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Towards a model theory of almost complex manifolds

by

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Abstract

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We develop notions of "almost complex analytic subsets" of almost complex manifolds, modelled after complex analytic subsets of complex manifolds. Basic analytic-geometric results are presented, including an identity principle for almost complex maps, and a proof that the singular locus of an almost complex analytic set is itself an "equational" almost complex analytic set, under certain conditions.

This work is partly motivated by geometric model theory. B. Zilber observed that a compact complex manifold, equipped with the logico-topological structure given by its complex analytic subsets, satisfies the axioms for a so-called "Zariski geometry", kick-starting a fruitful model-theoretic study of complex manifolds. Our results point towards a natural generalization of Zilber's theorem to almost complex manifolds, using our notions of almost complex analytic subset. We include a discussion of progress towards this goal.

Our development draws inspiration from Y. Peterzil and S. Starchenko's theory of non-standard complex analytic geometry. We work primarily in the real analytic setting.

To 媽媽, 爸爸, and 珩珩.

Contents

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Chapter 1

Introduction

In this thesis, we develop various notions of *almost complex analytic subsets* of real analytic almost complex manifolds, with the goal of emulating complex analytic subsets of complex manifolds. Almost complex manifolds are natural generalizations of complex manifolds, and our notions of almost complex analytic subsets are largely based on taking level sets of almost complex maps. The basic outlook is in the spirit of real and complex analytic geometry, although partial differential equations and differential geometry both play important roles in adapting to the *almost* complex setting.

Lurking in the background, however, is a more fundamental motivation, stemming from major developments in geometric model theory. We will spend the remainder of the present introductory section explaining this perspective, although our work does not, in the end, definitively settle the questions that arise in this way. This background context is also not necessary for understanding the main results, so readers interested only in the latter can skip ahead to the description of the contents below.

Model theory studies structures through the lens of mathematical logic. In the past three decades, the field has experienced a "geometric turn" of sorts: geometric methods have inspired developments in pure model theory, model theorists have made strides in their understanding of geometric examples, and finally, these efforts have paid of in the form of deep applications to arithmetic geometry and dynamics.

One core concept in this geometric turn is that of the *Zariski geometry*, a model-theoretic structure augmented with topological and geometric data, of which a prototypical example is the structure of a smooth algebraic variety over an algebraically closed field. A fundamental theorem of Hrushovski and Zilber's foundational paper, [HZ96], states that Zilber's conjecture—that any non-locally-modular strongly minimal set interprets an algebraically closed field—holds for Zariski geometries. Knowing that the conjecture is true in specific contexts often leads to meaningful classification theorems in those contexts.

Case in point is the model theory of compact complex manifolds, a microcosm of geometric model theory. The subject began with Zilber's observation that any compact complex manifold forms a Zariski geometry, under the structure induced by its complex analytic subsets [Zil93]. This was followed by the classification of connected groups interpretable in compact complex manifolds, as extensions of complex tori by linear algebraic groups, by Pillay and Scanlon [PS03b]. Nonstandard compact complex manifolds have been studied successfully, yielding, for instance, a nonstandard version of the Riemann existence theorem, by Moosa [Moo04]. The isolation of the "canonical base property" in compact complex manifolds has led to fruitful developments in other parts of pure and applied model theory [Cha12, MP08]. Finally, recent progress has been made on the understanding of compact complex manifolds equipped with a generic automorphism [BHM17].

Given the richness of the model theory of compact complex manifolds, one is led naturally to ask about generalizations to a wider geometric category, the most immediate being the *almost complex manifolds*: real manifolds M equipped with tangent bundle automorphisms $J_M : TM \to TM$, satisfying $J_M^2 = -\operatorname{id}_{TM}$. The basic question is whether a compact almost complex manifold can be given a structure, analogous to the one induced by the class of complex analytic subsets of a compact complex manifold, which forms a Zariski geometry. This was answered affirmatively by [Kes11], for a restricted class of almost complex manifolds, by considering the structure given by images of almost complex maps into the given almost complex manifold.

We take a approach dual to Kessler's, and look at the structure induced on an almost complex manifold M by level sets of maps *out* of M. This is formalized with the notion—alluded to at the beginning of the introduction—of an almost complex analytic subset of M, along with some variants. Zilber's proof that the complex analytic subsets of a complex manifold induce a Zariski geometry essentially collects a number of results from complex analytic geometry, including deep results such as Remmert's proper mapping theorem. That theorem, in turn, relies on an edifice of knowledge about complex analytic maps, the singular loci of complex analytic sets, and the ideal theory underlying the geometry—all outlined, for instance, in the classic survey [GR65].

In the present work, we seek to emulate this analytic-geometric theory in the almost complex setting. While we ultimately come out agnostic on deeper results like the proper mapping theorem, and the overarching issue of whether almost complex analytic subsets of almost complex manifolds yield Zariski geometries, these questions remain the guiding star of our investigations. Towards this end, we develop the rudiments of almost complex maps and almost complex analytic sets, revolving around a proof of an almost complex "identity principle". We also achieve an understanding of the singular loci of almost complex analytic sets, under certain hypotheses. The details of these results will be highlighted in the following précis of the thesis.

Description of the contents

Chapter 1 is the introduction, which also includes a description of the contents. The description contains details which can be skipped by readers looking only for an outline of each chapter, or pursued carefully by readers wishing to avoid reading the thesis in full.

In Chapter 2, we present the basic theory of almost complex manifolds. We introduce a major technical tool, the pseudoholomorphic curve, and prove or state a number of important facts. Some of these developments come from geometric folklore, but we also cite deeper results.

In Chapter 3, we prove the main analytic-geometric facts, centred around versions of the "identity principle" for almost complex maps. The chief notions of almost complex analytic sets are formally defined, and basic properties are deduced, using the identity principle.

Here are two sample results from Chapter 3. Note that an *almost complex map* $f : M \to N$ is one satisfying $df \circ J_M = J_N \circ df$.

Theorem (an almost complex identity principle; simplified from Theorem 3.2.1)**.** *Suppose that M and N are real analytic almost complex manifolds, with $\dim_{\mathbb{R}} M = 2m$, and $U \subset M$ is a connected open set. Let $f : U \to N$ be a real analytic almost complex map, and let $Z = f^{-1}(c)$ for some $c \in N$. Then $\dim_{\mathbb{R}} Z \geq 2m - 1$ implies that $Z = U$.*

Theorem (Corollary 3.3.6)**.** *If A is an "almost complex analytic set", then $\dim_{\mathbb{R}} A$ is even.*

In the complex analytic setting, where an algebraic theory exists, these are straightforward facts, but the almost complex analogues are not as easy. Theorem 3.2.1 in the text further addresses functions which are continuous extensions of almost complex maps, which aids the analysis of almost complex analytic sets at their boundaries. This viewpoint, as well as the proof of the theorem, are heavily inspired by the work of Peterzil and Starchenko on a model-theoretic version of complex analysis over algebraic closures of arbitrary real closed fields [PS01, PS03a, PS08, PS09]. In broad strokes, our work is a kind of parallel to theirs, as both projects seek to recreate complex analytic geometry in a setting where the underlying objects are "real". Their work goes much deeper, and hence serves as blueprint for ours.

One of the main techniques in our proofs is to use images of pseudoholomorphic curves, in place of $\mathbb{C}$-slices of $\mathbb{C}^n$ (or K-slices of K^n for Peterzil and Starchenko). Originally, our results were set in the $\mathbb{R}_{\mathrm{an}}$-definable category, but we were able to remove this condition. Nonetheless, working in the real analytic category is essential. We do not address the *nonstandard* definable setting, which could be an interesting avenue to explore.

Chapter 4 contains a description of the singular locus of an almost complex analytic set. Variations of the concept of almost complex analytic sets are introduced, partly as a concession to the limitations of the analysis. The main idea is to formulate an almost complex version of the Jacobian condition, by constructing an almost complex manifold structure on a complex vector bundle associated with the Jacobian of an almost complex map.

To give a more detailed description, we state the main result, and then describe its proof.

Theorem (Theorem 4.1.2)**.** *If X is a "well-presented" almost complex analytic subset of an almost complex manifold M, then $\operatorname{sing} X$ is an "equationally pseudoanalytic" subset of X.*

The "well-presented" condition implies that X is given locally as the level set of almost complex maps $f : M \to N$, in such a way that allows for the singularities to be analyzed by the Jacobian condition, i.e. $x \in X$ is singular if and only if $\operatorname{rank}_{\mathbb{R}} df_y < \operatorname{codim}_{\mathbb{R}} X$. By linear algebra, this is equivalent to $\bigwedge_{\mathbb{C}}^{\frac{1}{2}\operatorname{codim}_{\mathbb{R}} X} df_y = 0$. The following fact then shows that this condition picks out points y with the same value under two almost complex maps—the "lift" map and the zero section—which is essentially the definition of "equationally pseudoanalytic".

Theorem (Theorem 4.3.7 and Theorem 4.4.1)**.** *Given a almost complex map $f : M \to N$, the bundle $\bigwedge^k \hom_f(TM, TN)$ can be given the structure of an almost complex manifold, under which the following* lift *map is an almost complex map:*

$$\begin{aligned} \bigwedge_{\mathbb{C}}^{k} df : M &\to \bigwedge^{k} \hom_f(TM, TN) \\ y &\mapsto \bigwedge_{\mathbb{C}}^{k} df_y. \end{aligned}$$

The bulk of the chapter is actually devoted to the rigorous differential-geometric construction of the almost complex manifold structure on $\bigwedge^k \hom_f(TM, TN)$, closely following similar work by Gauduchon in [Gau94].

Chapter 5 is a short, largely self-contained chapter, which raises a simple question about "almost complex ideals". The main point is to record an interesting proof of a partial answer to the question, in the classical complex analytic setting.

Finally, Chapter 6 takes stock of the progress towards the goal of building Zariski geometries out of almost complex manifolds. The gist is that the work in Chapter 3 gives one access to a reasonable topology of closed sets, and that Chapter 4 and Chapter 5 give hints of how some of more substantial geometric axioms might be obtained.

We also consider the possibility of the full Zariski geometry result being false, and ask about whether this might be explained by a dichotomy theorem. We finish the chapter by speculatively proposing yet another notion of almost complex analytic set, one which could aid our understanding of the former notions, or alternatively, could itself turn out to be the correct notion for the purposes of Zariski geometries.

Terminology and prerequisites

The term "almost complex analytic subset" has been used loosely thus far. It will be replaced, in the main text, by a raft of more precise concepts, principally *pseudoanalytic subset* and *analytic subset*. Similarly, the term "almost complex map" will be replaced by *pseudoholomorphic map* or *holomorphic map*, depending on the context. Furthermore, we will be more precise about the underlying real category (differentiable, smooth, analytic, etc.) going forward.

Our presentation relies heavily on the basic theory of differentiable manifolds, as exposited, for instance, in the first half of [Lee13]. We also use some analysis, mostly results from partial differential equations, which are cited when needed. The language and perspective of elementary algebraic (and analytic) geometry, on the level of Chapter 1 of [Har77], is pervasive. Without it, a reader might be able to follow the reasoning, but it is hard to imagine that they will find any motivation for doing so. Model theory shows up in the final chapter, but even there, it is not essential.

Chapter 2

Almost complex manifolds

This chapter introduces almost complex manifolds and pseudoholomorphic curves. For the most part, only basic differential geometry is assumed, although various facts are cited without proof.

2.1 Basic definitions

Definition 2.1.1. *An **almost complex manifold** (M, J) is a real differentiable manifold M, together with a real differentiable vector bundle endomorphism $J : TM \to TM$, such that for each $p \in M$,*

$$(J_p)^2 = -\operatorname{id}_p : T_pM \to T_pM.$$

*Such a map J is called an **almost complex structure** on M.*

In an almost complex manifold (M, J), $J_p : T_pM \to T_pM$ induces a unique complex vector space structure on T_pM. Hence, an almost complex structure on M is equivalent to a complex vector bundle structure on TM. It follows that M has even real dimension, and that J is a vector bundle automorphism.

We will often denote an almost complex manifold simply by M. In this case, the associated almost complex structure is usually called J_M.

Example 2.1.2. *Complex manifolds can be viewed as almost complex manifolds. The induced almost complex structure is given by multiplication by i in each tangent space.*

Definition 2.1.3. *An almost complex manifold M is **integrable** if its almost complex structure is induced by a complex manifold structure on M.*

Fact 2.1.4. *All almost complex manifolds of 2 real dimensions are integrable.*

Later in this section, we will see how this follows from a much deeper fact. A simpler direct proof is also possible, as in, for instance, Theorem 4.16 of [MS98].

Loosely speaking, "generic" almost complex manifolds of real dimension greater than 2 are nonintegrable. Integrability is related to the existence of local "almost complex maps" to the complex field: an integrable almost complex manifold has holomorphic manifold charts, and conversely, the local existence of sufficiently many independently many "almost complex maps" to $\mathbb{C}$ implies local integrability. Because of this connection, we will give the formal definition of maps in our category, before presenting concrete examples of nonintegrable almost complex manifolds.

Definition 2.1.5. *If M and N are almost complex manifolds, then a **pseudoholomorphic map** between M and N is a real differentiable manifold map $f : M \to N$ such that the following diagram of real vector bundle maps*

$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[d, "J_M"'] & TN \arrow[d, "J_N"] \\ TM \arrow[r, "df"] & TN \end{tikzcd}$$

commutes.

*A pseudoholomorphic map with pseudoholomorphic inverse is called a **bipseudoholomorphism**. A pseudoholomorphic map with a codomain $N = \mathbb{C}^n$, for some n, is called a **holomorphic map**.*

Pseudoholomorphic maps are closed under composition. We will see more properties of them in Section 2.2, and we will also study pseudoholomorphic maps out of Riemann surfaces quite carefully. We will see that almost complex manifolds support an abundance of pseudoholomorphic images of Riemann surfaces, at least locally. By contrast, pseudoholomorphic and holomorphic maps *out* of higher-dimensional manifolds are rare, as the equations defining them are "overdetermined".

We now present a family of examples of nonintegrable almost complex manifolds. Note that $\mathcal{C}^k$ refers to the differentiability class.

Example 2.1.6 (Examples by Calabi from [Cal58])**.** *Let M be a real $\mathcal{C}^1$ orientable manifold of dimension 6, and suppose that $M \to \mathbb{R}^7$ is a real $\mathcal{C}^2$ immersion, i.e. a real $\mathcal{C}^2$ map of constant rank 6. We endow M with an almost complex structure, as follows.*

Identify $\mathbb{R}^7$ with the imaginary part of the octonions $\mathbb{O}$, which yields

$$M \to \mathbb{R}^7 \cong \operatorname{im} \mathbb{O}.$$

For any $p \in M$, we view the tangent space T_pM as embedded inside $\operatorname{im} \mathbb{O}$ as well, via the immersion. We define the almost complex structure by setting, for $v \in T_pM$,

$$J_p v = v \times \eta_p,$$

where η_p is the unit normal vector to M at p under the immersion, and $\times$ is the cross product in $\operatorname{im} \mathbb{O}$.

We give two instructive subclasses:

1. *If M is a compact manifold without boundary, and U is any neighbourhood of M, then the only holomorphic maps $U \to \mathbb{C}$ are the constant maps. (This is stated in the remarks following the Corollary to Theorem 4, in [Cal58].) This implies that M is not integrable, since complex manifolds have local holomorphic manifold charts everywhere.*

2. *On the other hand, consider the case of*

$$M = \Sigma \times \mathbb{R}^4 \subset \mathbb{R}^7,$$

where Σ is a 2 real dimensional manifold which is not compact, and thus M is also not compact. The immersion here is simply the inclusion map. Then, by Theorem 6 in [Cal58], the almost complex structure induced on M is integrable if and only if Σ is a minimal surface.

Almost complex structures on spheres are of particular historical interest.

Example 2.1.7. *The only real spheres S^{2n} of real dimension $2n$ that have almost complex structures are S^2 and S^6. (See [Aud94], Footnote 3 on p. 49.) By Fact 2.1.4, S^2 is integrable, and is hence the Riemann sphere, $\mathbb{CP}^1$.*

A nonintegrable almost complex structure on S^6 is induced by the octonions via the standard embedding $S^6 \subset \mathbb{R}^7$ as the unit circle, as described in Example 2.1.6. It is a famous open question whether S^6 has an integrable almost complex structure.

More mundane examples of almost complex manifolds are easy to obtain. However, proving nonintegrability by showing the nonexistence of local holomorphic functions is hard in general, so we first present a criterion that reduces the question of integrability to a straightforward computation.

Definition 2.1.8. *Let (M, J) be an almost complex manifold. The* ***Nijenhuis tensor*** *N_J is a tensor field of rank (1,2), defined by the equation*

$$N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

where X and Y are vector fields on M, and $[\cdot, \cdot]$ is the Lie bracket.

Remark 2.1.9. *One can verify that N_J does indeed define a tensor, meaning that for $x \in M$, $(N_J(X, Y))(x)$ depends only on the values of $X(x)$ and $Y(x)$.*

One can also verify that if $f : M \to N$ is a pseudoholomorphic map, then

$$N_{J_N}(df(X), df(Y)) = df(N_{J_M}(X, Y)).$$

Note that $df(X)$ and $df(Y)$ are not necessarily vector fields, but since N_J is a tensor, this equation makes sense at the level of individual vectors $df(X(x))$ and $df(Y(x))$.

Theorem 2.1.10 (Newlander-Nirenberg)**.** *An almost complex manifold (M, J) is integrable if and only if N_J vanishes identically on M.*

The forward direction is straightforward, but the converse in full generality is a deep result [NN57]. In the case of a real analytic almost complex manifold (M, J), there is an easier proof using the Frobenius integrability theorem, as explained in Section 2.2.3 of [Voi02].

