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Performance analysis of multi-carrier modulation systems

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Performance Analysis of Multi-Carrier Modulation Systems

A dissertation submitted in partial satisfaction of the
requirements for the degree
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in

Electrical Engineering (Communication Theory and Systems)

by

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2008

The dissertation of Andrew Shin-Chiun Ling is approved, and it is acceptable in quality and form for publication on microfilm:

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To God be the glory.

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ABSTRACT OF THE DISSERTATION

Performance Analysis of Multi-Carrier Modulation Systems

by

Andrew Shin-Chiun Ling

Doctor of Philosophy in Electrical Engineering

(Communication Theory and Systems)

University of California, San Diego, 2008

Professor Laurence B. Milstein, Chair

This dissertation considers the following question: "Should the sub-carriers in a multi-carrier system be spread or non-spread?" We approach this problem by comparing the theoretical bit error rates of multi-carrier direct sequence code division multiple access (MC-DS-CDMA) and multi-carrier code division multiple access (MC-CDMA) for an asynchronous uplink. To ensure a fair comparison, we constrain both schemes to the same bandwidth, information rate, and energy-per-bit. Since MC-DS-CDMA uses direct sequence spreading at each sub-carrier, while each MC-CDMA sub-carrier is unspread, MC-CDMA employs a larger number of sub-carriers than MC-DS-CDMA over a given bandwidth. As a result, there potentially exists between the two schemes a trade-off between diversity gain and channel estimation errors. Initially, we compare MC-DS-CDMA and MC-CDMA under a single-input single-output (SISO) system and for two different channel scenarios (i.e., two different cases for the coherence bandwidth of the channel), but we then extend the comparison to a multiple-input multiple-output (MIMO) system

employing dual transmit/receive diversity and space-time block coding. For both cases, we derive closed-form expressions of the bit error rates for both multi-carrier schemes, and we compare the numerical results for different information rates, number of users, and number of pilot symbols per channel estimate to determine the performance trade-offs that may exist between the two schemes. We also compare the MIMO results against those of the SISO system to determine the impact of additional diversity on these trade-offs.

Chapter 1

Introduction

Multi-carrier systems have received considerable interest over the years because of their ability to support high data rates and to effectively combat a frequency-selective multipath fading channel. Several multi-carrier schemes[1] have been proposed, but two of which have been studied extensively are multi-carrier direct sequence code division multiple access (MC-DS-CDMA) [1–7] and multi-carrier code division multiple access (MC-CDMA) [8–16]. MC-DS-CDMA involves the parallel transmission of several narrowband DS-CDMA waveforms over multiple sub-carriers. These sub-carriers can be used to achieve either (1) frequency division multiplexing (i.e., each sub-carrier is used to transmit a different data symbol), (2) frequency diversity (i.e., each sub-carrier is used to transmit the same data symbol) [2, 5], or (3) some combination of the two [1, 3, 4]. At each sub-carrier, each input data symbol is modulated by multiple chips of a pseudonoise (PN) sequence or spreading sequence,[2] where the chip rate is higher than the symbol rate. This operation results in an increase in the sub-carrier bandwidth by a factor known as the processing gain. In MC-CDMA, on the other hand, each symbol in the input data sequence is transmitted

[1] In this dissertation we will use the words "system" and "scheme" interchangeably to describe MC-DS-CDMA and MC-CDMA.

[2] In the literature, this is sometimes referred to as time-domain spreading.

over multiple sub-carriers. Each repetition of a symbol is modulated by a different chip in the spreading sequence,[3] such that the processing gain is defined as the number of sub-carriers over which a bit is repeated. Since the input symbol at each sub-carrier is modulated by only a single chip, the chip rate is equal to the symbol rate, and the "spreading" operation leaves the sub-carrier bandwidth unchanged. As in MC-DS-CDMA, multiple data symbols may also be transmitted in parallel over the sub-carriers, but this requires a reduction in the processing gain per data symbol.

1.1 MC-DS-CDMA vs. MC-CDMA

Given the above discussion, we conclude that the main difference between MC-DS-CDMA and MC-CDMA, from the perspective of the frequency domain, is in the widths of their sub-bands. Any systems engineer looking to incorporate multi-carrier signaling into his/her design is thus faced with the decision of whether to use spread or non-spread sub-carriers. This decision can be aided by examining plots comparing the bit error rates of MC-DS-CDMA and MC-CDMA. There exist several works in the literature [17–23] which provide such performance comparison studies of the two schemes, but to the best of our knowledge, the numerical results of these works are all based on computer simulations.

Therefore, in this dissertation, we take an analytical approach to comparing the error probability performances of MC-DS-CDMA and MC-CDMA for an asynchronous uplink. In order for this to be an "apples-to-apples" comparison, we constrain both multi-carrier schemes to the same bandwidth, information rate, and energy-per-bit. The implications of these constraints are as follows. First, given that MC-DS-CDMA uses spread sub-carriers, while the MC-CDMA sub-carriers are unspread, MC-CDMA employs a larger number of sub-carriers than MC-DS-

[3] In the literature, this is sometimes referred to as spreading in the frequency domain.

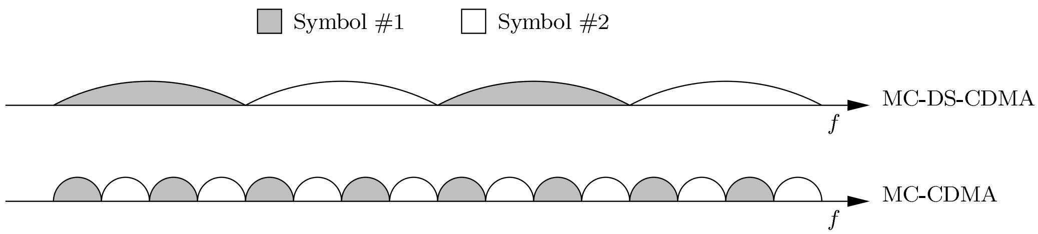

Figure 1.1: Spectra of the MC-DS-CDMA and MC-CDMA signals.

CDMA over a given bandwidth. Since both schemes are also forced to transmit at the same information rate (that is, to transmit the same number of data symbols in parallel), each data symbol is repeated across a larger number of sub-carriers in MC-CDMA than in MC-DS-CDMA (see Fig. 1.1). While, on the one hand, this means that the achievable frequency diversity gain is potentially higher in MC-CDMA, on the other hand, this also implies that the operating signal-to-noise ratio (SNR) at each sub-carrier is lower in MC-CDMA, because the energy of each data symbol is spread out over a larger number of sub-carriers.[4] This, in turn, implies that MC-CDMA is more susceptible to channel estimation errors if the complex channel gains at each sub-carrier frequency in both schemes have to be estimated. Thus, given the imposed system constraints, there potentially exists between these two multi-carrier schemes a trade-off between diversity gain and channel estimation errors.

Since many variations of MC-DS-CDMA and MC-CDMA have been proposed in the literature, we now highlight the differences between conventional implementations of these two schemes and the versions used in this dissertation. Most MC-CDMA systems consist of orthogonal sub-carriers[5] with overlapping sub-bands. In contrast, there are two main types of MC-DS-CDMA systems: ones which use disjoint (non-overlapping) sub-bands [2], and ones in which the sub-bands overlap [1, 7]. In this dissertation, we consider the MC-DS-CDMA system proposed

[4] We assume the distribution of energy over the sub-carriers is uniform (i.e., there is no water-filling).
[5] The phrase "orthogonal sub-carriers" normally implies minimum sub-carrier spacing.

in [2] and studied further in [3–6]. This version of MC-DS-CDMA employs bandlimited, non-overlapping sub-bands, which are obtained by using waveform shaping at each sub-carrier and by using a larger frequency separation between successive sub-carriers. As in [3, 4], we allow for the transmission of multiple parallel data streams, wherein higher information rates can be achieved at the expense of frequency diversity. At the receiver, the diversity branches of each of these data streams are combined via maximal ratio combining (MRC). It is well-known that MRC is the optimal combining scheme for a single user operating in an additive white Gaussian noise (AWGN) channel with no interference; in the presence of multiple access interference (MAI), however, MRC yields sub-optimal performance.[6] This latter detail is actually not a concern to us in this work, because our main goal is not to optimize the performances of the two multi-carrier schemes, but rather to analyze the bit error rates of MC-DS-CDMA and MC-CDMA under equal bandwidth, information rate, and energy-per-bit constraints, and to study the performance trade-offs that may exist between the two schemes. With this in mind, we construct an MC-CDMA system that matches our MC-DS-CDMA system as closely as possible, yet maintains the bare essence of MC-CDMA—that each sub-carrier is modulated by only a single chip in the spreading sequence, and that each repetition of a data symbol is modulated by a different chip. Thus, like the MC-DS-CDMA system we just described, our version of MC-CDMA also consists of bandlimited, non-overlapping sub-bands and employs MRC at the receiver as well.

The final element in our "apples-to-apples" comparison between MC-DS-CDMA and MC-CDMA that we need to consider is the channel estimation scheme. Given the use of MRC in both systems, knowledge of the complex channel gains at each sub-carrier frequency is required at the receiver. In practice, these channel gains have to be estimated, and the most common approach is to derive the channel estimates from transmitted pilot symbols. The penalty incurred

[6] Multi-user detection schemes can attain performances which approach those of the single-user case [24], but such schemes are difficult to analyze without resorting to computer simulations.

from transmitting these pilot symbols is a loss in the data rate. This loss can be minimized by making the distribution of pilot symbols in the time-frequency domain as sparse as possible, and by using interpolation to obtain estimates of the channel gains at the remaining frequencies and time slots [14–16]. Such a scheme, however, assumes that the channel is correlated in both frequency and time, and will not be able to give accurate channel estimates in a scenario where either the sub-bands fade independently or the channel changes rapidly. In this dissertation, we assume that neither system has knowledge of the coherence bandwidth of the channel; therefore, both systems are forced to transmit pilot symbols at each sub-carrier frequency. We also assume the channel is slowly-varying. Since we are not concerned with maximizing the throughput of the multi-carrier systems, we can improve the quality of the channel estimates by simply transmitting additional pilot symbols at each sub-carrier (i.e., use a denser distribution of pilot symbols in the time-frequency domain). At the receiver, we then take a sample average of the pilot outputs at each sub-carrier to form the corresponding estimate of the channel gain.

1.2 Outline of Dissertation

In Chapter 2, we compare MC-DS-CDMA and MC-CDMA for a single-input single-output (SISO) system. Two different channel scenarios are considered—we assume the coherence bandwidth of the channel is equal to the bandwidth of one MC-CDMA sub-band in Scenario #1, and is equal to the bandwidth of one MC-DS-CDMA sub-band in Scenario #2. Scenarios #1 and #2 represent the most commonly analyzed channel scenarios in the literature for MC-CDMA and MC-DS-CDMA, respectively, because they allow for the assumption of independent fading across the MC-CDMA sub-bands (Scenario #1) or the MC-DS-CDMA sub-bands (Scenario #2). We derive closed-form expressions of the bit error probabilities for both schemes under both channel scenarios, and we compare the numerical results for different information rates, number of users, and number of pilot symbols per channel estimate to determine the performance trade-offs that may exist between the two schemes.

In Chapter 3, we extend the comparison to a multiple-input multiple-output (MIMO) system employing dual transmit/receive diversity. We use Alamouti space-time block coding [25,26] at each sub-carrier of both schemes to achieve transmit diversity in the absence of channel state information (CSI) at the transmitter. By assuming (1) the coherence bandwidth of the channel is equal to the bandwidth of one MC-DS-CDMA sub-band, and (2) the number of data symbols transmitted in parallel, M, is such that $M > 1$, we consider only those cases where MC-CDMA has higher frequency diversity than MC-DS-CDMA, because these are the only instances in which there is a performance trade-off between the two schemes. Since increases in diversity yield diminishing gains, the addition of spatial diversity to the multi-carrier comparison will benefit MC-DS-CDMA more than MC-CDMA. To determine whether these gains for MC-DS-CDMA are enough to offset the difference in frequency diversity between the two schemes, we derive closed-form expressions of their bit error probabilities, and we compare the MIMO results against those of the SISO system for different information rates, number of users, and number of pilot symbols per channel estimate.

Finally, in Appendices A through W, we present all of the mathematical derivations used to arrive at the closed-form bit error rate expressions given in Chapters 2 and 3. Two appendices that are worthy of note are Appendix A, in which we derive closed-form expressions for the coefficients of a general partial fraction expansion, and Appendix D, in which we derive a closed-form expression for the probability of error of a general Hermitian quadratic form in *zero-mean* complex Gaussian random variables.

1.3 Notation

Throughout the rest of this dissertation, we use the subscript '1' to denote variables associated with MC-CDMA, and the subscript '2' to denote variables associated with MC-DS-CDMA. For a complex variable z, its complex conjugate is denoted by z^*. We denote column vectors by lowercase boldface letters (e.g., $\mathbf{x}$), and matrices by uppercase boldface letters (e.g., $\mathbf{X}$). The transpose and complex conjugate transpose (Hermitian) of $\mathbf{X}$ are denoted by $\mathbf{X}^T$ and $\mathbf{X}^H$, respectively. We define $\mathbf{I}_N$ and $\mathbf{0}_N$ as the $N \times N$ identity and zero matrices, respectively; also, we define $\mathbf{A} \otimes \mathbf{B}$ as the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$. Finally, we have $\delta_{n,m} \triangleq \delta[n-m]$, where $\delta[n]$ is the Kronecker delta function.

Chapter 2

MC-DS-CDMA vs. MC-CDMA in the Absence of Spatial Diversity

2.1 Introduction

In this chapter, we compare the error probability performances of MC-DS-CDMA and MC-CDMA for a SISO system. Two different channel scenarios are considered. In Scenario #1, the coherence bandwidth of the channel is assumed to equal the bandwidth of one MC-CDMA sub-band, while in Scenario #2, the coherence bandwidth of the channel is assumed to equal the bandwidth of one MC-DS-CDMA sub-band. We derive closed-form expressions for the bit error rates of both multi-carrier schemes for both channel scenarios. Using these closed-form expressions, we compare the numerical results of the two schemes for different information rates,

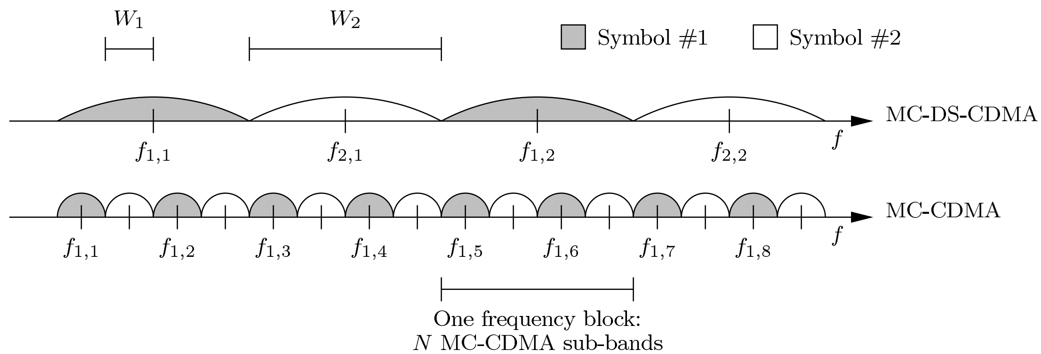

Figure 2.1: Spectra of the MC-DS-CDMA and MC-CDMA signals. In this example, $M = 2$ symbols are each repeated across $R_1 = 8$ and $R_2 = 2$ sub-carriers in MC-CDMA and MC-DS-CDMA, respectively. For Scenario #1, $(\Delta f)_c = W_1$, while for Scenario #2, $(\Delta f)_c = W_2$.

number of users, and number of pilot symbols per channel estimate to determine the performance trade-offs that may exist between the two schemes.

This chapter is organized as follows. Section 2.2 presents the system models for both MC-CDMA and MC-DS-CDMA, while Section 2.3 analyzes their performances for the two channel scenarios. We compare their numerical results in Section 2.4, and we summarize the main findings in Section 2.5.

2.2 System Models

2.2.1 Overview

Consider an asynchronous uplink with K users. In the MC-CDMA system, M BPSK symbols[1] are transmitted simultaneously in parallel across MR_1 disjoint (non-overlapping) frequency sub-bands, with each symbol repeated over R_1 sub-carriers (i.e., processing gain equals R_1). The bandwidth of each sub-band is $W_1 = 1/T_s$, where $1/T_s$ represents the symbol rate. In the MC-DS-CDMA system, the same M symbols are each repeated over R_2 sub-carriers, giving a total of MR_2 disjoint frequency sub-bands. The bandwidth of each MC-DS-CDMA sub-band

[1] Unless otherwise noted, the general term ‘symbol’ refers to both pilot and data symbols.

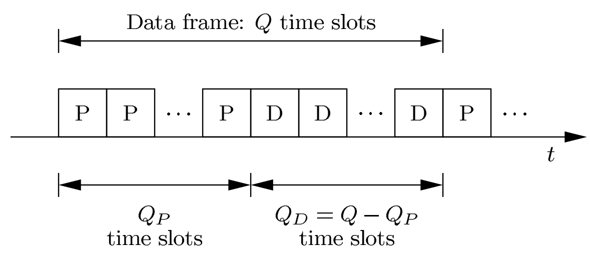

Figure 2.2: Data frame structure (P—pilot symbol, D—data symbol).

is $W_2 = 1/T_c = NW_1$, where $1/T_c = N/T_s$ is the chip rate and N is the spreading factor or processing gain in each sub-band. Constraining both systems to the same overall bandwidth, we have $MR_1W_1 = MR_2W_2$, which implies that

$$R_1 = NR_2. \tag{2.1}$$

An example of how these two systems compare in the frequency domain is given in Fig. 2.1.

In the time domain, for both systems, the transmissions made at each sub-carrier are divided into frames (see Fig. 2.2). Each frame consists of Q time slots, where the duration of a time slot is equal to T_s. Q_P of these slots are dedicated to the transmission of pilot symbols, while the remaining $Q_D = Q - Q_P$ slots are used to transmit actual data. It is assumed that the channel remains constant over the total duration of a frame. Without loss of generality, we assume the pilot symbols are transmitted during the first Q_P time slots of each frame.

2.2.2 Transmitter

The general transmitter block diagram for both systems is shown in Fig. 2.3. The input data for the k-th user is a random binary sequence with bit rate $1/T$, where each bit is equally likely to be $+1$ or -1. The data sequences for the K users are assumed to be independent. The concatenation of the pilot insertion, serial-to-parallel (S/P) conversion, and symbol mapper blocks allows for the transmission of pilot symbols ($+1$'s) at each sub-carrier during the first Q_P

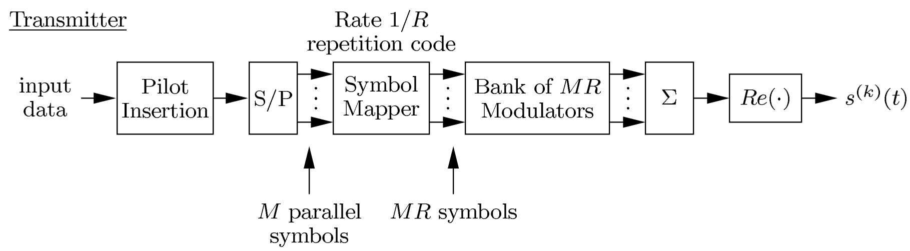

Figure 2.3: General transmitter block diagram.

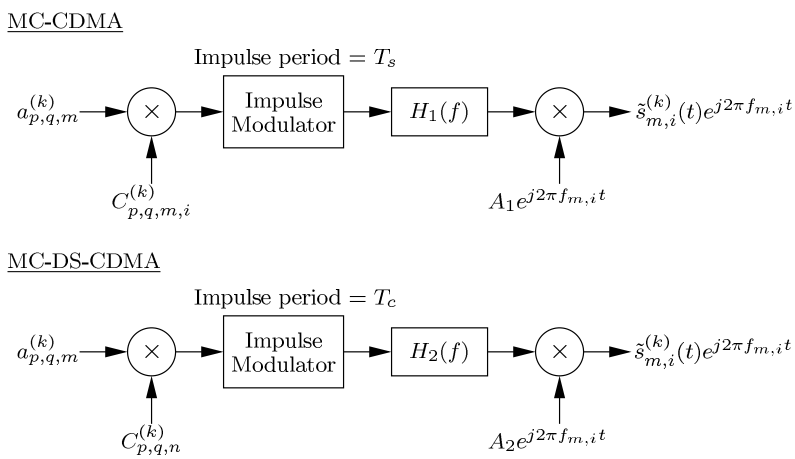

Figure 2.4: MC-CDMA and MC-DS-CDMA modulators.

time slots of each frame, as well as for the transmission of M parallel data symbols over MR sub-carriers during the remaining Q_D time slots, where $R = R_1$ for MC-CDMA and $R = R_2$ for MC-DS-CDMA. We label the M symbols at the output of the S/P converter as $\{a^{(k)}_{p,q,m}\}^M_{m=1}$, where p is the frame index, q is the time slot index, and $T_s = (Q_D/Q)MT$. Note that $a^{(k)}_{p,q,m}$ represents a pilot symbol for $q = 1, \ldots, Q_P$ and a data symbol for $q = Q_P + 1, \ldots, Q$. The R repetitions of each of the M symbols are mapped to the sub-carriers in such a way that the frequency separation between adjacent repetitions of a symbol is maximized, with the i-th repetition of the m-th symbol mapped to frequency $f_{m,i}$ $(m = 1, \ldots, M; i = 1, \ldots, R)$.

For both systems, the block diagrams of the modulators at frequency $f_{m,i}$ are shown in

Fig. 2.4. During each time slot, $a^{(k)}_{p,q,m}$ is multiplied by N chips from the k-th user's spreading sequence, $\{C^{(k)}_{p,q,n}\}^{N}_{n=1}$, in MC-DS-CDMA; in MC-CDMA, $a^{(k)}_{p,q,m}$ is multiplied by only a single chip, $C^{(k)}_{p,q,m,i}$. After passing through an impulse modulator and a chip wave-shaping filter, denoted by either $H_1(f)$ or $H_2(f)$, the signal is then modulated onto the corresponding sub-carrier. The transmitted signal for the k-th user is given by

$$s^{(k)}(t) = \sum_{m=1}^{M} \sum_{i=1}^{R} \mathrm{Re}\left\{ \tilde{s}^{(k)}_{m,i}(t) e^{j2\pi f_{m,i} t} \right\}, \tag{2.2}$$

where

$$\tilde{s}^{(k)}_{m,i}(t) = A_1 \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} a^{(k)}_{p,q,m}\, C^{(k)}_{p,q,m,i}\, h_1\big[t - (pQ + q - 1)T_s\big], \tag{2.3}$$

$$\tilde{s}^{(k)}_{m,i}(t) = A_2 \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} \sum_{n=1}^{N} a^{(k)}_{p,q,m}\, C^{(k)}_{p,q,n}\, h_2\big\{t - [(pQ + q - 1)N + n - 1]T_c\big\}, \tag{2.4}$$

for MC-CDMA and MC-DS-CDMA, respectively. In the above equations, A_1 and A_2 are the transmit amplitudes, and $h_1(t)$ and $h_2(t)$ are the impulse responses of the chip wave-shaping filters. All K users are assumed to use long spreading sequences. The spreading sequence of the desired user ($k = 1$) is known, while the spreading sequences of the $K - 1$ interfering users are modeled as independent random binary sequences of ± 1's.

2.2.3 Channel: Scenario #1

MC-CDMA

For Scenario #1, we have $(\Delta f)_c = W_1$. We assume frequency non-selective (flat) fading in each sub-band, with independent fading between the different sub-bands. Since each output of the serial-to-parallel converter is repeated across R_1 sub-carriers, each data symbol experiences frequency diversity of order R_1. Assuming perfect power control, the received signal is given by

$$r(t) = \sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{i=1}^{R_1} \mathrm{Re}\left\{ \tilde{r}^{(k)}_{m,i}(t) e^{j2\pi f_{m,i} t} \right\} + n_W(t), \tag{2.5}$$

where

$$\tilde{r}_{m,i}^{(k)}(t) = A_1 \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} a_{p,q,m}^{(k)}\, C_{p,q,m,i}^{(k)}\, g_{m,i,q}^{(k)}\, h_1 \left[t - (pQ + q - 1)T_s - \tau^{(k)}\right]. \tag{2.6}$$

In the above equations, the $\{\tau^{(k)}\}$, which represent the relative time delays of each user with respect to the desired user ($k = 1$), are independent and identically distributed (i.i.d.) uniform random variables over $[0, T_s)$, $n_W(t)$ is additive white Gaussian noise (AWGN) with power spectral density $N_0/2$, and the zero-mean circularly symmetric complex Gaussian random variable $g_{m,i,q}^{(k)} \triangleq \alpha_{m,i,q}^{(k)} e^{j\theta_{m,i,q}^{(k)}}$ represents the transfer function of the channel for the k-th user at frequency $f_{m,i}$ during the q-th time slot, where the $\{\alpha_{m,i,q}^{(k)}\}$ are i.i.d. Rayleigh random variables with unit second moment, and the $\{\theta_{m,i,q}^{(k)}\}$ are i.i.d. uniform random variables over $[0, 2\pi)$. Since the channel is assumed to remain constant over the duration of a frame, we will drop the index q from $g_{m,i,q}^{(k)}$ and simply write $g_{m,i}^{(k)} = \alpha_{m,i}^{(k)} e^{j\theta_{m,i}^{(k)}}$.

MC-DS-CDMA

With $(\Delta f)_c = W_1 < W_2$, the fading in each sub-band is assumed to be frequency-selective, where the number of independent, resolvable paths arriving at the receiver in each sub-band is given by [27, p. 841]

$$L \approx \frac{W_2}{(\Delta f)_c} = N. \tag{2.7}$$

With a combination of both path and frequency diversity, the overall order of diversity experienced by each of the M data symbols is equal to $R_2 L = R_2 N = R_1$. The performance will depend on the multipath intensity profile[2] (MIP) of the channel in each sub-band,

$$\Omega_{m,i,l}^{(k)} \triangleq E[(\alpha_{m,i,l}^{(k)})^2], \quad l = 1, \ldots, L, \tag{2.8}$$

where $\sum_{l=1}^{L} \Omega_{m,i,l}^{(k)} = 1$. In particular, we will consider the uniform (flat) MIP,

$$\Omega_{m,i,l}^{(k)} = \frac{1}{L}, \quad l = 1, \ldots, L, \tag{2.9}$$

[2] Also commonly referred to as the power delay profile.

and the exponential (exponentially-decaying) MIP,

$$\Omega^{(k)}_{m,i,l} = \Omega^{(k)}_{m,i,1} e^{-r_d(l-1)}, \quad l = 1, \ldots, L, \tag{2.10}$$

where

$$\Omega^{(k)}_{m,i,1} = \frac{1 - e^{-r_d}}{1 - e^{-r_d L}} \tag{2.11}$$

is the average power of the initial path, and r_d is the decay rate of the average power.

For the k-th user, the complex lowpass equivalent impulse response of the channel at frequency $f_{m,i}$ can be written as

$$\zeta^{(k)}_{m,i}(t) = \sum_{l=1}^{L} g^{(k)}_{m,i,l}\, \delta[t - (l-1)T_c], \tag{2.12}$$

where $g^{(k)}_{m,i,l} \triangleq \alpha^{(k)}_{m,i,l} e^{j\theta^{(k)}_{m,i,l}}$ represents the complex channel gain along the l-th path and is a zero-mean circularly symmetric complex Gaussian random variable. The $\{\alpha^{(k)}_{m,i,l}\}$ are independent Rayleigh random variables whose distributions depend on the MIP in each sub-band, and the $\{\theta^{(k)}_{m,i,l}\}$ are i.i.d. uniform random variables over $[0, 2\pi)$. With perfect power control, the received signal is given by

$$r(t) = \sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{i=1}^{R_2} \sum_{l=1}^{L} \mathrm{Re}\left\{ \tilde{r}^{(k)}_{m,i,l}(t) e^{j2\pi f_{m,i} t} \right\} + n_W(t), \tag{2.13}$$

where

$$\begin{aligned} \tilde{r}^{(k)}_{m,i,l}(t) = A_2 \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} \sum_{n=1}^{N} a^{(k)}_{p,q,m}\, C^{(k)}_{p,q,n}\, g^{(k)}_{m,i,l} \\ \cdot h_2 \left\{ t - [(pQ + q - 1)N + n - 1]T_c - (l-1)T_c - \tau^{(k)} \right\}. \end{aligned} \tag{2.14}$$

Here, the $\{\tau^{(k)}\}$ are i.i.d. uniform random variables over $[0, T_c)$.

2.2.4 Channel: Scenario #2

MC-CDMA

Since $(\Delta f)_c = W_2 > W_1$, there is also flat fading in each sub-band as in Scenario #1, but the fading between different sub-bands may be highly correlated. For this scenario, we apply a *correlated block fading* model. In this model, the MR_1 MC-CDMA sub-bands are divided into MR_1/N frequency blocks, where each block of N contiguous sub-bands corresponds to one MC-DS-CDMA sub-band, and there are

$$R_b \triangleq \frac{N}{M} \tag{2.15}$$

repetitions of each symbol within a block (see Fig. 2.1). The fading is assumed to be flat across each block, such that the fade amplitudes associated with the N sub-bands are identical (i.e., perfectly correlated). On the other hand, the fade amplitudes associated with two sub-bands from different blocks are assumed to be independent. As a result, even though each of the M symbols is repeated across R_1 sub-carriers, the effective order of frequency diversity per data symbol is only MR_1/N, the number of frequency blocks. Assuming perfect power control, the expression for the received signal is also given by (2.5)–(2.6), but the joint distribution of the $\{g_{m,i}^{(k)}\}$ is different from that of Scenario #1.

MC-DS-CDMA

With $(\Delta f)_c = W_2$, we assume flat fading in each sub-band and independent fading between different sub-bands. Since the R_2 repetitions of each symbol fade independently, each of the M data symbols experiences frequency diversity of order R_2. With only one resolvable path arriving at the receiver in each sub-band, the expression for the received signal can be obtained from (2.13)–(2.14) by making the substitution $L = 1$ and by dropping the dependence on the

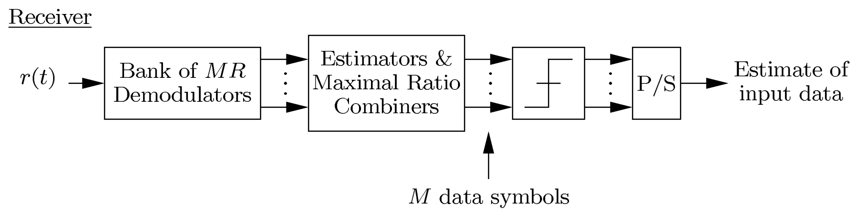

Figure 2.5: General receiver block diagram.

path index l in all of the variables. That is, we have

$$r(t) = \sum_{k=1}^{K} \sum_{m=1}^{M} \sum_{i=1}^{R_2} \mathrm{Re}\left\{ \tilde{r}_{m,i}^{(k)}(t) e^{j2\pi f_{m,i} t} \right\} + n_W(t), \tag{2.16}$$

where

$$\begin{aligned} \tilde{r}_{m,i}^{(k)}(t) = A_2 \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} \sum_{n=1}^{N} a_{p,q,m}^{(k)} \, C_{p,q,n}^{(k)} \, g_{m,i}^{(k)} \\ \cdot h_2 \left\{ t - [(pQ + q - 1)N + n - 1]T_c - \tau^{(k)} \right\}. \end{aligned} \tag{2.17}$$

2.2.5 Receiver

From Fig. 2.5, the receiver for both systems first consists of a bank of MR demodulators, where $R = R_1$ for MC-CDMA and $R = R_2$ for MC-DS-CDMA. At each of these demodulators, the received signal is bandpass matched filtered, down-converted to baseband, lowpass filtered, sampled, and then de-spread (see Fig. 2.6). The de-spreading operation at each sub-carrier involves multiplication by a single chip in MC-CDMA and multiplication by N chips in MC-DS-CDMA. For the MC-CDMA system, the same demodulator is used for both channel scenarios, because the fading in each sub-band is flat in either case. On the other hand, for the MC-DS-CDMA system, a RAKE demodulator with L fingers is used at each sub-carrier in Scenario #1 to capture the energy in the L paths, but only one finger is needed in Scenario #2 to capture the energy in the single path. It is assumed that each finger in the RAKE is perfectly synchronized with one of the arriving paths. Even though the receiver has no knowledge of the coherence

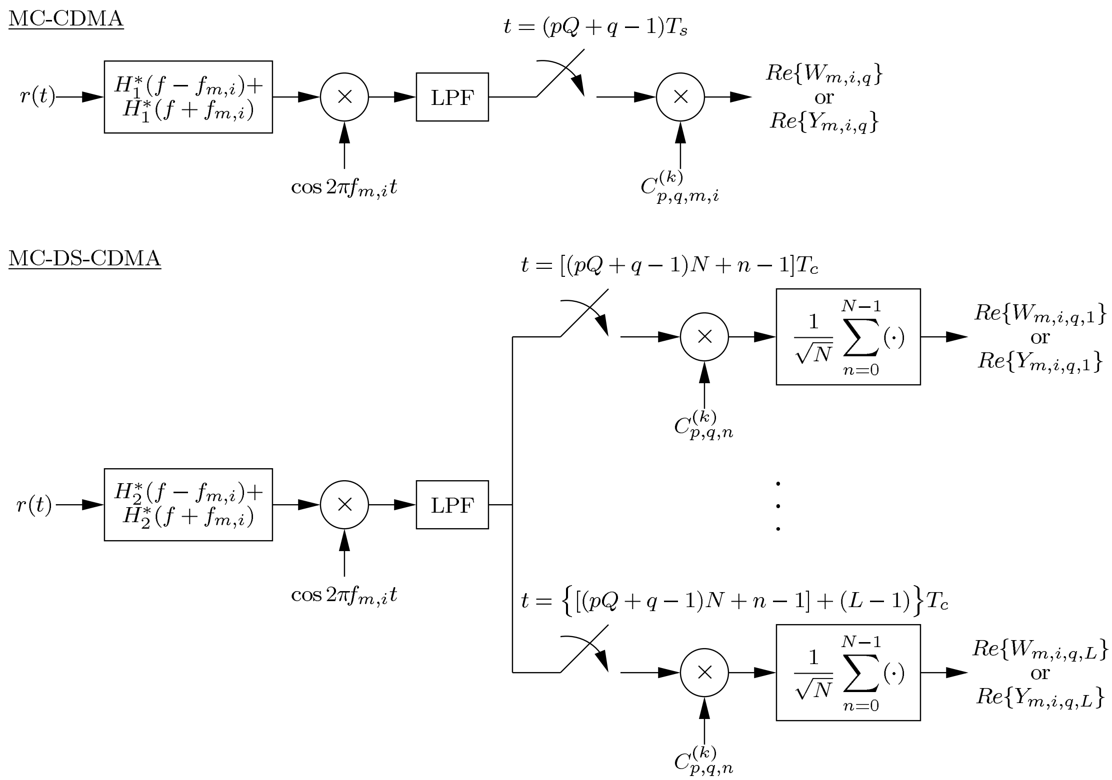

Figure 2.6: MC-CDMA and MC-DS-CDMA demodulators. The MC-DS-CDMA demodulator for Scenario #2 consists of only one finger.

bandwidth of the channel, we assume that the RAKE receiver is able to determine the number of paths that are present by using a procedure as described in [28, Ch. 6], the key step of which is to compare the strength of the signal in each finger to a predetermined threshold.

