Title
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Publication Date
2012

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division
of the
University of California, Berkeley

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Spring 2012

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Abstract

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For manifolds M with a specific rational homotopy type, I study a non-commutative Landau-Ginzburg model whose underlying ring is the differential-graded algebra (dga) $B = C_*(\Omega(M))$, that is chains on the based loop space with Pontryagin product and with potential W in B. For $M = \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \ldots \mathbb{C}P^{n_k}$ or $S^{n_1} \times S^{n_2} \times \ldots S^{n_k}$, we explain how the field theories we define have a Fukaya category interpretation.

À Diane

Contents

Acknowledgments

I would like to thank my advisor Constantin Teleman for sharing with me his perspective on mathematics and suggesting the topic of this dissertation. Denis Auroux introduced me to symplectic geometry and mirror symmetry, which plays an unexpectedly large role in this work. Conversations with Mohammed Abouzaid, Yasha Eliashberg and Toly Preygel have also been very helpful. My friendship with Dan Gardiner and Daniel Halpern-Leistner have left a lasting impact on my personality, both mathematically and otherwise. From a young age my mother and father raised me in such a way that it was inevitable that I would get a Ph.D. I thank them for that. Enfin et surtout je terminerai en remerciant ma femme Diane pour m'avoir soutenu dans cette épreuve difficile.

0.1 Introduction

In this thesis, we construct new examples of two-dimensional topological quantum field theories (TQFTs) over the prop $C_*(\overline{\mathcal{M}_{g,n}})$. Our primary methods are algebraic: we make use of the well known theorem of Kontsevich and Soibelman (KonSoi) that

Given a compact and smooth $\mathbb{Z}/2\mathbb{Z}$ graded Calabi-Yau A_∞ algebra B for which the Hodge to De-Rham spectral sequence degenerates, a choice of splitting for this spectral sequence gives rise to a TQFT . In section 1.1 of this thesis we will recall these notions and results in more detail.

The most basic example of such a category is the (dg-enhanced) derived category of quasicoherent sheaves $QCoh(\mathcal{X})$ on a compact and smooth Calabi-Yau variety. This category satisfies all of the above conditons, and the resulting field theory is known as the *B-model* for this Calabi-Yau variety. Homological Mirror Symmetry (Kon) predicts that the associated TQFT is expected to be equivalent to Gromov-Witten TQFT on the mirror CY variety $\mathcal{X}^\vee$.

Now consider $\mathcal{Y}$ to be a smooth but non-compact Calabi-Yau variety. Then $QCoh(\mathcal{Y})$ is a non-compact Calabi-Yau category, and by a modified version of the theorem of Kontsevich and Soibelman, we can get a so-called *positive-output* TQFT. The Landau-Ginzburg model uses deformation theory to compactify these theories by deforming the above category by a superpotential w, which is an algebraic function with a proper critical set. Recent work (Pre; LinPom) shows that this gives rise to a TQFT. For more on curved algebras, see section 1.2.

Section 1.1 of the present paper recalls that a similar situation occurs in topology. Namely, there is a positive output TQFT called *string topology* for a compact oriented manifold $\mathcal{Q}$ associated to the dg-category of dg-modules $mod(C_*(\Omega\mathcal{Q}))$ over the dg algebra $C_*(\Omega\mathcal{Q})$ (Lur), where $\Omega\mathcal{Q}$ denotes the based loop space of $\mathcal{Q}$ at some arbitrary point, pt. Throughout this thesis, all coefficients are taken to be $\mathbb{C}$, the field of complex numbers. As we explain below, this category is a smooth but not compact category. The relationship with string topology is revealed by the following calculation for the Hochschild homology:

$$\mathbf{HH}_*(C_*(\Omega\mathcal{Q})) \cong C_*(\mathcal{L}\mathcal{Q})$$

There is also a natural compact Calabi-Yau category associated to such a manifold, the category of modules over $C^*(\mathcal{Q})$, which however is not smooth. Such categories give rise to TQFT's with *positive-input*. When $\mathcal{Q}$ is simply connected, these two algebras are related via Koszul duality. Namely, the inclusion $pt \to Q$, induces a module structure:

$$C^*(\mathcal{Q}) \to \mathbb{C}$$

The vector space $\mathbb{C}$ can also be thought of as a module over $C_*(\Omega\mathcal{Q})$ by regarding it as the trivial local system. As discussed in (BluCohTel), the following isomorphisms hold:

$$\mathbb{R}Hom_{C^*(\mathcal{Q})}(\mathbb{C}, \mathbb{C}) \cong C_*(\Omega\mathcal{Q})$$

$$C^*(\mathcal{Q}) \cong \mathbb{R}Hom_{C_*(\Omega\mathcal{Q})}(\mathbb{C}, \mathbb{C})$$

and in fact this gives rise to fully faithful functors:

$$perf(C_*(\Omega\mathcal{Q}) \to mod(C^*(\mathcal{Q}))$$

and

$$perf(C^*(\mathcal{Q})^{op}) \to mod(C_*(\Omega\mathcal{Q})^{op})$$

Here $perf(C_*(\Omega\mathcal{Q})$ or $perf(C^*(\mathcal{Q})^{op})$ denotes the subcategory of perfect modules, which is defined for the reader below. Nevertheless, $\mathbb{C}$ is not a compact generator in the category $mod(C_*(\Omega\mathcal{Q}))$ which means that Koszul duality does not give rise to an equivalence of the full derived categories. Following (Abou), Section 1.1 also reviews the relationship between string topology and the Fukaya category of $T^*\mathcal{Q}$, which provides a geometric way of thinking about this Koszul duality.

The case where $\mathcal{Q}$ is $T^n = S^1 \times S^1 \times \cdots \times S^1$, served as motivation for the present work. Dyckerhoff (Dyc) proved the following theorem:

Theorem 0.1.1. *Let w be a function on $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$ with isolated singularities. The object $\mathbb{C}$ is a compact generator for $MF(\mathbb{C}[[x_1, x_2, \ldots, x_n]], w)$. Otherwise stated, $Hom_{MF(\mathbb{C}[[x_1,x_2,\ldots,x_n]],w)}(\mathbb{C}, -)$ defines an equivalence of categories:*

$$MF(\mathbb{C}[[x_1, x_2, \ldots, x_n]], w) \to mod(Hom_{MF(\mathbb{C}[[x_1,x_2,\ldots,x_n]],w)}(\mathbb{C}, \mathbb{C}))$$

Here $MF(\mathbb{C}[[x_1, x_2, \ldots, x_n]], w)$ denotes the category of matrix factorizations, whose definition occupies much of section 3. The relationship between this theorem and the previous discussion is that $C_*(\Omega T^n)$ is isomorphic to $\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots z_n, z_n^{-1}]$, the Laurent polynomial ring in several variables. As $T^n = S^1 \times S^1 \ldots \times S^1$ is not simply connected, we complete at the augmentation ideal of this ring to obtain $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$. In such cases, $MF(\mathbb{C}[[x_1, x_2, \ldots, x_n]], w)$ defines a quantum field theory. This result can be viewed as a deformed Koszul duality in the sense that $Hom_{MF(\mathbb{C}[[x_1,x_2,\ldots,x_n]],w)}(\mathbb{C}, \mathbb{C}) \cong H^*(T^n)$ with a deformed A_∞ structure :

$$m_\ell : H^*(T^n)^{\otimes \ell} \to H^*(T^n)$$

the coefficients of which can be derived from w in a straightforward manner (Dyc).

In this thesis, we will consider simply connected manifolds $\mathcal{Q}$ whose minimal models are pure Sullivan algebras, which are generalizations of complete intersection rings (see section 3 for the precise definition). Section 1.3 of our paper makes precise and then gives an answer to the following question:

Question 0.1.2. *If $C^*(\mathcal{Q})$ is a pure Sullivan algebra and given an element $w \in Z(C_*(\Omega\mathcal{Q}))$, when is $\mathbb{C}$ a compact generator of $MF(C_*(\Omega\mathcal{Q}), w)$ defining an equivalence with $mod(H^*(C^*(\mathcal{Q})), m_\ell)$?*

In section 1.4, we make the Hochschild cohomology of $MF(C_*(\Omega\mathcal{Q}), w)$ explicit, prove that the deformed category is still Calabi-Yau and deduce the degeneration of the aforementioned Hodge-de Rham spectral sequence. We will examine our condition in the special case that the differential of our pure Sullivan algebra is quadratic. In section 1.5, we give some comments on the pure Sullivan condition.

As mentioned earlier, morally, one can think of a potential w as "compactifying" the field theory. In section 2.1 of our paper, inspired by a program of (Sei), we explain how the simplest of our theories, such as when $\mathcal{Q} = \mathbb{C}P^n$ or S^n, arises by geometrically compactifying the cotangent bundles $T^*\mathbb{C}P^n$ and T^*S^n inside of a certain root stack. The definition of the root stack is explained at the beginning of section 7. A slightly more precise statement of our result is that we realize our category $MF(C_*(\Omega\mathcal{Q}), w)$ as being equivalent to the subcategory of the Fukaya category of the root stack generated by the zero section.

Section 2.2 discusses the above compactifications from the point of view of Homological Mirror Symmetry and SYZ fibrations, which allows us to refine the equivalence in the previous section for some low dimensional examples.

Section 2.3 rounds out this thesis by again using SYZ fibrations to generalize the above results to A_n plumbings of cotangent bundles of S^2. We begin by proving homological mirror symmetry for these plumbings. Then we move on to studying the curved deformations of their wrapped Fukaya categories.

All gradings referred to below will follow homological grading conventions. Given a pre-triangulated dg-category $\mathcal{C}$, we denote its associated triangulated category by $[\mathcal{C}]$. For invariants derived from these categories, such as $\mathbf{HH}^*(\mathcal{C})$ or $\mathbf{HH}_*(\mathcal{C})$, bold font will be used when it is important that the construction be carried out at the chain level. Any use of functors such as Hom or $\otimes$ is always assumed to be derived.

Chapter 1

Algebra

1.1 Background and Algebraic setup

Recall that a dg-module (or A_∞-module) N over a dg-algebra (or A_∞-algebra) $\mathcal{A}$ is *perfect* if it is contained in the smallest idempotent-closed triangulated subcategory of $[mod(\mathcal{A})]$ generated by $\mathcal{A}$. In general, unless explicitly stated otherwise, we will use the term "generation" to mean what symplectic geometers usually call "split- generation". That is to say, given a collection of objects $\mathbf{O}_i$ in a triangulated category $\mathcal{C}$, the subcategory generated by $\mathbf{O}_i$ will be the smallest idempotent-closed triangulated subcategory containing $\mathbf{O}_i$.

Definition 1.1.1. A dg-algebra $\mathcal{A}$ over $\mathbb{C}$ is *compact* if $\mathcal{A}$ is perfect as a $\mathbb{C}$ module (in this special case this simply says that $\mathcal{A}$ is equivalent to a finite dimensional vector space). A dg-algebra $\mathcal{A}$ is *smooth* if $\mathcal{A}$ is perfect as an $\mathcal{A}-\mathcal{A}$ bimodule.

A very useful criterion for smoothness is given by the notion of finite-type of Toën and Vaquie (ToëVaq).

Definition 1.1.2. A dg-algebra $\mathcal{A}$ is of finite type if it is a homotopy retract in the homotopy category of dg-algebras of a free algebra $(\mathbb{C}\langle v_1, v_2, \ldots, v_n\rangle, d)$ with $dv_j \in \mathbb{C}\langle v_1, v_2, \ldots, v_{j-1}\rangle$

Lemma 1.1.3. *If $\mathcal{A}$ is of finite type then $\mathcal{A}$ is smooth. The converse is also true if $\mathcal{A}$ is assumed to be compact.*

Lemma 1.1.4. *With the notation of the previous section, the dg-algebra $C_*(\Omega\mathcal{Q})$ is smooth.*

In the simply connected case, this follows from the classical Adams-Hilton construction (AdaHil) and the above theorem of Toën-Vaquie. Consider a cellular model for $\mathcal{Q}$ with cells in dimension $\leq dim(\mathcal{Q})$ and no 1-cells. Let $\mathcal{A}$ denote the tensor algebra generated by variables e_i^b, $deg(e_i^b) = b-1$, for all n and i, where e_i^b are in bijection with cells of dimension b in the cell decomposition.

Let A_b denote the algebra generated by cells of dimension $\leq b$. For a given cell, $d(e_i^b) = z$, where z is defined as the pushforward the canonical class in $H_{b-2}(\Omega S^{b-1})$ under the attaching map $f : S^{b-1} \to \mathcal{Q}$. Thus, we can see that the differential d maps $d : A_b \to A_{b-1}$ and that the algebra is of finite type. The theorem remains true in the non-simply connected case, but the proof is more complicated (Kon2).

Definition 1.1.5. A dg-algebra $\mathcal{A}$ is Calabi-Yau of dimension n if $Hom_{\mathbb{C}}(\mathcal{A}, \mathbb{C}) \cong \mathcal{A}[-n]$ as $\mathcal{A}$-$\mathcal{A}$ bimodules.

Example 1.1.6. Given a manifold $\mathcal{Q}$, the algebra $C^*(\mathcal{Q})$ is Calabi-Yau by Poincare duality.

Definition 1.1.7. A smooth dg-algebra $\mathcal{A}$ is non-compact Calabi-Yau if

$$Hom_{(\mathcal{A}^e)^{op}}(\mathcal{A}, \mathcal{A}^e) \cong \mathcal{A}[n]$$

as $\mathcal{A}$-$\mathcal{A}$ bimodules.

Example 1.1.8. Given a manifold $\mathcal{Q}$, we can again let $\mathcal{A} = C_*(\Omega\mathcal{Q})$. This is a non-campact Calabi-Yau. To see this, note that by smoothness we have that:

$$Hom_{\mathcal{A}^e}(Hom_{(\mathcal{A}^e)^{op}}(\mathcal{A}, \mathcal{A}^e), \mathcal{A}[n]) \cong \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A}[n]$$

As noted above we have that:

$$\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A}[n] \cong C_*(\mathcal{L}\mathcal{Q})[n]$$

The fundamental class of $\mathcal{Q}$ provides the desired isomorphism

$$Hom_{(\mathcal{A}^e)^{op}}(\mathcal{A}, \mathcal{A}^e) \cong \mathcal{A}[n]$$

Next we recall a tiny bit about how the duality between $C^*(\mathcal{Q})$ and $C_*(\Omega\mathcal{Q})$ for compact simply connected manifolds is reflected in the symplectic geometry of their cotangent bundles. Consider $T^*\mathcal{Q}$ with its standard symplectic form $d\theta$. The classical Fukaya category, $Fuk(T^*\mathcal{Q})$, consists of (twisted complexes of) compact exact Lagrangian submanifolds. For any two Lagrangian submanifolds $\mathcal{L}_1$ and $\mathcal{L}_2$, their morphisms are defined by the Floer homology groups, $HF^*(\mathcal{L}_1, \mathcal{L}_2)$ (Sei2). The zero section $\mathcal{Q}$ defines such an object. We have the following description of its endomorphisms:

$$HF^*(\mathcal{Q}, \mathcal{Q}) \cong C^*(\mathcal{Q})$$

For Liouville symplectic manifolds such as $T^*\mathcal{Q}$, it is convenient to consider a version of the Fukaya category, known as the *wrapped Fukaya category* , $WFuk(T^*\mathcal{Q})$, which allows us to incorporate non-compact Lagrangians into the Fukaya category. An important example of such an object is the cotangent fibre to a point q, denoted as T_q. For its definition see (Abou). One very important property of the wrapped Fukaya category is that we have a natural fully faithful functor:

$$i : Fuk(T^*\mathcal{Q}) \to WFuk(T^*\mathcal{Q})$$

Abouzaid proves the following theorem:

Theorem 1.1.9. *The cotangent fibre strongly generates the wrapped Fukaya category of $T^*\mathcal{Q}$ with background class $b \in H^*(T^*\mathcal{Q}, \mathbb{Z}_2)$ given by the pullback of the second Stieffel-Whitney class of $\mathcal{Q}$. The triangulated closure of the wrapped Fukaya category is equivalent to the category $perf(C_*(\Omega\mathcal{Q}))$.*

The second sentence follows from the first because of the following description of the endomorphisms of the cotangent fibre:

$$WHF^*(T_q, T_q) \cong C_*(\Omega\mathcal{Q})$$

1.2 Pure Sullivan algebras and Curved algebras

We consider Pure Sullivan dg-algebras $\mathcal{B}$ of the form:

$$(\wedge V, d) = (\mathbb{C}[x_1, ...x_n] \otimes \bigwedge(\beta_1, ...\beta_m), d(\beta_i) = f_i(x_1, \ldots, x_n), d(x_j) = 0)$$

where the $deg(x_i)$ are even and negative, the functions f_i have no linear term, and the $deg(\beta_i)$ are odd > 1. We further assume that $dim(H^*(\mathcal{B})) < \infty$.