The Newlander-Nirenberg theorem implies Fact 2.1.4 about all almost complex manifolds of 2 real dimensions being integrable, as follows. At any point $p \in M$, we can pick a vector field X that is supported on a neighbourhood of p, and hence $\{X, JX\}$ forms a basis of vector fields in that neighbourhood. Then, a quick calculation shows that shows that $N_J(X, JX) = 0$, and we always have $N_J(X, X) = 0$ and $N_J(JX, JX) = 0$, so N_J vanishes identically on the neighbourhood. Repeating this for every $p \in M$, we conclude that N_J vanishes identically on M.

We end the section with an elementary example of an almost complex manifold. The Newlander-Nirenberg theorem is used to prove nonintegrability.

Example 2.1.11. *Let $M = \mathbb{R}^{2m}$, and choose an isomorphism $TM \cong \mathbb{R}^{2m} \times \mathbb{R}^{2m}$. From this perspective, an almost complex structure J on $\mathbb{R}^{2m}$ is a matrix-valued map $A_J : \mathbb{R}^{2m} \to M_{2m\times 2m}(\mathbb{R})$ with $(A_J)^2(p) = -I$ for all $p \in \mathbb{R}^{2m}$. Any choice of A_J which is "generic" enough will yield a nonintegrable (M, J).*

Constant A_J, or any 2×2 matrix A_J, will of course result in an integrable (M, J). The direct product of two integrable almost complex structures is also integrable. However, the following "twisted product" of two integrable almost complex structures is already nonintegrable.

Let $M = \mathbb{R}^4$ have standard variables (x_1, y_1, x_2, y_2), and let J be defined by

$$A_J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-x_1} \\ 0 & 0 & e^{-x_1} & 0 \end{pmatrix}.$$

Working with the standard frame $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\right)$, one can compute that

$$N_J\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = \frac{\partial}{\partial x_2},$$

and hence (M, J) is not integrable.

Far more complicated matrices A_J are possible, and reflecting on difficulty of having N_J vanish on all pairs of basis elements, one gets heuristic sense of the genericness of nonintegrability.

2.2 Almost complex manifold theory

In this section, we present basic manifold-theoretic results for almost complex manifolds. Everything here is a straightforward extension of real differentiable manifold theory. Our presentation is heavily influenced by [Lee13], and conversations with Ben McMillan about the almost complex case were helpful to the author.

Theorem 2.2.1 (inverse function theorem). *Suppose that M and N are almost complex manifolds and $f : M \to N$ is a pseudoholomorphic map. If*

$$df_p : T_pM \to T_{f(p)}N$$

is a bijection for some $p \in M$, then there are neighbourhoods U of p and V of $f(p)$ such that $f|_U : U \to V$ is a bipseudoholomorphism.

Proof. We can find U and V such that $f|_U : U \to V$ is a diffeomorphism by the real inverse function theorem. It remains to show that the inverse $g : V \to U$ is pseudoholomorphic.

Since f is pseudoholomorphic,

$$J_N \circ df = df \circ J_M,$$

and pre- and post-composing with dg,

$$dg \circ J_N = J_M \circ dg.$$

□

Lemma 2.2.2 (rank theorem). *Suppose that M and N are almost complex manifolds of real dimensions $2m$ and $2n$, respectively, and $f : M \to N$ is a pseudoholomorphic map, with df of constant real rank $2r$. Then for any $p \in M$, there are exist* pseudoholomorphic charts, *meaning*

1. *neighbourhoods U of p and V of $f(p)$,*
2. *neighbourhoods of the origins $\hat{U} \subset \mathbb{R}^{2m}$ and $\hat{V} \subset \mathbb{R}^{2n}$,*
3. *almost complex structures J_1 on $\hat{U}$ and J_2 on $\hat{V}$, and*
4. *bipseudoholomorphisms $\phi : U \to \hat{U}$ and $\psi : V \to \hat{V}$;*

under which $\hat{f} = \psi \circ f \circ \phi^{-1}$ has the form

$$\hat{f}(x_1, \ldots, x_{2r}, \ldots, x_{2m}) = (x_1, \ldots, x_{2r}, 0, \ldots, 0).$$

Proof. By the rank theorem for real manifolds, we have real manifold charts $\phi : U \to \mathbb{R}^{2m}$ and $\psi : V \to \mathbb{R}^{2n}$ centred at p and $f(p)$, respectively, under which $\hat{f}$ has the desired form. We can use ϕ and ψ to induce almost complex structures

$$J_1 := d\phi \circ J_M \circ d\phi^{-1}$$

on $\hat{U} := \phi(U)$, and

$$J_2 := d\psi \circ J_N \circ d\psi^{-1},$$

on $\hat{V} = \psi(V)$, under which $\hat{f}$ is pseudoholomorphic. □

Definition 2.2.3. *Suppose A is a real embedded submanifold of an almost complex submanifold M. We call A an* ***almost complex submanifold*** *if for all $p \in A$, $J_p(T_pA) \subset T_pA$.*

Lemma 2.2.4 (constant rank level set theorem)**.** *Suppose that M and N are almost complex manifolds, and $f : M \to N$ is a pseudoholomorphic map, with df of constant real rank $2r$. Then each level set Z of f is a closed almost complex submanifold of real codimension $2r$ in M, with the property that for each $p \in M$,*

$$T_pZ = \ker df_p.$$

Proof. Let $2m$ and $2n$ be the dimensions of M and N, respectively. For a given $p_0 \in f^{-1}(c)$, by Lemma 2.2.2, we can find pseudoholomorphic charts (U, ϕ) and (V, ψ) centred around p_0 and $f(p_0) = c$ under which under which $\hat{f} = \psi \circ f \circ \phi^{-1}$ has the form

$$\hat{f}(x_1, \dots, x_{2r}, \dots, x_{2m}) = (x_1, \dots, x_{2r}, 0, \dots, 0).$$

Thus $Z := f^{-1}(c)$ is locally the preimage of a "slice" of euclidean space:

$$U \cap Z = \phi^{-1}\Big(\hat{f}^{-1}(0)\Big) = \phi^{-1}\Big(\phi(U) \cap \{(0, \dots, 0)\} \times \mathbb{R}^{2m-2r}\Big),$$

and hence is an embedded submanifold of M of codimension $2r$.

Furthermore, from the coordinate representation it is easy to see that

$$T_x\hat{f}^{-1}(0) = \ker d\hat{F}_x$$

for any $x \in \phi(U)$, and hence

$$T_pZ = \ker df_p$$

for any $p \in U$.

For any pseudoholomorphic map, J_M preserves $\ker df_p$: if $df_pv = 0$, then $df_p(J_Mv) = J_N(df_pv) = 0$. Thus J_M preserves T_pZ for all $p \in U$. This is true in a neighbourhood around every $p \in Z$, so we can conclude that Z is an almost complex submanifold of M.

Finally, by continuity of f, $Z = f^{-1}(c)$ is closed. □

Definition 2.2.5. *Let $f : M \to N$ be a map of real differentiable manifolds. A point $c \in N$ is called a* ***regular value*** *of f if for all $p \in f^{-1}(c)$, $df_p : T_pM \to T_{f(p)}N$ is surjective.*

Theorem 2.2.6 (regular level set theorem)**.** *Suppose $f : M \to N$ is a pseudoholomorphic map between almost complex manifolds of real dimensions $2m$ and $2n$, respectively, and $c \in N$ is a regular value of f. Then $Z := f^{-1}(c)$ is a closed almost complex submanifold of real codimension $2n$, with the property that for each $p \in M$,*

$$T_pZ = \ker df_p.$$

Proof. The set of points U in M where the real rank of df equals $2n$ is open, since being a submersion is an open property (e.g. by Proposition 4.1 of [Lee13]). We have $f^{-1}(c) \subset U$ by hypothesis.

Considering $U \subset M$ as an almost complex submanifold, and applying Lemma 2.2.4 to $f|_U$, we conclude that $f^{-1}(c)$ is a closed almost complex submanifold of U, and hence of M, of codimension equal to the rank, $2n$. □

This shows that level sets of regular values under pseudoholomorphic maps are almost complex submanifolds. In classical settings, conversely, submanifolds are given locally as level sets of regular values, but it is not clear whether this is true in the almost complex setting. We expect not: local almost complex submanifolds are abundant, as we will see in the next section, but pseudoholomorphic maps are rare. For instance, the compact almost complex manifolds from Example 2.1.6 do not admit local holomorphic maps, although we are not sure whether they admit local pseudoholomorphic maps.

2.3 Pseudoholomorphic curves

In this section, we start off in the real differentiable category, but move quickly to the real smooth and real analytic categories. An almost complex manifold (M, J) is said to be of differentiability class $\mathcal{C}^k$ if both the manifold M and the real vector bundle map J are of class $\mathcal{C}^k$.

Definition 2.3.1. *A **pseudoholomorphic curve** is a pseudoholomorphic map between a Riemann surface and an almost complex manifold. In other words, a pseudoholomorphic curve is a real differentiable map $u : \Sigma \to M$ between a complex manifold (Σ, i) of complex dimension 1, and an almost complex manifold (M, J), satisfying*

$$J \circ du = du \circ i.$$

Such curves are also commonly known as **J-holomorphic curves** or **J-curves**. They are analogues of holomorphic curves in complex manifolds, and played a crucial role in Gromov's foundational work in symplectic topology [Gro85]. For us, what is important is (i) that they exist, which we will prove over the course of this chapter, and (ii) that they satisfy a two basic properties, which we will now list.

The basic properties are natural and easy to state, but their proofs in the smooth setting are highly nontrivial. They rely on results about partial differential equations proved by Hartman and Wintner [HW53], and independently by Aronszajn [Aro57], as well as further analytic work as exposited or developed in [MS12]. We take the statements from [Wen14].

The first fact is a "unique continuation" or "identity" principle. One of the main tools in Chapter 3 is a version of the identity principle for real analytic pseudoholomorphic maps in general.

Fact 2.3.2 (Theorem 2.82 in [Wen14]). *Suppose that $u, v : \Sigma \to M$ are two smooth pseudoholomorphic curves, between a Riemann surface Σ, and a smooth almost complex manifold M. Let $p \in \Sigma$. If $u(p) = v(p)$, and u and v share partial real derivatives of all orders at p, then u and v are equal on some neighbourhood of p.*

This particular identity principle is, of course, trivial in the real analytic case. We will state and use a slight modification of it later (Fact 3.1.2).

The second fact that we will need is a local representation theorem for pseudoholomorphic curves.

Fact 2.3.3 (Theorem 2.117 in [Wen14]). *Suppose that $u : \Sigma \to M$ is a smooth pseudoholomorphic curve between a Riemann surface Σ and a smooth almost complex manifold M. For any $p \in \Sigma$, there is a neighbourhood U of p, a holomorphic map $\phi : U \to V \subset \mathbb{C}$, and a smooth injective pseudoholomorphic curve $v : V \to M$, such that $u|_U = v \circ \phi$.*

$$\begin{tikzcd} U \arrow[r, "u|_U"] \arrow[d, "\phi"'] & M \\ V \arrow[ur, "v"'] & \end{tikzcd} .$$

Remark 2.3.4. *This local representation theorem follows from Fact 2.3.2, which is trivial in the real analytic case, as well as certain existence theorems, which we will prove in the real analytic case shortly. Thus, it seems possible that there is an easier proof of Fact 2.3.3 in the real analytic setting, which bypasses the difficult analysis alluded to above. This would also free the results of the present work (particularly, the results in Chapter 3) from this dependency.*

Turning to the existence theorems, there are two related results, both following from the Cauchy-Kovalevskaya existence theorem for partial differential equations. As required by the Cauchy-Kovalevskaya theorem, everything in the rest of this section will be real analytic. Theorem 2.3.5 and its consequence, Theorem 2.3.8, were pointed out to the author by Robert Bryant, in public [Bry] and private communication.

Theorem 2.3.5 (unique extension to pseudoholomorphic curves). *Let $\gamma : \mathbb{R} \to (M, J)$ be a real analytic curve into a real analytic almost complex manifold. Then there exists a neighbourhood $U \subset \mathbb{C}$ of 0 and a pseudoholomorphic curve $u : U \to M$ extending $\gamma|_{\mathbb{R}\cap U}$. This is unique, in the sense that if $u' : U' \to M$ is another such extension, then u and u' agree in some neighbourhood of 0.*

Theorem 2.3.6 (uniform existence of pseudoholomorphic curves). *Let M be a real analytic almost complex manifold of real dimension $2m$, and let $p \in M$. Then there are neighbourhoods $U_1 \times U_2 \times U_3 \subset \mathbb{R}^{2m} \times \mathbb{R}^{2m} \times \mathbb{C}$ of $(0, 0, 0)$ and V of p, and a real analytic family*

$$\Upsilon : U_1 \times U_2 \times U_3 \to M$$

of pseudoholomorphic curves

$$u_{a_1,a_2} := \Upsilon(a_1, a_2, \cdot) : U_3 \to M,$$

such that for each $q \in V$ *and* $v \in T_qV$*, there is* $(a_1, a_2) \in U_1 \times U_2$ *with*

$$u_{a_1,a_2}(0) = q \ \ and \ \ du_{a_1,a_2}\left(\frac{d}{dx}\Big|_{a_1,a_2}\right) = v.$$

A simple special case of both of these theorems is the following.

Corollary 2.3.7. *Let* M *be a real analytic almost complex manifold. Choose* $p \in M$ *and* $v \in T_pM$*. Then there exists a pseudoholomorphic curve* $u : \Sigma \to M$ *with* $u(0) = p$ *and* $du_0\left(\frac{d}{dx}\big|_0\right) = v$*.*

A less trivial consequence concerns the intersections of almost complex submanifolds.

Theorem 2.3.8. *Let* X *and* Y *be two real analytic almost complex submanifolds of a real analytic almost complex manifold* (M, J)*. If* $X \cap Y$ *is a real submanifold of* M*, then it is an almost complex submanifold of* M*.*

Proof. Let $p \in X \cap Y$, and $v \in T_p(X \cap Y)$; we want to show that $J_pv \in T_p(X \cap Y)$.

Choose a real analytic curve $\gamma : \mathbb{R} \to (X \cap Y, J)$, with $\gamma(0) = p$ and $\gamma'(0) = d\gamma_0\left(\frac{d}{dx}\big|_0\right) = v$. We can view γ simultaneously as a curve in X, a curve in Y, and a curve in M. In each case, we apply the unique extension theorem to get pseudoholomorphic curves $u_1 : U_1 \to X$, $u_2 : U_2 \to Y$, and $u : U \to M$, all extending γ in neighbourhoods of 0. By uniqueness, $u = u_1$ on neighbourhood of 0, and hence, by shrinking U if necessary, we can assume that $u(U) \subset X$. Similarly, we can assume $u(U) \subset Y$, and hence, $u(U) \subset X \cap Y$.

Then, since u is pseudoholomorphic,

$$J_pv = J_p\left(du_0\left(\frac{d}{dx}\Big|_0\right)\right) = du_0\left(i \cdot \frac{d}{dx}\Big|_0\right) \in T_p(X \cap Y).$$

□

We now prove the existence theorems using the Cauchy-Kovalevskaya theorem, which guarantees existence and uniqueness for a very general class of analytic partial differential equations all at once. We state the latter first. It concerns vector solutions $u : \mathbb{R}^M \to \mathbb{R}^N$ near the origin to partial differential equations, subject to boundary conditions that specify the values and derivatives of u on the hyperplane $\mathbb{R}^{M-1} \times \{0\}$.

Theorem 2.3.9 (Cauchy-Kovalevskaya, c.f. [Fol95], Theorem 1.25)**.** *Consider the partial differential equation and boundary conditions*

$$\begin{cases} \partial_t^k u = G(x, t, (\partial_x^\alpha \partial_t^j u^i)_{|\alpha|+j \le k, j<k, i \le N},) \\ \partial_t^j u(x, 0) = \phi_j \quad 0 \le j < k, \end{cases}$$

where: (x,t) are variables in $\mathbb{R}^{M-1}\times\mathbb{R}$; $u=(u_1\ldots,u_n):\mathbb{R}^M\to\mathbb{R}^N$ is an unknown function; α is a multi-index in x, and ∂_x^α the corresponding partial derivative; and $G,\phi_0,\ldots,\phi_{k-1}$ are real analytic functions in a neighbourhood of 0.

For this problem, there exists an analytic solution u defined on some neighbourhood of 0. This is unique, in the sense that another such solution agrees with the first in some common neighbourhood of 0.

Proof of Theorem 2.3.5. Since the existence result is local, we can work in a coordinate chart for M, and hence without loss assume that $M\subset\mathbb{R}^{2m}$.

The main task is to translate the geometric data into differential equations in Euclidean space. We will use the variable $x+iy\cong(x,y)$ for the domain $\mathbb{C}\cong\mathbb{R}^2$, with x parameterizing the original copy of $\mathbb{R}$.