While the bandpass versions of the demodulators are presented in Fig. 2.6, we will actually work with complex equivalent lowpass signals in our analysis. The complex equivalent lowpass impulse responses of the bandpass filters are given by

$$\tilde{h}_{m,i,1}(t) = h_1^*(-t), \tag{2.18}$$

$$\tilde{h}_{m,i,2}(t) = h_2^*(-t). \tag{2.19}$$

The wave-shaping filters $H_1(f)$ and $H_2(f)$ are assumed to satisfy the Nyquist criterion and are of unit energy, i.e.,

$$\int_{-\infty}^{\infty} |H_1(f)|^2 \, df = 1, \tag{2.20}$$

$$\int_{-\infty}^{\infty} |H_2(f)|^2 \, df = 1. \tag{2.21}$$

We define

$$x_1(t) \Leftrightarrow X_1(f) \triangleq |H_1(f)|^2, \tag{2.22}$$

$$x_2(t) \Leftrightarrow X_2(f) \triangleq |H_2(f)|^2, \tag{2.23}$$

and assume

$$X_1(f) = \begin{cases} T_s, & -\frac{1}{2T_s} < f < \frac{1}{2T_s}, \\ 0, & \text{otherwise}, \end{cases} \Leftrightarrow \; x_1(t) = \frac{\sin \pi t/T_s}{\pi t/T_s}, \tag{2.24}$$

$$X_2(f) = \begin{cases} T_c, & -\frac{1}{2T_c} < f < \frac{1}{2T_c}, \\ 0, & \text{otherwise}, \end{cases} \Leftrightarrow \; x_2(t) = \frac{\sin \pi t/T_c}{\pi t/T_c}. \tag{2.25}$$

Since each sub-band is band-limited and there is no overlap between adjacent sub-bands, interference from adjacent channels is ignored. Furthermore, since $X_1(f)$ and $X_2(f)$ also satisfy the Nyquist criterion, inter-chip and inter-symbol interferences are ignored as well.

Going back to the receiver block diagram in Fig. 2.5, the total number of test statistics at the output of the bank of demodulators is equal to MR_1 for MC-CDMA, $MR_2L = MR_1$ for MC-DS-CDMA in Scenario #1, and MR_2 for MC-DS-CDMA in Scenario #2. For each of these outputs, the estimate of the corresponding channel gain is obtained by taking a sample average of the output test statistics during the first Q_P time slots of the frame. These channel estimates are then used as the weighting coefficients in the M maximal ratio combiners (MRCs) during the remaining Q_D time slots of the frame when data is actually transmitted. Hard decisions are made on the M MRC outputs, and parallel-to-serial conversion is finally performed to obtain the estimate of the input data sequence.

2.3 Performance Analysis

2.3.1 Output Test Statistic: MC-CDMA (Scenarios #1 and #2)

We derive the final test statistic for the m-th symbol of the desired user ($k = 1$), assuming a $+1$ was transmitted. Also, we assume perfect carrier and symbol synchronization. At frequency $f_{m,i}$, we label the complex equivalent lowpass versions of the demodulator outputs associated with the pilot and data symbols as $W_{m,i,q}$ ($q = 1, \ldots, Q_P$) and $Y_{m,i,q}$ ($q = Q_P + 1, \ldots, Q$), respectively. We can show that

$$W_{m,i,q} = A_1\, g_{m,i}^{(1)} + I_{W_{m,i,q}}^{(P)} + I_{W_{m,i,q}}^{(D)} + N_{W_{m,i,q}}, \quad q = 1, \ldots, Q_P, \tag{2.26}$$

$$Y_{m,i,q} = A_1\, g_{m,i}^{(1)} + I_{Y_{m,i,q}}^{(P)} + I_{Y_{m,i,q}}^{(D)} + N_{Y_{m,i,q}}, \quad q = Q_P + 1, \ldots, Q, \tag{2.27}$$

where the four terms in both equations represent the desired signal, multiple access interference (MAI) due to the pilot symbols of the interfering users, MAI due to the data of the interfering users, and AWGN, respectively. The MAI and AWGN terms for $q = 1, \ldots, Q_P$ are given by

$$\begin{aligned} I_{W_{m,i,q}}^{(P)} &= A_1\, C_{p,q,m,i}^{(1)} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q_P} C_{p',q',m,i}^{(k)}\, g_{m,i}^{(k)} \\ &\quad \cdot x_1 \left[t - (p'Q + q' - 1)T_s - \tau^{(k)}\right] \Big|_{t=[(pQ+q-1)T_s]}, \end{aligned} \tag{2.28}$$

$$\begin{aligned} I_{W_{m,i,q}}^{(D)} &= A_1\, C_{p,q,m,i}^{(1)} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=Q_P+1}^{Q} a_{p',q',m}^{(k)}\, C_{p',q',m,i}^{(k)}\, g_{m,i}^{(k)} \\ &\quad \cdot x_1 \left[t - (p'Q + q' - 1)T_s - \tau^{(k)}\right] \Big|_{t=[(pQ+q-1)T_s]}, \end{aligned} \tag{2.29}$$

$$N_{W_{m,i,q}} = C_{p,q,m,i}^{(1)} \cdot \left[\tilde{n}_W(t) * \tilde{h}_{m,i,1}(t)\right] \Big|_{t=[(pQ+q-1)T_s]}. \tag{2.30}$$

Similarly, for $q = Q_P + 1, \ldots, Q$, we have

$$I^{(P)}_{Y_{m,i,q}} = A_1 \, C^{(1)}_{p,q,m,i} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q_P} C^{(k)}_{p',q',m,i} \, g^{(k)}_{m,i} \cdot x_1 \left[t - (p'Q + q' - 1)T_s - \tau^{(k)} \right] \Big|_{t=[(pQ+q-1)T_s]}, \tag{2.31}$$

$$I^{(D)}_{Y_{m,i,q}} = A_1 \, C^{(1)}_{p,q,m,i} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=Q_P+1}^{Q} a^{(k)}_{p',q',m} \, C^{(k)}_{p',q',m,i} \, g^{(k)}_{m,i} \cdot x_1 \left[t - (p'Q + q' - 1)T_s - \tau^{(k)} \right] \Big|_{t=[(pQ+q-1)T_s]}, \tag{2.32}$$

$$N_{Y_{m,i,q}} = C^{(1)}_{p,q,m,i} \cdot \left[\tilde{n}_W(t) * \tilde{h}_{m,i,1}(t) \right] \Big|_{t=[(pQ+q-1)T_s]}. \tag{2.33}$$

In the above equations, $\tilde{n}_W(t)$ is zero-mean complex white Gaussian noise with two-sided power spectral density N_0, and we have assumed, without loss of generality, that $\tau^{(1)} = 0$. The MAI and AWGN terms are assumed to be independent, and for large K, the MAI terms are approximated as zero-mean complex Gaussian random variables. Since the desired signal and AWGN terms are zero-mean complex Gaussian as well, we therefore approximate $W_{m,i,q}$ and $Y_{m,i,q}$ as zero-mean complex Gaussian random variables. Note that $W_{m,i,q}$ and $Y_{m,i,q}$ are exactly Gaussian when $K = 1$.

From Appendix F, the minimum variance unbiased (MVU) estimate of $g^{(1)}_{m,i}$ is given by

$$\hat{W}_{m,i} = \frac{1}{A_1} \left(\frac{1}{Q_P} \sum_{q=1}^{Q_P} W_{m,i,q} \right). \tag{2.34}$$

Since $\hat{W}_{m,i}$ is a linear combination of the $\{W_{m,i,q}\}_{q=1}^{Q_P}$, $\hat{W}_{m,i}$ is also (approximately) zero-mean complex Gaussian. It is shown in Appendix F that the minimum mean square error (MMSE) estimate of $g^{(1)}_{m,i}$ is just a scaled version of $\hat{W}_{m,i}$, and that both the MVU and MMSE estimators yield the same probability of error. Henceforth, we will only consider the MVU estimator. The estimates of $g^{(1)}_{m,i}$ are used as the weights in the MRC, such that the MRC output during the q-th time slot ($q = Q_P + 1, \ldots, Q$) is equal to

$$Z_{m,q} = \sum_{i=1}^{R_1} \mathrm{Re} \left\{ \hat{W}^*_{m,i} Y_{m,i,q} \right\} = \sum_{i=1}^{R_1} \left\{ \frac{1}{2} \hat{W}^*_{m,i} Y_{m,i,q} + \frac{1}{2} \hat{W}_{m,i} Y^*_{m,i,q} \right\}, \tag{2.35}$$

Note that (2.35) represents the final test statistic for both channel scenarios. For a given i, $\hat{W}_{m,i}$ and $Y_{m,i,q}$ are correlated because the same fade amplitude is associated with both variables. The R_1 pairs $\{\hat{W}_{m,i}, Y_{m,i,q}\}_{i=1}^{R_1}$ are i.i.d. in Scenario #1 since the fading is independent between different sub-bands, but they may be correlated in Scenario #2 due to the correlated block fading.

2.3.2 Output Test Statistic: MC-DS-CDMA (Scenario #1)

For the analysis of the MC-DS-CDMA system, perfect carrier, chip, and path synchronization are assumed. The pilot and data outputs of the l-th RAKE finger at frequency $f_{m,i}$ are given by

$$W_{m,i,q,l} = A_2\sqrt{N}\, g_{m,i,l}^{(1)} + K_{W_{m,i,q,l}} + I_{W_{m,i,q,l}}^{(P)} + I_{W_{m,i,q,l}}^{(D)} + N_{W_{m,i,q,l}}, \quad q = 1, \ldots, Q_P, \quad (2.36)$$

and

$$Y_{m,i,q,l} = A_2\sqrt{N}\, g_{m,i,l}^{(1)} + K_{Y_{m,i,q,l}} + I_{Y_{m,i,q,l}}^{(P)} + I_{Y_{m,i,q,l}}^{(D)} + N_{Y_{m,i,q,l}}, \quad q = Q_P + 1, \ldots, Q, \quad (2.37)$$

respectively, where the terms $K_{W_{m,i,q,l}}$ and $K_{Y_{m,i,q,l}}$ represent the self-interference due to the other $L-1$ paths. For $q = 1, \ldots, Q_P$, the interference and noise terms are given by

$$\begin{aligned} K_{W_{m,i,q,l}} &= \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C_{p,q,n}^{(1)} \sum_{l'=1, l'\neq l}^{L} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q} \sum_{n'=1}^{N} a_{p',q',m}^{(1)}\, C_{p',q',n'}^{(1)}\, g_{m,i,l'}^{(1)} \\ &\quad \cdot x_2\big\{t - [(p'Q + q' - 1)N + n' - 1]T_c - (l'-1)T_c\big\}\Big|_{t=\{[(pQ+q-1)N+n-1]+(l-1)\}T_c}, \end{aligned} \quad (2.38)$$

$$\begin{aligned} I_{W_{m,i,q,l}}^{(P)} &= \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C_{p,q,n}^{(1)} \sum_{k=2}^{K} \sum_{l'=1}^{L} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q_P} \sum_{n'=1}^{N} C_{p',q',n'}^{(k)}\, g_{m,i,l'}^{(k)} \\ &\quad \cdot x_2\big\{t - [(p'Q + q' - 1)N + n' - 1]T_c - (l'-1)T_c - \tau^{(k)}\big\}\Big|_{t=\{[(pQ+q-1)N+n-1]+(l-1)\}T_c}, \end{aligned} \quad (2.39)$$

$$I^{(D)}_{W_{m,i,q,l}} = \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \sum_{k=2}^{K} \sum_{l'=1}^{L} \sum_{p'=-\infty}^{\infty} \sum_{q'=Q_P+1}^{Q} \sum_{n'=1}^{N} a^{(k)}_{p',q',m}\, C^{(k)}_{p',q',n'}\, g^{(k)}_{m,i,l'}$$
$$\cdot\, x_2 \left\{ t - [(p'Q + q' - 1)N + n' - 1]T_c - (l' - 1)T_c - \tau^{(k)} \right\} \Big|_{t=\{[(pQ+q-1)N+n-1]+(l-1)\}T_c}, \tag{2.40}$$

$$N_{W_{m,i,q,l}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \cdot \left[\tilde{n}_W(t) * \tilde{h}_{m,i,2}(t) \right] \Big|_{t=\{[(pQ+q-1)N+n-1]+(l-1)\}T_c}. \tag{2.41}$$

Identical expressions are derived for $K_{Y_{m,i,q,l}}$, $I^{(P)}_{Y_{m,i,q,l}}$, $I^{(D)}_{Y_{m,i,q,l}}$, and $N_{Y_{m,i,q,l}}$. We consider the self-interference to be negligible when either (1) there are a small number of users but N is large, or (2) the number of users is large such that the MAI will dominate. As in the MC-CDMA analysis, we approximate $W_{m,i,q,l}$ and $Y_{m,i,q,l}$ as zero-mean complex Gaussian random variables when K is large.

The MVU estimate of the channel gain along the l-th path, $g^{(1)}_{m,i,l}$, is given by

$$\hat{W}_{m,i,l} = \frac{1}{A_2\sqrt{N}} \left(\frac{1}{Q_P} \sum_{q=1}^{Q_P} W_{m,i,q,l} \right). \tag{2.42}$$

With maximal ratio combining on a total of R_2L paths, the final test statistic during the q-th time slot ($q = Q_P + 1, \ldots, Q$) is equal to

$$Z_{m,q} = \sum_{i=1}^{R_2} \sum_{l=1}^{L} \mathrm{Re}\left\{ \hat{W}^*_{m,i,l} Y_{m,i,q,l} \right\} = \sum_{i=1}^{R_2} \sum_{l=1}^{L} \left\{ \frac{1}{2} \hat{W}^*_{m,i,l} Y_{m,i,q,l} + \frac{1}{2} \hat{W}_{m,i,l} Y^*_{m,i,q,l} \right\}. \tag{2.43}$$

For a given l, the R_2 pairs $\{\hat{W}_{m,i,l}, Y_{m,i,q,l}\}_{i=1}^{R_2}$ are assumed to be i.i.d. Also, for a given i, the L pairs $\{\hat{W}_{m,i,l}, Y_{m,i,q,l}\}_{l=1}^{L}$ are assumed to be independent, but their distribution depends on the MIP.

2.3.3 Output Test Statistic: MC-DS-CDMA (Scenario #2)

For Scenario #2, to obtain expressions for the variables $W_{m,i,q}$, $Y_{m,i,q}$, etc., we simply substitute $L = 1$ into the corresponding equations for Scenario #1, drop the dependence on l in all of the variables, and ignore the self-interference terms. That is, we have

$$W_{m,i,q} = A_2\sqrt{N}\, g^{(1)}_{m,i} + I^{(P)}_{W_{m,i,q}} + I^{(D)}_{W_{m,i,q}} + N_{W_{m,i,q}}, \quad q = 1, \ldots, Q_P, \tag{2.44}$$

and

$$Y_{m,i,q} = A_2\sqrt{N}\, g^{(1)}_{m,i} + I^{(P)}_{Y_{m,i,q}} + I^{(D)}_{Y_{m,i,q}} + N_{Y_{m,i,q}}, \quad q = Q_P + 1, \ldots, Q, \tag{2.45}$$

where

$$\begin{aligned} I^{(P)}_{W_{m,i,q}} &= \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q_P} \sum_{n'=1}^{N} C^{(k)}_{p',q',n'}\, g^{(k)}_{m,i} \\ &\quad \cdot x_2\{t - [(p'Q + q' - 1)N + n' - 1]T_c - \tau^{(k)}\}\Big|_{t=[(pQ+q-1)N+n-1]T_c}, \end{aligned} \tag{2.46}$$

$$\begin{aligned} I^{(D)}_{W_{m,i,q}} &= \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=Q_P+1}^{Q} \sum_{n'=1}^{N} a^{(k)}_{p',q',m}\, C^{(k)}_{p',q',n'}\, g^{(k)}_{m,i} \\ &\quad \cdot x_2\left\{t - [(p'Q + q' - 1)N + n' - 1]T_c - \tau^{(k)}\right\}\Big|_{t=[(pQ+q-1)N+n-1]T_c}, \end{aligned} \tag{2.47}$$

$$N_{W_{m,i,q}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \cdot \left[\tilde{n}_W(t) * \tilde{h}_{m,i,2}(t)\right]\Big|_{t=[(pQ+q-1)N+n-1]T_c}. \tag{2.48}$$

Again, the expressions for $I^{(P)}_{Y_{m,i,q}}$, $I^{(D)}_{Y_{m,i,q}}$, and $N_{Y_{m,i,q}}$ are identical to (2.46)–(2.48). The final test statistic is given by

$$Z_{m,q} = \sum_{i=1}^{R_2} \mathrm{Re}\left\{\hat{W}^*_{m,i} Y_{m,i,q}\right\} = \sum_{i=1}^{R_2} \left\{\frac{1}{2}\hat{W}^*_{m,i} Y_{m,i,q} + \frac{1}{2}\hat{W}_{m,i} Y^*_{m,i,q}\right\}, \tag{2.49}$$

where

$$\hat{W}_{m,i} = \frac{1}{A_2\sqrt{N}} \left(\frac{1}{Q_P} \sum_{q=1}^{Q_P} W_{m,i,q}\right), \tag{2.50}$$

and the R_2 pairs $\{\hat{W}_{m,i}, Y_{m,i,q}\}_{i=1}^{R_2}$ are i.i.d.

2.3.4 Probability of Error: Scenario #1

MC-CDMA

From Appendix G, the probability of error for the MC-CDMA system in Scenario #1 is equal to

$$P_e = \left(\frac{1-\mu}{2}\right)^{R_1} \sum_{k=0}^{R_1-1} \binom{R_1-1+k}{k} \left(\frac{1+\mu}{2}\right)^k, \tag{2.51}$$

where

$$\mu = \frac{\sqrt{Q_P}\,\bar{\gamma}}{\sqrt{(1+Q_P\bar{\gamma})(1+\bar{\gamma})}}, \tag{2.52}$$

$$\bar{\gamma} = \left(\frac{1}{R_1}\frac{E_{b1}}{N_0}\right) \frac{1}{1+(K-1)\left(\frac{1}{R_1}\frac{E_{b1}}{N_0}\right)}. \tag{2.53}$$

In the above equations, μ represents the cross-correlation coefficient between $\hat{W}_{m,i}$ and $Y_{m,i,q}$, and

$$E_{b1} = \frac{R_1 A_2^2}{2} \tag{2.54}$$

is the energy per bit (see Appendix P.2). This result is identical to the probability of error of an ideal maximal ratio combiner with R_1-th order diversity [27, Eq. (14.4-15)], except that [27, Eq. (14.4-16)] has been replaced by our expression for μ. Note that when $Q_P = 1$, we have $\mu = \frac{\bar{\gamma}}{1+\bar{\gamma}}$, which matches the cross-correlation coefficient given in Table C-1 of [27] for differential phase signaling.

MC-DS-CDMA

Assuming a uniform MIP in each sub-band, the probability of error for the MC-DS-CDMA system is equal to (see Appendix G)

$$P_e = \left(\frac{1-\mu}{2}\right)^{R_2 L} \sum_{k=0}^{R_2 L-1} \binom{R_2 L-1+k}{k} \left(\frac{1+\mu}{2}\right)^k, \tag{2.55}$$

where

$$\mu = \frac{\sqrt{Q_P}\,\bar{\gamma}}{\sqrt{(1+Q_P\bar{\gamma})(1+\bar{\gamma})}}, \tag{2.56}$$

$$\bar{\gamma} = \frac{1}{L}\left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right)\frac{1}{1+\frac{K-1}{N}\left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right)}. \tag{2.57}$$

In the above equations, μ represents the cross-correlation coefficient between $\hat{W}_{m,i,l}$ and $Y_{m,i,q,l}$, and

$$E_{b2} = \frac{NR_2A_2^2}{2} \tag{2.58}$$

is the energy-per-bit (see Appendix P.1).

If the MIP in each sub-band is exponentially-decaying, for $Q_P = 1$, the probability of error is derived in Appendix H as

$$P_e = \sum_{l=1}^{L}\sum_{k=1}^{R_2}\tilde{b}_{l,(R_2-k+1)}\cdot\left(\frac{1-\mu_l}{2}\right)^k\sum_{n=0}^{R_2L-1}\binom{k-1+n}{n}\left(\frac{1+\mu_l}{2}\right)^n, \tag{2.59}$$

where

$$\mu_l = \frac{\bar{\gamma}_l}{1+\bar{\gamma}_l}, \tag{2.60}$$

$$\bar{\gamma}_l = \left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right)\frac{1}{1+\frac{K-1}{N}\left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right)}\,\Omega^{(1)}_{m,i,1}e^{-r_d(l-1)}, \tag{2.61}$$

and the $\{\tilde{b}_{l,k}\}$ are the coefficients of the partial fraction expansion given in (H.27). For $Q_P > 1$, we have from Appendix H that

$$\begin{aligned} P_e = &\sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2}\tilde{c}_{l_1,(R_2-k_1+1)}\tilde{d}_{l_2,(R_2-k_2+1)} \\ &\cdot\left(\frac{1-\xi_{l_1,l_2}}{2}\right)^{k_1}\sum_{n=0}^{k_2-1}\binom{k_1-1+n}{n}\left(\frac{1+\xi_{l_1,l_2}}{2}\right)^n, \end{aligned} \tag{2.62}$$

where

$$\xi_{l_1,l_2} = \frac{e^{-r_dl_1}\left(1+\frac{1}{\mu_{l_1}}\right)+e^{-r_dl_2}\left(1-\frac{1}{\mu_{l_2}}\right)}{e^{-r_dl_1}\left(1+\frac{1}{\mu_{l_1}}\right)-e^{-r_dl_2}\left(1-\frac{1}{\mu_{l_2}}\right)}, \tag{2.63}$$

$$\mu_l = \frac{\sqrt{Q_P}\,\bar{\gamma}_l}{\sqrt{(1+Q_P\bar{\gamma}_l)(1+\bar{\gamma}_l)}}, \tag{2.64}$$

and $\bar{\gamma}_l$ is as defined in (2.61). The $\{\tilde{c}_{l,k}\}$ and $\{\tilde{d}_{l,k}\}$ are the coefficients of the PFEs given in (H.38) and (H.39), respectively.

2.3.5 Probability of Error: Scenario #2

MC-CDMA

For $Q_P = 1$, the probability of error of the MC-CDMA system in Scenario #2 is equal to (see Appendix J)

$$\begin{aligned} P_e = &\left(\frac{1}{2}\right)^{R_1 - MR_1/N} \sum_{n_1=0}^{R_1-1} \binom{R_1 - MR_1/N - 1 + n_1}{n_1} \left(\frac{1}{2}\right)^{n_1} \\ &\cdot \left(\frac{1-\mu}{2}\right)^{MR_1/N} \sum_{n_2=0}^{R_1-1-n_1} \binom{MR_1/N - 1 + n_2}{n_2} \left(\frac{1+\mu}{2}\right)^{n_2}, \end{aligned} \tag{2.65}$$

where

$$\mu = \frac{\bar{\gamma}}{1+\bar{\gamma}}, \tag{2.66}$$

$$\bar{\gamma} = \left(\frac{1}{MR_1/N} \frac{E_{b1}}{N_0}\right) \frac{1}{1 + (K-1)\left(\frac{1}{R_1}\frac{E_{b1}}{N_0}\right)}. \tag{2.67}$$

For $Q_P > 1$, from Appendix J, we have

$$\begin{aligned} P_e = &\left(\frac{1}{2}\right)^{R_1 - MR_1/N} \sum_{n_1=0}^{R_1 - MR_1/N - 1} \binom{R_1 - MR_1/N - 1 + n_1}{n_1} \left(\frac{1}{2}\right)^{n_1} \\ &\cdot \left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N} \sum_{n_2=0}^{R_1 - MR_1/N - 1 - n_1} \binom{MR_1/N - 1 + n_2}{n_2} \left(\frac{1+\mu_{31}}{2}\right)^{n_2} \\ &\cdot \left(\frac{1-\mu_{41}}{2}\right)^{MR_1/N} \sum_{n_3=0}^{R_1 - MR_1/N - 1 - n_1 - n_2} \binom{MR_1/N - 1 + n_3}{n_3} \left(\frac{1+\mu_{41}}{2}\right)^{n_3} \\ &+ \left(\frac{1-\mu_{13}}{2}\right)^{R_1 - MR_1/N} \sum_{n_1=0}^{MR_1/N - 1} \binom{R_1 - MR_1/N - 1 + n_1}{n_1} \left(\frac{1+\mu_{13}}{2}\right)^{n_1} \\ &\cdot \left(\frac{1-\mu_{23}}{2}\right)^{R_1 - MR_1/N} \sum_{n_2=0}^{MR_1/N - 1 - n_1} \binom{R_1 - MR_1/N - 1 + n_2}{n_2} \left(\frac{1+\mu_{23}}{2}\right)^{n_2} \\ &\cdot \left(\frac{1-\mu_{43}}{2}\right)^{MR_1/N} \sum_{n_3=0}^{MR_1/N - 1 - n_1 - n_2} \binom{MR_1/N - 1 + n_3}{n_3} \left(\frac{1+\mu_{43}}{2}\right)^{n_3}, \end{aligned} \tag{2.68}$$

where

$$\mu_{31} = \frac{(\sqrt{Q_P}\,\bar{\gamma} - 1) + \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}{(\sqrt{Q_P}\,\bar{\gamma} + 1) + \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{2.69}$$

$$\mu_{41} = \frac{(\sqrt{Q_P}\,\bar{\gamma} - 1) - \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}{(\sqrt{Q_P}\,\bar{\gamma} + 1) - \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{2.70}$$

$$\mu_{13} = \frac{(1 - \sqrt{Q_P}\,\bar{\gamma}) + \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}{(1 + \sqrt{Q_P}\,\bar{\gamma}) - \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{2.71}$$

$$\mu_{23} = \frac{(1 + \sqrt{Q_P}\,\bar{\gamma}) - \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}{(1 - \sqrt{Q_P}\,\bar{\gamma}) + \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{2.72}$$

$$\mu_{43} = \frac{\sqrt{Q_P}\,\bar{\gamma}}{\sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{2.73}$$

and $\bar{\gamma}$ is as defined in (2.67).

MC-DS-CDMA

The probability of error for the final test statistic given in (2.49) is equal to (see Appendix G)

$$P_e = \left(\frac{1-\mu}{2}\right)^{R_2} \sum_{k=0}^{R_2-1} \binom{R_2 - 1 + k}{k} \left(\frac{1+\mu}{2}\right)^k, \tag{2.74}$$

where

$$\mu = \frac{\sqrt{Q_P}\,\bar{\gamma}}{\sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{2.75}$$

$$\bar{\gamma} = \left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right) \frac{1}{1 + \frac{K-1}{N}\left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right)}. \tag{2.76}$$

2.3.6 Asymptotic Results: Perfect CSI

Consider the limiting case where the channel remains constant and an arbitrarily long data frame can be used, i.e., $Q \to \infty$. If we let $Q_P \to \infty$, the SNR of each channel estimate goes to infinity as well, resulting in perfect channel state information (CSI). From the results for MC-CDMA in Scenario #1 and for MC-DS-CDMA in both Scenario #1 (uniform MIP) and Scenario #2, it is observed that (2.51), (2.55), and (2.74) are of the same general form as [27, Eq. (14.4-15)], with

$$\mu = \frac{\sqrt{Q_P}\,\bar{\gamma}}{\sqrt{(1+Q_P\bar{\gamma})(1+\bar{\gamma})}}, \tag{2.77}$$

and orders of diversity equal to R_1, R_2L, and R_2, respectively. Since

$$\lim_{Q_P\to\infty} \mu = \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}} \cdot \lim_{Q_P\to\infty} \sqrt{\frac{1}{1+\frac{1}{Q_P\bar{\gamma}}}} = \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}}, \tag{2.78}$$

which matches [27, Eq. (14.4-16)], we see that the probability of error for these three cases indeed converges to the probability of error given in [27] for an ideal MRC as $Q_P \to \infty$.

For the MC-CDMA system in Scenario #2, we first take the limit $Q_P \to \infty$ in each of the expressions in (2.69)–(2.73) to obtain

$$\lim_{Q_P\to\infty} \mu_{31} = 1, \tag{2.79}$$

$$\lim_{Q_P\to\infty} \mu_{41} = 1, \tag{2.80}$$

$$\lim_{Q_P\to\infty} \mu_{13} = -1, \tag{2.81}$$

$$\lim_{Q_P\to\infty} \mu_{23} = -1, \tag{2.82}$$

$$\lim_{Q_P\to\infty} \mu_{43} = \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}}. \tag{2.83}$$

Using these results, we can show that the limit of (2.68) as $Q_P \to \infty$ is equal to

$$\lim_{Q_P\to\infty} P_e = \left(\frac{1-\mu}{2}\right)^{MR_1/N} \sum_{n=0}^{MR_1/N-1} \binom{MR_1/N-1+n}{n} \left(\frac{1+\mu}{2}\right)^n, \tag{2.84}$$

where

$$\mu = \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}}. \tag{2.85}$$

When compared with [27, Eqs. (14.4-15), (14.4-16)], this is just the probability of error of an ideal MRC with (MR_1/N)-th order diversity.

Finally, for the MC-DS-CDMA system in Scenario #1 with an exponential MIP in each sub-band, the limit of (2.62) as $Q_P \to \infty$ is difficult to simplify, because both the $\{\tilde{c}_{l,k}\}$ and $\{\tilde{d}_{l,k}\}$

depend on Q_P as well. However, we have been able to verify that this limit agrees numerically with the probability of error of a similar MC-DS-CDMA system where perfect CSI is assumed at the receiver [4].

2.4 Numerical Results and Discussion

Using the closed-form expressions for the probability of error presented in Section 2.3, we now compare the performances of MC-DS-CDMA and MC-CDMA for the two channel scenarios. We consider an MC-CDMA system with $MR_1 = 512$ sub-carriers and an MC-DS-CDMA system with $MR_2 = 8$ sub-carriers. Since $R_1 = NR_2$, this implies a processing gain of $N = 64$ in each MC-DS-CDMA sub-band. By fixing the number of sub-carriers and varying the number of parallel data symbols transmitted, M, we can examine both systems for different orders of frequency diversity. We set $E_{b1} = E_{b2} = E_b$ and consider both the single- and multi-user cases: $K = 1$ and $K = 16$.

For Scenario #1, since $R_2L = R_1$, the overall order of diversity for each of the M symbols is the same for both systems. Here, channel estimates are made at R_1 sub-carrier frequencies in MC-CDMA and on a total of R_1 paths in MC-DS-CDMA. Thus, we see from (2.51)–(2.53) and (2.55)–(2.57) that the two systems yield the same probability of error when the MIP for each MC-DS-CDMA sub-band is uniform. When the MIP for each MC-DS-CDMA sub-band is exponentially-decaying, however, the MC-DS-CDMA system performs worse at all E_b/N_0 values for both the single- and multi-user cases, as shown in Figs. 2.7 and 2.8. As expected, when $K = 16$, the presence of MAI creates an error floor in both systems, causing the curves to flatten out as E_b/N_0 increases.