One of the underlying ideas of Chevalley-Eilenberg theory is that such algebras determine an L_∞ model $\mathfrak{g}$ for $\mathcal{B}$. We can define an algebra $\mathcal{A}$ which is the universal enveloping algebra $U\mathfrak{g}$ of these Lie algebras. We briefly explain certain ideas from rational homotopy theory which will be used extensively below. For more details, the reader is encouraged to consult (FelHalTho). To a simply connected space of finite type, $\mathcal{M}$, one can assign an L_∞ algebra $\mathfrak{g} = \pi_*(\Omega(\mathcal{M})) \otimes \mathbb{Q}$, with Whitehead-Samelson bracket. To recover $C^*(\mathcal{M})$, one considers the Chevalley-Eilenberg complex $C^*(\mathfrak{g})$, a canonical complex that computes Lie-algebra cohomology with coefficients in the trivial module. We have a quasi-isomorphism,

$$C^*(\mathfrak{g}) \to C^*(\mathcal{M})$$

Furthermore, one can show that $U\mathfrak{g}$ is then quasi-isomorphic to $C_*(\Omega(\mathcal{M}))$. To go in the other direction, the theory of rational homotopy allows us to assign a space, $\mathcal{M}$, well defined up to rational homotopy equivalence, to a 1-connected dga or equivalently a connected dg-Lie algebras of finite type.

In the case of pure Sullivan algebras $\mathcal{B}$, there is a concrete description of the universal enveloping A_∞ algebra $\mathcal{A}$. Using the homological perturbation lemma, we have an explicit A_∞ model for $\mathcal{A}$ of the form

$$(Sym(\mathfrak{g}_{even}) \otimes \Lambda(\mathfrak{g}_{odd}), m_n)$$

A formula for the higher multiplications appears in section 3 of (Bar). For our purposes, we note the following facts. First, the strict morphism of the abelian Lie algebra $\pi_{even}(\Omega(Q)) \to \mathfrak{g}$ corresponds to the inclusion of $Sym(\mathfrak{g}_{even}) \cong \mathbb{C}[u_1, \ldots, u_m] \to \mathcal{A}$. The higher multiplications m_n are multi-linear in these variables for $n \geq 3$. Finally, we have that the A_∞ algebra is strictly unital and the augmentation $U\mathfrak{g} \to \mathbb{C}$ defined by killing $\mathfrak{g}U\mathfrak{g}$ is also a strict morphism.

The reader should be warned that in the presence of quadratic terms in the f_i, the above identification with $Sym(\mathfrak{g}_{even}) \otimes \Lambda(\mathfrak{g}_{odd})$ is only an identification of vector spaces. In other words, there can be a non-trivial Lie bracket $B : \mathfrak{g}_{odd} \otimes \mathfrak{g}_{odd} \to \mathfrak{g}_{even}$, which means that forgetting higher products, $U\mathfrak{g}$ is a Clifford algebra over $Sym(\mathfrak{g}_{even})$. It also seems worth pointing out that the even variables u_i can be thought of as being Koszul dual to the odd variables β_i. Meanwhile the variables in $\mathfrak{g}_{odd}$, from here on denoted as e_j, are dual to the even variables x_j above.

Next, we discuss how to define an appropriate category of matrix factorizations. This section adopts the ideas of the foundational work (Pre) to our non-commutative context. For concreteness, let us consider as before the above A_∞-algebra $\mathcal{A}$, and an element $w \in \mathbb{C}[u_1, \dots, u_m]$ of degree $2j-2$. For example, if $M = \mathbb{C}P^n$, we have the following specific model:

$$U\mathfrak{g} = \mathbb{C}[u] \otimes \Lambda(e), m_{n+1}(e, e, e, \dots, e) = u$$

We can then consider potentials of the form $w = u^d$.

We define a variable x of degree $2j-2$. The element w defines a mapping from

$$w : \mathbb{C}[x] \to \mathcal{A}$$

and we can consider the A_∞ algebra $\mathcal{A}_0 = (\mathcal{A}[e], de = w)$, where e now has degree $2j-1$.

Definition 1.2.1. We define $Pre(MF(\mathcal{A}, w))$, to be the full subcategory of $mod(\mathcal{A}_0)$ consisting of modules which are perfect over $\mathcal{A}$.

This category is equipped with a natural $\mathbb{C}[[t]]$ (degree $t = -2j$) linear structure which we will now describe.

Remark 1.2.2. Of course, $\mathbb{C}[[t]]$ as a graded ring is usually denoted $\mathbb{C}[t]$. The notation $\mathbb{C}[[t]]$ is simply to note that it should be treated as a topological ring. For example, given a graded vector space V,

$$V[[t]]_n = \prod_{k \geq 0} V_{m+2jk}$$

This will be distinct from $V[t]$ if V is not homologically bounded from above.

We begin with the description of the $\mathbb{C}[[t]]$-linear structure from an abstract point of view and then give more concrete descriptions. We observe that:

$$Pre(MF(\mathcal{A}, w)) \cong RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathcal{A}))$$

the category of colimit preserving functors(Toë). We describe this construction a bit more below. For the reader who is become disoriented with the notation, notice that the $\mathbb{C}[x]$ structure on the right-hand side comes from the above algebra map w.

This category of functors is acted upon the category $RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathbb{C})$ by convolution. Let α denote a variable of degree $2j-1$. Then there is an isomorphism:

$$RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathbb{C}) \cong D_{fin}(\mathbb{C}[\alpha]/\alpha^2)$$

Here $D_{fin}(\mathbb{C}[\alpha]/\alpha^2)$ denotes the subcategory category of modules over $\mathbb{C}[\alpha]/\alpha^2$ which are homologically finite over $\mathbb{C}$. Next we notice that Koszul duality provides an equivalence :

$$D_{fin}(\mathbb{C}[\alpha]/\alpha^2) \cong Perf(\mathbb{C}[[t]])$$

The aforementioned $\mathbb{C}[[t]]$(degree $t = -2n$) linear structure now arises in view of the natural equivalence between (idempotent complete, pre-triangulated) module categories over $Perf(\mathbb{C}[[t]])$ and ordinary $\mathbb{C}[[t]]$-linear, (idempotent complete, pre-triangulated) dg categories.

This description of the action of t is relatively obscure and so we now aim to unravel it and make it more concrete. The object $\mathbb{C}$ in $D_{fin}(\mathbb{C}[\alpha]/\alpha^2)$ acts via the identity. The action for the module $\mathbb{C}[\alpha]/\alpha^2$ can be described by considering the composition of the two adjoint-functors:

$$i_* : RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathcal{A})) \to RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}[x]), Perf(\mathcal{A}))$$

$$i^* : RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}[x]), Perf(\mathcal{A})) \to RHom_{\mathbb{C}[x]}(Perf(\mathbb{C}), Perf(\mathcal{A}))$$

In view of the fact that $i^* \circ i_*(N) \cong \mathbb{C} \otimes_{\mathbb{C}[x]} N$, and the fact that N and $\mathbb{C}$ are perfect over $\mathcal{A}$ and $\mathbb{C}[x]$ respectively, it follows that $i^* \circ i_*(N)$ is perfect over $\mathcal{A}_0$.

One can resolve the module $\mathbb{C}$ over $\mathbb{C}[\alpha]/\alpha^2$ by the standard Koszul resolution:

$$\mathbb{C} \cong \bigoplus_k \frac{\mathbb{C}[\alpha]}{\alpha^2}[u^k/k!], du = \alpha$$

Applying this resolution to an object in M in $Pre(MF(\mathcal{A}, w))$, we conclude that:

$$Hom_{Pre(MF(\mathcal{A},w))}(M, N) = (Hom_{Perf(\mathcal{A})}(M, N)[[t]], d)$$

where

$$d : \phi \to d_{\mathcal{A}}(\phi) + t(\phi \circ e \wedge + e \wedge \circ \phi))$$

The differential $d_{\mathcal{A}}$ denotes the differential on $Hom_{Perf(\mathcal{A})}(M, N)$. In this equation t acts in the natural way.

The above construction generalizes the construction of the category of singularities for ordinary commutative rings (Orl). It is natural to ask how this $\mathbb{C}[[t]]$ linear structure arises from deformation theory or how it can be expressed in a way that resembles the usual category of matrix factorizations. We note that the element tw also defines a Maurer-Cartan element in $\mathbf{HH}^*(\mathcal{A}, \mathcal{A})[[t]]$. Such a Maurer-Cartan solution allows us to twist the differential on

$$(\bigoplus_n \mathcal{A}^{\otimes n}[[t]], d_A)$$

by the differential determined by the formula:

$$td_w : a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^{i+1} t a_0 \otimes a_1 \otimes \cdots \otimes a_i \otimes W \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

giving rise to a topological coalgebra

$$\mathbf{C} = (\bigoplus \mathcal{A}^{\otimes n}[[t]], d_{\mathcal{A}} + td_w)$$

.

We can look now at modules over this coalgebra which are topologically free over $\mathbb{C}[[t]]$, topologically cofree as modules over the underlying coalgebra, and are perfect over A when $t = 0$. We denote this category by $comod(\mathbf{C})$.

Lemma 1.2.3. *The functor $\mathcal{F} : M \to ((\bigoplus_n \mathcal{A}^{\otimes n} \otimes M)[[t]], d_{M/\mathcal{A}} + te\wedge)$ defines a fully faithful functor:*

$$Pre(MF(\mathcal{A}, w)) \to comod(\mathbf{C})$$

Ignoring differentials for ease of notation we have that:

$$Hom(\mathcal{F}(M), \mathcal{F}(N)) = Hom^{top}_{\mathbb{C}[[t]]}((\bigoplus_n \mathcal{A}^{\otimes n} \otimes M)[[t]], N[[t]])$$

We can identify this with:

$$Hom_{\mathbb{C}}(\bigoplus_n \mathcal{A}^{\otimes n} \otimes M, N)[[t]]$$

The differential on this complex is again the differential $d_{\mathcal{A}} + t(\phi \circ e \wedge + e \wedge \circ \phi)$.
Finally, we define

$$MF(\mathcal{A}, w) = Pre(MF(\mathcal{A}, w)) \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t)) \cong comod(\mathbf{C}) \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t))$$

The fact that $i^* \circ i_*(N)$ is perfect for any object in $Pre(MF(\mathcal{A}, w))$ implies just as in the usual case that

$$[MF(\mathcal{A}, w)] \cong [Pre(MF(\mathcal{A}, w))]/[Perf(\mathcal{A}_0)]$$

It is often convenient to work with the formal Ind-completion $Ind(MF(\mathcal{A}, w))$ which we shall denote by $MF^{\infty}(\mathcal{A}, w)$.

We have constructed a category of curved modules for a curved A_∞ algebra which arises as a deformation of an uncurved A_∞ algebra. It is worth pointing out that there is a more general notion of a curved A_∞ algebra, a notion which is most developed in the case of dg-algebras.

Definition 1.2.4. A triple $B = (\mathcal{A}, w, d)$ consisting of a $\mathbb{Z}/2j\mathbb{Z}$ graded algebra $\mathcal{A}$, a function of even degree w, and an derivation d of odd degree is called a graded curved dg-algebra if $d^2 = [w, a]$

Definition 1.2.5. A (left) curved module over a curved dg-algebra is a $\mathbb{Z}/2j\mathbb{Z}$ graded (left) module over $\mathcal{A}$ together with an odd derivation d such that $d^2 = w$.

There is a $\mathbb{Z}/2j\mathbb{Z}$ graded dg-category of modules, which we denote by $B-mod$. Positselski has studied curved Koszul duality extensively and in particular defined various versions of the derived category of curved modules over a curved dg-algebra. In particular, he considers:

Definition 1.2.6. We denote by $B-proj$ the $\mathbb{Z}/2j\mathbb{Z}$ graded dg-subcategory of $B-mod$ consisting of modules M whose underlying graded modules $M^\sharp$ are projective. As usual $[(B-proj)]$ is a triangulated category.

In many cases of interest, $B-proj$ coincides with the categories defined previously and is sometimes convenient to work with. In the case that $\mathcal{A}=R$ is a commutative ring and w is a non-zero function, $B-proj$ is nothing but the usual category of matrix factorizations.

In our situation, one can prove the following:

Theorem 1.2.7. *Let $\mathfrak{g}$ be a the Lie model for a pure Sullivan algebra and w a non-zero function. Let m be the ideal generated by $(u_1, u_2, \ldots, u_m)$. Let $B=(\widehat{U\mathfrak{g}}, w)$. Here $\widehat{U\mathfrak{g}}$ denotes the m-adic completion. $\mathcal{A}_0 \cong U\mathfrak{g}/(w)$. The functor $M \to \widehat{M}$ gives an equivalence between the category $MF(\mathcal{A}, w)$ and $B-proj$.*

To understand why we need to complete at m, we remind the reader of remark 3.2. For an explicit example, consider the case where u is a variable such that $deg(u)=2$ and $deg(t)=-2$. In this case, $\mathbb{C}[u]((t)) \cong \mathbb{C}[[u]]$ as two-periodic algebras.

Now we proceed with the argument. Because w acts by 0 on both categories it is easy to see that this functor is essentially surjective. Let M be an object and let P_M be a finitely generated dg-complex consisting of projective modules over $U\mathfrak{g}$ which is quasi-equivalent to M. There is a canonical system of homotopies $\sigma_n : P_M \to P_M$, such that $\sigma_0 = d_P$, $\sigma_0\sigma_1 + \sigma_1\sigma_0 = w$, $\sum_{i+j=n} \sigma_i\sigma_j = 0$

These homotopies give us an object of $B-proj$, $(\hat{P}_M, d)$. Using the definitions given earlier in this section, one can see that

$$Hom_{MF(\mathcal{A},w)}(M, N) \cong Hom_{B-proj}((\hat{P}_M, d), (\hat{P}_N, d))$$

Thus we are left to verify that $(\hat{P}_M, d) \cong \hat{M}$. The easiest way to do this is using the fact that we verify in later sections, that the category $MF(A, w)$ is generated by graded modules M with no differential. From here the argument is standard and follows exactly (LinPom) lemma 2.18.

