_\ell} \in \{0, 1\}$ a.e. by (3.10), satisfying (2.7). Now consider $\partial F_\ell / \partial x_k$. For (2.8), we consider four exhaustive cases:

- Case 1, $k \in (\mathcal{L}_\ell^{\text{down}} \cup \mathcal{L}_\ell^{\text{up}}) \backslash \{\ell\}$. From (3.8)–(3.10), link k may block the outflow of link ℓ when $k \in \mathcal{L}_\ell^{\text{down}}$, or link k may contribute to the inflow to link ℓ if $k \in \mathcal{L}_\ell^{\text{up}}$, thus we have $\frac{\partial F_\ell}{\partial x_k} \in \{0, -\frac{\partial f_\ell^{\text{out}}}{\partial x_k}, \beta_{k\ell} \frac{\partial f_k^{\text{out}}}{\partial x_k}, -\frac{\partial f_\ell^{\text{out}}}{\partial x_k} + \beta_{k\ell} \frac{\partial f_k^{\text{out}}}{\partial x_k}\}$ a.e. where the fourth possibility occurs only if $k \in \mathcal{L}_\ell^{\text{down}} \cap \mathcal{L}_\ell^{\text{up}}$. But $\frac{\partial f_\ell^{\text{out}}}{\partial x_k} \in \{0, -\alpha_{\ell k}^{s_{\eta(\ell)}} / \beta_{\ell k}\}$ a.e. and $\frac{\partial f_k^{\text{out}}}{\partial x_k} \in \{0, 1\}$ a.e., thus $\frac{\partial F_\ell}{\partial x_k} \geq 0$ a.e., satisfying (2.8).
- Case 2, $k = \ell$. We have $\frac{\partial f_\ell^{\text{out}}}{\partial x_\ell} \in \{0, 1\}$ a.e. and, for $j \in \mathcal{L}_\ell^{\text{up}}$, $\frac{\partial f_j^{\text{out}}}{\partial x_\ell} \in \{0, \alpha_{j\ell}^{s_{\eta(j)}} / \beta_{j\ell}\}$ a.e., however, Assumption 3.4.1 ensures that, a.e., either $\frac{\partial f_\ell^{\text{out}}}{\partial x_\ell} = 0$ or $\frac{\partial f_\ell^{\text{in}}}{\partial x_\ell} = 0$, *i.e.*, $\frac{\partial f_j^{\text{out}}}{\partial x_\ell} = 0$ for all $j \in \mathcal{L}_\ell^{\text{up}}$. Thus $\frac{\partial F_\ell}{\partial x_\ell} \in \{0, 1, 1 + \sum_{j \in \mathcal{L}_\ell^{\text{up}}} \beta_{j\ell} \frac{\partial f_j^{\text{out}}}{\partial x_\ell}\}$ a.e. But $\sum_{j \in \mathcal{L}_\ell^{\text{up}}} \beta_{j\ell} \frac{\partial f_j^{\text{out}}}{\partial x_\ell} \geq -\sum_{j \in \mathcal{L}_\ell^{\text{up}}} \alpha_{j\ell}^{s_{\eta(j)}} = -1$ by (3.7) (recall that $\eta(j) = \tau(\ell)$ for all $j \in \mathcal{L}_\ell^{\text{up}}$), that is, $\partial f_\ell^{\text{in}} / \partial x_\ell \geq -1$, thus $\frac{\partial F_\ell}{\partial x_\ell} \geq 0$ a.e., satisfying (2.8).
- Case 3, $k \in \mathcal{L}_\ell^{\text{adj}}$. In this case, inadequate capacity of link k may block flow to link ℓ, as discussed above. We have $\frac{\partial F_\ell}{\partial x_k} = \sum_{j \in \mathcal{L}_\ell^{\text{up}}} \beta_{j\ell} \frac{\partial f_j^{\text{out}}}{\partial x_k}$. Since $\frac{\partial f_j^{\text{out}}}{\partial x_k} \in \{0, -\alpha_{jk}^{s_{\eta(j)}} / \beta_{jk}\}$ a.e., we have $\frac{\partial F_\ell}{\partial x_k} \leq 0$ a.e., satisfying (2.8).
- Case 4, $k \notin \mathcal{L}_\ell^{\text{loc}}$. Then $\frac{\partial F_\ell}{\partial x_k} = 0$, trivially satisfying (2.8).

□

The following corollary implies that the one-step reachable set of the traffic dynamics from a (closed) box $\mathcal{I}$ for any given signaling input $\boldsymbol{s}$ is over-approximated by the union of boxes, one box for each $i = 1, \ldots n_{\mathcal{D}}$, where each of these boxes is efficiently computed by evaluating F_ℓ at two particular points for each $\ell \in \mathcal{L}$. The obtained over-approximation is denoted with the $\overline{\texttt{Post}}$ operator. This critical result allows efficient computation of a finite state representation of the traffic dynamics, as detailed in Section 3.5.

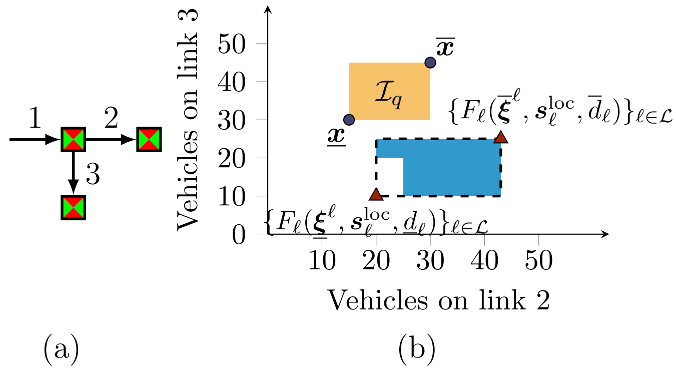


Figure 3.2: Approximating the one-step reachable set using mixed monotonicity properties. (a) A simple network with three links. (b) We are able to bound the one-step reachable set from the box $\mathcal{I}_q$ under control input $\boldsymbol{s} = \{1, 2, 3\}$ by evaluating the network dynamics of each link at two particular extreme points which depend on the topology of the network. The actual reach set is shaded in blue, the approximation is outlined with a dashed line, and the results are projected in the x_2 vs. x_3 plane.

Corollary 3.4.1. *Consider the set $\mathcal{I} = \{\boldsymbol{x} \mid \underline{\boldsymbol{x}} \le \boldsymbol{x} \le \bar{\boldsymbol{x}}\}$ for $\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \in \mathcal{X}$, and for each $\ell \in \mathcal{L}$, define $\underline{\boldsymbol{\xi}}^\ell(\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}) = \{\underline{\xi}_k^\ell(\underline{x}_k, \overline{x}_k)\}_{k\in\mathcal{L}_\ell^{loc}}$, $\overline{\boldsymbol{\xi}}^\ell(\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}) = \{\overline{\xi}_k^\ell(\underline{x}_k, \overline{x}_k)\}_{k\in\mathcal{L}_\ell^{loc}}$ where*

$$\underline{\xi}_k^\ell(\underline{x}_k, \overline{x}_k) = \begin{cases} \underline{x}_k & \text{if } k \in \mathcal{L}_\ell^{down} \cup \mathcal{L}_\ell^{up} \\ \overline{x}_k & \text{if } k \in \mathcal{L}_\ell^{adj} \end{cases} \tag{3.17}$$

$$\overline{\xi}_k^\ell(\underline{x}_k, \overline{x}_k) = \begin{cases} \overline{x}_k & \text{if } k \in \mathcal{L}_\ell^{down} \cup \mathcal{L}_\ell^{up} \\ \underline{x}_k & \text{if } k \in \mathcal{L}_\ell^{adj}. \end{cases} \tag{3.18}$$

Then for all $\boldsymbol{s} \in \mathcal{S}$, $\texttt{Post}(\mathcal{I}, \boldsymbol{s}) \subseteq \overline{\texttt{Post}}(\mathcal{I}, \boldsymbol{s})$ where

$$\overline{\texttt{Post}}(\mathcal{I}, \boldsymbol{s}) := \bigcup_{i=1}^{n_\mathcal{D}} \{\boldsymbol{x}' \mid F_\ell(\underline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{loc}, \underline{d}_\ell^i) \le x'_\ell \le F_\ell(\overline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{loc}, \overline{d}_\ell^i) \quad \forall \ell \in \mathcal{L}\}. \tag{3.19}$$

Proof. By substituting $\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}$ for x_1, x_2 and $\underline{d}_\ell^i, \overline{d}_\ell^i$ for d_1, d_2 in Theorem 2.2.1 and defining $f(\boldsymbol{x}, \bar{d}) \triangleq F(\boldsymbol{x}, \boldsymbol{s}, \bar{d})$, we obtain $\{\boldsymbol{x}' = F(\boldsymbol{x}, \boldsymbol{s}, \bar{d}) \mid \boldsymbol{x} \in \mathcal{I}, \bar{d} \in \mathcal{D}_i\} \subseteq \{\boldsymbol{x}' \mid F_\ell(\underline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{\mathrm{loc}}, \underline{d}_\ell^i) \le x'_\ell \le F_\ell(\overline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{\mathrm{loc}}, \overline{d}_\ell^i)\ \forall \ell \in \mathcal{L}\}$ for all $i = 1, \ldots, n_\mathcal{D}$. The corollary follows from the trivial fact that $\texttt{Post}(\mathcal{I}, \boldsymbol{s}) = \cup_{i=1}^{n_\mathcal{D}} \{\boldsymbol{x}' = F(\boldsymbol{x}, \boldsymbol{s}, \bar{d}) \mid \boldsymbol{x} \in \mathcal{I}, d \in \mathcal{D}_i\}$. □

Example 3.4.1. *Consider the network shown in Fig 3.2(a) with $\mathcal{L} = \{1, 2, 3\}$, with the parameters $x_\ell^{cap} = 50$ for all ℓ, $(c_1, c_2, c_3) = (20, 5, 30)$, $\beta_{12} = \beta_{13} = 1/2$, $\mathcal{S} = 2^\mathcal{L}$, $\alpha_{12}^{\{1\}} = \alpha_{13}^{\{1\}} = 1$, and $\mathcal{D} = \{\bar{d} \mid \underline{\bar{d}} \le \bar{d} \le \overline{\bar{d}}\}$ where $\underline{\bar{d}} = [0\ 5\ 0]^T$ and $\overline{\bar{d}} = [0\ 8\ 5]^T$. Let $\mathcal{I}_q = \{\boldsymbol{x} \mid \underline{\boldsymbol{x}} \le$*

$\boldsymbol{x} \leq \overline{\boldsymbol{x}}\}$ *where* $\underline{\boldsymbol{x}} = [40\ 15\ 30]^T$ *and* $\overline{\boldsymbol{x}} = [40\ 30\ 45]^T$. *Applying Corollary 3.4.1, we have*

$$\begin{aligned} \texttt{Post}(\mathcal{I}_q, \{1,2,3\}) &\subseteq \{\boldsymbol{x}' \mid [20\ 20\ 10]^T \leq \boldsymbol{x}' \leq [30\ 43\ 25]^T\} \\ &= \overline{\texttt{Post}}(\mathcal{I}_q, \{1,2,3\}). \end{aligned} \tag{3.20}$$

Fig. 3.2(b) plots $\mathcal{I}_q$ *and* $\texttt{Post}(\mathcal{I}_q, \{1,2,3\})$ *projected in the* x_2 *vs.* x_3 *plane. It is interesting to note that the network dynamics do not constitute a monotone system; this is apparent from the fact that* $\{F_\ell(\underline{\boldsymbol{\xi}}^\ell, \{1,2,3\}, \underline{d}_\ell)\}_{\ell\in\mathcal{L}} \notin \texttt{Post}(\mathcal{I}, \{1_q, 2, 3\})$ *and is attributable to a filled queue on link 3 blocking flow to link 2 from link 1.*

3.5 Finite State Representation

To apply the powerful tools of LTL synthesis, we require a finite state representation of the traffic network model. In general, obtaining finite state abstractions is a difficult problem and existing techniques do not scale well. In this section, we exploit the mixed monotonicity properties developed above and propose an efficient method for determining a finite state representation of the traffic network dynamics.

3.5.1 Finite State Abstraction

Definition 3.5.1 (Box partition). *For finite index set* $\mathcal{Q}$*, the set* $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$ *is a* box partition *of* $\mathcal{X}$ *(or simply a* box partition*), if each* $\mathcal{I}_q \subseteq \mathcal{X}$ *is a box,* $\cup_{q\in\mathcal{Q}}\mathcal{I}_q = \mathcal{X}$*, and* $\mathcal{I}_q \cap \mathcal{I}_{q'} = \emptyset$ *for all* $q, q' \in \mathcal{Q}$*. For* $q \in \mathcal{Q}$*, let* $\underline{\boldsymbol{x}}_q = \{\underline{x}_{q,\ell}\}_{\ell\in\mathcal{L}}$*,* $\overline{\boldsymbol{x}}_q = \{\overline{x}_{q,\ell}\}_{\ell\in\mathcal{L}}$ *denote the lower and upper corners, respectively, of* $\mathcal{I}_q$*, that is,* $\mathcal{I}_q = \{\boldsymbol{x} \mid \underline{\boldsymbol{x}}_q \prec_q^1 \boldsymbol{x} \prec_q^2 \overline{\boldsymbol{x}}_q\}$ *where* $\prec_q^1 = \{\prec_{q,\ell}^1\}_{\ell\in\mathcal{L}}$*,* $\prec_q^2 = \{\prec_{q,\ell}^2\}_{\ell\in\mathcal{L}}$*, and* $\prec_{q,\ell}^1, \prec_{q,\ell}^2 \in \{<, \leq\}$*.*

For a box partition $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$ of $\mathcal{X}$, let $\pi : \mathcal{X} \to \mathcal{Q}$ be uniquely defined by the condition $x \in \mathcal{I}_{\pi(x)}$, that is, $\pi(\cdot)$ is the natural projection from the domain $\mathcal{X}$ to the (index set of) boxes. A special case of a box partition of a rectangular domain is the following:

Definition 3.5.2 (Gridded box partition). *For* $\mathcal{X} = \{\boldsymbol{x} = \{x_\ell\}_{\ell\in\mathcal{L}} \mid \underline{x}_\ell \leq x_\ell \leq \bar{x}_\ell\}$*, a box partition* $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$ *of* $\mathcal{X}$ *is a* gridded *box partition if for each* $\ell \in \mathcal{L}$*, there exists* $N_\ell \in \{1, 2, \ldots\}$ *and a set of intervals* $\{I_1^\ell, \ldots, I_{N_\ell}^\ell\}$ *such that* $\cup_{i=1}^{N_\ell} I_i^\ell = [\underline{x}_\ell, \bar{x}_\ell]$ *and for each* $q \in \mathcal{Q}$*, there exists indices* $q_\ell \in \{1, \ldots, N_\ell\}$ *such that* $\mathcal{I}_q = \prod_{\ell\in\mathcal{L}} I_{q_\ell}^\ell$*. For gridded box partitions, we make the identification* $\mathcal{Q} \cong \prod_{\ell\in\mathcal{L}}\{1, \ldots, N_\ell\}$ *for all* $\ell \in \mathcal{L}$*.*

When a box partition is not a gridded box partition, we say it is *nongridded*. Fig. 3.3 shows two examples of box partitions, one of which is a gridded box partition. From a box partition of the traffic network domain $\mathcal{X}$, we obtain a finite state representation, or *abstraction*, of the traffic network model as follows. Each element of the box partition corresponds to a single state in the resulting finite state transition system, and to obtain a computationally tractable approach, we propose a method for efficiently obtaining a finite state abstraction using the mixed monotonicity properties developed above:

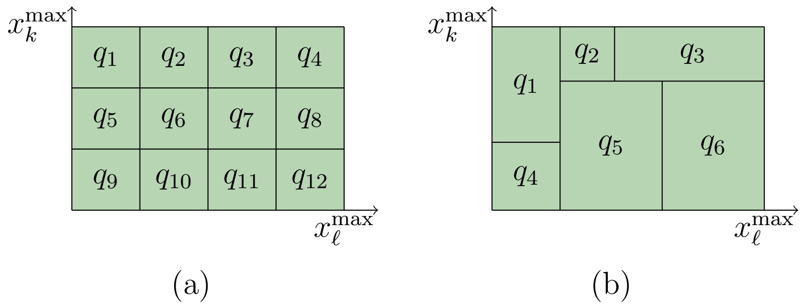


Figure 3.3: Stylized depictions of two box partitions. (a) A gridded box partition with regularly sized intervals. (b) A nongridded box partition.

Definition 3.5.3 (MM-induced finite state abstraction)**.** *Given a box partition* $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$ *of* $\mathcal{X}$*, the nondeterministic* mixed monotonicity-induced (MM-induced) finite state abstraction*, or simply the* finite state abstraction*, of the traffic model is the transition system* $\mathcal{T} = (\mathcal{Q}, \mathcal{S}, \rightarrow)$ *where* $\mathcal{Q}$ *is the index set of the box partition,* $\mathcal{S}$ *is the available signaling inputs, and* $\rightarrow$ *is defined by:*

$$(q, \boldsymbol{s}, q') \in\rightarrow \qquad \textit{if and only if} \quad \mathcal{I}_{q'} \cap \overline{\texttt{Post}}(\mathbf{cl}(\mathcal{I}_q), \boldsymbol{s}) \neq \emptyset. \tag{3.21}$$

Remark 3. *We must take the closure of* $\mathcal{I}_q$ *in* (3.21) *as the* $\overline{\texttt{Post}}$ *operator and relevant properties (*e.g., (3.19)*) assume a closed box. This allows efficient algorithms for constructing* $\rightarrow$ *via* (3.21) *as detailed below.*

Note that the MM-induced finite state abstraction is nondeterministic. Nondeterminism arises from the disturbance input $\bar{d}$ and from the fact that a collection of continuous states is abstracted to one discrete state.

By the definition of the finite state abstraction above, for any trajectory $\boldsymbol{x}[t]$, $t \in \mathbb{N}$ generated by the traffic model under input sequence $\boldsymbol{s}[t]$, $t \in \mathbb{N}$, there exists a unique sequence $q[t]$, $t \in \mathbb{N}$ with each $q[t] \in \mathcal{Q}$ such that $x[t] \in \mathcal{I}_{q[t]}$ and $q[t] \overset{\boldsymbol{s}[t]}{\rightarrow} q[t+1]$. A transition system satisfying this property is said to be a *discrete abstraction* of the dynamical system (3.14). A controller synthesized from the abstraction to satisfy an LTL formula as described in Section 3.4 can be applied to the original traffic network with the same guarantees because the abstraction *simulates* the original traffic network [BK08]. However, abstractions generally result in unavoidable conservatism, that is, nonexistence of an appropriate control strategy from the abstraction does not imply nonexistence of a control strategy for the original traffic network.

The following corollary to Proposition 2.2.2 implies that the finite state abstraction suggested in Definition 3.5.3 does not introduce excessive conservatism; specifically, Corollary 3.5.1 tells us that if $(q, \boldsymbol{s}, q') \in\rightarrow$, then for each link ℓ, it is possible for the state of link ℓ to transition from a state in box $\mathcal{I}_q$ to a state in $\mathcal{I}_{q'}$.

Corollary 3.5.1. *For the MM-induced finite state abstraction defined above,* $(q, \boldsymbol{s}, q') \in \rightarrow$ *if and only if*

$$\exists \bar{d} = \{d_\ell\}_{\ell \in \mathcal{L}} \in \mathcal{D}, \exists \boldsymbol{x}' = \{x'_\ell\}_{\ell \in \mathcal{L}} \in \mathcal{I}_{q'} \text{ such that} \tag{3.22}$$

$$\forall \ell \in \mathcal{L}, \exists \boldsymbol{x} \in \mathbf{cl}(\mathcal{I}_q) \text{ s.t. } x'_\ell = F_\ell(\boldsymbol{x}_\ell^{loc}[t], \boldsymbol{s}^{loc}[t], d_\ell[t]). \tag{3.23}$$

Proof. (if). Suppose (3.22)–(3.23) holds for some $q, q' \in \mathcal{Q}$ and $\boldsymbol{s} \in \mathcal{S}$, and let $\bar{d} \in \mathcal{D}$ and $\boldsymbol{x}' \in \mathcal{I}_{q'}$ be a particular solution such that (3.23) holds for all ℓ. We will show that $\boldsymbol{x}' \in \overline{\texttt{Post}}(\mathbf{cl}(\mathcal{I}_q), \boldsymbol{s})$. Let i^* be such that $\bar{d} \in \mathcal{D}_{i^*}$, and let $\underline{\boldsymbol{\xi}}^\ell$, $\overline{\boldsymbol{\xi}}^\ell$ be as in Corollary 3.4.1. We must have

$$F_\ell(\underline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{\text{loc}}, \underline{d}_\ell^{i^*}) \leq x'_\ell \leq F_\ell(\overline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{\text{loc}}, \overline{d}_\ell^{i^*}) \tag{3.24}$$

by Theorem 2.2.1. By (3.19), it follows that $\boldsymbol{x}' \in \overline{\texttt{Post}}(\mathbf{cl}(\mathcal{I}_q), \boldsymbol{s})$, and thus $(q, \boldsymbol{s}, q') \in \rightarrow$.

(only if). Suppose $(q, \boldsymbol{s}, q') \in \rightarrow$, it follows that $\mathcal{I}_{q'} \cap \overline{\texttt{Post}}(\mathbf{cl}(\mathcal{I}_q), \boldsymbol{s}) \neq \emptyset$, let $\boldsymbol{x}' \in \mathcal{I}_{q'} \cap \overline{\texttt{Post}}(\mathbf{cl}(\mathcal{I}_q), \boldsymbol{s})$ and let $i^* \in \{1, \ldots, n_\mathcal{D}\}$ be such that $F_\ell(\underline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{\text{loc}}, \underline{d}_\ell^{i^*}) \leq x'_\ell \leq F_\ell(\overline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{\text{loc}}, \overline{d}_\ell^{i^*})$ for all $\ell \in \mathcal{L}$. Proposition 2.2.2 implies that for each ℓ, there exists $\boldsymbol{x} \in \mathbf{cl}(\mathcal{I}_q)$ and $d_\ell^\dagger \in [\underline{d}_\ell^{i^*}, \overline{d}_\ell^{i^*}]$ such that $x'_\ell = F_\ell(\boldsymbol{x}_\ell^{\text{loc}}, \boldsymbol{s}_\ell^{\text{loc}}, d_\ell^\dagger)$. Indeed, suppose not, then

$$\tilde{x}_\ell \triangleq \sup_{x \in \mathbf{cl}(\mathcal{I}_q), d_\ell \in [\underline{d}_\ell^{i^*}, \overline{d}_\ell^{i^*}]} F_\ell(\boldsymbol{x}_\ell^{\text{loc}}, \boldsymbol{s}_\ell^{\text{loc}}, d_\ell) < x'_\ell, \quad \text{or} \tag{3.25}$$

$$\underset{\sim}{x}_\ell \triangleq \inf_{x \in \mathbf{cl}(\mathcal{I}_q), d_\ell \in [\underline{d}_\ell^{i^*}, \overline{d}_\ell^{i^*}]} F_\ell(\boldsymbol{x}_\ell^{\text{loc}}, \boldsymbol{s}_\ell^{\text{loc}}, d_\ell) > x'_\ell. \tag{3.26}$$

If (3.25) holds, then $F_\ell(\boldsymbol{x}, \boldsymbol{s}, \bar{d}) \leq \tilde{x}_\ell < x'_\ell \leq F_\ell(\overline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{\text{loc}}, \overline{d}_\ell^{i^*})$ for all $\underline{\boldsymbol{x}}_q \leq \boldsymbol{x} \leq \overline{\boldsymbol{x}}_q$ and all $\underline{d}^{i^*} \leq \bar{d} \leq \overline{\bar{d}}^{i^*}$, which implies the upper bound in (2.47) is not achieved, contradicting the first statement of Proposition 2.2.2. A symmetric argument shows that if (3.26) holds, then Proposition 2.2.2 is again contradicted. Defining $\bar{d} = \{d_\ell^\dagger\}_{\ell \in \mathcal{L}}$ for the particular collection $\{d_\ell^\dagger\}_{\ell \in \mathcal{L}}$ above implies that (3.22)–(3.23) holds, completing the proof. □

We remark that, in (3.23), the same choice of $x \in \mathbf{cl}(\mathcal{I}_q)$ will generally not work for all $\ell \in \mathcal{L}$ due to the over-approximation of the reachable set; see [Coo+15a] for further discussion.

3.5.2 Constructing the Transition System $\mathcal{T}$

We begin with the primary algorithm for calculating $\mathcal{T}$ shown in Fig. 3.4, which relies on Corollary 3.4.1 to compute $\overline{\texttt{Post}}$ and to construct the finite state abstraction as defined in Definition 3.5.3. This algorithm requires a function called SUCCESSORS that takes the lower and upper corners of a box Y as input, as well as a box partition of $\mathcal{X}$, and returns the indices of the box partitions which intersects Y. We first present a generic algorithm for SUCCESSORS applicable to any box partition. To this end, consider the nonempty box

function ABSTRACTION(*network model*, $\mathcal{D}$, $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$) **returns** $\mathcal{T}$
  inputs: *network model*, a traffic network model with
    update functions $\{F_\ell\}_{\ell\in\mathcal{L}}$ with domain $\mathcal{X}$
    and signal input set $\mathcal{S}$
   $\mathcal{D}$, the disturbance set $\mathcal{D} = \cup_{i=1}^{n_\mathcal{D}} \mathcal{D}^i$
   $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$, a box partition $\mathcal{X}$
  $\rightarrow := \emptyset$
  for each $\boldsymbol{s} \in \mathcal{S}$ **do**
    for each $q \in \mathcal{Q}$ **do**
      for $i := 1$ **to** $n_\mathcal{D}$ **do**
        $\underline{\boldsymbol{\xi}}^\ell :=$ as in (3.17)
        $\overline{\boldsymbol{\xi}}^\ell :=$ as in (3.18)
        $\underline{\boldsymbol{y}} := F_\ell(\underline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{\text{loc}}, \underline{d}_\ell^i)$
        $\overline{\boldsymbol{y}} := F_\ell(\overline{\boldsymbol{\xi}}^\ell, \boldsymbol{s}_\ell^{\text{loc}}, \overline{d}_\ell^i)$
        $\mathcal{Q}' :=$ SUCCESSORS($\underline{\boldsymbol{y}}, \overline{\boldsymbol{y}}, \{\mathcal{I}_q\}_{q\in\mathcal{Q}}$)
        $\rightarrow := \rightarrow \cup (q \times \boldsymbol{s} \times \mathcal{Q}')$
      end for
    end for
  end for
  return $\mathcal{T} := (\mathcal{Q}, \mathcal{S}, \rightarrow)$
end function

Figure 3.4: Algorithm for computing a finite state abstraction of the traffic dynamics. The algorithm requires function Successors, which can be implemented using different algorithms, depending on the structure of the box partition.