The condition that u extends γ requires

$$u(x,0)=\gamma(x),$$

and pseudoholomorphicity requires

$$J\circ du=du\circ i\cdot.$$

Expanding the latter using coordinate representations $J=(J_{ij})$ gives us

$$\begin{pmatrix} J_{1,1} & \cdots & J_{1,2m}\\ \vdots & \ddots & \vdots\\ J_{2m,1} & \cdots & J_{2m,2m}\end{pmatrix}\begin{pmatrix}\partial_x u_1 & \partial_y u_1\\ \vdots & \vdots\\ \partial_x u_{2m} & \partial_y u_{2m}\end{pmatrix}=\begin{pmatrix}\partial_x u_1 & \partial_y u_1\\ \vdots & \vdots\\ \partial_x u_{2m} & \partial_y u_{2m}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and hence

$$\begin{pmatrix}\sum_{j=1}^{2m} J_{1,j}\partial_x u_j & \sum_{j=1}^{2m} J_{1,j}\partial_y u_j\\ \vdots & \vdots\\ \sum_{j=1}^{2m} J_{2m,j}\partial_x u_j & \sum_{j=1}^{2m} J_{2m,j}\partial_y u_j\end{pmatrix}=\begin{pmatrix}\partial_y u_1 & -\partial_x u_1\\ \vdots & \vdots\\ \partial_y u_{2m} & -\partial_x u_{2m}\end{pmatrix}.$$

Reading off the entries yields a system of $4m$ differential equations, but using $J^2=-\operatorname{id}$ shows that half of them are redundant (e.g either the left or the right half of the matrices). Hence, overall, we have the system of equations

$$\begin{cases}\partial_y u_i=-\displaystyle\sum_{j=1}^{2m} J_{i,j}\partial_x u_j & 1\le i\le 2m\\ u(x,0)=\gamma(x).\end{cases}$$

This system is of the form specified in Theorem 2.3.9, with $k=1$ and $t=y$. Hence, we can conclude that $\tilde{\gamma}$ exists in a neighbourhood of 0, and is unique in the sense described by the statement. $\square$

The proof of the other existence theorem, Theorem 2.3.6, is very similar, but it is not needed in the subsequent developments, so we will only provide a sketch.

Proof of Theorem 2.3.6. We use $(q, v, (x, y))$ to denote coordinates of $\mathbb{R}^{2m} \times \mathbb{R}^{2m} \times \mathbb{C}$.

As in the proof of Theorem 2.3.5, we translate the geometric condition from the hypothesis to a partial differential equation condition on Υ, which yields:

$$\begin{cases} \partial_y \Upsilon_i = -\sum_{j=1}^{2m} J_{i,j} \partial_x \Upsilon_j \quad 1 \leq i \leq 2m \\ \Psi(q, v, 0, 0) = q \\ \partial_x \Upsilon(q, v, 0, 0) = v. \end{cases} \tag{2.1}$$

This system falls under the rubric of Theorem 2.3.9, with $k = 2$ and $t = y$. However, fewer partial differential equations are specified (only with respect to ∂_y, not ∂_{q_i} or ∂_{v_i}, and the initial conditions are only specified at 0, not on the entire hyperplane $y = 0$). Hence, the Cauchy-Kovalevskaya theorem guarantees existence of local solutions, but not uniqueness. □

It is also possible to prove a version of Theorem 2.3.6 where one can uniformly specify derivatives up to a fixed finite order. One needs to modify the statement to reflect that, thanks to the generalized Cauchy-Riemann equations, it is only possible to realize arbitrary *complex* derivatives up to finite order.

2.4 Appendix: Regularity of pseudoholomorphic curves

In this appendix, we record a pair of regularity results for pseudoholomorphic curves. The first uses classical arguments to show that smooth pseudoholomorphic curves into real analytic almost complex manifolds are themselves real analytic. The second, taken from [MS12], says that "Sobolev" class pseudoholomorphic curves into smooth almost complex manifolds are themselves smooth.

Combined, these theorems allow us to transfer results about Sobolev class pseudoholomorphic curves to the real analytic setting. In symplectic topology, the Sobolev setting is required for advanced techniques, as it guarantees that certain infinite-dimensional spaces (like the moduli space of all pseudoholomorphic curves in an almost complex manifold) form Banach spaces or manifolds. Thus, much of the modern literature, including [MS12] and [Wen14], works in the Sobolev setting. A version of our uniform existence theorem, Theorem 2.3.6, was originally obtained by transferring over a Sobolev analogue from [Wen14].

In retrospect, this was unnecessary—backwards, even—since the simple results that we need follow directly from the powerful Cauchy-Kovalevskaya machinery, as we have seen.

Nonetheless, it seems possible that deeper results could be more easily obtained via transfer principles, so we append them to this chapter here for the record.

Analytic regularity

The first transfer result will follow from a regularity theorem for elliptic systems of partial differential equations, so we begin with a discussion of such systems. Further details can be found in [Joh81].

Suppose we have a function $u = (u_1, \ldots, u_N) : \mathbb{R}^M \to \mathbb{R}^N$ of class C^d, satisfying a system of equations

$$\sum_{k=1}^{N} L_{j,k}(u_k) = 0 \quad j = 1, \ldots, N, \tag{2.2}$$

with linear partial differential operators $L_{j,k}$ of order at most d. More precisely, the operators $L_{j,k}$ are given by polynomials of degree at most d in the operator variables $\partial_1, \ldots, \partial_M$, with coefficients which are functions $\mathbb{R}^M \to \mathbb{R}$ in some fixed function space specified by the context. Note that the number of unknowns u_k and the number of equations are both the same, N.

Viewing an operator $L = L(p, \partial_1, \ldots, \partial_M)$ as a polynomial in $\partial_1, \ldots, \partial_M$, and with coefficients which are functions in the variable $p \in \mathbb{R}^M$, we refer to the highest-degree form as the **characteristic form**, and denote it by $Q(p, \partial_1, \ldots, \partial_M)$.

Definition 2.4.1. *A system of the form in Equation* (2.2) *is called* ***elliptic*** *if the matrix* $(Q_{j,k}(p, \xi_1, \ldots, \xi_M))$ *is invertible, for all* p *and all* $(\xi_1, \ldots, \xi_M) \neq (0, \ldots, 0)$.

We will appeal to the following regularity theorem.

Theorem 2.4.2 (from [Joh81], p. 144)**.** *Suppose a system as in Equation* (2.2) *is elliptic, and that the coefficients of polynomial operators* $L_{j,k}$ *are real analytic functions in a domain* D*, with orders no greater than* d*. Then any system of functions* u_k *which are of class* C^d*, and solve Equation* (2.2)*, are real analytic.*

The following theorem is simply a reflection of the fact that pseudoholomorphic curves satisfy elliptic systems of equations.

Theorem 2.4.3. *Suppose that* (N, J) *is a real analytic almost complex manifold. If* $u : \Sigma \to N$ *is a pseudoholomorphic curve of class* C^2 *from a Riemann surface* Σ *into* N*, then* u *is real analytic.*

Proof. Being real analytic is a local property, so we can assume that both the domain and range are affine open sets: $\Sigma \subset \mathbb{C}$ and $N \subset \mathbb{R}^{2n}$.

We will use the variable $x+iy \cong (x,y)$ for the domain $\mathbb{C} \cong \mathbb{R}^2$, and we will use coordinates $u = (u_j)$. In this notation, we have

$$du = \begin{pmatrix} \partial_x u_1 & \partial_y u_1 \\ \vdots & \vdots \\ \partial_x u_{2n} & \partial_y u_{2n}, \end{pmatrix}$$

and hence the pseudoholomorphic condition $J \circ du = du \circ i$ becomes

$$\begin{pmatrix} \sum_{k=1}^{2n} J_{1,k}\partial_x u_k & \sum_{k=1}^{2n} J_{1,k}\partial_y u_k \\ \vdots & \vdots \\ \sum_{k=1}^{2n} J_{2n,k}\partial_x u_k & \sum_{k=1}^{2n} J_{2n,k}\partial_y u_k \end{pmatrix} = \begin{pmatrix} u_y^1 & -u_x^1 \\ \vdots & \vdots \\ u_y^{2n} & -u_x^{2n} \end{pmatrix}.$$

Rewriting, this yields the system of equations

$$\begin{cases} \partial_y u_j = \displaystyle\sum_{k=1}^{2m} J_{j,k}\partial_x u_k \quad j = 1, \ldots, 2n \\ -\partial_x u_j = \displaystyle\sum_{k=1}^{2m} J_{j,k}\partial_y u_k \quad j = 1, \ldots, 2n. \end{cases} \tag{2.3}$$

As an aside, notice that the equations on the second line follow from those on the first:

$$\sum_{k=1}^{2m} J_{j,k}\partial_y u_k = \sum_{k,l=1}^{2m} J_{j,k}J_{k,l}\partial_x u_l = -\sum_{l=1}^{2m} \delta_{j,l}\partial_x u_l = -\partial_x u_j,$$

where $\delta_{j,l}$ is the Kronecker delta, and $j = 1, \ldots, 2n$. Indeed, a similar calculation shows that the two lines are equivalent.

Nonetheless, we proceed using both lines of Equation (2.3). Recall that we are assuming that u is of class C^2, so we can take second order partial derivatives. Then, for $j = 1, \ldots, 2n$, we have

$$\begin{aligned} 0 &= \partial_x\partial_y u_j - \partial_y\partial_x u_j \\ &= \partial_x \left(\sum_{k=1}^{2m} J_{j,k}\partial_x u_k \right) - \partial_y \left(-\sum_{k=1}^{2m} J_{j,k}\partial_y u_k \right) \\ &= \sum_{k=1}^{2m} J_{j,k}(\partial_x^2 + \partial_y^2)u_k + (\partial_x J_{j,k})(\partial_x u_k) + (\partial_y J_{j,k})(\partial_y u_k). \end{aligned}$$

We can view this as a system of linear partial differential equations of the form

$$\sum_{k=1}^{2m} L_{j,k}(u_k) = 0 \quad j = 1, \ldots, 2n, \tag{2.4}$$

with linear operators

$$L_{j,k} := J_{j,k}(\partial_x^2 + \partial_y^2) + (\partial_x J_{j,k})\partial_x + (\partial_y J_{j,k})\partial_y.$$

This is an example of a system of the form in Equation (2.2), with $M = 2$, $N = 2n$, and order $d = 2$.

We will now verify that the system is elliptic. Note that for all j and k, $L_{j,k}$ is a second-order differential operator, and hence, the characteristic form is given by

$$Q_{j,k}(p, \xi_x, \xi_y) := J_{j,k}(p)\left(\xi_x^2 + \xi_y^2\right).$$

Then, for any p and any $(\xi_x, \xi_y) \neq (0, 0)$, since $J(p)$ is invertible,

$$\begin{aligned} \det\left(Q_{j,k}(p, \xi_x, \xi_y)\right) &= \det\left(J_{j,k}(p)(\xi_x^2 + \xi_y^2)\right) \\ &= (\xi_x^2 + \xi_y^2) \cdot \det\left(J_{j,k}(p)\right) \\ &\neq 0, \end{aligned}$$

which shows that Equation (2.4) is elliptic. Thus, Theorem 2.4.2 applies, and we can conclude that the solutions u_k are all real analytic on Σ. $\square$

Smooth regularity

The second transfer result, taken from the literature, is much deeper. It relies on concepts from functional analysis, which we will introduce only very lightly here. Two good sources for additional details are [MS12], Appendix B, and [Wen14], Chapter 2.

Let $2 < p < \infty$, $0 \leq k < \infty$, and let Ω be an open subset of $\mathbb{R}^M$. We consider the *Sobolev space* $W^{k,p}(\Omega, \mathbb{R}^N)$ of functions $\Omega \to \mathbb{R}^N$ with bounded L^p-norm, such that for each multi-index α with $|\alpha| \leq k$, the α^{th}*-weak partial derivative* also has bounded L^p-norm. $W^{k,p}(\Omega, \mathbb{R}^N)$ forms a Banach space under a natural norm, called the $W^{k,p}$-norm. As mentioned earlier, many results about pseudoholomorphic curves are proved in the $W^{k,p}$ setting, since Banach spaces are a natural setting for studying infinite-dimensional geometric objects, like spaces of pseudoholomorphic curves.

Theorem 2.4.4 (Theorem B.4.2 from [MS12])**.** *Suppose Σ is a Riemann surface, (M, J) is a smooth almost complex manifold, and $u : \Sigma \to M$ is a $W^{k,p}$-function for some $k \geq 1$, such that $J \circ du = du \circ i$. Then u is a smooth pseudoholomorphic curve.*

Being a pseudoholomorphic curve simply means being differentiable and satisfying $J \circ du = du \circ i$, so this follows trivially from the smoothness.

Of course, we can combine Theorem 2.4.3 and Theorem 2.4.4.

Corollary 2.4.5. *Suppose Σ is a Riemann surface, (M, J) is a real analytic almost complex manifold, and $u : \Sigma \to M$ is a $W^{k,p}$-function for some $k \geq 1$, such that $J \circ du = du \circ i$. Then u is real analytic pseudoholomorphic curve.*

Chapter 3

Identity principles and pseudoanalytic sets

In this chapter, we prove a number of "identity principles", which say that pseudoholomorphic functions which are constant on large enough sets are constant everywhere. We then derive consequences for the behaviour of "pseudoanalytic subsets" of almost complex manifolds, which are those given locally as level sets of pseudoholomorphic maps. The main results are framed in the real analytic setting.

The work here was inspired by Peterzil and Starchenko's development of o-minimal complex analytic geometry, in [PS01, PS03a, PS08, PS09]. However, we work purely in the standard real analytic setting, and furthermore, assumptions regarding definability in an o-minimal structure turned out to be superfluous. Nonetheless, as a successful generalization of complex analytic geometry in a setting where the underlying objects are "real", their work remains a strong influence.

The original impetus was to prove an almost complex version of the proper mapping theorem. This proved to be out of reach, but our emphasis on studying not only level sets, but their closures, stems from a view towards more sophisticated applications, like the proper mapping theorem à la Peterzil and Starchenko in [PS09].

3.1 Identity principles in lower dimensions

Almost complex manifolds of real dimension 2 are essentially complex manifolds, and hence an identity principle carries over for free.

Proposition 3.1.1. *Suppose that (M, J_M) and (N, J_N) are smooth almost complex manifolds of real dimension 2, and M is connected. Let $f, g : M \to N$ be two smooth pseudoholomorphic maps, and suppose that are equal on a set containing a limit point. Then $f = g$.*

Proof. Applying Fact 2.1.4, M and N can be viewed as as complex manifolds. More precisely, there are integrable almost complex manifolds (M', i) and (N', i), and bipseudoholomorphisms $M \cong M'$ and $N \cong N'$. The induced maps $f', g' : M' \to N'$ between the corresponding complex manifolds are holomorphic, and thus, the classical identity principle for Riemann surfaces (e.g. see [For81], Theorem 1.11) implies that $f' = g'$. Thus $f = g$. $\square$

In the integrable case, this result can be used to prove an identity principle for maps between complex manifolds of arbitrary dimensions. Our goal is to prove an identity principle for maps between almost complex manifolds of arbitrary dimension, but this technique does not work, as almost complex manifolds do not typically have simple decompositions that allow pseudoholomorphic maps to be broken down into components.

Instead, we will need to do some work, relying on existing versions of the identity principle for pseudoholomorphic curves. An important, generic version was stated earlier, as Fact 2.3.2, but we also need something slightly stronger.

Fact 3.1.2 (Theorem 1.1 from [Ahn05]). *Let Σ be a connected Riemann surface with smooth manifold boundary $\partial\Sigma$, and (M, J) be a smooth almost complex manifold. If $f : \Sigma \to M$ is a pseudoholomorphic map which is continuous on $\Sigma \cup \partial\Sigma$, $\hat{f} : \Sigma \cup \partial\Sigma \to M$ is the continuous extension, and $\hat{f}$ is constant on an open arc of $\partial\Sigma$, then f is constant on Σ.*

This version of the identity principle is proved by using a kind of reflection principle at the boundary, which reduces it to Fact 2.3.2.

We collect everything we need from the identity principles for pseudoholomorphic curves into a single statement, which looks very similar to the version that we will prove in higher dimensions.

Proposition 3.1.3. *Let $U \subset \mathbb{C}$ be an open connected set with a smooth boundary ∂U. Suppose that $u : U \to (M, J)$ is a smooth pseudoholomorphic curve into a smooth almost complex manifold (M, J). Let $\hat{U}$ be the set of points in the closure of U where u can be extended by continuity, and let $\hat{u} : \hat{U} \to M$ be this extension. If $Z = \hat{u}^{-1}(c)$ for some $c \in M$ satisfies $\dim_{\mathbb{R}} Z \geq 1$, then $Z \supset U$, i.e. u is constant.*

Remark. *Note that Z is a real analytic subset of the real analytic manifold M. Hence, Z can be assigned a real analytic dimension, $\dim_{\mathbb{R}} Z$.*

Proof. We will consider two cases, depending on whether Z has nonzero real dimension inside of U, or whether that occurs on the boundary.

Case 1: $\dim_{\mathbb{R}} Z \cap U \geq 1$. Pick a point $p \in Z \cap U$ witnessing this dimension locally.

Assume for a contradiction that $\hat{u}$ is not constant. Then we can apply the local representation of pseudoholomorphic curves (Fact 2.3.3), around the point p, and obtain a factorization (perhaps after shrinking U)

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "\phi"'] & M \\ V, \arrow[ur, "v"'] & \end{tikzcd}$$

where $\phi : U \to V \subset \mathbb{C}$ is a holomorphic map and h is an injective pseudoholomorphic curve. Locally, ϕ itself can be represented in coordinates as $z \mapsto z^k$ for some k, and thus the level sets of ϕ are discrete. Hence, the same is true of u, contradicting $\dim_{\mathbb{R}} Z \geq 1$.

Note that the smoothness of ∂U was not needed in this case.

Case 2: $\dim_{\mathbb{R}} Z \cap U = 0$. This implies that $\dim_{\mathbb{R}} Z \cap (\hat{U} \setminus U) \geq 1$, and hence Z contains a subarc of the boundary ∂U. This case is taken care of by Fact 3.1.2. □

3.2 The identity principle in higher dimensions

Now we now state and prove an analogue to the identity principle for maps between almost complex manifolds of arbitrary dimensions. For the rest of the chapter, we will work in the setting of real analytic geometry, i.e. all manifolds and maps (including the almost complex structures J) will be assumed to be real analytic, and hence we can take advantage of the corresponding facts in that setting. Pseudoholomorphic curves were similar enough to holomorphic curves that we were able to avoid this assumption in the last section.