Example 1.2.8. One can also construct a resolution of $\mathbb{C}$ as an object of $B-mod$ by considering the Chevalley-Eilenberg complex for $U\mathfrak{g}$. The completed Chevalley-Eilenberg complex for the Lie-algebra g is given by

$$V_i(\mathfrak{g}) = \widehat{U\mathfrak{g}} \otimes \wedge^i \mathfrak{g}, d_0$$

with a differential

$$d_0(u \otimes g_1 \wedge g_2 \wedge \ldots \wedge g_n) = \Sigma(-1)^{i+1} u g_i \otimes g_1 \wedge \ldots \hat{g}_i \wedge \ldots g_n + \Sigma(-1)^{i+j} u \otimes [g_i, g_j] \wedge g_1 \ldots \hat{g}_i \ldots \hat{g}_j \ldots \wedge g_n$$

If we write $w = \sum u_i w_i$, $\wedge w_i du_i$ defines a map

$$d_1 : V_i \to V_{i+1}$$

We consider the curved complex

$$\bigoplus V_i, d = d_0 + d_1$$

Using the fact that the algebra elements w_i are central it is possible to prove that $d^2 = w * id$ and thereby producing a resolution of $\mathbb{C}$.

Remark 1.2.9. To illustrate the above discussion, it might be useful to consider the case of ordinary matrix factorizations, namely pairs (R, w) where R is a commutative ring as above. If one considers it as a two-periodic curved A_∞-algebra, various authors (CalTu) have noted that the category of comodules over the two-periodic bar algebra:

$$\bigoplus_n R^{\otimes n}, d_R + d_w$$

is always zero. Thus it is important to consider the bar complex over $\mathbb{C}[[t]]$, calculate the corresponding Hom-sets and then invert t. As an example, consider the case when the function w has isolated singularities. The naive Hochshild complex

$$(\prod Hom_{\mathbb{C}}(R^{n+1}, R), d_{Hoch} + [w,])$$

is always zero. However we have a quasi-isomorphism as two periodic complexes:

$$(\prod Hom_{\mathbb{C}}(R^{n+1}, R)((t)), d_{Hoch} + t[w,]) \cong (\bigoplus Hom_{\mathbb{C}}(R^{n+1}, R), d_{Hoch} + [w,])$$

This latter complex computes the Jacobian ring as one would expect.

1.3 The criterion for generation

In this section we discuss a criterion for smoothness and properness of the category $MF(\mathcal{A}, w)$. To state the criterion, we must consider the category of curved bimodules

$$MF(\mathcal{A} \otimes \mathcal{A}^{op}, w \otimes 1 - 1 \otimes w)$$

and we define $\mathbf{HH}^*(MF(\mathcal{A}, w))$ to be $Hom_{MF(\mathcal{A}\otimes\mathcal{A}^{op}, w\otimes 1 - 1\otimes w)}(\mathcal{A}, \mathcal{A})$. Using either description of our category, this can be computed explicitly as:

$$\mathbf{HH}^*(MF(\mathcal{A}, w)) \cong (\mathbf{HH}^*(\mathcal{A}, \mathcal{A})((t)), d_{\mathcal{A}} + [tw,])$$

The following is the analogue of Dyckerhoff's theorem for our situation:

Theorem 1.3.1. *If $\mathbf{HH}^*(MF(\mathcal{A}, w))$ is finite over $\mathbb{C}((t))$, then $\mathbb{C}((t))$ generates the category $MF(\mathcal{A}, w)$.*

We have an action of $\mathbb{C}[u_1, \ldots, u_m]$ on $DSing(\mathcal{A}_0)$ which factors through the complex $\mathbf{HH}^*(MF(\mathcal{A}, w))$. For any u in $\mathbb{C}[u_1, \ldots, u_m]$, we let K_u be the diagram

$$\mathbb{C}[u_1, \ldots, u_m] \xrightarrow{u} \mathbb{C}[u_1, \ldots, u_m]$$

Finally, for the sequence $\bar{u} = (u_1, \ldots, u_m)$ we define

$$K_{\bar{u}^p} = \otimes K_{u_i^p}$$

.

With this notation in hand, we consider the colimit of the diagram:

$$K_{\bar{u}} \to K_{\bar{u}^2} \to K_{\bar{u}^3} \ldots$$

which we denote by $R\Gamma_m$. For any object $\mathbf{O}$ in $MF(\mathcal{A}, w)$, we have an augmentation

$$R\Gamma_m \otimes_{\mathbb{C}[u_1,\ldots,u_m]} \mathbf{O} \to \mathbf{O} \to cone(e)$$

Because the action of $\mathbb{C}[u_1, \ldots, u_m]$ factors as above, we see that such an m-equivalence is in fact an equivalence and can conclude that $cone(e)$ is zero.

Now the objects $K_{\bar{u}^i} \otimes \mathbf{O}$ are in the triangulated subcategory generated by $\mathbb{C}$ because $\mathcal{A}$ is finitely generated as a module over $\mathbb{C}[u_1, \ldots, u_m]$. The objects $\mathbf{O}$ are compact in $MF^\infty(\mathcal{A}, w)$ and can be expressed as a colimit of $K_{\bar{u}^i} \otimes \mathbf{O}$. Therefore we can conclude that $\mathbf{O}$ is a direct summand of one of the $K_{\bar{u}^i}(\mathbf{D}) \otimes \mathbf{O}$ generated by $\mathbb{C}$ as well.

Remark 1.3.2. The same argument goes through if the ideal of the kernel of the above ring homomorphism is I. Namely, in this situation, the category is generated by $\mathbb{C}[u_1, \ldots u_m]/I \otimes_{\mathbb{C}[u_1,\ldots u_m]} \mathcal{A}$. We isolate the case where $\mathbb{C}[u_1, \ldots u_m]/I$ is finite dimensional because it has the most relevance to topological field theories.

To discuss homological smoothness, we must consider the category:

$$RHom^c_{\mathbb{C}((t))}(MF^\infty(\mathcal{A}, w), MF^\infty(\mathcal{A}, w))$$

the category of continuous endofunctors in the sense of (Toë), which we now describe. Given two dg-categories $\mathcal{C}_1$ and $\mathcal{C}_2$, the naive category of dg-functors Hom($\mathcal{C}_1$,$\mathcal{C}_2$) is not well behaved with respect to quasi-equivalence of dg-categories.

Toën (Toë) proved that there is a model structure on the category of dg-categories where weak equivalences are given by quasi-equivalences and the category $RHom(\mathcal{C}_1, \mathcal{C}_2)$ is a derived functor with respect to this model structure. We have a natural inclusion $RHom^c(\mathcal{C}_1, \mathcal{C}_2) \subset RHom(\mathcal{C}_1, \mathcal{C}_2)$ of all functors which commute with arbitrary colimits.

Toën proves that for if a co-complete dg-category $\mathcal{C}$ has a compact generator $\mathbf{O}$, and is thus equivalent to the category of modules $mod(Hom(\mathbf{O}, \mathbf{O})^{op})$, then we have that (Toë):

$$RHom^c(\mathcal{C},\mathcal{C}) \cong mod(Hom(\mathbf{O},\mathbf{O}) \otimes Hom(\mathbf{O},\mathbf{O})^{op})$$

Thus we have the following theorem:

Theorem 1.3.3. $RHom^c_{\mathbb{C}((t))}(MF^\infty(\mathcal{A},w), MF^\infty(\mathcal{A},w)) \cong MF^\infty(\mathcal{A}\otimes\mathcal{A}^{op}, w\otimes 1 - 1\otimes w)$

This follows because $\mathbb{C}$ also generates the category $MF^\infty(\mathcal{A}\otimes\mathcal{A}^{op}, w\otimes 1 - 1\otimes w)$. Let t_1 be the deformation parameter corresponding to $Pre(MF(\mathcal{A},w))$ and t_2 be the deformation parameter corresponding to $Pre(MF(\mathcal{A}^{op},-w))$. We know that $Hom_{\mathcal{A}}(\mathbb{C},\mathbb{C})$ is homologically bounded above. This implies that the mapping:

$$Hom_A(\mathbb{C},\mathbb{C})[[t_1]] \otimes_{\mathbb{C}[[t]]} Hom_{\mathcal{A}}(\mathbb{C},\mathbb{C})[[t_2]] \to Hom_{\mathcal{A}\otimes\mathcal{A}^{op}}(\mathbb{C},\mathbb{C})[[t_1,t_2]] \otimes \mathbb{C}[[t]]$$

is an equivalence. We achieve the result by inverting t.

Remark 1.3.4. One of the downsides of the formalism that we have chosen is that the above theorem will fail almost uniformly if the generating object $\mathbf{O}$ in $MF^\infty(A,w)$ does not have the property that $Hom(\mathbf{O},\mathbf{O})$ is homologically bounded from above, essentially because the above mapping will almost never be an equivalence. This situation might be improved by giving a definition of the monoidal category of dg-categories where one makes use of completed tensor products.

1.4 The Calabi-Yau property, Hochschild cohomology and the Degeneration Conjecture

Next we will show that the Calabi-Yau condition for our category $MF(\mathcal{A},w)$ follows from $\mathcal{A}$ being non-compact Calabi-Yau. We first briefly recall why $\mathcal{A}$ is non-compact Calabi-Yau. Recall that this means that there is an isomorphism of $\mathcal{A}-\mathcal{A}$- bimodules

$$RHom_{\mathcal{A}^e}(\mathcal{A},\mathcal{A}^e) \cong \mathcal{A}[n]$$

The dg-algebra $\mathcal{B}$ is rationally elliptic. By results in Chapter [35] of the book (FelHalTho) the algebra $H^*(\mathcal{B})$ is a Poincare duality algebra. Now results in (Cos) prove that the deformation theory of C_∞ algebras and Frobenius C_∞ algebras with a fixed trace coincide. By applying the perturbation lemma and viewing the dg-algebra $\mathcal{B}$ as a deformation of $H^*(\mathcal{B})$, this implies that the Frobenius structure on $H^*(\mathcal{B})$ enhances naturally to a Calabi-Yau structure on $\mathcal{B}$.

Next we have the following theorem proved in (VDBergh):

Theorem 1.4.1. *Let $\mathcal{A}$ be a homologically smooth algebra concentrated in degree ≥ 0. Then a cyclic A_∞ structure on its Koszul dual algebra gives rise to a non-compact Calabi-Yau structure on $\mathcal{A}$.*

Now that $\mathcal{A}$ is seen to be Calabi-Yau, we show that this implies the property for $MF(\mathcal{A},w)$. To prove the Calabi-Yau property for $MF(\mathcal{A},w)$, we note that we have a relative dualizing functor:

$$D : MF(\mathcal{A}, w) \to MF(\mathcal{A}^{op}, -w)^{op}$$
$$M \to RHom_{\mathcal{A}}(M, \mathcal{A})$$

D is manifestly an equivalence. Note that for any object O in $MF(\mathcal{A} \otimes \mathcal{A}^{op}, w \otimes 1 - 1 \otimes w)$ we have that

$$Hom_{MF(\mathcal{A}\otimes\mathcal{A}^{op}, w\otimes 1 - 1\otimes w)}(\mathbb{C}, O) \cong Hom_{MF(\mathcal{A}\otimes\mathcal{A}^{op}, w\otimes 1 - 1\otimes w)}(D(O), D(\mathbb{C}))$$

We know that

$$D(\mathbb{C}) \cong \mathbb{C}[n]$$

and because $\mathcal{A}$ is non-compact Calabi-Yau, we have that

$$D(\Delta) \cong \Delta[n]$$

From here we learn that for the diagonal

$$Hom_{MF}(\Delta, \mathbb{C}) \cong Hom_{MF}(\mathbb{C}, \Delta)$$

Next we denote the complex $Hom_{MF(\mathcal{A},w)}(\mathbb{C}, \mathbb{C})$ by $\mathcal{D}$. This means that there is an isomorphism of $\mathcal{D} - \mathcal{D}$- bimodules

$$RHom_{\mathcal{D}^e}(\mathcal{D}, \mathcal{D}^e) \cong \mathcal{D}[n]$$

which implies the Calabi-Yau condition for $MF(\mathcal{A}, w)$.

We can make our condition on finiteness of $\mathbf{HH}^*(MF(\mathcal{A}, w))$ more tractable by considering the deformation theory of the pure Sullivan algebra $\mathcal{B}$ itself. As noted in the introduction, for any simply connected space of finite type, we have fully faithful functors induced by the $C^*(\mathcal{Q}) - C_*(\Omega\mathcal{Q})$ bimodule $\mathbb{C}$. It then follows from a result of Keller (Kel) that for such a fully faithful functor there is a canonical equivalence in the homotopy category of $B(\infty)$ algebras:

$$\mathbf{HH}^*(C^*(\mathcal{Q}), C^*(\mathcal{Q})) \cong \mathbf{HH}^*(C_*(\Omega\mathcal{Q}), C_*(\Omega\mathcal{Q}))$$

In particular these two Koszul dual algebras have equivalent formal deformation theories. Suppose that, more generally, we consider a commutative algebra free- graded commutative model $(\bigwedge V, d)$ where V is a finite dimensional graded vector space. There is a very explicit complex quasi-isomorphic as a dg-Lie algebra to $\mathbf{HH}^*((\bigwedge V, d), (\bigwedge V, d))$. Recall that $T^{poly}(V)$ is the Lie-algebra of polyvector fields on $\bigwedge V$ with Schouten bracket. Part of Kontsevich's formality theorem says that the HKR map:

$$T^{poly}(V) \to \mathbf{HH}^*(\bigwedge V)$$

is the first Taylor coefficient in an L_∞ quasi-isomorphism between the two.

We can think of the derivation d as corresponding to a vector-field v. It follows from a spectral sequence argument that the HKR map gives a quasi-isomorphism:

$$(T^{poly}(V), [v, -]) \to \mathbf{HH}^*((\bigwedge V, d), (\bigwedge V, d))$$

Lemma 1.4.2. *This map can be corrected to an L_∞ quasi-isomorphism. In the case of a pure Sullivan algebra, the first Taylor coefficient agrees with the HKR map.*

Denote by f_n the Taylor coefficients of the Kontsevich formality morphism. We consider a modified L_∞ map

$$\widetilde{f_n} : (T^{poly}(V), [v, -]) \to \mathbf{HH}^*(\bigwedge(V, d))$$

given by the formula

$$\widetilde{f_n}(x_1, \cdots, x_n) = \sum 1/k! f_{k+n}(v, v, \cdots, v, x_1, x_2, \cdots, x_n)$$

This sum, seemingly a sum consisting of infinitely many terms, makes sense in this case for the following reason: for all $\gamma_1, \dots \gamma_m$ in $T^{poly}(V)$,

$$f_{n+k}(v, v, \cdots, v, x_1, x_2, \cdots, x_n)(\gamma_1, \dots \gamma_m) \text{ vanishes unless } 2(n+k) + m - 2 = k + \sum(|x_i|)$$

It is easy to check that this defines an L_∞ map between the two complexes in question. We must further convince ourselves that the map $\widetilde{f_1}$ remains a quasi-isomorphism. By the above equation, we know that for any $x \in T^{poly}(V)$ such that $f_1(x) \in HH^m$, we have that

$$\widetilde{f_1}(x) = f_1(x) + \alpha$$

where $\alpha \in HH^{<m}$ Assuming the map is surjective onto $HH^{<m}$, we learn by induction that the map is surjective onto $HH^{\leq m}$ as well. Injectivity of the map on homology is also clear.