In Scenario #2, each symbol in MC-DS-CDMA experiences frequency diversity of order R_2, while the effective order of frequency diversity in MC-CDMA is equal to MR_1/N. We have

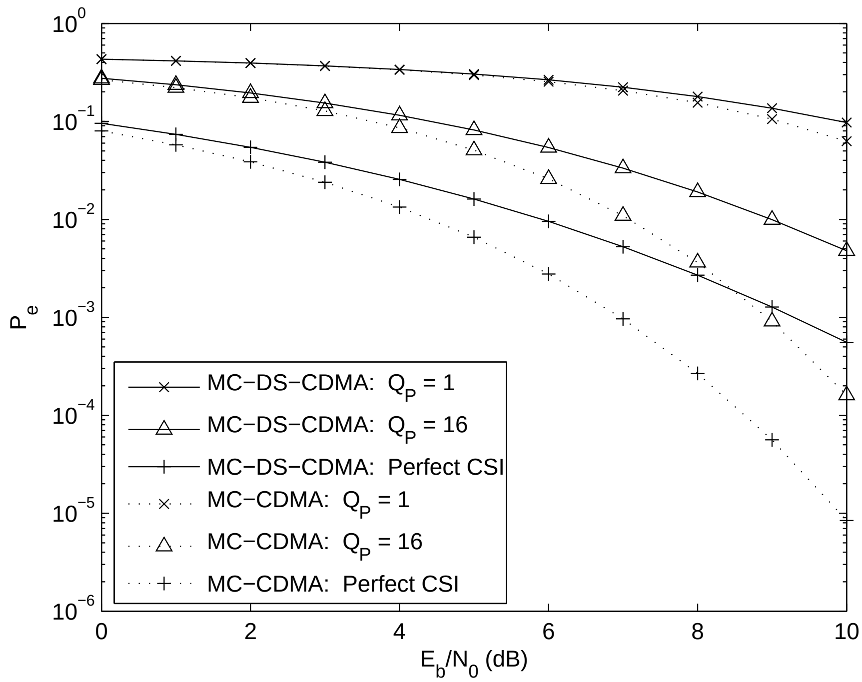

Figure 2.7: Scenario #1 (Exponential MIP): $M = 8$, $R_1 = 64$, $R_2 = 1$, $r_d = 0.5$, $K = 1$. Each data symbol experiences an overall diversity of order $R_1 = R_2 L = 64$ in both systems. The dotted curves are also the curves for MC-DS-CDMA when the MIP in each sub-band is uniform.

$MR_1/N \geq R_2$, with equality if $M = 1$. First, consider the case where $M = 1$ (see Figs. 2.9 and 2.10). Under perfect CSI, both systems exhibit equivalent performance for any number of users, since both systems have equal diversity. With noisy channel estimation, however, the corresponding curves for the two systems no longer coincide in both the single- and multi-user cases, with MC-DS-CDMA giving a lower probability of error at all E_b/N_0 values. The reason for this disparity in performance under imperfect CSI is because the power transmitted in each MC-CDMA sub-band is only $(1/N)$-th the power transmitted in each MC-DS-CDMA sub-band. Hence, at the receiver, the estimate of the channel gain at each sub-carrier is made, on average, at a lower SNR—or signal-to-interference-plus-noise ratio (SINR), in the case of multiple users—in MC-CDMA than in MC-DS-CDMA. However, since the two systems behave identically with

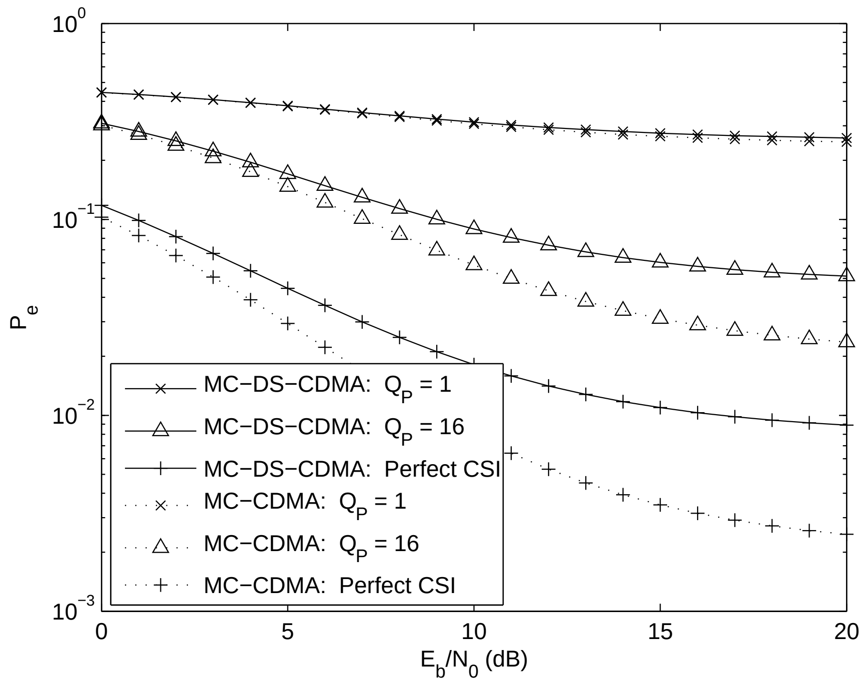

Figure 2.8: Scenario #1 (Exponential MIP): $M = 8$, $R_1 = 64$, $R_2 = 1$, $r_d = 0.5$, $K = 16$. Each data symbol experiences an overall diversity of order $R_1 = R_2L = 64$ in both systems. The dotted curves are also the curves for MC-DS-CDMA when the MIP in each sub-band is uniform.

perfect CSI ($Q_P \to \infty$), this gap in performance decreases as Q_P increases. In summary, since both systems exhibit the same order of frequency diversity when $M = 1$, only the effects of channel estimation errors are observed, and MC-DS-CDMA gives a lower P_e.

Now, consider the case where $M = 4$. Since the order of frequency diversity for each of the M symbols is higher in the MC-CDMA system, the average probability of error is always lower in MC-CDMA when there is perfect CSI, as shown in Figs. 2.11 and 2.12. With noisy channel estimation, however, we notice some interesting trade-offs between the two systems. With $K = 1$ and $Q_P = 16$, we observe in Fig. 2.11 that MC-DS-CDMA gives a lower P_e for $E_b/N_0 < 9.5$dB, while MC-CDMA performs better for $E_b/N_0 > 9.5$dB. This shows that the effects of channel estimation errors dominate at lower E_b/N_0 values, while the effects of frequency

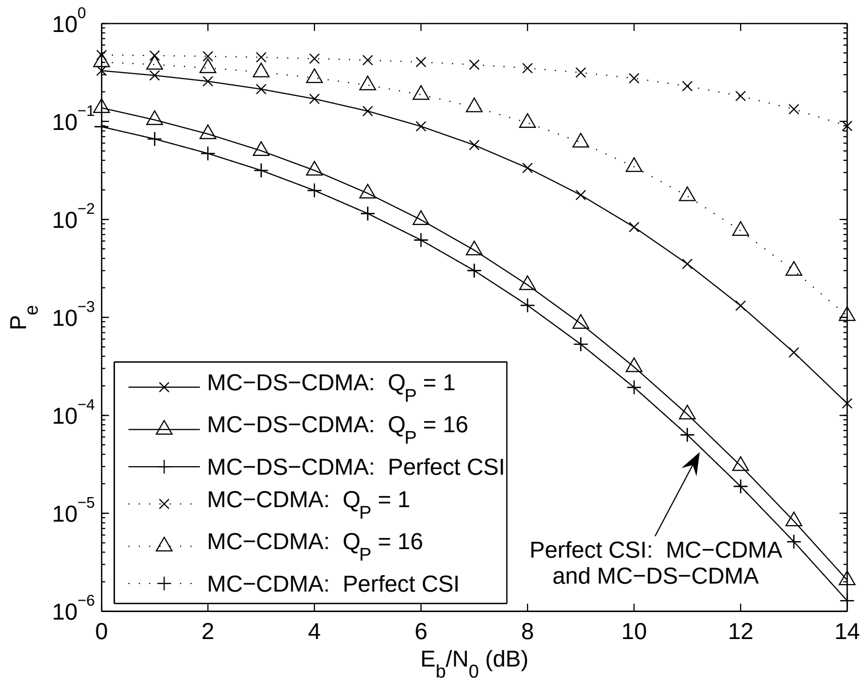

Figure 2.9: Scenario #2: $M = 1$, $R_1 = 512$, $R_2 = 8$, $K = 1$. The order of frequency diversity per data symbol is equal to $\frac{MR_1}{N} = R_2 = 8$ in both systems.

diversity dominate at higher E_b/N_0 values. The same trade-off is observed when $Q_P = 1$, but the point at which the two curves intersect occurs at a much higher E_b/N_0 value that is outside the range of this plot. Thus, we conclude that the SNR range for which MC-DS-CDMA is the preferred system shrinks as Q_P increases. It is important to note that in this single-user case, despite the presence of noisy channel estimation, one can always attain better performance with the MC-CDMA system by simply using a large enough E_b/N_0. With $K = 16$, however, this is not always the case. Once again, due to the presence of MAI, the curves for both systems exhibit an error floor (see Fig. 2.12). As a result, for both $Q_P = 1$ and $Q_P = 16$, we see that MC-DS-CDMA not only performs better at lower E_b/N_0 values, as expected, it actually gives a lower P_e at high E_b/N_0 values as well. But if we increase the number of pilot symbols to $Q_P = 32$, we then obtain the same trade-off that was seen in the single-user case, where MC-CDMA eventually fares better

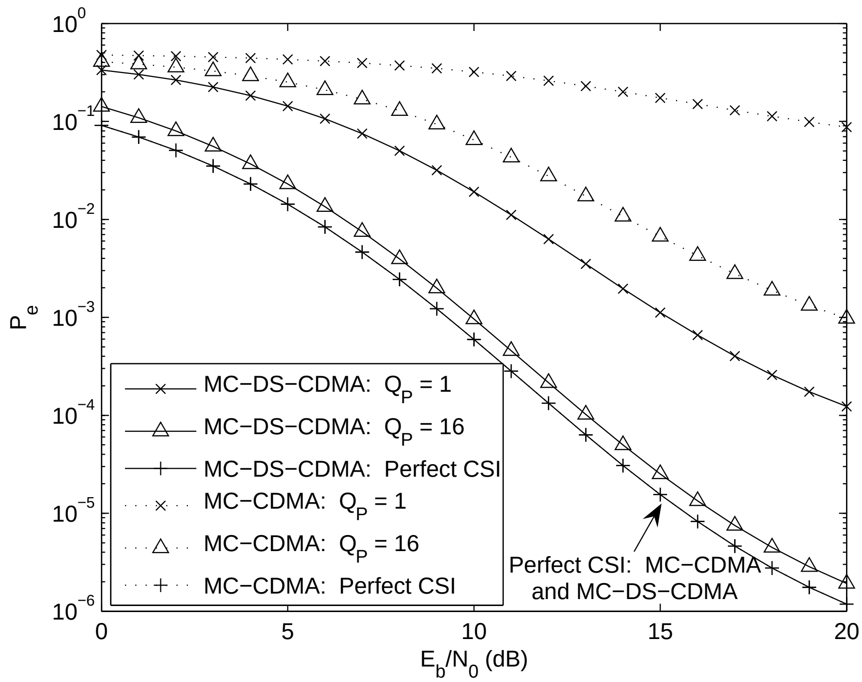

Figure 2.10: Scenario #2: $M = 1$, $R_1 = 512$, $R_2 = 8$, $K = 16$. The order of frequency diversity per data symbol is equal to $\frac{MR_1}{N} = R_2 = 8$ in both systems.

past a certain E_b/N_0 threshold. Thus, when $K > 1$ and $M > 1$, increasing E_b/N_0 alone will not guarantee better performance for the MC-CDMA system—we must increase Q_P as well.

As we noted before, for the analysis and the results presented above, it is assumed that neither system has knowledge of the coherence bandwidth of the channel. Thus, in Scenario #2, the MC-CDMA system is forced to estimate the channel at every sub-carrier, despite the presence of correlated fading in each frequency block. If the coherence bandwidth was known, channel estimates would only have to be made at one of the sub-carriers within each block. Assuming the total energy allocated for channel estimation was kept constant, this would then lead to less noisy channel estimates and, hence, better performance.

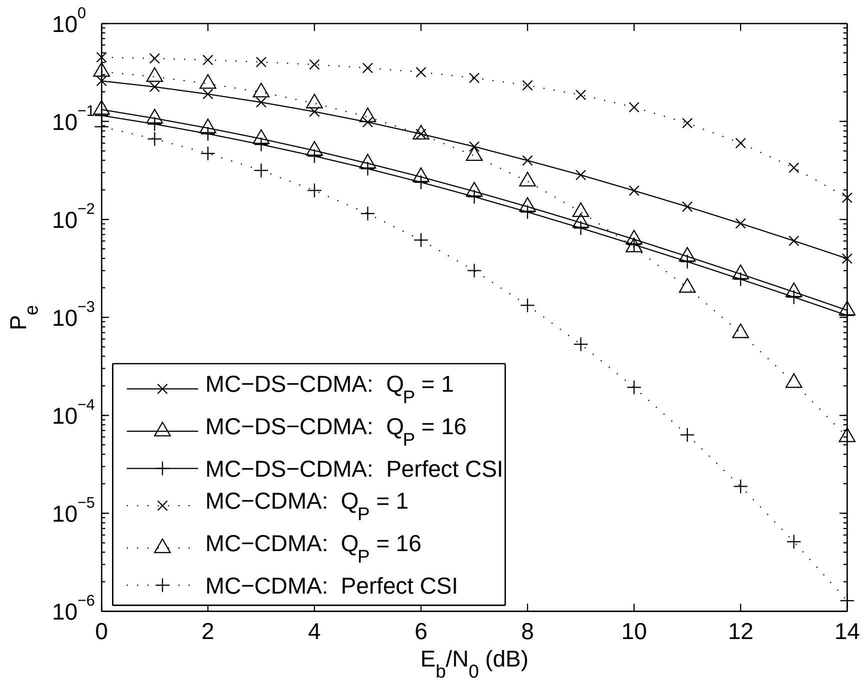

Figure 2.11: Scenario #2: $M = 4$, $R_1 = 128$, $R_2 = 2$, $K = 1$. Each data symbol experiences frequency diversity of order $\frac{MR_1}{N} = 8$ and $R_2 = 2$ in MC-CDMA and MC-DS-CDMA, respectively.

2.5 Conclusion

We constructed a comparison between MC-DS-CDMA and MC-CDMA, and we derived closed-form expressions for the probability of error of both systems under two different cases for the coherence bandwidth of the channel. The opposing effects of diversity and channel estimation errors on the probability of error were examined by varying M (number of symbols transmitted simultaneously in parallel), Q_P (number of pilot symbols used in forming each channel estimate), and K (number of users). In Scenario #1, the two systems gave equivalent performance when the MIP in each MC-DS-CDMA sub-band was uniform (flat), but the MC-DS-CDMA system experienced a loss in performance when the MIP was exponentially-decaying. In Scenario #2, when $M = 1$, the MC-DS-CDMA system always gave a lower probability of error under imperfect

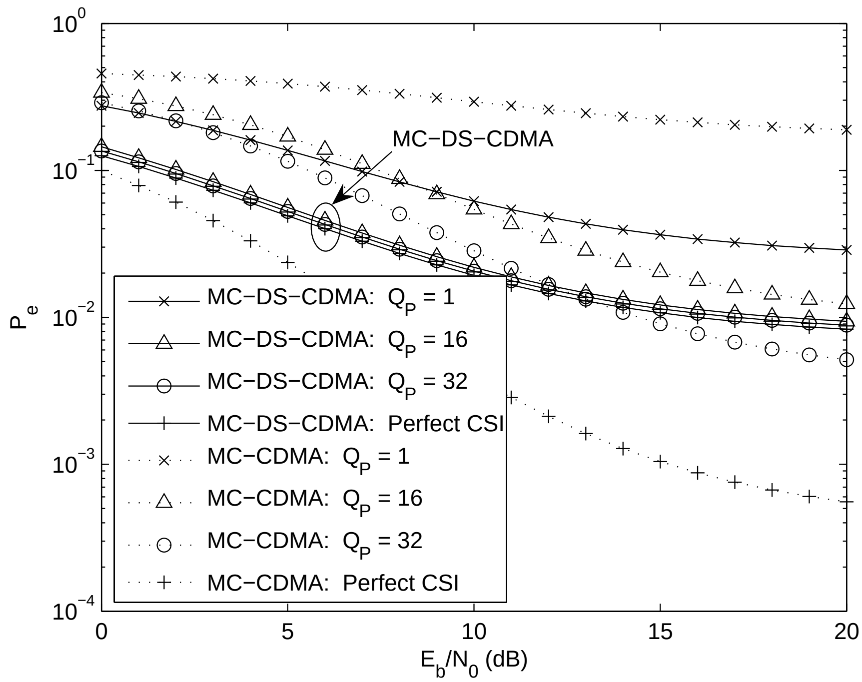

Figure 2.12: Scenario #2: $M = 4$, $R_1 = 128$, $R_2 = 2$, $K = 16$. Each data symbol experiences frequency diversity of order $\frac{MR_1}{N} = 8$ and $R_2 = 2$ in MC-CDMA and MC-DS-CDMA, respectively.

channel estimation. For $M > 1$ and $K = 1$, a trade-off existed where MC-DS-CDMA performed better at lower E_b/N_0 values, while MC-CDMA fared better at higher E_b/N_0 values. Finally, a similar trade-off occurred when $M > 1$ and $K > 1$ only if Q_P was sufficiently large. Otherwise, MC-DS-CDMA outperformed MC-CDMA at all E_b/N_0 values.

2.6 Acknowledgements

This chapter is essentially a reprint of the paper: A. S. Ling and L. B. Milstein, "Trade-Off Between Diversity and Channel Estimation Errors in Asynchronous MC-DS-CDMA and MC-CDMA," *IEEE Transactions on Communications*, vol. 56, no. 4, pp. 584–597, Apr. 2008. Portions of this chapter have also appeared in: A. S. Ling and L. B. Milstein, "Comparison of Multi-Carrier

Modulation Techniques," in *Proceedings of the 2005 IEEE Military Communications Conference (MILCOM '05)*, vol. 2, Atlantic City, NJ, Oct. 17–20, 2005, pp. 731–739. The dissertation author was the primary author of both of these papers.

Chapter 3

MC-DS-CDMA vs. MC-CDMA in the Presence of Spatial Diversity

3.1 Introduction

In this chapter, we extend the comparison between MC-DS-CDMA and MC-CDMA to a MIMO system employing two antennas at both the transmitter and the receiver. For both multi-carrier schemes, we use Alamouti space-time block coding at each sub-carrier frequency to achieve transmit diversity in the absence of CSI at the transmitter. By assuming (1) the coherence bandwidth of the channel is equal to the bandwidth of one MC-DS-CDMA sub-band, and (2) the number of data symbols transmitted in parallel, M, is such that $M > 1$, we restrict the comparison only to those cases where MC-CDMA has higher frequency diversity than MC-DS-

CDMA, because these are the only instances in which there is a performance trade-off between the two multi-carrier schemes. Since increases in diversity yield diminishing gains, and MC-CDMA starts off with higher frequency diversity than MC-DS-CDMA, the addition of spatial diversity to the multi-carrier comparison should benefit MC-DS-CDMA more than MC-CDMA. To determine whether these gains for MC-DS-CDMA are enough to offset the difference in frequency diversity between the two schemes, we use an approach based on quadratic forms (see Appendix D) to derive closed-form expressions of their bit error probabilities, and we compare the MIMO results against those of the SISO system for different information rates, number of users, and number of pilot symbols per channel estimate.

This chapter is organized as follows. In Section 3.2, we present the system model assuming a general multi-carrier signaling scheme. We apply MC-CDMA and MC-DS-CDMA signaling to this system model in Section 3.3, and we analyze their error probability performances in Section 3.4. The numerical results of the MIMO system are then compared against those of the SISO system in Section 3.5, and we summarize the main findings in Section 3.6.

3.2 General MIMO Multi-Carrier System Model

3.2.1 Overview

Consider an uplink scenario involving K asynchronous users. Assume two antennas at each transmitter and two antennas at the receiver. The available bandwidth is the same for each user and is divided into S bandlimited, non-overlapping frequency sub-bands of equal width.[1] We assume each sub-band is associated with a sub-carrier frequency and has a bandwidth of W.

In the time domain, the transmissions made at each sub-carrier are structured into

[1] Again, spectral efficiency is not a concern in this comparison, so we do not insist on using minimum sub-carrier spacing and having the sub-bands overlap.

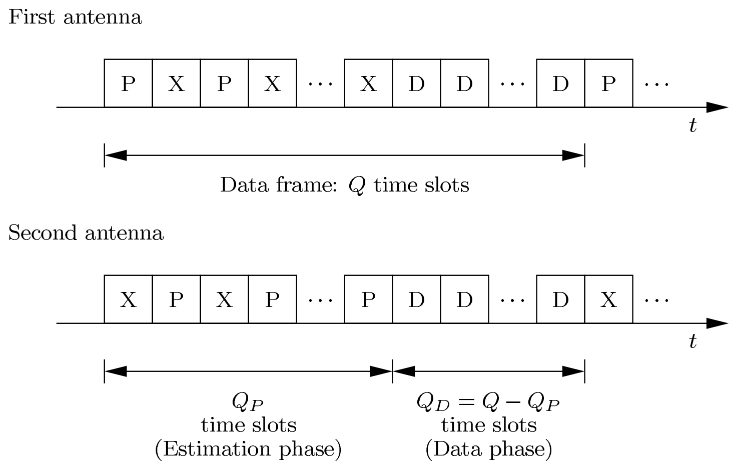

Figure 3.1: Data frame structure (P—pilot symbol, D—data symbol, X—no transmission).

frames, each consisting of Q time slots. As in Chapter 2, we assume a slowly-varying channel which remains constant over the duration of a frame, and we designate the first Q_P time slots of each frame as the *estimation phase* and the remaining $Q_D = Q - Q_P$ time slots as the *data phase* (see Fig. 3.1). During each time slot of the data phase, both antennas simultaneously transmit M distinct binary data symbols in parallel across the S sub-bands, where each symbol is repeated on R sub-carriers. Thus, we have $S = MR$, and the overall system bandwidth is equal to $SW = MRW$. Similarly, during the estimation phase, we transmit M pilot symbols (+1's) in parallel and we spread the energy of each pilot symbol across R sub-bands. However, pilot symbols are transmitted on the first antenna only during the odd time slots and on the second antenna only during the even time slots. As a result, only the odd (even) time slots are used to estimate the channel gains associated with the first (second) transmit antenna, and each estimate is formed from $Q_P/2$ pilot symbols. To keep the total energy transmitted from the two antennas the same over all of the time slots, we also use a different transmit amplitude during the estimation phase. Our motivation for imposing this energy constraint, as well as for using the on-off pilot transmission scheme, is to maintain the same energy-per-pilot symbol between

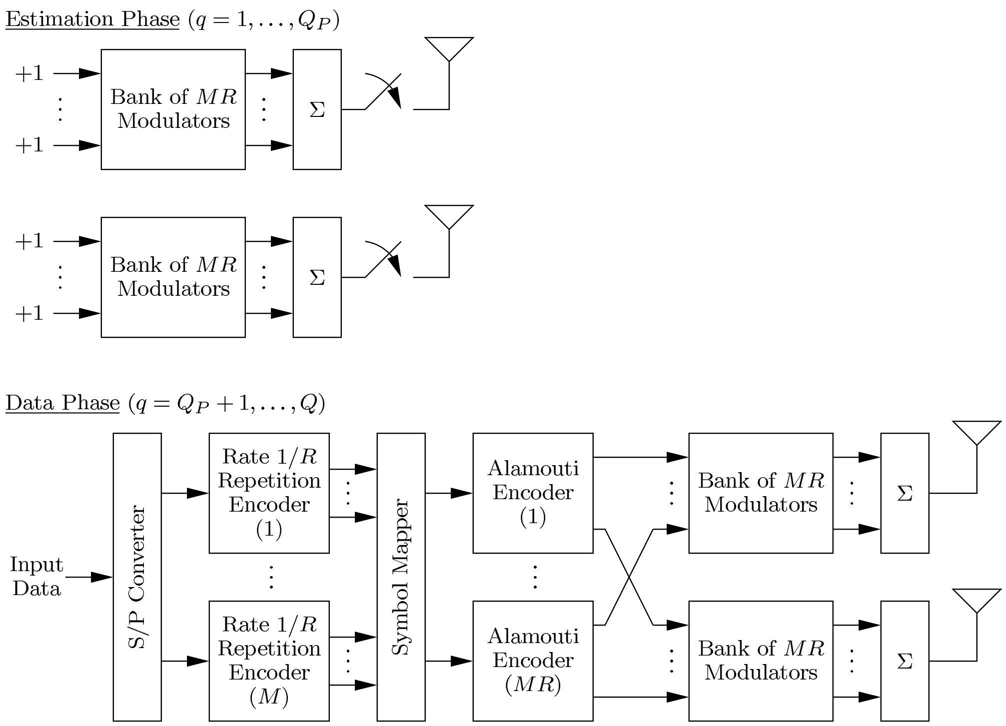

Figure 3.2: Transmitter block diagram.

this system and the SISO system in Chapter 2. By doing so, channel estimates formed from the same number of pilot symbols will be of the same quality in both the SISO and MIMO systems.

3.2.2 Transmitter

The transmitter block diagram is shown in Fig. 3.2. During the estimation phase, the on-off transmission of pilot symbols on all of the sub-carriers is modeled using the switches at the two antennas. The switch for the first (second) antenna is closed during the odd (even) time slots and open during the even (odd) times slots. The details of the multi-carrier modulation blocks will be given in Section 3.3 when we consider MC-CDMA and MC-DS-CDMA.

During the data phase, the input data is modeled as an independent random binary sequence with bit rate $1/T$. The serial-to-parallel (S/P) converter produces M parallel sequences,

Table 3.1: Definition of $b^{(k)}_{p,q,u,m}$

Estimation phase ($q = 1, \ldots, Q_P$):	$b^{(k)}_{p,q,u,m} = \begin{cases} \delta_{u,1}, & q \text{ odd} \\ \delta_{u,2}, & q \text{ even} \end{cases}$
Data phase ($q = Q_P + 1, \ldots, Q$):	$b^{(k)}_{p,q,u,m} = \begin{cases} a^{(k)}_{p,q,m}, & u = 1,\ q \text{ odd} \\ a^{(k)}_{p,q+1,m}, & u = 2,\ q \text{ odd} \\ -a^{(k)}_{p,q,m}, & u = 1,\ q \text{ even} \\ a^{(k)}_{p,q-1,m}, & u = 2,\ q \text{ even} \end{cases}$

and we label each group of M symbols as $\{a^{(k)}_{p,q,m}\}^{M}_{m=1}$, where p is the frame index, q is the time slot index, and k is the user index. A rate $1/R$ repetition code is applied to each of these M symbols, and the resulting MR symbols are then mapped to the MR sub-carriers in such a way that maximizes the frequency separation between adjacent repetitions of a symbol. We denote the frequency to which the i-th repetition of the m-th symbol is mapped by $f_{m,i}$.[2] After symbol mapping, Alamouti space-time block coding is performed at each sub-carrier frequency, and the outputs of the encoders are fed through the two banks of modulators. Finally, the signals are combined and transmitted over the two antennas.

To represent the transmitted signals at each frequency during the estimation and data phases under a common mathematical framework, we define $b^{(k)}_{p,q,u,m}$ as the k-th user's input to the modulators at frequencies $\{f_{m,i}\}_{i=1,\ldots,R}$ of the u-th transmit antenna during the q-th time slot of the p-th frame. As shown in Table 3.1, for $q = 1, \ldots, Q_P$, $b^{(k)}_{p,q,u,m}$ is equal to either $+1$ or 0, while for $q = Q_P + 1, \ldots, Q$, $b^{(k)}_{p,q,u,m}$ is equal to the output of the Alamouti encoder. The transmitted signal for the k-th user is given by the 2×1 vector

$$\mathbf{s}^{(k)}(t) = \sum_{m=1}^{M} \sum_{i=1}^{R} \mathrm{Re}\left\{ \tilde{\mathbf{s}}^{(k)}_{m,i}(t) e^{j2\pi f_{m,i} t} \right\}, \tag{3.1}$$

where

$$\tilde{\mathbf{s}}^{(k)}_{m,i}(t) = [\tilde{s}^{(k)}_{1,m,i}(t)\ \tilde{s}^{(k)}_{2,m,i}(t)]^T. \tag{3.2}$$

[2]Hence, there exists a one-to-one correspondence between the pair (m, i) and the sub-carrier frequency $f_{m,i}$.

The expression for $\tilde{s}^{(k)}_{u,m,i}(t)$ $(u = 1, 2)$ depends on the multi-carrier scheme used.

3.2.3 Channel

Assume the two antennas at both the transmitter and receiver are sufficiently spaced, such that the channels between different transmit-receive antenna pairs are independent. Each of these channels is frequency-selective with respect to the overall system bandwidth, but each sub-band is assumed to be frequency non-selective with Rayleigh-distributed fade amplitudes. Consequently, our use of Alamouti space-time block coding yields fourth-order spatial diversity at each sub-carrier frequency; the theoretical frequency diversity gain, however, depends on the correlation of the channel gains across the different sub-bands, which is a function of the coherence bandwidth of the channel.

In the absence of Doppler effects, there is no inter-channel interference between adjacent sub-bands since they do not overlap. Assuming perfect power control, the complex lowpass equivalent received signal at frequency $f_{m,i}$ is then given by the 2×1 vector

$$\tilde{\mathbf{r}}_{m,i}(t) = [\tilde{r}_{1,m,i}(t)\ \tilde{r}_{2,m,i}(t)]^T, \tag{3.3}$$

where

$$\tilde{r}_{v,m,i}(t) = \sum_{k=1}^{K} \sum_{u=1}^{2} g^{(k)}_{v,u,m,i} \tilde{s}^{(k)}_{u,m,i}(t - \tau^{(k)}) + \tilde{n}_v(t), \quad v = 1, 2. \tag{3.4}$$

In the above equation, $g^{(k)}_{v,u,m,i} \triangleq \alpha^{(k)}_{v,u,m,i} e^{j\theta^{(k)}_{v,u,m,i}}$ is a zero-mean circularly symmetric complex Gaussian random variable representing the k-th user's channel gain at frequency $f_{m,i}$ between the u-th transmit antenna and the v-th receive antenna, where $\alpha^{(k)}_{v,u,m,i}$ is a Rayleigh random variable with unit second moment, and $\theta^{(k)}_{v,u,m,i}$ is a uniform random variable over $[0, 2\pi)$. The $\{g^{(k)}_{v,u,m,i}\}$ are identically distributed, but their joint statistics depend on the coherence bandwidth of the channel. Also, $\tau^{(k)}$ represents the relative time delay between the k-th user and the desired user, and $\tilde{n}_v(t)$ is a zero-mean complex additive white Gaussian noise (AWGN) random process

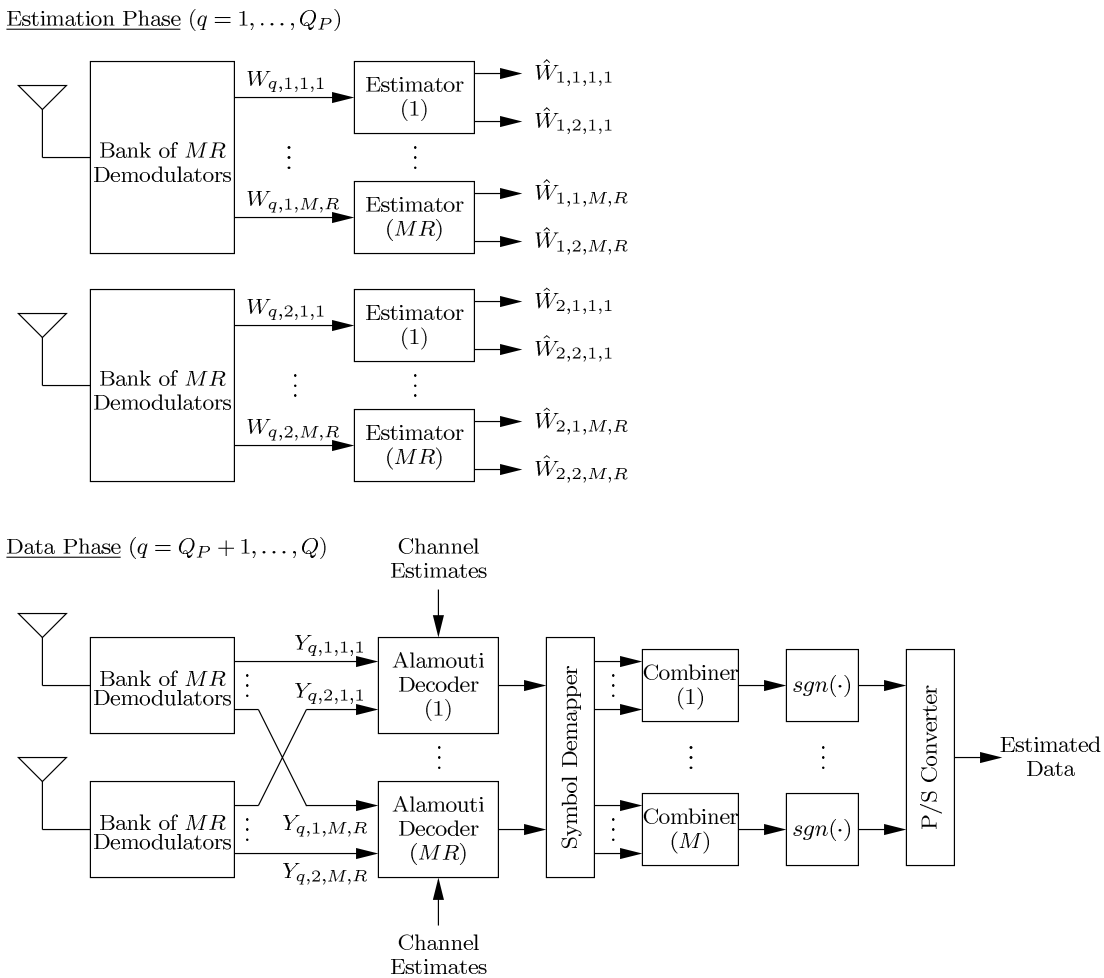

Figure 3.3: Receiver block diagram.

with two-sided power spectral density N_0. The noise processes at the two receive antennas are assumed to independent.