The proof is modelled on the analogous results of Peterzil and Starchenko, namely Theorem 2.13(1) in [PS03a] and Theorem 3.1 in [PS09]. The basic idea of our proof is that we replace Peterzil and Starchenko's use of K-slices of K^n—for which there is no corresponding concept for general almost complex manifolds, as they are not always decomposable—with images of pseudoholomorphic curves, using Corollary 2.3.7.

Theorem 3.2.1. *Suppose that M and N are real analytic almost complex manifolds, with $\dim_{\mathbb{R}} M = 2m$, and $U \subset M$ is a connected open set. Let $f : U \to N$ be a real analytic pseudoholomorphic map, and pick $c \in N$.*

Extend f by continuity into a subset $\hat{U}$ of the topological closure of U where possible: $\hat{f} : \hat{U} \to N$. Let $Z := \hat{f}^{-1}(c)$. Then if $\dim_{\mathbb{R}} Z \geq 2m - 1$, then $Z = \hat{U}$.

Proof. We will argue that $\dim_{\mathbb{R}} Z = 2m$. From this, it follows that Z has nonempty interior in $\hat{U}$, and since Z and real analytic, and U is connected, we can conclude that $Z = \hat{U}$.

So suppose for a contradiction that $\dim_{\mathbb{R}} Z = 2m - 1$. Let $p \in Z$ be a point where the full dimension $2m - 1$ is locally realized.

Fix an element $w \in (T_pM \setminus T_pZ) \cap B^{2m}_{\epsilon}$, and apply Corollary 2.3.7 to obtain a pseudoholomorphic curve $u : B \to M$, from the unit ball $B \subset \mathbb{C}$ into M, with $u(0) = p$ and $du_0 \left(\frac{d}{dx}\big|_0 \right) = w$. Since $du_0 \left(\frac{d}{dx}\big|_0 \right) = w \neq 0$, we can assume, by shrinking B if necessary, that $u(B)$ is an almost complex submanifold of M.

Claim. $\dim_{\mathbb{R}} u(B) \cap Z = 1$.

Proof. To prove this claim we will treat $u(B)$ as a real analytic manifold, and argue purely in the real analytic setting, forgetting about the almost complex structure.

First, apply a real analytic change coordinates in a neighbourhood around p so that $u(B)$ is a linear 2-plane. (Formally, this is a real analytic version of [Lee13], for instance.)

After linearizing, we have $w \in u(B) \setminus T_pZ$. Using a manifold chart we can work inside of $\mathbb{R}^{2n}$ (restricting the following Euclidean spaces to neighbourhoods around p where necessary), and define a map $\pi : \mathbb{R}^{2n} \to T_pZ \cong \mathbb{R}^{2n-1}$ which is the linear projection along w.

When restricted to Z, this induces a local real analytic diffeomorphism near p, π_Z. By choice of π, $\pi\big(u(B)\big) \subset T_pZ$ is an infinite 1-dimensional line, and hence

$$\pi_Z^{-1}\Big(\pi\big(u(B)\big)\Big) = u(B) \cap Z.$$

is also 1-dimensional. □

Now, since $du_0\left(\frac{d}{dx}\big|_0\right) = w$, the inverse function theorem for almost complex manifolds (Theorem 2.2.1) implies that we can assume, after possibly shrinking B, that u is a bipseudoholomorphism from B to the 2-dimensional almost complex manifold $u(B)$. We will pull everything down to $B \subset \mathbb{C}$ under u^{-1}.

Concretely, this means that we have:

1. an open, connected, 2-real-dimensional almost complex manifold $V := u^{-1}(U) \subset B \subset \mathbb{C}$,
2. a pseudoholomorphic curve $g : V \to N$ given by $g = f|_{u(B)\cap U} \circ u|_V$,
3. a continuous extension $\hat{g} : \hat{V} \to N$, where $\hat{V} = u^{-1}(u(B) \cap \hat{U})$, and
4. a level set $Y = \hat{g}^{-1}(c) = u^{-1}(Z)$ with $\dim_{\mathbb{R}} Z = 1$.

Note that we have all of this for any $p \in Z$ where the local dimension is $2m - 1$. By choosing a "generic" such p, and shrinking domains if necessary, we can assume that the resulting level set Y has a a boundary ∂Y which is given by a real analytic curve with no self-intersections. (Formally, we can rely on real analytic cell decomposition or Whitney stratifications, and choose p as a generic element in the cell, or stratification.) This puts us in the situation of Proposition 3.1.3, and thus we can conclude that $\dim Y = \dim_{\mathbb{R}} Z = 2$, a contradiction.

□

3.3 Pseudoanalytic sets and singular points

We will now define "almost complex analytic subsets" of almost complex manifolds, which are generalizations of complex analytic sets in complex manifolds. Using the identity principle, and other tools, we prove that certain basic properties from the complex case carry over.

We continue to work entirely in the real analytic setting.

Definition 3.3.1. *A subset A of a almost complex manifold M is called **locally pseudoanalytic** if for every $p \in A$ there exists a pseudoholomorphic function $f : U \to N$ from a neighbourhood $U \subset M$ of p to an almost complex manifold N such that*

$$U \cap A = f^{-1}(c)$$

*for some $c \in N$. We say that A is **pseudoanalytic** if the same condition holds for every $p \in M$ (not just every $p \in A$).*

*We define **locally analytic** and **analytic** subsets similarly, except that the functions f are required to be* holomorphic*. (Correspondingly, the choices for N are $N = \mathbb{C}^n$ for some n.)*

Of course, (locally) analytic implies (locally) pseudoanalytic. The complex analytic subsets of a complex manifold coincide with its analytic subsets, although in principle it could have more pseudoanalytic subsets. Thus, analytic sets are perhaps the more natural notion, but pseudoanalytic sets are often just as easy to work with.

For a general almost complex manifold, the two notions do not always coincide. As a trivial example, the compact 6-manifolds in Example 2.1.6 do not admit any nonconstant holomorphic maps, and hence do not have any analytic subsets. But individual points are pseudoanalytic, as they are level sets of the identity map, which is pseudoholomorphic.

Other examples of pseudoanalytic sets include fibres of direct products of almost complex manifolds. If the manifolds involved are nonintegrable, then the fibres will be nonintegrable almost complex manifolds.

Example 3.3.2. *The following is an explicitly presented almost complex manifold, which is not a direct product, and in which analytic sets arise as fibres of a holomorphic map. We expand on and follow the conventions of Example 2.1.11. In particular, we continue to present almost complex structures J on $\mathbb{R}^{2m}$ as $2m \times 2m$ matrices A_J, depending on the variables $(x_1, y_1, \ldots, x_m, y_m)$ on $\mathbb{R}^{2m}$. We describe other maps using matrices as well, using similar identifications.*

We define $(\mathbb{R}^6, J)$ by slightly modifying the explicit example from Example 2.1.11:

$$A_J = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e^{-(x_1+x_2)} \\ 0 & 0 & 0 & 0 & e^{x_1+x_2} & 0 \end{pmatrix}.$$

Let $\pi : \mathbb{R}^6 \to \mathbb{R}^2$ be projection down to the (x_1, y_1) coordinates, identifying $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\}^4$. The codomain is an almost complex submanifold, whose induced almost complex structure is the standard complex structure, $(\mathbb{R}^2, i)$. One easily verifies that π is holomorphic.

For each $(a,b) \in \mathbb{R}^2$, $\pi^{-1}(a,b) = \{(a,b)\} \times \mathbb{R}^4$ *forms a 4-dimensional almost complex submanifold of* $(\mathbb{R}^6, J)$, *with an induced complex structure* J' *which depends on* a. *One computes, as before,*

$$N_{J'}\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \frac{\partial}{\partial x_3},$$

and hence $(\pi^{-1}(a,b), J')$ *is nonintegrable. This implies that* $(\mathbb{R}^6, J)$ *is nonintegrable, but in addition, one can perform essentially the same Nijenhuis computation to confirm this. Because of the dependency of* J *on* x_1, *it is not decomposable as a direct product of almost complex manifolds.*

In the rest of the section, we will discuss some basic properties of pseudoanalytic sets. The results apply equally well to analytic sets, either directly, or via the same proofs.

Proposition 3.3.3. *A set* $A \subset M$ *is pseudoanalytic if and only if it is closed and locally pseudoanalytic. A pseudoanalytic set is both a Euclidean closed, locally pseudoanalytic set, and also the Euclidean closure of a locally pseudoanalytic set, but neither of the two latter conditions imply pseudoanalyticity, nor do they imply each other. In other words, the following is a complete diagram of implications:*

$$\begin{tikzcd}
\textit{pseudoanalytic} \arrow[d, leftrightarrow] & \\
\textit{closed \& locally pseudoanalytic} \arrow[d] \arrow[dr] & \\
\textit{locally pseudoanalytic} & \textit{closure of locally pseudoanalytic set.}
\end{tikzcd}$$

Proof. If A is pseudoanalytic then it is locally pseudoanalytic by definition. If $p \in M \backslash A$, then there is a neighbourhood of M in which A is a level set, which is closed in the neighbourhood, and hence some (possibly smaller) neighbourhood of p in M will not contain A at all. Thus A is closed.

Conversely, if A is a closed, locally pseudoanalytic set, then to show it is pseudoanalytic we need to check that A is given locally by a level set for points $p \in M \setminus A$. But since A is closed, we can find neighbourhoods around such points that do not intersect with A at all, in which case A is trivially represented as a level set.

This proves the equivalence, and from it, both of the two remaining implications follow easily.

Examples of locally pseudoanalytic sets which do not satisfy any of the other conditions include, for instance, the open unit ball inside of the complex plane, or the punctured complex plane. These fail the condition for pseudoholomorphicity at points on the boundaries.

An example of a closure of a locally pseudoanalytic set which does not satisfy any of the other conditions is the closed unit ball inside of the complex plane. ∎

We now wish to understand the singular loci of pseudoanalytic sets. Generally, in classical settings, regular and singular loci are defined in terms of the Jacobian condition, but since we do not have access to a robust algebraic theory, we start our discussion with a manifold-theoretic definition. In Chapter 4, we discuss the extent to which we have been able to harness the Jacobian condition, in the almost complex setting.

Definition 3.3.4. *Let M be a $\mathcal{C}$ manifold, where $\mathcal{C}$ is a fixed category of differentiable manifolds (e.g. the category of C^∞ manifolds, or the category of real analytic manifolds). If $A \subset M$, then $p \in A$ is called* ***$\mathcal{C}$-regular*** *if there is a neighbourhood of p in A which is an embedded $\mathcal{C}$ submanifold of M. We denote the set of such points by $\operatorname{reg}_{\mathcal{C}} A$, called the* ***$\mathcal{C}$-regular locus****. Points that are not $\mathcal{C}$ regular are called* ***$\mathcal{C}$-singular****, and the set of such points are denoted $\operatorname{sing}_{\mathcal{C}} A$, called the* ***$\mathcal{C}$-singular locus****.*

In the category of (real analytic) almost complex manifolds, we will drop the $\mathcal{C}$, and simply write ***regular****,* ***singular****, $\operatorname{reg} A$, and $\operatorname{sing} A$, etc.*

Our first result shows how two of these notions are related, for pseudoanalytic sets.

Proposition 3.3.5. *Suppose that A is a locally pseudoanalytic subset of a (real analytic) almost complex manifold M. Then a point in A is regular if and only if it is regular in the real analytic sense, i.e.*

$$\operatorname{reg} A = \operatorname{reg}_{\mathbb{R}} A.$$

Proof. The containment $\operatorname{reg} A \subset \operatorname{reg}_{\mathbb{R}} A$ holds by definition, because almost complex submanifolds are required to be real analytic submanifolds.

Conversely, suppose that $p \in \operatorname{reg}_{\mathbb{R}} A$. We know from real analytic geometry that $\operatorname{sing}_{\mathbb{R}} A$ is a closed subset of A, and hence there is some neighbourhood $U \subset M$ of p such that $U \cap A = U \cap \operatorname{reg}_{\mathbb{R}} A$. We wish to show that $p \in \operatorname{reg} A$, and because this is a local property, there is no harm in restricting to the case where $U = M$, so we will assume that $A = \operatorname{reg}_{\mathbb{R}} A$. Furthermore, what follows, we restrict to smaller neighbourhoods of p where necessary, without explicit mention.

Let $v \in T_pA \setminus \{0\}$. Since A is a real analytic submanifold, there is a real analytic curve $\gamma : (-1, 1) \subset \mathbb{R} \to A$ with $\gamma(0) = p$ and $\gamma'(0) = v$. By Theorem 2.3.5 there exists a unique pseudoholomorphic curve $\tilde{\gamma} : B \subset \mathbb{C} \to M$ inside M extending γ. Note that $\tilde{\gamma}$ does not *a priori* live inside A, but we will now show that it does.

Suppose that A is described locally in M as the level set at c of a pseudoholomorphic function f. Consider the restriction $g := f|_{\tilde{\gamma}(B)}$, which is a pseudoholomorphic curve. (Since $v \neq 0$, $\tilde{\gamma}(B)$ is an almost complex submanifold of M by the almost complex inverse function theorem, Theorem 2.2.1.)

Now, since

$$g^{-1}(c) = \tilde{\gamma}(B) \cap A \supset \gamma\Big((-1, 1)\Big),$$

the level set $g^{-1}(c)$ is at least 1-dimensional. Thus, the identity principle, Theorem 3.2.1, implies $g^{-1}(c) = \tilde{\gamma}(B)$, so $\tilde{\gamma}(B) \subset f^{-1}(c) = A$. Thus the curve $\tilde{\gamma}$ lives in A, and we can perform the usual calculation

$$J_p v = J_p \left(u_{*0} \left(\frac{d}{dx}\Big|_{z=0} \right) \right) = u_{*0} \left(i \cdot \frac{d}{dx}\Big|_{z=0} \right) \in T_p(A),$$

which shows that $p \in \operatorname{reg} A$. □

This is useful because real analytic geometry facts about $\operatorname{reg}_{\mathbb{R}}$ carry over directly to reg automatically.

Corollary 3.3.6. *If A is a locally pseudoanalytic set, then $\dim_{\mathbb{R}} A$ is even.*

Proof. We have

$$\dim_{\mathbb{R}} A = \dim_{\mathbb{R}} \operatorname{reg}_{\mathbb{R}} A = \dim_{\mathbb{R}} \operatorname{reg} A,$$

and since $\operatorname{reg} A$ is an almost complex submanifold, it has even dimension. □

Note that if $X = f^{-1}(c)$ is a regular level set of a pseudoholomorphic map f (i.e. if for all $x \in X$, df_x is surjective), then by the constant rank theorem (Lemma 2.2.4), X is an almost complex submanifold, and hence even-dimensional. In the integrable case, or in classical algebraic geometry, we can always find generators of the ideal of a variety X which generically present X regularly, but we are not aware of an analogue to this algebraic fact in the almost complex setting. Hence, with our current understanding, Corollary 3.3.6 was not obvious *a priori*.

We will highlight one consequence of the corollary, which specifies how pseudoanalytic sets can overlap. It is an analogue of Theorem 3.2 in [PS09], and is essentially another version of the identity principle. We make a simple definition first.

Definition 3.3.7. *A nonempty pseudoanalytic set is* ***irreducible*** *if it cannot be expressed as the union of two proper pseudoanalytic subsets. A locally pseudoanalytic set is* ***irreducible*** *if its topological closure is.*

Theorem 3.3.8. *Suppose M is an almost complex manifold. Let $A_1 \subset M$ be an irreducible locally pseudoanalytic set, with $\dim_{\mathbb{R}} A_1 = 2d$, and let $A_2 \subset M$ be a pseudoanalytic set. Then either $A_1 \subset A_2$ or $\dim_{\mathbb{R}}(\operatorname{cl}(A_1) \cap A_2) \leq 2d - 2$.*

Proof. We assume that

$$\dim_{\mathbb{R}} (\operatorname{cl}(A_1) \cap A_2) \geq 2d - 1$$

and will show $A_1 \subset A_2$. Note that topological operators without subscript are implicitly assumed to be relative to M, e.g. $\operatorname{cl} = \operatorname{cl}_M$.

Since $\operatorname{reg} A_1 = \operatorname{reg}_{\mathbb{R}} A_1$ is dense in A_1, we have that $\operatorname{cl}(\operatorname{reg}_{\mathbb{R}} A_1) = \operatorname{cl}(A_1)$, and hence if we replaced A_1 with $\operatorname{reg} A_1$, the above dimension formula would still hold. Furthermore, if we showed that $\operatorname{reg} A_1 \subset A_2$, then it would follow that $A_1 \subset A_2$, since A_2 is closed. Thus we can assume without loss of generality that $A_1 = \operatorname{reg} A_1$, that is, that A_1 is a connected almost complex submanifold without boundary of M of pure dimension $2d$.

Choose $p \in \operatorname{cl} A_1$ where we have local dimension

$$\dim_{\mathbb{R},p}\left(\operatorname{cl}(A_1) \cap A_2\right) \geq 2d - 1.$$

Let U be a neighbourhood of p, $f : U \to N$ a real analytic pseudoholomorphic map, and $c \in N$ such that

$$U \cap A_2 = f^{-1}(c).$$

We will view f as a function $U \cap A_1 \to N$. Now we apply Theorem 3.2.1 to the continuation $\hat{f}$ of (the restriction of) f and conclude that

$$U \cap A_1 \subset \hat{f}^{-1}(c) = U \cap A_2.$$

Note that Theorem 3.2.1 requires the domain of f to be connected, which we can achieve by shrinking U if necessary, even to the point where p is no longer in U, but only in the domain $\hat{U}$ of the continuous extension $\hat{f}$.