$\mathcal{I}_q = \{\boldsymbol{x} \mid \underline{\boldsymbol{x}} \prec_q^1 \boldsymbol{x} \prec_q^2 \overline{\boldsymbol{x}}\}$ and let $Y \triangleq \{\boldsymbol{x} \mid \underline{\boldsymbol{y}} \le \boldsymbol{x} \le \overline{\boldsymbol{y}}\}$. It is straightforward to show that $\mathcal{I}_q \cap Y \neq \emptyset$ if and only if $\underline{\boldsymbol{x}} \prec_q^1 \overline{\boldsymbol{y}}$ and $\underline{\boldsymbol{y}} \prec_q^2 \overline{\boldsymbol{x}}$.

The algorithm in Fig. 3.5 utilizes this fact to compute $\mathcal{Q}'$, the indices of the partitions that intersect a box defined by the corners $\underline{\boldsymbol{y}}$ and $\overline{\boldsymbol{y}}$. The algorithm is convenient because it works for any box partition of $\mathcal{X}$, however it requires comparing the corners $\underline{\boldsymbol{y}}, \overline{\boldsymbol{y}}$ to the corners of each box $\mathcal{I}_q$, $q \in \mathcal{Q}$. Thus, computing $\mathcal{T}$ scales quadratically with $|\mathcal{Q}|$ since we must determine if $\overline{\texttt{Post}}(\boldsymbol{s}, \mathcal{I}_q)$ intersects each box $\mathcal{I}_{q'}$, $q' \in \mathcal{Q}$ for each $q \in \mathcal{Q}$.

However, the general algorithm in Fig. 3.5 fails to take into account any structure in the partition itself. For example, for gridded box partitions, we can identify $\mathcal{Q}'$ by comparing the corners $\underline{\boldsymbol{y}}, \overline{\boldsymbol{y}}$ mixed to the partition's constituent coordinate intervals. For simplicity of presentation, we consider gridded box partitions $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$ where, for each $\ell \in \mathcal{L}$, there exists a set of intervals $\{I_1^\ell, \dots, I_{N_\ell}^\ell\}$ of the form

$$I_1^\ell = [\eta_0^\ell, \eta_1^\ell], \quad I_j^\ell = (\eta_{j-1}^\ell, \eta_j^\ell], \; j = 2, \dots, N_\ell \tag{3.27}$$

function SUCCESSORS($\underline{\boldsymbol{y}}$, $\overline{\boldsymbol{y}}$, $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$) **returns** $\mathcal{Q}'$
  inputs: $\underline{\boldsymbol{y}}$ and $\overline{\boldsymbol{y}}$, points in domain $\mathcal{X}$
      $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$, an interval partition of $\mathcal{X}$
  initialize: $\mathcal{Q}' = \emptyset$
  for each $q' \in \mathcal{Q}$ **do**
    if $(\underline{\boldsymbol{x}}_q \prec_q^1 \overline{\boldsymbol{y}}) \wedge (\underline{\boldsymbol{y}} \prec_q^2 \overline{\boldsymbol{x}}_q)$ **then**
      $\mathcal{Q}' := \mathcal{Q}' \cup \{q'\}$
    end if
  end for
  return $\mathcal{Q}'$
end function

Figure 3.5: A generic algorithm for overapproximating successor states applicable to any box partition.

for $0 = \eta_0^\ell \leq \eta_1^\ell < \eta_2^\ell < \ldots < \eta_{N_\ell-1}^\ell < \eta_{N_\ell}^\ell = x_\ell^{\text{cap}}$ such that $\mathcal{I}_q = \prod_{\ell\in\mathcal{L}} I_{q_\ell}^\ell$ for $q = \{q_\ell\}_{\ell\in\mathcal{L}} \in \mathcal{Q} \cong \prod_{\ell\in\mathcal{L}}\{1,\ldots,N_\ell\}$. Define

$$\overline{j}_\ell = \begin{cases} 1 & \text{if } \overline{y}_\ell = 0 \\ \max\limits_{j\in\{1,\ldots,N_\ell\}} j \text{ s.t. } \eta_{j-1}^\ell < \overline{y}_\ell & \text{else} \end{cases} \tag{3.28}$$

$$\underline{j}_\ell = \min_{j\in\{1,\ldots,N_\ell\}} j \text{ s.t. } \underline{y}_\ell \leq \eta_j^\ell \tag{3.29}$$

and let $\mathcal{Q}' = \{\{q_\ell\}_{\ell\in\mathcal{L}} \mid q_\ell \in \{\underline{j}_\ell, \underline{j}_\ell + 1, \ldots, \overline{j}_\ell\}\}$. Then $\mathcal{I}_q \cap Y \neq \emptyset$ if and only if $q' \in \mathcal{Q}'$. Thus, to determine the partitions $\mathcal{Q}'$ that intersect a given box Y, we simply identify the indices of the intervals that intersects Y along each dimension. Finding $\underline{j}_\ell$ and $\overline{j}_\ell$ can be done in $O(N_\ell)$ time for each ℓ, thus solving for $\mathcal{Q}'$ requires $O(|\mathcal{L}| \max_{\ell\in\mathcal{L}}\{N_\ell\})$ time. Thus, for gridded box partitions, we can instead use the implementation of SUCCESSORS found in Fig. 3.6.

The algorithm in Fig. 3.6 may be applied to nongridded box partitions with some modification. In particular, a nongridded box partition $\{\mathcal{I}_q\}_{q\in Q}$ can be *refined* to obtain the coarsest possible gridded box partition with the property that each box $\mathcal{I}_q$ is the union of boxes from the refinement. This refinement is used as an index set; to compute the possible transitions from $\mathcal{I}_q$ for $q \in \mathcal{Q}$ under signaling $\boldsymbol{s} \in \mathcal{S}$, we compute $\underline{\boldsymbol{y}}$ and $\overline{\boldsymbol{y}}$ as in lines 11 and 12 of the algorithm in Fig. 3.4, and then use the refinement along with the algorithm in Fig. 3.6 to determine $\mathcal{Q}'$, the set of intersected boxes. The refinement does not introduce additional states in the transition system or require addition reach computations; it is only used to efficiently determine $\mathcal{Q}'$. For example, the coarsest refinement of Fig. 3.3(b) partitions the box labeled q_5 into four boxes, which are all labeled q_5. This method will be faster if the total number of intervals in the refinement is less than $|\mathcal{Q}|$.

function SUCCESSORS($\underline{\boldsymbol{y}}$, $\overline{\boldsymbol{y}}$, $\{\mathcal{I}_q\}_{q\in Q}$) **returns** $\mathcal{Q}'$
    inputs: $\underline{\boldsymbol{y}} = \{\underline{y}_\ell\}_{\ell\in\mathcal{L}}$ and $\overline{\boldsymbol{y}} = \{\overline{y}_\ell\}_{\ell\in\mathcal{L}}$,
            points in domain $\mathcal{X}$
            $\mathcal{Q}$, a grid interval partition of $\mathcal{X}$
    for each $\ell \in \mathcal{L}$ **do**
        $\overline{j}_\ell :=$ as in (3.28)
        $\underline{j}_\ell :=$ as in (3.29)
    end for
    return $\mathcal{Q}' := \{(j_\ell)_{\ell\in\mathcal{L}} \mid j_\ell \in \{\underline{j}_\ell, \dots, \overline{j}_\ell\}\ \forall \ell \in \mathcal{L}\}$
end function

Figure 3.6: An algorithm for identifying successor states when $\mathcal{Q}$ is a gridded box partition.

3.5.3 Augmenting the State Space with Signaling

To capture control objectives that include the state of the signals themselves (which are modeled as inputs in the finite state abstraction $\mathcal{T}$), we augment the discrete state space. Examples of specifications that require this augmention include φ_2 and φ_3 above or the specifications "the state of an intersection cannot change more than once per $n^{\min}$ time steps" or "an input signal cannot remain unchanged for $n^{\max}$ time steps." In particular, we propose augmenting the finite state abstraction to encompass both the current state of the finite state abstraction and the current state of the traffic signals.

Definition 3.5.4 (Augmented finite state abstraction)**.** *The* augmented finite state abstraction *of the traffic network is the transition system $\mathcal{T}_{aug} = (\mathbb{Q}, \mathcal{S}, \rightarrow_{aug})$ where*

- *$\mathbb{Q} = \mathcal{Q} \times \mathcal{S}$ is the set of discrete states consisting of the box partition index set and the set of allowed input signals,*
- *$\mathcal{S}$ is the set of allowed input signals,*
- *$\rightarrow_{aug} \subseteq \mathbb{Q} \times \mathcal{S} \times \mathbb{Q}$ is the set of transitions given by $((q, \boldsymbol{\sigma}), \boldsymbol{s}, (q', \boldsymbol{\sigma}')) \in \rightarrow_{aug}$ for $(q, \boldsymbol{\sigma}), (q', \boldsymbol{\sigma}') \in \mathbb{Q}$ if and only if $(q, \boldsymbol{s}, q') \in \rightarrow$ and $\boldsymbol{\sigma}' = \boldsymbol{s}$.*

3.6 Synthesizing Controllers from LTL Specifications

3.6.1 Synthesis Summary

We omit the details of how a control strategy is synthesized from the nondeterministic transition system $\mathcal{T}_{\text{aug}}$ for a given LTL control objective, as this is well-documented in the literature, see *e.g.* [Yor+12; Hor05]. Instead, we summarize the main steps of this synthesis as follows: from the LTL control objective, we obtain a deterministic Rabin automaton that accepts all and only trajectories that satisfy the LTL specification using off-the-shelf software. We then construct the synchronous product of the Rabin automaton and $\mathcal{T}_{\text{aug}}$ in Definition 3.5.4, resulting in a nondeterministic product Rabin automaton from which a control strategy is found by solving a Rabin game [Hor05]. The result is a control strategy for which trajectories of the traffic network are guaranteed to satisfy the LTL specification.

As the discrete state space is finite, the signaling control strategy takes the form of a collection of "lookup" tables over the discrete states of the system, $\mathbb{Q}$, and there is one such table for each state in the Rabin automaton. Thus, implementing the control strategy requires implementing the underlying deterministic transition system of the specification Rabin automaton, which is interpreted as a finite memory controller that "tracks" progress of the LTL specification and updates at each time step. Given the current state of the Rabin transition system, the controller chooses the signaling input dictated by the current state of the augmented system $\mathbb{Q}$. Thus, we obtain a state feedback, finite memory controller. Additionally, the controller update only requires knowledge of the currently occupied partition of $\mathcal{Q}$, and thus does not require precise knowledge of the state $\boldsymbol{x}$.

3.6.2 Computational Requirements

For each $q \in \mathcal{Q}$ and each $\boldsymbol{s} \in \mathcal{S}$, determining the set $\{q' \mid q \xrightarrow{\boldsymbol{s}} q'\}$ requires first computing $\overline{\texttt{Post}}(\mathcal{I}_q, \boldsymbol{s})$, which requires computing $F_\ell(\cdot)$ at $2n_\mathcal{D}$ points for each $\ell \in \mathcal{L}$. Since $F_\ell(\cdot, \boldsymbol{s}, \cdot)$ is only a function of the links in $\mathcal{L}^{\text{loc}}$, each computation of this function requires time $O(1)$ assuming the average number of links at an intersection does not change with network size. Thus $\overline{\texttt{Post}}(\mathcal{I}_q, \boldsymbol{s})$ is computed in time $O(|\mathcal{L}|n_\mathcal{D})$. Then, we identify the set $\mathcal{Q}'$ of boxes that intersect $\overline{\texttt{Post}}(\mathcal{I}_q, \boldsymbol{s})$. As described in Section 3.5.2, this requires $2|\mathcal{Q}|$ comparisons of vectors of length $|\mathcal{L}|$ and thus is done in time $O(|\mathcal{Q}||\mathcal{L}|)$ via the algorithm in Fig. 3.5. However, for gridded box partitions, $\mathcal{Q}'$ is computed in time $O(|\mathcal{L}| \max_{\ell\in\mathcal{L}}\{N_\ell\})$ by the algorithm in Fig. 3.6. Even for nongridded box partitions, $\mathcal{Q}'$ can be computed in time $O(|\mathcal{L}| \max_{\ell\in\mathcal{L}}\{N_\ell\})$ where N_ℓ is interpreted as the number of intervals of link ℓ resulting from the coarsest refinement of the box partition that results in a gridded box partition. For a gridded partition, $|\mathcal{Q}| = \prod_{\ell\in\mathcal{L}} N_\ell$ and thus the number of boxes grows exponentially with the number of links in the network. For a nongridded box partition, the number of partitions can be substantially lower. Since $\{q' \mid q \xrightarrow{\boldsymbol{s}} q'\}$ must be computed for each q and $\boldsymbol{s}$, constructing $\mathcal{T}$ requires time $O(|\mathcal{Q}|^2|\mathcal{S}||\mathcal{L}|^2 n_\mathcal{D})$ when using the algorithm in Fig. 3.5 or time $O(|\mathcal{Q}||\mathcal{S}| \max_{\ell\in\mathcal{L}}\{N_\ell\}|\mathcal{L}|^2 n_\mathcal{D})$ for the algorithm in Fig. 3.6.

We briefly compare these computational requirements to that of polyhedral methods such as those in [Yor+12]. As the dynamics in (3.8)–(3.10) are piecewise affine, such methods can in principle be applied here. Computing $\texttt{Post}(\mathcal{I}_q, \boldsymbol{s})$ requires polyhedral affine transformations and polyhedral geometric sums, operations that scale exponentially in $|\mathcal{L}|$ [KV10b; Her+13]. To determine if $\texttt{Post}(\mathcal{I}_q, \boldsymbol{s})$ intersects another polytope, geometric differences are required, which again scales exponentially with $|\mathcal{L}|$.

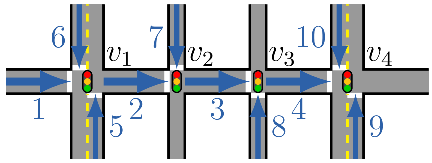


Figure 3.7: Signalized network consisting of a major corridor road (links 1, 2, 3, and 4) which intersects minor cross streets (links 5, 6, 7, 8, 9, and 10).

3.7 Case Study

We consider the example network in Fig. 3.7 which consists of a main corridor (links 1, 2, 3, and 4) with intersecting cross streets (links 5, 6, 7, 8, 9, and 10) and four intersections, a commonly encountered network configuration. The gray links exit the network and are not explicitly modeled. The network parameters are $(x_1^{\text{cap}}, \dots, x_{10}^{\text{cap}}) = (40, 50, 50, 50, 40, 40, 40, 40, 40, 40)$, $(c_1, \dots, c_{10}) = (20, 20, 20, 20, 10, 10, 10, 10, 10, 10)$, $\beta_{12} = \beta_{23} = \beta_{34} = \beta_{62} = \beta_{52} = 0.5$, $\beta_{73} = \beta_{84} = 0.9$, $\alpha_{62}^{\{1\}} = \alpha_{52}^{\{1\}} = 0.5$, and all other supply ratios are one, where the time step is 15 seconds. We assume

$$\begin{aligned}\mathcal{D} =&\{\bar{d} \mid \mathbf{0} \leq \bar{d} \leq [10\ 0\ 0\ 0\ 10\ 10\ 0\ 0\ 10\ 10]\}\\ &\cup \{\bar{d} \mid \mathbf{0} \leq \bar{d} \leq [10\ 0\ 0\ 0\ 10\ 10\ 10\ 10\ 0\ 0]\}.\end{aligned} \tag{3.30}$$

We further assume the available signals are $\mathcal{S}_{v_1} = \{\{1\}, \{5, 6\}\}$, $\mathcal{S}_{v_2} = \{\{2\}, \{7\}\}$, $\mathcal{S}_{v_3} = \{\{3\}, \{8\}\}$, and $\mathcal{S}_{v_4} = \{\{4\}, \{9, 10\}\}$. We wish to find a control policy for the four signalized intersections that satisfies the LTL property $\varphi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4$ where

$$\begin{aligned}\varphi_1 =& \square\lozenge(\boldsymbol{s}_{v_1} = \{5, 6\}) \wedge \square\lozenge(\boldsymbol{s}_{v_2} = \{7\})\\ &\wedge \square\lozenge(\boldsymbol{s}_{v_3} = \{8\}) \wedge \square\lozenge(\boldsymbol{s}_{v_4} = \{9, 10\})\end{aligned} \tag{3.31}$$

"Each signal actuates cross street traffic infinitely often"

$$\varphi_2 = \lozenge\square\big((x_1 \leq 30) \wedge (x_2 \leq 30) \wedge (x_3 \leq 30) \wedge (x_4 \leq 30)\big) \tag{3.32}$$

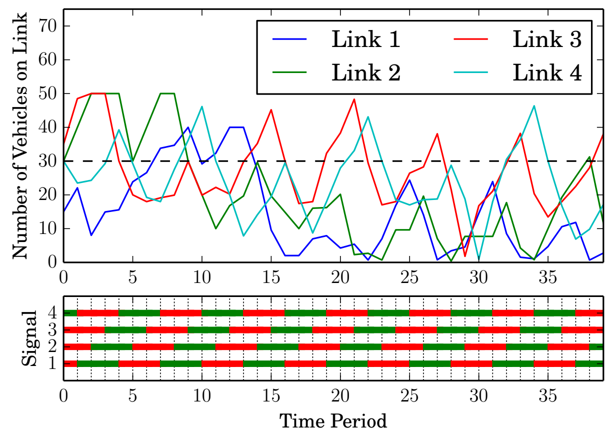
"Eventually, links 1, 2, 3, and 4 have fewer than 30 vehicles on each link and this remains true for all time"

$$\varphi_3 = \square\big(\neg(\boldsymbol{s}_{v_4} = \{4\}) \wedge \bigcirc(\boldsymbol{s}_{v_4} = \{4\}) \rightarrow \bigcirc\bigcirc(\boldsymbol{s}_{v_4} = \{4\})\big) \tag{3.33}$$

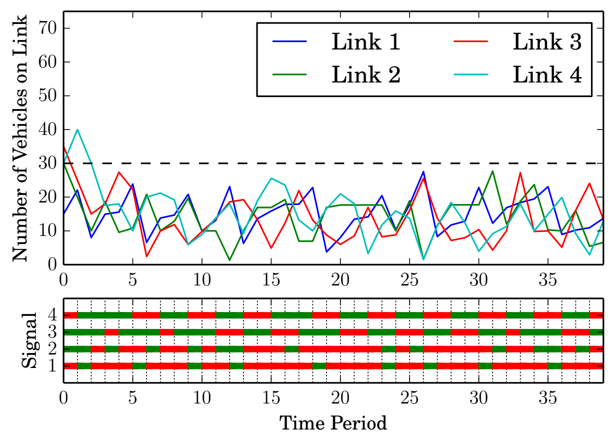
$$\begin{aligned}\varphi_4 =& \square\big(\neg(\boldsymbol{s}_{v_4} = \{9, 10\}) \wedge \bigcirc(\boldsymbol{s}_{v_4} = \{9, 10\})\\ &\rightarrow \bigcirc\bigcirc(\boldsymbol{s}_{v_4} = \{9, 10\})\big)\end{aligned} \tag{3.34}$$

For φ_3 (resp. φ_4), "The signal at intersection v_4 must actuate corridor traffic (resp. cross street traffic) for at least two sequential time-steps."

Thus, φ_2 reflects our preference for actuating corridor traffic and ensures that eventually, links 2, 3, and 4 have "adequate supply" because if the number of vehicles on these links is



(a)



(b)

Figure 3.8: (a) A sample trajectory of a naïve strategy that actuates corridor traffic in a fixed "green wave" pattern. This policy does not satisfy the desired control objective, in particular, (3.32) is not satisfied. (b) A sample trajectory resulting from the synthesized control policy that is guaranteed to satisfy the LTL policy (3.31)–(3.34). In the lower plots of (a) and (b), green (resp., red) for the signal trace indicates corridor traffic (resp., cross street traffic) is actuated.

less than 30, then these links can always accept upstream demand, thus avoiding *congestion* (congestion occurs when demand is greater than supply). Condition φ_1 ensures that, despite the preference for facilitating traffic along the corridor, we must infinitely often actuate traffic at the cross streets. Conditions φ_3 and φ_4 are needed if, *e.g.* there exists crosswalks at intersection v_4 and a minimum amount of time is required to allow pedestrians to cross. Note that repeated application of the $\bigcirc$ ("next") operator allows us to consider finite time horizons as in (3.33) and (3.34).

We partition the state space into 408 boxes that favors larger boxes when there are fewer total vehicles in the network. There are 16 signaling inputs, and thus, the number of states in the transition system $\mathcal{T}_{\text{aug}}$ is $|\mathbb{Q}| = 6528$. The Rabin automaton generated from φ contains 62 states and one acceptance pair. Computing the finite state abstraction $\mathcal{T}$ took 22.4 seconds. In contrast, the computation would be intractable using polyhedral methods. Computing the product automaton took 30.9 minutes and computing the control strategy took 15.5 minutes on a Macbook Pro with a 2.3 GHz processor where we use the Rabin game solver in `conPAS2` [Yor+12], however `conPAS2` is written in MATLAB and the synthesis process is likely to be much more efficient if implemented in C or C++ and optimized. Furthermore, all computations can be performed offline and some are parallelizable, such as computing the product automaton. Finally, we note that the computed control strategy is implemented with minimal online costs.

Fig. 3.8(a) shows a sample trajectory of the network using a naïve coordinated signaling strategy whereby each intersection actuates corridor traffic for three time steps and then cross traffic for three time steps, and the actuation times are offset to facilitate a "green wave", a commonly employed strategy for coordinating signaling along a corridor. The exogenous disturbance is generated uniformly randomly from $\mathcal{D}$. The trajectories are not guaranteed to satisfy the control objective, in particular, φ_2 is violated. Fig. 3.8(b) shows a sample trajectory of the system with a control strategy synthesized using the finite state abstraction augmented with signal history and the LTL requirement above. The control strategy is correct-by-construction and thus guaranteed to satisfy φ from any initial state.

We see that the synthesized controller reacts to increased vehicles on the corridor by actuating the corridor links, thereby preventing congestion (inadequate supply) along the corridor. At the same time, the controller actuates cross streets when doing so does not adversely affect conditions on the corridor (*i.e.*, cause congestion). In contrast, the fixed time controller in Fig 3.8(a) is not able to react to the current conditions of the network and fails to prevent congestion along the corridor; in fact, links 2, 3, and 4 periodically reach full capacity.

3.8 Discussion

We have proposed a framework for synthesizing a control strategy for a traffic network that ensures the resulting traffic dynamics satisfy a control objective expressed in linear temporal logic (LTL).

This large collection of possible control objectives is well-suited for modern transportation infrastructure where there are many, sometimes competing, objectives. For example, the desire for maintaining large throughput of a main arterial road corridor must be balanced with the need to actuate minor cross streets. Modern transportation infrastructure also requires consideration of pedestrian movements and other transportation such as bicycles, which leads to additional constraints, *e.g.* "minimimum green times" that allow adequate time for pedestrians to cross a street. In addition, as population densities increase, control approaches that actively mitigate congestion are increasingly necessary and difficult to design. Finally, increased demand also means that traffic infrastructure is often operated near capacity, and thus it is critical that control methods are robust to unpredictable conditions such as traffic accidents, inclement weather, or "one-off" events such as concerts or sports events.

In addition to offering a novel domain for applying formal methods tools in a control theory setting, we have identified and exploited key properties of traffic networks to allow efficient computation of a finite state abstraction. Future research will investigate systematic methods for determining an appropriate box partition to further reduce the number of states in the computed abstraction. Additionally, traffic networks are often composed of tightly coupled neighborhoods and towns connected by sparse longer roads, and such networks may be amenable to a compositional formal methods approach using an assume-guarantee framework [CGP99].

Chapter 4

A Compartmental Model for Traffic Flow Networks

Despite the large number of traffic modeling approaches considered in the literature (see [Hel01; HB01] for reviews), little attention has been paid to the qualitative dynamical properties of traffic flow models for general network topologies. For example, models such as [PBH90; MP90; Leb96] and the celebrated *Cell Transmission Model (CTM)* of Daganzo [Dag94; Dag95] were primarily developed for simulation with few analytical results available. The primary exception is [Gom+08] which provides a thorough investigation of the CTM when modeling a stretch of highway. The authors characterize equilibria and stability properties for this specific network class, but the results are not extended to more general networks, the authors assume a specific class of linear supply and demand functions, and the dynamics resulting from infeasible onramp demands are not fully analyzed.

In Chapter 3, we focused on control synthesis for a discrete-time model of traffic flow along an arterial network. We identified properties of the flow model that enabled efficient abstraction and control synthesis. In this chapter, we propose a general model that encompasses the CTM as defined in [Dag94; Dag95; Gom+08] and extends the model to general nonlinear supply and demand functions and to more general network topologies. Using this model, we significantly extend the few existing results on equilibria and convergence such as [Gom+08] and we present a simple linear program for obtaining a *ramp metering* control strategy that achieves the maximum possible steady-state network throughput.

Our work is related to the *dynamical flow networks* recently proposed in [Com+13a; Com+13b] and further studied in [CLS15]. In [Com+13a; Com+13b], downstream supply is not considered and thus downstream congestion does not affect upstream flow, an unrealistic assumption for traffic modeling. In [CLS15], the authors allow flow to depend on the density of downstream links, but the paper focuses on throughput optimality of a particular class of routing policies that ensure the resulting dynamics are *monotone* [Hir85; AS03]. In contrast, the model proposed here is generally not monotone. Furthermore, the adaptation to the CTM described briefly in [CLS15, Section II.C] differs from our model in the following important respects: the model as discussed in [CLS15, Section II.C] assumes a path graph network topology, requires identical links (*i.e.*, identical supply and demand functions), and only considers trajectories in the region in which supply does not restrict flow (that is, $\alpha^v(x) = 1$ for all $v \in \mathcal{V}$ in our model), which is shown to be positively invariant given their assumptions. In this work, we generalize each of these restrictions.

In a separate direction of research, many network models attempt to apply single road PDE models such as [LW55; Ric56] directly to networks, see [GP06] for a thorough treatment. Recent results such as [CC12] and [Han+12] provide analytical tools for traffic network estimation and modeling using PDE models. The CTM and related models, including our proposed model, can be considered to be a discretization of an appropriate PDE model [Leb96]. Alternatively, these models and the model we propose in this work fit into the broad class of *compartmental systems* that model the flow of a substance among interconnected "compartments" [San78; MKO78; JS93].