In general the formula for our map $\widetilde{f_1}$ can be computed explicitly from work of (Cal) but is very complicated, the coefficients of the map being given in terms of Bernoulli numbers. In the case of interest, we actually know more, namely, we have the following:

Lemma 1.4.3. *If the dg algebra is pure Sullivan the map $\widetilde{f_1}$ agrees with the map f_1.*

Following [Ca], we denote our coordinates by u_k, and write and $v = \sum v_i \partial_i$. There is a matrix valued one form given by

$$\Gamma_{ji} = \sum_k \partial_i \partial_k v_j du_k$$

and define

$$\theta = \sum_{n>0} c_n i_{tr}(\Gamma^n)$$

where the c_n are certain rational coefficients.

Calaque proves that

$$\widetilde{f_1} = f_1(e^\theta)$$

Now if $v = \sum f(x_1, , x_n)d/de_i$ as above, then the matrix is strictly upper triangular, the trace of any power of it is therefore zero, and thus $\widetilde{f_1} = f_1$.

In the pure Sullivan case, potentials tw in $\mathbf{HH}^*(\mathcal{A}, \mathcal{A})[[t]]$ correspond to odd-polyvector fields

$$tw(d/de_1, d/de_2, \ldots d/de_m) \in T^{poly}(B)[[t]]$$

The general theory of deformations of Koszul dual algebras (Efi) shows that the polyvector-field $v = \sum f(x_1, , x_n)d/de_i$ in $T^{poly}(\bigwedge(V^*[1]))$ gives rise to the Koszul dual of $(\bigwedge V, d)$ e.g. $\mathcal{A}$ (one can as above pass from formal deformation theory to actual deformation theory). Since the Kontsevich formality map has the property that

$$f_{n+1}(w, x_1, x_2, \ldots, x_n) = 0, n > 1$$

this proves that tw corresponds to the polyvector field we have claimed.

After passing to the generic fiber, the Hochschild cohomology is given by :

$$(T^{poly}(V)((t)), [v + tw(d/de_1, \ldots, d/de_m),])$$

Definition 1.4.4. By analogy with the case of ordinary matrix factorizations, we will say that w has an *isolated singularity* if the homology of this complex is finite dimensional.

Next we discuss the degeneration conjecture. For $(\mathcal{A}, w)$ as above with isolated singularities, we can see from the formula for Hochschild cohomology that $HH^*(\mathcal{A}, w)$ will always be concentrated in even degree. To see this, note that if we replace the variables d/de_i by u_i then we have a sequence of $\mathbb{Z}/2j\mathbb{Z}$ graded complexes

$$(T^{poly}(C[[u_1, u_2, \ldots]][x_1, \ldots, x_n]), [w + \sum f_i u_i,]) \subset HH^*(\mathcal{A}, w)$$

and

$$HH^*(\mathcal{A}, w) \subset (T^{poly}(C[x_1, \ldots, x_n][[u_1, u_2, \ldots u_m]]), [w + \sum f_i u_i,])$$

In the special case when the cohomology of the complex is finite dimensional this implies that $w + \sum f_i u_i$ has isolated singularites. The degeneration conjecture is then automatic because the Hochschild cohomology is automatically concentrated in even degrees. As the category $MF(\mathcal{A}, w)$ is Calabi-Yau, the Hochschild homology will all be concentrated in the either even or odd degree (depending upon the parity of the Calabi-Yau structure) and the degeneration conjecture thus follows for these algebras without any additional work.

Example 1.4.5. For $\prod S^{2n_j}$ the condition that w has an isolated singularity is similar to the usual Jacobian condition and states that $\mathbb{C}[u_1, \ldots, u_m]/(u_i dw/du_i)$ be finite dimensional. The proof follows from the more general statement below.

More generally, we again discuss the case when $\mathfrak{g}$ is formal. In this case, recall that $\mathfrak{g}$ is determined by a bilinear form:

$$B : \mathfrak{g}_{odd} \otimes \mathfrak{g}_{odd} \to \mathfrak{g}_{even}$$

and $U\mathfrak{g}$ is a graded Clifford algebra over $\mathbb{C}[u_1, \ldots, u_m]$. We let D_k be the closed subvariety of $\mathbb{C}[u_1, \ldots u_m]$ for which $rank(B) \leq k$ and assume further that the $D_k - D_{k-1}$ is smooth. Let R denote $U\mathfrak{g}/(w)$.

Theorem 1.4.6. *Let $\mathcal{B}$ be a pure Sullivan algebra, whose Lie model $\mathfrak{g}$ is formal and as above. Let w be a potential which intersects the varieties D_k transversally at every point. Then:*

(a) w has isolated singularities

(b) $Proj(R)$ has finite homological dimension as an abelian category.

The first statement is a calculation, so we explain the second one. Consider the exact functor between derived categories

$$\pi : D^b(Gr - R) \to D^b(Proj(R))$$

We can consider the abelian subcategory of $Gr - R$, denoted $Gr - R_{\geq i}$ which consists of modules M such that $M_p = 0$ for $p \leq i$ Restricted to this subcategory,

$$\pi_{\geq i} : D^b_{\geq i}(Gr - R) \to D^b(Proj(R))$$

has a right adjoint

$$R\omega : D^b(Proj(R)) \to D^b_{\geq i}(Gr - R)$$

Thus we will show that for any $M, N \in D^b(Gr - R)$, $\mathrm{Ext}^i(M, R\omega \circ \pi(N))$ vanishes for large i.

Suppose that Q is a graded prime ideal different from the maximal ideal and lying in a component of D_k, but not D_{k-1}. We denote $R/rad(QR)$ by B. Now denote by P the prime ideal corresponding to the irreducible component of D_k which Q is in. One can prove that the correspondence $P \mapsto rad(PR)$ gives a bijection between (graded) prime ideals in $\mathbb{C}[u_1, \ldots, u_m]$ and (graded) prime ideals of $U\mathfrak{g}$ (Mus). We have a short exact sequence:

$$0 \to S \to R/(rad(PR), Q) \to B \to 0$$

where S is B torsion by the assumption that the prime Q lie in a component of D_k but not D_{k-1}. Now we know by our condition, that $\mathbb{C}[u_1, \ldots, u_m]/Q[l]$ has a finite resolution as a $\mathbb{C}[u_1, \ldots, u_m]/P$ module and thus so does $R/(rad(PR), Q)$[l] as a $R/rad(PR)$ module.

The above exact sequence reveals that $\mathrm{Ext}^i_{R/(rad(PR))}(B[l], M)$ is B torsion for $i > m$. It is also easy to show from the transversality hypothesis that $R/rad(PR)[l]$ has finite homological dimension over R. Next, we note the following lemma, which is proved for ungraded rings in (Bro)

Lemma 1.4.7. *Let R be a graded FBN ring. Given a bounded complex C in $D(Gr-R)$ if* $\mathrm{Ext}^i(R/P[l], C)$ *is R/P torsion for $i >> j$ for every two-sided prime ideal P then* $\mathrm{Ext}^i(M, C)$ *vanishes for $i >> 0$.*

To finish the argument, we use a change of ring spectral sequence. Namely, we have a spectral sequence:

$$E^{pq} = Ext^p_{R/(rad(PR))}(R/rad(QR), Ext^q_R(R/rad(PR), M))$$

By the above discussion, E^{pq} is $R/rad(QR)$ torsion for $p > m$. Because $R/rad(PR)$ has finite homological dimension over R, E^{pq} vanishes for q sufficiently high, depending only on P. Therefore for large enough i only depending on P, $Ext^i_R(R/rad(QR), M)$ is torsion. Since there are only finitely many P that arise, the result follows from the previous lemma.

1.5 Comments on the Pure Sullivan Condition

The condition that our dg-algebra be pure Sullivan may seem like a restrictive condition. To get a better feeling for why this a natural condition if we are to expect a full open-closed field theory, we look at two examples, one where the rational homotopy type is hyperbolic and one where it is elliptic, but not pure Sullivan.

Example 1.5.1. Suppose now $\mathcal{Q}$ is $(S^3 \times S^3 \times S^3)\#(S^3 \times S^3 \times S^3)$. A standard calculation in rational homotopy theory proceeds as follows:

Let N be the wedge $(S^3 \times S^3 \times S^3) \vee (S^3 \times S^3 \times S^3)$.

Then it is clear that

$$\pi_*(\Omega N) \otimes \mathbb{Q} \cong Ab(x_1, x_2, x_3) * Ab(x_4, x_5, x_6)$$

In this formula, $Ab(x_i, x_j, x_k)$ denotes the abelian Lie algebra generated by three even variables and * denotes the free product of Lie algebras. Next consider the manifold given by $\mathcal{U} = S^3 \times S^3 \times S^3 - D$, where D is a small open disc in $S^3 \times S^3 \times S^3$.

$$\pi_*(\Omega U) \otimes \mathbb{Q} \cong Ab(x_1, x_2, x_3) * Free(x)$$

Here $Free(x)$ denotes the free Lie algebra on one generator and $deg(x) = 7$, which corresponds to the Whitehead triple product of three dimensional spheres. Next we have the following general formula in (FelHalTho), Theorem 24.7, for the rational homotopy Lie algebra of the connected sum of two manifolds M, N.

$$\pi_*(\Omega(M\#N)) \otimes \mathbb{Q} \cong \pi(\Omega M') * \pi(\Omega N')/(\alpha + \beta)$$

Here M' and N' are M and N with small discs removed and α and β are the attaching maps for the top cell.

In the case under consideration, the top cell is attached along the Whitehead product. This leads to the following calculation of homotopy groups for $\mathcal{Q}$:

$$\pi_*(\Omega\mathcal{Q}) \otimes \mathbb{Q} \cong Ab(x_1, x_2, x_3) * Ab(x_4, x_5, x_6) * Free(x)$$

The center of the universal enveloping algebra can be seen to be $\mathbb{C}$ because the radical $R(\mathfrak{g})$ of the above Lie algebra is zero.

Lemma 1.5.2. *For any graded Lie algebra $\mathfrak{g}$ in characteristic zero, such that each graded piece $dim(\mathfrak{g}_i) < \infty$, there is a containment $Z(U\mathfrak{g}) \subset U(R(\mathfrak{g}))$*

Thus even on the homological level, the center of $H_*(\Omega\mathcal{Q})$ is given by $\mathbb{C}$. In view of the fact that the image of the map

$$HH^*(C_*(\Omega\mathcal{Q})) \to H_*(\Omega\mathcal{Q})$$

is contained in the center, there is no possibility for non-trivial curved deformations.

Even when the algebra is rationally elliptic and the image of the above morphism $HH^*(C_*(\Omega\mathcal{Q})) \to H_*(\Omega\mathcal{Q})$ is non-empty, there may be no compactifying deformation. Let $\mathfrak{g}$ be a nilpotent finite dimensional lie algebra concentrated in even degree. Let $\mathfrak{h}$ denote its center.

Lemma 1.5.3. *Suppose the natural morphism $Sym(\mathfrak{h}) \to Z(U\mathfrak{g})$ is surjective. Then for any potential, w, the curved category $(\widehat{\mathcal{A}}, w) - proj$ is either empty or non-compact.*

If w has any linear component, then one can compute that the Hochschild cohomology vanishes. If w is non-linear, let I denote the ideal $\mathfrak{g}U\mathfrak{g} \cap Sym(\mathfrak{h})$. Next, consider the curved module $M = \widehat{U\mathfrak{g}/I}$. We have that

$$Hom(M, M) \cong \widehat{U\mathfrak{g}/I} \otimes \Lambda(h)$$

To see this let $h_1, \ldots h_j$ denote a basis for $\mathfrak{h}$. We can write $w = \sum h_i w_i$. We let $(K_{(h_1,\ldots,j_j)}(\widehat{U\mathfrak{g}}), d_0)$ denote the Koszul complex associated to the ideal I. Then the map $\wedge w_i dh_i$ defines a map:

$$d_1 : K^i_{(h_1,\ldots,j_j)}(\widehat{U\mathfrak{g}}) \to K^{i+1}_{(h_1,\ldots,j_j)}(\widehat{U\mathfrak{g}})$$

One can then see that $d_1 + d_0$ turns the $K_{(h_1,\ldots,j_j)}(\widehat{U\mathfrak{g}})$ into a matrix factorization $\mathcal{P}$.

$$Hom(M, M) \cong Hom(\mathcal{P}, M) \cong \widehat{U\mathfrak{g}/I} \otimes \Lambda(h)$$

We have $dim_{\mathbb{C}}(Hom(M, M)) = \infty$ and thus the category is not compact.

Even if the map $Sym(\mathfrak{h}) \to Z(U\mathfrak{g})$ is not surjective, the above observation can be used to put strong restrictions on the possible curvings that can compactify the category. We do not pursue this further for reasons of space and interest.

Example 1.5.4. Let $\mathfrak{g}$ be a nilpotent finite dimensional lie algebra of rank three, with product given by $[x_1, x_2] = x_3$ and all other brackets are zero. The universal enveloping algebra is the algebra $\mathcal{A} = \mathbb{C}\{x, y, z\}/(xy - yx = z)$. Here the center of the universal enveloping algebra is a polynomial ring $\mathbb{C}[z]$ and we consider curved modules $(\mathcal{A}, z^n)$. One can check that the category vanishes when $n = 1$. One could also try to deform using not only curved deformation of $\mathcal{A}$ but also deform the higher multiplications. However, in this example, this does not affect the result.

Lemma 1.5.5. *There is no proper $\mathbb{Z}/2\mathbb{Z}$-graded deformation of $\mathcal{A}$.*

Any such Maurer-Cartan solution would necessarily be of the form

$$p(z) + p_{12}(x, y, z)d/dx \wedge d/dy + p_{13}(x, y, z)d/dx \wedge d/dz + p_{23}(x, y, z)d/dy \wedge d/dz$$

where $p(z)$ is in $\mathbb{C}[z]((t))$ and $p_{ij}(x, y, z)$ are in $\mathbb{C}[x, y, z]((t))$. The fact that this satisfies the Maurer-Cartan equation implies that

$$p_{13}(x, y, z) = p_{23}(x, y, z) = 0$$

One can then compute that the $p_{12}(x, y, z)d/dx \wedge d/dy$ terms are exact and conclude that there are no proper deformations.

Chapter 2

Symplectic Geometry and Mirror Symmetry

2.1 Tangential Fukaya categories

Given the close connection between string topology and the Floer theory of the cotangent bundle $T^*\mathcal{Q}$ explained in the introduction, we aim to give a Floer theoretic interpretation of our curved deformations of $C_*(\Omega\mathcal{Q})$. This construction was introduced independently by Nick Sheridan in his thesis (She) and the author in (Pom).

For motivation, let us consider the easiest case of a symplectic mirror to a Landau-Ginzburg model, that of S^2. We think of a sphere as being the (open) disk bundle of the cotangent bundle, $D^*(S^1)$, compactified by the points at 0 and ∞. This is then mirror to

$$(\mathbb{C}[z, z^{-1}], w = z + 1/z)$$

Work of (Sei3) proves that if we want to understand mirror symmetry for the Landau-Ginzburg model

$$(\mathbb{C}[z, z^{-1}], w = z^d + 1/z^d)$$

we can either consider the Fukaya category of the orbifold $S^2//(\mathbb{Z}/d\mathbb{Z})$, where $\mathbb{Z}/d\mathbb{Z}$ acts by rotations that fix the two points, or more concretely a Fukaya category where we require disks to intersect the compactifying divisor with ramification of order d.