This shows that an open set inside A_1 is contained inside A_2, and hence $\operatorname{cl}(A_1) \cap A_2$ has nonempty interior in $\operatorname{cl} A_1$. Since $\operatorname{cl} A_1$ is irreducible and A_2 is closed, $\operatorname{cl}(A_1) \setminus A_2$ must in turn have empty interior in $\operatorname{cl} A_1$, so $\operatorname{cl} A_1 \subset A_2$.

Thus, we have shown that $A_1 \subset A_2$ as desired. $\square$

Chapter 4

Bundles, lifts, and singular loci

In this chapter, we study singular sets of pseudoanalytic sets via an abstract version of the Jacobian criterion. The language needed to make the statement involves developing some technical machinery, involving almost complex bundles and lifts, and this takes up the bulk of the chapter. The statement itself, together with the motivation, and some remaining open questions, is described in the first section.

4.1 Singular loci of almost complex analytic sets

Consider an affine complex algebraic variety $V \subset \mathbb{C}^n$. The set of singular points $\operatorname{sing}_{\mathbb{C}} V$ of V was defined to be those where V is not locally a complex manifold (Definition 3.3.4). A key fact about singular points is the following.

Fact 4.1.1. *The singular locus* $\operatorname{sing}_{\mathbb{C}} V$ *of a complex algebraic variety* V *forms a proper Zariski closed subset of* V*.*

Indeed, in this classical setting, there is an alternative characterization of the singular locus which makes it clear that $\operatorname{sing}_{\mathbb{C}} V$ is Zariski closed. In particular, if we choose generators $I(V) = (f_1, \ldots, f_n)$, then $x \in V$ is singular if and only if the Jacobian matrix

$$(\operatorname{Jac}_{\mathbb{C}} f)(z) := \left(\frac{\partial f_i}{\partial z_j}(z)\right)$$

has rank less than $\operatorname{codim}_{\mathbb{C}} V = n - \dim_{\mathbb{C}} A$. Because the rank of a matrix being less than d is equivalent to the $d \times d$ minors all having zero determinant, this condition is algebraic.

An almost complex version of these facts would be extremely useful for the analysis of pseudoanalytic sets, but we are constrained by a lack of a corresponding ring theory for pseudoholomorphic functions. This topic is discussed briefly in Chapter 5, but more work is needed before it would be helpful here. For now, we make a strong assumption that allows us to set these algebraic issues aside. Definitions and explanations will be given after the statement.

Theorem 4.1.2. *If X is a well-presented subset of an almost complex manifold M, then* $\operatorname{sing} X$ *is equationally pseudoanalytic, and a proper subset of X.*

Definition 4.1.3. *A Euclidean closed subset X of an almost complex manifold M is* ***well-presented*** *if for all $x \in X$, there is a neighbourhood U, an almost complex manifold N, and a pseudoholomorphic map $f = (f_1, \dots, f_{2n}) : U \to N$, such that $X \cap U = f^{-1}(c)$ for some $c \in N$, and*

$$([f_1]_x, \dots, [f_{2n}]_x)\,,$$

the real ideal generated by germs at x of the components f_i of f, is a radical ideal in the ring $O_x^{\mathbb{R}}$ of germs at x of real analytic functions to $\mathbb{R}$.

A Euclidean closed subset X of an almost complex manifold M is ***equationally pseudoanalytic*** *if for each $x \in X$, there is a neighbourhood U and two pseudoholomorphic maps $f, g : U \to N$ to a common almost complex manifold N, such that*

$$X \cap U = \{y \in U \mid f(y) = g(y)\}.$$

It follows from the definitions that

$$X \text{ well-presented} \implies X \text{ pseudoanalytic} \implies X \text{ equationally pseudoanalytic.}$$

The proof of the theorem works with the real Jacobian of a pseudoholomorphic map f, but the vanishing of the determinant minors is not usually a pseudoholomorphic condition, so we must replace this with a more abstract construction. This involves making sense of the "lift" of f, which is loosely given by

$$\begin{aligned} \tilde{f} : M &\to \bigwedge^k \hom_f(TM, TN) \\ x &\mapsto \left(x, \bigwedge^k df\right), \end{aligned}$$

and showing that $\tilde{f}$ is pseudoholomorphic. We will use this to show that a point x is singular if and only if $\tilde{f}(x) = 0(x)$, where 0 is the 0-section. The nature of "almost complex bundles" like $\bigwedge^k \hom_f(TM, TN)$ is that generally, if the underlying manifolds are not integrable, there is no pseudoholomorphic projection down to a fixed complex vector space, and hence the resulting condition is only equationally pseudoanalytic.

Rigorously describing these objects will occupy us for the next three sections. Our construction is a mild extension of the ones presented in [Kru07] and in the appendix of [Aud94], although we have tried to be more systematic and detailed. Readers willing to take the existence of these objects for granted can skip ahead and read the proof in Section 4.5. This may also provide motivation for the intervening technical developments.

4.2 Linear connections

Throughout the chapter, we work in the real smooth category. As a starting point, we note that almost complex manifolds already come equipped with a natural structure on the tangent bundle:

Definition 4.2.1. *A* ***complex vector bundle*** (E, M, i) *is a real vector bundle* E*, over a real manifold* M*, equipped with a real vector bundle endomorphism* $i_E : M \to M$ *satisfying* $(i_{E_p})^2 = -\operatorname{id}_{E_p}$ *for all* $p \in M$*.*

Remark 4.2.2. *By definition, an almost complex structure* (M, J_M) *on a real manifold* M *is equivalent to a complex vector bundle structure* (TM, M, i_{TM}) *on its real tangent bundle* TM*, and the two structures are related by* $i_{TM} = J_M$*.*

In order to work with bundles, we will need to make use of connections. Intuitively, these give a way of transporting vectors in the bundle in a parallel manner over the underlying manifold.

Definition 4.2.3. *Let* M *be a real smooth manifold, and* $(E, M, \pi : E \to M)$ *be a real vector bundle over a real manifold* M*. A* ***(real linear) connection*** *is a real linear map* $\nabla : \Gamma_M(E) \to \Gamma_M(hom(TM, E))$ *satisfying the Leibniz rule: for* $X \in \Gamma_M(TM)$*,* $Y \in \Gamma_M(E)$*, and* $f \in C^\infty(M) = \Gamma_M(M \times \mathbb{R})$*, we have*

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y,$$

where $\nabla_X(\cdot) : \Gamma_M(E) \to \Gamma_M(E)$ *denotes the operator induced by* ∇ *and* X*.*

Note that a connection is $C^\infty(M)$-linear in the subscript slot, but only $\mathbb{R}$-linear in the upper slot.

We recall some preliminaries from the theory of vector bundles and connections. The following perspective can be found in [Gau94] or [KMS93].

Given a real vector bundle E over M, we can define a subbundle VE of TE over TM, called the **vertical bundle**, by $V_\xi E := \ker(d\pi : TE \to TM)$. A **horizontal bundle** is a choice of another subbundle $HE \subset TE$ such that TE splits as $T_\xi E \cong H_\xi E \oplus V_\xi E$. It turns out that the choice of a horizontal bundle is equivalent to a choice of a connection on E over M. Furthermore, given this choice of horizontal bundle or connection, we have the identifications

$$T_\xi E \cong H_\xi E \oplus V_\xi E \cong T_x M \oplus T_\xi E_x \cong T_x M \oplus E_x,$$

for $x \in M$ and $\xi \in E_x$. The identification of the outside terms, $T_\xi E \cong T_x M \oplus E_x$, is referred to as the induced **splitting**.

Given a vector $U \in T_\xi E$, we define its **vertical part**, $v^\nabla(U) \in E_x$, as image of the projection $v^\nabla : T_\xi E \to E_x$ given by the splitting. For U of the form $ds(X)$, where $s \in \Gamma_M(E)$ and $X \in T_x M$, we have

$$v^\nabla(ds(X)) = \nabla_X s. \tag{4.1}$$

Remark 4.2.4. *It is worth making a general point about notation in differential geometry. Given a map $f : M_1 \to M_2$, and a section $X \in \Gamma_{M_1}(TM_1)$ (i.e. a vector field on M_1), we can compose and get a map $df \circ X : M_1 \to TM_2$. This is does* not *induce a well-defined section of $TM_2 \to M_2$, both because f may not be onto, and also because $X(x_1) = X(x_2)$ does not imply $df(X(x_1)) = df(X(x_2))$. Nonetheless, it is common to see $df(X)$ used to denote an element of $\Gamma_{M_2}(TM_2)$. The justification is that this often appears in formulas (like Equation* (4.1)*) with the implicit understanding that X is only defined locally, and that $df(X)$ is an arbitrary local extension of the partial section $TM_2 \to M_2$ induced by $df \circ X$, to a full neighbourhood.*

Definition 4.2.5. *Let E be a complex vector bundle over a real manifold M. A (real linear) connection ∇ on E is called **complex linear** if $\nabla i_E = i_E \nabla$.*

*In the case where M is an almost complex manifold, and $E = TM$, by Remark 4.2.2, a connection is complex linear if and only if $\nabla J_M = J_M \nabla$. In this case, we further say that ∇ is **minimal** if its torsion is equal to one quarter of the Nijenhuis tensor of J_M, i.e. $T_\nabla = \frac{1}{4} N_M$.*

Fact 4.2.6 (Theorem 32 in Appendix A of [Kru07])**.** *If (M, J_M) is an almost complex manifold, then there exist minimal connections on its tangent bundle.*

Here are some basic observations about almost complex and minimal connections. In general, the difference $A := \nabla' - \nabla$ between two real linear connections is a 1-form, i.e. $A(fY) = fA(Y)$ for all $Y \in \Gamma(E)$ and $f \in C^\infty(M) = \Gamma(M \times \mathbb{R})$. If A is the difference between two minimal connections, then unwinding the definitions, we see further that A is symmetric, meaning for all $x \in M$ and $X, Y \in T_x M$,

$$A_X Y = A_Y X. \tag{4.2}$$

This, together with the fact that ∇' and ∇ are almost complex, imply that for all $x \in M$, and $X \in T_x M$,

$$A_{J_M X} = J_M \circ A_X. \tag{4.3}$$

4.3 Almost complex tangent bundles and constructed bundles

The first step in our construction is to find an almost complex structure on the tangent bundle.

Theorem 4.3.1 (Theorem 1 of [Kru07])**.** *Let (M, J_M) be an almost complex manifold, and $(TM, M, \pi : TM \to M)$ be the real tangent bundle of the underlying real manifold M. Then there exists an almost complex manifold structure $(TM, J_{TM} : TTM \to TTM)$ on TM, such that:*

1. *For each minimal almost complex connection ∇ on TM, under the induced splitting $T_\xi TM \cong T_x M \oplus T_x M$, we have $J_{TM,\xi} \cong J_{M,x} \oplus J_{M,x} = i_{TM,x} \oplus i_{TM,x}$, where $x = \pi(\xi)$.*

2. *The projection $\pi : (TM, J_{TM}) \to (M, J_M)$ is pseudoholomorphic.*

Proof. This result is essentially contained in Theorem 1 of [Kru07], but the proof is worth repeating here, because it is the model for how we will treat more complicated bundles.

To construct the almost complex structure, we first choose any minimal connection ∇ on TM. This yields the splitting described in (1), and indeed, we *define* J_{TM} by the formula there. What needs to be shown is that if we make the same definition with a different minimal connection, ∇', we get the same result.

So let $A := \nabla' - \nabla$. Using Equation (4.1), and evaluating the sections, we have that

$$\left(v^{\nabla'} - v^{\nabla}\right)(ds_x(X)) = A_X(s(x)) \;\forall s \in \Gamma_M(TM), x \in M.$$

Fix $X \neq 0$. Then all $U \in T_\xi TM$ is of the form $U = ds_x X$, for some s and x with $s(x) = \xi$. Hence we have

$$(v^{\nabla'} - v^{\nabla})(U) = A_X \xi \tag{4.4}$$

for all $U \in T_\xi TM$ with $d\pi(U) = X$.

Let J'_{TM} be defined relative to the splitting ∇' as described above. Our goal is to show that $J'_{TM}U = J_{TM}U$, which we will do in the coordinates given by the splitting ∇', i.e. we must show

$$v^{\nabla'}(J'_{TM}U) = v^{\nabla'}(J_{TM}U)$$

for all U.

By continuity, it suffices to show this for all U with $X := d\pi(U) \neq 0$, where Equation (4.4) holds. We compute:

$$\begin{aligned} v^{\nabla'}(J'_{TM}U) - v^{\nabla'}(J_{TM}U) &= v^{\nabla'}(J'_{TM}U) - v^{\nabla}(J_{TM}U) - A_{J_M X}\xi \\ &= J_M v^{\nabla'} U - J_M v^{\nabla} U - A_{J_M X}\xi \\ &= J_M A_X \xi - A_{J_M X}\xi \\ &= 0, \end{aligned}$$

using Equation (4.3) in the last step.

Thus, J_{TM} is well defined, and (1) is also proved.

Under the splitting induced by ∇, $d\pi_\xi : T_\xi TM \to T_x M$ is simply the coordinate projection down to the first factor. The definition of J_{TM} implies that $J_M \circ d\pi = d\pi \circ J_{TM}$, proving (2). $\square$

At least one other almost complex structure on TM can be found in the literature, originally in [YK66], and again in [LS01]. This other construction is known to be different from the one presented above, unless (M, J_M) is integrable (c.f. Theorem 3 in [Kru07]).

We now wish to build bundles out of tangent bundles of almost complex manifolds, such as T^*M, $TM_1 \otimes_{\mathbb{C}} TM_2$, $\hom_{\mathbb{C}}(TM_1, TM_2)$, etc., and realize them as almost complex manifolds. To do this systematically, we first define a general class of bundles that these examples belong to.

Definition 4.3.2. *A **constructed bundle** is a complex vector bundle E which is either a tangent bundle $E = TM$ over an almost complex manifold (M, J_M) (with complex vector bundle structure as described in Remark 4.2.2), or, inductively, one obtained from previously defined constructed bundles via one of the following operations:*

1. *Dual: $E = F^*$ over M is the complex dual bundle of a constructed bundle F over M.*
2. *Tensor product: $E = E_1 \otimes E_2 := E_1 \otimes_{\mathbb{C}} E_2$ over M, where E_i are constructed bundles over M, and $\otimes_{\mathbb{C}}$ is the tensor product of complex vector bundles.*
3. *Exterior power: $E = \bigwedge F := \bigwedge_{\mathbb{C}} F$ over M, where F is a constructed bundle over M, and $\bigwedge_{\mathbb{C}}$ is the exterior power of complex vector bundles.*
4. *Pullback: Let F over N be a constructed bundle, M be another almost complex manifold, and $f : M \to N$ be a pseudoholomorphic map. Then $E = f^*F$ over M is the pullback via f of F over N.*

$$\begin{tikzcd} E = f^*F \arrow[r] \arrow[d] & F \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

   We often conflate E and F, notationally.

5. *Base change: This is a common special case of the pullback, with $M = M_1 \times M_2$, $N = M_1$,$f = \pi_1$, the projection.*

Note that the definition explicitly includes information about how the constructed bundle is inductively built up from its underlying tangent bundles.

Remark 4.3.3. *Since the complex vector bundle operations in the last definition are standard, we did not need to write out the explicit formulas for i_E at the inductive steps. Nonetheless, we will want to work with them going forward, so give them here for completeness.*

For duals:

$$i_{F^*}(\lambda)(X) := \lambda(i_F X);$$

for tensor products:

$$i_{F \otimes F}(X_1 \otimes X_2) := (i_F X_1);$$

for exterior powers:

$$i_{\bigwedge^n F}(X_1 \wedge \cdots \wedge X_n) := (i_F X_1) \wedge X_2 \wedge \cdots \wedge X_n;$$

and for pullbacks:

$$i_{f^*F,x} := i_{F,f(x)}.$$

Next, we will define connections that arise naturally on constructed bundles.

Definition 4.3.4. *Suppose E is a constructed bundle over (M, J_M), built up from tangent bundles (TM_i, M_i, π_i) in a specified manner. Choose minimal connections ∇_i on TM_i over M_i. Then we define the **induced connection** ∇ on E as the one which is built up from the ∇_i in a parallel way, meaning that corresponding to the inductive steps in the construction of E, we apply the appropriate construction to the connection, from the following:*

1. *Dual: Given ∇_F on F over M, define ∇ on $E = F^*$ over M by the "product rule" formula*
$$(\nabla_X \lambda)(\xi) := X(\lambda(\xi)) - \lambda(\nabla_{F,X}\xi),$$
for sections λ of E, ξ of F, and X of TM.

2. *Tensor product: Given ∇_{E_i} on E_i over M for $i = 1, 2$, define ∇ on $E = E_1 \otimes E_2$ by the formula*
$$\nabla(\xi_1 \otimes \xi_2) := (\nabla_{E_1}\xi_1) \otimes \xi_2 + \xi_1 \otimes (\nabla_{E_2}\xi_2),$$
for sections ξ_i of E_i. For products with more factors, proceed inductively.

3. *Exterior power: Given ∇_E on E over M, define ∇ on $E = \bigwedge^n E$ by the formula*
$$\nabla(\xi_1 \wedge \cdots \wedge \xi_n) := \sum_{i=1}^{n} (\cdots \xi_{i-1} \wedge \nabla_E(\xi_i) \wedge \xi_{i+1} \wedge \cdots),$$
for sections ξ_i of E_i. (This is induced, for instance, by viewing $\bigwedge^n E \subset \bigotimes^n E$.)