In Section 4.1, we propose the traffic network model. In Section 4.2, we discuss conditions under which our model is and is not monotone. In Section 4.3, we characterize existence and uniqueness of equilibrium flows. We demonstrate how the preceding analysis can be used for ramp metering in Section 4.4. Some of the proofs are collected at the end in an appendix.

4.1 Dynamic Model of Traffic

4.1.1 Network Structure

A traffic network consists of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{O})$ with *junctions* $\mathcal{V}$ and *ordinary links* $\mathcal{O}$ along with a set of *onramps* $\mathcal{R}$ which serve as entry points into the network. For $\ell \in \mathcal{O}$, let $\eta(\ell)$ denote the head vertex of link ℓ and let $\tau(\ell)$ denote the tail vertex of link ℓ, and traffic flows from $\tau(\ell)$ to $\eta(\ell)$. Each onramp $\ell \in \mathcal{R}$ directs an exogenous input flow onto $\mathcal{G}$ via a junction, and $\eta(\ell) \in \mathcal{V}$ for $\ell \in \mathcal{R}$ denotes the entry junction for onramp ℓ. By convention, $\tau(\ell) = \emptyset$ for all $\ell \in \mathcal{R}$. Ordinary links (resp., onramps) are denoted with a solid (resp., dashed) arrow in figures.

Assumption 4.1.1. *The traffic network graph is acyclic.*

Acyclicity is a reasonable assumption when modeling a portion of the road network of particular interest. For example, the road network leading out of a metropolitan area during the evening commute may be modeled as an acyclic graph where road links leading towards the metropolitan area are not modeled due to low utilization by commuters.

Let $\mathcal{L} \triangleq \mathcal{O} \cup \mathcal{R}$. For each $v \in \mathcal{V}$, we denote by $\mathcal{L}_v^{\text{in}} \subset \mathcal{L}$ the set of incoming links to node v and by $\mathcal{L}_v^{\text{out}} \subset \mathcal{L}$ the set of outgoing links, *i.e.*, $\mathcal{L}_v^{\text{in}} = \{\ell : \eta(\ell) = v\}$ and $\mathcal{L}_v^{\text{out}} = \{\ell : \tau(\ell) = v\}$. We assume $\mathcal{L}_v^{\text{in}} \neq \emptyset$ for all $v \in \mathcal{V}$, thus the network flows start at onramps. Furthermore, we assume $\mathcal{L}_{\eta(\ell)}^{\text{out}} \neq \emptyset$ for all $\ell \in \mathcal{R}$, *i.e.*, onramps always flow into at least one ordinary link downstream.

We define $\mathcal{R}^{\text{start}} \triangleq \{\ell \in \mathcal{R} : \mathcal{L}^{\text{in}}_{\eta(\ell)} \cap \mathcal{O} = \emptyset\}$ to be the set of links that lead to junctions that have only onramps as incoming links, and $\mathcal{V}^{\text{sink}} \triangleq \{v \in \mathcal{V} : \mathcal{L}^{\text{out}}_v = \emptyset\}$ to be the set of junctions that have no outgoing links.

4.1.2 Link supply and demand

For each link $\ell \in \mathcal{O}$, we associate the time-varying density $x_\ell(t) \in [0, x^{\text{jam}}_\ell]$ where $x^{\text{jam}}_\ell \in (0, \infty)$ is the *jam density* of link ℓ. For $\ell \in \mathcal{R}$, we associate the time-varying density $x_\ell(t) \in [0, \infty)$, thus onramps have no maximum density, that is, they act as "queues". We define $x \triangleq \{x_\ell\}_{\ell \in \mathcal{L}}$.

Furthermore, we assume each $\ell \in \mathcal{L}$ possesses a *demand* function $\Phi^{\text{out}}_\ell(x_\ell)$ that quantifies the amount of traffic wishing to flow downstream, and we assume each $\ell \in \mathcal{O}$ possesses a *supply* function $\Phi^{\text{in}}_\ell(x_\ell)$. We make the following assumption on the supply and demand functions:

Assumption 4.1.2. *For each $\ell \in \mathcal{O}$:*

A1. The demand function $\Phi^{out}_\ell(x_\ell) : [0, x^{jam}_\ell] \to \mathbb{R}_{\geq 0}$ is strictly increasing and continuously differentiable[1] *on $(0, x^{jam}_\ell)$ with $\Phi^{out}_\ell(0) = 0$, and $\frac{d}{dx_\ell}\Phi^{out}_\ell(x_\ell)$ is bounded above.*

A2. The supply function $\Phi^{in}_\ell(x_\ell) : [0, x^{jam}_\ell] \to \mathbb{R}_{\geq 0}$ is strictly decreasing and continuously differentiable on $(0, x^{jam}_\ell)$ with $\Phi^{in}_\ell(x^{jam}_\ell) = 0$, and $\frac{d}{dx_\ell}\Phi^{in}_\ell(x_\ell)$ is bounded below.

For each $\ell \in \mathcal{R}$:

A3. In addition to A1, $\Phi^{out}_\ell(x_\ell)$ is bounded above with supremum $\overline{\Phi}^{out}_\ell \triangleq \sup \Phi^{out}_\ell(x_\ell)$ and there exists $M_\ell > 0$ such that[2] *$\frac{d}{dx_\ell}\Phi^{out}_\ell(x_\ell) \leq M_\ell(1 + x_\ell)^{-2}$ for all x_ℓ.*

Assumption 4.1.2 implies that for each $\ell \in \mathcal{O}$, there exists unique x^{crit}_ℓ such that $\Phi^{\text{out}}_\ell(x^{\text{crit}}_\ell) = \Phi^{\text{in}}_\ell(x^{\text{crit}}_\ell) =: \Phi^{\text{crit}}_\ell$. Fig. 4.1 depicts examples of supply and demand functions.

4.1.3 Dynamic Model

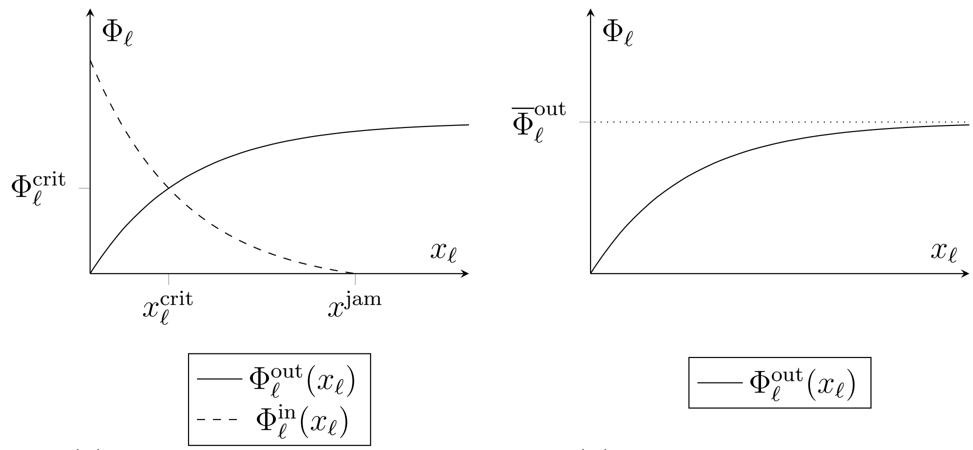
We now describe the time evolution of the densities on each link. The domain of interest is

$$\mathcal{X} \triangleq \{x : x_\ell \in [0, \infty)\ \forall \ell \in \mathcal{R} \text{ and } x_\ell \in [0, x^{\text{jam}}_\ell]\ \forall \ell \in \mathcal{O}\}. \tag{4.1}$$

Let $\mathcal{X}^\circ$ denote the interior of $\mathcal{X}$.

[1] These assumptions are made to simplify the exposition but can be relaxed to Lipschitz continuity and nonstrict monotonicty beyond the critical density; such functions are considered in the examples.

[2] The bound on the derivative of $\Phi^{\text{out}}_\ell(x_\ell)$ is a very mild technical condition used in the proofs of some propositions. For example, the condition is satisfied when $\Phi^{\text{out}}_\ell(x_\ell)$ attains its maximum.



(a) Ordinary link, $\ell \in \mathcal{O}$ (b) Onramp link, $\ell \in \mathcal{R}$

Figure 4.1: Plot of prototypical supply and demand functions $\Phi^{\text{in}}(x)$ and $\Phi^{\text{out}}(x)$ for (a) an ordinary road link, and prototypical demand function $\Phi^{\text{out}}(x)$ for (b) an onramp link.

For each onramp $\ell \in \mathcal{R}$, we assume there exists exogenous *input flow* $d_\ell(t)$. Furthermore, for each $\ell \in \mathcal{L}$ we subsequently define an output flow function $f_\ell^{\text{out}}(x)$, and for each $\ell \in \mathcal{O}$ we define an input flow function $f_\ell^{\text{in}}(x)$, such that

$$\dot{x}_\ell = F_\ell(x,t) \triangleq \begin{cases} d_\ell(t) - f_\ell^{\text{out}}(x) & \text{if } \ell \in \mathcal{R} \\ f_\ell^{\text{in}}(x) - f_\ell^{\text{out}}(x) & \text{if } \ell \in \mathcal{O} \end{cases} \tag{4.2}$$

where the functions $f_\ell^{\text{in}}(x)$ and $f_\ell^{\text{out}}(x)$ are defined below. When $d_\ell(t) \equiv d_\ell$ for constant d_ℓ for all $\ell \in \mathcal{R}$, the dynamics are autonomous and we write $F_\ell(x)$ instead. We define $F(x,t) \triangleq \begin{bmatrix} F_1(x,t) & \cdots F_{|\mathcal{L}|}(x,t) \end{bmatrix}'$ for some enumeration of $|\mathcal{L}|$ where $'$ denotes transpose, and we similarly define $F(x)$ when the dynamics are autonomous.

For each $\ell, k \in \mathcal{L}$,

$$\beta_{\ell k} \in [0,1] \tag{4.3}$$

is the *split* ratio describing the fraction of vehicles flowing out of link ℓ that are routed to link k. It follows that $\beta_{\ell k} > 0$ only if $\eta(\ell) = \tau(k)$. We require that $\sum_{k \in \mathcal{L}_v^{\text{out}}} \beta_{\ell k} \leq 1$ for all $\ell \in \mathcal{L}_v^{\text{in}}$ and $1 - \sum_{k \in \mathcal{L}_{\eta(\ell)}^{\text{out}}} \beta_{\ell k}$ is interpreted to be the fraction of the outflow on link ℓ that is routed *off* the network via, *e.g.*, an infinite capacity offramp. To ensure continuity of $f_\ell^{\text{out}}(\cdot)$, we make the following assumption:

Assumption 4.1.3. *If $v \notin \mathcal{V}^{sink}$, then $\beta_{\ell k} > 0$ for all $\ell \in \mathcal{L}_v^{in}$ and all $k \in \mathcal{L}_v^{out}$.*

A large variety of phenomenological rules for determining the outflows of road links have been proposed in the literature; see [Leb05; LK04] for several examples. We employ the *proportional priority, first-in-first-out (PP/FIFO)* rule for junctions adapted from [KV10a]:

PP/FIFO Rule. *For* $v \in \mathcal{V}^{sink}$*,* $f_\ell^{out}(x) \triangleq \Phi_\ell^{out}(x_\ell)$ *for all* $\ell \in \mathcal{L}_v^{in}$*. For each* $v \in \mathcal{V} \backslash \mathcal{V}^{sink}$*, we must ensure that the inflow of each outgoing link does not exceed the link supply. Define*

$$\alpha^v(x) \triangleq \max_{\alpha \in [0,1]} \quad \alpha \tag{4.4}$$

$$s.t. \quad \alpha \sum_{j \in \mathcal{L}_v^{in}} \beta_{jk} \Phi_j^{out}(x_j) \leq \Phi_k^{in}(x_k) \quad \forall k \in \mathcal{L}_v^{out}. \tag{4.5}$$

By scaling the demand of each link by $\alpha^v(x)$*, we ensure that the supply of each downstream link is not violated:*

$$f_\ell^{out}(x) \triangleq \alpha^v(x) \Phi_\ell^{out}(x_\ell) \quad \forall \ell \in \mathcal{L}_v^{in}. \tag{4.6}$$

To complete the model, we determine $f_\ell^{in}(x)$ *from conservation of flow:*

$$f_\ell^{in}(x) = \sum_{k \in \mathcal{L}_{\tau(\ell)}^{in}} \beta_{k\ell} f_k^{out}(x) \quad \forall \ell \in \mathcal{O}. \tag{4.7}$$

The format of (4.4) emphasizes the fact that the outflow of a link is the largest possible flow such that neither link demand nor downstream supply is exceeded and such that the outflow of all incoming links at a junction is proportional to the demand of these links. This proportionality constraint gives rise to the *proportional priority* terminology. The fixed turn ratios along with the supply and demand restrictions implies that a lack of supply of an outgoing link restricts flow to other outgoing links, a phenomenon known in the transportation literature as a *first-in-first-out (FIFO)* property [KV10a; Dag95].

4.1.4 Basic Properties of the PP/FIFO rule

We first note two properties captured by the proposed network flow model.

Lemma 4.1.1. *A simple consequence of the PP/FIFO rule is for all* $x \in \mathcal{X}$*,*

$$f_\ell^{in}(x) \leq \Phi_\ell^{in}(x_\ell) \qquad \forall \ell \in \mathcal{O} \tag{4.8}$$

$$f_\ell^{out}(x) \leq \Phi_\ell^{out}(x_\ell) \qquad \forall \ell \in \mathcal{L}. \tag{4.9}$$

We note that the domain $\mathcal{X}$ in (4.1) is easily seen to be positively invariant. Furthermore, it is not difficult to establish Lipschitz continuity of (4.2) which ensures global existence and uniqueness of solutions for piecewise continuous input flows $\{d_\ell(t)\}_{\ell \in \mathcal{R}}$ [Hal80, Chapter I]. Now suppose $d_\ell(t) \equiv d_\ell$ for some constant d_ℓ for all $\ell \in \mathcal{R}$ so that the dynamics are autonomous. From the PP/FIFO rule, we conclude that $f_\ell^{\text{out}}(x)$, and thus $F_\ell(x)$, is a continuous selection of differentiable functions determined by the constraints (4.5), that is, $F(x)$ is *piecewise differentiable* [Sch12, Section 4.1].

4.2 Lack of Monotonicity and Its Advantages

The traffic network with constant input flows is *monotone* [Hir85] if, for all links $\ell \in \mathcal{L}$ and $x \in \mathcal{X}^\circ$, we have

$$\frac{\partial F_\ell(x)}{\partial x_k} \geq 0 \quad \forall k \neq \ell \tag{4.10}$$

where, when $F_\ell(x)$ is not differentiable (which occurs on a set of measure zero), we interpret the partial derivative in an appropriate directional sense, see [Sch12, Scetion 4.1.2] for details. Cooperative systems are order preserving systems with respect to the standard order defined by the positive orthant and are a special class of *monotone* systems [Hir85; AS03]. We show that traffic networks are, in general, not cooperative. In the following example, increased demand of an incoming link at a junction causes a decrease in the inflow entering an outgoing link.

Example 4.2.1. *Consider a road network with two onramps labeled* $\{1,2\}$ *and two ordinary links labeled* $\{3,4\}$ *as shown in Fig. 4.2(b). Suppose that* $\beta_{13} = \beta_{14} = \frac{1}{2}$, $\beta_{23} = \frac{2}{3}$, *and* $\beta_{24} = \frac{1}{3}$ *and* $\{v_2, v_3\} = \mathcal{V}^{sink}$. *Furthermore, suppose* $\Phi_i^{out}(\rho_i) = \max\{\rho_i, c\}$, *for* $i = 1, 2$ *and* $c \in \mathbb{R}_{>0}$. *Now consider* $x = (x_1, x_2, x_3, x_4)$ *such that* $x_1 = x_2 < \frac{2}{9}c$ *and suppose the supply of link 3 is the limiting factor for the flow through junction* v_1 *so that* $f_3^{in}(x) = \Phi_3^{in}(x) = \alpha^{v_1}(x)\left(\frac{1}{2}x_1 + \frac{2}{3}x_2\right) = \frac{7}{6}\alpha^{v_1}(x)x_1$ *and* $f_4^{in}(x) = \frac{5}{6}\alpha^{v_1}(x)x_1$. *Now consider* $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) \triangleq (x_1, \frac{9}{2}x_2, x_3, x_4)$. *The supply of link 3 is unchanged, but the total demand from links 1 and 2 for link 3 has tripled so that* $\alpha^{v_1}(\bar{x}) = \frac{1}{3}\alpha^{v_1}(x)$. *Then*

$$f_4^{in}(\bar{x}) = \alpha^{v_1}(\bar{x})\left(\tfrac{1}{2}\bar{x}_1 + \tfrac{1}{3}\bar{x}_2\right) = \tfrac{2}{3}\alpha^{v_1}(x)x_1 < f_4^{in}(x) \tag{4.11}$$

Since $f_4^{out}(x) = f_4^{out}(\bar{x}) = \Phi_4^{out}(x_4)$, *we have* $\dot{\bar{x}}_4(0) < \dot{x}_4(0)$ *and thus there exists* $\epsilon > 0$ *such that* $x_4(\epsilon) > \bar{x}_4(\epsilon)$, *showing the system is not cooperative.*

Far from being a negative property of the model, lack of cooperativity is the main reason why ramp metering can increase network throughput or decrease average travel time. By metering the outflow of onramp 2 in Example 4.2.1, it would be possible to increase the inflow to link 4, thereby increasing throughput.

4.3 Equilibria and Stability With Constant Input Flows

We now characterize the equilibria possible from the above model with constant input flow $\{d_\ell\}_{\ell \in \mathcal{R}}$. We will investigate the case where $\lim_{t\to\infty} F_\ell(x(t)) = 0$ for all $\ell \in \mathcal{L}$, and, when input flow exceeds network capacity, the case where $\lim_{t\to\infty} F_\ell(x(t)) = 0$ for all $\ell \in \mathcal{O}$ and $\lim_{t\to\infty} f_\ell^{\text{out}}(x(t)) = c_\ell \leq d_\ell$ for some constant c_ℓ for all $\ell \in \mathcal{R}$. In the latter case, the density of some onramps (specifically, those with $c_\ell < d_\ell$) will diverge to infinity, but we will see that

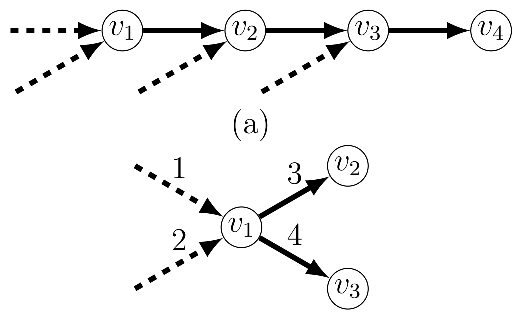


Figure 4.2: (a) A network that models a stretch of highway with onramps. Each junction is such that $|\mathcal{L}_v^{\text{out}}| \leq 1$, *i.e.*, each junction is a merge. "Offramps" are only modeled through the split ratios at junctions. This system is cooperative. (b) A network with two onramps $\{1, 2\}$ and two ordinary links $\{3, 4\}$. This example system is not cooperative due to the proportional priority assumption at junction v_1. In particular, increased density on link 2 can decrease the flow entering link 4.

a meaningful definition of equilibrium nonetheless exists. From a practical point of view, such a characterization is useful, *e.g.*, during "rush hour" when the input flow of a traffic network may exceed network capacity for a limited but extended period of time.

Define $f_{\mathcal{R}}^{\text{out}}(x) \triangleq \left[f_1^{\text{out}}(x) \quad \dots \quad f_{|\mathcal{R}|}^{\text{out}}(x)\right]'$, and likewise for $f_{\mathcal{O}}^{\text{in}}(x)$ and $f_{\mathcal{R}}^{\text{out}}(x)$ for some enumeration of $\mathcal{O}$ and $\mathcal{R}$. The dynamics (4.7) have the form

$$f_{\mathcal{O}}^{\text{in}}(x) = Af_{\mathcal{O}}^{\text{out}}(x) + Bf_{\mathcal{R}}^{\text{out}}(x). \tag{4.12}$$

where $A_{\ell k} = \beta_{k\ell}$ for $\ell, k \in \mathcal{O}$ and $B_{\ell k} = \beta_{k\ell}$ for $\ell \in \mathcal{O}$, $k \in \mathcal{R}$. Acyclicity ensures $(I - A)$ is invertible: this can be seen by noting that the only solution to the equation $f = Af$ is $f = 0$, which follows by a cascading argument since $f_\ell = 0$ for $\ell \in L_1 \triangleq \{\ell \in \mathcal{O} \mid \mathcal{L}_{\tau(\ell)}^{\text{in}} \subset \mathcal{R}\}$ since $A_{\ell k} = 0$ for all k for $\ell \in L_1$, then $f_\ell = 0$ for $\ell \in L_2 \triangleq \{\ell \in \mathcal{O} \mid \mathcal{L}_{\tau(\ell)}^{\text{in}} \subset L_1 \cup \mathcal{R}\}$ since $A_{\ell k} = 0$ for $k \notin L_1$ for $\ell \in L_2$, *etc.* That is, a nonzero solution implies vehicles remain in the network indefinitely, implying existence of a cycle in the network.

4.3.1 Feasible input flows

Definition 4.3.1. *The constant input flow* $\{d_\ell\}_{\ell \in \mathcal{R}}$ *is* feasible *if there exists density* $x^e \triangleq \{x_\ell^e\}_{\ell \in \mathcal{L}} \in \mathcal{X}$ *such that*

$$f_\ell^{out}(x^e) = d_\ell \qquad \forall \ell \in \mathcal{R} \tag{4.13}$$

$$f_\ell^{out}(x^e) = f_\ell^{in}(x^e) \qquad \forall \ell \in \mathcal{O}. \tag{4.14}$$

We define $f_\ell^e \triangleq f_\ell^{out}(x^e)$ *for all* $\ell \in \mathcal{L}$, *and the set* $\{f_\ell^e\}_{\ell \in \mathcal{L}}$ *is called an* equilibrium flow.

If the input flow is not feasible, it is said to be *infeasible*. It is clear that for a feasible input flow $\{d_\ell\}_{\ell\in\mathcal{R}}$, we must have for all $\ell \in \mathcal{R}$:

$$d_\ell \le \overline{\Phi}_\ell^{\text{out}} \qquad \begin{array}{l}\text{if there exists } x_\ell^* < \infty \text{ such that}\\ \Phi_\ell^{\text{out}}(x_\ell) = \overline{\Phi}_\ell^{\text{out}} \text{ for all } x_\ell \ge x_\ell^*\end{array} \tag{4.15}$$

$$\text{or} \quad d_\ell < \overline{\Phi}_\ell^{\text{out}} \qquad \text{if } \Phi_\ell^{\text{out}}(x_\ell) < \overline{\Phi}_\ell^{\text{out}} \text{ for all } x_\ell \in [0,\infty). \tag{4.16}$$

Proposition 4.3.1. *An equilibrium flow $\{f_\ell^e\}_{\ell\in\mathcal{L}}$ with corresponding equilibrium densities $\{x_\ell^e\}_{\ell\in\mathcal{L}}$ satisfies*

$$f_\ell^e \le \Phi_\ell^{crit} \qquad \forall \ell \in \mathcal{O}. \tag{4.17}$$

Proof. Suppose there exists $\ell \in \mathcal{O}$ such that $f_\ell^{\text{e}} > \Phi_\ell^{\text{crit}}$. By the definition of equilibrium flow, we have $f_\ell^{\text{out}}(x^{\text{e}}) = f_\ell^{\text{in}}(x^{\text{e}}) = f_\ell^{\text{e}}$. Since $f_\ell^{\text{out}}(x) \le \Phi_\ell^{\text{out}}(x_\ell)$ for all x by (4.8), we have $\Phi_\ell^{\text{out}}(x_\ell^{\text{e}}) > \Phi_\ell^{\text{crit}}$. But by Assumption 4.1.2, for all x_ℓ such that $\Phi_\ell^{\text{out}}(x_\ell) > \Phi_\ell^{\text{crit}}$, it must be $\Phi_\ell^{\text{in}}(x_\ell) \le \Phi_\ell^{\text{crit}}$ and thus $f_\ell^{\text{in}}(x^{\text{e}}) = f_\ell^{\text{e}} > \Phi_\ell^{\text{in}}(x_\ell)$, which contradicts (4.9). □

Proposition 4.3.2. *Assume (4.15)–(4.16). An input flow $\{d_\ell\}_{\ell\in\mathcal{R}}$ is feasible if and only if*

$$(I-A)^{-1}Bd \le \Phi^{crit} \tag{4.18}$$

where $d \triangleq \begin{bmatrix} d_1 & \dots & d_{|\mathcal{R}|}\end{bmatrix}'$, $\Phi^{crit} \triangleq \begin{bmatrix}\Phi_1^{crit} & \cdots & \Phi_{|\mathcal{O}|}^{crit}\end{bmatrix}'$, and $\le$ denotes elementwise inequality. Furthermore, for feasible input flows, the equilibrium flow $\{f_\ell^e\}_{\ell\in\mathcal{O}}$ is unique.

Proof. (uniqueness) By (4.12), an equilibrium flow of a feasible input flow satisfies

$$f_{\mathcal{O}}^{\text{e}} = Af_{\mathcal{O}}^{\text{e}} + Bd \tag{4.19}$$

where $f_{\mathcal{O}}^{\text{e}}$ is the vector of equilibrium flows for $\mathcal{O}$. Thus $f_{\mathcal{O}}^{\text{e}} = (I-A)^{-1}Bd$ is the unique solution to (4.19).

(only if) Applying Proposition 4.3.1 to the unique $f_{\mathcal{O}}^{\text{e}}$ above gives necessity.