3.2.4 Receiver

The block diagram for the receiver is shown in Fig. 3.3. At each antenna, the received signal first passes through a bank of MR demodulators. We label the output of the demodulator at frequency $f_{m,i}$ of the v-th receive antenna during the estimation and data phases as $W_{q,v,m,i}$

$(q = 1, \ldots, Q_P)$ and $Y_{q,v,m,i}$ $(q = Q_P + 1, \ldots, Q)$, respectively. During the estimation phase, the estimate of $g^{(k)}_{v,u,m,i}$ is formed by taking a sample average of the $\{W_{q,v,m,i}\}$ during the odd or even time slots:

$$\hat{W}_{v,u,m,i} = \begin{cases} \frac{1}{A_0}\left(\frac{1}{Q_P/2}\sum_{\substack{q=1\\ q \text{ odd}}}^{Q_P} W_{q,v,m,i}\right), & u = 1, \\ \frac{1}{A_0}\left(\frac{1}{Q_P/2}\sum_{\substack{q=1\\ q \text{ even}}}^{Q_P} W_{q,v,m,i}\right), & u = 2, \end{cases} \tag{3.5}$$

where the normalizing constant A_0 ensures that $\hat{W}_{v,u,m,i}$ is an unbiased estimate of $g^{(k)}_{v,u,m,i}$. During the data phase, the $\{Y_{q,v,m,i}\}$ at each sub-carrier frequency over two consecutive time slots feed into a linear Alamouti decoder, which outputs the 2×1 vector

$$\begin{aligned} \mathbf{z}_{q,m,i} &= [Z_{q,m,i}\ Z_{q+1,m,i}]^T, \\ &= \hat{\mathbf{G}}^H_{m,i}\mathbf{y}_{q,m,i}, \qquad q = Q_P + 1, \ldots, Q;\ q \text{ odd}, \end{aligned} \tag{3.6}$$

where

$$\hat{\mathbf{G}}_{m,i} = \begin{bmatrix} \hat{W}_{1,1,m,i} & \hat{W}_{1,2,m,i} \\ \hat{W}_{2,1,m,i} & \hat{W}_{2,2,m,i} \\ \hat{W}^*_{1,2,m,i} & -\hat{W}^*_{1,1,m,i} \\ \hat{W}^*_{2,2,m,i} & -\hat{W}^*_{2,1,m,i} \end{bmatrix} \equiv [\hat{\mathbf{w}}_{1,m,i}\ \hat{\mathbf{w}}_{2,m,i}], \tag{3.7}$$

$$\mathbf{y}_{q,m,i} = \begin{bmatrix} Y_{q,1,m,i} \\ Y_{q,2,m,i} \\ Y^*_{q+1,1,m,i} \\ Y^*_{q+1,2,m,i} \end{bmatrix}. \tag{3.8}$$

Thus, we have

$$Z_{q,m,i} = \hat{\mathbf{w}}^H_{1,m,i}\mathbf{y}_{q,m,i}, \tag{3.9}$$

$$Z_{q+1,m,i} = \hat{\mathbf{w}}^H_{2,m,i}\mathbf{y}_{q,m,i}, \tag{3.10}$$

and we focus on the detection of $Z_{q,m,i}$. After Alamouti decoding, the demapper groups together the R variables associated with each of the M parallel data symbols and feeds them into a bank of M combiners. The output of the m-th combiner is expressed as

$$\begin{aligned}\hat{Z}_{q,m} &= \mathrm{Re}\left\{\sum_{i=1}^{R} Z_{q,m,i}\right\} \\ &= \sum_{i=1}^{R}\left\{\frac{1}{2}\hat{\mathbf{w}}_{1,m,i}^{H}\mathbf{y}_{q,m,i} + \frac{1}{2}\mathbf{y}_{q,m,i}^{H}\hat{\mathbf{w}}_{1,m,i}\right\}, \quad q = Q_P+1,\ldots,Q;\ q \text{ odd}, \qquad (3.11)\end{aligned}$$

and the m-th data symbol during the q-th time slot is decoded as $\mathrm{sgn}\left\{\hat{Z}_{q,m}\right\}$. Finally, parallel-to-serial (P/S) conversion gives the estimated data sequence.

3.2.5 Probability of Error

By defining

$$\mathbf{w}_m = \left[\tilde{\mathbf{w}}_{m,1}^T \ \cdots \ \tilde{\mathbf{w}}_{m,R}^T\right]^T, \qquad (3.12)$$

$$\mathbf{y}_{q,m} = \left[\tilde{\mathbf{y}}_{q,m,1}^T \ \cdots \ \tilde{\mathbf{y}}_{q,m,R}^T\right]^T, \qquad (3.13)$$

where

$$\tilde{\mathbf{w}}_{m,i} = \left[\hat{W}_{1,1,m,i} \quad \hat{W}_{2,1,m,i} \quad \hat{W}_{1,2,m,i} \quad \hat{W}_{2,2,m,i}\right]^T, \qquad (3.14)$$

$$\tilde{\mathbf{y}}_{q,m,i} = \left[Y_{q,1,m,i} \quad Y_{q,2,m,i} \quad Y_{q+1,1,m,i} \quad Y_{q+1,2,m,i}\right]^T, \qquad (3.15)$$

we can re-express $\hat{Z}_{q,m}$ in (3.11) as a Hermitian quadratic form in zero-mean complex Gaussian random variables,[3]

$$\hat{Z}_{q,m} = \mathbf{v}_{q,m}^H \mathbf{F} \mathbf{v}_{q,m}, \quad q = Q_P+1,\ldots,Q;\ q \text{ odd}, \qquad (3.16)$$

[3] We will show later that the random variables in (3.14)–(3.15) are zero-mean complex Gaussian.

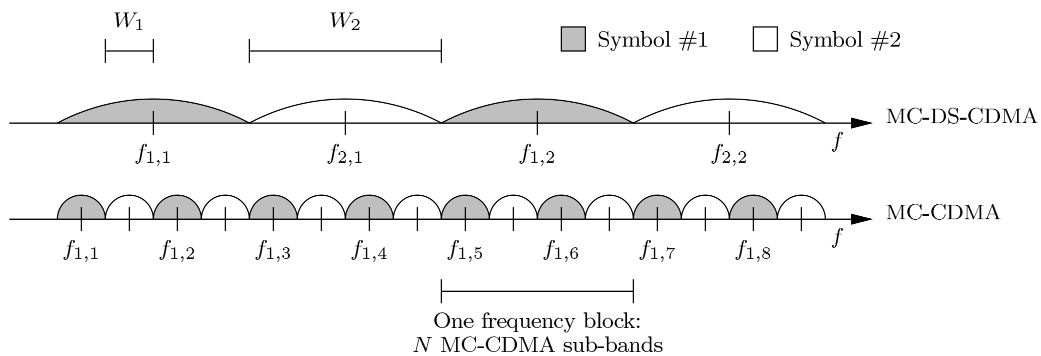

Figure 3.4: Spectra of the MC-DS-CDMA and MC-CDMA signals. In this example, $M = 2$ parallel data symbols are each repeated across $R_1 = 8$ and $R_2 = 2$ sub-carriers in MC-CDMA and MC-DS-CDMA, respectively, where $N = 4$ is the processing gain in each MC-DS-CDMA sub-band. We assume $(\Delta f)_c = W_2$.

where

$$\mathbf{v}_{q,m} = \begin{bmatrix} \mathbf{w}_m \\ \mathbf{y}_{q,m} \end{bmatrix}, \tag{3.17}$$

$$\mathbf{F} = \frac{1}{2}\begin{bmatrix} \mathbf{0}_{4R} & \mathbf{I}_{4R} \\ \mathbf{I}_{4R} & \mathbf{0}_{4R} \end{bmatrix}. \tag{3.18}$$

Assuming a +1 was transmitted, the bit error probability is given by

$$P_e = P\left\{\hat{Z}_{q,m} < 0\right\}. \tag{3.19}$$

A general method for obtaining a closed-form expression of (3.19) is outlined in Appendix D.

3.3 MC-CDMA vs. MC-DS-CDMA

3.3.1 Overview

We now apply MC-CDMA and MC-DS-CDMA signaling to the system model presented in Section 3.2. To distinguish the system parameters between the two schemes, we write $S = S_1$, $W = W_1$, and $R = R_1$ for MC-CDMA and $S = S_2$, $W = W_2$, and $R = R_2$ for MC-DS-CDMA.

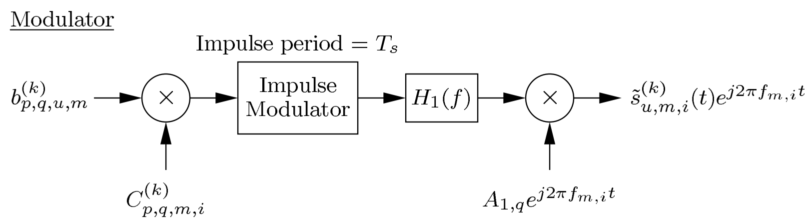

Figure 3.5: MC-CDMA modulator at frequency $f_{m,i}$ of the u-th transmit antenna.

Each MC-CDMA sub-band is non-spread and has a symbol rate of $1/T_s$, where $T_s = (Q_D/Q)MT$. Therefore, the bandwidth of each sub-band is $W_1 = 1/T_s$. On the other hand, each MC-DS-CDMA sub-band has a processing gain (spreading factor) of N, so $W_2 = NW_1 = 1/T_c$, where $1/T_c = N/T_s$ is the chip rate. Since the two multi-carrier schemes are constrained to the same information rate and bandwidth, we have $MR_1W_1 = MR_2W_2$, which implies

$$R_1 = NR_2. \tag{3.20}$$

An example of how the two schemes compare in the frequency domain is given in Fig. 3.4.

In the comparison that follows, we assume the following: (1) the channel is the same for both schemes, and the coherence bandwidth of the channel, $(\Delta f)_c$, is equal to W_2; (2) perfect carrier, chip, and symbol synchronization are established at the receiver; and (3) all K users use long spreading sequences, which are modeled as independent random binary sequences of ±1's.

3.3.2 MC-CDMA

Transmitter

The block diagram of the MC-CDMA modulator at frequency $f_{m,i}$ of the u-th transmit antenna, as shown in Fig. 3.5, is identical to the one in Fig. 2.4. Each symbol at the input of the MC-CDMA modulator is multiplied by only a single chip from the k-th user's spreading sequence, $C^{(k)}_{p,q,m,i}$. The resulting sequence modulates an impulse train and passes through a chip wave-shaping filter denoted by $H_1(f)$. Finally, the output signal of the filter modulates the corresponding sub-carrier.

The transmitted signal for the k-th user is given by (3.1), where

$$\tilde{s}^{(k)}_{u,m,i}(t) = \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} A_{1,q}\, b^{(k)}_{p,q,u,m}\, C^{(k)}_{p,q,m,i}\, h_1\big[t - (pQ + q - 1)T_s\big]. \tag{3.21}$$

In the above equation, $h_1(t)$ is the impulse response of the chip wave-shaping filter, and $A_{1,q}$ is the transmit amplitude during the q-th time slot:

$$A_{1,q} = \begin{cases} A_{1e}, & q = 1, \ldots, Q_P, \\ A_{1d}, & q = Q_P + 1, \ldots, Q. \end{cases} \tag{3.22}$$

To keep the total energy transmitted from the two antennas the same over all time slots, we choose

$$A_{1d} = \frac{A_{1e}}{\sqrt{2}}. \tag{3.23}$$

As a result, the energy per bit can be shown to equal (see Appendix P.4)

$$E_{b1} = \frac{1}{2} R_1 A_{1e}^2 = R_1 A_{1d}^2. \tag{3.24}$$

Channel

Given $(\Delta f)_c = W_2 = NW_1 \gg W_1$, the coherence bandwidth of the channel spans multiple MC-CDMA sub-bands, which means that the sub-bands may be highly correlated. We assume a *correlated block fading* model in which the MR_1 sub-bands are grouped into MR_1/N frequency blocks—each block of N sub-bands corresponds to one MC-DS-CDMA sub-band, and the number of repetitions-per-data symbol within a block is equal to (see Fig. 3.4)

$$R_b \triangleq \frac{N}{M}. \tag{3.25}$$

We assume flat fading across each block, such that the N fade amplitudes associated with each block are identical (i.e., perfectly correlated). Also, we assume the fade amplitudes associated

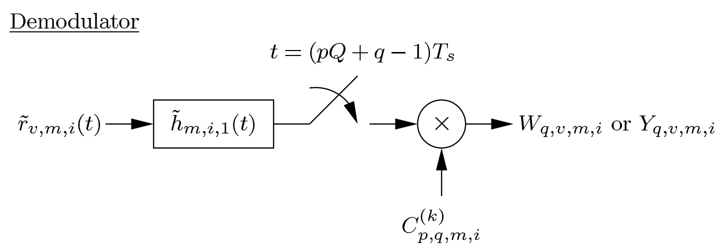

Figure 3.6: Complex equivalent lowpass version of the MC-CDMA demodulator at frequency $f_{m,i}$ of the v-th receive antenna.

with any two sub-bands from different blocks are independent. Thus, even though each symbol is transmitted across R_1 sub-carriers, the effective order of frequency diversity per data symbol is only MR_1/N, and the overall diversity gain per data symbol is $4MR_1/N$, where the factor of 4 comes from the spatial diversity. Note that the frequency diversity gain for MC-CDMA does not change when we vary M, since MR_1 and N are both fixed.

Receiver

The complex equivalent lowpass version of the MC-CDMA demodulator in Fig 2.6 is shown in Fig. 3.6, where

$$\tilde{h}_{m,i,1}(t) = h_1^*(-t) \tag{3.26}$$

is the complex equivalent lowpass impulse response of the bandpass filter. The despreading operation in MC-CDMA simply involves multiplication by a single chip. Assume $H_1(f)$ is of unit energy, i.e.,

$$\int_{-\infty}^{\infty} |H_1(f)|^2 \, df = 1. \tag{3.27}$$

We define

$$x_1(t) \Leftrightarrow X_1(f) \triangleq |H_1(f)|^2 \tag{3.28}$$

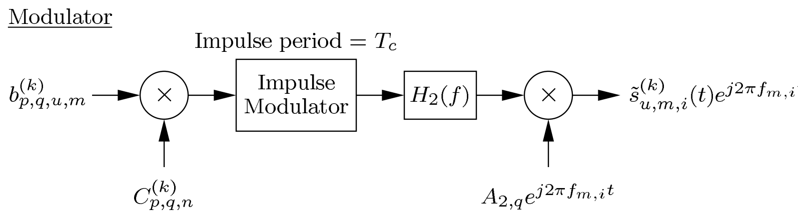

Figure 3.7: MC-DS-CDMA modulator at frequency $f_{m,i}$ of the u-th transmit antenna.

and assume

$$X_1(f) = \begin{cases} T_s, & -\frac{1}{2T_s} < f < \frac{1}{2T_s}, \\ 0, & \text{otherwise}, \end{cases} \;\Leftrightarrow\; x_1(t) = \frac{\sin \pi t/T_s}{\pi t/T_s}. \tag{3.29}$$

Since $X_1(f)$ satisfies the Nyquist criterion, inter-symbol interference is not present.

3.3.3 MC-DS-CDMA

Transmitter

The block diagram of the MC-DS-CDMA modulator at frequency $f_{m,i}$ of the u-th transmit antenna is identical to the one in Fig. 2.4 (see Fig. 3.7). During each time slot, $b^{(k)}_{p,q,u,m}$ is multiplied by N chips in the spreading sequence, $\{C^{(k)}_{p,q,n}\}_{n=1}^{N}$. The transmitted signal for the k-th user is also given by (3.1), but with

$$\tilde{s}^{(k)}_{u,m,i}(t) = \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} \sum_{n=1}^{N} A_{2,q}\, b^{(k)}_{p,q,u,m}\, C^{(k)}_{p,q,n}\, h_2\big\{t - [(pQ + q - 1)N + n - 1]T_c\big\}, \tag{3.30}$$

where $h_2(t)$ is the impulse response of the chip wave-shaping filter, and $A_{2,q}$ is the transmit amplitude during the q-th time slot, which is defined as

$$A_{2,q} = \begin{cases} A_{2e}, & q = 1, \ldots, Q_P, \\ A_{2d}, & q = Q_P + 1, \ldots, Q. \end{cases} \tag{3.31}$$

We choose

$$A_{2d} = \frac{A_{2e}}{\sqrt{2}}, \tag{3.32}$$

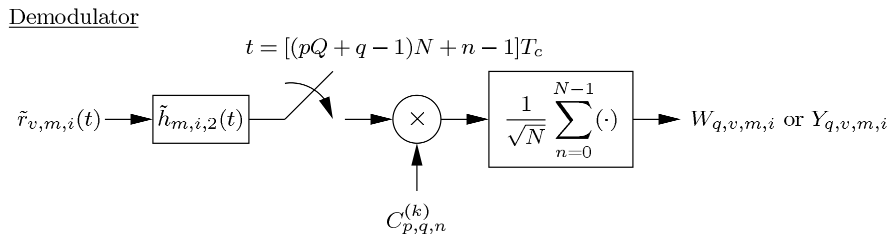

Figure 3.8: Complex equivalent lowpass version of the MC-DS-CDMA demodulator at frequency $f_{m,i}$ of the v-th receive antenna.

such that the energy per bit is equal to (see Appendix P.3)

$$E_{b2} = \frac{1}{2} R_2 A_{2e}^2 N = R_2 A_{2d}^2 N. \tag{3.33}$$

Channel

With $(\Delta f)_c = W_2$, we assume flat fading in each MC-DS-CDMA sub-band and independent fading between different sub-bands. As a result, the order of frequency diversity seen by each data symbol is R_2, and the overall diversity gain per data symbol is $4R_2$. Since $R_1 = NR_2$, we have $4MR_1/N = 4MR_2 \geq 4R_2$, which implies that *MC-CDMA has M times the diversity of MC-DS-CDMA*.

Receiver

We use the complex equivalent lowpass version of the MC-DS-CDMA demodulator in Fig. 2.6, where

$$\tilde{h}_{m,i,2}(t) = h_2^*(-t) \tag{3.34}$$

(see Fig. 3.8). In contrast to MC-CDMA, the despreading operation in MC-DS-CDMA involves multiplication by N chips. We define

$$x_2(t) \Leftrightarrow X_2(f) \triangleq |H_2(f)|^2, \tag{3.35}$$

where the expression for $X_2(f)$ is given by

$$X_2(f) = \begin{cases} T_c, & -\frac{1}{2T_c} < f < \frac{1}{2T_c}, \\ 0, & \text{otherwise}, \end{cases} \Leftrightarrow \; x_2(t) = \frac{\sin \pi t/T_c}{\pi t/T_c}. \tag{3.36}$$

Since $X_2(f)$ also satisfies the Nyquist criterion, inter-symbol interference is not present.

3.4 Performance Analysis

In this section, we present the expressions for $W_{q,v,m,i}$, $Y_{q,v,m,i}$, and $\hat{W}_{v,u,m,i}$ as they apply to MC-CDMA and MC-DS-CDMA. We also give closed-form results for the bit error probabilities of both schemes, assuming the desired user is $k = 1$ (hence, $\tau^{(1)} = 0$).

3.4.1 MC-CDMA

Demodulator Output

The output of the demodulator at frequency $f_{m,i}$ of the v-th antenna during the q-th time slot is given by

$$W_{q,v,m,i} = A_{1e}\, g^{(1)}_{v,u,m,i} + I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}}, \quad q = 1, \ldots, Q_P, \tag{3.37}$$

$$Y_{q,v,m,i} = A_{1d} \sum_{u=1}^{2} b^{(1)}_{p,q,u,m}\, g^{(1)}_{v,u,m,i} + I_{Y_{q,v,m,i}} + N_{Y_{q,v,m,i}}, \quad q = Q_P + 1, \ldots, Q, \tag{3.38}$$

where

$$I_{W_{q,v,m,i}} = C^{(1)}_{p,q,m,i} \sum_{k=2}^{K} \sum_{u=1}^{2} g^{(k)}_{v,u,m,i} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q} A_{1,q'} b^{(k)}_{p',q',u,m} C^{(k)}_{p',q',m,i} \cdot x_1[t - (p'Q + q' - 1)T_s - \tau^{(k)}]\Big|_{t=(pQ+q-1)T_s} \tag{3.39}$$

and

$$N_{W_{q,v,m,i}} = C^{(1)}_{p,q,m,i} \left(\tilde{n}_v(t) * \tilde{h}_{m,i,1}(t)\right)\Big|_{t=(pQ+q-1)T_s} \tag{3.40}$$

represent the multiple access interference (MAI) due to the other $K-1$ users and the contribution of the AWGN to the test statistic, respectively. The expressions for $I_{Y_{q,v,m,i}}$ and $N_{Y_{q,v,m,i}}$ are identical to (3.39) and (3.40), respectively, except that they are associated with a different range of values for q. Note that the desired signal and AWGN terms are both zero-mean complex Gaussian. Thus, in the single-user case, $W_{q,v,m,i}$ and $Y_{q,v,m,i}$ are both zero-mean complex Gaussian random variables. When K is large, the MAI terms are approximated as zero-mean complex Gaussian random variables, and therefore, $W_{q,v,m,i}$ and $Y_{q,v,m,i}$ are approximated as zero-mean complex Gaussian random variables as well.

Channel Estimation

Substituting (3.37) into (3.5) and setting $A_0 = A_{1e}$, the estimate of $g^{(1)}_{v,u,m,i}$ is given by

$$\hat{W}_{v,u,m,i} = \begin{cases} g^{(1)}_{v,u,m,i} + \dfrac{1}{A_{1e}} \dfrac{1}{Q_P/2} \displaystyle\sum_{\substack{q=1 \\ q \text{ odd}}}^{Q_P} \left(I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}} \right), & u = 1, \\ g^{(1)}_{v,u,m,i} + \dfrac{1}{A_{1e}} \dfrac{1}{Q_P/2} \displaystyle\sum_{\substack{q=1 \\ q \text{ even}}}^{Q_P} \left(I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}} \right), & u = 2. \end{cases} \tag{3.41}$$

Since $\hat{W}_{v,u,m,i}$ is a linear combination of the $\{W_{q,v,m,i}\}$, it is also a zero-mean complex Gaussian random variable when $K = 1$, and is approximated as a zero-mean complex Gaussian random variable when K is large.

From estimation theory, both $\{W_{q,v,m,i}\}_{q \text{ odd}}$ and $\{W_{q,v,m,i}\}_{q \text{ even}}$ represent *linear models*. The classical approach assumes $g^{(k)}_{v,u,m,i}$ is deterministic, and we can show that $\hat{W}_{v,u,m,i}$ is the minimum variance unbiased (MVU) estimator [29]. On the other hand, the Bayesian approach uses the fact that $g^{(1)}_{v,u,m,i}$ is zero-mean complex Gaussian. It can be shown that the minimum mean square error (MMSE) estimate of $g^{(1)}_{v,u,m,i}$ is just a scaled version of $\hat{W}_{v,u,m,i}$, and that both the MVU and MMSE estimators yield the same probability of error (see Appendix F).

Probability of Error

The probability of error for MC-CDMA is derived as (see Appendix M)

$$
\begin{aligned}
P_e = &\left(\frac{1}{2}\right)^{4(R_1-MR_1/N)} \sum_{i_2=0}^{4(R_1-MR_1/N)-1} \binom{4(R_1-MR_1/N)-1+i_2}{i_2} \left(\frac{1}{2}\right)^{i_2} \\
&\cdot \left(\frac{1-\mu_{31}}{2}\right)^{4MR_1/N} \sum_{i_3=0}^{4(R_1-MR_1/N)-1-i_2} \binom{4MR_1/N-1+i_3}{i_3} \left(\frac{1+\mu_{31}}{2}\right)^{i_3} \\
&\cdot \left(\frac{1-\mu_{41}}{2}\right)^{4MR_1/N} \sum_{r=0}^{4(R_1-MR_1/N)-1-i_2-i_3} \binom{4MR_1/N-1+r}{r} \left(\frac{1+\mu_{41}}{2}\right)^{r} \\
+ &\left(\frac{1-\mu_{13}}{2}\right)^{4(R_1-MR_1/N)} \sum_{i_1=0}^{4MR_1/N-1} \binom{4(R_1-MR_1/N)-1+i_1}{i_1} \left(\frac{1+\mu_{13}}{2}\right)^{i_1} \\
&\cdot \left(\frac{1-\mu_{23}}{2}\right)^{4(R_1-MR_1/N)} \sum_{i_2=0}^{4MR_1/N-1-i_1} \binom{4(R_1-MR_1/N)-1+i_2}{i_2} \left(\frac{1+\mu_{23}}{2}\right)^{i_2} \\
&\cdot \left(\frac{1-\mu_{43}}{2}\right)^{4MR_1/N} \sum_{r=0}^{4MR_1/N-1-i_1-i_2} \binom{4MR_1/N-1+r}{r} \left(\frac{1+\mu_{43}}{2}\right)^{r},
\end{aligned}
\tag{3.42}
$$

where

$$
\mu_{31} = \frac{R_b - \sqrt{d\left(\frac{E_{b1}}{N_0}\right)} - \sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right)}{R_b - \sqrt{d\left(\frac{E_{b1}}{N_0}\right)} + \sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right)}, \tag{3.43}
$$

$$
\mu_{41} = \frac{R_b + \sqrt{d\left(\frac{E_{b1}}{N_0}\right)} - \sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right)}{R_b + \sqrt{d\left(\frac{E_{b1}}{N_0}\right)} + \sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right)}, \tag{3.44}
$$

$$
\mu_{13} = \frac{\sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right) - \left(R_b - \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right)}{\sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right) + \left(R_b + \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right)}, \tag{3.45}
$$

$$
\mu_{23} = \frac{\sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right) + \left(R_b - \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right)}{\sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right) - \left(R_b + \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right)}, \tag{3.46}
$$

$$
\mu_{43} = \frac{R_b}{\sqrt{d\left(\frac{E_{b1}}{N_0}\right)}}. \tag{3.47}
$$

In the above equations, we have

$$d\left(\frac{E_{b1}}{N_0}\right) = R_b^2 + 2R_b\left(1+\frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right) + 2\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2, \tag{3.48}$$

where

$$c_1\left(\frac{E_{b1}}{N_0}\right) = (K-1) + \frac{R_1}{E_{b1}/N_0}. \tag{3.49}$$

Asymptotic Analysis—Perfect CSI

While the complexity of the above expressions makes it difficult to gain an intuitive understanding of the probability of error, the usefulness of this closed-form result becomes apparent when we consider the asymptotic case of perfect CSI. Suppose we have a completely static channel, such that the fade amplitudes are constant over an infinitely long data frame (i.e., $Q \to \infty$). If we let $Q_P \to \infty$, then the error terms in (3.41) vanish, and the channel estimation becomes perfect. Since

$$\lim_{Q_P\to\infty} d\left(\frac{E_{b1}}{N_0}\right) = R_b\left(R_b + 2c_1\left(\frac{E_{b1}}{N_0}\right)\right), \tag{3.50}$$

we have

$$\lim_{Q_P\to\infty} \mu_{31} = 1, \tag{3.51}$$

$$\lim_{Q_P\to\infty} \mu_{41} = 1, \tag{3.52}$$

$$\lim_{Q_P\to\infty} \mu_{13} = -1, \tag{3.53}$$

$$\lim_{Q_P\to\infty} \mu_{23} = -1, \tag{3.54}$$

$$\lim_{Q_P\to\infty} \mu_{43} = \sqrt{\frac{\bar{\gamma}_P}{1+\bar{\gamma}_P}}, \tag{3.55}$$

where

$$\bar{\gamma}_P = \frac{R_b}{2c_1\left(\frac{E_{b1}}{N_0}\right)} = \frac{1}{2}\left(\frac{1}{MR_1/N}\frac{E_{b1}}{N_0}\right)\frac{1}{1+(K-1)\frac{1}{R_1}\frac{E_{b1}}{N_0}}. \tag{3.56}$$

Using these results, we finally obtain

$$\lim_{Q_P\to\infty} P_e = \left(\frac{1-\mu}{2}\right)^{4MR_1/N} \sum_{i=0}^{4MR_1/N-1} \binom{4MR_1/N-1+i}{i} \left(\frac{1+\mu}{2}\right)^i, \tag{3.57}$$

where

$$\mu = \sqrt{\frac{\bar{\gamma}_P}{1+\bar{\gamma}_P}}. \tag{3.58}$$

Note that (3.57) is simply the probability of error of an ideal maximal ratio combiner with $4MR_1/N$-th order diversity and an average SNR or SINR per diversity branch of $\bar{\gamma}_P$ [27, Eqs. (14.4-15)–(14.4-16)].

3.4.2 MC-DS-CDMA

Demodulator Output

The demodulator output at frequency $f_{m,i}$ of the v-th antenna during the q-th time slot is given by

$$W_{q,v,m,i} = A_{2e}\sqrt{N} g^{(1)}_{v,u,m,i} + I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}}, \quad q = 1, \ldots, Q_P, \tag{3.59}$$

$$Y_{q,v,m,i} = A_{2d}\sqrt{N} \sum_{u=1}^{2} b^{(1)}_{p,q,u,m}\, g^{(1)}_{v,u,m,i} + I_{Y_{q,v,m,i}} + N_{Y_{q,v,m,i}}, \quad q = Q_P+1, \ldots, Q, \tag{3.60}$$

where

$$\begin{aligned} I_{W_{q,v,m,i}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \sum_{k=2}^{K}\sum_{u=1}^{2} g^{(k)}_{v,u,m,i} \sum_{p'=-\infty}^{\infty}\sum_{q'=1}^{Q}\sum_{n'=1}^{N} A_{2,q'} b^{(k)}_{p',q',u,m} C^{(k)}_{p',q',n'} \\ \cdot\, x_2\{t - [(p'Q+q'-1)N+n'-1]T_c - \tau^{(k)}\}\Big|_{t=[(pQ+q-1)N+n-1]T_c} \end{aligned} \tag{3.61}$$

and

$$N_{W_{q,v,m,i}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \left(\tilde{n}_v(t) * \tilde{h}_{m,i,2}(t)\right)\Big|_{t=[(pQ+q-1)N+n-1]T_c} \tag{3.62}$$

correspond to the MAI and AWGN terms, respectively. Again, the expressions for $I_{Y_{q,v,m,i}}$ and $N_{Y_{q,v,m,i}}$ are identical to (3.61) and (3.62), respectively. As in the MC-CDMA analysis, $W_{q,v,m,i}$ and $Y_{q,v,m,i}$ are both zero-mean complex Gaussian random variables in the single-user case and are approximated as zero-mean complex Gaussian random variables when K is large.

Channel Estimation

The expression for $\hat{W}_{v,u,m,i}$ is obtained by substituting (3.59) into (3.5):

$$\hat{W}_{v,u,m,i} = \begin{cases} g^{(1)}_{v,u,m,i} + \dfrac{1}{A_{2e}\sqrt{N}} \dfrac{1}{Q_P/2} \displaystyle\sum_{\substack{q=1 \\ q \text{ odd}}}^{Q_P} \big(I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}}\big), & u = 1, \\ g^{(1)}_{v,u,m,i} + \dfrac{1}{A_{2e}\sqrt{N}} \dfrac{1}{Q_P/2} \displaystyle\sum_{\substack{q=1 \\ q \text{ even}}}^{Q_P} \big(I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}}\big), & u = 2. \end{cases} \tag{3.63}$$

For MC-DS-CDMA, we require a normalizing constant of $A_0 = A_{2e}\sqrt{N}$.

Probability of Error

From Appendix L, the probability of error is derived as

$$P_e = \left(\frac{1-\mu}{2}\right)^{4R_2} \sum_{i=0}^{4R_2-1} \binom{4R_2 - 1 + i}{i} \left(\frac{1+\mu}{2}\right)^i, \tag{3.64}$$

where

$$\mu = \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}}, \tag{3.65}$$

$$\bar{\gamma} = \frac{1}{2}\left\{\left(1 + \frac{1}{Q_P/2}\right) c_2\!\left(\frac{E_{b2}}{N_0}\right) + \frac{1}{Q_P/2}\left[c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]^2\right\}^{-1}, \tag{3.66}$$

and

$$c_2\!\left(\frac{E_{b2}}{N_0}\right) = \frac{K-1}{N} + \frac{R_2}{E_{b2}/N_0}. \tag{3.67}$$

Note that this is just the probability of error of an ideal maximal ratio combiner [27, Eqs. (14.4-15),(14.4-16)] with $4R_2$-th order diversity, except that the average SNR or SINR per diversity branch has been replaced by $\bar{\gamma}$ of (3.66).