This orbifold has a natural generalization. Consider a variety X and a collection of effective Cartier divisor $\mathcal{D}_i$, and d_i a collection of positive integers. The Cartier divisors define a natural morphism :

$$X \to [\mathbb{A}^n/(\mathbb{C}^*)^n]$$

Definition 2.1.1. The *root stack* $X_{(D_i,d_i)}$ is defined to be the fibre product

$$X \times_{[\mathbb{A}^n/(\mathbb{C}^*)^n]} [\mathbb{A}^n/(\mathbb{C}^*)^n]$$

where the map

$$[\mathbb{A}^n/(\mathbb{C}^*)^n] \to [\mathbb{A}^n/(\mathbb{C}^*)^n]$$

is the d_i-power map.

There are three important properties of the root stack:

(a) The root stack defines an orbifold, which has non-trivial orbifold stabilizers along the divisors

(b) The coarse moduli space is exactly X and away from D_i the map $X_{(\mathcal{D}_i,d_i)} \to X$ is an isomorphism.

(c) A map from a variety which is ramified to order d_i along the divisors $\mathcal{D}_i$ lifts uniquely to a map to $X_{(\mathcal{D}_i,d_i)}$.

Let $\mathcal{Q}$ be any simply connected manifold with a metric whose geodesic flow is periodic. There are three known families of examples, that of S^n $(n > 1)$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$. We will treat the first two cases here. $T^*\mathcal{Q} - \mathcal{Q}$ then acquires a Hamiltonian S^1 action by rotating the geodesics (which then give rise to Reeb orbits when restricted to the unit cotangent bundle). This induces a natural Hamiltonian action on $(T^*\mathcal{Q} - \mathcal{Q}) \times \mathbb{C}$. The moment map for this Hamiltonian S^1 action

$$(T^*\mathcal{Q} - \mathcal{Q}) \times \mathbb{C} \to \mathbb{R}$$

is given by

$$(x, z) \mapsto H(x) + 1/2|z|^2$$

Where $H(x) = |x|$ is the Hamiltonian associated to the Hamiltonian action on $T^*\mathcal{Q} - \mathcal{Q}$. We then take the reduced space, that is the preimage of a regular value quotiented out by the S^1 action. Finally, we glue back in the zero section to obtain a manifold X which is a symplectic compactification of the open disk bundle $D^*(\mathcal{Q})$ by the smooth divisor $\mathcal{D}$.

When $\mathcal{Q}$ is $\mathbb{C}P^n$, $X \cong \mathbb{C}P^n \times \mathbb{C}P^n$. Namely, we have an anti-holomorphic involution,

$$I : \mathbb{C}P^n \times \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n$$

given by

$$(z, w) \to (\bar{w}, \bar{z})$$

Its fixed point set: $\mathcal{L} : \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n$, is a Lagrangian submanifold, which corresponds to the zero section in the general construction. The divisor $\mathcal{D}$ parameterizes oriented closed geodesics and is embedded as a $(1, 1)$ hypersurface, the locus where

$$\sum z_i w_i = 0$$

When $\mathcal{Q}$ is S^n, we obtain the projective quadric Q_n and the divisor $\mathcal{D}$ is the projective quadric Q_{n-1}. The zero section in the general construction corresponds to a vanishing sphere $\mathcal{L}$ under a degeneration to a singular quadric.

In each of the cases we have

$$\pi_2(X, \mathcal{L}) \cong \mathbb{Z}$$

We want to consider three different moduli spaces of holomorphic disks.

Definition 2.1.2. We define the moduli space of tangential disks to be $\mathfrak{M}^d_{j,\ell}(X, \mathcal{L})$ the moduli space of holomorphic maps whith the following extra data:

(a) a map $u : (\mathbb{D}^2, S^1) \to (X, \mathcal{L})$

(b) a collection of j points on the boundary

(c) $u^{-1}(\mathcal{D}) = d(p_1 + p_2 + \cdots p_\ell)$ where p_j are points in $\mathbf{int}(\mathbb{D}^2)$

Definition 2.1.3. We define the auxiliary moduli space $MA^d_j(X, \mathcal{L})$ which parameterizes objects which consist of the following three pieces of data:

(a) a collection of j points $q_1, \dots, q_j$ on the boundary of a disk

(b) two points p_1, p_2 in $\mathbf{int}(\mathbb{D}^2)$ such that there is a biholomorphism

$$\mathbb{D}^2 \to \mathbb{D}^2$$

which sends

$$p_1 \to -r, \quad p_2 \to r, \quad q_1 \to i$$

(c) a map $(\mathbb{D}^2, S^1) \to (X, \mathcal{L})$ such that $u^{-1}(\mathcal{D}) = p_1 + (d-1)p_2$

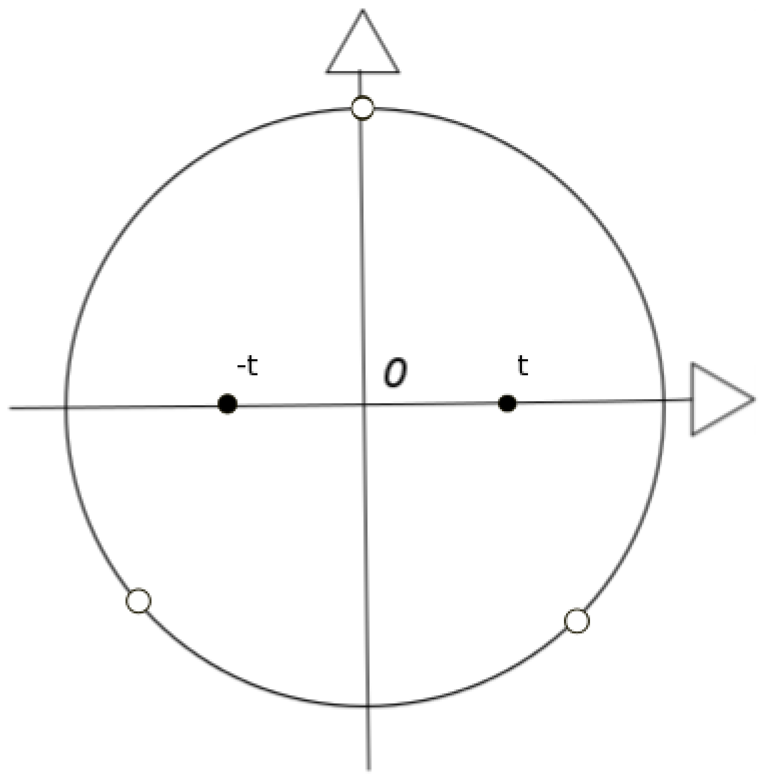
Definition 2.1.4. The Mickey Mouse moduli space $MM^d_{j,\ell}(X, \mathcal{L})$ parameterizes objects which consist of the following three pieces of data:

(a) A map from a nodal disk (u_1, u_2, u_3) with three components glued along marked points.

(b) a collection of marked points on the boundaries of each of the disks

(c) interior marked points p_1 and p_3 in u_1 and u_3 which intersect $\mathcal{D}$ with multiplicities $d-1$ and 1.

See the following figure for an example of an object in the auxiliary moduli space.



-t
0
t

Using the moduli spaces of tangential disks, we now define a version of Floer theory for the Lagrangian submanifold $\mathcal{L} \subset X_{(\mathcal{D},d)}$. We take the point of view that we will count in our theory holomorphic disks which intersect the boundary divisor with multiplicity d. As in relative Gromov-Witten theory, it is easy to acheive transversality for configurations of stable tangential disks, none of the components of which lie completely in the divisor. This is due to the following lemma in (CieMoh) which allows us to do all perturbations in a suitable open neighborhood V of our Lagrangian $\mathcal{L}$.

Lemma 2.1.5. *There is a Baire dense set of tamed almost complex structures $\mathcal{J}^{reg}(V)$ that agree with $\mathcal{J}_0$ outside V such that $\mathfrak{M}^d_{j,\ell}(X, \mathcal{L})$ is regular.*

The difficulty in defining such a theory in general is that under Gromov compactness, holomorphic curves may have components consisting of holomorphic spheres which live entirely in the divisor $\mathcal{D}$ and the moduli space of such objects can often be non-regular since we cannot deform the complex structure in a neighborhood of the divisor. In our situation, this problem is more manageable because such configurations have high codimension in the moduli space of J-holomorphic spheres. To be precise, we have the following lemma:

Lemma 2.1.6. *The moduli space of tangential J-holomorphic spheres is generically a pseudo-manifold. The configurations of tangential J-holomorphic spheres consisting of non-constant components living inside the divisor is codimension at least 4.*

We give the argument for $\mathcal{Q} = \mathbb{C}P^n$. The reader is encouraged to verify that the same argument is easily adapted to the case $\mathcal{Q} = S^n$. The real dimension of the moduli-space of tangential J-holomorphic spheres with l intersections with the divisor is:

$$2n + 2c_1(u) - 6 - 2l(d-1) = 2n + 2n(ld) - 6 + 2l$$

Meanwhile the moduli space of J-holomorphic spheres $P^1 \to D$ of symplectic area ld is regular and has dimension

$$2n - 2 + 2c_1^{TD}(u) - 6 = 2n - 2 + 2n(ld) - 6$$

Comparing, we realize that this space of spheres is codimension $2 + 2l \geq 4$ in the moduli space of tangential spheres as expected.

It is interesting to note that in the case $\mathcal{Q} = \mathbb{C}P^1$, we have the following calculation:

Lemma 2.1.7. *$\mathfrak{M}^d_{j,\ell}(X, \mathcal{L})$ has the structure of an oriented pseudo manifold.*

We make a transversality calculation for the open part of the moduli space in the case $\mathcal{Q} = \mathbb{C}P^1$. One can make a similar calculation for the various boundary components which arise under Gromov compactness. For this it is easier to work with the root stack $X_{(\mathcal{D},d)}$. We need to prove that the complex structure J is regular for disks in $X_{(\mathcal{D},d)}$. We have a map from $\pi : X_{(\mathcal{D},d)} \to X$, which gives rise to an exact sequence

$$0 \to TX_{(\mathcal{D},d)} \to \pi^* TX \to R \to 0$$

For any map $f : \mathbb{D}^2 \to X_{(\mathcal{D},d)}$, we get a sequence of sheaves on $\mathbb{D}^2$, and using the reflection principle, we can double this to a sequence of sheaves on $\mathbb{C}P^1$. On $\mathbb{C}P^1$, we know that R is a skyscraper sheaf of rank $2(d-1)\ell$, concentrated at the intersection point with $\mathcal{D}$ and its opposite and that that the double of the bundle f^*TX is of the form $\mathcal{O}(2d\ell) \oplus \mathcal{O}(2d\ell)$. We conclude that the double of $TX_{(\mathcal{D},d)}$ is also positive to deduce the desired result.

The author does not know whether this persists for S^{n+1} or $\mathbb{C}P^n$ for $n > 1$, though as noted above this is somewhat tangential to our main line of inquiry.

In any case, the theory can be defined along standard lines in two equivalent ways, either following (Sei2) or (FOOO).

Here we take the Morse Bott definition given in (FOOO) and we consider some model for chains, $C_*(X)((t))$, and using the evaluation maps

$$ev_i : \mathfrak{M}^d_{k+1,\ell}(X, \mathcal{L}) \to \mathcal{L}$$

to define a sequence of higher products

$$m_k(\alpha_1, ...\alpha_k) = \sum_{\ell} ev_{0,*}(\prod ev_i^*(\alpha_i))t^{\ell}$$

In our case, we are doing something slightly non-standard to our category by giving the Novikov-variable t a grading in order to relate it to the deformations we considered previously. This is valid here because the Lagrangian $\mathcal{L}$ is a *monotone* Lagrangian submanifold. A symplectic manifold X is monotone if $\omega(X) = \tau c_1(TX)$ for $\tau \geq 0$. Recall that a Lagrangian $\mathcal{L}$ is monotone if the two maps, corresponding to the action and Maslov index respectively:

$$A : \pi_2(X, \mathcal{L}) \to \mathbb{R},\ I : \pi_2(X, \mathcal{L}) \to \mathbb{Z}$$

satisfy the equation:

$$2A(u) = \tau I(u)\ \forall u \in \pi_2(X, \mathcal{L})$$

Using the perturbation lemma, we have defined an A_∞ deformation:

$$m_n : H^*(\mathcal{L})^{\otimes n}[[t]] \to H^*(\mathcal{L})[[t]]$$

the moduli spaces $\mathfrak{M}^d_{j,1}(X, \mathcal{L})$ in particular define classes e_d in $HH^*(C^*(\mathcal{Q}), C^*(\mathcal{Q}))$

Lemma 2.1.8. *The class e_d in $\mathfrak{M}^d_{j,1}(X, \mathcal{L})$ is gauge equivalent to the d-fold cup product e_1^d of the class defined by $\mathfrak{M}^1_{j,1}(X, \mathcal{L})$*

To prove this result for all d, we proceed by induction and consider the auxiliary moduli space. By standard Gromov compactness arguments, the boundary of the auxiliary moduli space consists of points where:

(a) $r \to 0$, the boundary is the moduli space $\mathfrak{M}^d_{j,1}(X, \mathcal{L})$.

(b) $r \to 1$ the boundary is the Mickey Mouse moduli space $MM^d(X, \mathcal{L})$.

The boundary as $r \to 1$ represents the Hochschild cup product of $e_{d-1} * e_1$

The boundary as $r \to 0$ is the class e_d.

This cobordism thus gives rise to the equation:

$$e_{d-1} * e_1 - e_d = \partial(MA)$$

where MA is the Hochschild cochain defined by the auxillary moduli space. This equation implies the result by induction.

It is interesting to note that the above proof follows the same line of reasoning as the proof in (FOOO) that bulk deformation :

$$H^*(X) \to \mathbf{HH}^*(Fuk(X))$$

is a ring homomorphism.

When $d = 1$ it is easy to calculate the relevant Hochschild cohomology class e_1. In all cases, the cohomology ring is monogenerated. Namely we have that

$$C^*(\mathcal{L}) \cong \mathbb{C}[x]/(x^j)$$

Lemma 2.1.9. *The class e_1 is that which corresponds to the deformation*

$$HF(\mathcal{L}, \mathcal{L}) \cong \mathbb{C}((t))[x]/(x^j - t)$$

For $\mathcal{Q} = \mathbb{C}P^n$ this follows from the fact that $HF^*(\mathcal{L}, \mathcal{L}) \cong QH^*(\mathbb{C}P^n)$, which is known to agree with the above ring (FOOO). For $\mathcal{Q} = S^n$, this result is contained in (Smi). We give a brief synopsis. To every *weakly unobstructed* Lagrangian in $Fuk(X)$, we assign a number m_0 (FOOO). Let $Fuk(X, 0)$ denote the subcategory generated by Lagrangians such that $m_0 = 0$. Smith shows that the Lagrangian $\mathcal{L}$ generates this subcategory. Let $QH^*(X, 0)$ denote the 0-eigenspace under the map.

$$c_1 \cup : QH^*(X) \to QH^*(X)$$

Smith proves that $QH^*(X, 0) \cong HH^*(Fuk(X, 0))$. It follows by Maslov index considerations that as a vector space $HF^*(\mathcal{L}, \mathcal{L}) \cong H^*(S^n)$. Beauville (Bea) calculated the quantum cohomology of X. The only A_∞ structure (up to gauge equivalence) on $H^*(S^n)$ which gives the correct HH^* is the one above.