(if) Let $f_{\mathcal{O}}^{\text{e}} = (I-A)^{-1}Bd$ be a candidate equilibrium flow. From (4.18), there exists unique $\{x_\ell^{\text{e}}\}_{\ell\in\mathcal{O}}$ such that $f_\ell^{\text{e}} = \Phi_\ell^{\text{out}}(x_\ell^{\text{e}})$ for which $f_\ell^{\text{e}} \le \Phi_\ell^{\text{in}}(x_\ell^{\text{e}})$ for all $\ell \in \mathcal{O}$. Furthermore, there exists $\{x_\ell^{\text{e}}\}_{\ell\in\mathcal{R}}$ such that $\Phi_\ell^{\text{out}}(x_\ell^{\text{e}}) = d_\ell$ for all $\ell \in \mathcal{R}$ by (4.15)–(4.16). We now show that these flows satisfy the PP/FIFO rule. We first show that $\alpha^v(x^{\text{e}}) = 1$. Considering (4.4)–(4.5), for all $v \in \mathcal{L}\backslash\mathcal{V}^{\text{sink}}$:

$$\sum_{\ell\in\mathcal{L}_v^{\text{in}}} \beta_{\ell k}^v \Phi_\ell^{\text{out}}(x_\ell^{\text{e}}) = \sum_{\ell\in(\mathcal{L}_v^{\text{in}}\cap\mathcal{O})} \beta_{\ell k}^v f_\ell^{\text{e}} + \sum_{\ell\in(\mathcal{L}_v^{\text{in}}\cap\mathcal{R})} \beta_{\ell k}^v d_\ell \qquad \forall k \in \mathcal{L}_v^{\text{out}} \tag{4.20}$$

$$= f_k^{\text{e}} \qquad \forall k \in \mathcal{L}_v^{\text{out}} \tag{4.21}$$

$$\le \Phi_k^{\text{in}}(x_k^{\text{e}}) \qquad \forall k \in \mathcal{L}_v^{\text{out}} \tag{4.22}$$

where (4.21) follows from (4.19), and thus (4.4)–(4.6) is satisfied when $\alpha^v(x^{\text{e}}) = 1$ for all $v \in \mathcal{V}$. Also, $f_\ell^{\text{out}}(x) = \Phi_\ell^{\text{out}}(x_\ell)$ for all $\ell \in \mathcal{L}_v^{\text{in}}$ for all $v \in \mathcal{V}^{\text{sink}}$ and (4.7) follows from (4.19), thus proving sufficiency. □

While the equilibrium flow for a feasible input flow is unique by Proposition 4.3.2, in general, multiple equilibria densities may support this equilibrium flow. However there does exists a unique equilibrium for which each link is in *freeflow*:

Definition 4.3.2. *An ordinary link $\ell \in \mathcal{O}$ is said to be in* freeflow *if $f_\ell^{out}(x) = \Phi_\ell^{out}(x_\ell)$. Otherwise, link ℓ is* congested.

Note that at equilibrium, if link $\ell \in \mathcal{O}$ is in freeflow, then necessarily $x_\ell \le x_\ell^{\text{crit}}$.

Corollary 4.3.1. *For a feasible input flow, there exists a unique equilibrium density $\{x_\ell^e\}_{\ell \in \mathcal{L}}$ such that each link $\ell \in \mathcal{O}$ is in freeflow.*

Proof. Such an equilibrium density is constructed in the proof of the "if" direction in the proof of Proposition 4.3.2. □

Furthermore, if the input flow is *strictly feasible*, then the equilibrium density is unique, and, moreover, it is asymptotically stable:

Definition 4.3.3. *A feasible input flow $\{d_\ell\}_{\ell \in \mathcal{R}}$ is said to be* strictly feasible *if the corresponding (unique) equilibrium flow satisfies $f_\ell^e < \Phi_\ell^{crit}$ for all $\ell \in \mathcal{O}$.*

Proposition 4.3.3. *If the input flow $\{d_\ell\}_{\ell \in \mathcal{R}}$ is strictly feasible, then the equilibrium density is unique and it is locally asymptotically stable.*

Proof. (uniqueness) We claim if $f_\ell^{\text{e}} < \Phi_\ell^{\text{out}}(x_\ell^{\text{e}})$ for any $\ell \in \mathcal{L}$, then there exists $k \in \mathcal{O}$ such that $f_k^{\text{e}} = \Phi_k^{\text{crit}}$. For now we take this claim to be true and note that it contradicts the hypothesis of strictly feasible flows. Thus we conclude $f_\ell^{\text{e}} = \Phi_\ell^{\text{out}}(x_\ell^{\text{e}})$ for all $\ell \in \mathcal{L}$. Furthermore, since $\Phi_\ell^{\text{out}}(\cdot)$ is strictly increasing, x_ℓ^{e} is unique.

To prove the claim, suppose $f_\ell^{\text{e}} < \Phi_\ell^{\text{out}}(x_\ell^{\text{e}})$ for some $\ell \in \mathcal{L}$ and $f_k^{\text{e}} < \Phi_k^{\text{crit}}$ for all $k \in \mathcal{O}$. There must exist $\ell' \in \mathcal{L}_{\eta(\ell)}^{\text{out}}$ such that $f_{\ell'}^{\text{e}} = \Phi_{\ell'}^{\text{in}}(x_{\ell'}^{\text{e}})$, *i.e.* there exists a lack of supply on link ℓ' since the flow on link ℓ is less than demand. Since $f_{\ell'}^{\text{e}} < \Phi_{\ell'}^{\text{crit}}$ by assumption, we then have $f_{\ell'}^{\text{e}} < \Phi_{\ell'}^{\text{out}}(x_{\ell'}^{\text{e}})$ and we find another $\ell'' \in \mathcal{L}_{\eta(\ell')}^{\text{out}}$ such that $f_{\ell''}^{\text{e}} = \Phi_{\ell''}^{\text{in}}(x_{\ell''}^{\text{e}})$, but this cannot continue indefinitely since the traffic network is acyclic and finite, thus there exists $k \in \mathcal{O}$ such that $f_k^{\text{e}} = \Phi_k^{\text{crit}}$.

(stability) Because the traffic network is directed and acyclic, it is a standard graph theoretic result that there exists a topological ordering on the junctions. From this topological ordering on the junctions, we enumerate the links with the numbering function $e(\cdot) : \mathcal{L} \to \{0, \ldots, |\mathcal{L}|\}$ where $e(\ell)$ is the enumeration of link ℓ such that for each v and $\ell \in \mathcal{L}_v^{\text{in}}$, $k \in \mathcal{L}_v^{\text{out}}$, we have $e(\ell) < e(k)$ (Such an enumeration can be accomplished by, *e.g.*, first enumerating $\mathcal{L}_{v_1}^{\text{in}}$, then $\mathcal{L}_{v_2}^{\text{in}}$, *etc.* where $v_1, \ldots, v_{|\mathcal{V}|}$ are the junctions in topological order). Furthermore, $f_\ell^{\text{out}}(x^{\text{e}}) = \Phi_\ell^{\text{out}}(x_\ell^{\text{e}})$ for all ℓ, and it is straightforward to show

$$\frac{\partial F_\ell}{\partial x_\ell}(x^{\text{e}}) < 0 \qquad \forall \ell \in \mathcal{L} \tag{4.23}$$

and

$$\frac{\partial F_\ell}{\partial x_k}(x^{\mathrm{e}}) = 0 \qquad \text{if } k \notin \mathcal{L}^{\mathrm{in}}_{\tau(\ell)}, \tag{4.24}$$

thus the Jacobian matrix of the traffic network flow evaluated at the equilibrium, $\frac{\partial F}{\partial x}(x^{\mathrm{e}})$, is lower triangular with strictly negative entries along the diagonal and therefore Hurwitz. Asymptotic stability within a neighborhood of the equilibrium follows from, *e.g.*, [Kha02, Theorem 4.7]. □

Note that Corollary 4.3.1 implies each link $\ell \in \mathcal{O}$ is in freeflow for the unique equilibrium in Proposition 4.3.3. Stability in Proposition 4.3.3 can also proved using the Lyapunov function $||F(x)||_1$. We remark that the dynamics are cooperative when all links are in freeflow. This fact allows us to conclude convergence to the equilibrium from the invariant box $\{x : 0 \le x_\ell \le x^{\mathrm{e}}_\ell \ \forall \ell \in \mathcal{L}\}$, and the argument can be extended to (not necessarily strictly) feasible input flows as in the following:

Proposition 4.3.4. *For a feasible input flow, all trajectories $x(t)$ such that $0 \le x_\ell(0) \le x^e_\ell$ for all $\ell \in \mathcal{L}$ converge to $\{x^e_\ell\}_{\ell\in\mathcal{L}}$ where $\{x^e_\ell\}_{\ell\in\mathcal{L}}$ is the unique equilibrium density in Corollary 4.3.1 for which all links $\ell \in \mathcal{O}$ are in freeflow. That is,*

$$\lim_{t\to\infty} x_\ell(t) = x^e_\ell \qquad \textit{if} \quad 0 \le x_\ell(0) \le x^e_\ell \ \forall \ell \in \mathcal{L}. \tag{4.25}$$

Proof. Let $\mathcal{A} \triangleq \{x : 0 \le x_\ell \le x^{\mathrm{e}}_\ell \ \forall \ell \in \mathcal{L}\}$ and note that $\alpha^v(x) = 1$ for all $x \in \mathcal{A}$, $v \in \mathcal{V}$ since $\mathcal{A}$ is contained within the freeflow region. In particular, $f^{\mathrm{out}}_\ell(x) = \Phi^{\mathrm{out}}_\ell(x_\ell)$ for all $\ell \in \mathcal{L}$ for $x \in \mathcal{A}$.

We now consider the general scalar system $\dot{x} = s(t) - g(x)$, $x(t) \in \mathbb{R}$ with $s(\cdot), g(\cdot)$ differentiable and monotone increasing in t and x, respectively. We claim that if $\dot{x}(0) \ge 0$, then $\dot{x}(t) \ge 0$ for all t. Indeed, suppose $\dot{x}(\tau) = 0$ for some $\tau \ge 0$. Then $\frac{d}{dt}\dot{x}\big|_{t=\tau} = \big(\dot{s}(t) + g'(x)\dot{x}\big)\big|_{t=\tau} = \dot{s}(\tau) \ge 0$.

Now consider $\ell \in \mathcal{R}^{\mathrm{start}}$. It follows that if $x_\ell(0) = 0$, then $\dot{x}_\ell = d_\ell - \Phi^{\mathrm{out}}_\ell(x_\ell)$ and $\dot{x}_\ell(0) \ge 0$. From the above analysis, we have $\dot{x}_\ell \ge 0$ for all $t \ge 0$. Furthermore, $f^{\mathrm{out}}_\ell(x(t)) = \Phi^{\mathrm{out}}_\ell(x_\ell(t))$ is monotonically increasing as a function of t. Since $\dot{x}_\ell = f^{\mathrm{in}}_\ell(x) - \Phi^{\mathrm{out}}_\ell(x_\ell)$ where $f^{\mathrm{in}}_\ell(x) = \sum_{k\in\mathcal{L}^{\mathrm{in}}_{\tau(\ell)}} \beta^{\tau(\ell)}_{k\ell}\Phi^{\mathrm{out}}_k(x_k)$, we proceed inductively to conclude that $f^{\mathrm{in}}_\ell(x(t))$ is monotonically increasing in time and $\dot{x}_\ell \ge 0$ for all $\ell \in \mathcal{L}$.

Finally, observe that $\frac{\partial F_\ell}{\partial x_k}(x) \ge 0$ for all $k \ne \ell$ and $x \in \mathcal{A}$. Therefore, $\dot{x} = F(x)$ is *cooperative* [Hir85] within $\mathcal{A}$. For such systems, if $\tilde{x}(0) \le x(0) \le \bar{x}(0)$ then $\tilde{x}(t) \le x(t) \le \bar{x}(t)$ for all $t \ge 0$ where $\tilde{x}(t), x(t)$, and $\bar{x}(t)$ are trajectories of $\dot{x} = F(x)$. Taking $\tilde{x}(0) = 0$ and $\bar{x}(t) \equiv x^{\mathrm{e}}$, we have that $\tilde{x}(t)$ is monotonically increasing in time and bounded above by x^{e} and therefore converges to an equilibrium. Corollary 4.3.1 implies x^{e} is the unique equilibrium in $\mathcal{A}$, thus $\lim_{t\to\infty}\tilde{x}(t) = x^{\mathrm{e}}$, concluding the proof. □

4.3.2 Infeasible input flows

We now wish to extend a notion of equilibrium to the case when the input flow is infeasible. We have already seen that the density of an ordinary link $\ell \in \mathcal{O}$ will not exceed the jam density x_ℓ^{jam} for any input flow. Thus any density accumulation due to the infeasible input flow must occur on the onramps $\mathcal{R}$. It is therefore reasonable to consider an equilibrium condition in which the densities, input flows, and output flows on the ordinary links, and the output flows on onramp links, approach a steady state while onramp densities may grow without bound.

Definition 4.3.4. *For any input flow* $\{d_\ell\}_{\ell\in\mathcal{R}}$*, the collection* $\{f_\ell^e\}_{\ell\in\mathcal{L}}$ *is called an* equilibrium flow *of the traffic network system if there exists a set* $\{x_\ell^e\}_{\ell\in\mathcal{L}}$ *with*

$$0 \le x_\ell^e \le x_\ell^{jam} \;\forall \ell \in \mathcal{O}, \quad \text{and} \quad 0\le x_\ell^e \le \infty \;\forall \ell \in \mathcal{R} \tag{4.26}$$

such that

$$f_\ell^e = f_\ell^{out}(x^e) = f_\ell^{in}(x^e) \;\forall \ell \in \mathcal{O} \quad \text{and} \quad f_\ell^e = f_\ell^{out}(x^e) \;\forall \ell \in \mathcal{R} \tag{4.27}$$

and, for all $\ell \in \mathcal{R}$*, either* $f_\ell^e = d_\ell$*, or* $f_\ell^e < d_\ell$ *and* $x_\ell^e = \infty$ *where* $\{f_\ell^{out}(x^e)\}_{\ell\in\mathcal{O}}$*,* $\{f_\ell^{in}(x^e)\}_{\ell\in\mathcal{O}}$*, and* $\{f_\ell^{out}(x^e)\}_{\ell\in\mathcal{R}}$ *are determined by the PP/FIFO rule and we interpret* $\Phi_\ell^{out}(\infty) \triangleq \overline{\Phi}_\ell^{out}$ *for all* $\ell \in \mathcal{R}$*. By a slight abuse of nomenclature, we call* $\{x_\ell^e\}_{\ell\in\mathcal{L}}$ *an* equilibrium density.

Definition 4.3.4 naturally extends the definition for equilibrium flow given in Definition 4.3.1 to the case when the input flow is infeasible.

Proposition 4.3.5. *For constant input flows* $\{d_\ell\}_{\ell\in\mathcal{R}}$*, an equilibrium flow exists.*

The proof is provided in the appendix.

We now consider the uniqueness of equilibrium flows. We first consider the case when the traffic network graph is a polytree:

Definition 4.3.5. *A* polytree *is a directed acyclic graph with exactly one undirected path between any two vertices.*

Equivalently, a polytree is a weakly connected directed acyclic graph for which the underlying undirected graph contains no cycles. Figs. 4.4(a), 4.2(a), and 4.2(b) depict polytrees.

Proposition 4.3.6. *Given constant infeasible input flow* $\{d_\ell\}_{\ell\in\mathcal{R}}$*. If the traffic network graph* $\mathcal{G}$ *is a polytree, then the equilibrium flow* $\{f_\ell^e\}_{\ell\in\mathcal{L}}$ *is unique.*

Proof of Proposition 4.3.6. Suppose there exists two equilibrium flows, $\{f_\ell^{\mathrm{e}}\}_{\ell\in\mathcal{L}}$ and $\{\tilde{f}_\ell^{\mathrm{e}}\}_{\ell\in\mathcal{L}}$ with corresponding equilibrium densities $\{x_\ell^{\mathrm{e}}\}_{\ell\in\mathcal{L}}$ and $\{\tilde{\rho}_\ell^{\mathrm{e}}\}_{\ell\in\mathcal{L}}$, and without loss of generality, assume $\tilde{f}_\ell^{\mathrm{e}} < f_\ell^{\mathrm{e}}$ for a particular link ℓ. If $\ell \in \mathcal{O}$, by conservation of flow and fixed turn ratios (4.7), there must exist $k \in \mathcal{L}_{\tau(\ell)}^{\mathrm{in}}$ such that $\tilde{f}_k^{\mathrm{e}} < f_k^{\mathrm{e}}$. Continuing by induction, we conclude that there must exist $j \in \mathcal{R}$ such that $\tilde{f}_j^{\mathrm{e}} < f_j^{\mathrm{e}}$, thus we assume, without loss of generality, $\ell \in \mathcal{R}$.

Observe that, since ℓ is an onramp and therefore $f_\ell^{\mathrm{e}} \leq d_\ell$, we must have $\tilde{f}_\ell^{\mathrm{e}} < d_\ell$ and thus $\Phi_\ell^{\mathrm{out}}(\tilde{\rho}_\ell^{\mathrm{e}}) > \tilde{f}_\ell^{\mathrm{e}}$. Furthermore, $\Phi_\ell^{\mathrm{out}}(\tilde{\rho}_\ell^{\mathrm{e}}) \geq \Phi_\ell^{\mathrm{out}}(x_\ell^{\mathrm{e}})$.

Let $\ell_1 \triangleq \ell$. It is the case that for any link $\ell_i \in \mathcal{L}$ for which $\Phi_{\ell_i}^{\mathrm{out}}(\tilde{\rho}_{\ell_i}^{\mathrm{e}}) > \tilde{f}_{\ell_i}^{\mathrm{e}}$, there must exist $\ell_{i+1} \in \mathcal{L}_{\eta(\ell_i)}^{\mathrm{out}}$ such that $\tilde{f}_{\ell_{i+1}}^{\mathrm{e}} = \Phi_{\ell_{i+1}}^{\mathrm{in}}(\tilde{\rho}_{\ell_{i+1}}^{\mathrm{e}})$, *i.e.* there must exist an outgoing link with insufficient supply to meet the demand.

Suppose, in this case, that $\tilde{f}_{\ell_{i+1}}^{\mathrm{e}} < f_{\ell_{i+1}}^{\mathrm{e}}$. Since $f_{\ell_{i+1}}^{\mathrm{e}} \leq \Phi_{\ell_{i+1}}^{\mathrm{in}}(x_{\ell_{i+1}}^{\mathrm{e}})$ by (4.8), we conclude $\Phi_{\ell_{i+1}}^{\mathrm{in}}(\tilde{\rho}_{\ell_{i+1}}^{\mathrm{e}}) < \Phi_{\ell_{i+1}}^{\mathrm{in}}(x_{\ell_{i+1}}^{\mathrm{e}})$ and since $\Phi_{\ell_{i+1}}^{\mathrm{in}}(\cdot)$ is decreasing, we must have $\tilde{\rho}_{\ell_{i+1}}^{\mathrm{e}} > x_{\ell_{i+1}}^{\mathrm{e}}$. Therefore $\Phi^{\mathrm{out}}(\tilde{\rho}_{\ell_{i+1}}^{\mathrm{e}}) > \Phi^{\mathrm{out}}(x_{\ell_{i+1}}^{\mathrm{e}})$. Furthermore it must be $\Phi_{\ell_{i+1}}^{\mathrm{out}}(\tilde{\rho}_{\ell_{i+1}}^{\mathrm{e}}) > \tilde{f}_{\ell_{i+1}}^{\mathrm{e}}$, and we conclude there exists $\ell_{i+2} \in \mathcal{L}_{\eta(\ell_{i+1})}^{\mathrm{out}}$ such that $\tilde{f}_{\ell_{i+2}}^{\mathrm{e}} = \Phi_{\ell_{i+2}}^{\mathrm{in}}(\tilde{\rho}_{\ell_{i+2}}^{\mathrm{e}})$. Continuing by induction, we create a sequence $\ell_1, \ell_2, \ldots, \ell_{i_0}$ until we reach link ℓ_{i_0} with $\tilde{f}_{\ell_{i_0}}^{\mathrm{e}} < f_{\ell_{i_0}}^{\mathrm{e}}$ and $\Phi_{\ell_{i_0}}^{\mathrm{out}}(\tilde{\rho}_{\ell_{i_0}}^{\mathrm{e}}) \geq \Phi_{\ell_{i_0}}^{\mathrm{out}}(x_{\ell_{i_0}}^{\mathrm{e}})$ for which there exists $\ell_{i_0+1} \in \mathcal{L}_{\eta(\ell_{i_0})}^{\mathrm{out}}$ such that $\tilde{f}_{\ell_{i_0+1}}^{\mathrm{e}} = \Phi_{\ell_{i_0+1}}^{\mathrm{in}}(\tilde{\rho}_{\ell_{i_0+1}}^{\mathrm{e}})$ but $\tilde{f}_{\ell_{i_0+1}}^{\mathrm{e}} \geq f_{\ell_{i_0+1}}^{\mathrm{e}}$(Such a i_0 must exist since the traffic network contains no directed cycles).

Now, due to fixed turn ratios, there must exist $\ell' \in \mathcal{L}_{\eta(\ell_{i_0})}^{\mathrm{in}}$ such that $\tilde{f}_{\ell'}^{\mathrm{e}} > f_{\ell'}^{\mathrm{e}}$ and, by the PP/FIFO rule,

$$\Phi_{\ell'}^{\mathrm{out}}(\tilde{\rho}_{\ell'}^{\mathrm{e}}) > \Phi_{\ell'}^{\mathrm{out}}(x_{\ell'}^{\mathrm{e}}). \tag{4.28}$$

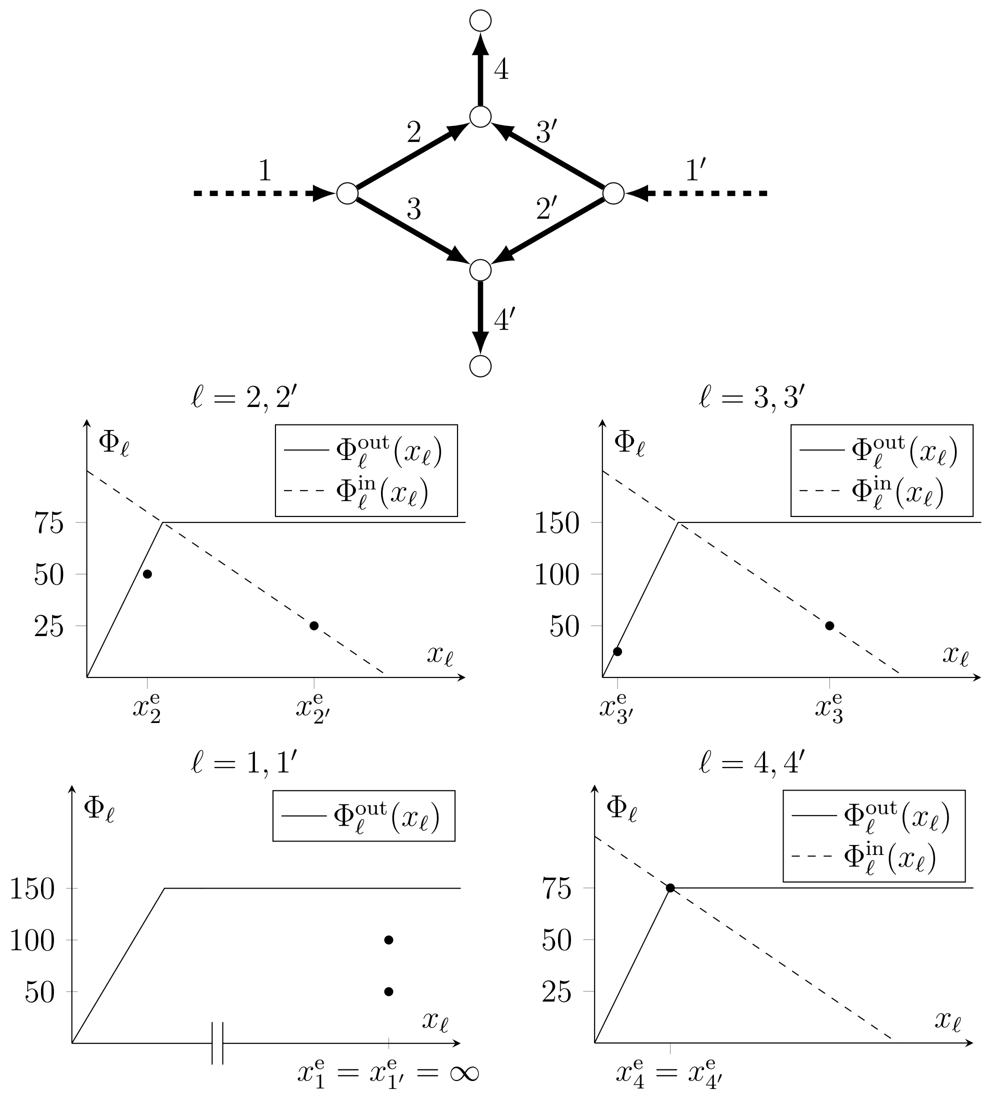
Arguing as before, we thus conclude there exists $k_1 \in \mathcal{R}$ such that $\tilde{f}_{k_1}^{\mathrm{e}} > f_{k_1}^{\mathrm{e}}$ for which $\Phi_{k_1}^{\mathrm{out}}(x_{k_1}^{\mathrm{e}}) \geq \Phi_{k_1}^{\mathrm{out}}(\tilde{\rho}_{k_1}^{\mathrm{e}})$. By a symmetric argument as above, we establish a sequence $k_1, \ldots, k_{j_0}$ until we reach link k_{j_0} with $\tilde{f}_{k_{j_0}}^{\mathrm{e}} > f_{k_{j_0}}^{\mathrm{e}}$ and $\Phi_{k_{j_0}}^{\mathrm{out}}(x_{k_{j_0}}^{\mathrm{e}}) \geq \Phi_{k_{j_0}}^{\mathrm{out}}(\tilde{\rho}_{k_{j_0}}^{\mathrm{e}})$ for which there exists $k_{j_0+1} \in \mathcal{L}_{\eta(k_{j_0})}^{\mathrm{out}}$ such that $f_{k_{j_0+1}}^{\mathrm{e}} = \Phi_{k_{j_0+1}}^{\mathrm{in}}(x_{k_{j_0+1}}^{\mathrm{e}})$ but $f_{k_{j_0+1}}^{\mathrm{e}} \geq \tilde{f}_{k_{j_0+1}}^{\mathrm{e}}$, leading to the existence of $k' \in \mathcal{L}_{\eta(k_{j_0})}^{\mathrm{in}}$ such that $f_{k'}^{\mathrm{e}} > \tilde{f}_{k'}^{\mathrm{e}}$ and $\Phi_{k'}^{\mathrm{out}}(x_{k'}^{\mathrm{e}}) > f_{k'}^{\mathrm{out}}(\tilde{\rho}_{k'}^{\mathrm{e}})$. Observe that we must have $k_{j_0} \neq \ell'$ by (4.28).