Asymptotic Analysis—Perfect CSI

Once again, the results for perfect CSI can be obtained by assuming both a static channel and an infinitely long data frame ($Q \to \infty$), and then computing the limits of the expressions in

(3.64)–(3.66) as $Q_P \to \infty$. Since Q_P is only present in $\bar{\gamma}$, we can simply replace $\bar{\gamma}$ in μ with

$$\bar{\gamma}_P \triangleq \lim_{Q_P \to \infty} \bar{\gamma} = \frac{1}{2c_2\left(\frac{E_{b2}}{N_0}\right)} = \frac{1}{2}\left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right)\frac{1}{1+\frac{K-1}{N}\frac{1}{R_2}\frac{E_{b2}}{N_0}}. \tag{3.68}$$

3.5 Numerical Results

3.5.1 Review of General Trends

We briefly summarize the general trends observed in Scenario #2 for the SISO system. When $M > 1$ and $K = 1$, the error probability curves for MC-CDMA and MC-DS-CDMA always cross. This cross-over point, which we denote by $(\Delta, P_{e,\Delta})$, implies that $P_{e,\text{MC-DS-CDMA}} < P_{e,\text{MC-CDMA}}$ for $E_b/N_0 < \Delta$ and $P_{e,\text{MC-DS-CDMA}} > P_{e,\text{MC-CDMA}}$ for $E_b/N_0 > \Delta$. Likewise, we also have $(E_b/N_0)_{\text{MC-DS-CDMA}} < (E_b/N_0)_{\text{MC-CDMA}}$ for $P_e > P_{e,\Delta}$ and $(E_b/N_0)_{\text{MC-DS-CDMA}} > (E_b/N_0)_{\text{MC-CDMA}}$ for $P_e < P_{e,\Delta}$. When $M > 1$ and $K > 1$, the MC-DS-CDMA and MC-CDMA curves cross only if Q_P is sufficiently large; otherwise, MC-DS-CDMA gives a lower P_e for all E_b/N_0. This is due to the fact that our detection schemes only involve matched filtering and lack a mechanism for combatting MAI. As a result, an error floor is present at high E_b/N_0, and the curves for both multi-carrier schemes flatten out.

In the figures to be presented shortly, we will see that these trends for the SISO system also apply to our MIMO system. This is because the multi-carrier comparisons in both cases are centered around the trade-off between diversity gain and channel estimation errors.

3.5.2 SISO vs. MIMO Comparison

Using the closed-form expressions for the bit error probabilities of MC-CDMA and MC-DS-CDMA presented in the previous section and in Section 2.3.5, we compare the MIMO results with those of Scenario #2 of the SISO system to study the impact of the additional diversity on the multi-carrier comparison. For both the SISO and MIMO systems, we assume $S_1 = MR_1 = 512$ and $S_2 = MR_2 = 8$ sub-carriers for MC-CDMA and MC-DS-CDMA, respectively, where the processing gain of each MC-DS-CDMA sub-band is $N = 64$. We set $E_{b1} = E_{b2} = E_b$ and consider both $K = 1$ and $K = 16$. The overall diversity gains per data symbol in MC-DS-CDMA and MC-CDMA are equal to R_2 and MR_1/N, respectively, in the SISO system, and $4R_2$ and $4MR_1/N$, respectively, in the MIMO system. As we pointed out earlier in Section 3.3, MC-CDMA has M times the diversity of MC-DS-CDMA. Thus, when $M = 1$, both multi-carrier schemes have equal diversity, and the probability of error was shown to be always lower in MC-DS-CDMA due to more reliable channel estimation (see Section 2.4). Henceforth, we only consider $M > 1$. Since the transmission of pilot symbols during the estimation phase occurs during every time slot in the SISO system, but only during every other time slot at each antenna in the MIMO system, we assume $Q_P = 16$ for the SISO system and consider both $Q_P = 16$ and $Q_P = 32$ for the MIMO system. When $Q_P = 16$, the MIMO system maintains the same throughput as the SISO system[4] (i.e., both systems transmit Q_D data symbols per frame at each sub-carrier), but when $Q_P = 32$, the MIMO system uses the same amount of total energy (i.e., same number of pilot symbols) as the SISO system to estimate each channel gain.

First, we consider the single-user scenario and compare the SISO and MIMO results against a target bit error rate of $P_e^* = 10^{-5}$. The case $M = 2$ is examined in Fig. 3.9. Here, MC-CDMA has twice the frequency diversity of MC-DS-CDMA. Note that the MC-DS-CDMA and MC-CDMA curves for the SISO system cross near P_e^*, which means that both multi-carrier schemes require approximately the same E_b/N_0 to achieve P_e^*. For the MIMO system, however, we observe that for both $Q_P = 16$ and $Q_P = 32$, the E_b/N_0 values required to achieve P_e^* are lower in MC-DS-CDMA by at least 3dB. Even if we reduce the target bit error rate to $P_e^* = 10^{-7}$, the required E_b/N_0 values are still lower in MC-DS-CDMA. Next, we consider $M = 8$. We observe

[4]We assume the SISO and MIMO systems use data frames of the same length (i.e., same Q).

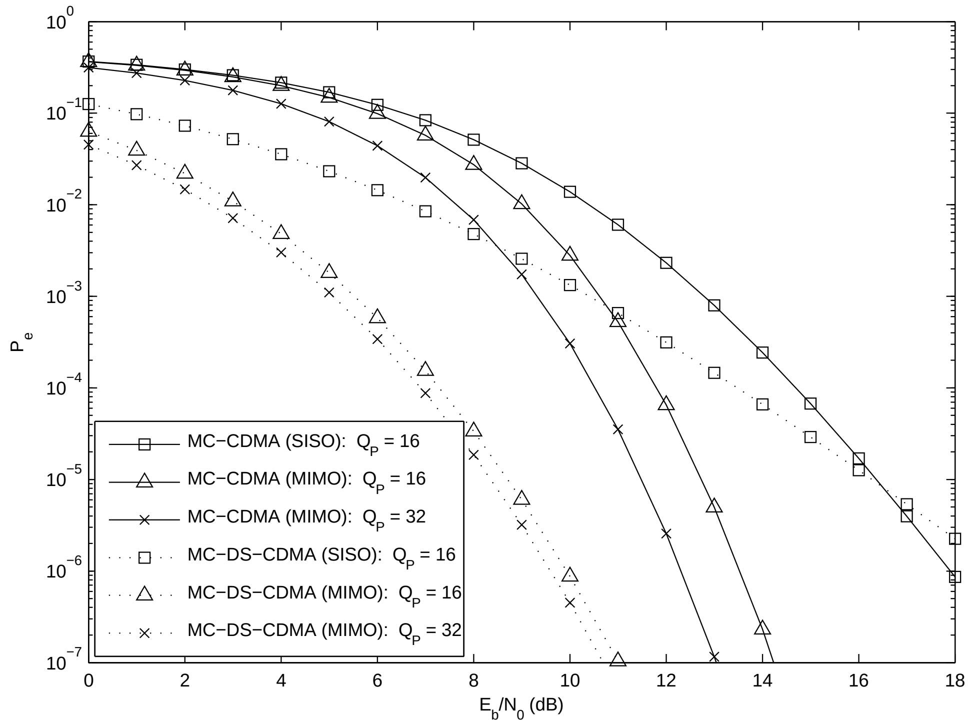

Figure 3.9: SISO vs. MIMO: $K = 1$, $M = 2$, $R_2 = 4$, $MR_1/N = 8$. The orders of diversity (per data symbol) in MC-DS-CDMA and MC-CDMA are equal to 4 and 8, respectively, in the SISO system, and 16 and 32, respectively, in the MIMO system.

in Fig. 3.10 that the cross-over points in the SISO case and in both instances of the MIMO case occur at P_e values which are greater than $P_e^* = 10^{-5}$. Thus, MC-CDMA achieves P_e^* at lower E_b/N_0 values than MC-DS-CDMA in both the SISO and MIMO cases, although the differences in the required E_b/N_0 values are much smaller in the two MIMO cases. Based on these results for $M = 2$ and $M = 8$, we conclude that for $K = 1$, the performance gains obtained by MC-DS-CDMA from the addition of spatial diversity are enough to offset the difference in frequency diversity between the two schemes when M is small, but not when M is large.

Now, consider a multi-user scenario with $K = 16$. When $M = 2$, we see in Fig. 3.11 that the MC-CDMA and MC-DS-CDMA curves do not cross in either the SISO or MIMO cases. Furthermore, at a given E_b/N_0, the differences in probability of error between the two schemes

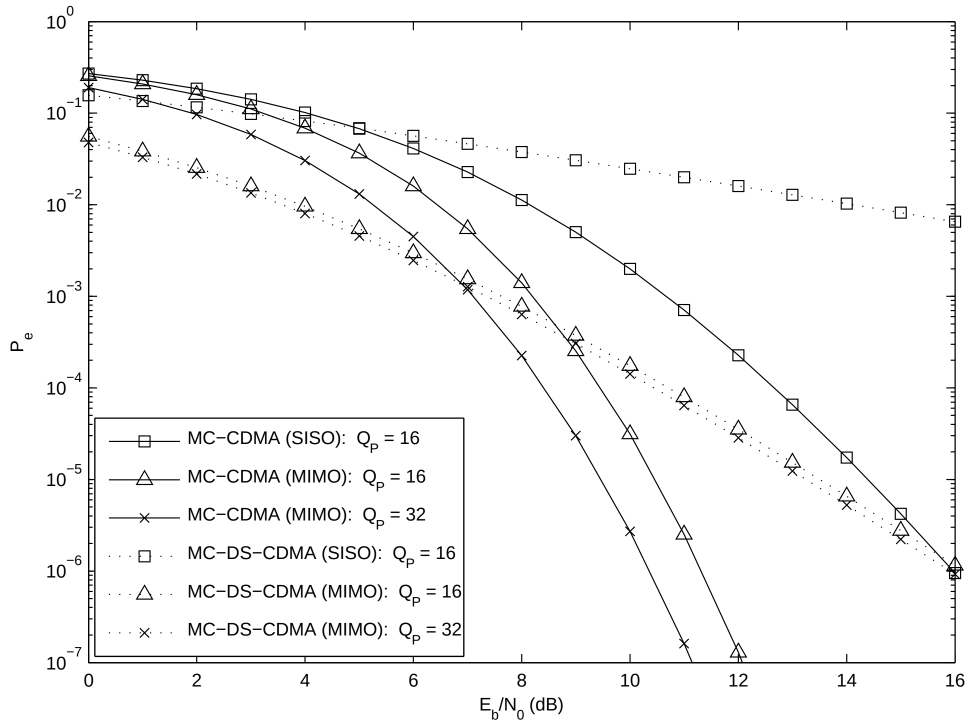

Figure 3.10: SISO vs. MIMO: $K = 1$, $M = 8$, $R_2 = 1$, $MR_1/N = 8$. The orders of diversity (per data symbol) in MC-DS-CDMA and MC-CDMA are equal to 1 and 8, respectively, in the SISO system, and 4 and 32, respectively, in the MIMO system.

are much larger in the two MIMO cases than in the SISO case. This implies that in order for the MC-CDMA and MC-DS-CDMA curves to cross, we will have to use a larger Q_P in the MIMO case than in the SISO case. When $M = 8$, the curves for the two schemes in Fig. 3.12 do cross in the case of the SISO system when $Q_P = 16$. In the MIMO system, however, a cross-over point does not exist for either $Q_P = 16$ or $Q_P = 32$; we must use $Q_P > 32$ in order for the curves to cross. Thus, we conclude that the additional diversity increases the value of Q_P required for the MC-CDMA and MC-DS-CDMA curves to cross and for there to be a trade-off in performance between the two multi-carrier schemes.

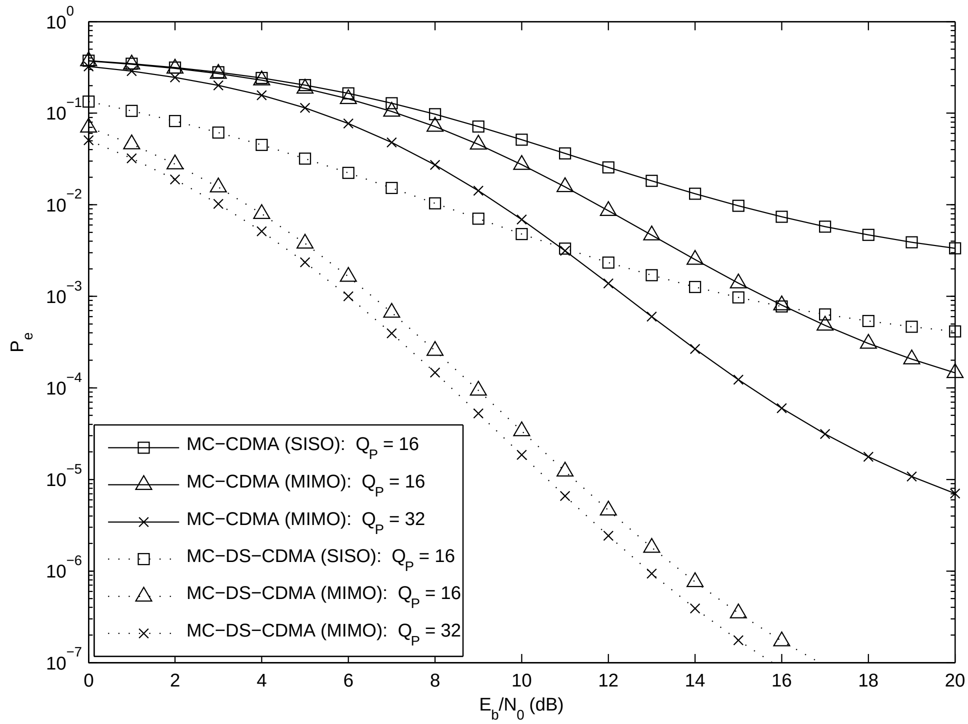

Figure 3.11: SISO vs. MIMO: $K = 16$, $M = 2$, $R_2 = 4$, $MR_1/N = 8$.

3.6 Conclusion

Our previous comparison between MC-DS-CDMA and MC-CDMA focused on the trade-off between diversity gain and channel estimation errors, but only a SISO system was considered. In this work, we incorporated spatial diversity into the comparison by using a MIMO system employing Alamouti space-time block coding. To quantify the effects of this additional diversity, we derived closed-form expressions for the bit error probabilities of the two multi-carrier schemes, and we compared the results of the MIMO system against those of the SISO system. Since we only considered those cases where MC-CDMA has higher frequency diversity than MC-DS-CDMA, we argued that the additional diversity benefits MC-DS-CDMA more than MC-CDMA. It was shown for the case of a single user that these gains for MC-DS-CDMA can offset the difference in

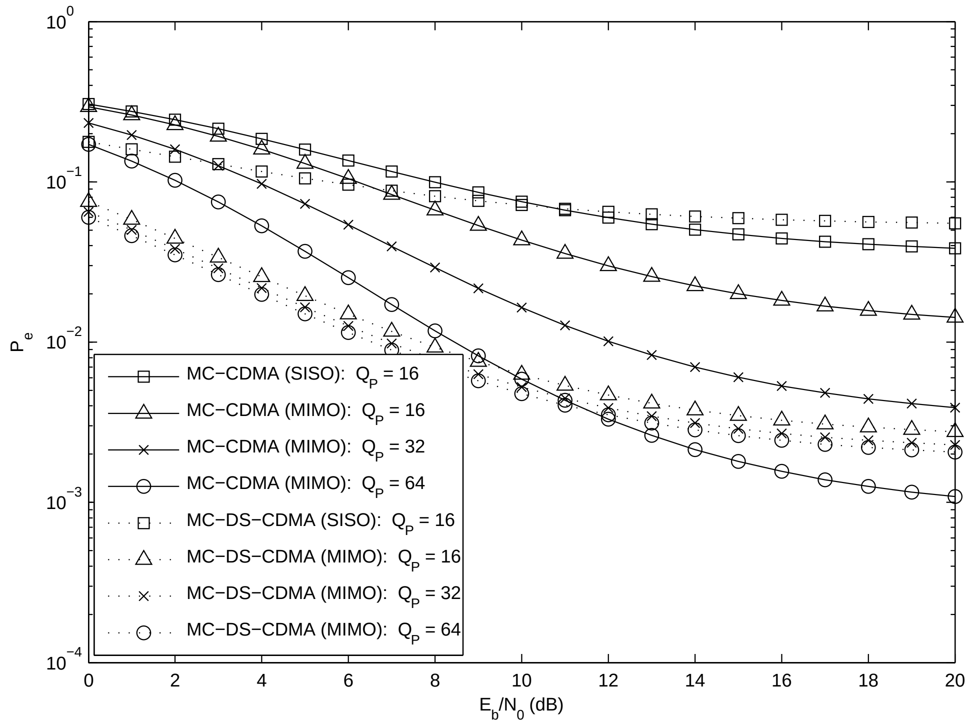

Figure 3.12: SISO vs. MIMO: $K = 16$, $M = 8$, $R_2 = 1$, $MR_1/N = 8$.

frequency diversity between the two schemes when M is small, but not when M is large, where M represents the number of parallel data streams. Also, when matched-filter-based detection is used for the case of multiple users, our results showed that the additional diversity increases the number of pilot symbols required to force a performance trade-off between the two schemes (i.e., to get the MC-CDMA and MC-DS-CDMA curves to cross).

3.7 Acknowledgements

The author is grateful to Dr. John Proakis for pointing out the correct definitions for the vectors $\tilde{\mathbf{w}}_{m,i}$ and $\tilde{\mathbf{y}}_{q,m,i}$ in Section 3.2.5.

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Effects of Spatial Diversity and Imperfect Channel Estimation on Wideband MC-DS-CDMA and MC-CDMA," submitted to *IEEE Transactions on Communications*. The dissertation author was the primary author of this paper.

Chapter 4

Conclusion

In this dissertation, we presented an analytical comparison of the error probability performances of MC-DS-CDMA and MC-CDMA. To ensure a fair comparison, both multi-carrier schemes were required to use (1) the same bandwidth, information rate, and energy-per-bit, (2) the same combining scheme at the receiver (i.e., maximal ratio combining), and (3) the same pilot-based scheme to estimate the channel gains at each sub-carrier frequency. We also forced both schemes to employ bandlimited, non-overlapping sub-bands. These constraints resulted in a possible trade-off between diversity gain and channel estimation errors between the two schemes.

In Chapter 2, we compared the two schemes for a SISO system under two different channel scenarios. We derived closed-form expressions of the probability of error for both schemes under both scenarios, and we compared the numerical results for different information rates, number of users, and number of pilot symbols per channel estimate. For the different cases considered, we showed that a performance trade-off between the two schemes *can* only occur under the following conditions:

1. The coherence bandwidth of the channel is equal to the bandwidth of one MC-DS-CDMA

sub-band (i.e., Scenario #2);

2. The number of parallel data streams, M, is such that $M > 1$.

This performance trade-off manifested itself in a cross-over point between the two curves; that is, there existed some Δ such that a lower bit error rate is achieved by using MC-DS-CDMA for $E_b/N_0 < \Delta$, and by using MC-CDMA for $E_b/N_0 > \Delta$. Such a trade-off implies that the effects of channel estimation errors dominate at low E_b/N_0 values, while the effects of diversity dominate at high E_b/N_0 values. When the above two conditions were satisfied, this trade-off always occurred in the single-user case; but in the multi-user case, when matched filtering was used at the receiver, this trade-off only occurred if the number of pilot symbols used in each channel estimate was large enough. In all other cases, we showed that the probability of error was uniformly lower in one multi-carrier scheme than the other (i.e., the bit error rate curves for the two schemes never cross).

In Chapter 3, we extended the comparison to a MIMO system employing dual transmit/receive diversity and Alamouti space-time block coding at each sub-carrier of both schemes. Here, we restricted the comparison to those cases where MC-CDMA has higher frequency diversity than MC-DS-CDMA (i.e., when the coherence bandwidth of the channel is equal to the bandwidth of one MC-DS-CDMA sub-band, and when $M > 1$). Once again, we derived closed-form expressions for the bit error rates of the two schemes, and we compared these MIMO results against those of the SISO system. For the single-user case, we chose a target bit error rate of $P_e^* = 10^{-5}$, and we showed that the performance gains in MC-DS-CDMA due to the additional diversity were enough to offset the difference in frequency diversity between the two schemes when M was small, but not when M was large. In the multi-user case, we observed that the additional diversity increased the number of pilot symbols required for there to be a trade-off in performance between the two multi-carrier schemes.

Aside from the trade-off studies involving MC-DS-CDMA and MC-CDMA, the other key contributions of this work include derivations of closed-form expressions for the coefficients of a partial fraction expansion and for the probability of error of a general Hermitian quadratic form in *zero-mean* complex Gaussian random variables. The combination of these two results suggests an alternative approach to performance analysis that is applicable to diversity systems using maximal ratio combining over Rayleigh fading channels with an arbitrary branch covariance matrix. The drawback of this approach is that it requires the evaluation of eigenvalues, but we showed in this dissertation that it is possible in certain cases (e.g., when the diversity channels are i.i.d.) to derive these eigenvalues in closed form.

Appendix A

Partial Fraction Expansion

In this appendix, we derive closed-form expressions for the coefficients of three different partial fraction expansions encountered in this dissertation. The approach we use is identical to the one in [30], but our results were obtained independently of [30]. We also present an iterative algorithm as an alternate method for computing the coefficients in the first of the three expansions.[1] This algorithm is essentially the same as the one given in [31], but our algorithm was derived independently of this work.

A.1 Expansion #1

A.1.1 Closed-Form Expression for the PFE Coefficients

Consider the partial fraction expansion (PFE)

$$\prod_{n=1}^{\Lambda} \frac{1}{(\lambda_n s + 1)^{m_n}} = \sum_{n=1}^{\Lambda} \sum_{k=1}^{m_n} \frac{c_{n,k}}{(\lambda_n s + 1)^{m_n - k + 1}}, \tag{A.1}$$

[1] This algorithm is easily extended to the other two expansions.

where m_n denotes the order of the pole at $s = -1/\lambda_n$, and the $\{c_{n,k}\}$ are the PFE coefficients. To derive a closed-form expression for $c_{n,k}$, we first multiply both sides of (A.1) by $(\lambda_n s + 1)^{m_n}$ to get

$$g_n(s) = h_n(s) + (\lambda_n s + 1)^{m_n} \sum_{\substack{i=1 \\ i \neq n}}^{\Lambda} \sum_{p=1}^{m_i} c_{i,p} \left(\lambda_i s + 1\right)^{-(m_i - p + 1)}, \quad n = 1, \dots, \Lambda, \tag{A.2}$$

where

$$g_n(s) = \prod_{\substack{i=1 \\ i \neq n}}^{\Lambda} \left(\lambda_i s + 1\right)^{-m_i}, \tag{A.3}$$

$$h_n(s) = \sum_{p=1}^{m_n} c_{n,p} \left(\lambda_n s + 1\right)^{p-1}. \tag{A.4}$$

Then, we evaluate the $(k-1)$-th derivative of (A.2) at $s = -1/\lambda_n$:

$$\left.\frac{d^{k-1}}{ds^{k-1}} g_n(s)\right|_{s=-1/\lambda_n} = \left.\frac{d^{k-1}}{ds^{k-1}} h_n(s)\right|_{s=-1/\lambda_n}, \quad k = 1, \dots, m_n. \tag{A.5}$$

To arrive at (A.5), we used the fact that

$$\left.\frac{d^{k-1}}{ds^{k-1}} \left((\lambda_n s + 1)^{m_n} \sum_{\substack{i=1 \\ i \neq n}}^{\Lambda} \sum_{p=1}^{m_i} c_{i,p} \left(\lambda_i s + 1\right)^{-(m_i - p + 1)} \right)\right|_{s=-1/\lambda_n} = 0, \quad k = 1, \dots, m_n. \tag{A.6}$$

This result is easily proved when we realize that each term in the derivative contains a factor of the form $(\lambda_n s + 1)^r$ $(r = 1, \dots, m_n)$, which evaluates to zero when $s = -1/\lambda_n$. Using the formula

$$\frac{d^k}{ds^k} (\lambda s + 1)^n = \frac{n!}{(n-k)!} \lambda^k (\lambda s + 1)^{n-k}, \quad n = 1, 2, \dots; \ k = 0, \dots, n, \tag{A.7}$$

which is easily proved by induction, and noting that

$$\left.\frac{d^{k-1}}{ds^{k-1}} \left(\lambda_n s + 1\right)^{p-1}\right|_{s=-1/\lambda_n} = 0, \quad k > p, \tag{A.8}$$

we have

$$\begin{aligned} \left.\frac{d^{k-1}}{ds^{k-1}} h_n(s)\right|_{s=-1/\lambda_n} &= \sum_{p=k}^{m_n} c_{n,p} \cdot \left.\frac{d^{k-1}}{ds^{k-1}} \left(\lambda_n s + 1\right)^{p-1}\right|_{s=-1/\lambda_n} \\ &= c_{n,k} \cdot (k-1)! \cdot \lambda_n^{k-1}, \quad k = 1, \dots, m_n. \end{aligned} \tag{A.9}$$

Substituting this result into (A.5) and using the expression for $g_n(s)$ in (A.3), we obtain

$$c_{n,k} = \frac{(1/\lambda_n)^{k-1}}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left\{ \prod_{\substack{i=1 \\ i\neq n}}^{\Lambda} (\lambda_i s + 1)^{-m_i} \right\} \Bigg|_{s=-1/\lambda_n} . \tag{A.10}$$

To evaluate the derivative in (A.10), we apply the theorem [32, Thm. 1]

$$\frac{d^n}{dx^n} \left[\prod_{i=1}^{m} f_i(x) \right] = n! \cdot \sum_{\Omega} \prod_{k=1}^{m} \frac{1}{i_k!} \frac{d^{i_k}}{dx^{i_k}} f_k(x), \quad \Omega = \left\{ i_1, \ldots, i_m \left| \sum_{p=1}^{m} i_p = n \right. \right\}, \tag{A.11}$$

and use the formula

$$\frac{d^k}{ds^k} (\lambda s + 1)^{-m} = (-1)^k \frac{(m-1+k)!}{(m-1)!} \lambda^k (\lambda s + 1)^{-(m+k)}, \quad m = 1, 2, \ldots, \tag{A.12}$$

which is also easily proved by induction. This yields

$$\frac{d^{k-1}}{ds^{k-1}} \left\{ \prod_{\substack{i=1 \\ i\neq n}}^{\Lambda} (\lambda_i s + 1)^{-m_i} \right\} = (k-1)! \sum_{\Omega_{n,k}} \prod_{\substack{q=1 \\ q\neq n}}^{\Lambda} (-1)^{i_q} \binom{m_q - 1 + i_q}{i_q} \lambda_q^{i_q} (\lambda_q s + 1)^{-(m_q + i_q)} , \tag{A.13}$$

where $\Omega_{n,k} = \left\{ i_1, \ldots, i_{n-1}, i_{n+1}, \ldots, i_\Lambda \left| \sum_{q=1, q\neq n}^{\Lambda} i_q = k-1 \right. \right\}$. Finally, we substitute (A.13) into (A.10) and use the relation

$$\lambda_n^{k-1} = \prod_{\substack{q=1 \\ q\neq n}}^{\Lambda} \lambda_n^{i_q}$$

to get

$$\begin{aligned} c_{n,k} &= (1/\lambda_n)^{k-1} \sum_{\Omega_{n,k}} \prod_{\substack{q=1 \\ q\neq n}}^{\Lambda} (-1)^{i_q} \binom{m_q - 1 + i_q}{i_q} \lambda_q^{i_q} (\lambda_q s + 1)^{-(m_q + i_q)} \Bigg|_{s=-1/\lambda_n} \\ &= \sum_{\Omega_{n,k}} \prod_{\substack{q=1 \\ q\neq n}}^{\Lambda} \binom{m_q - 1 + i_q}{i_q} \left(-\frac{\lambda_q}{\lambda_n} \right)^{i_q} \left(1 - \frac{\lambda_q}{\lambda_n} \right)^{-(m_q + i_q)} \\ &= \prod_{\substack{r=1 \\ r\neq n}}^{\Lambda} \left(1 - \frac{\lambda_r}{\lambda_n} \right)^{-m_r} \sum_{\Omega_{n,k}} \prod_{\substack{q=1 \\ q\neq n}}^{\Lambda} \binom{m_q - 1 + i_q}{i_q} \left(1 - \frac{\lambda_n}{\lambda_q} \right)^{-i_q} . \end{aligned} \tag{A.14}$$

If we define

$$\mu_{mn} = \frac{\lambda_m + \lambda_n}{\lambda_m - \lambda_n}, \quad m \neq n, \tag{A.15}$$

such that

$$\left(1-\frac{\lambda_m}{\lambda_n}\right)^{-1} = \frac{1-\mu_{mn}}{2},$$
$$\left(1-\frac{\lambda_n}{\lambda_m}\right)^{-1} = \frac{1+\mu_{mn}}{2},$$

then $c_{n,k}$ is re-written as

$$c_{n,k} = \prod_{\substack{r=1 \\ r\neq n}}^{\Lambda} \left(\frac{1-\mu_{rn}}{2}\right)^{m_r} \sum_{\Omega_{n,k}} \prod_{\substack{q=1 \\ q\neq n}}^{\Lambda} \binom{m_q-1+i_q}{i_q} \left(\frac{1+\mu_{qn}}{2}\right)^{i_q}. \tag{A.16}$$

We make a few important observations. First, we see from (A.14) that $c_{n,k}$ only depends on the ratios $\{\lambda_n/\lambda_q\}_{q\neq n}$ and not on the exact values of the $\{\lambda_n\}$. Therefore, for any constant b, we expect all expansions of the form

$$\prod_{n=1}^{\Lambda} \frac{1}{(b\lambda_n s+1)^{m_n}} = \sum_{n=1}^{\Lambda}\sum_{k=1}^{m_n} \frac{c_{n,k}}{(b\lambda_n s+1)^{m_n-k+1}} \tag{A.17}$$

to have the same set of PFE coefficients, since all of the $\{\lambda_n\}$ are multiplied by the same constant. Second, when $\Lambda = 2$, (A.16) yields simple expressions which are conducive to efficient numerical evaluation:

$$c_{1,k} = \left(\frac{1-\mu_{21}}{2}\right)^{m_2} \binom{m_2-1+(k-1)}{k-1} \left(\frac{1+\mu_{21}}{2}\right)^{k-1}, \quad k=1,\ldots,m_1, \tag{A.18}$$
$$c_{2,k} = \left(\frac{1-\mu_{12}}{2}\right)^{m_1} \binom{m_1-1+(k-1)}{k-1} \left(\frac{1+\mu_{12}}{2}\right)^{k-1}, \quad k=1,\ldots,m_2. \tag{A.19}$$

The expressions for $\{c_{n,k}\}$ when $\Lambda > 2$, however, are much more complicated, and we can calculate the $\{c_{n,k}\}$ more efficiently by using iterative techniques (see Appendix A.1.2 for an example). Finally, note that (A.1), when evaluated at $s = 0$, leads to the result

$$\sum_{n=1}^{\Lambda}\sum_{k=1}^{m_n} c_{n,k} = 1. \tag{A.20}$$

This relation can be used to ensure that no round-off errors have been made when the $\{c_{n,k}\}$ are calculated numerically.

A.1.2 Iterative Algorithm for the PFE Coefficients

We now derive an iterative algorithm for calculating the $\{c_{n,k}\}$. From (A.3) and (A.10), we have

$$c_{n,k} = \frac{(1/\lambda_n)^{k-1}}{(k-1)!} \left. \frac{d^{k-1}}{ds^{k-1}} g_n(s) \right|_{s=-1/\lambda_n} . \tag{A.21}$$

Since the first derivative of $g_n(s)$ is given by

$$\begin{aligned} \frac{d}{ds} g_n(s) &= -\sum_{\substack{i=1\\ i\neq n}}^{\Lambda} m_i \lambda_i \left(\lambda_i s + 1\right)^{-m_i - 1} \prod_{\substack{r=1\\ r\neq n, r\neq i}}^{\Lambda} \left(\lambda_r s + 1\right)^{-m_r} \\ &= -\sum_{\substack{i=1\\ i\neq n}}^{\Lambda} m_i \lambda_i \left(\lambda_i s + 1\right)^{-1} g_n(s), \end{aligned} \tag{A.22}$$

for $k = 1, \ldots, m_n - 1$, we expand the k-th derivative of $g_n(s)$ as

$$\begin{aligned} \frac{d^k}{ds^k} g_n(s) &= \frac{d^{k-1}}{ds^{k-1}} \left[\frac{d}{ds} g_n(s) \right] \\ &= -\sum_{\substack{i=1\\ i\neq n}}^{\Lambda} m_i \lambda_i \frac{d^{k-1}}{ds^{k-1}} \left[\left(\lambda_i s + 1\right)^{-1} g_n(s) \right] \\ &= -\sum_{\substack{i=1\\ i\neq n}}^{\Lambda} m_i \lambda_i \sum_{p=0}^{k-1} \binom{k-1}{p} \frac{d^p}{ds^p} g_n(s) \cdot \frac{d^{k-1-p}}{ds^{k-1-p}} \left(\lambda_i s + 1\right)^{-1}, \end{aligned} \tag{A.23}$$

where the expansion rule [33, Sec. 5.1.4(s)]

$$\frac{d^n}{dx^n}(uv) = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + \binom{n}{n} \frac{d^n v}{dx^n} u \tag{A.24}$$

was applied in the last step. Using (A.12) to obtain

$$\frac{d^{k-1-p}}{ds^{k-1-p}} \left(\lambda_i s + 1\right)^{-1} = (-1)^{k-1-p} (k-1-p)! \cdot \lambda_i^{k-1-p} \left(\lambda_i s + 1\right)^{-(k-p)}, \tag{A.25}$$

we show that the k-th derivative of $g_n(s)$, evaluated at $s = -1/\lambda_n$, is simply a function of all of its lower-order derivatives evaluated at $s = -1/\lambda_n$:

$$\left. \frac{d^k}{ds^k} g_n(s) \right|_{s=-1/\lambda_n} = \sum_{\substack{i=1\\ i\neq n}}^{\Lambda} m_i \sum_{p=0}^{k-1} \frac{(k-1)!}{p!} \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_i} \right)^{-(k-p)} \left. \frac{d^p}{ds^p} g_n(s) \right|_{s=-1/\lambda_n} . \tag{A.26}$$

For notational simplicity, define

$$g_n^{(k)} = \frac{d^k}{ds^k} g_n(s)\bigg|_{s=-1/\lambda_n}. \tag{A.27}$$

Thus, we can re-write (A.26) as

$$g_n^{(k)} = \sum_{\substack{i=1 \\ i \neq n}}^{\Lambda} m_i \sum_{p=0}^{k-1} \frac{(k-1)!}{p!} \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_i} \right)^{-(k-p)} g_n^{(p)}. \tag{A.28}$$

The iterative algorithm is summarized as follows:

1. Calculate $g_n^{(0)} = \prod_{\substack{i=1 \\ i \neq n}}^{\Lambda} \left(1 - \frac{\lambda_i}{\lambda_n} \right)^{-m_i}$.
2. Calculate $g_n^{(1)}$, $g_n^{(2)}$, ..., $g_n^{(m_n - 1)}$ using (A.28).
3. Calculate $c_{n,1}$, $c_{n,2}$, ..., c_{n,m_n} using (A.21).