4. *Pullback: Given ∇_F on F over N and $f : M \to N$, define ∇ on $E = f^*F$ by the formula*
$$\nabla_X(f^*\xi) := f^*(\nabla_{F,df(X)}\xi),$$
for sections ξ of F, and X of TM.

5. *Base change: If E over $M_1 \times M_2$ is the pullback of F over M_1, and ∇_F is given, we define ∇ on E by*
$$\nabla_{(X_1,X_2)} := (\nabla_F)_{X_1},$$
for sections X_i of TM_i. That is, simply act by "ignoring" the M_2 component.

This definition depends on the original choice of minimal connections underlying the tangent bundles.

These are standard constructions for complex vector bundles, so the straightforward proofs that these do indeed define connections (i.e. that the Leibniz rules are satisfied) are omitted. Further straightforward calculations using the relations from Remark 4.3.3 yield:

Lemma 4.3.5. *An induced connection on a constructed bundle is complex linear.*

In Theorem 4.3.1, the fact that the difference between two minimal connections satisfies Equation (4.2) and Equation (4.3) allows us to canonically define an almost complex structure on the tangent bundle of an almost complex manifold. To do the same for arbitrary constructed bundles, we need an analogue for constructed bundles.

Lemma 4.3.6. *Suppose that E is a constructed bundle over M, and TM_i are the underlying tangent bundles on M_i. Choose minimal connections ∇_i and ∇'_i on TM_i over M_i, and let ∇ and ∇' be the respective induced connections on E over M. Then $A := \nabla' - \nabla$ satisfies*

$$A_{J_M X} = i_E \circ A_X,$$

where $i_E : E \to E$ is the complex vector bundle structure on E.

Again, the proof consists of straightforward inductive computations.

Finally, we can define the almost complex manifold structure on a constructed bundle.

Theorem 4.3.7. *Let E be a constructed bundle over an almost complex manifold (M, J_M). Then E can be given the structure $(E, J_E : TE \to TE)$ of an almost complex manifold such that π is pseudoholomorphic.*

Moreover, let ∇ be a connection on E induced by its construction, which in turn induces the splitting $T_\xi E \cong T_x M \oplus E_x$. Then relative to this splitting, we have $J_E \cong J_M \oplus i_E$.

With all of our preliminaries, the proof of this theorem is now exactly analogous to that of tangent bundle case, Theorem 4.3.1, with the various properties we have proved about constructed bundles and induced connections replacing facts about the tangent bundle and minimal connections. Nonetheless, for completeness, the details are presented.

Proof. We choose an induced connection ∇ and *define* $J_E : TE \to TE$ by the formula in the statement of the theorem. We then need to check that this definition is independent of the chosen induced connection (i.e. the choice of the underlying minimal connections).

Let ∇' be another induced connection, and J'_E be the associated almost complex connection. We want to show that for all $U \in T_\xi E$, $J_E U = J'_E U$.

We will do this in the coordinates induced by the connection ∇', which comes down to showing that

$$v^{\nabla'}(J_E U) = v^{\nabla'}(J'_E U).$$

By continuity, it suffices to verify this for all U with $X := d\pi(U) \neq 0$, where we know that Equation (4.4) holds.

We compute:

$$\begin{aligned} v^{\nabla'}(J'_E U) - v^{\nabla'}(J_E U) &= v^{\nabla'}(J'_E U) - v^{\nabla}(J_E U) - A_{J_M X}\xi \\ &= i_E v^{\nabla'} U - i_E v^{\nabla} U - A_{J_M X}\xi \\ &= i_E A_X \xi - A_{J_M X}\xi \\ &= 0, \end{aligned}$$

using Lemma 4.3.6 in the last step. This shows that J_E is defined independently of the induced connection, and is given by the prescribed equation under any induced splitting.

Finally, under a fixed splitting, $d\pi : TE \to TM$ is simply the first coordinate projection, and the way that J_E was defined makes it clear that $J_M \circ d\pi = d\pi \circ J_E$. □

Remark 4.3.8. *J_E is defined in terms of i_E, which itself is inductively defined via Definition 4.3.2.*

4.4 Hom bundles and lifts of pseudoholomorphic maps

Given two constructed bundles (E_1, M_1) and (E_2, M_2), and a pseudoholomorphic map $f : M_1 \to M_2$, we define the **hom bundle** $\hom_f(M_1, M_2)$ over M_1 by

$$\hom_f(E_1, E_2) := \hom(E_1, f^*E_2) := (E_1)^* \otimes f^*E_2.$$

The fibres are thus given by

$$\hom_f(M_1, M_2)_x = \hom_{\mathbb{C}}(E_{1,x}, E_{2,f(x)}).$$

We will be interested in the constructed bundle

$$E := \bigwedge^k \hom_f(TM_1, TM_2).$$

There are a number of maps induced by f, many of which are similar, which we would like to name. The situation is described by the following diagram:

$$\begin{tikzcd}
TM_1 \arrow[dr, "df^*"] \arrow[drr, bend left, "df"] \arrow[ddr, bend right] & & \\
& f^*TM_2 \arrow[r] \arrow[d] & TM_2 \arrow[d] \\
& M_1 \arrow[r, "f"] & M_2
\end{tikzcd}$$

First, $f : M_1 \to M_2$ induces as usual a map $df : TM_1 \to TM_2$. Then, by the universal property of the pullback f^*TM_2 (essentially, as a fibre product), the maps $df : TM_1 \to TM_2$ and the natural projection $TM_1 \to M_1$ induce a map $df^* : TM_1 \to f^*TM_2$. It is common to conflate df and df^*, but we will keep the distinction to make our main computation clearer.

One relation that we will need is

$$f^*(df X) = df^* X, \tag{4.5}$$

for $X \in \Gamma_{M_1}(TM_1)$. For this to make sense, we must make use of the conventions discussed in Remark 4.2.4.

Finally, we have an induced map

$$\begin{aligned} \tilde{f} : M_1 &\to \bigwedge^k \hom_f(TM_1, TM_2) \\ x &\mapsto \left(x, \bigwedge^k df^* \right), \end{aligned}$$

for each k, called the **lift** of f. (This is the kth alternating power of the 1-*jet* of f.)

As a constructed bundle, the codomain $E = \bigwedge^k \hom_f(TM_1, TM_2)$ is itself an almost complex manifold.

Theorem 4.4.1. *Given a pseudoholomorphic map $f : M_1 \to M_2$, its lift $\tilde{f} : M_1 \to \bigwedge^k \hom_f(E_1, E_2)$ is pseudoholomorphic.*

Proof. Let ∇ be the connection on E over M_1, induced by the underlying minimal connections ∇_i on TM_i. This, in turn, induces a splitting

$$T_\xi E \cong T_x(M_1) \oplus \bigwedge^k \hom(TM_{1,x}, TM_{2,f(x)}),$$

where $\xi \in E_x$. By Equation (4.1), given a local section $X \in \Gamma_{M_1}(TM_1)$, $d\tilde{f}(X) \in \Gamma_{M_1}(E)$ is represented in the splitting by

$$d\tilde{f}(X) = \left(X, \nabla_X \bigwedge^k df^* \right).$$

Recall from Theorem 4.3.7 that we also have, under the splitting, $J_E \cong J_{M_1} \oplus i_E$, where $i_E : E \to E$ is the complex vector bundle structure on E. We will shortly prove that

$$\left(\nabla_X \bigwedge^k df^* \right)(Y) = \left(\nabla_Y \bigwedge^k df^* \right)(X), \tag{4.6}$$

and this, together with the fact that many of our objects are already complex linear, shows that

$$\begin{aligned} \left(\nabla_{J_{M_1} X} \bigwedge^k df^* \right)(Y) &= \left(\nabla_Y \bigwedge^k df^* \right)(J_{M_1} X) \\ &= i_E \left((\nabla_Y \bigwedge^k df^*)(X) \right) \\ &= i_E \left((\nabla_X \bigwedge^k df^*)(Y) \right), \end{aligned}$$

i.e. we have complex linearity in the subscript slot of ∇ as well.

With these facts in mind, we can now easily see that $\tilde{f}$ is pseudoholomorphic:

$$\begin{aligned} J_E(d\tilde{f}(X)) &= \left(J_{M_1}X, i_E\left(\nabla_X \bigwedge^k df^*\right)\right) \\ &= \left(J_{M_1}X, \nabla_{J_{M_1}X} \bigwedge^k df^*\right) \\ &= d\tilde{f}(J_{M_1}X). \end{aligned}$$

It remains to verify Equation (4.6). One should bear in mind Remark 4.2.4 regarding the use of expressions like $df(X)$ and $df^*(X)$.

First, using Definition 4.3.4, we have

$$\begin{aligned} \left(\nabla_X \bigwedge^k df^*\right)(Y) - \left(\nabla_Y \bigwedge^k df^*\right)(X) &= \cdots + (\cdots \wedge df^* \wedge \nabla_X df^* \wedge df^* \wedge \cdots)(Y) + \cdots \\ &\quad \cdots - (\cdots \wedge df^* \wedge \nabla_Y df^* \wedge df^* \wedge \cdots)(X) - \cdots, \end{aligned}$$

where ∇ also denotes the connection on $\hom_f(TM_1, TM_2)$. Thus, it suffices to show

$$(\nabla_X df^*)(Y) - (\nabla_Y df^*)(X) = 0.$$

We will expand the left side using explicit expressions for ∇ with respect to the underlying minimal connections ∇_i on TM_i, using the inductive steps in Definition 4.3.4. Specifically, we use the general equations $(\nabla_{\hom(E_1,E_2)}(\lambda))(s) = \nabla_{E_2}(\lambda(s)) - \lambda(\nabla_{E_1}s)$, and $(g^*\nabla_Z)(g^*s) = g^*(\nabla_{dg(Z)}s)$.

$$\begin{aligned} (\nabla_X(df^*))(Y) - (\nabla_Y(df^*))(X) &= (f^*\nabla_2)_X(df^*Y) - df^*(\nabla_{1,X}Y) \\ &\quad -(f^*\nabla_2)_Y(df^*X) + df^*(\nabla_{1,Y}X) \\ &\overset{(4.5)}{=} (f^*\nabla_2)_X(f^*(dfY)) - df^*(\nabla_{1,X}Y) \\ &\quad -(f^*\nabla_2)_Y(f^*(dfX)) + df^*(\nabla_{1,Y}X) \\ &= f^*(\nabla_{2,dfX}dfY) - df^*(\nabla_{1,X}Y) \\ &\quad -f^*(\nabla_{2,dfY}dfX) + df^*(\nabla_{1,Y}X) \\ &\overset{(4.5)}{=} f^*(\nabla_{2,dfX}dfY) - f^*(\nabla_{2,dfY}dfX) \\ &\quad -f^*(df(\nabla_{1,X}Y)) + f^*(df(\nabla_{1,Y}X)). \end{aligned}$$

We will show that this last expression is the pullback of the zero section, and hence itself vanishes. Since ∇_i are minimal, they are related to the Nijenhuis tensors N_i of M_i via $T_{\nabla_i} = \frac{1}{4}N_i$. As the Nijenhuis tensor is preserved by differentials of pseudoholomorphic maps, we have

$$\begin{aligned} 0 &= \frac{1}{4}\big(N_2(df X, df Y) - df(N_1(X,Y))\big) \\ &= T_{\nabla_2}(df X, df Y) - df(T_1(X,Y)) \\ &= \nabla_{2,df X} df Y - \nabla_{2,df Y} df X - [df X, df Y] \\ &\quad -df(\nabla_{1,X} Y - \nabla_{1,Y} X - [X,Y]) \\ &= \nabla_{2,df X} df Y - \nabla_{2,df Y} df X \\ &\quad -df(\nabla_{1,X} Y) + df(\nabla_{1,Y} X), \end{aligned}$$

so indeed, applying the pullback f^* finishes the proof. □

4.5 Proof of the theorem

We now restate and prove Theorem 4.1.2. A sketch of the idea was given in introductory section.

Theorem. *If X is a well-presented subset of an almost complex manifold M, then* $\operatorname{sing} X$ *is equationally pseudoanalytic, and a proper subset of X.*

Proof. Since X is a pseudoanalytic set, Proposition 3.3.5 implies that $\operatorname{sing} X = \operatorname{sing}_{\mathbb{R}} X$, which is Euclidean closed, by the real analytic analogue of the present theorem (or of Fact 4.1.1). Let $2d = \operatorname{codim}_{\mathbb{R}} X$, where d is a whole number, by Corollary 3.3.6.

To verify equational pseudoanalyticity, pick any $x \in \operatorname{sing} X$. Since X is well-presented, we find a neighbourhood U, an almost complex manifold N, and a pseudoholomorphic $f = (f_1, \dots, f_{2n}) : U \to N$, with $X \cap U = f^{-1}(0)$, and

$$\sqrt{(f_1, \dots, f_{2n})} = (f_1, \dots, f_{2n})$$

as real ideals, where the components f_i are conflated with their germs at x.

Working at the level of germs at x, and viewing X as a subset of the real manifold underlying M, we have

$$I^{\mathbb{R}}(X) = \sqrt{(f_1, \dots, f_{2n})} = (f_1, \dots, f_{2n}).$$

Thus, by the Jacobian characterization of singular locus in the real analytic setting, we have that in some neighbourhood of x,

$$y \in \operatorname{sing} X \iff y \in \operatorname{sing}_{\mathbb{R}} X \iff \operatorname{rank}_{\mathbb{R}} df_y < 2d \iff \cdots .$$

We can view df_y as a map of complex vector spaces, and hence continue the equivalencies:

$$\cdots \iff \operatorname{rank}_{\mathbb{C}} df_y < d \iff \bigwedge_{\mathbb{C}}^{d} df_y = 0 \iff \tilde{f}(y) = 0(y),$$

where $\tilde{f}$ is of course the lift corresponding to the dth alternating power, and 0 is the 0-section. (Note that while df and df^* differ as bundle maps, on the level of a specific fibre, they can be identified.)

This holds in every neighbourhood of $x \in X$, and hence $\operatorname{sing} X$ is equationally pseudo-analytic. $\square$

Chapter 5

Real and almost complex ideals

In this chapter, we raise a simple question about "almost complex ideals". We provide some answers in the integrable case, under further conditions. The hope is that the proof technique, which analyses linear properties of partial differential operators, might be adaptable more generally. Many other questions could be asked about ideals in the almost complex setting, in relation to pseudoanalytic sets, and the general investigations of this thesis. But even restricting ourselves as we have, the chapter is quite speculative.

Only basic definitions, which can be found in Section 2.2, are required here.

5.1 Comparing real and almost complex ideals

Suppose that $(\mathbb{R}^{2m}, J)$ is an almost complex manifold. Let $\mathcal{O}_p^{\mathbb{R}}$ be the ring of germs of real analytic functions $\mathbb{R}^{2m} \to \mathbb{R}$ at $p \in \mathbb{R}^{2m}$, and let $\mathcal{O}_p$ be the ring of germs of pseudoholomorphic maps $(\mathbb{R}^{2m}, J) \to (\mathbb{C}, i)$ at p. To compare these two rings, we define the following operations.

Definition 5.1.1. *Let $I \subset \mathcal{O}_p$ be an ideal. We define $I^{\mathbb{R}} \subset \mathcal{O}_p^{\mathbb{R}}$ to be the ideal generated by the real and imaginary parts of elements of I, i.e.*

$$I^{\mathbb{R}} := \langle u, v \in \mathcal{O}_p^{\mathbb{R}} \mid \exists\, f \in I \text{ such that } f = u + iv \rangle$$

Now let $J \subset \mathcal{O}_p^{\mathbb{R}}$ be an ideal. We define the ideal $J^{\mathrm{ac}} \subset \mathcal{O}_p$ by

$$J^{\mathrm{ac}} := \{ f \in \mathcal{O}_p \mid \exists\, u, v \in J \text{ such that } f = u + iv \text{ and } f \text{ is pseudoholomorphic.} \}$$

For any ideal $I \in \mathcal{O}_p$, we have $(I^{\mathbb{R}})^{\mathrm{ac}} \supset I$. Given our interest in pseudoanalytic sets, it is natural to ask whether equality always holds.

Question 5.1.2. *Is it the case that for all ideals $I \in \mathcal{O}_p$, we have $(I^{\mathbb{R}})^{\mathrm{ac}} = I$?*

For instance, the class of ideals satisfying this property satisfies the ascending chain condition:

Proposition 5.1.3. *Let C be the class of ideals of $\mathcal{O}_p$ satisfying $(I^{\mathbb{R}})^{\mathrm{ac}} = I$. Then C satisfies the ascending chain condition, i.e. if*

$$I_0 \subset I_1 \subset I_2 \subset \cdots$$

is a chain of ideals from C, then there is some N such that

$$I_N = I_{N+1} = I_{N+2} = \cdots .$$

Proof. Consider the chain of ideals

$$I_0^{\mathbb{R}} \subset I_1^{\mathbb{R}} \subset I_2^{\mathbb{R}} \subset \cdots .$$

Because $\mathcal{O}_p^{\mathbb{R}}$ satisfies the ascending chain condition for all ideals, we have N such that

$$I_N^{\mathbb{R}} = I_{N+1}^{\mathbb{R}} = I_{N+2}^{\mathbb{R}} = \cdots .$$

Then for $i \geq N$, we have

$$I_i = (I_i^{\mathbb{R}})^{\mathrm{ac}} = (I_{i+1}^{\mathbb{R}})^{\mathrm{ac}} = I_{i+1}.$$

□

Thus, if we had an affirmative answer to Question 5.1.2, we would know that the ascending chain condition holds for all ideals. Unfortunately, we only have partial answers, in very classical settings. In such settings, the ascending chain condition is already known, and indeed, robust algebraic theories already exist. Nonetheless, in the next section, we present these partial answers, in the hope that the proofs can be adapted to more general cases.