Returning to our main calculation, we have a "finite determinacy" lemma. We state it for $\mathbb{C}P^n$, but the obvious adaptation of the theorem to the cases where $\mathcal{Q} = S^n$ also holds.

We first explain what we know about the $A(\infty)$ structure on $HF^*_{X_{D,d}}(\mathcal{L},\mathcal{L}) \cong \mathbb{C}[e]/e^{n+1}((t))$ We can write

$$m = tm_{2d} + \sum t^k \widetilde{m}^k$$

where $k \geq 2$ and the arity of $\widetilde{m}^k$ is even, larger than $2d$, and increasing with k. Note that for $d = 1$ the above isomorphism is not an algebra map as the multiplication m_2 is deformed.

We have the following formula for m_{2d}

$$m_{2d}(e^{a_1}, e^{a_2}, \ldots, e^{a_{2d}}) = t, \quad if \quad \sum(a_i) = (n+1)d$$

Lemma 2.1.10. *The $A(\infty)$ structure on $HF^*_{X_{D,d}}(\mathcal{L},\mathcal{L}) \cong \mathbb{C}[e]/e^{n+1}((t))$ is determined by the fact that $m_j = 0$, $2 < j < 2d$ and $m_{2d}(e^{a_1}, e^{a_2}, \ldots, e^{a_{2d}}) = t$, if $\sum(a_i) = (n+1)d$*

For the lowest k appearing in the sum above, it follows by a calculation of the Gerstenhaber algebra structure on $HH^*(C^*(\mathbb{C}P^n), C^*(\mathbb{C}P^n))$ that $\widetilde{m}^k$ is a Hochschild class for the $A(\infty)$ algebra on the same vector space but with just one higher multiplication $m_{2d}(e^{a_1}, e^{a_2}, \ldots, e^{a_{2d}}) = t$, if $\sum(a_i) = (n+1)d$. The calculation of the Gerstenhaber structure may be done using the formality theory developed in section 1.4.

By calculations similar to those presented at the beginning of this section, $\widetilde{m}^k$ is necessarily exact and thus there is a class p such that $b(p) = \widetilde{m}_k$. One can thus write down an $A(\infty)$ change of coordinates given by the formula $id + p$ which eliminates $\widetilde{m}_k$ and one can see that this only affects products of higher-arity. Continuing in this way, one can prove the desired lemma.

By a Kunneth theorem, we can get similar results for manifolds of the form $\mathcal{Q} = \prod \mathbb{C}P^{n_j}$.

One can give a uniform treatment of these results in the context of symplectic field theory as well. Seidel (Sei) has defined open-closed string maps from

$$S : SH^*(D^*(\mathcal{Q})) \to HH^*(Fuk(D^*(\mathcal{Q}))$$

$$S^{eq} : SH^*_{eq}(D^*(\mathcal{Q})) \to CC^*(Fuk(D^*(\mathcal{Q}))$$

This map is defined below.

Denote $D^*(\mathcal{Q})$ by W and by $\widehat{W}$ its symplectic completion, which is $T^*\mathcal{Q}$. We consider a class of Hamiltonians

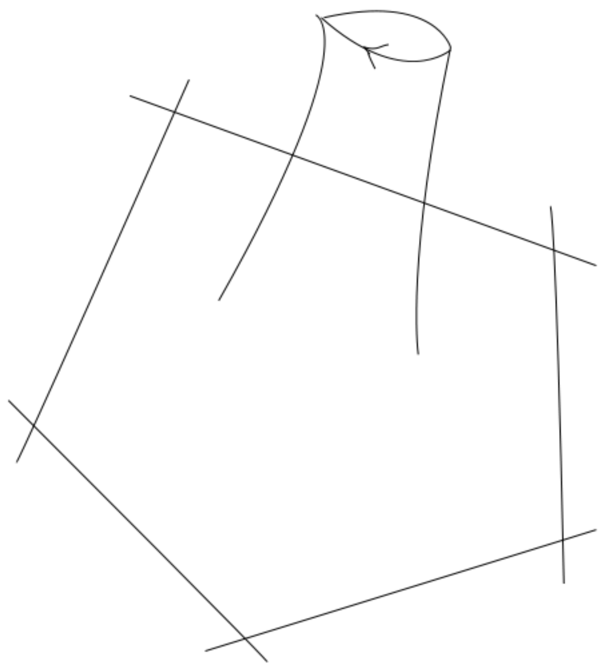
$$H : \widehat{W} \to \mathbb{R}$$

such that

(a) on W, the function H is $\mathcal{C}^2$ small and $H \leq 0$ in this region

(b) in the region $S^*(M) \times \mathbb{R}^+$, H(x,r) is a function $h(e^r)$, where h is a strictly increasing function, with $h = \alpha e^{2r} + \beta_0$ for $r > r_0$, α, β_0 , $r_0 \in \mathbb{R}^+$

A schematic drawing for the type of curves that arise in Seidel's map is given below.



Let $S(H)$ denote the set of time one periodic flows σ of the Hamiltonian H. To make things explicit, in the region $S^*(\mathcal{Q}) \times \mathbb{R}^+$, X_H has the form $h'(e^r)R$, where R is the Reeb vector field and time one orbits of X_H can be identified with orbits of the Reeb vector field of time $h'(e^r)$.

Assuming $S(H)$ is discrete, the chain complex is given by the vector space generated by two copies of

$S(H)$ (BouEkhElia),

$$\mathbb{C} < S(H)_m, S(H)_M >$$

The generators $S(H)_m$ and $S(H)_M$ come from the fact that because our Hamiltonian is time-independent each Reeb orbit gives rise to an S^1-family of 1-periodic orbits. For any Reeb orbit σ, we denote this moduli space by S_σ. Picking a generic Morse-function for each S_σ with exactly two critical points, the generators $S(H)_m$ and $S(H)_M$ correspond to the minimum and maximum of this Morse function.

In loose terms, the differential in $SH^*(W)$ is given by counting elements of the moduli space $M(S_{\sigma_+}, S_{\sigma_-})$ of cylinders which solve Floer's equation:

$$u : \mathbb{R} \times S^1 \to \hat{W}$$

(a) $\bar{\partial} u = \mathcal{J}(X_H)$

(b) $lim_{r \to \infty} u(r, \theta) \to \alpha \subset S_{\sigma_+}$

(c) $lim_{r \to -\infty} u(r, \theta) \to \beta \subset S_{\sigma_-}$

To be precise, we notice that given any two Reeb orbits σ_+ , σ_-, we have canonical evaluation maps

$$ev_+ : M(S_{\sigma_+}, S_{\sigma_-}) \to S_{\sigma_+}$$

$$ev_- : M(S_{\sigma_+}, S_{\sigma_-}) \to S_{\sigma_-}$$

The differential

$$d(\sigma_{+,m}) = \Sigma_{\sigma_-} N(\sigma_{+,m}, \sigma_{-,m})\sigma_{-,m} + N(\sigma_{+,m}, \sigma_{-,M})\sigma_{-,M}$$

Where $N(\sigma_{+,m}, \sigma_{-,M/m})$ counts curves u such that $ev_+(u)$ lies in the unstable locus of m and $ev_-(u)$ lies in the stable locus of M/m. The obvious variation of the above formula holds for $\sigma_{+,M}$.

The definition of SH^*_{eq} is similar except that the complex is generated by a single copy of $S(H)$, and the differential simply counts Floer trajectories between Reeb orbits.

In this situation, Seidel defined the above maps by first picking a radially symmetric function $p(r) : \mathbb{D}^2 \to \mathbb{R}$, which is 0 in a neighborhood of the boundary and 1 in a neighborhood of the origin. Denote by H_r the function $p(r)H$.

For any Lagrangian $\mathcal{L}$ in W, we consider Floer interpolation trajectories, that is maps

$$u : \mathbb{D}^2 - \{0\} \to \widehat{W}$$

which satisfy

(a) a deformed Floer equation $\bar{\partial} u = \mathcal{J}(X_{H_r})$

(b) $lim_{r \to 0} u(r, \theta) \to \alpha$

(c) $u(\partial \mathbb{D}^2) \in \mathcal{L}$

Seidel's maps are then defined by the obvious generalization of the ordinary A_∞ product in Lagrangian Floer theory using the moduli space of such trajectories.

In our case, the space of Hamiltonian orbits is not discrete, but one can define appropriate Morse-Bott versions of $SH^*(D^*(\mathcal{Q}))$, $SH^*_{eq}(D^*(\mathcal{Q}))$ and S^{eq}. For example, since the Reeb flow is periodic on $S^*(\mathcal{Q})$, we have a spectral sequence for $SH^*(D^*(\mathcal{Q}))$ (Sei)

$$E^{dq} = \begin{cases} H^*(W), & d = 0 \\ H^*(\partial W), & d < 0 \end{cases}$$

Here there is one copy of $H^*(\partial W)$ for each positive integer d corresponding to the d-fold cover of the simple time 1 geodesics. The number q is related to the grading conventions in SH^* and plays no role in our considerations. It can be extracted from d, the Maslov index of the simple geodesics and the degree of the corresponding cohomology class.

Likewise, we have a spectral sequence converging to $SH^*_{eq}(D^*(\mathcal{Q}))$ whose first page is given by

$$E^{dq} = \begin{cases} H^*(W), & d = 0 \\ H^*(\mathcal{D}), & d < 0 \end{cases}$$

Note that as $\alpha \to 0$, the time-one Hamiltonian orbits escape to $r = \infty$. Namely our curves limit to punctured $\mathcal{J}$-holomorphic half cylinders asymptotic to d-fold covers of Reeb orbits on $S^*(\mathcal{Q})$. By a removal of singularities argument, such holomorphic disks in $T^*(\mathcal{Q})$ are in bijection with with maps $\mathbb{D}^2 \to X$ tangent to $\mathcal{D}$ with tangency of order d.

The compactness results of (BouEkhElia) allow us to study the limit as $H \to 0$. In general, the limiting curves can be complicated SFT moduli spaces consisting of multi-level curves. In this particular case, however, the higher level curves vanish for reasons of index. We can therefore identify the classes e_d with the image in $SH^*_{eq}(D^*(\mathcal{Q}))$ of d-fold Reeb orbits, which in turn correspond to to the classes $H^0(\mathcal{D})$, which survive the above spectral sequence.

Remark 2.1.11. More generally, in the formalism above, if $\mathcal{X}$ is a projective variety and $\mathcal{D}$ is a smooth ample divisor, one could examine chains S in $\mathcal{D}$ representing classes of $H_*(\mathcal{D})$ such that $[S]_d$ defines a class in $SH^*_{eq}(D^*(\mathcal{Q}))$. We can then define an infinitesimal cyclic deformation by considering as above disks with a single point of intersection with $\mathcal{D}$ of multiplicity d, but additionally requiring that the point of tangency simultaneously lie in S.

In order to understand better the relationship between this section and the previous two, it is useful to note the following strong result due to McLean from a recent paper (McL).

Theorem 2.1.12. *If $T^*\mathcal{Q}$ is symplectomorphic to an affine variety A, then $\mathcal{Q}$ is (rationally) elliptic.*

Rephrasing, if there is a projective symplectic manifold X, and an ample normal crossings divisor $\cup\mathcal{D}_i$, such that $X - \cup\mathcal{D}_i \cong T^*\mathcal{Q}$, then $\mathcal{Q}$ is rationally elliptic. McLean proves this result by proving that under the existence of such a compactification, $rank(SH^k(A))$ has polynomial growth in k. The degree of the growth is bounded by the maximal number of components which have non-empty intersection.

In particular if $\mathcal{D}$ is a smooth divisor and we have, following the above notation, that $X - \mathcal{D} \cong T^*\mathcal{Q}$, then it follows from a theorem of [Bott] that $\mathcal{Q}$ is rationally equivalent to a sphere or a projective space.

The author does not know an example $\mathcal{Q}$, such that $T^*\mathcal{Q}$ is symplectomorphic to an affine variety A, where $C^*(\mathcal{Q})$ is not pure Sullivan. It seems interesting in the context of the analysis at the end of section 6 to see whether these results can be further strengthened.

2.2 Full Fukaya Categories of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and root stacks

The reader will notice that the results on the Fukaya category of e.g. $\mathbb{C}P^1 \times \mathbb{C}P^1$ were not fully satisfactory as they described only the subcategory of $Fuk(\mathbb{C}P^1 \times \mathbb{C}P^1)$ generated by the zero-section $\mathcal{L}$ as a curved deformation, rather than the full Fukaya category. In this section, we show that by regarding $C_*(\Omega S^2)$ as a $\mathbb{Z}/2\mathbb{Z}$-graded dga, we may realize the full Fukaya category as a curved deformation.

Specializing the formula given in section 1.2 for $C_*(\Omega\mathbb{C}P^n)$ to when $n = 1$, we obtain that:

$$C_*(\Omega S^2) \cong \mathbb{C}[u] \otimes \Lambda(e)$$

with $m_2(e, e) = u$. Here the $deg|u| = 2$ and $deg|e| = 1$.

Next we construct the mirror to T^*S^2. This construction is very similar to published work of (Aur) and in fact work in progress (Aur2) gives a mirror construction that generalizes what we are about to explain.

For ϵ a real constant, we view T^*S^2 as a hypersurface in $\mathbb{C}^3$ cut out by the equation:

$$Y_\epsilon = \{(x, y, z) \in \mathbb{C}^3 : xy - z^2 = \epsilon\}.$$

We have an action of S^1 on this manifold given by

$$\alpha * (x, y, z) = (\alpha x, \alpha^{-1} y, z)$$

We also have a mapping $z : Y_\epsilon \to \mathbb{C}$ which defines a Lefschetz fibration. Picking a real number δ larger than $\sqrt{\epsilon}$ and we work relative to the divisor

$$D_\delta = \{(x, y, z) \in Y_\epsilon, z = \delta\}.$$

There is a special Lagrangian fibration defined on the complement of this divisor given by

$$L_{r,\vec{\lambda}} = \{(x, y, z, w) \in Y_\epsilon : |z - \delta| = r, \mu(x, y, z) = \lambda\}.$$

Here $\mu(x, y, z)$ is the moment map for the above S^1 action. One can construct the SYZ mirror to this hypersurface and we obtain the following:

Let

$$X = \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C}^* : uv = (1 + w)^2\}$$

X has a canonical small resolution $\pi : \widetilde{X} \to X$ given by considering the subvariety of $\mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{C}P^1$ cut out by the following equations:

$$\widetilde{X} = \{(x, y, z, t_1, t_2) \in \mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{C}P^1 : vt_1 = (1 + w)t_2, ut_2 = (1 + w)t_1\}$$

Theorem 2.2.1. *The mirror to Y_ϵ is given by the LG-model $(\widetilde{X}, v)$.*

We consider two categories of matrix factorizations, which correspond to the $\mathbb{Z}$-graded and $\mathbb{Z}/2\mathbb{Z}$-graded modules over $C_*(\Omega S^2)$ respectively. To construct the $\mathbb{Z}/2\mathbb{Z}$-graded category of matrix factorizations in this non-affine setting, we now consider the dg category of curved complexes of coherent sheaves $\mathrm{Coh}(\widetilde{X}, v)$, that is, the category with objects:

$$E = (\ E_1 \underset{e_0}{\overset{e_1}{\rightleftarrows}} E_0\ )$$

where the E_i are coherent sheaves of $\mathcal{O}_{\widetilde{X}}$-modules and the e_i are morphisms of $\mathcal{O}_{\widetilde{X}}$-modules satisfying $e_{i+1} \circ e_i = v \cdot \mathrm{id}_{E_i}$. The morphism complexes are defined exactly as before except with $\mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}$ rather than Hom_A.