Again, we can confirm that round-off errors are not present by making sure the $\{c_{n,k}\}$ satisfy (A.20).

A.2 Expansion #2

Now, consider the PFE

$$\prod_{l=1}^{L} \frac{\upsilon_{1l}^R}{(\upsilon + j\upsilon_{1l})^R} = \sum_{l=1}^{L} \sum_{k=1}^{R} c_{l,k} \frac{\upsilon_{1l}^k}{(\upsilon + j\upsilon_{1l})^{R-k+1}}. \tag{A.29}$$

This equation can be re-written as

$$\prod_{l=1}^{L} \frac{1}{\left(\frac{\upsilon}{\upsilon_{1l}} + j \right)^R} = \sum_{l=1}^{L} \sum_{k=1}^{R} \frac{c_{l,k}}{\left(\frac{\upsilon}{\upsilon_{1l}} + j \right)^{R-k+1}}. \tag{A.30}$$

We follow the same steps as in Appendix A.1.1 to derive a closed-form expression for $c_{l,k}$.

First, we multiply both sides of (A.30) by $\left(\frac{\upsilon}{\upsilon_{1l}} + j \right)^R$ to get

$$g_l(\upsilon) = h_l(\upsilon) + \left(\frac{\upsilon}{\upsilon_{1l}} + j \right)^R \sum_{\substack{i=1 \\ i \neq l}}^{L} \sum_{p=1}^{R} c_{i,p} \left(\frac{\upsilon}{\upsilon_{1i}} + j \right)^{-(R-p+1)}, \quad n = 1, \ldots, L, \tag{A.31}$$

where

$$g_l(\upsilon) = \prod_{\substack{i=1 \\ i\neq l}}^{L} \left(\frac{\upsilon}{\upsilon_{1i}} + j\right)^{-R}, \tag{A.32}$$

$$h_l(\upsilon) = \sum_{p=1}^{R} c_{l,p} \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{p-1}. \tag{A.33}$$

Next, we take the $(k-1)$-th derivative of (A.31) and evaluate the resulting equation at $\upsilon = -j\upsilon_{1l}$ to obtain

$$\frac{d^{k-1}}{d\upsilon^{k-1}} g_l(\upsilon)\bigg|_{\upsilon=-j\upsilon_{1l}} = \frac{d^{k-1}}{d\upsilon^{k-1}} h_l(\upsilon)\bigg|_{\upsilon=-j\upsilon_{1l}}, \quad k = 1, \ldots, R, \tag{A.34}$$

where we used the fact that

$$\frac{d^{k-1}}{d\upsilon^{k-1}} \left[\left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{R} \sum_{\substack{i=1 \\ i\neq l}}^{L} \sum_{p=1}^{R} c_{i,p} \left(\frac{\upsilon}{\upsilon_{1i}} + j\right)^{-(R-p+1)}\right]\Bigg|_{\upsilon=-j\upsilon_{1l}} = 0, \quad k = 1, \ldots, R. \tag{A.35}$$

By using the formula

$$\frac{d^k}{d\upsilon^k} \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{m} = \frac{m!}{(m-k)!} \left(\frac{1}{\upsilon_{1l}}\right)^{k} \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{m-k}, \quad m = 1, 2, \ldots;\ k = 0, \ldots, m, \tag{A.36}$$

which is easily proved by induction, we can show that

$$\begin{aligned} \frac{d^{k-1}}{d\upsilon^{k-1}} h_l(\upsilon)\bigg|_{\upsilon=-j\upsilon_{1l}} &= \sum_{p=k}^{R} c_{l,p} \cdot \frac{d^{k-1}}{d\upsilon^{k-1}} \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{p-1}\Bigg|_{\upsilon=-j\upsilon_{1l}} \\ &= c_{l,k} \cdot \frac{(k-1)!}{\upsilon_{1l}^{k-1}}, \quad k = 1, \ldots, R. \end{aligned} \tag{A.37}$$

Substituting this result into (A.34), we get

$$c_{l,k} = \frac{\upsilon_{1l}^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\upsilon^{k-1}} \left\{\prod_{\substack{i=1 \\ i\neq l}}^{L} \left(\frac{\upsilon}{\upsilon_{1i}} + j\right)^{-R}\right\}\Bigg|_{\upsilon=-j\upsilon_{1l}}. \tag{A.38}$$

The derivative in (A.38) is evaluated by applying both (A.11) and the formula

$$\frac{d^k}{d\upsilon^k} \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{-m} = (-1)^k \frac{(m-1+k)!}{(m-1)!} \left(\frac{1}{\upsilon_{1l}}\right)^{k} \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{-(m+k)}, \quad m = 1, 2, \ldots, \tag{A.39}$$

which is also easily proved by induction. Thus, we have

$$c_{l,k} = \frac{\upsilon_{1l}^{k-1}}{(k-1)!} \cdot (k-1)! \sum_{\Omega_{l,k}} \prod_{\substack{q=1 \\ q \neq l}}^{L} (-1)^{i_q} \binom{R-1+i_q}{i_q} \left(\frac{1}{\upsilon_{1q}}\right)^{i_q} \left(\frac{\upsilon}{\upsilon_{1q}} + j\right)^{-(R+i_q)} \Bigg|_{\upsilon=-j\upsilon_{1l}}$$
$$= \upsilon_{1l}^{k-1} \sum_{\Omega_{l,k}} \prod_{\substack{p=1 \\ p \neq l}}^{L} \left(\frac{\upsilon}{\upsilon_{1p}} + j\right)^{-R} \prod_{\substack{q=1 \\ q \neq l}}^{L} (-1)^{i_q} \binom{R-1+i_q}{i_q} \left(\frac{1}{\upsilon_{1q}}\right)^{i_q} \left(\frac{\upsilon}{\upsilon_{1q}} + j\right)^{-i_q} \Bigg|_{\upsilon=-j\upsilon_{1l}}, \tag{A.40}$$

where $\Omega_{l,k} = \left\{ i_1, \ldots, i_{l-1}, i_{l+1}, \ldots, i_L \,\middle|\, \sum_{q=1, q\neq l}^{L} i_q = k-1 \right\}$. Using the relation

$$\upsilon_{1l}^{k-1} = \prod_{\substack{q=1 \\ q \neq l}}^{L} \upsilon_{1l}^{i_q},$$

we simplify (A.40) as

$$c_{l,k} = \prod_{\substack{p=1 \\ p \neq l}}^{L} \left(\frac{\upsilon}{\upsilon_{1p}} + j\right)^{-R} \sum_{\Omega_{l,k}} \prod_{\substack{q=1 \\ q \neq l}}^{L} (-1)^{i_q} \binom{R-1+i_q}{i_q} \left(\frac{\upsilon_{1l}}{\upsilon_{1q}}\right)^{i_q} \left(\frac{\upsilon}{\upsilon_{1q}} + j\right)^{-i_q} \Bigg|_{\upsilon=-j\upsilon_{1l}}$$
$$= j^{-R(L-1)} \cdot \prod_{\substack{p=1 \\ p \neq l}}^{L} \left(1 - \frac{\upsilon_{1l}}{\upsilon_{1p}}\right)^{-R} \sum_{\Omega_{l,k}} \prod_{\substack{q=1 \\ q \neq l}}^{L} (-1)^{i_q} \binom{R-1+i_q}{i_q} j^{-i_q} \frac{\left(\frac{\upsilon_{1l}}{\upsilon_{1q}}\right)^{i_q}}{\left(1 - \frac{\upsilon_{1l}}{\upsilon_{1q}}\right)^{i_q}}$$
$$= (-j)^{R(L-1)+k-1} \cdot \prod_{\substack{p=1 \\ p \neq l}}^{L} \left(1 - \frac{\upsilon_{1l}}{\upsilon_{1p}}\right)^{-R} \sum_{\Omega_{l,k}} \prod_{\substack{q=1 \\ q \neq l}}^{L} \binom{R-1+i_q}{i_q} \left(1 - \frac{\upsilon_{1q}}{\upsilon_{1l}}\right)^{-i_q}. \tag{A.41}$$

If we compare (A.29) and (A.41) against (A.1) and (A.14), respectively, we note that we can write $c_{l,k}$ as

$$c_{l,k} = (-j)^{R(L-1)+k-1} \cdot \tilde{c}_{l,k}, \tag{A.42}$$

where the $\{\tilde{c}_{l,k}\}$ are the coefficients in the PFE

$$\prod_{l=1}^{L} \left(\frac{\upsilon}{\upsilon_{1l}} + 1\right)^{-R} = \sum_{l=1}^{L} \sum_{k=1}^{R} \tilde{c}_{l,k} \left(\frac{\upsilon}{\upsilon_{1l}} + 1\right)^{-(R-k+1)}. \tag{A.43}$$

Also, if we substitute $\upsilon = 0$ into (A.30) and use the relation in (A.42), we have

$$\prod_{l=1}^{L} j^{-R} = \sum_{l=1}^{L} \sum_{k=1}^{R} c_{l,k} \cdot j^{-(R-k+1)}$$

$$j^{-RL} = \sum_{l=1}^{L}\sum_{k=1}^{R}(-j)^{R(L-1)+k-1}\cdot\tilde{c}_{l,k}\cdot j^{-(R-k+1)}$$
$$1 = \sum_{l=1}^{L}\sum_{k=1}^{R}\tilde{c}_{l,k}, \tag{A.44}$$

which agrees with the result in (A.20).

A.3 Expansion #3

Finally, we also consider the PFE

$$\prod_{l=1}^{L}\frac{\upsilon_{2l}^{R}}{(\upsilon - j\upsilon_{2l})^{R}} = \sum_{l=1}^{L}\sum_{k=1}^{R} d_{l,k}\frac{\upsilon_{2l}^{k}}{(\upsilon - j\upsilon_{2l})^{R-k+1}}, \tag{A.45}$$

which we can re-write as

$$\prod_{l=1}^{L}\frac{1}{\left(\frac{\upsilon}{\upsilon_{2l}} - j\right)^{R}} = \sum_{l=1}^{L}\sum_{k=1}^{R}\frac{d_{l,k}}{\left(\frac{\upsilon}{\upsilon_{2l}} - j\right)^{R-k+1}}. \tag{A.46}$$

To obtain a closed-form expression for $d_{l,k}$, we simply replace υ_{1l} with υ_{2l} and j with $-j$ in the derivation for $c_{l,k}$ (see Appendix A.2):

$$d_{l,k} = j^{R(L-1)+k-1}\cdot\prod_{\substack{p=1\\p\neq l}}^{L}\left(1-\frac{\upsilon_{2l}}{\upsilon_{2p}}\right)^{-R}\sum_{\Omega_{l,k}}\prod_{\substack{q=1\\q\neq l}}^{L}\binom{R-1+i_q}{i_q}\left(1-\frac{\upsilon_{2q}}{\upsilon_{2l}}\right)^{-i_q}. \tag{A.47}$$

We can express $d_{l,k}$ as

$$d_{l,k} = j^{R(L-1)+k-1}\cdot\tilde{d}_{l,k}, \tag{A.48}$$

where the $\{\tilde{d}_{l,k}\}$ are the coefficients in the PFE

$$\prod_{l=1}^{L}\left(\frac{\upsilon}{\upsilon_{2l}}+1\right)^{-R} = \sum_{l=1}^{L}\sum_{k=1}^{R}\tilde{d}_{l,k}\left(\frac{\upsilon}{\upsilon_{2l}}+1\right)^{-(R-k+1)}. \tag{A.49}$$

Substituting $\upsilon = 0$ into (A.46) and using (A.48), it is easily shown that

$$\sum_{l=1}^{L}\sum_{k=1}^{R}\tilde{d}_{l,k} = 1. \tag{A.50}$$

Appendix B

Kronecker Products

Given a $m \times n$ matrix $\mathbf{A} = \{a_{ij}\}$ and a $r \times s$ matrix $\mathbf{B} = \{b_{ij}\}$, the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is defined as the $mr \times ns$ partitioned matrix [33, Sec. 2.6.18]

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}.$$

B.1 Examples

Assuming $\mathbf{I}_p$ denotes a $p \times p$ identity matrix, we have the following:

1. $\mathbf{I}_p \otimes \mathbf{A} = \mathrm{diag}(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A})$.

2. $\mathbf{A} \otimes \mathbf{I}_p = \begin{bmatrix} a_{11}\mathbf{I}_p & a_{12}\mathbf{I}_p & \cdots & a_{1n}\mathbf{I}_p \\ a_{21}\mathbf{I}_p & a_{22}\mathbf{I}_p & \cdots & a_{2n}\mathbf{I}_p \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{I}_p & a_{m2}\mathbf{I}_p & \cdots & a_{mn}\mathbf{I}_p \end{bmatrix}.$

3. $\mathbf{I}_p \otimes \mathbf{I}_q = \mathbf{I}_{pq}$.

B.2 Properties

The following properties for the Kronecker product are taken from [33, Sec. 2.6.18]:

1. If $\mathbf{z}$ and $\mathbf{w}$ are vectors of appropriate dimensions, then $\mathbf{Az} \otimes \mathbf{Bw} = (\mathbf{A} \otimes \mathbf{B})(\mathbf{z} \otimes \mathbf{w})$.

2. If α is a scalar, then $(\alpha\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (\alpha\mathbf{B}) = \alpha(\mathbf{A} \otimes \mathbf{B})$.

3. The Kronecker product is distributive with respect to addition:

   (a) $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$,

   (b) $\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$.

4. The Kronecker product is associative: $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$.

5. $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$.

6. The *mixed product rule*: If the dimensions of the matrices are such that the following expressions exist, then $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$.

7. If the inverses exist, then $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$.

8. If $\{\lambda_i\}$ and $\{\mathbf{x}_i\}$ are the eigenvalues and the corresponding eigenvectors for $\mathbf{A}$, and $\{\mu_j\}$ and $\{\mathbf{y}_j\}$ are the eigenvalues and the corresponding eigenvectors for $\mathbf{B}$, then $\mathbf{A} \otimes \mathbf{B}$ has eigenvalues $\{\lambda_i \mu_j\}$ with corresponding eigenvectors $\{\mathbf{x}_i \otimes \mathbf{y}_j\}$.

9. If matrix $\mathbf{A}$ has size $n \times n$ and $\mathbf{B}$ has size $m \times m$, then $\det(\mathbf{A} \otimes \mathbf{B}) = (\det \mathbf{A})^m (\det \mathbf{B})^n$.

Now, assume $\mathbf{C}$ and $\mathbf{D}$ are both $p \times p$ matrices. Using the above properties, we prove the following corollaries:

Corollary 1.

$$(\mathbf{I}_n \otimes \mathbf{C} \otimes \mathbf{I}_m) + (\mathbf{I}_n \otimes \mathbf{D} \otimes \mathbf{I}_m) = \mathbf{I}_n \otimes (\mathbf{C} + \mathbf{D}) \otimes \mathbf{I}_m.$$

Proof. Apply property 3 twice. □

Corollary 2.

$$\det(\mathbf{I}_n \otimes \mathbf{C} \otimes \mathbf{I}_m) = (\det \mathbf{C})^{nm}.$$

Proof. Apply property 9 twice. □

Corollary 3.

$$(\mathbf{I}_n \otimes \mathbf{C} \otimes \mathbf{I}_m)^{-1} = \mathbf{I}_n \otimes \mathbf{C}^{-1} \otimes \mathbf{I}_m.$$

Proof. Apply property 7 twice. □

Corollary 4.

$$(\mathbf{I}_n \otimes \mathbf{C} \otimes \mathbf{I}_m)(\mathbf{I}_n \otimes \mathbf{D} \otimes \mathbf{I}_m) = \mathbf{I}_n \otimes \mathbf{CD} \otimes \mathbf{I}_m.$$

Proof. Apply property 6 twice. □

Corollary 5.

$$(\mathbf{I}_n \otimes \mathbf{C} \otimes \mathbf{I}_m)^H = \mathbf{I}_n \otimes \mathbf{C}^H \otimes \mathbf{I}_m.$$

Proof. Extend property 5 to the case of the Hermitian operator, $(\cdot)^H$, and apply it twice. □

Appendix C

Special Matrices

In this appendix, we first review some inverse formulas, as well as a few important concepts involving the algebraic multiplicity and geometric multiplicity of an eigenvalue. Then, we derive several identities for diagonal matrices and for the matrices $\mathbf{A}_n$, $\mathbf{B}_2(\epsilon)$, and $\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon)$.

C.1 Inverse Formulas

We list some important results for the inverse of a matrix.

1. Given a general 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{C.1}$$

   the inverse of $\mathbf{A}$ is equal to

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}, \tag{C.2}$$

   where $|\mathbf{A}| = a_{11}a_{22} - a_{21}a_{12}$.

2. The *matrix inversion lemma* is given by [29, Sec. A1.1.3]

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{DA}^{-1}\mathbf{B} + \mathbf{C}^{-1})^{-1}\mathbf{DA}^{-1}, \tag{C.3}$$

   where $\mathbf{A}$ is $n \times n$, $\mathbf{B}$ is $n \times m$, $\mathbf{C}$ is $m \times m$, and $\mathbf{D}$ is $m \times n$, and $\mathbf{A}$ and $\mathbf{C}$ are nonsingular.

3. A special case of the matrix inversion lemma, known as *Woodbury's identity*, occurs when $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{u}$ is an $n \times 1$ column vector [29, Sec. A1.1.3]:

$$(\mathbf{A} + \mathbf{uu}^T)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{uu}^T\mathbf{A}^{-1}}{1 + \mathbf{u}^T\mathbf{A}^{-1}\mathbf{u}}. \tag{C.4}$$

C.2 Algebraic Multiplicity and Geometric Multiplicity of an Eigenvalue

Given a matrix $\mathbf{A}$, we have the following:

Definition 1. [34, Sec. 1.4.3] *The dimension of the eigenspace of a matrix* $\mathbf{A}$ *corresponding to the eigenvalue* λ *is called the geometric multiplicity of the eigenvalue* λ*.*

Definition 2. [34, Sec. 1.4.3] *The multiplicity of* λ *as a zero of the characteristic polynomial* $\det(\lambda\mathbf{I} - \mathbf{A})$ *is called the algebraic multiplicity of the eigenvalue* λ*.*

Definition 3. [34, Sec. 1.4.4] *If the geometric multiplicity is the same as the algebraic multiplicity for each eigenvalue,* $\mathbf{A}$ *is said to be nondefective.*

Definition 4. [34, Sec. 2.5.1] *A matrix* $\mathbf{A}$ *is said to be normal if* $\mathbf{AA}^H = \mathbf{A}^H\mathbf{A}$*.*

Corollary 6. [34, p. 103] *A normal matrix is nondefective.*

In the work that follows, unless otherwise noted, the term multiplicity implies *algebraic* multiplicity.

C.3 Diagonal Matrices

Consider a general $n \times n$ diagonal matrix

$$\mathbf{D}_n \triangleq \begin{bmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn} \end{bmatrix} \equiv \mathrm{diag}(d_{11}, \ldots, d_{nn}). \tag{C.5}$$

For any scalar c, we have

$$\lambda \mathbf{I}_n - c\mathbf{D}_n = \mathrm{diag}(\lambda - cd_{11}, \ldots, \lambda - cd_{nn}). \tag{C.6}$$

The characteristic polynomial of $c\mathbf{D}_n$ is simply the determinant of (C.6):

$$\det(\lambda \mathbf{I}_n - c\mathbf{D}_n) = \prod_{i=1}^{n} (\lambda - cd_{ii}). \tag{C.7}$$

Assuming the diagonal entries of $\mathbf{D}_n$ are distinct, the n eigenvalues of $c\mathbf{D}_n$ are then given by

$$\lambda_i = cd_{ii}, \quad \text{multiplicity } 1, \quad i = 1, \ldots, n. \tag{C.8}$$

If $\mathbf{D}_n = \mathbf{I}_n$, then $d_{ii} = 1$ for all i, and $c\mathbf{I}_n$ only has one distinct eigenvalue:

$$\lambda = c, \quad \text{multiplicity } n. \tag{C.9}$$

The inverse of $\mathbf{D}_n$ is also a diagonal matrix:

$$\mathbf{D}_n^{-1} = \begin{bmatrix} d_{11}^{-1} & & \\ & \ddots & \\ & & d_{nn}^{-1} \end{bmatrix} = \mathrm{diag}(d_{11}^{-1}, \ldots, d_{nn}^{-1}). \tag{C.10}$$

We apply this result to obtain the inverse of (C.6) as

$$(\lambda \mathbf{I}_n - c\mathbf{D}_n)^{-1} = \mathrm{diag}[(\lambda - cd_{11})^{-1}, \ldots, (\lambda - cd_{nn})^{-1}]. \tag{C.11}$$

If $\mathbf{D}_n = \mathbf{I}_n$, then (C.11) reduces to

$$(\lambda \mathbf{I}_n - c\mathbf{I}_n)^{-1} = (\lambda - c)^{-1} \mathbf{I}_n. \tag{C.12}$$

C.4 The Matrix $\mathbf{A}_n$

Consider the $n \times n$ *all-ones* matrix

$$\mathbf{A}_n \triangleq \mathbf{a}_n \mathbf{a}_n^T, \tag{C.13}$$

where $\mathbf{a}_n = [1 \cdots 1]^T$ is the all-ones column vector of length n. Since $\mathbf{A}_n$ is the outer product of two column vectors, it is a *rank-one* matrix. Note that

$$\begin{aligned} \mathbf{A}_n \mathbf{A}_n &= \mathbf{a}_n (\mathbf{a}_n^T \mathbf{a}_n) \mathbf{a}_n^T \\ &= n \cdot \mathbf{a}_n \mathbf{a}_n^T \\ &= n \mathbf{A}_n, \end{aligned} \tag{C.14}$$

which follows from the fact that $\mathbf{a}_n^T \mathbf{a}_n = n$. We can easily verify that $\mathbf{A}_n$ is *normal*, i.e., $\mathbf{A}_n \mathbf{A}_n^H = \mathbf{A}_n^H \mathbf{A}_n$. Hence, $\mathbf{A}_n$ is *nondefective*, and the geometric multiplicity is the same as the algebraic multiplicity for each eigenvalue of $\mathbf{A}_n$ (see Section C.2). We use this to make the following claim.

Corollary 7. *The eigenvalues of $\mathbf{A}_n$ are given by*

$$\lambda_1 = n, \quad \textit{multiplicity } 1, \tag{C.15}$$

$$\lambda_2 = 0, \quad \textit{multiplicity } n-1. \tag{C.16}$$

As a result, the characteristic polynomial of $\mathbf{A}_n$ is equal to

$$\det(\lambda \mathbf{I}_n - \mathbf{A}_n) = \lambda^{n-1}(\lambda - n). \tag{C.17}$$

Proof. The vector $\mathbf{x}_1 = \mathbf{a}_n$ is an eigenvector of $\mathbf{A}_n$ with eigenvalue $\lambda_1 = n$, since $\mathbf{A}_n \mathbf{a}_n = n \cdot \mathbf{a}_n$. The remaining eigenvectors $\mathbf{x}_2, \ldots, \mathbf{x}_n$ span an eigenspace of dimension $n-1$ and are orthogonal to $\mathbf{a}_n$, i.e., $\mathbf{a}_n^T \mathbf{x}_i = 0$ ($\mathbf{x}_i \neq \mathbf{0}$) for $i = 2, \ldots, n$. This implies that the sum of the elements of $\mathbf{x}_i$ ($i = 2, \ldots, n$) must add up to 0. Since each row of $\mathbf{A}_n$ is equal to $\mathbf{a}_n^T$, we have $\lambda_i \mathbf{x}_i = \mathbf{A}_n \mathbf{x}_i = \mathbf{0}$

for $i = 2, \ldots, n$, which implies $\lambda_i = 0$ for $i = 2, \ldots, n$. Thus, the eigenvalues of $\mathbf{A}_n$ are equal to n and 0, with *geometric* multiplicities of 1 and $n-1$, respectively. Eq. (C.17) then follows from the fact that $\mathbf{A}_n$ is nondefective, so the algebraic multiplicity is equal to the geometric multiplicity for each eigenvalue. □

For any scalar c, the characteristic polynomial of $c\mathbf{A}_n$, $\det(\lambda\mathbf{I}_n - c\mathbf{A}_n)$, is obtained by using simple algebraic manipulations and by applying (C.17):

$$\begin{aligned}\det(\lambda\mathbf{I}_n - c\mathbf{A}_n) &= \det\left[c\left(\frac{\lambda}{c}\mathbf{I}_n - \mathbf{A}_n\right)\right] \\ &= c^n \det\left(\frac{\lambda}{c}\mathbf{I}_n - \mathbf{A}_n\right) \\ &= c^n\left(\frac{\lambda}{c}\right)^{n-1}\left(\frac{\lambda}{c} - n\right) \\ &= \lambda^{n-1}(\lambda - cn). \end{aligned} \tag{C.18}$$

To derive an expression for the inverse of $\lambda\mathbf{I}_n - c\mathbf{A}_n$, we make the substitutions $\mathbf{A} = \lambda\mathbf{I}_n$ and $\mathbf{u} = j\sqrt{c}\,\mathbf{a}_n$ in (C.4), where $j = \sqrt{-1}$, to get

$$(\lambda\mathbf{I_n} - c\mathbf{A}_n)^{-1} = \frac{1}{\lambda}\mathbf{I}_n + \frac{\frac{1}{\lambda^2}c\mathbf{A}_n}{1 - \frac{1}{\lambda}c\mathbf{a}_n^T\mathbf{a}_n} = \frac{1}{\lambda}\left(\mathbf{I}_n + \frac{c}{\lambda - cn}\mathbf{A}_n\right). \tag{C.19}$$

C.5 The Matrix $\mathbf{B}_2(\epsilon)$

Consider the 2×2 matrix

$$\mathbf{B}_2(\epsilon) \triangleq \begin{bmatrix} 1 & -\epsilon \\ \epsilon^* & 1 \end{bmatrix}. \tag{C.20}$$

Since

$$\mathbf{B}_2^H(\epsilon)\mathbf{B}_2(\epsilon) = (1 + |\epsilon|^2)\mathbf{I}_2 = \mathbf{B}_2(\epsilon)\mathbf{B}_2^H(\epsilon), \tag{C.21}$$

the two columns of $\mathbf{B}_2(\epsilon)$ are orthogonal. The determinant of $\mathbf{B}_2(\epsilon)$ is equal to

$$\det[\mathbf{B}_2(\epsilon)] = 1 + |\epsilon|^2 = \det[\mathbf{B}_2^H(\epsilon)], \tag{C.22}$$

while the inverse of $\mathbf{B}_2(\epsilon)$ is evaluated as (see (C.2))

$$\mathbf{B}_2^{-1}(\epsilon) = \frac{1}{\det[\mathbf{B}_2(\epsilon)]}\begin{bmatrix} 1 & \epsilon \\ -\epsilon^* & 1 \end{bmatrix} = \frac{1}{1+|\epsilon|^2}\,\mathbf{B}_2^H(\epsilon). \tag{C.23}$$

Now consider the matrix

$$\lambda\mathbf{I}_2 - c\mathbf{B}_2(\epsilon) = \begin{bmatrix} \lambda - c & c\epsilon \\ -c\epsilon^* & \lambda - c \end{bmatrix}, \tag{C.24}$$

where c is a scalar. The determinant of (C.24) is just the characteristic polynomial of $c\mathbf{B}_2(\epsilon)$:

$$\det[\lambda\mathbf{I}_2 - c\mathbf{B}_2(\epsilon)] = (\lambda - c)^2 + c^2|\epsilon|^2 = \lambda^2 - 2c\lambda + c^2(1+|\epsilon|^2). \tag{C.25}$$

Similarly, we have

$$\det[\lambda\mathbf{I}_2 - c\mathbf{B}_2^H(\epsilon)] = \lambda^2 - 2c\lambda + c^2(1+|\epsilon|^2). \tag{C.26}$$

Since $c\mathbf{B}_2(\epsilon)$ and $c\mathbf{B}_2^H(\epsilon)$ have equal characteristic polynomials, their eigenvalues must also be the same. Using the quadratic formula to solve for the roots of (C.25), we get

$$\lambda = \frac{2c \pm \sqrt{4c^2 - 4c^2(1+|\epsilon|^2)}}{2} = c \pm jc|\epsilon|. \tag{C.27}$$

Therefore, the eigenvalues of $c\mathbf{B}_2(\epsilon)$ and $c\mathbf{B}_2^H(\epsilon)$ are given by

$$\lambda_1 = c - jc|\epsilon|, \qquad \text{multiplicity 1}, \tag{C.28}$$

$$\lambda_2 = c + jc|\epsilon|, \qquad \text{multiplicity 1}. \tag{C.29}$$

Using (C.2), we evaluate the inverse of $\lambda\mathbf{I}_2 - c\mathbf{B}_2(\epsilon)$ as

$$\begin{aligned}[\lambda\mathbf{I}_2 - c\mathbf{B}_2(\epsilon)]^{-1} &= \frac{1}{\det[\lambda\mathbf{I}_2 - c\mathbf{B}_2(\epsilon)]}\begin{bmatrix} \lambda - c & -c\epsilon \\ c\epsilon^* & \lambda - c \end{bmatrix} \\ &= \frac{1}{\lambda^2 - 2c\lambda + c^2(1+|\epsilon|^2)}[\lambda\mathbf{I}_2 - c\mathbf{B}_2^H(\epsilon)]. \end{aligned} \tag{C.30}$$

We can similarly show that

$$[\lambda \mathbf{I}_2 - c\mathbf{B}_2^H(\epsilon)]^{-1} = \frac{1}{\lambda^2 - 2c\lambda + c^2(1+|\epsilon|^2)}[\lambda \mathbf{I}_2 - c\mathbf{B}_2(\epsilon)]. \tag{C.31}$$

Finally, consider the matrix

$$\lambda \mathbf{I}_2 - c\mathbf{B}_2^{-1}(\epsilon) = \lambda \mathbf{I}_2 - \frac{c}{1+|\epsilon|^2}\mathbf{B}_2^H(\epsilon). \tag{C.32}$$

By replacing c with $\frac{c}{1+|\epsilon|^2}$ in (C.31), we obtain

$$[\lambda \mathbf{I}_2 - c\mathbf{B}_2^{-1}(\epsilon)]^{-1} = \frac{1}{\lambda^2 - \frac{2c}{1+|\epsilon|^2}\lambda + \frac{c^2}{1+|\epsilon|^2}}\left(\lambda \mathbf{I}_2 - \frac{c}{1+|\epsilon|^2}\mathbf{B}_2(\epsilon)\right). \tag{C.33}$$

C.6 The Matrix $\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon)$

We now examine the $2n \times 2n$ matrix $c\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon)$, where $\otimes$ denotes the Kronecker product. From (C.15)–(C.16), the eigenvalues of $\mathbf{A}_n$ are given by

$$\mu_1 = n, \quad \text{multiplicity } 1, \tag{C.34}$$

$$\mu_2 = 0, \quad \text{multiplicity } n-1, \tag{C.35}$$

so the eigenvalues of $c\mathbf{A}_n$ must equal

$$\mu_1' = cn, \quad \text{multiplicity } 1, \tag{C.36}$$

$$\mu_2' = 0, \quad \text{multiplicity } n-1. \tag{C.37}$$

Also, substituting $c = 1$ into (C.28)–(C.29), the eigenvalues of $\mathbf{B}_2(\epsilon)$ are

$$\lambda_1' = 1 - j|\epsilon|, \quad \text{multiplicity } 1, \tag{C.38}$$

$$\lambda_2' = 1 + j|\epsilon|, \quad \text{multiplicity } 1. \tag{C.39}$$

Therefore, by Property 8 in Appendix B.2, the $2n$ eigenvalues of $c\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon)$ are given by $\{\lambda'_p \mu'_q\}$:

$$\lambda_1 = 0, \quad \text{multiplicity } 2n-2, \tag{C.40}$$

$$\lambda_2 = cn(1 - j|\epsilon|), \quad \text{multiplicity } 1, \tag{C.41}$$

$$\lambda_3 = cn(1 + j|\epsilon|), \quad \text{multiplicity } 1. \tag{C.42}$$

Since these eigenvalues are the roots of the characteristic polynomial of $c\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon)$, we must have

$$\begin{aligned} \det[\lambda \mathbf{I}_{2n} - (c\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon))] &= \lambda^{2n-2}[\lambda - cn(1 - j|\epsilon|)] \cdot [\lambda - cn(1 + j|\epsilon|)] \\ &= \lambda^{2n-2}\left[\lambda^2 - 2cn\lambda + (cn)^2(1 + |\epsilon|^2)\right]. \end{aligned} \tag{C.43}$$

Given that $c\mathbf{B}_2(\epsilon)$ and $c\mathbf{B}_2^H(\epsilon)$ have the same eigenvalues (see (C.28)–(C.29)), we must also have

$$\det[\lambda \mathbf{I}_{2n} - (c\mathbf{A}_n \otimes \mathbf{B}_2^H(\epsilon))] = \lambda^{2n-2}[\lambda^2 - 2cn\lambda + (cn)^2(1 + |\epsilon|^2)]. \tag{C.44}$$