5.2 Real and complex ideals

First, as pointed out by Clifton Ealy, there is a simple answer in the case of radical ideals.

Proposition 5.2.1. *Let $\mathcal{O}$ be the ring of germs of holomorphic functions $\mathbb{C}^n \to \mathbb{C}$ at $p \in \mathbb{C}$, and $\mathcal{O}^{\mathbb{R}}$ the ring of germs of real analytic functions $\mathbb{R}^{2n} \to \mathbb{R}$ at $p \in \mathbb{R}^2 \cong \mathbb{C}$. If $I \subset \mathcal{O}$ is a radical ideal, then $(I^{\mathbb{R}})^{\mathrm{ac}} = I$.*

Proof. We first reason with an ideal I which is not necessarily radical. We always have $I \subset (I^{\mathbb{R}})^{\mathrm{ac}}$, and hence $V(I) \supset V((I^{\mathbb{R}})^{\mathrm{ac}})$ as germs of subsets of $\mathbb{C}^n$.

Indeed, the reverse containment is true as well: if $p \in V(I)$, and $f = u + iv \in (I^{\mathbb{R}})^{\mathrm{ac}}$, then $p \in V(f)$. Thus $V(I) = V((I^{\mathbb{R}})^{\mathrm{ac}})$.

Applying the nullstellensatz, we obtain the top line of the diagram

$$\begin{array}{ccc} \sqrt{I} & = & \sqrt{(I^{\mathbb{R}})^{\mathrm{ac}}} \\ \cup & & \cup \\ I & \subset & (I^{\mathbb{R}})^{\mathrm{ac}}, \end{array}$$

with the other containments true by definition.

Finally, if I is radical, the diagram collapses, and we can conclude that $I = (I^{\mathbb{R}})^{\mathrm{ac}}$. □

If the ideals in question are not radical, then we can still get the same conclusion, if we restrict to the case where the domain is 1-dimensional, and the ideal is principal. A fair bit of work seems to be required.

The main ideas of this proof were suggested to the author by his thesis advisor.

Theorem 5.2.2. *Let $\mathcal{O}$ be the ring of germs of holomorphic functions $\mathbb{C} \to \mathbb{C}$ at $p \in \mathbb{C}$, and $\mathcal{O}^{\mathbb{R}}$ the ring of germs of real analytic functions $\mathbb{R}^2 \to \mathbb{R}$ at $p \in \mathbb{R}^2 \cong \mathbb{C}$. If $I = (g)$ is a principal ideal generated by $g : \mathbb{C} \to \mathbb{C}$, then $(I^{\mathbb{R}})^{\mathrm{ac}} = I$.*

Proof. Let $g = u + iv$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$. Then $I^{\mathbb{R}} = (u, v)$. We will analyze the situation using partial differential operators.

Let $\mathbb{G}$ be $\mathcal{O}^{\mathbb{R}}$ viewed as an additive group. We will define group homomorphisms Λ, Θ, Δ, and Ψ, which fit into the following diagram:

$$\begin{tikzcd} \mathbb{G}^2 \arrow[r, "\Lambda"] \arrow[d, "\Theta"'] & \mathbb{G} \arrow[d, "\Delta"] \\ \mathbb{G}^2 \arrow[r, "\Psi"] & \mathbb{G}. \end{tikzcd}$$

Three of the definitions can be given immediately. Let x and y be the standard coordinates of $\mathbb{R}^2$. We set:

1. $\Lambda(\alpha, \beta) := \alpha u - \beta v$. ($\Lambda$ for **l**inear.)

2. $\Delta(\xi) := \xi_{xx} + \xi_{yy}$. Note that $\ker \Delta$ consists of germs of harmonic functions, or equivalently, the germs of real parts of holomorphic functions.

3. $\Theta(\alpha, \beta) := (\alpha_x - \beta_y, \alpha_y + \beta_x)$. Note that $\ker \Theta$ consists of harmonic pairs of functions, i.e. pairs satisfying the Cauchy-Riemann equations, i.e. pairs which are respectively the real and imaginary parts of a single holomorphic function.

We can think of $\ker \Lambda$ as parameterizing $\mathbb{G}$-linear combinations of u and v, and hence $\ker \Delta \circ \Lambda$ as parameterizing such combinations that are real parts of holomorphic functions. The theorem is proved by analyzing pairs in this kernel in terms of the other path in the diagram above.

The reader can get a better sense of the overall picture at first by skipping the proofs of the claims.

Claim 5.2.3. *There exists an operator $\Psi : \mathbb{G}^2 \to \mathbb{G}$ making the above diagram commute. Indeed, we have an explicit formula*

$$\begin{aligned} \Psi(\gamma, \delta) &= 2u_x\gamma + 2u_y\delta + u(\gamma_x + \delta_y) + v(\gamma_y - \delta_x) \\ &= (u\gamma - v\delta)_x + (u\delta + v\gamma)_y. \end{aligned}$$

Proof. This can be verified by plugging the given formula into the diagram and verifying that it commutes, but it is more enlightening to see where the formula comes from.

Looking at the diagram, since Λ has degree 0 as a differential operator, Δ has degree 2, and Θ has degree 1, if Ψ exists we would expect it to have degree 1.

So we can define an arbitrary degree 1 operator,

$$\Psi'(\gamma, \delta) = A\gamma + B\delta + C\gamma_x + D\delta_x + E\gamma_y + F\delta_y,$$

and plug it into the commutative diagram in place of Ψ.

Along the top, we get

$$\begin{aligned} \Delta(\Lambda(\alpha, \beta)) &= \alpha_{xx}u + 2\alpha_x u_x + \alpha u_{xx} \\ &\quad +\alpha_{yy}u + 2\alpha_y u_y + \alpha u_{yy} \\ &\quad -\beta_{xx}v - 2\beta_x v_x - \beta v_{xx} \\ &\quad -\beta_{yy}v - 2\beta_y v_y - \beta v_{yy}. \end{aligned}$$

Note that last column of four terms cancels out, since u and v are harmonic.

On the bottom, we have

$$\begin{aligned} \Psi'(\theta(\alpha, \beta)) &= A(\alpha_x - \beta_y) + B(\alpha_y + \beta_x) \\ &\quad +C(\alpha_{xx} - \beta_{xy}) + D(\alpha_{xy} + \beta_{xx}) \\ &\quad +E(\alpha_{xy} - \beta_{yy}) + F(\alpha_{yy} + \beta_{xy}). \end{aligned}$$

Comparing both sides, we find a unique solution, given by

$$\begin{aligned} A &= 2u_x = 2v_y \\ B &= 2u_y = -2v_x \\ C &= F = u \\ D &= -E = -v. \end{aligned}$$

(This is consistent, by the Cauchy-Riemann equations.)

Hence, taking $\Psi = \Psi'$ will make the diagram commute, and we have derived the first formula for Ψ. The second formula can be derived using the Cauchy-Riemann equations. $\square$

Claim 5.2.4. $\ker \Psi = \Theta(\ker \Lambda)$.

Proof. $\Theta(\ker \Lambda) \subset \ker \Psi$ follows from the diagram. We wish to show the reverse containment, i.e. assuming that

$$0 = \Psi(\gamma, \delta) = (\gamma u - \delta v)_x + (\delta u + \gamma v)_y,$$

we want to find an (α, β) satisfying

$$\begin{cases} \alpha u - \beta v = 0, \\ \alpha_x - \beta_y = \gamma \\ \alpha_y + \beta_x = \delta. \end{cases}$$

We will search for solutions (α, β) of the form

$$\begin{cases} \alpha = \xi v \\ \beta = \xi u, \end{cases}$$

for some ξ, so that the first equation $\alpha u - \beta v = 0$ is automatically satisfied. Plugging this form into the two other equations in the system, and using the Cauchy-Riemann equations, yields the system

$$\begin{cases} A(\xi) := \xi_x v - \xi_y u + 2\xi v_x - \gamma = 0 \\ B(\xi) := \xi_y v + \xi_x u + 2\xi u_x - \delta = 0, \end{cases}$$

which we wish to solve.

Rearranging, this is equivalent to

$$\begin{cases} vA(\xi) + uB(\xi) = 0 \\ -uA(\xi) + vB(\xi) = 0. \end{cases}$$

Expanding and further rearranging, we can rewrite this system as

$$\begin{cases} (\xi(u^2 + v^2))_x = \delta u + \gamma v \\ (\xi(u^2 + v^2))_y = -\gamma u + \delta v. \end{cases}$$

To solve this system, we look for solutions ξ of the form $\xi = \frac{\eta}{(u^2+v^2)}$, so η needs to satisfy

$$\begin{cases} \eta_x = \delta u + \gamma v \\ \eta_y = -\gamma u + \delta v. \end{cases}$$

Finally, the existence of solutions η to systems of this form is dictated by Poincaré's Lemma. We have a fixed 1-form

$$\omega := (\delta u + \gamma v)\, dx + (-\gamma u + \delta v)\, dy,$$

and are seeking a 0-form η such that

$$\omega = d\eta := \eta_x\, dx + \eta_y\, dy,$$

i.e. we want to show that ω is exact. Poincaré's Lemma states that any form is exact if and only if it is closed (for germs, where we do not have to worry about neighbourhoods). Thus, we must check that $d\omega = 0$.

We compute:

$$\begin{aligned} d\omega &= (\delta u + \gamma v)_y\, dy \wedge dx + (-\gamma u + \delta v)_x\, dx \wedge dy \\ &= \left(-(\delta u + \gamma v)_y + (-\gamma u + \delta v)_x\right)\, dx \wedge dy \\ &= -\Psi(\gamma, \delta)\, dx \wedge dy \\ &= 0, \end{aligned}$$

using the condition on (γ, δ) from the beginning of the proof of the claim.

$\square$

Claim 5.2.5. $\ker \Psi \circ \Theta = \ker \Theta + \ker \Lambda$.

Proof. The $\supset$ containment follows from the commutative diagram. Conversely, if $(\alpha, \beta) \in \ker \Psi \circ \Theta$, then by Claim 5.2.4, $\Theta(\alpha, \beta) \in \ker \Psi = \Theta(\ker \Lambda)$, so (α, β) is in the coset of $\ker \Theta$ given by an element of $\ker \Lambda$, i.e. $(\alpha, \beta) \in \ker \Theta + \ker \Lambda$. $\square$

We will now prove the theorem. Suppose that $f \in ((g)^{\mathbb{R}})^{\mathrm{ac}}$. We wish to show $f \in (g)$.

Write $\mathrm{re}(f) = \alpha u - \beta v = \Lambda(\alpha, \beta)$ for $\alpha, \beta \in \mathcal{O}^{\mathbb{R}} = \mathbb{G}$. Since f is holomorphic, $(\alpha, \beta) \in \ker \Delta \circ \Lambda$, and hence by Claim 5.2.5, $(\alpha, \beta) \in \ker \Theta + \ker \Lambda$.

Choose $(\lambda, \delta) \in \ker \Theta$—which are thus conjugate pairs—such that $(\alpha, \beta) = (\lambda, \delta) + \ker \Lambda$, and hence, applying Λ,

$$\mathrm{re}\, f = \alpha u - \beta v = \gamma u - \delta v.$$

The real part of a holomorphic function determines the whole function up to an imaginary constant, so there is $c \in \mathbb{R}$ with

$$f = (\gamma + i\delta)(u + iv) + ic.$$

Since $f \in ((g)^{\mathbb{R}})^{\mathrm{ac}}$,

$$\mathrm{im}\, f = \gamma v + \delta u + c \in (g)^{\mathbb{R}} = (u, v),$$

so $c \in (g)^{\mathbb{R}}$.

We can assume $(g) \subsetneq \mathcal{O}$, since otherwise the theorem is trivial. Now, if $c \neq 0$, then $1 \in (g)^{\mathbb{R}}$, and hence $1 \in (g)$, contradicting our assumption. Thus $c = 0$, and the theorem is proved. $\square$

This proof does not directly extend to the case of holomorphic functions of several variables, let alone to the almost complex case. However, it seems possible that the methods could be adapted, in some cases. We record partial progress in various directions in the next section.

5.3 Obstacles in general

First, one might attempt to extend Theorem 5.2.2 to the where I is finitely generated. The proof breaks down, however, at the application of Poincaré's lemma. In place of the form ω, we have a finite sum $\sum_i \omega_i$, and while it is still true that $\sum_i \omega_i$ is closed, we don't know that each ω_i is closed.

A second natural extension of the theorem would be to the setting of holomorphic functions of several complex variables. We sketch some details of how this might go.

Our set up is that now $\mathcal{O}$ is the ring of holomorphic functions $\mathbb{C}^m \to \mathbb{C}$ at $p \in \mathbb{C}^m$, and $\mathcal{O}^{\mathbb{R}}$ the ring of germs of real analytic functions $\mathbb{R}^{2m} \to \mathbb{R}$ at $p \in \mathbb{R}^{2m} \cong \mathbb{C}^m$.

Let $I = (g)$ be a principal ideal generated by $g : \mathbb{C}^m \to \mathbb{C}$. The goal would be to show, as before, that $(I^{\mathbb{R}})^{\mathrm{ac}} = I$.

Let $g = u + iv$, where $u, v : \mathbb{R}^{2m} \to \mathbb{R}$. As before, we have, $I^{\mathbb{R}} = (u, v)$. Let $\mathbb{G}$ be $\mathcal{O}^{\mathbb{R}}$ viewed as an additive group.

We will again define maps in the diagram

$$\begin{tikzcd} \mathbb{G}^2 \arrow[r, "\Lambda"] \arrow[d, "\Theta"'] & \mathbb{G} \arrow[d, "\Delta"] \\ \mathbb{G}^{2m} \arrow[r, "\Psi"] & \mathbb{G}^{m(m+1)}, \end{tikzcd}$$

as follows:

1. $\Lambda(\alpha, \beta) := \alpha u - \beta v$. This is the same as before.
2. $\Delta(\xi) := (\ldots, \xi_{x_i x_j} + \xi_{y_i y_j}, \xi_{x_i y_j} - \xi_{y_i x_j}, \ldots)$. Here, $\ker \Delta$ consists of germs of *pluriharmonic* functions, or equivalently, the germs of real parts of holomorphic functions $\mathbb{C}^m \to \mathbb{C}$. Because of this equivalence, this is the natural generalization of the Δ from above.
3. $\Theta(\alpha, \beta) := (\ldots, \alpha_{x_i} - \beta_{y_i}, \alpha_{y_i} + \beta_{x_i}, \ldots)$. Note that $\ker \Theta$ consists of harmonic pairs of functions, i.e. pairs satisfying the several variable Cauchy-Riemann equations, i.e. pairs which are respectively the real and imaginary parts of a single holomorphic function.

These three maps again determine a unique map Ψ that makes the diagram commute, which we will now give in coordinates. If we write

$$\Psi(\ldots, \gamma_i, \delta_i, \ldots) = \big(\ldots, \Psi^{i,j,0}(\ldots, \gamma_i, \delta_i, \ldots), \Psi^{i,j,1}(\ldots, \gamma_i, \delta_i, \ldots), \ldots\big),$$

then we have

$$\Psi^{i,j,0}(\ldots, \gamma_i, \delta_i, \ldots) = (\gamma_i u - \delta_i v)_{x_j} + (\delta_j u + \gamma_j v)_{y_i},$$

and

$$\begin{aligned} \Psi^{i,j,1}(\ldots, \gamma_i, \delta_i, \ldots) &= -(\delta_i u + \gamma_i v)_{x_j} + (\delta_j u + \gamma_j v)_{x_i} \\ &= (\gamma_i u - \delta_i v)_{y_j} - (\gamma_j u - \delta_j v)_{y_i}. \end{aligned}$$

We again want to understand the kernel of the composite maps, and in place of Claim 5.2.4, the key is:

Goal 5.3.1. *Show that* $\ker \Psi \cap \operatorname{im} \Theta = \Theta(\ker \Lambda)$.

Again, $\supset$ follows from the diagram. However, the argument for $\subset$ could not be replicated: the conditions for $(\ldots, \gamma_i, \delta_i, \ldots)$ being in $\ker \Psi \cap \operatorname{im} \Theta$, do not seem to imply all of the following exactness conditions needed to show membership in $\Theta(\ker \Lambda)$, obtained from generalizing the proof of Claim 5.2.4:

$$\begin{cases} \forall\, i \neq j,\ (\gamma_i v + \delta_i u)_{x_j} = 0 \\ \forall\, i \neq j,\ (-\gamma_i u + \delta_i v)_{y_j} = 0 \\ \forall\, i, j,\ (\gamma_i u - \delta_i v)_{x_j} + (\delta_j u + \gamma_j v)_{y_i} = 0 \end{cases}.$$

The third condition is equivalent to $\Psi^{i,j,0}(\ldots,\gamma_i,\delta_i,\ldots)=0$, but the remaining ones do not seem to follow. Attempts to resolve this question using brute force manipulations, or using abstract rank computations (e.g. via the model theory of partial differential equations), fell short.

Chapter 6

Towards Zariski geometries

In this chapter, we formulate questions about the structure of almost complex manifolds, inspired by model theory. The motivation for this line of inquiry comes from Zilber's observation that a compact complex manifold, equipped with the class of its complex analytic subsets, forms a structure known as a "Zariski geometry" [Zil93]. This sets the scene for a fruitful model-theory of compact complex manifolds. The natural question for us, then, is whether a compact almost complex manifold can be seen as a Zariski geometry, in a similar way.