Definition 2.2.2. Denote by $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{Coh}(\widetilde{X}, v)] \subset [\mathrm{Coh}(\widetilde{X}, v)]$ the thick triangulated subcategory generated by the total curved complexes of exact triples of curved quasi-coherent $\mathcal{O}_X$-modules. Objects of $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{Coh}(\widetilde{X}, v)]$ are called *acyclic*. The triangulated category $\mathrm{MF}(\widetilde{X}, v)$ is defined to be the quotient triangulated category

$$[\mathrm{Coh}(\widetilde{X}, v)]/\,\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{Coh}(\widetilde{X}, v)]$$

This definition is also used in (Pos; Orl).

There is an embedding $i : \mathbb{A}^1 \hookrightarrow \widetilde{X}$ defined by restricting to the locus where $t_2 = 0$. This identifies with the singular locus of the function v.

It is easy to see using results of (LinPom) that the structure sheaf for $\mathbb{A}^1$ generates the category $MF(\widetilde{X}, v)$. Furthermore, a calculation reveals that

$$Hom(\mathbb{A}^1, \mathbb{A}^1) \cong C_*(\Omega S^2)$$

Here is how that calculation goes. The singular locus is contained in the part where $t_1 \neq 0$. Therefore we can consider a new variable $q = t_2/t_1$. The neighborhood $t_1 \neq 0$ is an affine space with variables q, u. In these coordinates, the potential $v = q^2 u$. The object that we want to consider is the brane defined by $q = 0$. As it is no different, and this will be useful later, we consider this brane in the category of modules over $(\mathbb{C}[u,q], q^{p+1}u^p)$. Following the model category structure introduced in [LinPomerleano], we can construct a matrix factorization which resolves this coherent matrix factorization as follows:

$$P = (\ \mathbb{C}[u,q] \underset{u^p q^p}{\overset{q}{\rightleftarrows}} \mathbb{C}[u,q]\ )$$

From here we have that $Hom(P,P) \cong Hom(P, \mathbb{C}[u,q]/q)$. The differential vanishes on this latter complex and we have that

$$Hom(P, \mathbb{C}[u,q]/q) \cong \mathbb{C}[u] \otimes \Lambda(e)$$

Chasing through the isomorphism $Hom(P,P) \cong Hom(P, \mathbb{C}[u,q]/q)$, we get that $m_{p+1}(e,e,\ldots,e) = u^p$ as claimed.

One point to remember is that we must view $C_*(\Omega S^2)$ as a $\mathbb{Z}/2\mathbb{Z}$ graded object.

In the above notation, now just considering the category of matrix factorizations over $(\mathbb{C}[u,q], q^2u)$, it will be helpful to record the endomorphisms of the brane defined by $u = 0$. The corresponding matrix factorization is given by:

$$P' = (\ \mathbb{C}[u,q] \underset{q^2}{\overset{u}{\rightleftarrows}} \mathbb{C}[u,q]\ )$$

Then we have that $Hom(P', P') \cong Hom(P', \mathbb{C}[u,q]/u) \cong \mathbb{C}[q]/q^2$

This corresponds to the fact that the exceptional $\mathbb{C}P^1$ is mirror in $\widetilde{X}$ to the zero section in T^*S^2.

Notice that there is $\mathbb{C}^*$ action on $\widetilde{X}$, given by,

$$\alpha * (u, v, w) = (\alpha^2 u, \alpha^{-2} v, w)$$

Notice that v has weight -2 with respect to this $\mathbb{C}^*$ action.

We now describe how to use this $\mathbb{C}^*$- action to define a graded refinement of this category.

Definition 2.2.3. A *graded coherent D-brane* is an equivariant coherent sheaf E, equivariant with respect to a $\mathbb{C}^*$ action, equipped with an endomorphism d_E of weight one, such that $d_E^2 = v$. The term *graded matrix factorization* is reserved for the case when E is a vector bundle.

The category of graded coherent D-branes has a triangulated structure given by shifting the weights. As usual, given two graded coherent D-branes E and F, the complex of morphisms

$$Hom(E,F), d \circ f - (-1)^{|f|} f \circ d$$

makes the category into a differential graded category.

We write the formula for the differential so that the reader can note that it has degree 1 and the category $[Coh_{\mathbb{C}^*}(\widetilde{X}, v)]$ is $\mathbb{Z}$-graded triangulated category. As above, we can define the category $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{Coh}_{\mathbb{C}^*}(\widetilde{X}, v)]$ and we define $MF_{\mathbb{C}^*}(\widetilde{X}, v)$ as above.

Lemma 2.2.4. *The structure sheaf* $\mathbb{A}^1$ *generates the category* $MF_{\mathbb{C}^*}(\widetilde{X}, v)$.

We have an obvious functor

$$F : MF_{\mathbb{C}^*}(\widetilde{X}, v) \to MF(\widetilde{X}, v)$$

In [Shipman] it is proven that the idempotent completion of the category $MF_{\mathbb{C}^*}(\widetilde{X}, v)$ embeds as the compact objects into the absolute derived category of graded *quasicoherent* D-branes. Therefore, it is enough to show that if P is a graded matrix factorization resolving $\mathbb{A}^1$ for any quasi-coherent curved module $\mathcal{J}$, $Hom(P, \mathcal{J})$ is never zero (BonVDB). Given the graded matrix factorization P and any graded quasi-coherent D-brane, $\mathcal{J}$, it is proven in (Shi) and (LinPom) that the space of morphisms $Hom(P, J)$ is simply a $\mathbb{Z}$-graded refinement of the space of morphisms of the space of morphisms between $F(P)$ and $F(\mathcal{J})$. If P is an equivariant matrix factorization resolving $\mathbb{A}^1$, then the fact that $Hom(F(P), F(\mathcal{J}))$ is never zero, implies that $Hom(P, \mathcal{J})$ is never zero.

Notice that with this grading, the variable u has degree 2, which makes the isomorphism:

$$Hom(\mathbb{A}^1, \mathbb{A}^1) \cong C_*(\Omega S^2)$$

into a graded isomorphism.

Following Auroux's paper, one can determine the deformation which corresponds to further compactification of Y_ϵ to $\mathbb{C}P^1 \times \mathbb{C}P^1$. The standard Hori-Vafa mirror for $\mathbb{C}P^1 \times \mathbb{C}P^1$ is given by

$$((\mathbb{C}^*)^2, z_1 + z_2 + 1/z_1 + 1/z_2)$$

Our mirror is a partial compactification of the toric mirror, and there is a coordinate transformation given by

$$v = z_1 + z_2, u = (z_1 + z_2)/z_1^2, w = z_2/z_1$$

Therefore we conclude that:

Lemma 2.2.5. *Compactification corresponds to the deformation* u/w.

Notice that this is not a deformation of the category $MF_{\mathbb{C}^*}(\widetilde{X}, v)$ only of the category $MF(\widetilde{X}, v)$. We denote the $\mathbb{Z}/2\mathbb{Z}$ graded Hochschild cohomology by $HH^*_{\mathbb{Z}/2\mathbb{Z}}(C_*(\Omega S^2)$. The classes u and u/w are equivalent as Hochschild classes in $HH^*_{\mathbb{Z}/2\mathbb{Z}}(C_*(\Omega S^2)$, but not as Maurer-Cartan classes in $HH^*_{\mathbb{Z}/2\mathbb{Z}}(C_*(\Omega S^2)((t))$.

We note that this claim does not contradict the formality lemma in the previous section. Note that $HH^*_{\mathbb{Z}/2\mathbb{Z}}(C_*(\Omega S^2)((t))$ is *not* isomorphic to $HH^*_{\mathbb{Z}/2\mathbb{Z}}(C^*(S^2))((t))$. To be precise, we can see from the analysis in section 1.2 that :

$$HH^*_{\mathbb{Z}/2\mathbb{Z}}(C_*(\Omega S^2) \cong \frac{\mathbb{C}[u, a] \otimes \Lambda b}{(a^2, ab, au)}$$

$$HH^*_{\mathbb{Z}/2\mathbb{Z}}(C^*(S^2)) \cong \frac{\mathbb{C}[[u]][a] \otimes \Lambda b}{(a^2, ab, au)}$$

For both deformed categories mentioned above, $End(\mathbb{C}((t)), \mathbb{C}((t)))$ are the same. Geometrically, the equivariant category $MF_{\mathbb{C}^*}(\widetilde{X}, v)$ is mirror to the Wrapped Fukaya category consisting of exact Maslov-index zero Lagrangians.

The mirror to the category $MF(\widetilde{X}, v)$ is given by considering a larger category which includes compact *weakly unobstructed* Lagrangians, such as the Lagrangian tori used to define our Lagrangian torus fibration. This category is now only $\mathbb{Z}/2\mathbb{Z}$-graded.

The cotangent fiber is still a generator for this larger category. Following [Auroux] there are exotic non-exact Lagrangian tori, contained T^*S^2 which are non-zero in the Fukaya category of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and which are Floer theoretically disjoint from the zero-section. These correspond to the other objects of the deformed category $MF(\widetilde{X}, v + u/w)$

We have the following calculation:

Lemma 2.2.6. *$(u/w)t \cong ut + u^3t^2$ as Maurer-Cartan classes in $HH^*(MF(\widetilde{X}, v))((t))$*

In view of the previous discussion it is interesting to discuss the root stacks $\mathbb{C}P^1 \times \mathbb{C}P^1_{(D,d)}$ as well. For generic t, the LG-models $(\widetilde{X}, v + t(u/w)^d)$ all have isolated critical locus. Let Z denote the scheme theoretic critical locus of this function.

Recall that the orbifold cohomology $H^*(X_{(\mathcal{D},d)} \times_{(X_{(\mathcal{D},d)} \times X_{(\mathcal{D},d)})} X_{(\mathcal{D},d)})$ is the space of states for orbifold Gromov-Witten theory.

Calculating the orbifold cohomology of $X_{(\mathcal{D},d)}$ we have the following lemma:

Lemma 2.2.7. *$dim\ H_{orb}(\mathbb{C}P^1 \times \mathbb{C}P^1_{(D,d)}) = length(Z)$*

This observation should imply homological mirror symmetry for the orbifold once the appropriate definitions for the entire Fukaya category of $\mathbb{C}P^1 \times \mathbb{C}P^1_{(D,d)}$ are in place.

More precisely, the above lemma appears to be the shadow of a homomorphism:

$$H^*(X_{(\mathcal{D},d)} \times_{X_{(\mathcal{D},d)} \times X_{(\mathcal{D},d)}} X_{(\mathcal{D},d)}) \to \mathbf{HH}^*(Fuk(X_{(\mathcal{D},d)})$$

for some to be defined Fukaya category of the orbifold $X_{(\mathcal{D},d)}$.

In joint work with Kevin Lin, we will extend this picture to higher dimensions. For example, we examine the case of T^*S^3. We again view T^*S^3 as

$$Y_\epsilon = \{(x, y, z, w) \in \mathbb{C}^4 : xy - zw = \epsilon\}$$

Notice that there is an action of T^2 which survives on Y_ϵ when ϵ is nonzero. Then the moment map for this T^2 action on Y_ϵ is

$$\mu_2 : (x, y, z, w) \mapsto (|x|^2 - |y|^2, |z|^2 - |w|^2).$$

Let δ be a constant; then in Y_ϵ, we have the anticanonical divisor D_δ which is given by the locus $\{xy = \delta\}$. Note that the divisor is preserved by the natural T^2 action. Letting $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $r \in \mathbb{R}_{>0}$, we put

$$L_{r,\vec{\lambda}} = \{(x, y, z, w) \in Y_\epsilon : |xy - \delta| = r, \mu_2(x, y, z, w) = \vec{\lambda}\}.$$

One can show that these $L_{r,\vec{\lambda}}$, when they are smooth, are special Lagrangian tori in Y_ϵ. The mirror is the small resolution of the singularity:

$$uv = (1 + w_1)(1 + w_2)$$

with potential $w = v$.

We consider as before the variety, Q_3, the projective quadric in $\mathbb{C}P^4$, which is again a compactification of Y_ϵ. The compactification of the affine quadric in projective space corresponds to the deformation

$$w = v + t(u^2/w_1 w_2)$$

We will also generalize the above discussion and discuss the Fukaya category of the root stacks in this case.

2.3 Mirror Symmetry for the A_n plumbings and curved deformations of plumbings

Previously, we have discussed the simplest sort of Stein manifold, cotangent bundles. In this section, we look at the next simplest case, which is the A_n plumbing of cotangent bundles of S^2. Here the situation is considerably more complicated, and there is no known homotopical description of the wrapped Fukaya category. In this section, we compute the category for plumbings of spheres and then examine its curved deformation theory.

Again we consider the A_n plumbing of spheres as a hypersurface:

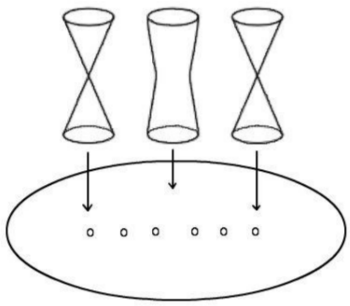
$$Y_\epsilon = \{(x, y, z) \in \mathbb{C}^3 : xy - z^{n+1} = \epsilon\}$$

Theorem 2.3.1. *$WFuk(Y_\epsilon) \cong MF_{\mathbb{C}^*}(\widetilde{X}, v)$*

The space $(\widetilde{X}, v)$ is the mirror of Y_ϵ, whose construction follows closely the ideas of the previous section. More precisely, the map

$$z : Y_\epsilon \to \mathbb{C}$$

defines a Lefschetz fibration as shown in the figure below.



Furthermore, we have as before an action of S^1 on this manifold given by

$$\alpha * (x, y, z) = (\alpha x, \alpha^{-1} y, z)$$

One can then construct a special Lagrangian fibration on Y_ϵ using the same recipe as before. For a detailed exposition of how to construct the mirror using this special Lagrangian fibration, the reader may consult (Chan), based upon an earlier talk of Auroux (Aur2).