The derivation for $[\lambda \mathbf{I}_{2n} - (c\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon))]^{-1}$ is a little more involved. First, using (C.13) and the *mixed product rule* (Property 6 in Appendix B.2), we can re-write $\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon)$ as

$$\begin{aligned} \mathbf{A}_n \otimes \mathbf{B}_2(\epsilon) &= (\mathbf{a}_n \mathbf{a}_n^T) \otimes (\mathbf{I}_2 \mathbf{B}_2(\epsilon)) \\ &= (\mathbf{a}_n \otimes \mathbf{I}_2)(\mathbf{a}_n^T \otimes \mathbf{B}_2(\epsilon)) \\ &= (\mathbf{a}_n \otimes \mathbf{I}_2)[(1 \cdot \mathbf{a}_n^T) \otimes (\mathbf{B}_2(\epsilon)\mathbf{I}_2)] \\ &= (\mathbf{a}_n \otimes \mathbf{I}_2)(1 \otimes \mathbf{B}_2(\epsilon))(\mathbf{a}_n^T \otimes \mathbf{I}_2) \\ &= (\mathbf{a}_n \otimes \mathbf{I}_2)\mathbf{B}_2(\epsilon)(\mathbf{a}_n^T \otimes \mathbf{I}_2) \\ &\equiv \mathbf{C}_{n,2}\mathbf{B}_2(\epsilon)\mathbf{C}_{n,2}^T, \end{aligned} \tag{C.45}$$

where $\mathbf{C}_{n,m}$ is defined as the $nm \times m$ matrix

$$\mathbf{C}_{n,m} \triangleq \mathbf{a}_n \otimes \mathbf{I}_m = \begin{bmatrix} \mathbf{I}_m \\ \vdots \\ \mathbf{I}_m \end{bmatrix}. \tag{C.46}$$

Note that

$$\begin{aligned}\mathbf{C}_{n,m}\mathbf{C}_{n,m}^T &= (\mathbf{a}_n \otimes \mathbf{I}_m)(\mathbf{a}_n \otimes \mathbf{I}_m)^T \\ &= (\mathbf{a}_n \otimes \mathbf{I}_m)(\mathbf{a}_n^T \otimes \mathbf{I}_m) \\ &= (\mathbf{a}_n\mathbf{a}_n^T) \otimes (\mathbf{I}_m\mathbf{I}_m) \\ &= \mathbf{A}_n \otimes \mathbf{I}_m \end{aligned} \tag{C.47}$$

and

$$\begin{aligned}\mathbf{C}_{n,m}^T\mathbf{C}_{n,m} &= (\mathbf{a}_n \otimes \mathbf{I}_m)^T(\mathbf{a}_n \otimes \mathbf{I}_m) \\ &= (\mathbf{a}_n^T \otimes \mathbf{I}_m)(\mathbf{a}_n \otimes \mathbf{I}_m) \\ &= (\mathbf{a}_n^T\mathbf{a}_n) \otimes (\mathbf{I}_m\mathbf{I}_m) \\ &= n\mathbf{I}_m. \end{aligned} \tag{C.48}$$

Since $\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon) = \mathbf{C}_{n,2}\mathbf{B}_2(\epsilon)\mathbf{C}_{n,2}^T$, we make the substitutions $\mathbf{A} = \lambda\mathbf{I}_{2n}$, $\mathbf{B} = j\sqrt{c}\,\mathbf{C}_{n,2}$, $\mathbf{C} = \mathbf{B}_2(\epsilon)$, and $\mathbf{D} = j\sqrt{c}\,\mathbf{C}_{n,2}^T$ in the matrix inversion lemma (see (C.3)) and use (C.48) to get

$$\begin{aligned}[\lambda\mathbf{I}_{2n} - (c\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon))]^{-1} &= [\lambda\mathbf{I}_{2n} + (j\sqrt{c}\,\mathbf{C}_{n,2})\mathbf{B}_2(\epsilon)(j\sqrt{c}\,\mathbf{C}_{n,2}^T)]^{-1} \\ &= \frac{1}{\lambda}\mathbf{I}_{2n} + \frac{c}{\lambda^2}\mathbf{C}_{n,2}\left(-\frac{c}{\lambda}\mathbf{C}_{n,2}^T\mathbf{C}_{n,2} + \mathbf{B}_2^{-1}(\epsilon)\right)^{-1}\mathbf{C}_{n,2}^T \\ &= \frac{1}{\lambda}\mathbf{I}_{2n} + \frac{c}{\lambda^2}\mathbf{C}_{n,2}\left(-\frac{cn}{\lambda}\mathbf{I}_2 + \mathbf{B}_2^{-1}(\epsilon)\right)^{-1}\mathbf{C}_{n,2}^T. \end{aligned} \tag{C.49}$$

By setting $c = -1$ and replacing λ with $-\frac{cn}{\lambda}$ in (C.33), we obtain

$$\begin{aligned}\left(-\frac{cn}{\lambda}\mathbf{I}_2 + \mathbf{B}_2^{-1}(\epsilon)\right)^{-1} &= \frac{1}{\frac{(cn)^2}{\lambda^2} - \frac{2cn}{1+|\epsilon|^2}\frac{1}{\lambda} + \frac{1}{1+|\epsilon|^2}}\left[-\frac{cn}{\lambda}\mathbf{I}_2 + \frac{1}{1+|\epsilon|^2}\mathbf{B}_2(\epsilon)\right] \\ &= \frac{(1+|\epsilon|^2)\lambda^2}{\lambda^2 - 2cn\lambda + (cn)^2(1+|\epsilon|^2)}\left[-\frac{cn}{\lambda}\mathbf{I}_2 + \frac{1}{1+|\epsilon|^2}\mathbf{B}_2(\epsilon)\right] \\ &= \frac{\lambda}{\lambda^2 - 2cn\lambda + (cn)^2(1+|\epsilon|^2)}\left[\lambda\mathbf{B}_2(\epsilon) - cn(1+|\epsilon|^2)\mathbf{I}_2\right]. \end{aligned} \tag{C.50}$$

Finally, we substitute this result back into (C.49) and use (C.45) and (C.47) to show that

$$\begin{aligned}
&[\lambda \mathbf{I}_{2n} - (c\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon))]^{-1} \\
&\quad = \frac{1}{\lambda}\mathbf{I}_{2n} + \frac{1}{\lambda}\frac{c}{\lambda^2 - 2cn\lambda + (cn)^2(1+|\epsilon|^2)} \cdot \mathbf{C}_{n,2}\left[\lambda \mathbf{B}_2(\epsilon) - cn(1+|\epsilon|^2)\mathbf{I}_2\right]\mathbf{C}_{n,2}^T \\
&\quad = \frac{1}{\lambda}\left\{\mathbf{I}_{2n} + \frac{c}{\lambda^2 - 2cn\lambda + (cn)^2(1+|\epsilon|^2)}\left[\lambda(\mathbf{A}_n \otimes \mathbf{B}_2(\epsilon)) - cn(1+|\epsilon|^2)(\mathbf{A}_n \otimes \mathbf{I}_2)\right]\right\} \\
&\quad = \frac{1}{\lambda}\left\{\mathbf{I}_{2n} + \frac{c}{\lambda^2 - 2cn\lambda + (cn)^2(1+|\epsilon|^2)}\left[\mathbf{A}_n \otimes \left(\lambda \mathbf{B}_2(\epsilon) - cn(1+|\epsilon|^2)\mathbf{I}_2\right)\right]\right\}. \qquad \text{(C.51)}
\end{aligned}$$

Appendix D

General Hermitian Quadratic Forms in Zero-Mean Complex Gaussian Random Variables

We consider a communication system in which the final test statistic is expressed as a Hermitian quadratic form in *zero-mean* complex Gaussian random variables.[1] Assuming binary signaling, where each bit is equally likely to be $+1$ or -1, we derive a closed-form expression for the probability of error (see (D.8)). This result is applicable to multi-channel communication systems which use, for example, maximal ratio combining on Rayleigh fading channels with an arbitrary branch covariance matrix. While (D.8) was independently derived by the authors in [49], our result represents an extension of their work in that we are able to derive a closed-form expression for the partial fraction expansion (PFE) coefficients that is valid for any set of

[1] See [35, Appendix B], [36, Ch. 9], and [37–49] for other works which deal with Hermitian quadratic forms in complex Gaussian random variables.

eigenvalues.[2] Furthermore, we consider three general examples in this appendix, and we derive the required eigenvalues in closed form for each case.

D.1 Probability of Error

Consider the general Hermitian quadratic form

$$Z = \mathbf{v}^H \mathbf{F} \mathbf{v}, \tag{D.1}$$

where $\mathbf{v} = [v_1 \cdots v_N]^T$ is a column vector of N jointly distributed zero-mean complex Gaussian random variables, and $\mathbf{F}$ is an $N \times N$ Hermitian matrix. Define the $N \times N$ covariance matrix of $\mathbf{v}$ as

$$\mathbf{R} = \frac{1}{2} E\left[\mathbf{v}\mathbf{v}^H\right]. \tag{D.2}$$

The characteristic function of Z, $\Phi_Z(jv) \triangleq E[e^{jvZ}]$, is given by [35, Eq. (B-3-16)]

$$\Phi_Z(jv) = \frac{1}{\det\left(\mathbf{I}_N - 2jv\mathbf{R}\mathbf{F}\right)} = \prod_{n=1}^{\Lambda} \left(1 - 2jv\lambda_n\right)^{-m_n}, \tag{D.3}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\lambda_1, \ldots, \lambda_\Lambda$ are the $\Lambda \leq N - m_0$ *distinct non-zero eigenvalues* of the matrix $\mathbf{RF}$ with multiplicities $m_1, \ldots, m_\Lambda$, such that $\sum_{i=1}^{\Lambda} m_i = N - m_0$, and m_0 is the number of eigenvalues of $\mathbf{RF}$ equal to zero.[3] Expanding the right side of (D.3) in terms of partial fractions, we obtain

$$\Phi_Z(jv) = \prod_{n=1}^{\Lambda} \left(1 - 2jv\lambda_n\right)^{-m_n} = \sum_{n=1}^{\Lambda} \sum_{k=1}^{m_n} \frac{c_{n,(m_n-k+1)}}{\left(1 - 2jv\lambda_n\right)^k}. \tag{D.4}$$

By replacing N with Λ and λ_n with $-2j\lambda_n$ in (A.16), we can write $c_{n,k}$ in closed form as

$$c_{n,k} = \prod_{\substack{r=1 \\ r \neq n}}^{\Lambda} \left(\frac{1 - \mu_{rn}}{2}\right)^{m_r} \sum_{\Omega_{n,k}} \prod_{\substack{p=1 \\ p \neq n}}^{\Lambda} \binom{m_p - 1 + i_p}{i_p} \left(\frac{1 + \mu_{pn}}{2}\right)^{i_p}, \tag{D.5}$$

[2] The authors in [49] were only able to derive the PFE coefficients for the case of distinct eigenvalues.

[3] The eigenvalues equal to zero only contribute a factor of 1 to $\Phi_Z(jv)$. A zero eigenvalue (i.e., a singular $\mathbf{RF}$) can be shown to occur when there is perfect CSI.

where

$$\mu_{mn} \triangleq \frac{\lambda_m + \lambda_n}{\lambda_m - \lambda_n}, \quad m \neq n, \tag{D.6}$$

and $\Omega_{n,k}$ denotes the set of non-negative integers

$$\Omega_{n,k} : \left\{ i_1, \ldots, i_{n-1}, i_{n+1}, \ldots, i_\Lambda \,\middle|\, \sum_{\substack{p=1 \\ p \neq n}}^{\Lambda} i_p = k - 1 \right\}. \tag{D.7}$$

If we assume binary signaling, the probability of error is derived as [49, Eqs. (46)–(48)]

$$\begin{aligned} P_e &= P[Z < 0] \\ &= \int_{-\infty}^{0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_Z(j\upsilon) e^{-j\upsilon x} \, d\upsilon \, dx \\ &= \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,k}. \end{aligned} \tag{D.8}$$

An alternative derivation of P_e that is based on the probability density function (PDF) of Z is given in Appendix E.

In order to use (D.8), we need the eigenvalues of $\mathbf{RF}$. In general, these eigenvalues will need to be evaluated numerically; but if the number of distinct eigenvalues is small, then it may be possible to derive them in closed-form by solving for the roots of the characteristic polynomial of $\mathbf{RF}$, $\det(\lambda \mathbf{I}_N - \mathbf{RF})$. Also, as we pointed out in Appendix A.1.1, when $\Lambda > 2$, the $\{c_{n,k}\}$ can be evaluated more efficiently by using iterative techniques than by using (D.5).

D.2 Examples

Using the approach of the previous section, we derive some general results that are used in this dissertation. Consider the Hermitian quadratic form in the *zero-mean* complex Gaussian random variables $\{X_i\}_{i=1}^{M}$ and $\{Y_i\}_{i=1}^{M}$,

$$Z = \sum_{i=1}^{M} \left(A|X_i|^2 + B|Y_i|^2 + C X_i^* Y_i + C^* X_i Y_i^* \right), \tag{D.9}$$

where A, B, and C are constants. We assume that X_i and Y_i are correlated for all i, and that the M pairs $\{X_i, Y_i\}_{i=1}^{M}$ are identically distributed but not necessarily independent. This quadratic form was examined in [27, Appendix B] and [40], but both works, in contrast, allowed X_i and Y_i to have non-zero means and assumed the M pairs $\{X_i, Y_i\}_{i=1}^{M}$ are independent and identically distributed (i.i.d.). By defining the $2M \times 1$ column vector

$$\mathbf{v} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \tag{D.10}$$

where

$$\mathbf{x} = [X_1 \cdots X_M]^T, \tag{D.11}$$

$$\mathbf{y} = [Y_1 \cdots Y_M]^T, \tag{D.12}$$

along with the $2M \times 2M$ Hermitian matrix

$$\mathbf{F} = \begin{bmatrix} A\mathbf{I}_M & C\mathbf{I}_M \\ C^*\mathbf{I}_M & B\mathbf{I}_M \end{bmatrix}, \tag{D.13}$$

we can re-write Z as

$$Z = \mathbf{v}^H \mathbf{F} \mathbf{v}. \tag{D.14}$$

The $2M \times 2M$ covariance matrix of $\mathbf{v}$ is given by

$$\mathbf{R} = \frac{1}{2} E\left[\mathbf{v}\mathbf{v}^H\right] = \begin{bmatrix} \mathbf{\Sigma}_{xx} & \mathbf{\Sigma}_{xy} \\ \mathbf{\Sigma}_{xy}^H & \mathbf{\Sigma}_{yy} \end{bmatrix}, \tag{D.15}$$

where the $M \times M$ matrices $\mathbf{\Sigma}_{xx}$, $\mathbf{\Sigma}_{yy}$, and $\mathbf{\Sigma}_{xy}$ are defined as

$$\mathbf{\Sigma}_{xx} \triangleq \frac{1}{2} E\left[\mathbf{x}\mathbf{x}^H\right], \tag{D.16}$$

$$\mathbf{\Sigma}_{yy} \triangleq \frac{1}{2} E\left[\mathbf{y}\mathbf{y}^H\right], \tag{D.17}$$

$$\mathbf{\Sigma}_{xy} \triangleq \frac{1}{2} E\left[\mathbf{x}\mathbf{y}^H\right]. \tag{D.18}$$

Thus, we have

$$\mathbf{RF} = \begin{bmatrix} A\boldsymbol{\Sigma}_{xx} + C^*\boldsymbol{\Sigma}_{xy} & B\boldsymbol{\Sigma}_{xy} + C\boldsymbol{\Sigma}_{xx} \\ A\boldsymbol{\Sigma}_{xy}^H + C^*\boldsymbol{\Sigma}_{yy} & B\boldsymbol{\Sigma}_{yy} + C\boldsymbol{\Sigma}_{xy}^H \end{bmatrix}. \tag{D.19}$$

In the following examples, we consider three special cases for the matrices $\boldsymbol{\Sigma}_{xx}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{xy}$, and we derive the probability of error for each case.

D.2.1 Example #1

As in [27, Appendix B] and [40], we assume that X_i and Y_i are correlated, and that the M pairs $\{X_i, Y_i\}_{i=1}^M$ are i.i.d. Thus, the matrices $\boldsymbol{\Sigma}_{xx}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{xy}$ must be of the form

$$\boldsymbol{\Sigma}_{xx} = \mu_{xx}\mathbf{I}_M, \tag{D.20}$$

$$\boldsymbol{\Sigma}_{yy} = \mu_{yy}\mathbf{I}_M, \tag{D.21}$$

$$\boldsymbol{\Sigma}_{xy} = \mu_{xy}\mathbf{I}_M, \tag{D.22}$$

where

$$\mu_{xx} = \frac{1}{2}E\left[|X_i|^2\right], \tag{D.23}$$

$$\mu_{yy} = \frac{1}{2}E\left[|Y_i|^2\right], \tag{D.24}$$

$$\mu_{xy} = \frac{1}{2}E\left[X_i Y_i^*\right]. \tag{D.25}$$

Substituting (D.20)–(D.22) into (D.19), we get

$$\mathbf{RF} = \begin{bmatrix} (A\mu_{xx} + C^*\mu_{xy})\mathbf{I}_M & (B\mu_{xy} + C\mu_{xx})\mathbf{I}_M \\ (A\mu_{xy}^* + C^*\mu_{yy})\mathbf{I}_M & (B\mu_{yy} + C\mu_{xy}^*)\mathbf{I}_M \end{bmatrix} \equiv \mathbf{K} \otimes \mathbf{I}_M, \tag{D.26}$$

where

$$\mathbf{K} = \begin{bmatrix} A\mu_{xx} + C^*\mu_{xy} & B\mu_{xy} + C\mu_{xx} \\ A\mu_{xy}^* + C^*\mu_{yy} & B\mu_{yy} + C\mu_{xy}^* \end{bmatrix}, \tag{D.27}$$

and $\otimes$ denotes the Kronecker product.

Characteristic Polynomial of RF

Using the relation $\mathbf{I}_{2M} = \mathbf{I}_2 \otimes \mathbf{I}_M$, along with Properties 2, 3, and 9 from Appendix B.2, we have

$$\begin{aligned}\det(\lambda \mathbf{I}_{2M} - \mathbf{RF}) &= \det[\lambda(\mathbf{I}_2 \otimes \mathbf{I}_M) - (\mathbf{K} \otimes \mathbf{I}_M)] \\ &= \det[(\lambda \mathbf{I}_2 - \mathbf{K}) \otimes \mathbf{I}_M] \\ &= [\det(\lambda \mathbf{I}_2 - \mathbf{K})]^M. \end{aligned} \tag{D.28}$$

We substitute the expression for $\mathbf{K}$ in (D.27) into (D.28) to obtain

$$\begin{aligned}\det(\lambda \mathbf{I}_{2M} - \mathbf{RF}) &= \begin{vmatrix} \lambda - (A\mu_{xx} + C^*\mu_{xy}) & -(B\mu_{xy} + C\mu_{xx}) \\ -(A\mu_{xy}^* + C^*\mu_{yy}) & \lambda - (B\mu_{yy} + C\mu_{xy}^*) \end{vmatrix}^M \\ &= \Big\{[\lambda - (A\mu_{xx} + C^*\mu_{xy})][\lambda - (B\mu_{yy} + C\mu_{xy}^*)] \\ &\quad - (A\mu_{xy}^* + C^*\mu_{yy})(B\mu_{xy} + C\mu_{xx})\Big\}^M \\ &= (\lambda^2 - b\lambda - c)^M, \end{aligned} \tag{D.29}$$

where

$$b = A\mu_{xx} + B\mu_{yy} + C\mu_{xy}^* + C^*\mu_{xy} = A\mu_{xx} + B\mu_{yy} + 2\mathrm{Re}\{C^*\mu_{xy}\}, \tag{D.30}$$

$$c = (|C|^2 - AB)(\mu_{xx}\mu_{yy} - |\mu_{xy}|^2) = (|C|^2 - AB)\mu_{xx}\mu_{yy}(1 - |\rho|^2), \tag{D.31}$$

and

$$\rho \triangleq \frac{\mu_{xy}}{\sqrt{\mu_{xx}\mu_{yy}}} \tag{D.32}$$

is the complex correlation coefficient of X_i and Y_i. Note that b and c are both real quantities.

Eigenvalues of RF

Using the quadratic formula to solve for the roots of (D.29), we see that the matrix **RF** has $\Lambda = 2$ distinct eigenvalues, both of multiplicity M:

$$\lambda_1 = \frac{b - \sqrt{b^2 + 4c}}{2}, \quad m_1 = M, \tag{D.33}$$

$$\lambda_2 = \frac{b + \sqrt{b^2 + 4c}}{2}, \quad m_2 = M. \tag{D.34}$$

If we assume both $|C|^2 > AB$ and $|\rho| < 1$, such that $c > 0$, then we have $\lambda_1 < 0$ and $\lambda_2 > 0$.

Probability of Error

Substituting $\Lambda = 2$ and $m_1 = m_2 = M$ into (D.8), and using the fact that λ_1 is the only negative eigenvalue, we obtain

$$P_e = \sum_{k=1}^{M} c_{1,k}. \tag{D.35}$$

Making the same substitutions in (D.5), we find that $c_{1,k}$ is equal to

$$c_{1,k} = \left(\frac{1 - \mu_{21}}{2}\right)^M \binom{M - 1 + (k-1)}{k-1} \left(\frac{1 + \mu_{21}}{2}\right)^{k-1}, \tag{D.36}$$

where

$$\mu_{21} = \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}. \tag{D.37}$$

Thus, P_e becomes

$$\begin{aligned} P_e &= \left(\frac{1 - \mu_{21}}{2}\right)^M \sum_{k=1}^{M} \binom{M - 1 + (k-1)}{k-1} \left(\frac{1 + \mu_{21}}{2}\right)^{k-1} \\ &= \left(\frac{1 - \mu_{21}}{2}\right)^M \sum_{i=0}^{M-1} \binom{M - 1 + i}{i} \left(\frac{1 + \mu_{21}}{2}\right)^{i}. \end{aligned} \tag{D.38}$$

Finally, we substitute the expressions for λ_1 and λ_2 from (D.33)–(D.34) into (D.37) to obtain

$$\mu_{21} = \frac{b}{\sqrt{b^2 + 4c}} \equiv \sqrt{\frac{\bar{\gamma}}{1 + \bar{\gamma}}}, \tag{D.39}$$

where

$$\bar{\gamma} = \frac{b^2}{4c}. \tag{D.40}$$

Note that (D.38) and (D.39) are identical in form to [27, Eqs. (14.4-15),(14.4-16)], which is the probability of error of a maximal ratio combiner with perfect channel estimation and i.i.d. Rayleigh fading channels. In fact, the special case of maximal ratio combining is obtained by making the substitutions $A = 0$, $B = 0$, and $C = \frac{1}{2}$ in the above results, which yields

$$b = \mathrm{Re}(\mu_{xy}), \tag{D.41}$$

$$c = \frac{1}{4}\mu_{xx}\mu_{yy}(1 - |\rho|^2), \tag{D.42}$$

and

$$\bar{\gamma} = \frac{[\mathrm{Re}(\rho)]^2}{1 - |\rho|^2}. \tag{D.43}$$

If ρ is real, then $\bar{\gamma}$ becomes

$$\bar{\gamma} = \frac{\rho^2}{1 - \rho^2}, \tag{D.44}$$

which implies

$$\mu_{21} = |\rho|. \tag{D.45}$$

The fact that μ_{21} is always given by (D.39), regardless of whether the channel estimation is perfect or imperfect, is in agreement with the findings in [50] for the maximal ratio combiner.

D.2.2 Example #2

We now generalize Example #1 and assume that the M pairs $\{X_i, Y_i\}_{i=1}^{M}$ are independent but *non-identically* distributed. That is, we assume the $M \times M$ matrices $\mathbf{\Sigma}_{xx}$, $\mathbf{\Sigma}_{yy}$, and

$\boldsymbol{\Sigma}_{xy}$ are of the form

$$\boldsymbol{\Sigma}_{xx} = \begin{bmatrix} \mathbf{D}_{xx} & & \\ & \ddots & \\ & & \mathbf{D}_{xx} \end{bmatrix} = \mathbf{I}_N \otimes \mathbf{D}_{xx}, \tag{D.46}$$

$$\boldsymbol{\Sigma}_{yy} = \begin{bmatrix} \mathbf{D}_{yy} & & \\ & \ddots & \\ & & \mathbf{D}_{yy} \end{bmatrix} = \mathbf{I}_N \otimes \mathbf{D}_{yy}, \tag{D.47}$$

$$\boldsymbol{\Sigma}_{xy} = \begin{bmatrix} \mathbf{D}_{xy} & & \\ & \ddots & \\ & & \mathbf{D}_{xy} \end{bmatrix} = \mathbf{I}_N \otimes \mathbf{D}_{xy}, \tag{D.48}$$

where $\mathbf{D}_{xx}$, $\mathbf{D}_{yy}$, and $\mathbf{D}_{xy}$ are $L \times L$ diagonal matrices, and

$$N \triangleq \frac{M}{L} \tag{D.49}$$

is an integer (i.e., M is divisible by L). Note that when $L = 1$, $\boldsymbol{\Sigma}_{xx}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{xy}$ reduce to scaled identity matrices, and we obtain the model in Example #1. Substituting (D.46)–(D.48) into (D.19) and using Properties 2 and 3 from Appendix B.2, we obtain

$$\mathbf{RF} = \begin{bmatrix} \mathbf{I}_N \otimes (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy}) & \mathbf{I}_N \otimes (B\mathbf{D}_{xy} + C\mathbf{D}_{xx}) \\ \mathbf{I}_N \otimes (A\mathbf{D}_{xy}^H + C^*\mathbf{D}_{yy}) & \mathbf{I}_N \otimes (B\mathbf{D}_{yy} + C\mathbf{D}_{xy}^H) \end{bmatrix}. \tag{D.50}$$

Characteristic Polynomial of RF

The characteristic polynomial of $\mathbf{RF}$ is given by

$$\begin{aligned} &\det(\lambda \mathbf{I}_{2M} - \mathbf{RF}) \\ &= \begin{vmatrix} \lambda(\mathbf{I}_N \otimes \mathbf{I}_L) - [\mathbf{I}_N \otimes (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy})] & -\mathbf{I}_N \otimes (B\mathbf{D}_{xy} + C\mathbf{D}_{xx}) \\ -\mathbf{I}_N \otimes (A\mathbf{D}_{xy}^H + C^*\mathbf{D}_{yy}) & \lambda(\mathbf{I}_N \otimes \mathbf{I}_L) - [\mathbf{I}_N \otimes (B\mathbf{D}_{yy} + C\mathbf{D}_{xy}^H)] \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} \mathbf{I}_N \otimes [\lambda \mathbf{I}_L - (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy})] & -\mathbf{I}_N \otimes (B\mathbf{D}_{xy} + C\mathbf{D}_{xx}) \\ -\mathbf{I}_N \otimes (A\mathbf{D}_{xy}^H + C^*\mathbf{D}_{yy}) & \mathbf{I}_N \otimes [\lambda \mathbf{I}_L - (B\mathbf{D}_{yy} + C\mathbf{D}_{xy}^H)] \end{vmatrix}, \tag{D.51}$$

where we used the relation $\mathbf{I}_M = \mathbf{I}_N \otimes \mathbf{I}_L$ and Properties 2 and 3 from Appendix B.2. We can apply the formula for the determinant of a partitioned matrix [29, Sec. A1.1.3],

$$\begin{vmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{vmatrix} = \det(\mathbf{A}_{11}) \det(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}), \tag{D.52}$$

to expand (D.51) as

$$\det(\lambda \mathbf{I}_{2M} - \mathbf{RF}) = \det\big\{\mathbf{I}_N \otimes [\lambda \mathbf{I}_L - (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy})]\big\} \cdot \det(\mathbf{\Phi}_1 - \mathbf{\Phi}_2), \tag{D.53}$$

where

$$\mathbf{\Phi}_1 \triangleq \mathbf{I}_N \otimes [\lambda \mathbf{I}_L - (B\mathbf{D}_{yy} + C\mathbf{D}_{xy}^H)], \tag{D.54}$$

$$\mathbf{\Phi}_2 \triangleq [\mathbf{I}_N \otimes (A\mathbf{D}_{xy}^H + C^*\mathbf{D}_{yy})] \cdot \{\mathbf{I}_N \otimes [\lambda \mathbf{I}_L - (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy})]\}^{-1} \cdot [\mathbf{I}_N \otimes (B\mathbf{D}_{xy} + C\mathbf{D}_{xx})]. \tag{D.55}$$

Using Property 9 in Appendix B.2, along with the result in (C.7), we evaluate the first determinant in (D.53) as

$$\begin{aligned} \det\{\mathbf{I}_N \otimes [\lambda \mathbf{I}_L - (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy})]\} &= \{\det[\lambda \mathbf{I}_L - (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy})]\}^N \\ &= \prod_{i=1}^{L} [\lambda - (A[\mathbf{D}_{xx}]_{ii} + C^*[\mathbf{D}_{xy}]_{ii})]^N, \end{aligned} \tag{D.56}$$

where $[\mathbf{D}]_{ii}$ denotes the i-th diagonal entry of $\mathbf{D}$. As for the second determinant in (D.53), we first note that

$$\begin{aligned} &[\lambda \mathbf{I}_L - (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy})]^{-1} \\ &\quad = \operatorname{diag}\Big\{[\lambda - (A[\mathbf{D}_{xx}]_{11} + C^*[\mathbf{D}_{xy}]_{11})]^{-1}, \ldots, [\lambda - (A[\mathbf{D}_{xx}]_{LL} + C^*[\mathbf{D}_{xy}]_{LL})]^{-1}\Big\}, \end{aligned} \tag{D.57}$$

which follows from (C.11). Next, using Property 7 from Appendix B.2 and applying Corollary 4 from Appendix B.2 twice, we can re-write $\mathbf{\Phi}_2$ in (D.55) as

$$\begin{aligned}\mathbf{\Phi}_2 &= [\mathbf{I}_N \otimes (A\mathbf{D}_{xy}^H + C^*\mathbf{D}_{yy})] \cdot \left\{\mathbf{I}_N \otimes [\lambda\mathbf{I}_L - (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy})]^{-1}\right\} \cdot [\mathbf{I}_N \otimes (B\mathbf{D}_{xy} + C\mathbf{D}_{xx})] \\ &= \mathbf{I}_N \otimes \left[(A\mathbf{D}_{xy}^H + C^*\mathbf{D}_{yy})[\lambda\mathbf{I}_L - (A\mathbf{D}_{xx} + C^*\mathbf{D}_{xy})]^{-1}(B\mathbf{D}_{xy} + C\mathbf{D}_{xx})\right] \\ &= \mathbf{I}_N \otimes \operatorname{diag}\left(\frac{(A[\mathbf{D}_{xy}]_{11}^* + C^*[\mathbf{D}_{yy}]_{11})(B[\mathbf{D}_{xy}]_{11} + C[\mathbf{D}_{xx}]_{11})}{\lambda - (A[\mathbf{D}_{xx}]_{11} + C^*[\mathbf{D}_{xy}]_{11})}, \ldots \right. \\ &\qquad \left. \ldots, \frac{(A[\mathbf{D}_{xy}]_{LL}^* + C^*[\mathbf{D}_{yy}]_{LL})(B[\mathbf{D}_{xy}]_{LL} + C[\mathbf{D}_{xx}]_{LL})}{\lambda - (A[\mathbf{D}_{xx}]_{LL} + C^*[\mathbf{D}_{xy}]_{LL})}\right). \end{aligned} \tag{D.58}$$

Since

$$\lambda\mathbf{I}_L - (B\mathbf{D}_{yy} + C\mathbf{D}_{xy}^H) = \operatorname{diag}[\lambda - (B[\mathbf{D}_{yy}]_{11} + C[\mathbf{D}_{xy}]_{11}^*), \ldots, \lambda - (B[\mathbf{D}_{yy}]_{LL} + C[\mathbf{D}_{xy}]_{LL}^*)], \tag{D.59}$$

we use (D.54) and (D.58) to get

$$\begin{aligned}&\mathbf{\Phi}_1 - \mathbf{\Phi}_2 \\ &= \left(\mathbf{I}_N \otimes \operatorname{diag}[\lambda - (B[\mathbf{D}_{yy}]_{11} + C[\mathbf{D}_{xy}]_{11}^*), \ldots, \lambda - (B[\mathbf{D}_{yy}]_{LL} + C[\mathbf{D}_{xy}]_{LL}^*)]\right) \\ &\quad - \left[\mathbf{I}_N \otimes \operatorname{diag}\left(\frac{(A[\mathbf{D}_{xy}]_{11}^* + C^*[\mathbf{D}_{yy}]_{11})(B[\mathbf{D}_{xy}]_{11} + C[\mathbf{D}_{xx}]_{11})}{\lambda - (A[\mathbf{D}_{xx}]_{11} + C^*[\mathbf{D}_{xy}]_{11})}, \ldots \right.\right. \\ &\qquad \left.\left. \ldots, \frac{(A[\mathbf{D}_{xy}]_{LL}^* + C^*[\mathbf{D}_{yy}]_{LL})(B[\mathbf{D}_{xy}]_{LL} + C[\mathbf{D}_{xx}]_{LL})}{\lambda - (A[\mathbf{D}_{xx}]_{LL} + C^*[\mathbf{D}_{xy}]_{LL})}\right)\right]. \\ &= \mathbf{I}_N \otimes \operatorname{diag}\left\{\lambda - (B[\mathbf{D}_{yy}]_{11} + C[\mathbf{D}_{xy}]_{11}^*) - \frac{(A[\mathbf{D}_{xy}]_{11}^* + C^*[\mathbf{D}_{yy}]_{11})(B[\mathbf{D}_{xy}]_{11} + C[\mathbf{D}_{xx}]_{11})}{\lambda - (A[\mathbf{D}_{xx}]_{11} + C^*[\mathbf{D}_{xy}]_{11})}, \ldots \right. \\ &\qquad \left. \ldots, \lambda - (B[\mathbf{D}_{yy}]_{LL} + C[\mathbf{D}_{xy}]_{LL}^*) - \frac{(A[\mathbf{D}_{xy}]_{LL}^* + C^*[\mathbf{D}_{yy}]_{LL})(B[\mathbf{D}_{xy}]_{LL} + C[\mathbf{D}_{xx}]_{LL})}{\lambda - (A[\mathbf{D}_{xx}]_{LL} + C^*[\mathbf{D}_{xy}]_{LL})}\right\} \\ &= \mathbf{I}_N \otimes \operatorname{diag}\left\{\frac{e_{11}(\lambda)}{\lambda - (A[\mathbf{D}_{xx}]_{11} + C^*[\mathbf{D}_{xy}]_{11})}, \ldots, \frac{e_{LL}(\lambda)}{\lambda - (A[\mathbf{D}_{xx}]_{LL} + C^*[\mathbf{D}_{xy}]_{LL})}\right\}, \end{aligned} \tag{D.60}$$

where

$$\begin{aligned} e_{ii}(\lambda) \triangleq\ & [\lambda - (A[\mathbf{D}_{xx}]_{ii} + C^*[\mathbf{D}_{xy}]_{ii})][\lambda - (B[\mathbf{D}_{yy}]_{ii} + C[\mathbf{D}_{xy}]_{ii}^*)] \\ & - (A[\mathbf{D}_{xy}]_{ii}^* + C^*[\mathbf{D}_{yy}]_{ii})(B[\mathbf{D}_{xy}]_{ii} + C[\mathbf{D}_{xx}]_{ii}). \end{aligned} \tag{D.61}$$

It is straightforward to show that $e_{ii}(\lambda)$ reduces to

$$e_{ii}(\lambda) = \lambda^2 - b_i\lambda - c_i, \tag{D.62}$$

where

$$b_i = A[\mathbf{D}_{xx}]_{ii} + B[\mathbf{D}_{yy}]_{ii} + C[\mathbf{D}_{xy}]^*_{ii} + C^*[\mathbf{D}_{xy}]_{ii}, \tag{D.63}$$

$$c_i = (|C|^2 - AB)([\mathbf{D}_{xx}]_{ii}[\mathbf{D}_{yy}]_{ii} - |[\mathbf{D}_{xy}]_{ii}|^2). \tag{D.64}$$

The determinant of (D.60) is then evaluated by applying Property 9 from Appendix B.2:

$$\begin{aligned}\det(\mathbf{\Phi}_1 - \mathbf{\Phi}_2) &= \left\{\det\left[\operatorname{diag}\left(\frac{e_{11}(\lambda)}{\lambda - (A[\mathbf{D}_{xx}]_{11} + C^*[\mathbf{D}_{xy}]_{11})}, \ldots\right.\right.\right.\\ &\qquad \left.\left.\left.\ldots \frac{e_{LL}(\lambda)}{\lambda - (A[\mathbf{D}_{xx}]_{LL} + C^*[\mathbf{D}_{xy}]_{LL})}\right)\right]\right\}^N\\ &= \prod_{i=1}^{L}\left(\frac{e_{ii}(\lambda)}{\lambda - (A[\mathbf{D}_{xx}]_{ii} + C^*[\mathbf{D}_{xy}]_{ii})}\right)^N. \end{aligned} \tag{D.65}$$

Finally, we substitute the results of (D.56) and (D.65) into (D.53) to show that the characteristic polynomial of $\mathbf{RF}$ is equal to

$$\det(\lambda\mathbf{I}_{2M} - \mathbf{RF}) = \prod_{i=1}^{L}[e_{ii}(\lambda)]^N = \prod_{i=1}^{L}(\lambda^2 - b_i - c_i)^N. \tag{D.66}$$

Eigenvalues of RF

Using the quadratic formula to solve for the roots of $\det(\lambda\mathbf{I}_{2M} - \mathbf{RF})$, we obtain

$$\lambda_{2i-1} = \frac{b_i - \sqrt{b_i^2 + 4c_i}}{2}, \quad m_{2i-1} = N, \tag{D.67}$$

$$\lambda_{2i} = \frac{b_i + \sqrt{b_i^2 + 4c_i}}{2}, \quad m_{2i} = N, \tag{D.68}$$

for $i = 1, \ldots, L$. If we assume both $|C|^2 > AB$ and $[\mathbf{D}_{xx}]_{ii}[\mathbf{D}_{yy}]_{ii} > |[\mathbf{D}_{xy}]_{ii}|^2$, such that $c_i > 0$, then we have $\lambda_{2i-1} < 0$ and $\lambda_{2i} > 0$.