By another parallel argument, this then implies the existence of a link m for which $\tilde{f}_m^{\mathrm{e}} < f_m^{\mathrm{e}}$ and $\Phi_m^{\mathrm{out}}(\tilde{\rho}_m^{\mathrm{e}}) \geq \Phi_m^{\mathrm{out}}(x_m^{\mathrm{e}})$ for which there exists $m_1 \in \mathcal{L}_{\eta(m)}^{\mathrm{out}}$ such that $\tilde{f}_{m_1}^{\mathrm{e}} = \Phi_{m_1}^{\mathrm{in}}(\tilde{\rho}_{m_1}^{\mathrm{e}})$ and $\tilde{f}_{m_1}^{\mathrm{e}} \geq f_{m_1}^{\mathrm{e}}$, and likewise we have $m \neq k'$. Furthermore, $m_1 \neq \ell_{i_0}$, as this would imply a cycle in the underlying undirected graph. However, this process cannot continue indefinitely since the traffic network is finite.

□

If the undirected traffic network does contain cycles, then equilibrium flows may not be unique when the input flow is infeasible. Such examples with nonunique equilibrium flows are not difficult to construct, as the following example shows.

Example 4.3.1. *Consider the traffic network shown at the top of Fig. 4.3 with supply and demand curves given below in Fig. 4.3, and assume $d_1 = d_{1'} = 125$. The table at the bottom of Fig. 4.3 gives values for an equilibrium flow for this infeasible demand. But, by symmetry, we could switch f_ℓ^e and $f_{\ell'}^e$ for $\ell \in \{1, 2, 3, 4\}$ and thus the equilibrium flow is not unique.*



$$f_1^{\mathrm{e}} = 100 \qquad f_2^{\mathrm{e}} = 50 \qquad f_3^{\mathrm{e}} = 50 \qquad f_4^{\mathrm{e}} = 75$$

$$\Phi_1^{\mathrm{out}}(x_1^{\mathrm{e}}) = 150 \qquad \Phi_2^{\mathrm{out}}(x_2^{\mathrm{e}}) = 60 \qquad \Phi_3^{\mathrm{out}}(x_3^{\mathrm{e}}) = 150 \qquad \Phi_4^{\mathrm{out}}(x_4^{\mathrm{e}}) = 75$$

$$\Phi_2^{\mathrm{in}}(x_2^{\mathrm{e}}) = 80 \qquad \Phi_3^{\mathrm{in}}(x_3^{\mathrm{e}}) = 50 \qquad \Phi_4^{\mathrm{in}}(x_4^{\mathrm{e}}) = 75$$

$$f_{1'}^{\mathrm{e}} = 50 \qquad f_{2'}^{\mathrm{e}} = 25 \qquad f_{3'}^{\mathrm{e}} = 25 \qquad f_{4'}^{\mathrm{e}} = 75$$

$$\Phi_1^{\mathrm{out}}(x_{1'}^{\mathrm{e}}) = 150 \qquad \Phi_2^{\mathrm{out}}(x_{2'}^{\mathrm{e}}) = 75 \qquad \Phi_3^{\mathrm{out}}(x_{3'}^{\mathrm{e}}) = 30 \qquad \Phi_4^{\mathrm{out}}(x_{4'}^{\mathrm{e}}) = 75$$

$$\Phi_2^{\mathrm{in}}(x_{2'}^{\mathrm{e}}) = 25 \qquad \Phi_3^{\mathrm{in}}(x_{3'}^{\mathrm{e}}) = 190 \qquad \Phi_4^{\mathrm{in}}(x_{4'}^{\mathrm{e}}) = 75$$

Figure 4.3: An example demonstrating nonuniqueness of flows. In the setup, for each $\ell \in \{1, 2, 3, 4\}$, the supply and demand curves are equal for ℓ and ℓ', however we identify an equilibrium flow in which $f_\ell^{\mathrm{e}} \neq f_{\ell'}^{\mathrm{e}}$. By symmetry, we could switch f_ℓ^{e} and $f_{\ell'}^{\mathrm{e}}$ and obtain an alternative equilibrium.

We now consider convergence properties for infeasible flows. For a specific class of networks, we conclude global convergence to the equilibrium flow:

Proposition 4.3.7. *Given constant input flow $\{d_\ell\}_{\ell \in \mathcal{R}}$. If*

(1) $|\mathcal{L}_v^{out}| \leq 1$*, for all* $v \in \mathcal{V}$*,*

(2) For all $v \in \mathcal{V}$*, there exists* Γ_v *s.t.* $\gamma_\ell = \Gamma_v \; \forall \ell \in \mathcal{L}_v^{in}$

then there exists a unique equilibrium flow $\{f_\ell^e\}_{\ell \in \mathcal{L}}$ *and*

$$\lim_{t \to \infty} f_\ell^{in}(x(t)) = f_\ell^e \qquad \forall \ell \in \mathcal{O} \tag{4.29}$$

$$\lim_{t \to \infty} f_\ell^{out}(x(t)) = f_\ell^e \qquad \forall \ell \in \mathcal{L} \tag{4.30}$$

for any initial condition $x(0) \in \mathcal{X}$*.*

The condition $|\mathcal{L}_v^{\text{out}}| \leq 1$, for all $v \in \mathcal{V}$ implies that each junction is a merging junction and consists of only one outgoing link or no outgoing links. The constraint $\gamma_\ell = \Gamma_v \; \forall \ell \in \mathcal{L}_v^{\text{in}}$ for some Γ_v implies that the fraction of flow exiting a link that is routed off the network is the same for each incoming link at a particular junction.

4.4 Ramp Metering

Ramp metering is an active and rich area of research; see [PK00] for a review of approaches to ramp metering. In this section, we leverage the results on equilibria and convergence established above to design ramp metering strategies that achieve the maximum possible network throughput.

Definition 4.4.1. *A* ramp metering strategy *is a collection of functions* $\{m_\ell(t)\}_{\ell \in \mathcal{R}}$*,* $m_\ell(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *that modifies the demand function of onramps. In particular, we introduce the* metered demand function

$$\Phi_\ell^{out,m}(x_\ell(t)) \triangleq \min\{\Phi_\ell^{out}(x_\ell(t)), m_\ell(t)\} \quad \forall \ell \in \mathcal{R}. \tag{4.31}$$

The traffic network dynamics are exactly as above with the metered demand function $\Phi_\ell^{out,m}(x_\ell)$ *replacing the demand function* $\Phi_\ell^{out}(x_\ell)$ *for all* $\ell \in \mathcal{R}$*.*

In the following, we assume constant metering strategies, *i.e.*, $m_\ell(t) \triangleq m_\ell$, and thus the network flow model remains autonomous for constant input flows. We then have $\bar{\Phi}_\ell^{\text{out},m} \triangleq \min\{\overline{\Phi}_\ell^{\text{out}}, m_\ell\}$ is the maximum outflow of link $\ell \in \mathcal{R}$. The metering objective we consider is network throughput at equilibrium where *network throughput* of an equilibrium flow $\{f_\ell^{\text{e}}\}_{\ell \in \mathcal{L}}$ is defined to be $\sum_{\ell \in \mathcal{R}} f_\ell^{\text{e}}$.

The main result of this section is Theorem 4.4.1 which states that there exists a ramp metering strategy, obtained via a linear program, that induces an equilibrium with optimal network throughput. This result follows from the following proposition, which states that the resulting equilibrium flow from any ramp metering strategy is induced by a suitable choice of a (potentially different) ramp metering strategy and the assumption that each link is in freeflow.

Proposition 4.4.1. *Consider a constant ramp metering strategy $\{m_\ell\}_{\ell\in\mathcal{R}}$ that induces an equilibrium flow $\{f_\ell^e\}_{\ell\in\mathcal{L}}$ with a corresponding equilibrium density $\{x_\ell^e\}_{\ell\in\mathcal{L}}$. Then there exists another constant ramp metering strategy $\{\tilde{m}_\ell\}_{\ell\in\mathcal{R}}$ with the same equilibrium flow $\{f_\ell^e\}_{\ell\in\mathcal{L}}$ and new equilibrium density $\{\tilde{\rho}_\ell^e\}_{\ell\in\mathcal{L}}$ such that $\tilde{\rho}_\ell^e \leq x_\ell^{crit}$ for all $\ell \in \mathcal{O}$.*

Proof. We construct such an alternative metering strategy explicitly. For each $\ell \in \mathcal{R}$, define $\tilde{m}_\ell \triangleq f_\ell^{\mathrm{e}}$ (if $f_\ell^{\mathrm{e}} = d_\ell$, then we can in fact choose any $\tilde{m}_\ell \geq d_\ell$). If $d_\ell < \overline{\Phi}_\ell^{\mathrm{out}}$ and $f_\ell^{\mathrm{e}} = d_\ell$, let $\tilde{\rho}_\ell^{\mathrm{e}}$ be such that $\Phi_\ell^{\mathrm{out}}(\tilde{\rho}_\ell^{\mathrm{e}}) = f_\ell^{\mathrm{e}}$. Otherwise, let $\tilde{\rho}_\ell^{\mathrm{e}} = \infty$.

For $\ell \in \mathcal{O}$, let $\tilde{\rho}_\ell^{\mathrm{e}} \in [0, x_\ell^{\mathrm{crit}}]$ be such that $\Phi_\ell^{\mathrm{out}}(\tilde{\rho}_\ell^{\mathrm{e}}) = f_\ell^{\mathrm{e}}$ (such $\tilde{\rho}_\ell^{\mathrm{e}}$ always exists since $f_\ell^{\mathrm{e}} \leq \Phi_\ell^{\mathrm{crit}}$ by Proposition 4.3.1). Observe that $\tilde{\rho}_\ell^{\mathrm{e}} \leq x_\ell^{\mathrm{e}}$, and thus $\Phi_\ell^{\mathrm{in}}(\tilde{\rho}_\ell^{\mathrm{e}}) \geq \Phi_\ell^{\mathrm{in}}(x_\ell^{\mathrm{e}})$. It is easy to verify that $\{f_\ell^{\mathrm{e}}\}_{\ell\in\mathcal{L}}$ satisfies the definition of an equilibrium flow for the metered networked flow system where $\alpha^v(\tilde{\rho}^{\mathrm{e}}) = 1$ for all $v \in \mathcal{V}$. □

Given infeasible demand $\{d_\ell\}_{\ell\in\mathcal{R}}$, consider the following linear program:

$$\max_{\{s_\ell\}_{\ell\in\mathcal{R}},\{f_\ell^{\mathrm{e}}\}_{\ell\in\mathcal{O}}} \quad \sum_{\ell\in\mathcal{R}} s_\ell \tag{4.32}$$

$$\text{subject to} \quad f_{\mathcal{O}}^{\mathrm{e}} = A f_{\mathcal{O}}^{\mathrm{e}} + Bs \tag{4.33}$$

$$0 \leq s_\ell \leq \min\{d_\ell, \overline{\Phi}_\ell^{\mathrm{out}}\} \quad \forall \ell \in \mathcal{R} \tag{4.34}$$

$$0 \leq f_\ell^{\mathrm{e}} \leq \Phi_\ell^{\mathrm{crit}} \quad \forall \ell \in \mathcal{O} \tag{4.35}$$

where $s = \begin{bmatrix} s_1 & \dots & s_{|\mathcal{R}|}\end{bmatrix}'$. The feasible set (4.33)–(4.35) is compact and thus the convex program attains its maximum. From a solution to (4.32)–(4.35) we construct an optimal metering strategy:

Theorem 4.4.1. *Let $\{s_\ell^\star\}_{\ell\in\mathcal{R}}, \{f_\ell^{e\star}\}_{\ell\in\mathcal{O}}$ be a maximizer of (4.32)–(4.35). Then any metering strategy $\{m_\ell\}_{\ell\in\mathcal{R}}$ satisfying $m_\ell = s_\ell^\star$ if $s_\ell^\star < d_\ell$ and $m_\ell \geq s_\ell^\star$ if $s_\ell^* = d_\ell$ induces the equilibrium flow $\{f_\ell^e\}_{\ell\in\mathcal{L}}$ given by $f_\ell^e = f_\ell^{e\star}$ for all $\ell \in \mathcal{O}$ and $f_\ell^e = s_\ell^\star$ for all $\ell \in \mathcal{R}$. Furthermore, $\{f_\ell^e\}_{\ell\in\mathcal{L}}$ achieves the maximum possible network throughput.*

Proof. Proposition 4.4.1 demonstrates the sufficiency of only considering ramp metering strategies that induce an equilibrium for which each ordinary link $\ell \in \mathcal{O}$ is in freeflow. The program (4.32)–(4.35) maximizes throughput subject to this free flow. □

We remark that, as in Proposition 4.3.4, $\lim_{t\to\infty} x_\ell(t) = \tilde{\rho}_\ell^{\mathrm{e}}$ for all trajectories such that $x_\ell(0) \leq \tilde{\rho}_\ell^{\mathrm{e}}$ for all $\ell \in \mathcal{L}$ where $\{\tilde{\rho}^{\mathrm{e}}\}_{\ell\in\mathcal{L}}$ is the equilibrium density construction in Proposition 4.4.1.

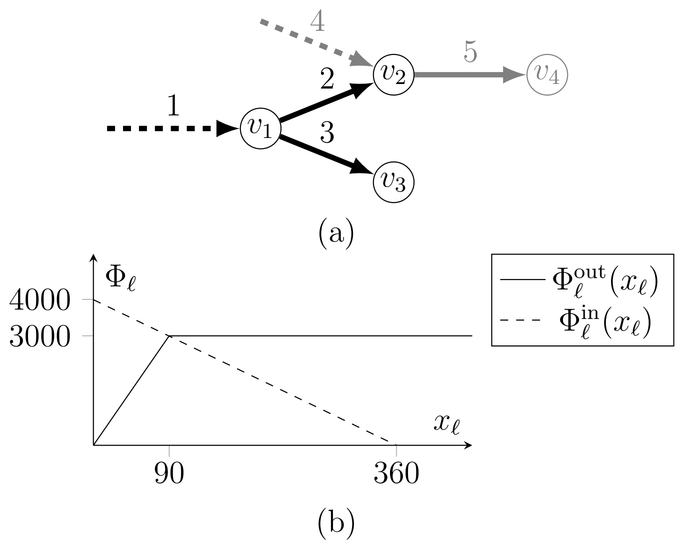


Figure 4.4: A network with two onramps $\{1, 4\}$ and three ordinary links $\{2, 3, 5\}$. The dark links and nodes are enough to demonstrate that the system is not cooperative due to the FIFO junction assumption: increased density on link 2 could restrict the outflow from link 1, and due to fixed split ratios, the inflow to link 3 also decreases. The additional shaded links and node illustrate how ramp metering increases network throughput: by restricting outflow from onramp 4, the density on link 2 may be reduced, leading ultimately to increased flow on link 3. (b) The supply and demand functions for links $\{2, 3, 5\}$.

Example 4.4.1. *Consider a road network with two onramp links labeled* $\{1, 4\}$ *and three ordinary links labeled* $\{2, 3, 5\}$ *as shown in Fig. 4.4(a). This network is not cooperative; in particular,* $\partial F_3(x)/\partial x_2 < 0$ *when link* 2 *does not have adequate supply, see [KV12, pp. 14–15] for a similar example. We assume links* $\{2, 3, 5\}$ *each have the supply and demand functions as shown in Fig. 4.4(b) and that* $\overline{\Phi}_1^{out} = 3000$ *and* $\overline{\Phi}_4^{out} = 6000$ *(units are vehicles per hour), and we assume* $\beta_{12} = \beta_{13} = \frac{1}{2}$ *and* $\beta_{25} = \beta_{45} = 1$*. With input flows* $d_1 = d_4 = 2500$*, it can be verified that the equilibrium with no ramp metering is*

$$\{f_1^e, f_2^e, f_3^e, f_4^e, f_5^e\} = \{2000, 1000, 1000, 2000, 3000\} \tag{4.36}$$

$$\{x_1^e, x_2^e, x_3^e, x_4^e, x_5^e\} = \{\infty, 270, 30, \infty, 90\} \tag{4.37}$$

and therefore the total network throughput is $f_1^e + f_4^e = 4000$*.*

Solving (4.32)–(4.35) *and applying Theorem 4.4.1, we choose* $m_1 \geq 2500, m_4 = 1750$*, and then*

$$\{f_1^e, f_2^e, f_3^e, f_4^e, f_5^e\} = \{2500, 1250, 1250, 1750, 3000\} \tag{4.38}$$

$$\{x_1^e, x_2^e, x_3^e, x_4^e, x_5^e\} = \{\infty, 37.5, 37.5, \infty, 90\} \tag{4.39}$$

with network throughput $f_1^e + f_4^e = 4250$*.*

4.A Proofs

4.A.1 Proof of Proposition 4.3.5

Proof of Proposition 4.3.5. We introduce a change of coordinates which allows us to capture "equilibrium" conditions in which onramps have infinite density.

Let

$$t(x) \triangleq \frac{x}{1+x}, \quad t^{-1}(\hat{x}) = \frac{\hat{x}}{1-\hat{x}} \tag{4.40}$$

and define the change of coordinates

$$T(x) \triangleq \begin{bmatrix} t(x_1) & \dots & t(x_{|\mathcal{R}|}) & x'_{\mathcal{O}} \end{bmatrix}' \qquad T^{-1}(\hat{x}) = \begin{bmatrix} t^{-1}(\hat{x}_1) & \dots & t^{-1}(\hat{x}_{|\mathcal{R}|}) & \hat{x}'_{\mathcal{O}} \end{bmatrix}' \tag{4.41}$$

where we assume onramps $\mathcal{R}$ are enumerated $1, \dots, |\mathcal{R}|$ and $x_{\mathcal{O}}$ is the vector of densities for links $\mathcal{O}$ for some enumeration of $\mathcal{O}$. Let $\hat{x} = \begin{bmatrix} \hat{x}'_{\mathcal{R}} & \hat{x}'_{\mathcal{O}} \end{bmatrix}' \triangleq T(x)$ and observe

$$\begin{aligned} \dot{\hat{x}}_\ell &= (1-\hat{x}_\ell)^2 F_\ell(x) \\ &= (1-\hat{x}_\ell)^2 F_\ell\left(T^{-1}(\hat{x})\right) && (4.42) \\ &=: \hat{F}_\ell(\hat{x}) \qquad\qquad \forall \ell \in \mathcal{R}. && (4.43) \end{aligned}$$

Note that only onramps undergo a coordinate change, that is, $x_{\mathcal{O}} = \hat{x}_{\mathcal{O}}$.

Similarly define

$$\hat{F}_\ell(\hat{x}) \triangleq F_\ell(T^{-1}(\hat{x})) \quad \forall \ell \in \mathcal{O} \tag{4.44}$$

so that $\dot{x}_\ell = \dot{\hat{x}}_\ell = \hat{F}_\ell(\hat{x})$ for all $\ell \in \mathcal{O}$. Let $\hat{F}(\hat{x}) \triangleq \begin{bmatrix} \hat{F}_1(\hat{x}) & \dots & \hat{F}_{|\mathcal{L}|}(\hat{x}) \end{bmatrix}'$, then $\dot{\hat{x}} = \hat{F}(\hat{x})$.

We introduce the change of coordinates so that $\hat{x}_\ell$ remains bounded even as $x_\ell \to \infty$ for $\ell \in \mathcal{R}$. Furthermore, the definition of $\hat{F}(\hat{x})$ can be suitably extended to the case where $\hat{x}_\ell = 1$ for all $\ell \in \mathcal{R}'$ for some subset $\mathcal{R}' \subseteq \mathcal{R}$, even though $T^{-1}(\hat{x})$ is not defined for such $\hat{x}$. In particular, let

$$\hat{\mathcal{X}}' \triangleq \{\hat{x} : \hat{x}_\ell \in [0,1) \ \forall \ell \in \mathcal{R} \text{ and } \hat{x}_\ell \in [0, x_\ell^{\text{jam}}] \ \forall \ell \in \mathcal{O}\} \tag{4.45}$$

and $\hat{\mathcal{X}} \triangleq \mathbf{cl}(\hat{\mathcal{X}}')$, $S \triangleq \hat{\mathcal{X}} \backslash \hat{\mathcal{X}}'$ where $\mathbf{cl}(\cdot)$ denotes closure. Observe that (4.43) and (4.44) define $\hat{F}(\cdot)$ on $\hat{\mathcal{X}}'$. Furthermore, Assumption 1 ensures that $\hat{F}(\cdot)$ is Lipschitz continuous on $\hat{\mathcal{X}}'$. Lipschitz continuity implies uniform continuity, and thus there exists a unique extension of $\hat{F}(\cdot)$ to $\hat{\mathcal{X}}$ given by [Rud76, Chapter 4]

$$\hat{F}(\hat{x}) \triangleq \lim_{\substack{\hat{y} \to \hat{x} \\ \hat{y} \in \hat{\mathcal{X}}'}} \hat{F}(\hat{x}) \quad \forall \hat{x} \in S. \tag{4.46}$$

This definition is equivalent to interpreting $\Phi_\ell^{\text{out}}(t^{-1}(1)) \triangleq \overline{\Phi}_\ell^{\text{out}}$ for $\ell \in \mathcal{R}$ in the PP/FIFO rule. Existence and uniqueness of solutions to $\dot{\hat{x}} = \hat{F}(\hat{x})$ initialized in $\hat{\mathcal{X}}$ follows readily. We now define $\bar{F}(\cdot) : \hat{\mathcal{X}} \to \mathbb{R}^{|\mathcal{L}|}$ as follows:

$$\bar{F}(\hat{x}) \triangleq \begin{cases} F(T^{-1}(\hat{x})) & \text{if } \hat{x} \in \hat{\mathcal{X}}' \\ \lim_{\hat{y}\to\hat{x},\hat{y}\in\hat{\mathcal{X}}'} \bar{F}(\hat{y}) & \text{if } \hat{x} \in S \end{cases} \tag{4.47}$$

where again the limit is guaranteed to exist because $\bar{F}(\cdot)$ is Lipschitz continuous on $\hat{\mathcal{X}}'$.

We have $\hat{\mathcal{X}}$ is compact, convex, and positively invariant. It follows that there exists a stationary point $\hat{x}^{\mathrm{e}}$ such that $\bar{F}(\hat{x}^{\mathrm{e}}) = 0$ [BM08, Theorem 4.20]. This stationary point corresponds to an equilibrium density x^{e} in the original coordinates via the map $T^{-1}(\cdot)$ in (4.41) and thus gives an equilibrium flow as defined in Definition 4.3.4. In particular, $\hat{x}^{\mathrm{e}}_\ell = 1$ for $\ell \in \mathcal{R}$ implies $x^{\mathrm{e}}_\ell = \infty$. □

4.A.2 Piecewise Differentiability

Assume $d_\ell(t) \equiv d_\ell$ for some constant d_ℓ for all $\ell \in \mathcal{R}$ so that the dynamics are autonomous. Note that the solution of (4.4)–(4.5) is

$$\alpha^v(x) = \begin{cases} \min\left\{1, \min_{k\in\mathcal{L}_v^{\mathrm{out}}}\left\{\left(\sum_{j\in\mathcal{L}_v^{\mathrm{in}}} \beta^v_{jk}\Phi^{\mathrm{out}}_j(x_j)\right)^{-1}\Phi^{\mathrm{in}}_k(x_k)\right\}\right\} & \text{if } \exists \ell \in \mathcal{L}_v^{\mathrm{in}} \text{ s.t. } x_\ell > 0 \\ 1 & \text{otherwise,} \end{cases} \tag{4.48}$$

thus $\{f^{\mathrm{out}}_\ell(x)\}_{\ell\in\mathcal{L}_v^{\mathrm{in}}}$ is uniquely defined in (4.6). Furthermore, by considering the finite set of functions possible for $\alpha^v(x)$ determined by the minimizing $k \in \mathcal{L}_v^{\mathrm{out}}$ in (4.48), we conclude that $f^{\mathrm{out}}_\ell(x)$ and thus $F_\ell(x)$ is a continuous selection of differentiable functions. Indeed, at each junction, either every outgoing link has adequate supply to accomodate the demand of incoming links, or there exists at least one link that does not have adequate supply. If more than one link does not have adequate supply, the most restrictive link determines the flow through the junction. Thus, for each $v \in \mathcal{V}$, there are $|\mathcal{L}_v^{\mathrm{out}}| + 1$ functions possible for $\alpha^v(x)$ in (4.48). We then consider $F(x)$ to be selected from $\prod_{v\in\mathcal{V}}(|\mathcal{L}_v^{\mathrm{out}}|+1)$ *modes* of the network.

Let $\mathcal{I}$ denote an index set of these possible modes, and let $F^{(i)}(x)$ for $i \in \mathcal{I}$ denote the particular mode defined implicitly by the corresponding minimizers of (4.48) for each $v \in \mathcal{V}$. The function $F(x)$ is then piecewise differentiable. Let $J^{(i)}(x)$ denote the Jacobian of $F^{(i)}(x)$, which is well defined on $\{x \in \mathcal{X}^\circ \mid F(x) = F^{(i)}(x)\}$. Consider the directional derivative

$$F'(x_0; y) \triangleq \lim_{\substack{h\to 0\\ h>0}} \frac{F(x_0+hy) - F(x_0)}{h}. \tag{4.49}$$

A key property of piecewise differentiable functions is that the derivative (4.49) exists for all $x_0 \in \mathcal{X}^\circ$ and $y \in \mathbb{R}^{|\mathcal{L}|}$, and $F'(x_0; y) \in \{J^{(i)}(x_0)y \mid i \in \mathcal{I}\}$. It follows [Sch12] that

$$\dot{F}(x) \triangleq \frac{d}{dt}F(x(t)) \in \{J^{(i)}(x)F(x) \mid i \in \mathcal{I}\}. \tag{4.50}$$

We use (4.50) below when we consider stability of the traffic network.

4.A.3 Cooperativity and Convergence in Networks with only Merging Junctions

To prove Proposition 4.3.7, we introduce the following definition from [JS93]:

Definition 4.A.1. *A matrix $A \in \mathbb{R}^{n\times n}$ is a* compartmental matrix *if $[A]_{ij} \geq 0$ for all $i \neq j$ and $\sum_{i=1}^{n}[A]_{ij} \leq 0$ for all j where $[A]_{ij}$ is the ij-th entry of A.*

Equivalently, A is a compartmental matrix if and only if A is *Metzler* [BP94] and $\mu_1(A) \leq 0$ where $\mu_1(A) \triangleq \lim_{h\to 0^+} \frac{1}{h}(||I + hA||_1 - 1)$ is the *logarithmic norm* of A and $||A||_1$ is the matrix norm induced by the vector one-norm [DH72]. This observation provides a connection to *contraction theory* for non-Euclidean norms [Son10]. In particular, Lemma 4.A.1 below shows that $F(x)$ is nonexpansive in a region of the state-space relative to a weighted one-norm.