Kessler showed that certain almost complex manifolds, equipped with structures "generated" by images of compact complex manifolds under pseudoholomorphic maps, do form Zariski geometries [Kes11]. However, her results are restricted to almost complex manifolds M with the property that pseudoholomorphic maps from complex manifolds to M factor through $\mathbb{C}$, and hence essentially only support images of pseudoholomorphic curves.

Our perspective is dual to Kessler's: we ask about the existence of Zariski geometries stemming from the level sets of pseudoholomorphic maps *out* of almost complex manifolds, rather than the images of maps into them. The hope is that this more naturally captures the spirit of complex analytic subsets of compact complex manifolds, without restrictions.

Working at this level of generality, however, presents significant challenges. For complex manifolds, Zilber had access to a rich theory of complex analytic geometry, underpinned by a robust algebraic counterpart. But even basic facts that are taken for granted in the complex case do not automatically carry over to almost complex manifolds. These kinds of results—preliminaries of an almost complex analytic theory—are the focus of this thesis. Although we fall short of the deeper theory needed to show that compact almost complex manifolds form Zariski geometries, we record here our progress in towards this goal, to explain the impetus for our work. Naturally, this chapter will contain more open questions than closed ones.

6.1 The axioms for Zariski geometries

Zariski geometries are model-theoretic structures augmented with topological and geometric data, emulating the behaviour of the prototypical example: the Zariski topology and structure of a smooth algebraic variety over an algebraically closed field. Fundamental theorems about Zariski geometries, from [HZ96], classify "1-dimensional" Zariski geometries, and have led to significant applications in algebraic and diophantine geometry. In the case of compact complex manifolds, and hence for almost complex manifolds, we work with "higher-dimensional" Zariski geometries. The presentation here follows [Zil10] very closely.

Let $(M, \mathcal{C})$ be a set M equipped with a set $\mathcal{C} := \bigsqcup C_i$, where each $\mathcal{C}_i$ is a collection of subsets of the Cartesian power $M^i = M \times \cdots \times M$.

Definition 6.1.1. *A pair $(M, \mathcal{C})$ is called a **topological structure** if each collection $\mathcal{C}_i$ coincides with the closed sets of a topology on M^i, and $\mathcal{C}$ coincides with the collection of positive quantifier-free definable sets in an **associated language** on the universe M. In other words, $(M, \mathcal{C})$ is a topological structure if the elements of $\mathcal{C}$, called the **closed sets** of M, are closed under*

1. *arbitrary intersections,*
2. *finite unions,*
3. *Cartesian products,*
4. *permutations of coordinates (i.e. if $X \subset M^k$ is closed, and X^σ is the image of X under the permutation of the k components of M^k under $\sigma \in S_k$, then X^σ is closed), and*
5. *fibres (i.e. if $X \subset M^k \times M^l$ is closed, and $a \in M^k$, then ${}_aX := \{b \in M^l \mid (a, b) \in X\}$ is closed in M^l);*

and furthermore, $\mathcal{C}$ includes

1. *the set M,*
2. *all singletons $\{a\}$ for $a \in M$, and*
3. *the graph of equality in M^2.*

Definition 6.1.2. *Let $(M, \mathcal{C})$ be a topological structure. We call Boolean combinations of elements of $\mathcal{C}$ the **constructible sets**. These are the quantifier-free definable sets in the corresponding language. Sets formed by finitely many applications of Boolean operations and coordinate projections, to the elements of $\mathcal{C}$, are called the **definable sets**. These are the same as the definable sets in the associated language.*

*A definable set $X \in \mathcal{C}$ is **irreducible** if there are no proper relatively closed subsets X_1 and X_2 of X with $X = X_1 \cup X_2$.*

A topological structure $(M, \mathcal{C})$ *is* ***Noetherian*** *if its closed sets satisfy the descending chain condition, i.e. for each* n*, every descending chain of closed subsets of* M^n *stabilizes at a finite stage.*

Noetherian topological structures are the logico-topological framework for Zariski geometries. We now mix in the geometric structure.

Definition 6.1.3. *Let* $(M, \mathcal{C})$ *be a Noetherian topological structure.*

We say that $(M, \mathcal{C})$ *is* ***semiproper*** *if the projection maps are semiproper, i.e. if* $X \subset M^n$ *is closed and irreducible, and* $\pi : M^n \to M^k$ *is a projection map, then there is a closed* $F \subsetneq \overline{\pi X}$ *with* $\overline{\pi X} \setminus F \subset \pi X$*.*

We say that $(M, \mathcal{C})$ *has a* ***good notion of dimension*** *if there exists a* ***dimension*** *function* $\dim : \mathcal{C} \to \{0\} \cup \mathbb{N}$ *with the following properties:*

1. *the dimension of a point is 0,*
2. *for closed sets* S_1 *and* S_2*,* $\dim(S_1 \cup S_2) = \max\{\dim S_1, \dim S_2\}$*,*
3. *(strong irreducibility) for any irreducible closed* $S \subset M^n$*, if* $S_1 \subsetneq S$ *is closed, then* $\dim S_1 < \dim S$*,*
4. *(addition formula) for any irreducible closed* $S \subset M^n$*, and any projection map* $\pi : M^n \to M^k$*,*
$$\dim S = \dim \pi S + \min_{a \in \pi S} \dim(\pi^{-1}(a) \cap S),$$
and
5. *(fibre condition) for any irreducible closed* $S \subset M^n$*, and any projection map* $\pi : M^n \to M^k$*, there is* $V \subset \pi S$ *relatively open such that*
$$\min_{a \in \pi(S)} \dim(\pi^{-1}(a) \cap S) = \dim(\pi^{-1}(v) \cap S)$$
for any $v \in V \cap \pi S$*.*

A ***Zariski structure*** $(M, \mathcal{C}, \dim)$ *is a Noetherian topological structure* $(M, \mathcal{C})$*, which is semi-proper, and which has a good notion of dimension* $\dim$*.*

Zariski geometries satisfy two more conditions.

Definition 6.1.4. *A Zariski structure* $(M, \mathcal{C}, \dim)$ *is called* ***essentially uncountable*** *if every closed set* $A \subset M^n$ *which is the union of countably many closed subsets is actually the union of finitely many of them.*

A Zariski structure $(M, \mathcal{C}, \dim)$ *is called* ***presmooth*** *if for any closed and irreducible* $X_1, X_2 \subset M^n$*, if* $X \subset X_1 \cap X_2$ *is an irreducible component, then*
$$\dim X \geq \dim X_1 + \dim X_2 - \dim M^n.$$

A ***Zariski geometry*** *is a Zariski structure* $(M, \mathcal{C}, \dim)$ *which is essentially uncountable and pre-smooth.*

In [Zil10], Zilber requires a stronger form of presmoothness ("strong presmoothness") in his definition of Zariski geometry, but we have adopted this slightly weaker version for simplicity.

6.2 Notions of closed sets

We will now define topological structures that arise from our investigations, and evaluate the extent to which they form Zariski geometries. As mentioned in the chapter introduction, our work largely focuses on the development of a basic theory of almost complex analytic geometry. The main geometric axioms are currently out of reach, and this section can be seen as a prospectus for future research towards proving them.

First, we recall some notions from Definition 3.3.1 and Definition 4.1.3. Let (M, J) be a real analytic almost complex manifold, and take $X \subset M$ a real analytic subset. We say that X is *equationally pseudoanaltyic* if in each neighbourhood, it is given as the set of points which take the same value under two pseudoholomorphic maps. We define X to be *pseudoanalytic* if in each neighbourhood, it is given as the level set of a pseudoholomorphic map. And finally, X is *analytic* if in each neighbourhood, it is given as the zero set of a holomorphic map, i.e. a pseudoholomorphic map to $\mathbb{C}^k$ for some k. We do not consider well-presented sets here, in the hopes that they are merely a temporary crutch.

In general, for $X \subset M$, we have

$$X \text{ analytic} \implies X \text{ pseudoanalytic} \implies X \text{ equationally analytic.}$$

There are trivial counterexamples to the converse of the first implication. For instance, let (M, J) be a compact real manifold of dimension 6 with an almost complex structure induced by the octonions, as in Example 2.1.6. There, we saw that such an M does not support any local nonconstant holomorphic maps, and hence, all analytic subsets have real dimension 6. In particular, singletons, which are level sets of the identity map, are pseudoanalytic subsets of M which are not analytic. The converse of the second implication seems unlikely to be true, although we do not have a counterexample. Questions about these converses can also be asked in the integrable case, where the answer might be different.

Question 6.2.1. *Is there an example of an equationally analytic subset of an almost complex manifold which is not pseudoanalytic?*

In an integrable almost complex manifold, do the three concepts agree?

In any case, for integrable almost complex manifolds, the analytic subsets coincide by definition with the complex analytic subsets, making the notion of analytic subset perhaps the most natural of the three. On the other hand, in Chapter 4, we were only able to show, even under ideal circumstances, that the set of singular points of an analytic subset forms a equationally pseudoanalytic set. Thus, we may be forced to analyze equationally pseudoanalytic sets, even if we care ultimately about analytic sets.

Once we have a suitable generalization of the notion of a complex analytic subset of a complex manifold, the natural model-theoretic goal would be to prove an analogue of Zilber's theorem.

Theorem 6.2.2 (Zilber, c.f. Theorem 3.4.3 in [Zil10])**.** *A compact complex manifold, together with the classes of complex analytic subsets of each Cartesian power, and equipped with complex analytic dimension, satisfies the Zariski geometry axioms.*

One immediate issue is that none of the equationally pseudoanalytic, pseudoanalytic, or analytic subsets of a general almost complex manifold, by themselves, are known to form topological structures in the sense of Definition 6.1.1. Again, the octonion-induced compact 6-manifolds from Example 2.1.6 provide obvious examples: their analytic subsets do not include singletons or diagonals. We summarize what is known about the closure of these classes under the operations from the definition of topological structures, in the following proposition.

Proposition 6.2.3. *The following table indicates whether the specified classes of subsets of a real analytic almost complex manifold (and its Cartesian powers) is* always *preserved under the corresponding property. That is,* ✓ *indicates that the property is preserved in all real analytic almost complex manifolds, and* ✗ *indicates that it is not preserved in at least some cases. Blanks indicate incomplete knowledge.*

	eq. pseudoanalytic	*pseudoanalytic*	*analytic*
arbitrary $\cap$	✓	✓	✓
finite $\cup$			✓
Cartesian $\times$	✓	✓	✓
permutations	✓	✓	✓
fibres	✓	✓	✓
contains M	✓	✓	✓
contains singletons	✓	✓	✗
contains diagonal			✗

Proof. These are mostly straightforward from the definitions, together with underlying facts about real analytic subsets. We elaborate with some details.

Closure under arbitrary intersections follows from local Noetherianity for real analytic subsets. It can be checked locally, and hence does not require full Noetherianity or compactness.

Closure under finite unions for analytic subsets follows by taking products of the defining analytic maps, which is not available for the other two classes.

Equational pseudoanalytic and pseudoanalytic sets contain singletons, via the identity map. The identity map is not in general a holomorphic map, so this does not work for analytic sets.

Finally, as mentioned earlier, the compact real 6-manifolds with octonion-induced almost complex structures from Example 2.1.6 do not support nonconstant local holomorphic maps, and hence singletons and diagonals are not analytic sets. □

Thus, to formulate the question about Zariski geometries, we must close off under the operations from Definition 6.1.1.

Definition 6.2.4. *Let (M, J) be a real analytic almost complex manifold. We define the classes of* ***equationally pseudoanalytic closed****,* ***pseudoanalytic closed****, and* ***analytic closed****, subsets of M, and its Cartesian powers, to be the coarsest topological structures containing, respectively, the classes of equationally pseudoanalytic, pseudoanalytic, and analytic subsets of M.*

Question 6.2.5. *Let (M, J) be a compact real analytic almost complex manifold, let $\mathcal{C}$ be the class of analytic closed subsets of M, and let* dim *be real analytic dimension. Is is the case that $(M, \mathcal{C}, \dim)$ satisfies the Zariski geometry axioms? What if $\mathcal{C}$ is the class of pseudoanalytic closed subsets, or the class of equationally closed subsets?*

We now briefly touch on the prospects of proving each axiom. The first is essentially built into the definition.

Proposition 6.2.6. *If (M, J) is a compact real analytic almost complex manifold, and $\mathcal{C}$ is the class of equationally pseudoanalytic closed sets, or the class of pseudoanalytic closed sets, or the class of analytic closed sets, then $(M, \mathcal{C})$ is a Noetherian topological structure.*

Proof. All equationally pseudoanalytic, pseudoanalytic, and analytic subsets are real analytic subsets, by definition, and hence by construction, so are equationally pseudoanalytic closed, pseudoanalytic closed, and analytic closed subsets. Then use the fact that the class of real analytic subsets of a compact real analytic manifold are Noetherian. □

Beyond that, our results are merely suggestive.

Semi-properness is very much an open question. In the integrable case, compactness of M is critical, and semi-properness follows from Remmert's proper mapping theorem. The proof of Remmert's theorem uses the fact that the singular part of a complex analytic subset is itself complex analytic, and also relies heavily on the algebraic and analytic theory corresponding to the geometry. In Chapter 4, we showed that singular parts of analytic subsets satisfying certain conditions are equationally pseudoanalytic, and in Chapter 5, we initiated a study of the almost complex ideal theory, but a proof of the proper mapping theorem would likely require far deeper results. We also do not have strong evidence indicating whether or not it might be true. We say a few words below about what might happen if it is not.

Turning to the axioms for a good notion of dimension, we first note that real analytic dimension is defined for all real analytic subsets, and hence for all sets belonging to the topological structures mentioned above. The first two properties of a *good* notion of dimension hold automatically, but the status of the other properties is less clear. Strong irreducibility would follow if we knew that, for pseudoanalytic or analytic sets, irreducibility is equivalent to real analytic irreducibility of the underlying real analytic set. A proof of this equivalence might stem from a deeper analysis of the kind initiated in Chapter 5, and such a result could allow the remaining dimension axioms to be deduced from facts about real analytic geometry.

A nice property of our notion of dimension comes from Corollary 3.3.6, which implies that pseudoanalytic sets are even-dimensional, another almost complex analogue of a fact which can be taken for granted in the complex analytic setting. The tools involved in its proof might be helpful with the deeper dimension axioms. If so, it would be useful to fill the following gap in our knowledge.

Question 6.2.7. *Is the real analytic dimension of any equationally pseudoanalytic set even?*

This would follow from an equational version of the identity principle, which seems obtainable, although we have not explored this question in detail.

Lastly, we have the two Zariski geometry conditions. As usual, the Baire category theorem implies essential uncountablity. The presmoothness condition is not known. It is not true in the real analytic setting, but nonetheless, it might also follow from real analytic geometry, and basic facts about pseudoanalytic and analytic sets as in Chapter 3 or Chapter 4.

To cap off our discussion, we note that while some progress has been made towards the Zariski geometry axioms, we do not have strong evidence to suggest that Question 6.2.5 will be answered affirmatively. It could well turn out to be false, but the tools we have developed would likely still be helpful in classifying the resulting landscape. We highlight one particularly attractive outcome.

Question 6.2.8. *Suppose* $(M, \mathcal{C}, \dim)$ *as in Question 6.2.5 does not satisfy the Zariski geometry axioms. Does the model-theoretic structure* $(M, \mathcal{C})$ *interpret the field* $(\mathbb{R}, +, \times)$*?*

6.3 Stratifications

We record here one last speculative suggestion for studying the analytic structure of almost complex manifolds.

As noted before, there are generally few or no maps from a given almost complex manifold to other almost complex manifolds, let alone to $\mathbb{C}^n$, because the differential equations governing these maps are generally overdetermined. Our usual examples, the compact almost complex manifolds of real dimension 6 from Example 2.1.6, have no local holomorphic functions. However, thanks to the existence theorem (Corollary 2.3.7) and the inverse function theorem (Theorem 2.2.1), all almost complex manifolds have almost complex (possibly non-closed) submanifolds of real dimension 2.

It might be useful to include these kinds of sets in our topological structure. Kessler's approach, mentioned earlier, is to work with images of compact complex manifolds [Kes11]. We suggest another approach here, which is more intrinsic to the underlying manifold, but also harder to wrangle with.

Definition 6.3.1. *Let (M, J) be a real analytic almost complex manifold. A real analytic subset $A \subset M^n$ is said to be* ***almost complex stratified*** *if there exists a decomposition*

$$A = \bigsqcup_{i=1}^{k} A_i$$

into real analytic subsets A_i which are almost complex submanifolds of M^n.

Two particular decompositions of interest are the ***regular decomposition****, given by*

$$A_i := \operatorname{reg}(\operatorname{sing}^{i-1} A), \quad i = 1, \dots, k,$$

and the ***real regular decomposition****, given by*

$$A_i := \operatorname{reg}_{\mathbb{R}}(\operatorname{sing}_{\mathbb{R}}^{i-1} A), \quad i = 1, \dots, k.$$

If A has a (resp. real) regular decomposition consisting of A_i which are almost complex manifolds, then we say that A is ***(resp. real) regularly stratified****.*

Subsets A that are regularly stratified or real regularly stratified are, by definition, almost complex stratified, but in general there are no other implications. However, if A is pseudoanalytic, then Proposition 3.3.5 suggests that the regular and real regular decompositions might coincide. These stratifications could conceivably aid our study of equationally pseudoanalytic, pseudoanalytic, and analytic subsets.

The final possibility, largely unexplored so far, is that the topological structures generated by the classes of stratified sets might themselves have the structure of Zariski geometries. While *a priori* distinct, this question could turn out to be closely related with the Zariski geometry questions from Section 6.2.

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