We again consider the minimal resolution $\widetilde{X}$ of the A_n singularity,

$$X = \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C}^* : uv - (1 + w)^{n+1} = 0\}$$

The mirror is best described as an open part of a toric variety. Let

$$X_{tor} = \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C} : uv - (1 + w)^{n+1} = 0\}$$

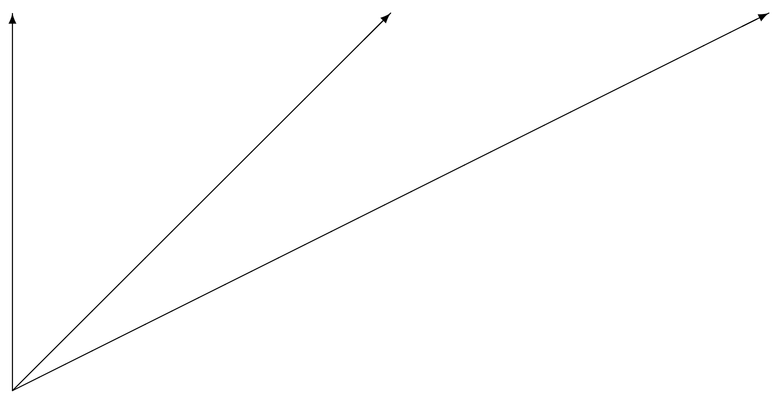
The fan for this toric singularity is given by Δ in $N_{\mathbb{R}}$ for $N = \mathbb{Z}^2$ consisting of the cone generated by the vectors:

$$v_{\rho_1} = (0, 1),\ v_{\rho_2} = (n, 1),$$

The toric resolution is given by the fan Δ in $N_{\mathbb{R}}$ for $N = \mathbb{Z}^2$ consisting of all cones generated by no more than two of the vectors:

$$v_{\rho_1} = (0, 1),\ v_{\rho_2} = (1, 1),\ v_{\rho_3} = (2, 1),\ \ldots v_{\rho_{n+1}} = (n, 1)$$

For $n = 2$ this gives the following picture:



The mirror is equipped with a natural $\mathbb{C}^*$ action:

$$\alpha * (u, v, w) = (\alpha^2 u, \alpha^{-2} v, w)$$

It is with respect to this action that we consider the category of graded D-Branes. The zero fiber of v is an A_n configuration of $\mathbb{C}P^1$ in the fiber of the resolution over $(0, 0, -1)$ in $\widetilde{X} \to X$, the components of which we denote by Z_i, $i = 1, \ldots, n$ glued to an $\mathbb{A}^1$, which we denote by A at a single point along

Z_1. The reduced scheme structure on the singular locus of v coincides with $Z_i \cup A$ for $i = 1, \ldots, n-1$. Under the mirror map, each $\mathbb{C}P^1$ in the zero fiber corresponds to a matching sphere Lagrangian and the $\mathbb{A}^1$ corresponds to a Lefschetz thimble at the beginning of the chain.

We will need to study the Fukaya-Seidel category $Fuk(Y_\epsilon, z)$. We recall its definition here. Objects consist of (twisted complexes of) exact Lagrangians in Y_ϵ and an ordered collection of Lefschetz thimbles Δ_1, Δ_2, $\ldots \Delta_{n+1}$. Let V_1, V_2, $\ldots V_{n+1}$ denote the corresponding vanishing cycles. In his book (Sei2), Seidel defines the Floer cohomology for the thimbles as:

$$Hom(\Delta_i, \Delta_j) = \begin{cases} \mathbb{C}, & i = j \\ HF^*(V_i, V_j), & i < j \\ 0, & i > j \end{cases}$$

We apply the following theorem of Abouzaid and Seidel, which enables us to compute the wrapped Fukaya category of the Lefschetz fibration in terms of the inclusion of a smooth fibre and the Fukaya-Seidel category $Fuk(Y_\epsilon, z)$ (Abou2; Sei4) :

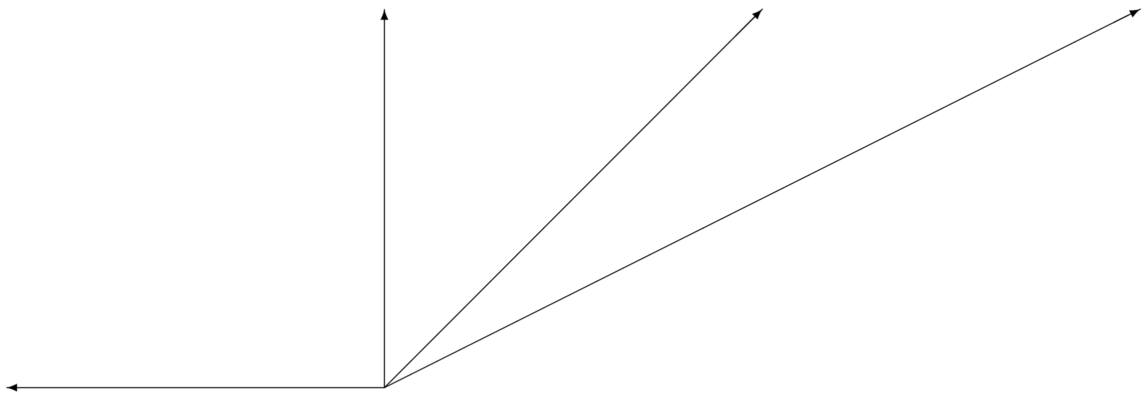
Theorem 2.3.2. *There is a natural transformation from Serre: $[Fuk(Y_\epsilon, z)] \to [Fuk(Y_\epsilon, z)]$ to the id : $[Fuk(Y_\epsilon, z)] \to [Fuk(Y_\epsilon, z)]$ such that $WFuk(Y_\epsilon)$ is isomorphic to the localization of of $Fuk(Y_\epsilon, z)$ with respect to this natural transformation.*

We describe how this works. We denote by B the exceptional algebra associated to the thimbles in $Fuk(Y_\epsilon, z)$. Let E be the algebra associated to the vanishing cycles in the fiber $Fuk(Y_z)$. We have an inclusion $B \to E$ of A_∞ algebras. We have an exact sequence of $B - B$ bimodules

$$B \to E \to E/B$$

taking the boundary morphism $E/B \to B$ gives rise to the above natural transformation.

The strategy we will follow is to first produce a mirror to $[Fuk(Y_\epsilon, z)]$ and then use the above theorem of Seidel and Abouzaid to deduce the result. We prove the result by partially compactifying $\widetilde{X}$ by adding an extra divisor D, which corresponds to adding the vector $(-1, 0)$ to the above toric fan.



This is mirror to the Lefschetz fibration (Y_ϵ, z). Notice that the divisor D is isomorphic to $\mathbb{C}^*$ which is mirror to the fibre $Y_z \to Y_\epsilon$ (also a $\mathbb{C}^*$).

We denote this compactified space by $\overline{X}$. Notice that the divisor D is an anti-canonical divisor for $\overline{X}$. The singular locus of the potential v when extended to the space $\overline{X}$ is a chain of $\mathbb{C}P^1$, whose irreducible components consist of the compactification of A denoted by $\bar{A}$ and Z_i.

Let L denote $\mathcal{O}_{\bar{A}} \oplus_i \mathcal{O}_{Z_i}$, $i = 1, \ldots n$ as objects in $MF(\overline{X}, v)$. The object L generates the category because it is proven in (LinPom) that a generator of the category of coherent sheaves on the singular locus generates the category $MF(\overline{X}, v)$. Picking a point p away from the singular locus on Z_n, we have that the collection $p \oplus \mathcal{O}_{Z_n}$ generates $Coh(Z_n)$. In particular, it generates the skyscraper sheaf at the point $q = Z_n \cap Z_{n-1}$. Inductively, we can now generate the category $Coh(\cup Z_i)$, where $i = 1, \ldots n-1$. The object $[p]$ is zero in $MF(\overline{X}, v)$, since it avoids the critical locus.

It is interesting to notice that in the case of $MF(\widetilde{X}, v)$, the category of matrix factorizations is actually generated by $\mathcal{O}_A \oplus_i \mathcal{O}_{Z_i}$ $i = 1, \ldots n-1$. The reason is that A is affine so we can generate the skycraper sheaf at $q' = A \cap Z_1$. Again using an inductive argument, we can generate the category of coherent sheaves on the entire critical locus.

Similarly, let $\mathcal{L}$ denote the sum of Δ_1 and the matching spheres $\mathcal{L}_j$. Seidel proves that the Lefschetz thimbles split generate the Fukaya-Seidel category (Sei2). As noted above, in our setting it is more appropriate to consider a different generating set for the Fukaya category. The thimble Δ_1 and the matching spheres $\mathcal{L}_j$ also generate the Fukaya-Seidel category $Fuk(Y_\epsilon, z)$. This can be seen because given a thimble at the beginning of the chain and all of the matching spheres, the other thimbles arise as a Dehn twist (Sei2) of the preceeding thimble and the matching sphere.

It is easy to verify on the homology level that:

$$Hom_{MF(\overline{X},v)}(L,L) \cong Hom_{Fuk(Y_\epsilon,z)}(\mathcal{L},\mathcal{L})$$

The homology $Hom_{MF(\overline{X},v)}(L,L)$ has the following quiver presentation.

$$Q = (\ \alpha_0 \rightleftarrows \alpha_1 \rightleftarrows \alpha_2 \rightleftarrows \cdots \rightleftarrows \alpha_{n-1} \rightleftarrows \alpha_n\ )$$

The quiver is given by taking the path ring of the above graph modulo the following relations

$$(\alpha_0|\alpha_1|\alpha_0) = 0, (\alpha_i|\alpha_{i+1}|\alpha_i) = (\alpha_i|\alpha_{i-1}|\alpha_i)$$

and

$$(\alpha_{i-1}|\alpha_i|\alpha_{i+1}) = (\alpha_{i+1}|\alpha_i|\alpha_{i-1}) = 0$$

We next explain how to verify this. The calculation of $Hom_{MF(\overline{X},v)}(\mathcal{O}_{Z_i},\mathcal{O}_{Z_i})$ proceeds by analogy with the corresponding calculation in the ordinary derived category. As was proven in (LinPom), we can compute $Hom(\mathcal{O}_{Z_i},\mathcal{O}_{Z_i})$ by $R\Gamma(R\mathcal{H}om(\mathcal{O}_{Z_i},\mathcal{O}_{Z_i}))$. The local calculation in the previous section demonstrates that $R\mathcal{H}om^0(\mathcal{O}_{Z_i},\mathcal{O}_{Z_i}) \cong \mathcal{O}_{Z_i}$. Thus $Hom^0(\mathcal{O}_{Z_i},\mathcal{O}_{Z_i}) \cong \mathbb{C}$.

The fact that the branes $\mathcal{O}_{Z_i}$ do not intersect the compactifying divisor implies that:

$$Hom^2(\mathcal{O}_{Z_i},\mathcal{O}_{Z_i}) \cong \mathbb{C}$$

For $\mathcal{O}_L$, Serre duality implies that $Hom^2(\mathcal{O}_L,\mathcal{O}_L)$ vanishes. The calculations that

$$Hom_{MF(\overline{X},v)}(\mathcal{O}_{Z_i},\mathcal{O}_{Z_{i+1}}) \cong \mathbb{C}$$

and

$$Hom_{MF(\overline{X},v)}(\mathcal{O}_{Z_{i+1}},\mathcal{O}_{Z_i}) \cong \mathbb{C}$$

are also purely local.

We also have the following formality lemma for the $Hom_{MF(\overline{X},v)}(L,L)$ in the matrix factorization category, whose proof is an adaptation of an argument of Seidel and Thomas (SeiTho):

Lemma 2.3.3. *The quiver algebra $Hom_{MF(\overline{X},v)}(L,L)$ is intrinsically formal.*

We summarize the main point. Lemma 4.21 of the paper by Seidel and Thomas prove that the A_{n+1} quiver algebra, which we denote by A is formal. We denote the quiver that we care about by $\widetilde{A}$. For degree reasons we have the exact sequence:

$$\mathbf{HH}^{q-1}(A, A[2-q]) \to \mathbf{HH}^{q-1}(A, A[2-q]) \to 0$$

Using their notation, we define $\phi_{i_0,i_1,\ldots,i_0} \in \mathbf{HH}^q(A, A[2-q])$ to be

$$\phi_{i_0,i_1,\ldots,i_0}(c) = \begin{cases} (i_0|i_0+1|i_0), & c = (i_0|i_1) \otimes \ldots \otimes (i_{q-1}|i_0) \\ 0, & \text{all other basis elements c} \end{cases}$$

These form a basis for $\mathbf{HH}^q(A, A[2-q])$. Seidel and Thomas use two sets of classes of Hochschild cochains in $\mathbf{HH}^{q-1}(A, A[2-q])$ to generate enough relations in $HH^q(A, A[2-q])$ to prove that this group is zero. These classes are:

$$\phi'(c) = \begin{cases} (i_0|i_{q-2}), & c = (i_0|i_1|) \otimes (i_2|i_3) \ldots \otimes (i_{q-3}|i_{q-2}) \\ 0, & \text{all other basis elements c} \end{cases}$$

$$\phi''(c) = \begin{cases} (i_0|i_1|i_0), & c = (i_0|i_1|i_0) \otimes (i_3|i_4) \ldots \otimes (i_{q-2}|i_0) \\ 0, & \text{all other basis elements c} \end{cases}$$

The first set of classes are unaffected by annihilating $(i_0|i_1|i_0)$. The second set of classes are precisely those classes which annihilate $\phi_{i_0,i_1,\ldots,i_0}$, which also vanish when we annihilate $(i_0|i_1|i_0)$. Thus, the same argument implies that $HH^2(\widetilde{A}, \widetilde{A}) = 0$.

It follows that after idempotent completion, $Fuk(Y_\epsilon, z) \cong MF(\overline{X}, v)$. From here, we note that the divisor D defines a section of the anti-canonical line bundle $\mathcal{K}^{-1}$ on $\overline{X}$ and hence a natural transformation. This natural transformation is given by taking a matrix factorization P, tensoring it with $\mathcal{K}$, and using the section defining our divisor $\mathcal{D}$ to define a morphism:

$$P \otimes \mathcal{K} \to P$$

Lemma 2.3.4. *$MF(\widetilde{X}, v)$ is the localization of $MF(\overline{X}, v)$ with respect to this natural transformation. This natural transformation is mirror to that induced by the inclusion of the fibre $Y_z \subset Y_\epsilon$.*

To begin, we observe that the functor

$$\pi : MF(\overline{X}, v) \to MF(\widetilde{X}, v)$$

is essentially surjective because coherent sheaves extend from open subsets. Next, let $\mathcal{C}$ be a category, F a functor and $N : F \to id$ a natural transformation. Let π denote the functor

$$\mathcal{C} \to \mathcal{C}_{loc}$$

where $\mathcal{C}_{loc}$ denotes the localization with respect to N. Then we have that:

$$\varinjlim Hom_{\mathcal{C}}(F^p(X), Y) \cong Hom_{\mathcal{C}_{loc}}(\pi(X), \pi(Y))$$

Recall that given two matrix factorizations, we can compute $Hom(E, F)$ by $R\Gamma(\mathcal{H}om(E, F))$. The result now follows since:

$$R\Gamma(\mathcal{H}om(E,F)|_{\tilde{X}}) \cong \varinjlim R\Gamma(\mathcal{H}om(E, F \otimes K^{-p}))$$

To show that the natural trasformations are equivalent, note that the natural transformations are trivial restricted to the $\mathcal{O}_{Z_i}$. Thus, we only have to consider the object $\mathcal{O}_{\bar{A}}$. By the matrix factorization calculations mentioned above, we have $Hom^0(\mathcal{O}_{\bar{A}}, \mathcal{O}_{\bar{A}}(1)) \cong \mathbb{C}^2$. Up to scalar multiplication, non-zero morphisms in this space are paramaterized by $\mathbb{C}P^1$. All the different points give rise to isomorphic localizations except the point corresponding to $\bar{A} \cap Z_1$. On the mirror side, there is a morphism in wrapped Floer cohomology from the thimble to the first matching sphere, which rules out this possibility.

To demonstrate the non-trivial nature of the wrapped Fukaya category of the plumbing, one can compute directly the endomorphisms of the Lefschetz thimble in the wrapped category. For example, the thimble at the beginning of the chain discussed above has endomorphism algebra:

$$\mathbb{C}[u] \otimes \Lambda(e) \quad m_{n+1}(e, e, e..., e) = u^n$$

We conclude our thesis by stating the following proposition, which the reader can check by direct computation.

Lemma 2.3.5. *The curved deformations corresponding to u^j compactify the category.*

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