Lemma 4.A.1. *Given $\Omega \subseteq \mathcal{X}$ and diagonal matrix W with positive entries on the diagonal such that $WJ^{(i)}(x)$ is a compartmental matrix for all $i \in \mathcal{I}$ and all $x \in \Omega^\circ$ such that $F^{(i)}(x) = F(x)$ where $J^{(i)}(x)$ denotes the Jacobian of $F^{(i)}(x)$ and Ω° denotes the interior of Ω. Then $V(x) \triangleq ||WF(x)||_1$ is decreasing along trajectories $x(t)$ of the traffic network when $x(t) \in \Omega$. Moreover, if Ω is positively invariant, then the flows of the network converge to an equilibrium flow as defined in Definition 4.3.4.*

Proof. The following proof is adapted from the proof of [MKO78, Theorem 2]. Consider the change of coordinates constructed in the proof of Proposition 4.3.5 in Section 4.A.1, and define $\bar{V}(\cdot) : \hat{\mathcal{X}} \to \mathbb{R}$ by

$$\bar{V}(\hat{x}) \triangleq ||W\bar{F}(\hat{x}))||_1 \tag{4.51}$$

where $\bar{F}(\cdot)$ is given in (4.47) so that $\bar{V}(\hat{x}) = V(x)$ when $\hat{x} = T(x)$. Let $\{F^{(i)}(\cdot) \mid i \in \mathcal{I}\}$ be the collection of modes as described in Section 4.A.2. It follows that

$$\dot{\bar{F}}(\hat{x}) \in \{\bar{J}^{(i)}(\hat{x})\bar{F}(\hat{x}) \mid i \in \mathcal{I}\} \tag{4.52}$$

where, defining $\hat{\mathcal{X}}^\circ$ to be the interior of $\hat{\mathcal{X}}$,

$$\bar{J}^{(i)}(\hat{x}) \triangleq \begin{cases} J^{(i)}(T^{-1}(\hat{x})) & \text{if } \hat{x} \in \hat{\mathcal{X}}^\circ \\ \lim_{\hat{y}\to\hat{x},\hat{y}\in\hat{\mathcal{X}}^\circ} \bar{J}^{(i)}(\hat{y}) & \text{if } \hat{x} \in \hat{\mathcal{X}}\backslash\hat{\mathcal{X}}^\circ. \end{cases} \tag{4.53}$$

By assumption and the above analysis, $W\bar{J}^{(i)}(\hat{x})$ is a compartmental matrix for all $\hat{x} \in \hat{\Omega}$ where $\hat{\Omega} \triangleq \{T(x) : x \in \Omega\}$ and all i such that $\bar{F}(\hat{x}) = \bar{J}^{(i)}(\hat{x})\bar{F}(\hat{x})$, *i.e.*, the selected index in (4.52).

There exists a vector $\nu(\hat{x}) \in \{-1, 0, 1\}^{|L|}$ such that $V(\hat{x}) = \nu(\hat{x})'W\bar{F}(\hat{x})$ and $\dot{V}(\hat{x}(t)) = \nu(\hat{x})'W\dot{\bar{F}}(\hat{x})$ with the property $\bar{F}_\ell > 0$ implies $\nu_\ell = 1$ and $\bar{F}_\ell < 0$ implies $\nu_\ell = -1$.

We drop the supscript (i) and time dependence (t) notation for clarity. Let

$$I = \{\ell \mid \bar{F}_\ell > 0, \text{ or } \bar{F}_\ell = 0, \dot{\bar{F}}_\ell > 0\} \tag{4.54}$$

$$J = \{\ell \mid \bar{F}_\ell < 0, \text{ or } \bar{F}_\ell = 0, \dot{\bar{F}}_\ell < 0\} \tag{4.55}$$

$$K = \{\ell \mid \bar{F}_\ell = 0 \text{ and } \dot{\bar{F}}_\ell = 0\}. \tag{4.56}$$

We partition $\bar{Q} \triangleq W\bar{J}$ into blocks such that

$$\begin{bmatrix} W_I \dot{\bar{F}}_I \\ W_J \dot{\bar{F}}_J \\ W_K \dot{\bar{F}}_K \end{bmatrix} = \begin{bmatrix} \bar{Q}_{II} & \bar{Q}_{IJ} & \bar{Q}_{IK} \\ \bar{Q}_{JI} & \bar{Q}_{JJ} & \bar{Q}_{JK} \\ \bar{Q}_{KI} & \bar{Q}_{KJ} & \bar{Q}_{KK} \end{bmatrix} \begin{bmatrix} \bar{F}_I \\ \bar{F}_J \\ \bar{F}_K \end{bmatrix} \tag{4.57}$$

where $\bar{Q}_{IJ} = [q_{ij}]_{i\in I, j\in J}$, $\bar{F}_I = \{\bar{F}_\ell\}_{\ell\in I}$, *etc.* and W_I, W_J, W_K are the diagonal blocks of W. Then

$$\dot{\bar{V}} = \mathbf{1}^T W_I \dot{\bar{F}}_I - \mathbf{1}^T W_J \dot{\bar{F}}_J \tag{4.58}$$

$$= \mathbf{1}^T \bar{Q}_{II} \bar{F}_I + \mathbf{1}^T \bar{Q}_{IJ} \bar{F}_J - \mathbf{1}^T \bar{Q}_{JI} \bar{F}_I - \mathbf{1}^T \bar{Q}_{JJ} \bar{F}_J \tag{4.59}$$

$$\begin{aligned} = &- (2\mathbf{1}^T \bar{Q}_{JI} + \mathbf{1}^T \bar{Q}_{KI} + \alpha_I)\bar{F}_I \\ &+ (2\mathbf{1}^T \bar{Q}_{IJ} + \mathbf{1}^T \bar{Q}_{KJ} + \alpha_J)\bar{F}_J \end{aligned} \tag{4.60}$$

where

$$\alpha_I = -(\mathbf{1}^T \bar{Q}_{II} + \mathbf{1}^T \bar{Q}_{JI} + \mathbf{1}^T \bar{Q}_{KI}) \tag{4.61}$$

$$\alpha_J = -(\mathbf{1}^T \bar{Q}_{IJ} + \mathbf{1}^T \bar{Q}_{JJ} + \mathbf{1}^T \bar{Q}_{KJ}) \tag{4.62}$$

and $-\alpha_I \le 0$ and $-\alpha_J \le 0$ (where $\le$ is interpreted elementwise) because the column sums of $\bar{Q}$ are less than or equal to zero since $\bar{Q}$ is assumed to be a compartmental matrix. Note that, additionally, the entries of $\bar{Q}_{IJ}$, $\bar{Q}_{JI}$, $\bar{Q}_{IK}$, $\bar{Q}_{KI}$, $\bar{Q}_{JK}$, and $\bar{Q}_{KJ}$ are nonnegative since $\bar{Q}$ is nonnegative for entries not on the diagonal. It then follows that (4.60) is nonpositive, *i.e.*, $\dot{\bar{V}} \le 0$.

Supposing Ω is positively invariant, we show convergence to an equilibrium flow via LaSalle's invariance principle. To that end, define $E \triangleq \{\hat{x} : \dot{\bar{V}}(\hat{x}) = 0\}$. Let $\hat{z}(t)$ be a trajectory completely contained in E and consider $L_+(t) = \{\ell \mid \bar{F}_\ell(\hat{z}(t)) > 0\}$ and $L_-(t) = \{\ell \mid \bar{F}_\ell(\hat{z}(t)) < 0\}$. We have $\bar{V}(\hat{z}) \equiv V_\infty$ for some V_∞, and thus

$$\mathbf{1}' W_{L_+} \bar{F}_{L_+} - \mathbf{1}' W_{L_-} \bar{F}_{L_-} \equiv V_\infty. \tag{4.63}$$

We now claim the sets $L_+(t)$ and $L_-(t)$ are monotonically increasing with respect to set inclusion. To prove the claim, observe that since $\dot{\bar{V}} \equiv 0$ and considering (4.59), we have $\mathbf{1}^T \bar{Q}_{JI} \bar{F}_I \equiv 0$ and $\mathbf{1}^T \bar{Q}_{IJ} \bar{F}_J \equiv 0$. This implies

$$\sum_{k\in I} q_{\ell k} \bar{F}_k \equiv 0 \quad \text{for all } \ell \in J \tag{4.64}$$

$$\sum_{k\in J} q_{\ell k} \bar{F}_k \equiv 0 \quad \text{for all } \ell \in I \tag{4.65}$$

since the entries of $\bar{Q}_{JI}$, $\bar{Q}_{IJ}$, $\bar{F}_I$ are nonnegative and the entries of $\bar{F}_J$ are nonpositive and thus each entry of $\bar{Q}_{JI}\bar{F}_I$ and $\bar{Q}_{IJ}\bar{F}_J$ must be zero. Since $L_+ \subset I$ and $L_- \subset J$, and $\bar{F}_\ell = 0$ for all $\ell \in I \backslash L_+$ and for all $\ell \in J \backslash L_-$, we have

$$\sum_{k \in L_+} q_{\ell k}\bar{F}_k \equiv 0 \text{ for all } \ell \in L_- \tag{4.66}$$

$$\sum_{k \in L_-} q_{\ell k}\bar{F}_k \equiv 0 \text{ for all } \ell \in L_+. \tag{4.67}$$

Consider $\ell \in L_+(\tau)$ at some time τ, and from (4.57), we have

$$W_\ell \dot{\bar{F}}_\ell = q_{\ell\ell}\bar{F}_\ell + \sum_{k \in L_+ \backslash \{\ell\}} q_{\ell k}\bar{F}_k + \sum_{k \in L_-} q_{\ell k}\bar{F}_k \tag{4.68}$$

$$= q_{\ell\ell}\bar{F}_\ell + \sum_{k \in L_+ \backslash \{\ell\}} q_{\ell k}\bar{F}_k \tag{4.69}$$

$$\geq q_{\ell\ell}\bar{F}_\ell \tag{4.70}$$

where $q_{\ell\ell} \leq 0$. It follows that since $\bar{F}_\ell(\hat{z}(\tau)) > 0$, then

$$\bar{F}_\ell(\hat{z}(t)) \geq \bar{F}_\ell(\hat{z}(\tau))e^{(t-\tau)q_{\ell\ell}/W_\ell} > 0 \tag{4.71}$$

for all $t \geq \tau$, and thus $\bar{F}_\ell(\hat{z}(t)) > 0$ for all $t \geq \tau$, implying that L_+ is monotonically increasing. A similar analysis holds for $L_-(t)$, proving the claim.

Furthermore, because $\dot{\bar{V}}(\hat{z}) \equiv 0$, we must have $\alpha_I\bar{F}_I \equiv 0$ and $\alpha_J\bar{F}_J \equiv 0$ since, in (4.60), each term $\bar{Q}_{JI}$, $\bar{Q}_{KI}$, α_I, $\bar{Q}_{IJ}$, $\bar{Q}_{KJ}$, and α_J are nonnegative. Since $\mathbf{1}^T\bar{Q}\bar{F} = -(\alpha_I\bar{F}_I + \alpha_J\bar{F}_J)$, we have that $\mathbf{1}^T\bar{Q}\bar{F} = \mathbf{1}'W\bar{J}\bar{F}_\ell \equiv 0$ and thus $\mathbf{1}'\dot{\bar{F}} \equiv 0$. Therefore

$$\mathbf{1}'W_{L_+}\bar{F}_{L_+} + \mathbf{1}'W_{L_-}\bar{F}_{L_-} \equiv C \tag{4.72}$$

for C some constant.

Combining (4.63) and (4.72), we have that $2\mathbf{1}'W_{L_-}\bar{F}_{L_-} \equiv C - V_\infty$. If $C - V_\infty \neq 0$, then $\int_0^\infty \mathbf{1}'W_{L_-}\bar{F}_{L_-}\, dt = -\infty$, but this is a contradiction since $\dot{z} = \bar{F}(\hat{z})$ where $z(t) \triangleq T^{-1}(\hat{z})$ and z is bounded below. Thus $L_-(t) \equiv \emptyset$.

Let $\mathcal{R}^\infty \triangleq \{\ell \in \mathcal{R} \mid \exists c > 0 \text{ s.t. } \bar{F}_\ell(\hat{z}) \equiv c\}$, and let $M_+ \triangleq L_+ \backslash \mathcal{R}^\infty$. Since $2\mathbf{1}'W_{L_+}\bar{F}_{L_+} = V_\infty + C$, it follows that $\mathbf{1}'W_{M_+}\bar{F}_{M_+}(\hat{x}) \equiv C_2$ for some constant $C_2 \geq 0$. If $C_2 > 0$, then $\int_0^{-\infty} \mathbf{1}'W_{M_+}\bar{F}_{M_+}\, dt = \infty$, which is also a contradiction since with $y(t) \triangleq z(-t)$ and $\hat{y}(t) \triangleq \hat{z}(-t)$, we have $\dot{y}_\ell = -\bar{F}_\ell(\hat{y})$ for all $\ell \in M_+$ and $y_\ell(t)$ is bounded below. Therefore $C_2 = 0$, and we have shown $\bar{F}_\ell(\hat{z}) \equiv 0$ for all $\ell \in \mathcal{L} \backslash \mathcal{R}^\infty$. Combined with the definition of $\mathcal{R}^\infty$, this implies that $\hat{z}(t) \equiv \hat{z}^{\mathrm{e}} \in \hat{\mathcal{X}}$. Furthermore, these are exactly the conditions required such that $z^{\mathrm{e}} \triangleq T^{-1}(\hat{z}^{\mathrm{e}})$ is an equilibrium density as defined in Definition 4.3.4.

□

We now turn our attention to the class of networks considered in Proposition 4.3.7. For networks satisfying condition 1 of Proposition 4.3.7, $\mathcal{V}^{\text{sink}}$ is a singleton, suppose $\mathcal{V}^{\text{sink}} = \{v_{\text{sink}}\}$. Furthermore, for each $\ell \in \mathcal{L}$ there exists a unique path $\{\ell_1, \dots, \ell_{n_\ell}\} \subset \mathcal{L}$ with $\ell_1 = \ell$ such that $\eta(\ell_{n_\ell}) = v_{\text{sink}}$. Supposing 1) and 2) of Proposition 4.3.7, let

$$w_\ell \triangleq \begin{cases} 1 - \Gamma_{\eta(\ell)} & \text{if } \Gamma_{\eta(\ell)} < 1 \\ 1 & \text{if } \Gamma_{\eta(\ell)} = 1 \end{cases} \qquad \forall \ell \in \mathcal{L} \tag{4.73}$$

$$W_\ell \triangleq w_{\ell_1} \cdot \ldots \cdot w_{\ell_{n_\ell}}. \tag{4.74}$$

Lemma 4.A.2. *Given a traffic network with constant input flows $\{d_\ell\}_{\ell \in \mathcal{R}}$ satisfying the 1) and 2) of Proposition 4.3.7. Define w_ℓ and W_ℓ as in (4.73)–(4.74), and let $W \triangleq \text{diag}(W_1, \dots, W_{|\mathcal{L}|})$. Then $W\left(\frac{\partial F^{(i)}}{\partial x}(x)\right)$ is a compartmental matrix for all $i \in \mathcal{I}$ such that $F(x) = F^{(i)}(x)$ and $x \in \mathcal{X}^\circ$.*

Proof. Consider a particular link ℓ and the corresponding ℓth column of $\partial F^{(i)}/\partial x$ for some $i \in \mathcal{I}$. In the following, we omit the superscript (i) and all partial derivatives are assumed to correspond to the mode i. We have

$$\begin{aligned} \textstyle\sum_{k \in \mathcal{L}} W_k \frac{\partial F_k}{\partial x_\ell} = \frac{\partial}{\partial x_\ell}\Big(&- \textstyle\sum_{k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}} W_k f_k^{\text{out}} + W_\ell f_\ell^{\text{in}} \\ &- \textstyle\sum_{k \in \mathcal{L}^{\text{in}}_{\eta(\ell)}} W_k f_k^{\text{out}} + \textstyle\sum_{k \in \mathcal{L}^{\text{out}}_{\eta(\ell)}} W_k f_k^{\text{in}}\Big). \end{aligned} \tag{4.75}$$

It can be shown that, for networks such that $|\mathcal{L}_v^{\text{out}}| \leq 1$ for all $v \in \mathcal{V}$, we have $\frac{\partial F_k}{\partial x_\ell} \geq 0$ for all $\ell \neq k$, *i.e.*, the system is *cooperative* [Hir85]. Observe that $\beta_{k\ell}^{\tau(\ell)} = (1 - \Gamma_{\tau(\ell)})$ and $W_k = (1 - \Gamma_{\tau(\ell)}) W_\ell$ for all $k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}$ for all $\ell \in \mathcal{O}$. We subsequently show

$$\frac{\partial}{\partial x_\ell}\left(- \textstyle\sum_{k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}} W_k f_k^{\text{out}}(x) + W_\ell f_\ell^{\text{in}}(x)\right)(x) = 0 \tag{4.76}$$

$$\frac{\partial}{\partial x_\ell}\left(- \textstyle\sum_{k \in \mathcal{L}^{\text{in}}_{\eta(\ell)}} W_k f_k^{\text{out}}(x) + \textstyle\sum_{k \in \mathcal{L}^{\text{out}}_{\eta(\ell)}} W_k f_k^{\text{in}}(x)\right)(x) \leq 0 \tag{4.77}$$

for all $x \in \mathcal{X}^\circ$. Combining (4.76)–(4.77) with (4.75) gives $\sum_{k \in \mathcal{L}} W_k \frac{\partial F_k}{\partial x_\ell} \leq 0$ for all ℓ, thus proving the claim. To prove (4.76)–(4.77), consider a particular $x \in \mathcal{X}^\circ$:
(Flows at $\tau(\ell)$) If upstream demand exceeds the supply of link ℓ, that is, $\ell \in \mathcal{O}$ and $\Phi_\ell^{\text{in}}(x_\ell) < \sum_{k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta_{k\ell}^{\tau(\ell)} \Phi_k^{\text{out}}(x_k)$, then the PP/FIFO rule stipulates

$$f_k^{\text{out}}(y) = \Phi_\ell^{\text{in}}(y_\ell) \frac{\Phi_k^{\text{out}}(y_k)}{\sum_{j \in \mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta_{j\ell}^{\tau(\ell)} \Phi_j^{\text{out}}(y_j)} \; \forall k \in \mathcal{L}^{\text{in}}_{\tau(\ell)} \tag{4.78}$$

$$f_\ell^{\text{in}}(y) = \Phi_\ell^{\text{in}}(y_\ell) \tag{4.79}$$

for all $y \in \mathcal{B}_\epsilon(x)$ for some $\epsilon > 0$ where $\mathcal{B}_\epsilon(x)$ is the ball of radius ϵ centered at x. Then $\sum_{k\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} f^{\text{out}}_k(y) = \left(1-\Gamma_{\tau(\ell)}\right)^{-1}\Phi^{\text{in}}_\ell(y_\ell)$ and

$$-\sum_{k\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} W_k f^{\text{out}}_k(y) + W_\ell f^{\text{in}}_\ell(y) = 0 \;\forall y \in \mathcal{B}_\epsilon(x) \tag{4.80}$$

which implies (4.76).

If link ℓ has adequate supply, we have $f^{\text{out}}_k(x) = \Phi^{\text{out}}_k(x_k)$ for $k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}$ and $f^{\text{in}}_\ell(x) = \sum_{k\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta^{\tau(\ell)}_{k\ell}\Phi^{\text{out}}(x_k)$, neither of which is a function of x_ℓ, and thus also (4.76) holds.

(Flows at $\eta(\ell)$) By hypothesis, $\mathcal{L}^{\text{out}}_{\eta(\ell)}$ is either empty or a singleton. If it is empty, then $f^{\text{out}}_k(x) = \Phi^{\text{out}}_k(x_k)$ for all $k \in \mathcal{L}^{\text{in}}_{\eta(\ell)}$ and the lefthand side of (4.77) is

$$-\sum_{k\in\mathcal{L}^{\text{in}}_{\eta(\ell)}} W_k \frac{\partial}{\partial x_\ell} f^{\text{out}}_k(x) = -W_\ell \Phi'^{\text{out}}_\ell(x_\ell) < 0 \tag{4.81}$$

and (4.77) holds. If $\mathcal{L}^{\text{out}}_{\eta(\ell)}$ is nonempty, let $\mathcal{L}^{\text{out}}_{\eta(\ell)} = \{m\}$ and observe that $W_k = (1-\Gamma_{\eta(\ell)})W_m$ and $\beta^{\eta(\ell)}_{km} = 1-\Gamma_{\eta(\ell)}$ for all $k \in \mathcal{L}^{\text{in}}_{\eta(\ell)}$. Suppose link m has adequate supply for upstream demand so that $f^{\text{out}}_k(x) = \Phi^{\text{out}}_k(x_k)$ for all $k \in \mathcal{L}^{\text{in}}_{\eta(\ell)}$ and $f^{\text{in}}_m(x) = \sum_{k\in\mathcal{L}^{\text{in}}_{\eta(\ell)}} \beta^{\eta(\ell)}_{km}\Phi^{\text{out}}_k(x_k)$. Then the lefthand side of (4.77) is $-W_\ell\Phi'^{\text{out}}_\ell(x_\ell) + W_m\beta^{\eta(\ell)}_{\ell m}\Phi'^{\text{out}}_\ell(x_\ell) = 0$ and therefore (4.77) holds. If link m has inadequate supply, there exists $\epsilon > 0$ such that

$$\sum_{k\in\mathcal{L}^{\text{in}}_{\eta(\ell)}} f^{\text{out}}_k(y) = \left(1-\Gamma_{\eta(\ell)}\right)^{-1}\Phi^{\text{in}}_m(y_m) \tag{4.82}$$

$$\forall y \in \mathcal{B}_\epsilon(x) \tag{4.83}$$

$$f^{\text{in}}_m(y) = \Phi^{\text{in}}_m(y_m) \qquad \forall y \in \mathcal{B}_\epsilon(x). \tag{4.84}$$

Then $-\sum_{k\in\mathcal{L}^{\text{in}}_{\eta(\ell)}} W_k f^{\text{out}}_k(y) + W_m f^{\text{in}}_m(y) = 0$ for all $y \in \mathcal{B}_\epsilon(x)$ and (4.77) follows. □

Proof of Proposition 4.3.7. A network satisfying the 1) and 2) of the proposition consists of only merging junctions and is necessarily a polytree, thus Proposition 4.3.6 ensures uniqueness of the equilibrium flow.

By Lemma 4.A.2 above, $WJ^{(i)}(x)$ is a compartmental matrix for all $i \in \mathcal{I}$ such that $F(x) = F^{(i)}(x)$ and all $x \in \mathcal{X}^\circ$. Applying Lemma 4.A.1 with $\Omega \triangleq \mathcal{X}$ completes the proof.

□

4.B An Auxiliary Result on Norm-Based Lyapunov Functions via Contraction Analysis

In this appendix, we present an auxiliary result on contraction inspired by the results of this chapter. It is well know that for contractive systems with an equilibrium, the distance to the equilibrium evaluated along any trajectory decreases exponentially with time. We prove that, additionally, the magnitude of the velocity evaluated along any trajectory decreases exponentially with time, thus giving an alternative choice of Lyapunov function.

Consider the nonlinear system

$$\dot{x} = f(x) \tag{4.85}$$

for $x \in \mathbb{R}^n$ and continuously differentiable $f(\cdot)$. Denote the Jacobian as

$$J(x) \triangleq \frac{\partial f}{\partial x}(x). \tag{4.86}$$

We assume there exists a unique equilibrium of (4.85), and without loss of generality, we assume $x = 0$ is the equilibrium point. Then $f(0) = 0$ and $f(x) \neq 0$ for all $x \neq 0$.

Let $|\cdot|$ be a vector norm on $\mathbb{R}^n$, $\|\cdot\|$ its induced matrix norm, and $\mu(A) \triangleq \lim_{h\to 0^+} \frac{1}{h}(\|I + hA\| - 1)$ the associated matrix measure.

Theorem 4.B.1. *If there exists $c > 0$ such that $\mu(J(x)) \leq -c$ for all $x \in \mathbb{R}^n$ then $x = 0$ is globally asymptotically stable and*

$$|f(x(t))| \leq |f(x(0))|e^{-ct}. \tag{4.87}$$

Proof. Let $V(x) \triangleq |f(x)|$. $V(x(t))$ is then absolutely continuous as a function of t and therefore

$$\dot{V}(x(t)) \triangleq \lim_{h\to 0+} \frac{V(x(t+h)) - V(x(t))}{h} \tag{4.88}$$

for almost all t. Furthermore,

$$\lim_{h\to 0^+} \left| \frac{|f(x(t+h)))| - |f(x) + h\dot{f}(x)|}{h} \right| \leq \lim_{h\to 0^+} \left| \frac{f(x(t+h)) - f(x)}{h} - \dot{f}(x) \right| \tag{4.89}$$

$$= 0 \tag{4.90}$$

where we use the definition of $\dot{f}(x)$ and the fact $\big||x| - |y|\big| \leq |x - y|$. Since also $\dot{f}(x) = J(x)f(x)$, we combine (4.88)–(4.90) and obtain

$$\dot{V}(x(t)) = \lim_{h\to 0+} \frac{|f(x) + hJ(x)f(x)| - |f(x)|}{h} \tag{4.91}$$

$$\leq \lim_{h\to 0+} \frac{\|I + hJ(x)\| \cdot |f(x)| - |f(x)|}{h} \tag{4.92}$$

$$= \lim_{h\to 0+} \frac{\|I + hJ(x)\| - 1}{h} |f(x)| \tag{4.93}$$

$$= \mu(J(x))V(x). \tag{4.94}$$

By hypothesis, we then have $\dot{V}(x) \leq -cV(x)$, and (4.87) follows by integration. Furthermore, $f(x)$ is a diffeomorphism from $\mathbb{R}^n$ to itself [DV08, pp. 34–35], thus $V(x)$ is positive definite and radially unbounded. By standard Lyapunov theory, *e.g.* [Kha02], global asymptotic stability follows. □

It is well known that for contractive systems with $\mu(J(x)) \leq -c$ for all x, $|x(t) - \xi(t)| \leq |x(0) - \xi(0)|e^{-ct}$ for any pair of trajectories $x(\cdot)$ and $\xi(\cdot)$ [Son10]. Thus, taking $\xi(t) \equiv 0$, we see that for such systems, $V(x) = |x|$ is a Lyapunov function guaranteeing global asymptotic stability. Theorem 4.B.1 demonstrates that $V(x) = |f(x)|$ is an alternative choice of Lyapunov function. Additionally, it is straightfoward that for any continuously differentiable class-$\mathcal{K}$ function $\rho(\cdot)$,

$$x \mapsto \rho(|f(x)|) \tag{4.95}$$

is also a Lyapunov function.

Furthermore, Theorem 4.B.1 is a generalization of Krasovskii's well known sufficient condition for asymptotic stability [Kra63]. Indeed, Krasovskii's theorem is a special case of Theorem 4.B.1 when $|\cdot|$ is taken to be a weighted Euclidean norm, *i.e.* $|x| = (x^T Px)^{1/2}$ for some positive definite matrix P and therefore [DV08]

$$\mu(A) = \bar{\lambda}_i \left(\frac{B + B^T}{2} \right), \quad B \triangleq P^{1/2}AP^{-1/2} \tag{4.96}$$

where $\bar{\lambda}_i(M)$ denotes the largest eigenvalue of M, from which it follows

$$\mu(A) < -c \quad \iff \quad PA + A^T P < -2cP. \tag{4.97}$$

Using Theorem 4.B.1 with (4.97) and taking $\rho(y) \triangleq y^2$ in (4.95) gives a standard version of Krasovskii's theorem.

Chapter 5

Mixed Monotonicity in Traffic Flow Dynamics

We continue the study of the dynamical behavior of a compartmental system model of traffic flow initiated in Chapter 4. Recall that the flow of mass from a compartment, or *link*, to downstream links is governed by a local flow *demand* as well as downstream *supply* of capacity available to accomodate incoming flow. In this chapter, we focus on the mixed monotonicity property for traffic networks. In this chapter, we define mixed monotonicity for continuous-time systems, where as the mixed monotone property defined in Chapter 2 and applied to traffic networks in Chapter 5 was defined in discrete-time.

This chapter builds on recent results [GH06; Gom+08; CLS15; LCS14] which have utilized a compartmental systems approach [JS93] to analyze the dynamical behavior of transportation networks. In these prior works, the strongest results rely on the system dynamics being *monotone* whereby trajectories of the system preserve a partial ordering [Hir85; Smi95]. Yet, as detailed in Section 4.2 and elaborated on in Section 5.2.1, vehicular traffic networks with diverging junctions and fixed routing policies are not monotone, as noted in [KV12], since downstream congestion on one outgoing link blocks incoming flow to neighboring outgoing links [MD02]. Thus, a significant gap exists in the literature for understanding the dynamics of traffic flow networks. This chapter works towards filling this gap.

We first show that traffic dynamics possess a *mixed monotonicity* property, which is much weaker than monotonicity. Despite the lack of monotonicity in the standard form of the dynamics, such systems can be embedded into a higher dimensional monotone system, [GH94; KM06; ESS06; Smi08]. With this key observation, we bring the powerful tools of monotone system theory to this more general problem.

Next, we identify a certain class of *polytree* networks for which, using this embedding, we prove global asymptotic convergence to a unique equilibrium. In exchange for the generality offered by mixed monotonicity, we obtain only a sufficient condition that requires equilibria of the embedding system to be unique. We show that this condition is satisfied for polytree networks. We show through an example that the polytree restriction is necessary for the embedding system to have a unique equilibrium.

The remainder of the chapter is organized as follows: In Section 5.1, we define the network model. In Section 5.2, we show that the dynamics are not monotone but are mixed monotone, which allows the system to be embedded in a larger dimensional monotone system. In Section 5.3, we apply these results to establish global stability for a class of networks, and we consider two examples. In Section 5.4, we provide a discussion of the approach and comparisons to other techniques for stability analysis of flow networks, and we give concluding remarks.

5.1 A Compartmental Model of Traffic Flow

In this section, we recall the basic compartmental model presented in Chapter 4, with minor alterations. In particular, as noted below, split ratios are determined only by the outgoing link at a junction, and not by the incoming-outgoing link pair.

5.1.1 Notation

All inequalities are interpreted elementwise, *e.g.*, for $x, y \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$ for $i = 1, \dots n$ where x_i, y_i denote the ith element of x, y. We denote the vector of all zeros by 0 when its dimension is clear from context. We denote the set of nonnegative real numbers by $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$.

5.1.2 Network Topology

A traffic network consists of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{O})$ with *junctions* $\mathcal{V}$ and *ordinary links* $\mathcal{O}$ along with a set of *entry links* $\mathcal{R}$ which are entry points into the network. Physically, a link represents a segment of roadway, and we assume $\mathcal{G}$ is a connected graph. Let $\eta(\ell)$ and $\tau(\ell)$ denote the head and tail junction of link $\ell \in \mathcal{L}$, respectively, where we assume $\eta(\ell) \neq \tau(\ell)$, *i.e.*, no self-loops. Traffic flows from $\tau(\ell)$ to $\eta(\ell)$. By convention, $\tau(\ell) = \emptyset$ for all $\ell \in \mathcal{R}$.

Let $\mathcal{L} \triangleq \mathcal{O} \cup \mathcal{R}$. For each $v \in \mathcal{V}$, we denote by $\mathcal{L}_v^{\text{in}} \subset \mathcal{L}$ the set of input links to node v and by $\mathcal{L}_v^{\text{out}} \subset \mathcal{L}$ the set of output links, *i.e.* $\mathcal{L}_v^{\text{in}} = \{\ell : \eta(\ell) = v\}$ and $\mathcal{L}_v^{\text{out}} = \{\ell : \tau(\ell) = v\}$. We assume $\mathcal{L}_v^{\text{in}} \neq \emptyset$ for all $v \in \mathcal{V}$, thus the network flow starts at entry links. Furthermore, we assume $\mathcal{L}_{\eta(\ell)}^{\text{out}} \neq \emptyset$ for all $\ell \in \mathcal{R}$, *i.e.* entry links always flow into at least one ordinary link downstream. If $|\mathcal{L}_v^{\text{in}}| > 1$, then v is a *merging junction*, and if $|\mathcal{L}_v^{\text{out}}| > 1$, then v is a *diverging junction*.

Define $\mathcal{V}^{\text{sink}} \triangleq \{v \in \mathcal{V} \mid \mathcal{L}_v^{\text{out}} = \emptyset\}$ to be the set of junctions that have no outgoing links and

$$\mathcal{L}^{\text{sink}} \triangleq \{\ell \in \mathcal{L} \mid \eta(\ell) \in \mathcal{V}^{\text{sink}}\} \tag{5.1}$$

the corresponding set of input links to these junctions.

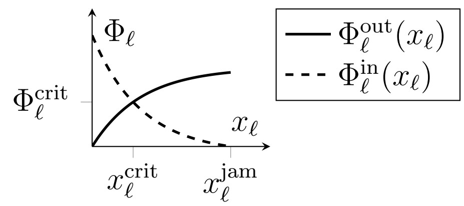


Figure 5.1: Plot of prototypical supply and demand functions $\Phi_\ell^{\mathrm{in}}(x_\ell)$ and $\Phi_\ell^{\mathrm{out}}(x_\ell)$.

5.1.3 Link supply and demand

Each link $\ell \in \mathcal{L}$ has state $x_\ell(t) \in [0, x_\ell^{\mathrm{jam}}]$ where $x_\ell^{\mathrm{jam}} \in (0, \infty)$ is the *jam density* of link ℓ. Furthermore, each link possesses a state-dependent *demand* function $\Phi_\ell^{\mathrm{out}}(x_\ell)$ and *supply* function $\Phi_\ell^{\mathrm{in}}(x_\ell)$ satisfying:

Assumption 5.1.1. *For each $\ell \in \mathcal{L}$:*

- *The demand function $\Phi_\ell^{out}(x_\ell) : [0, x_\ell^{jam}] \to \mathbb{R}_{\geq 0}$ is strictly increasing and Lipschitz continuous with $\Phi_\ell^{out}(0) = 0$.*
- *The supply function $\Phi_\ell^{in}(x_\ell) : [0, x_\ell^{jam}] \to \mathbb{R}_{\geq 0}$ is strictly decreasing and Lipschitz continuous with $\Phi_\ell^{in}(x_\ell^{jam}) = 0$.*

Assumption 5.1.1 implies that for each $\ell \in \mathcal{L}$, there exists unique x_ℓ^{crit} such that

$$\Phi_\ell^{\mathrm{out}}(x_\ell^{\mathrm{crit}}) = \Phi_\ell^{\mathrm{in}}(x_\ell^{\mathrm{crit}}) =: \Phi_\ell^{\mathrm{crit}}. \tag{5.2}$$

Figure 5.1 depicts examples of supply and demand functions satisfying Assumption 5.1.1.

5.1.4 Dynamic Model

At each junction $v \in \mathcal{V}$, there exists a collection of fixed *split* ratios $\{\beta_{\to\ell}\}_{\ell \in \mathcal{L}_v^{\mathrm{out}}}$ with each $\beta_{\to\ell} > 0$ describing how incoming flow is split among outgoing links. Conservation of flow implies

$$\sum_{\ell \in \mathcal{L}_v^{\mathrm{out}}} \beta_{\to\ell} \leq 1 \quad \forall v \in \mathcal{V}, \tag{5.3}$$

where strict inequality in (5.3) implies that a fraction of the flow is routed off the network via, *e.g.*, an unmodeled off-ramp.

Note that we associate a single split ratio with each output link rather than with each input/output link pair as was the case in the previous chapter. This implies that split ratios cannot differ for different incoming links and, as we will see in Section 5.2, leads to a mixed monotonicity property. If we assume each junction is either single-input or single-output as in [Dag95; LCS14], then there is no distinction between the two models.

The flow dynamics of the network are as follows:

$$\dot{x}_\ell = f_\ell^{\text{in}}(x) - f_\ell^{\text{out}}(x) \qquad \forall\ \ell \in \mathcal{L} \tag{5.4}$$

$$=: F_\ell(x) \tag{5.5}$$

$$\alpha^v(x) \triangleq \min\left\{1, \min_{\ell \in \mathcal{L}_v^{\text{out}}} \left\{ \frac{\Phi_\ell^{\text{in}}(x_\ell)}{\beta_{\to\ell} \sum_{k \in \mathcal{L}_v^{\text{in}}} \Phi_k^{\text{out}}(x_k)} \right\}\right\} \ \forall v \in \mathcal{V} \tag{5.6}$$

$$f_\ell^{\text{out}}(x) = \alpha^{\eta(\ell)}(x) \Phi_\ell^{\text{out}}(x_\ell) \tag{5.7}$$

$$f_\ell^{\text{in}}(x) = \begin{cases} \min\{d_\ell, \Phi_\ell^{\text{in}}(x_\ell)\} & \text{if } \ell \in \mathcal{R} \\ \beta_{\to\ell} \sum_{k \in \mathcal{L}_{\tau(\ell)}^{\text{in}}} f_k^{\text{out}}(x_k) & \text{if } \ell \in \mathcal{O}. \end{cases} \tag{5.8}$$

Above, $\alpha^v(x) \in [0, 1]$ defined in (5.6) is a factor that scales the outgoing flow of each input link at the junction v such that the incoming flow to each output link is less than its supply. Thus, the model (5.4)–(5.8) maximizes the flow through links while ensuring that the outgoing flow at each link does not exceed the link's demand and the incoming flow does not exceed the link's supply. For entry link $\ell \in \mathcal{R}$, the incoming flow is additionally restricted to not exceed the exogenous demand d_ℓ. The model further requires that, at each junction, the collection of outgoing flows for the input links is proportional to the collection of flow demand from the input links. This condition is referred to as *proportional-priority*, [KV10a; KV12], and differs from the constant priority model employed in Chapter 3. From (5.4)–(5.8), we have forward invariance of the domain

$$\mathcal{X} \triangleq \prod_{\ell \in \mathcal{L}} [0, x_\ell^{\text{jam}}]. \tag{5.9}$$

The model (5.4)–(5.8) is modified from the model proposed in Chapter 4 so that entry links have finite capacity. This is reasonable for traffic networks where entry links correspond to onramps with finite storage capacity.

Define the routing matrices $R_\mathcal{O} \in \mathbb{R}^{\mathcal{O} \times \mathcal{O}}$ and $R_\mathcal{R} \in \mathbb{R}^{\mathcal{O} \times \mathcal{R}}$ elementwise as follows:

$$[R_\mathcal{O}]_{k\ell} = \begin{cases} \beta_{\to k} & \text{if } k \in \mathcal{L}_{\eta(\ell)}^{\text{out}} \\ 0 & \text{otherwise} \end{cases} \quad \forall \ell, k \in \mathcal{O} \tag{5.10}$$

$$[R_\mathcal{R}]_{k\ell} = \begin{cases} \beta_{\to k} & \text{if } k \in \mathcal{L}_{\eta(\ell)}^{\text{out}} \\ 0 & \text{otherwise} \end{cases} \quad \forall k \in \mathcal{O}, \forall \ell \in \mathcal{R}. \tag{5.11}$$

Assumption 5.1.2. *The matrix $(I - R_\mathcal{O})$ is invertible.*

Assumption 5.1.2 is equivalent to the assertion that eventually all vehicles will leave the network and is thus a natural assumption on the split ratios [Var13b]. Let

$$P = (I - R_\mathcal{O})^{-1} R_\mathcal{R}, \tag{5.12}$$

that is, P describes how the flow from entry links is routed through the network. As $(I - R_{\mathcal{O}})^{-1} = I + R_{\mathcal{O}} + R_{\mathcal{O}}^2 + \ldots$, we have $P \geq 0$. Let

$$f_\ell^{\mathrm{e}} = \begin{cases} d_\ell & \text{if } \ell \in \mathcal{R} \\ [Pd]_\ell & \text{if } \ell \in \mathcal{O}. \end{cases} \tag{5.13}$$

where $[Pd]_\ell$ is the ℓth entry of Pd.

Assumption 5.1.3. *The input flow $d = \{d_\ell\}_{\ell \in \mathcal{R}}$ satisfies*

$$f_\ell^e < \Phi_\ell^{crit} \quad \forall \ell \in \mathcal{L}. \tag{5.14}$$

Assumption 5.1.3 states that the network has adequate capacity to accomodate the input flow d, that is, d is *strictly feasible* [Gom+08]. It follows from Assumption 5.1.3 that

$$x_\ell^{\mathrm{e}} \triangleq (\Phi_\ell^{\mathrm{out}})^{-1}(f_\ell^{\mathrm{e}}) < x_\ell^{\mathrm{crit}} \tag{5.15}$$

for all $\ell \in \mathcal{L}$ constitutes an equilibrium of the traffic network dynamics (5.4)–(5.8), let $x^{\mathrm{e}} = \{x_\ell^{\mathrm{e}}\}_{\ell \in \mathcal{L}}$. Indeed, for this case, $\alpha^v(x^{\mathrm{e}}) = 1$ for all v, that is, the outgoing flow on every link is equal to demand. A key result of this chapter is that this equilibrium is unique and globally asymptotically stable for a class of networks defined subsequently.

5.2 Nonmonotone Behavior of Traffic Networks

5.2.1 Lack of monotonicity

Consider the system $\dot{x} = G(x)$, $x \in X \subseteq \mathbb{R}^n$ where X is forward invariant and has convex interior. Suppose $G(\cdot)$ is locally Lipschitz and satisfies

$$\frac{\partial G_i}{\partial x_j}(x) \geq 0 \quad \forall x \in X, \forall i \neq j \tag{5.16}$$

whenever the derivative exists. Then the system $\dot{x} = G(x)$ is order-preserving with respect to the positive orthant $\mathbb{R}^n_{\geq 0}$, that is,

$$x(0) \leq y(0) \text{ implies } x(t) \leq y(t) \quad \forall t \geq 0 \tag{5.17}$$

where $x(t)$, $y(t)$ are solutions of the system with initial conditions $x(0)$, $y(0)$. A dynamical system $\dot{x} = G(x)$ satisfying (5.16) is said to be *monotone with respect to the positive orthant*, or simply *monotone* [Hir85; Smi95].

Traffic flow networks with no diverging junctions are monotone, as has been noted and studied in Chapter 4 and in [Gom+08]. However, networks with diverging junctions are not monotone. To see this, consider a diverging junction v and assume some link $\ell \in \mathcal{L}_v^{\text{out}}$ is the unique minimizer in (5.6) for some x so that, for all y in a neighborhood of x,

$$\alpha^v(y) = \Phi_\ell^{\text{in}}(y_\ell)\left(\beta_{\to\ell}\sum_{k\in\mathcal{L}_v^{\text{in}}}\Phi_k^{\text{out}}(y_k)\right)^{-1}. \tag{5.18}$$

It follows that, for all $k \in \mathcal{L}_v^{\text{out}}$, $k \neq \ell$ with $\eta(k) \neq \eta(\ell)$,

$$\frac{\partial F_k}{\partial x_\ell}(x) = \frac{\partial f_k^{\text{in}}}{\partial x_\ell}(x) = \frac{\partial}{\partial x_\ell}\left(\frac{\beta_{\to k}}{\beta_{\to\ell}}\Phi_\ell^{\text{in}}(x_\ell)\right) < 0, \tag{5.19}$$

and thus the system is not monotone.

Remark 4. *It is standard to generalize the condition* (5.16) *to partial orders with respect to arbitrary orthants [HS05a], and one may wonder if the traffic dynamics are monotone with respect to some alternative orthant order. The answer is negative; indeed, the relationship* (5.19) *holds for any pair of output links, and for a junction with at least three output links, this implies that the system is not monotone with respect to any orthant via the graphical condition in,* e.g., *[AS04, Proposition 2].*

The interpretation of (5.19) is that, due to congestion (lack of supply) on link ℓ, an increase in the number of vehicles on link ℓ would worsen the congestion (decrease supply), and vehicles destined for link ℓ would further block flow to other outgoing links, causing a reduction in the incoming flow to these links. Thus, lack of monotonicity is indeed expected for traffic networks and is relevant for transportation engineering because it is a primary explanation for why traffic control methods such as ramp metering are able to increase throughput.

The phenomenon of downstream traffic blocking flow to other downstream links at a diverging junction is referred to as the *first-in-first-out (FIFO)* property, [Dag95; KV10a], and it is a feature of traffic flow that has been observed even on wide freeways with many lanes, [MD02; CAH02]. Some of the recent literature in dynamical flow models propose alternative modeling choices for diverging junctions, *e.g.*, [CLS15; LCS14], which ensures that the resulting dynamics are monotone and therefore do not exhibit this FIFO property.

5.2.2 A weaker property: Mixed monotonicity

The main result of this chapter is that, while vehicular traffic networks are not monotone, they possess a weaker *mixed monotonicity* property. This property allows the traffic network dynamics to be embedded within a larger monotone system amenable to techniques for stability analysis of such systems.

We begin with a general characterization of mixed monotone systems which is a continuous-time analogue of the characterization in [Smi08] and is closely related to recent results for nonmonotone interconnections of monotone systems, *e.g.*, [AES14], as we discuss in Section 5.4.

Definition 5.2.1 (Mixed Monotone)**.** *The system $\dot{x} = G(x)$, $x \in X \subseteq \mathbb{R}^n$ where X has convex interior and G is locally Lipschitz is* mixed monotone *if there exists a locally Lipshitz continuous function $g(x, y)$ satisfying:*

1. $g(x, x) = G(x)$ *for all* $x \in X$

2. $\dfrac{\partial g_i}{\partial x_j}(x, y) \geq 0$ *for all* $x, y \in X$ *and all* $i \neq j$ *whenever the derivative exists*

3. $\dfrac{\partial g_i}{\partial y_j}(x, y) \leq 0$ *for all* $x, y \in X$ *and all* i, j *whenever the derivative exists.*

The function $g(x, y)$ is called a decomposition function *for the system.*

For a mixed monotone system with decomposition function $g(x, y)$, it follows that the symmetric system

$$\dot{x} = g(x, y) \tag{5.20}$$

$$\dot{y} = g(y, x) \tag{5.21}$$

is order-preserving with respect to the orthant $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\leq 0}$. This system plays a key role in the analysis to follow.

For all $\ell, k \in \mathcal{L}$, define

$$s_{\ell k} = \begin{cases} 1 & \text{if } \tau(k) = \tau(\ell) \text{ and } k \neq \ell \\ 0 & \text{else} \end{cases} \tag{5.22}$$

and for each $\ell \in \mathcal{L}$ and $x, y \in \mathbb{R}^{\mathcal{L}}$, let

$$\xi^\ell_k(x_k, y_k) = s_{\ell k} y_k + (1 - s_{\ell k}) x_k \quad \forall k \in \mathcal{L}, \tag{5.23}$$

$$\xi^\ell(x, y) = \{\xi^\ell_k(x_k, y_k)\}_{k \in \mathcal{L}}. \tag{5.24}$$

Theorem 5.2.1. *The traffic network model (5.4)–(5.8) is mixed monotone with decomposition function*

$$g_\ell(x, y) = f^{in}_\ell(\xi^\ell(x, y)) - f^{out}_\ell(x). \tag{5.25}$$

Proof. We first note that $F_\ell(x)$ is Lipschitz continuous for each $\ell \in \mathcal{L}$; in the following, statements involving derivatives are interpreted to hold whenever the derivative exists.

For ease of notation, we interpret $f^{\text{in}}_\ell(x, y) = f^{\text{in}}_\ell(\xi^\ell(x, y))$.

It holds trivially that $g_\ell(x,x) = F_\ell(x)$ for all $\ell \in \mathcal{L}$. We now show that

$$\frac{\partial f_\ell^{\text{out}}}{\partial x_k}(x) \le 0 \quad \forall x \in \mathcal{X}, \forall \ell \ne k \tag{5.26}$$

$$\frac{\partial f_\ell^{\text{in}}}{\partial x_k}(x,y) \ge 0 \quad \forall x, y \in \mathcal{X}, \forall \ell \ne k \tag{5.27}$$

$$\frac{\partial f_\ell^{\text{in}}}{\partial y_k}(x,y) \le 0 \quad \forall x, y \in \mathcal{X}, \forall k \tag{5.28}$$

which implies that $g_\ell(x,y)$ is indeed a valid decomposition function for the system and the system is mixed monotone, completing the proof.

To this end, first consider $k \in \mathcal{L}^{\text{out}}_{\eta(\ell)}$. We have

$$\frac{\partial f_\ell^{\text{out}}}{\partial x_k}(x) = \frac{\partial \alpha^{\eta(\ell)}}{\partial x_k}(x)\Phi_\ell^{\text{out}}(x_\ell) \in \left\{0, \frac{\Phi_\ell^{\text{out}}(x_\ell)}{\sum_{j\in\mathcal{L}^{\text{in}}_{\eta(\ell)}} \beta_{\to k}\Phi_j^{\text{out}}(x_j)} \frac{d\Phi_k^{\text{in}}}{dx_k}(x_k)\right\} \le 0. \tag{5.29}$$

If $k \in \mathcal{L}^{\text{in}}_{\eta(\ell)}$, then (5.29) still holds and whenever $\partial\alpha^{\eta(\ell)}/\partial x_k \ne 0$, there exists $m \in \mathcal{L}^{\text{out}}_{\eta(\ell)}$ such that

$$\alpha^{\eta(\ell)}(x) = \left(\sum_{j\in\mathcal{L}^{\text{in}}_{\eta(\ell)}} \beta_{\to m}\Phi_j^{\text{out}}(x_j)\right)^{-1} \Phi_m^{\text{in}}(x_m) \tag{5.30}$$

$$\frac{\partial \alpha^{\eta(\ell)}}{\partial x_k}(x) = -\frac{\frac{d\Phi_k^{\text{out}}}{dx_k}(x_k)\Phi_m^{\text{in}}(x_m)}{\beta_{\to m}\left(\sum_{j\in\mathcal{L}^{\text{in}}_{\eta(\ell)}} \Phi_j^{\text{out}}(x_j)\right)^2} \le 0, \tag{5.31}$$

and thus (5.26) holds. Next, we have

$$\frac{\partial f_\ell^{\text{in}}}{\partial x_k}(x,y) = \begin{cases} \beta_{\to\ell}\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \frac{\partial f_j^{\text{out}}}{\partial x_k}(\xi^\ell(x,y)) & \text{if } k \in \mathcal{L}^{\text{in}}_{\tau(\ell)} \\ 0 & \text{else} \end{cases} \tag{5.32}$$

by (5.22)–(5.24). For $k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}$, we have

$$\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \frac{\partial f_j^{\text{out}}}{\partial x_k}(\xi^\ell(x,y)) = \frac{\partial \alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x,y)) \sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \Phi_j^{\text{out}}(x_j) + \alpha^{\tau(\ell)}(\xi^\ell(x,y))\frac{d\Phi_k^{\text{out}}}{dx_k}(x_k). \tag{5.33}$$

If $\alpha^{\tau(\ell)} \neq 1$ on some neighborhood of $\xi^\ell(x,y)$, then there exists $m \in \mathcal{L}^{\text{out}}_{\tau(\ell)}$ such that

$$\alpha^{\tau(\ell)}(\xi^\ell(x,y)) = \frac{\Phi^{\text{in}}_m(y_m)}{\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta_{\to m}\Phi^{\text{out}}_j(x_j)}, \tag{5.34}$$

$$\frac{\partial \alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x,y)) = -\frac{\frac{d\Phi^{\text{out}}_k}{dx_k}(x_k)\Phi^{\text{in}}_m(y_m)}{\beta_{\to m}\left(\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \Phi^{\text{out}}_j(x_j)\right)^2}. \tag{5.35}$$

It follows that (5.33) evaluates to zero for this case. Therefore,

$$\frac{\partial f^{\text{in}}_\ell}{\partial x_k}(\xi^\ell(x,y)) \in \left\{0, \beta_{\to\ell}\frac{d\Phi^{\text{out}}_k}{dx_k}(x_k)\right\} \geq 0, \tag{5.36}$$

and thus (5.27) holds. Finally, we have

$$\frac{\partial f^{\text{in}}_\ell}{\partial y_k}(x,y) = \begin{cases} \beta_{\to\ell}\frac{\partial\alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x,y))\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \Phi^{\text{out}}_j(x_j) & \\ & \text{if } \tau(k)=\tau(\ell) \text{ and } k\neq\ell \\ 0 & \text{else.}\end{cases} \tag{5.37}$$

If $\frac{\partial\alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x,y)) \neq 0$ for some $k \neq \ell$ with $\tau(k)=\tau(\ell)$, then it must be that

$$\frac{\partial \alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x,y)) = \left(\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta_{\to k}\Phi^{\text{out}}_j(x_j)\right)^{-1}\frac{d\Phi^{\text{in}}_k}{dx_k}(y_k). \tag{5.38}$$

We conclude that (5.28) holds because

$$\frac{\partial f^{\text{in}}_\ell}{\partial y_k}(x,y) \in \left\{0, \frac{\beta_{\to\ell}}{\beta_{\to k}}\frac{d\Phi^{\text{in}}_k}{dx_k}(y_k)\right\} \leq 0. \tag{5.39}$$

□

5.3 Asymptotic Behavior

5.3.1 Main result

We now use Theorem 5.2.1 and the order-preserving properties of (5.20)–(5.21) to prove global stability of a particular class of traffic networks.

Definition 5.3.1. *The connected graph $\mathcal{G}$ is said to be a* polytree *graph if the underlying undirected graph is acyclic.*

The "underlying undirected graph" is the undirected graph that results from replacing each directed edge with an undirected edge between the same two nodes.

The class of networks that constitute polytree graphs is somewhat restrictive, as it does not allow cycles or multiple paths between two locations. However, polytrees still encompass a large class of relevant networks, such as a stretch of freeway with onramps and offramps, or a portion of a freeway network leading into (resp. out from) a large metropolitan area, which is useful for modeling the morning (resp. evening) commute patterns in the area. Furthermore, we show *via* a simple example in Section 5.3.2 that assuming the network is a polytree is necessary for the results presented here; this is an important observation in its own right as it demonstrates the limitations of the proposed approach and motivates additional techniques to overcome this limitation as discussed in Section 5.4.

Theorem 5.3.1. *The equilibrium x^{e} identified in* (5.15) *is globally asymptotically stable for polytree networks.*

Proof. Let $\bar{x} = \{\bar{x}_\ell\}_{\ell\in\mathcal{L}}$. We have

$$0 \leq g_\ell(0,\bar{x}) = \begin{cases} d_\ell & \text{if } \ell \in \mathcal{R} \\ 0 & \text{if } \ell \in \mathcal{O} \end{cases} \tag{5.40}$$

$$0 \geq g_\ell(\bar{x},0) = \begin{cases} 0 & \text{if } \ell \in \mathcal{L}\backslash\mathcal{L}^{\text{sink}} \\ -\Phi_\ell^{\text{out}}(\bar{x}_\ell) & \text{if } \ell \in \mathcal{L}^{\text{sink}}. \end{cases} \tag{5.41}$$

Let $\leq_C$ be the order relation with respect to $C \triangleq \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\leq 0}$, that is, $(x,y) \leq_C (\tilde{x},\tilde{y})$ if and only if $x \leq \tilde{x}$ and $\tilde{y} \leq y$. Trivially, $(0,\bar{x}) \leq_C (\bar{x},0)$ and (5.40)–(5.41) implies

$$(g(\bar{x},0), g(0,\bar{x})) \leq_C (0,0) \leq_C (g(0,\bar{x}), g(\bar{x},0)). \tag{5.42}$$

Now consider the solution $(x(t),y(t))$ to (5.20)–(5.21) with initial condition $(0,\bar{x})$. Since (5.20)–(5.21) is order-preserving with respect to C, the second inequality of (5.42) implies that $(x(t),y(t))$ is monotonically increasing with respect to $\leq_C$ [Smi95, Ch. 3, Prop. 2.1]. Symmetrically, $(y(t),x(t))$ is the solution to (5.20)–(5.21) with initial condition $(\bar{x},0)$ and is monotonically decreasing by the first inequality of (5.42). It follows that $(x(t),y(t)) \to (x^*,y^*)$ for some (x^*,y^*) an equilibrium of (5.20)–(5.21) satisfying

$$x^* \leq x^{\mathrm{e}} \leq y^*. \tag{5.43}$$

By symmetry, (y^*,x^*) is also an equilibrium of (5.20)–(5.21). That is, $f_\ell^{\text{in}}(\xi^\ell(x^*,y^*)) = f_\ell^{\text{out}}(y^*)$ for all $\ell \in \mathcal{L}$, and $f_\ell^{\text{in}}(\xi^\ell(y^*,x^*)) = f_\ell^{\text{out}}(x^*)$ for all $\ell \in \mathcal{L}$.

Consider a trajectory $z(t)$ of (5.4)–(5.8) with initial condition $z^0 \in \mathcal{X}$, and let $\omega(z^0)$ denote the corresponding omega limit set. This induces the corresponding trajectory $(z(t),z(t))$ of the symmetric system (5.20)–(5.21) with initial condition (z^0,z^0). Since $(0,\bar{x}) \leq_C (z^0,z^0) \leq_C (\bar{x},0)$, it follows that $(x(t),y(t)) \leq_C (z(t),z(t)) \leq_C (y(t),x(t))$ for all $t \geq 0$, that is,

$$x(t) \leq z(t) \leq y(t) \quad \forall t \geq 0. \tag{5.44}$$

We thus have $x^* \leq w \leq y^*$ for all $w \in \omega(z^0)$. We now show that $x^* = y^* = x^{\text{e}}$.

Suppose $y^* \neq x^{\text{e}}$. Recalling that $y^* \geq x^{\text{e}}$, this implies there exists $\ell \in \mathcal{L}$ such that $y^*_\ell > x^{\text{e}}_\ell$. By acyclicity of polytree networks, we assume, without loss of generality, that there does not exists $k \in \mathcal{L}^{\text{out}}_{\eta(\ell)}$ such that $y^*_k > x^{\text{e}}_k$ (otherwise we could choose this downstream link k instead of ℓ). This implies that $\Phi^{\text{in}}_k(y^*_k) = \Phi^{\text{in}}_k(x^{\text{e}}_k)$ for all $k \in \mathcal{L}^{\text{out}}_{\eta(\ell)}$. Without loss of generality, we further assume that $f^{\text{out}}_\ell(y^*) > f^{\text{e}}_\ell$. Indeed, if this were not the case, then there must exist some downstream link with inadequate supply since $\Phi^{\text{out}}_\ell(y^*_\ell) > \Phi^{\text{out}}_\ell(x^{\text{e}}_\ell) = f^{\text{e}}_\ell$. That is, there exists $k \in \mathcal{L}^{\text{out}}_{\eta(\ell)}$, for which $y^*_k = x^{\text{e}}_k$, such that

$$\sum_{j \in \mathcal{L}^{\text{in}}_{\eta(\ell)}} f^{\text{out}}_j(y^*) = (1/\beta_{\to k}) f^{\text{in}}_k(y^*_k) > \sum_{j \in \mathcal{L}^{\text{in}}_{\eta(\ell)}} f^{\text{e}}_j, \tag{5.45}$$

for which there must exist some $j \in \mathcal{L}^{\text{in}}_{\eta(\ell)}$, $j \neq \ell$ with $f^{\text{out}}_j(y^*) > f^{\text{e}}_j$, and we could choose j instead of ℓ.

We thus have $f^{\text{in}}_\ell(\xi^\ell(y^*, x^*)) = f^{\text{out}}_\ell(y^*) > f^{\text{e}}_\ell$. Define $\ell_0 \triangleq \ell$ and, starting from ℓ_0, choose inductively $\ell_1, \ell_2, \ldots, \ell_n$ to satisfy $\ell_i \in \mathcal{L}^{\text{in}}_{\tau(\ell_{i-1})}$ for all i (that is, link ℓ_i is upstream of link ℓ_{i-1}) such that $f^{\text{out}}_{\ell_i}(y^*) > f^{\text{e}}_{\ell_i}$, until no additional upstream link satisfying this condition exists. Note it is possible that $\ell_n = \ell_0 = \ell$. Thus

$$f^{\text{in}}_{\ell_n}(\xi^{\ell_n}(y^*, x^*)) = f^{\text{out}}_{\ell_n}(y^*) > f^{\text{e}}_{\ell_n} \tag{5.46}$$

and $f^{\text{out}}_j(y^*) \leq f^{\text{e}}_j$ for all $j \in \mathcal{L}^{\text{in}}_{\tau(\ell_n)}$. But then there must exist k with $\tau(k) = \tau(\ell_n)$, $k \neq \ell_n$ such that $\beta_{\to k} \sum_{j \in \mathcal{L}^{\text{in}}_{\tau(k)}} f^{\text{out}}_j(y^*) = \Phi^{\text{in}}_k(y^*_k)$ for which $\Phi^{\text{in}}_k(y^*_k) \leq f^{\text{e}}_k$. Define $k_0 = k$ and construct another sequence $k_1, k_2, \ldots, k_n$ satisfying $k_i \in \mathcal{L}^{\text{out}}_{\eta(k_{i-1})}$ for all i (that is, link k_i is downstream of link k_{i-1}) such that $\Phi^{\text{in}}_{k_i}(y^*_{k_i}) \leq f^{\text{e}}_{k_i}$, until no additional downstream link satisfying this condition exists. Note it is possible that $k_n = k_0 = k$. It thus holds that $\Phi^{\text{in}}_{k_n}(y^*_{k_n}) \leq f^{\text{e}}_{k_n}$ and $\Phi^{\text{in}}_j(y^*_j) > f^{\text{e}}_j$ for all $j \in \mathcal{L}^{\text{out}}_{\eta(k_n)}$. Recall $f^{\text{out}}_{k_n}(y^*) \leq \Phi^{\text{in}}_{k_n}(y^*_{k_n}) < \Phi^{\text{out}}_{k_n}(y^*_{k_n})$, where the second inequality follows because $\Phi^{\text{in}}_{k_n}(y^*_{k_n}) \leq f^{\text{e}}_{k_n}$ and thus $y^*_{k_n} > x^{\text{crit}}_{k_n}$. Therefore there exists $j \in \mathcal{L}^{\text{out}}_{\eta(k_n)}$ such that $\beta_{\to j} \sum_{m \in \mathcal{L}^{\text{in}}_{\eta(k_n)}} f^{\text{out}}_m(y^*) = \Phi^{\text{in}}_j(y^*_j)$, but then there must exist $m \in \mathcal{L}^{\text{in}}_{\eta(k_n)}$ such that $f^{\text{out}}_m(y^*) > f^{\text{e}}_m$ since $\Phi^{\text{in}}_j(y^*_j) > f^{\text{e}}_j$, for which $y^*_m > x^e_m$. However, $m \neq \ell_i$ for any $\ell_i \in \{\ell_0, \ldots, \ell_n\}$ chosen previously, as this would imply the graph is not a polytree. Taking $\ell = m$, we could begin the process again, continuing indefinitely; since the graph is finite, we arrive at a contradiction, and thus we must have $y^* = x^{\text{e}}$.

Now suppose $x^* \neq x^{\text{e}}$, that is, there exists ℓ such that $x^*_\ell < x^{\text{e}}_\ell$. Without loss of generality, assume there does not exist $k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}$ such that $x^*_k < x^{\text{e}}_k$. This implies

$$f^{\text{out}}_\ell(x^*) \leq \Phi^{\text{out}}_\ell(x^*_\ell) < f^{\text{e}}_\ell. \tag{5.47}$$

Since $f^{\text{in}}_\ell(\xi^\ell(x^*, y^*)) = f^{\text{out}}_\ell(x^*)$ and

$$\beta_{\to\ell} \sum_{k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}} \Phi^{\text{out}}_k(x^*_k) = \beta_{\to\ell} \sum_{k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}} \Phi^{\text{out}}_k(x^{\text{e}}_k) = f^{\text{e}}_\ell, \tag{5.48}$$

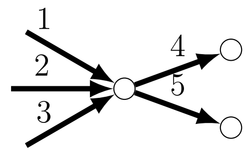


Figure 5.2: An example consisting of a primary junction with three incoming links and two outgoing links.

there must exists $j \in \mathcal{L}^{\text{out}}_{\tau(\ell)}$, $j \neq \ell$, such that $f^{\text{in}}_\ell(\xi^\ell(x^*, y^*)) = (\beta_{\to\ell}/\beta_{\to j})\Phi^{\text{in}}_j(y^*_j)$, for which we must have $y^*_j > x^{\text{e}}_j$, a contradiction since we have shown $y^* = x^{\text{e}}$. As $x^* = y^* = x^{\text{e}}$, we conclude that $\omega(z^0) = \{x^{\text{e}}\}$ for all $z^0 \in \mathcal{X}$, that is, the equilibrium x^{e} is globally attractive.

Finally, suppose the links are indexed $1, \dots, |\mathcal{L}|$ such that the index of link ℓ is less than the index of each $k \in \mathcal{L}^{\text{out}}_{\eta(\ell)}$ (such an indexing is always possible for polytree graphs). Then the Jacobian evaluated at the equilibrium, $(\partial F/\partial x)(x^{\text{e}})$, is upper triangular with respect to this indexing since $f^{\text{out}}_\ell(x^{\text{e}}) = \Phi^{\text{out}}_\ell(x^{\text{e}}_\ell)$ for all $\ell \in \mathcal{L}$. Additionally, the Jacobian contains strictly negative entries along the diagonal since $\Phi^{\text{out}}_\ell(\cdot)$ is strictly increasing, and it is therefore Hurwitz. Thus the equilibrium is locally asymptotically stable by, *e.g.*, [Kha02, Thereom 4.7] and therefore globally asymptotically stable since it is also globally attractive. □

5.3.2 Examples

We first consider an example consisting of a single primary junction with multiple inputs and outputs which satisfies the polytree assumption.

Example 5.3.1. *Let $\mathcal{R} = \{1, 2, 3\}$ and $\mathcal{O} = \{4, 5\}$, and suppose the network consists of one primary junction as in Figure 5.2. We assume each link has supply and demand functions of the form*

$$\Phi^{in}_\ell(x_\ell) = \zeta_\ell - m_\ell x_\ell \tag{5.49}$$

$$\Phi^{out}_\ell(x_\ell) = \gamma_\ell(1 - exp(-\delta_\ell x_\ell)) \tag{5.50}$$

where $\zeta_\ell \in [4, 10]$, $m_\ell \in [0.75, 1.25]$, $\gamma_\ell \in [4, 6]$, $\delta_\ell \in [0.25, 0.75]$, and each parameter is chosen uniformly randomly from the given set for the simulation. The split ratios $\beta_{\to 4}$ and $\beta_{\to 5}$ are also chosen randomly. The exogenous input satisfies $d_\ell \in [3, 5]$ and is chosen uniformly randomly, and we ensure $\{d_\ell\}_{\ell \in \mathcal{R}}$ satisfies Assumption 5.1.3. Figure 5.3 shows plots of the demand and supply functions, along with a sample trajectory $x(t)$ of the system. We see that the system converges to the unique equilibrium, and furthermore $f^{out}_\ell(x(t))$ is strictly less than $\Phi^{out}_\ell(x_\ell(t))$ for $\ell \in \mathcal{R}$ for most of the sample trajectory, indicating that the flow is constrained by a downstream link. In particular, link 4 restricts the flow of upstream links and results in nonmonotone behavior. We see this in the lower plot of the figure, which shows $x_4(t)$ and $x_5(t)$, along with $y_4(t)$ and $y_5(t)$ for another trajectory with initial condition satisfying $y_0 \leq x_0$. Since the trace of $x_5(t)$ and $y_5(t)$ cross, the system is not monotone.

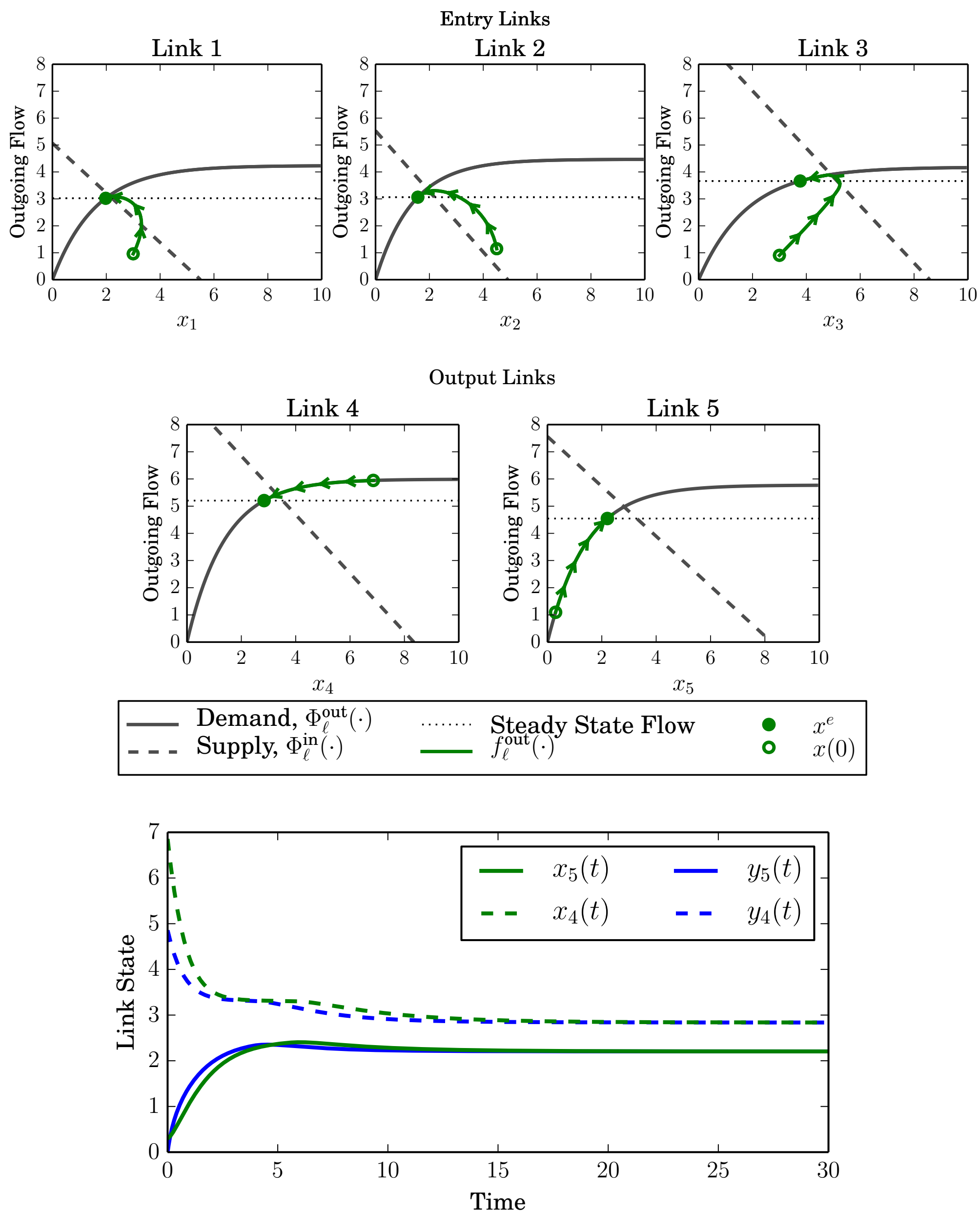


Figure 5.3: Demand and supply functions for Example 1, along with an example trajectory of the system. The bottom plot shows the trace of the state of links 4 and 5 for the example trajectory (green), along with traces corresponding to a different initial condition (blue). These traces demonstrates that the system is not monotone since the solid lines cross.

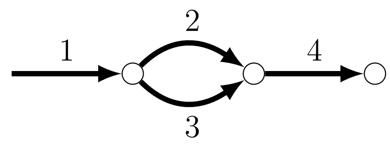


Figure 5.4: An example that is not a polytree network for which Theorem 5.3.1 does not apply. In particular, the symmetric system (5.20)–(5.21) admits additional equilibria that do not correspond to an equilibrium of the traffic network, demonstrating the necessity of the polytree condition and illuminating an important limitation of the proposed technique.

We next present an example network that is not a polytree, for which $y^* \neq x^{\mathrm{e}}$ if we attempt to apply the proof technique of Theorem 5.3.1. This example shows that the polytree condition in Theorem 5.3.1 is tight and illuminates the limitations of applying monotonicity results to establish convergence in nonmonotone traffic flow networks. We discuss these limitations in further detail below.

Example 5.3.2. *Consider the traffic network shown in Figure 5.4 and assume*

$$\Phi_\ell^{out}(x_\ell) = x_\ell, \qquad \ell \in \{1, \dots, 4\} \tag{5.51}$$

$$\Phi_\ell^{in}(x_\ell) = 30 - x_\ell, \qquad \ell \in \{1, 2, 4\} \tag{5.52}$$

$$\Phi_3^{in}(x_3) = 100 - x_\ell, \tag{5.53}$$

and $\beta_{\to 2} = \beta_{\to 3} = 1/2$, $\beta_{\to 4} = 1$, *and* $d_1 = 10$. *Then*

$$x^e = (10, 5, 5, 10), \quad f^e = (10, 5, 5, 10) \tag{5.54}$$

constitutes the unique equilibrium and equilibrium flow. However, taking $x^* = x^e$ *and* $y^* = (20, 25, 50, 15)$, *we have that* $g(x^*, y^*) = g(y^*, x^*) = 0$ *for* $g(x, y)$ *as given in Theorem 5.2.1, that is,* (x^*, y^*) *and* (y^*, x^*) *constitute equilibria of the symmetric system. Nonetheless, simulations indicate that* x^e *is indeed globally asymptotically stable.*

5.4 Discussion

In this chapter, we showed that traffic flow networks are not monotone due to multi-output junctions. Nevertheless, we observed in Theorem 5.2.1 that increasing the number of vehicles on one outgoing link can only decrease incoming flow to adjacent outgoing links, and thus the dynamics are mixed monotone and amenable to an embedding into a higher order, symmetric monotone system.

Theorem 5.3.1 used monotonicity of this embedding system and proved convergence to the equilibrium of the original nonmonotone system. In [ESS06; AS03; ES06; AES14], a similar technique is used to establish general small-gain theorems for nonmonotone interconnections of monotone subsystems. Other approaches to proving stability of flow networks rely on a one-norm contraction property that is partially due to conservation of mass in the network but requires monotone dynamics as in Chapter 4, see also [CLS15]. It is not clear how techniques that rely on contraction properties can be extended to mixed monotone systems, even after embedding such systems in a symmetric monotone system as in (5.20)–(5.21). However, this is a natural area to explore for overcoming the polytree condition required in Theorem 5.3.1.

Chapter 6

Conclusions

Formal techniques for provably correct verification and design of engineered systems is a critical challenge as these systems become more complex and our reliance on their safe and predictable operation grows. This dissertation contributes fundamental theory for analysis and control of a large class of dynamical systems that model networked physical processes. In addition, we focus on transportation networks to demonstrate how these fundamental contributions are applicable.

6.1 Key Contributions

In Chapter 2, we studied mixed monotone discrete-time systems and provided an algorithm for efficiently computing finite abstractions, or symbolic models, of these systems using rectangular partitions of the state space. Mixed monotonicity is a general property applicable to many physical systems of practical importance and naturally generalizes the familiar notation of monotonicity in dynamical systems. We demonstrated how the resulting finite abstractions are used for formally verifying the correct behavior of the original system or for synthesizing a control strategy that ensures correct behavior of the closed loop system.

In Chapter 3, we leveraged the abstraction algorithm of Chapter 2 and proposed a framework for synthesizing control strategies for traffic networks. Given a control objective expressed in linear temporal logic (LTL), the proposed correct-by-design approach ensures that the resulting control strategy satisfies the objective. Temporal logic is a formal specification language capable of encompassing many objectives relevant to transportation. For example, the objective may be to develop a ramp metering and signal coordination strategy that avoids congestion on the freeway but prevents onramp queues from blocking adjacent arterial traffic.

In Chapter 4, we proposed and analyzed a macroscopic traffic flow model that merges ideas from compartmental system theory and dynamical system theory with existing, validated traffic network models. This models extends the traffic flow ideas presented in Chapter 3 and captures important phenomenological properties of vehicular traffic flow such as fixed turn ratios and backwards propagation of congestion. In this chapter, we contributed a theoretical foundation and qualitative analysis of the resulting dynamical behavior. Chapter 5 continues the study of this traffic model and specifically investigates the mixed monotone properties that result.

6.2 Future Work

Future directions for research that emerge from this thesis may be divided into two prominent categories: scalable tools for correct-by-design approaches for engineering systems, and control theoretical approaches to analyzing and controlling next generation transportation systems. These two directions naturally overlap at the domain of cyber-physical systems.

6.2.1 Scalable Tools for Correct-By-Design Systems

It has long been noted that computational procedures for control and analysis of dynamical systems which work well in low dimensions often fail to scale due to the "curse of dimensionality", and this is particularly true for formal methods applied to dynamical systems. The limitations largely stem from the finite symbolic model of the original system which is required for computation. This symbolic model typically requires reachability computations and therefore is expensive to compute, and once constructed, the finite model may be prohibitively large to reason about.

To address the first issue, this dissertation showed that by exploiting certain physical properties of the system, we perform reachability computations quickly, which greatly aids the construction of the symbolic model. The key is to identify contraints, often induced by the physics and interconnection structure, that restrict behavior of the system. Such an approach is required to make significant inroads in scalable approaches for these systems. Indeed, the physical aspect of cyber-physical systems is what sets them apart and must be exploited to the fullest. Control and dynamical systems theory offer numerous tools for distilling complex physical phenomena to their essential features, and future research must draw on these domains to develop novel approaches for formal methods in networked systems engineering.

Additionally, future research should consider data-driven approaches that take advantage of measured data to reduce computation. For example, rather than computing reachable sets with a sufficiently detailed dynamical model, measured system traces may be used to estimate the future state. This approach could further allow specifications given in probabilistic temporal logic such as "with 95 percent probability, the freeway maintains throughput above a given threshold until onramp queues are empty." These systems may be modeled as, *e.g.*, Markov Decision Processes (MDP), which are amenable to combining the theory of probabilistic model-checking with machine learning to obtain algorithms that learn the MDP while determining provably correct control actions.

6.2.2 Control of Next Generation Transportation Systems

Not only do transportation systems embody key structural properties amenable to scalable tools for verification and design, they are a prime example of critical infrastructure requiring sustainable solutions and novel approaches. Due to lack of space and funding, building new infrastructure is not a viable option to meet future transportation needs. Thus, solutions must make more efficient use of the existing built environment. Furthermore, the next generation of transportation systems will include connected vehicles and infrastructure as well as automated systems that must interoperate with existing legacy systems.

Tomorrow's intelligent transportation systems will be equipped with a heterogeneous mix of sensors both within the infrastructure and with connected vehicles, offering unprecedented opportunities to improve safety and increase mobility. For example, it is a mundane fact that the most expensive component of properly timing traffic signals is obtaining accurate measurements of traffic movement. The impending ubiquity of wireless sensors will eliminate this difficulty. Furthermore, future research will enable systematic responses to anomalous scenarios such as major traffic incidents or inclement weather conditions. Currently, such scenarios are often addressed heuristically with control policies constructed manually. Increased complexity of transportation and other networked systems, as well as the dramatic increase in available data, demands automated and formal approaches to these problems.

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