Probability of Error

Suppose the $2L$ eigenvalues given by $\{\lambda_{2i-1}, \lambda_{2i}\}_{i=1}^{L}$ are distinct. We then substitute $\Lambda = 2L$ and $m_{2i-1} = m_{2i} = N$ $(i = 1, \ldots, L)$ into (D.8) and use the fact that the negative eigenvalues are given by $\{\lambda_{2i-1}\}_{i=1}^{L}$ to obtain

$$P_e = \sum_{\substack{n=1 \\ n \text{ odd}}}^{2L} \sum_{k=1}^{N} c_{n,k}, \tag{D.69}$$

where the $\{c_{n,k}\}$ are the coefficients in the partial fraction expansion (see (D.4))

$$\prod_{n=1}^{2L} (1 - 2j\upsilon\lambda_n)^{-N} = \sum_{n=1}^{2L} \sum_{k=1}^{N} \frac{c_{n,(N-k+1)}}{(1 - 2j\upsilon\lambda_n)^k}. \tag{D.70}$$

For $L > 1$, it is more efficient to calculate the $\{c_{n,k}\}$ using iterative techniques (see Appendix A.1.2 for an example) than using (D.5).

D.2.3 Example #3

In this example, we assume the matrices $\mathbf{\Sigma}_{xx}$, $\mathbf{\Sigma}_{yy}$, and $\mathbf{\Sigma}_{xy}$ are of the form

$$\mathbf{\Sigma}_{xx} = \begin{bmatrix} \mathbf{B}_{xx} & & \\ & \ddots & \\ & & \mathbf{B}_{xx} \end{bmatrix} = \mathbf{I}_N \otimes \mathbf{B}_{xx}, \tag{D.71}$$

$$\mathbf{\Sigma}_{yy} = \begin{bmatrix} \mathbf{B}_{yy} & & \\ & \ddots & \\ & & \mathbf{B}_{yy} \end{bmatrix} = \mathbf{I}_N \otimes \mathbf{B}_{yy}, \tag{D.72}$$

$$\mathbf{\Sigma}_{xy} = \begin{bmatrix} \mathbf{B}_{xy} & & \\ & \ddots & \\ & & \mathbf{B}_{xy} \end{bmatrix} = \mathbf{I}_N \otimes \mathbf{B}_{xy}, \tag{D.73}$$

where the $L \times L$ matrices $\mathbf{B}_{xx}$, $\mathbf{B}_{yy}$, and $\mathbf{B}_{xy}$ are given by

$$\mathbf{B}_{xx} = s_{xx}\mathbf{A}_L + t_{xx}\mathbf{I}_L, \tag{D.74}$$

$$\mathbf{B}_{yy} = s_{yy}\mathbf{A}_L + t_{yy}\mathbf{I}_L, \tag{D.75}$$

$$\mathbf{B}_{xy} = s_{xy}\mathbf{A}_L, \tag{D.76}$$

$\mathbf{A}_L$ is the $L \times L$ all-ones matrix, and

$$N \triangleq \frac{M}{L} \tag{D.77}$$

is an integer (i.e., M is divisible by L). The constants s_{xx}, s_{yy}, t_{xx}, and t_{yy} are all real and ≥ 0, but the constant s_{xy} may be complex. Note that this model reduces to the one in Example #1 when $L = 1$ and $t_{xx} = t_{yy} = 0$. Substituting (D.71)–(D.73) into (D.19) and using Properties 2 and 3 from Appendix B.2, we get

$$\mathbf{RF} = \begin{bmatrix} \mathbf{I}_N \otimes (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy}) & \mathbf{I}_N \otimes (B\mathbf{B}_{xy} + C\mathbf{B}_{xx}) \\ \mathbf{I}_N \otimes (A\mathbf{B}_{xy}^H + C^*\mathbf{B}_{yy}) & \mathbf{I}_N \otimes (B\mathbf{B}_{yy} + C\mathbf{B}_{xy}^H) \end{bmatrix}. \tag{D.78}$$

Characteristic Polynomial of RF

As in Example #2, we use the relation $\mathbf{I}_M = \mathbf{I}_N \otimes \mathbf{I}_L$, Properties 2 and 3 from Appendix B.2, and (D.52) to expand the characteristic polynomial of $\mathbf{RF}$ as

$$\det(\lambda\mathbf{I}_{2M} - \mathbf{RF}) = \det\big\{\mathbf{I}_N \otimes [\lambda\mathbf{I}_L - (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy})]\big\} \cdot \det(\mathbf{\Phi}_1 - \mathbf{\Phi}_2), \tag{D.79}$$

where

$$\mathbf{\Phi}_1 \triangleq \mathbf{I}_N \otimes [\lambda\mathbf{I}_L - (B\mathbf{B}_{yy} + C\mathbf{B}_{xy}^H)], \tag{D.80}$$

$$\mathbf{\Phi}_2 \triangleq [\mathbf{I}_N \otimes (A\mathbf{B}_{xy}^H + C^*\mathbf{B}_{yy})]\,\{\mathbf{I}_N \otimes [\lambda\mathbf{I}_L - (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy})]\}^{-1}\,[\mathbf{I}_N \otimes (B\mathbf{B}_{xy} + C\mathbf{B}_{xx})]. \tag{D.81}$$

To derive expressions for the two determinants in (D.79), we first note that

$$A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy} = (As_{xx} + C^*s_{xy})\mathbf{A}_L + At_{xx}\mathbf{I}_L, \tag{D.82}$$

$$A\mathbf{B}_{xy}^H + C^*\mathbf{B}_{yy} = (As_{xy}^* + C^*s_{yy})\mathbf{A}_L + C^*t_{yy}\mathbf{I}_L, \tag{D.83}$$

$$B\mathbf{B}_{xy} + C\mathbf{B}_{xx} = (Bs_{xy} + Cs_{xx})\mathbf{A}_L + Ct_{xx}\mathbf{I}_L, \tag{D.84}$$

$$B\mathbf{B}_{yy} + C\mathbf{B}_{xy}^H = (Bs_{yy} + Cs_{xy}^*)\mathbf{A}_L + Bt_{yy}\mathbf{I}_L, \tag{D.85}$$

which follow from (D.74)–(D.76). Thus, we have

$$\lambda \mathbf{I}_L - (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy}) = (\lambda - At_{xx})\mathbf{I}_L - (As_{xx} + C^*s_{xy})\mathbf{A}_L, \tag{D.86}$$

$$\lambda \mathbf{I}_L - (B\mathbf{B}_{yy} + C\mathbf{B}_{xy}^H) = (\lambda - Bt_{yy})\mathbf{I}_L - (Bs_{yy} + Cs_{xy}^*)\mathbf{A}_L. \tag{D.87}$$

The determinant of (D.86) is evaluated by replacing λ with $\lambda - At_{xx}$, c with $As_{xx} + C^*s_{xy}$, and n with L in (C.18):

$$\begin{aligned}\det[\lambda \mathbf{I}_L - (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy})] &= \det[(\lambda - At_{xx})\mathbf{I}_L - (As_{xx} + C^*s_{xy})\mathbf{A}_L] \\ &= (\lambda - At_{xx})^{L-1}[(\lambda - At_{xx}) - (As_{xx} + C^*s_{xy})L]. \end{aligned} \tag{D.88}$$

We then use Property 9 from Appendix B.2 to show that

$$\begin{aligned}\det\{\mathbf{I}_N \otimes [\lambda \mathbf{I}_L - (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy}]\} &= \{\det[\lambda \mathbf{I}_L - (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy})]\}^N \\ &= (\lambda - At_{xx})^{(L-1)N}[(\lambda - At_{xx}) - (As_{xx} + C^*s_{xy})L]^N. \end{aligned} \tag{D.89}$$

As for the second determinant in (D.79), we first substitute (D.87) into the expression for $\mathbf{\Phi}_1$ in (D.80) to get

$$\mathbf{\Phi}_1 = \mathbf{I}_N \otimes [(\lambda - Bt_{yy})\mathbf{I}_L - (Bs_{yy} + Cs_{xy}^*)\mathbf{A}_L]. \tag{D.90}$$

Next, we make the appropriate substitutions in (C.19) to obtain the inverse of (D.86),

$$\begin{aligned}[\lambda \mathbf{I}_L - (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy})]^{-1} &= [(\lambda - At_{xx})\mathbf{I}_L - (As_{xx} + C^*s_{xy})\mathbf{A}_L]^{-1} \\ &= \frac{1}{\lambda - At_{xx}}\left(\mathbf{I}_L + \frac{As_{xx} + C^*s_{xy}}{(\lambda - At_{xx}) - (As_{xx} + C^*s_{xy})L}\mathbf{A}_L\right), \end{aligned} \tag{D.91}$$

from which we also obtain

$$\begin{aligned}&\{\mathbf{I}_N \otimes [\lambda \mathbf{I}_L - (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy}]\}^{-1} \\ &\quad = \mathbf{I}_N \otimes [\lambda \mathbf{I}_L - (A\mathbf{B}_{xx} + C^*\mathbf{B}_{xy})]^{-1} \\ &\quad = \mathbf{I}_N \otimes \frac{1}{\lambda - At_{xx}}\left(\mathbf{I}_L + \frac{As_{xx} + C^*s_{xy}}{(\lambda - At_{xx}) - (As_{xx} + C^*s_{xy})L}\mathbf{A}_L\right) \end{aligned} \tag{D.92}$$

after applying Property 7 from Appendix B.2. We then substitute (D.92) into the expression for $\mathbf{\Phi}_2$ in (D.81), use the expressions in (D.82)–(D.85), and apply Corollary 4 from Appendix B.2 twice to get

$$\begin{aligned}\mathbf{\Phi}_2 = \mathbf{I}_N \otimes \frac{1}{\lambda - At_{xx}} \Big\{ &[(As^*_{xy} + C^* s_{yy})\mathbf{A}_L + C^* t_{yy}\mathbf{I}_L] \\ &\cdot \left(\mathbf{I}_L + \frac{As_{xx} + C^* s_{xy}}{(\lambda - At_{xx}) - (As_{xx} + C^* s_{xy})L} \mathbf{A}_L \right) \\ &\cdot [(Bs_{xy} + Cs_{xx})\mathbf{A}_L + Ct_{xx}\mathbf{I}_L] \Big\}.\end{aligned} \tag{D.93}$$

Multiplying out the terms in (D.93) and using the relation $\mathbf{A}_L \mathbf{A}_L = L\mathbf{A}_L$ from (C.14), we can show that $\mathbf{\Phi}_2$ is equal to

$$\begin{aligned}\mathbf{\Phi}_2 = \mathbf{I}_N \otimes \frac{1}{\lambda - At_{xx}} \Big\{ &\Big[(As^*_{xy} + C^* s_{yy})(Bs_{xy} + Cs_{xx})L + (As^*_{xy} + C^* s_{yy})Ct_{xx} \\ &+ (Bs_{xy} + Cs_{xx})C^* t_{yy} \Big] \mathbf{A}_L + |C|^2 t_{xx} t_{yy} \mathbf{I}_L \\ &+ \frac{(As_{xx} + C^* s_{xy})[(Bs_{xy} + Cs_{xx})L + Ct_{xx}]}{(\lambda - At_{xx}) - (As_{xx} + C^* s_{xy})L} \cdot [(As^*_{xy} + C^* s_{yy})L + C^* t_{yy}]\mathbf{A}_L \Big\}.\end{aligned} \tag{D.94}$$

Combining the results of (D.90) and (D.94), we have

$$\begin{aligned}&\mathbf{\Phi}_1 - \mathbf{\Phi}_2 \\ &= \mathbf{I}_N \otimes [(\lambda - Bt_{yy})\mathbf{I}_L - (Bs_{yy} + Cs^*_{xy})\mathbf{A}_L] \\ &\quad - \mathbf{I}_N \otimes \frac{1}{\lambda - At_{xx}} \Big\{ \Big[(As^*_{xy} + C^* s_{yy})(Bs_{xy} + Cs_{xx})L + (As^*_{xy} + C^* s_{yy})Ct_{xx} \\ &\quad + (Bs_{xy} + Cs_{xx})C^* t_{yy} \Big] \mathbf{A}_L + |C|^2 t_{xx} t_{yy} \mathbf{I}_L \\ &\quad + \frac{(As_{xx} + C^* s_{xy})[(Bs_{xy} + Cs_{xx})L + Ct_{xx}]}{(\lambda - At_{xx}) - (As_{xx} + C^* s_{xy})L} \cdot [(As^*_{xy} + C^* s_{yy})L + C^* t_{yy}]\mathbf{A}_L \Big\} \\ &= \mathbf{I}_N \otimes \frac{1}{\lambda - At_{xx}} [X(\lambda)\mathbf{I}_L - Y(\lambda)\mathbf{A}_L],\end{aligned} \tag{D.95}$$

where $X(\lambda)$ and $Y(\lambda)$ are given by

$$X(\lambda) \triangleq (\lambda - At_{xx})(\lambda - Bt_{yy}) - |C|^2 t_{xx} t_{yy}, \tag{D.96}$$

$$
\begin{aligned}
Y(\lambda) \triangleq\; & (\lambda - At_{xx})(Bs_{yy} + Cs_{xy}) + [(As_{xy}^* + C^*s_{yy})(Bs_{xy} + Cs_{xx})L \\
& + (As_{xy}^* + C^*s_{yy})Ct_{xx} + (Bs_{xy} + Cs_{xx})C^*t_{yy}] \\
& + \frac{(As_{xx} + C^*s_{xy})[(Bs_{xy} + Cs_{xx})L + Ct_{xx}]}{(\lambda - At_{xx}) - (As_{xx} + C^*s_{xy})L} \cdot [(As_{xy}^* + C^*s_{yy})L + C^*t_{yy}] \\
=\; & \frac{1}{(\lambda - At_{xx}) - (As_{xx} + C^*s_{xy})L} \Big\{ (\lambda - At_{xx})^2 (Bs_{yy} + Cs_{xy}^*) \\
& - (\lambda - At_{xx})(Bs_{yy} + Cs_{xy}^*)(As_{xx} + C^*s_{xy})L \\
& + (\lambda - At_{xx})(As_{xy}^* + C^*s_{yy})(Bs_{xy} + Cs_{xx})L \\
& + (\lambda - At_{xx})(As_{xy}^* + C^*s_{yy})Ct_{xx} \\
& + (\lambda - At_{xx})(Bs_{xy} + Cs_{xx})C^*t_{yy} \\
& + (As_{xx} + C^*s_{xy})|C|^2 t_{xx} t_{yy} \Big\},
\end{aligned}
\tag{D.97}
$$

respectively. We evaluate the determinant of (D.95) by first applying Property 9 from Appendix B.2 and then replacing λ with $X(\lambda)$, c with $Y(\lambda)$, and n with L in (C.18):

$$
\begin{aligned}
\det(\mathbf{\Phi}_1 - \mathbf{\Phi}_2) &= \left\{ \det\left(\frac{1}{\lambda - At_{xx}} [X(\lambda)\mathbf{I}_L - Y(\lambda)\mathbf{A}_L] \right) \right\}^N \\
&= \frac{1}{(\lambda - At_{xx})^{LN}} \Big\{ \det[X(\lambda)\mathbf{I}_L - Y(\lambda)\mathbf{A}_L] \Big\}^N \\
&= \frac{1}{(\lambda - At_{xx})^{LN}} [X(\lambda)]^{(L-1)N} \cdot [X(\lambda) - Y(\lambda)L]^N.
\end{aligned}
\tag{D.98}
$$

Finally, we substitute the results of (D.89) and (D.98) into (D.79) to obtain

$$
\begin{aligned}
& \det(\lambda \mathbf{I}_{2M} - \mathbf{RF}) \\
&= [X(\lambda)]^{(L-1)N} \cdot \left\{ \frac{1}{\lambda - At_{xx}} [X(\lambda) - Y(\lambda)L] \cdot [(\lambda - At_{xx}) - (As_{xx} + C^*s_{xy})L] \right\}^N .
\end{aligned}
\tag{D.99}
$$

Using (D.96) and (D.97), we can show that

$$
\begin{aligned}
& X(\lambda) \cdot [(\lambda - At_{xx}) - (As_{xx} + C^*s_{xy})L] \\
&= (\lambda - At_{xx})^2 (\lambda - Bt_{yy}) - (\lambda - At_{xx})(\lambda - Bt_{yy})(As_{xx} + C^*s_{xy})L \\
&\quad - (\lambda - At_{xx})|C|^2 t_{xx} t_{yy} + (As_{xx} + C^*s_{xy})|C|^2 t_{xx} t_{yy} L,
\end{aligned}
\tag{D.100}
$$

and

$$
\begin{aligned}
&Y(\lambda)L \cdot [(\lambda - At_{xx}) - (As_{xx} + C^* s_{xy})L] \\
&\quad = (\lambda - At_{xx})^2 (Bs_{yy} + Cs_{xy}^*)L - (\lambda - At_{xx})(Bs_{yy} + Cs_{xy}^*)(As_{xx} + C^* s_{xy})L^2 \\
&\qquad + (\lambda - At_{xx})(As_{xy}^* + C^* s_{yy})(Bs_{xy} + Cs_{xx})L^2 \\
&\qquad + (\lambda - At_{xx})(As_{xy}^* + C^* s_{yy})Ct_{xx}L \\
&\qquad + (\lambda - At_{xx})(Bs_{xy} + Cs_{xx})C^* t_{yy}L + (As_{xx} + C^* s_{xy})|C|^2 t_{xx} t_{yy} L.
\end{aligned}
\tag{D.101}
$$

We then apply these results to get

$$
\begin{aligned}
&\frac{1}{(\lambda - At_{xx})}[X(\lambda) - Y(\lambda)L] \cdot [(\lambda - At_{xx}) - (As_{xx} + C^* s_{xy})L] \\
&\quad = \Big\{(\lambda - At_{xx})(\lambda - Bt_{yy}) - (\lambda - Bt_{yy})(As_{xx} + C^* s_{xy})L - |C|^2 t_{xx} t_{yy}\Big\} \\
&\qquad - \Big\{(\lambda - At_{xx})(Bs_{yy} + Cs_{xy}^*)L - (Bs_{yy} + Cs_{xy}^*)(As_{xx} + C^* s_{xy})L^2 \\
&\qquad + (As_{xy}^* + C^* s_{yy})(Bs_{xy} + Cs_{xx})L^2 + (As_{xy}^* + C^* s_{yy})Ct_{xx}L + (Bs_{xy} + Cs_{xx})C^* t_{yy}L\Big\} \\
&\quad = \lambda^2 - b\lambda - c,
\end{aligned}
\tag{D.102}
$$

where

$$
b = (At_{xx} + Bt_{yy}) + (As_{xx} + Bs_{yy} + Cs_{xy}^* + C^* s_{xy})L, \tag{D.103}
$$

$$
\begin{aligned}
c &= (|C|^2 - AB)t_{xx}t_{yy} + [(As_{xy}^* + C^* s_{yy})C - (Bs_{yy} + Cs_{xy}^*)A]t_{xx}L \\
&\quad + [(Bs_{xy} + Cs_{xx})C^* - (As_{xx} + C^* s_{xy})B]t_{yy}L \\
&\quad + [(As_{xy}^* + C^* s_{yy})(Bs_{xy} + Cs_{xx}) - (Bs_{yy} + Cs_{xy}^*)(As_{xx} + C^* s_{xy})]L^2 \\
&= (|C|^2 - AB)[t_{xx}t_{yy} + (s_{xx}t_{yy} + s_{yy}t_{xx})L + (s_{xx}s_{yy} - |s_{xy}|^2)L^2].
\end{aligned}
\tag{D.104}
$$

Substituting (D.102) into (D.99), we obtain

$$
\det(\lambda \mathbf{I}_{2M} - \mathbf{RF}) = [X(\lambda)]^{(L-1)N} \cdot (\lambda^2 - b\lambda - c)^N. \tag{D.105}
$$

Eigenvalues of RF

The eigenvalues of $\mathbf{RF}$ are given by the solutions to $\det(\lambda \mathbf{I}_{2M} - \mathbf{RF}) = 0$. First, we solve

$$[X(\lambda)]^{(L-1)N} = 0, \tag{D.106}$$

which can be expanded as (see (D.96))

$$[\lambda^2 - (At_{xx} + Bt_{yy})\lambda - (|C|^2 - AB)t_{xx}t_{yy}]^{(L-1)N} = 0. \tag{D.107}$$

Using the quadratic formula, we obtain the following solutions:

$$\lambda_1 = \frac{1}{2}(At_{xx} + Bt_{yy}) - \frac{1}{2}\sqrt{(At_{xx} + Bt_{yy})^2 + 4(|C|^2 - AB)t_{xx}t_{yy}}, \quad m_1 = (L-1)N, \tag{D.108}$$

$$\lambda_2 = \frac{1}{2}(At_{xx} + Bt_{yy}) + \frac{1}{2}\sqrt{(At_{xx} + Bt_{yy})^2 + 4(|C|^2 - AB)t_{xx}t_{yy}}, \quad m_2 = (L-1)N. \tag{D.109}$$

We also apply the quadratic formula to the equation

$$(\lambda^2 - b\lambda - c)^N = 0 \tag{D.110}$$

to obtain the solutions

$$\lambda_3 = \frac{b - \sqrt{b^2 + 4c}}{2}, \quad m_3 = N, \tag{D.111}$$

$$\lambda_4 = \frac{b + \sqrt{b^2 + 4c}}{2}, \quad m_4 = N. \tag{D.112}$$

If $|C|^2 > AB$, $s_{xx}s_{yy} \geq |s_{xy}|^2$, and the other constants in (D.104) are such that $c > 0$, then we have $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, and $\lambda_4 > 0$. Depending on the values of the constants A, B, C, s_{xx}, etc., it may actually be possible for, say, λ_3 to equal λ_1, in which case the number of distinct non-zero eigenvalues is less than four (i.e., $\Lambda < 4$).

Probability of Error

From (D.8), the probability of error is given by

$$P_e = \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,k}, \tag{D.113}$$

where the $\{c_{n,k}\}$ are the coefficients in the partial fraction expansion (see (D.4))

$$\prod_{n=1}^{\Lambda} (1 - 2j\upsilon\lambda_n)^{-m_n} = \sum_{n=1}^{\Lambda} \sum_{k=1}^{m_n} \frac{c_{n,(m_n-k+1)}}{(1 - 2j\upsilon\lambda_n)^k}. \tag{D.114}$$

Appendix E

Alternative Derivation of P_e

We present an alternative derivation for the probability of error given in (D.8). Our approach involves taking the inverse Fourier transform of $\Phi_Z(j\upsilon)$ to obtain the PDF of Z, and then integrating the PDF over the negative values of Z.

E.1 PDF of Z

The PDF of Z is given by the inverse Fourier transform of its characteristic function [27, (2.1-72)]

$$f_Z(x) = \frac{1}{2\pi}\ \text{p.v.} \int_{-\infty}^{\infty} \Phi_Z(j\upsilon) e^{-j\upsilon x}\, d\upsilon, \tag{E.1}$$

where we have treated the improper integral as a *principal value* [51, Sec. 6.3]:

$$\text{p.v.} \int_{-\infty}^{\infty} f(z)\, dz \triangleq \lim_{\rho\to\infty} \int_{-\rho}^{\rho} f(z)\, dz. \tag{E.2}$$

Substituting the expression for $\Phi_Z(j\upsilon)$ from (D.4) into (E.1), we obtain

$$f_Z(x) = \sum_{n=1}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \frac{1}{2\pi}\ \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{-j\upsilon x}}{(1-2j\upsilon\lambda_n)^k}\, d\upsilon$$

$$
\begin{aligned}
&= \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{j}{2\lambda_n}\right)^k \cdot \frac{1}{2\pi} \text{ p.v.} \int_{-\infty}^{\infty} h_{n,k}(\upsilon)\, d\upsilon \\
&\quad + \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{j}{2\lambda_n}\right)^k \cdot \frac{1}{2\pi} \text{ p.v.} \int_{-\infty}^{\infty} h_{n,k}(\upsilon)\, d\upsilon,
\end{aligned} \tag{E.3}
$$

where

$$
h_{n,k}(\upsilon) \triangleq \frac{e^{-j\upsilon x}}{\left(\upsilon + \frac{j}{2\lambda_n}\right)^k}. \tag{E.4}
$$

The reason for treating the negative and positive eigenvalues separately will become apparent shortly.

To evaluate

$$
I = \text{p.v.} \int_{-\infty}^{\infty} h_{n,k}(\upsilon)\, d\upsilon = \lim_{\rho \to \infty} \int_{-\rho}^{\rho} h_{n,k}(\upsilon)\, d\upsilon, \tag{E.5}
$$

we first recognize that the integral

$$
I_\rho = \int_{-\rho}^{\rho} h_{n,k}(\upsilon)\, d\upsilon \tag{E.6}
$$

can be interpreted as a contour integral of the complex-valued function $h_{n,k}(z)$ along γ_ρ, the directed line segment along the real axis from $-\rho$ to ρ:

$$
I_\rho = \int_{\gamma_\rho} h_{n,k}(z)\, dz. \tag{E.7}
$$

We then construct a simple *closed* contour Γ_ρ, such that γ_ρ is one of its components, and we use residue theory[1] to solve the integral along Γ_ρ as $\rho \to \infty$. Note that $h_{n,k}(z)$ has a k-th order pole at $z = -j/2\lambda_n$. The cases $x < 0$ and $x > 0$ will need to be treated separately.

E.1.1 $x < 0$

For $x < 0$, we close the contour with expanding semi-circles in the *upper-half* complex plane. Define the contour $\Gamma_\rho = \gamma_\rho + C_\rho^+$, where γ_ρ is as previously defined, and C_ρ^+ is the

[1] An elementary treatment of contour integration and residue theory is given in [52, Appendix A], while [51] covers these topics more thoroughly.

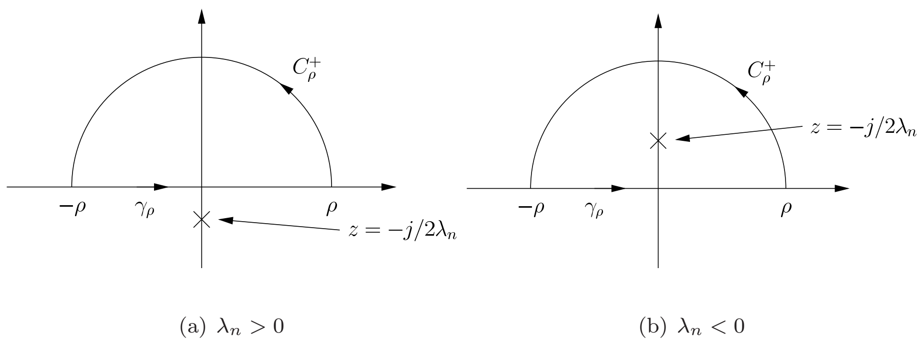


Figure E.1: For $x < 0$, we close the contour with a semi-circle in the upper-half complex plane.

positively oriented (counter-clockwise) semi-circle of radius ρ in the upper-half complex plane. We focus on the integral

$$\lim_{\rho\to\infty}\int_{\Gamma_\rho} h_{n,k}(z)\,dz = \lim_{\rho\to\infty}\int_{\gamma_\rho} h_{n,k}(z)\,dz + \lim_{\rho\to\infty}\int_{C_\rho^+} h_{n,k}(z)\,dz. \tag{E.8}$$

First, consider the case $\lambda_n > 0$. From Fig. E.1(a), we see that the contour Γ_ρ does not enclose any poles, and the function $h_{n,k}(z)$ is analytic inside the contour. Therefore, by Cauchy's Integral Theorem [51, Sec. 4.4, Theorem 9], we have

$$\lim_{\rho\to\infty}\int_{\Gamma_\rho} h_{n,k}(z)\,dz = 0, \quad \lambda_n > 0. \tag{E.9}$$

Using Jordan's Lemma [51, Sec. 6.4, Lemma 3], we also have

$$\lim_{\rho\to\infty}\int_{C_\rho^+} h_{n,k}(z)\,dz = 0, \quad \lambda_n > 0. \tag{E.10}$$

Substituting the results of (E.9) and (E.10) into (E.8), we conclude that

$$\text{p.v.}\int_{-\infty}^{\infty} h_{n,k}(\upsilon)\,d\upsilon = \lim_{\rho\to\infty}\int_{\gamma_\rho} h_{n,k}(z)\,dz = 0, \quad \lambda_n > 0. \tag{E.11}$$

As a result, $f_Z(x)$ in (E.3) reduces to

$$f_Z(x) = \sum_{\substack{n=1\\ \lambda_n < 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{j}{2\lambda_n}\right)^k \cdot \frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^{\infty} h_{n,k}(\upsilon)\,d\upsilon. \tag{E.12}$$

Now, consider the case $\lambda_n < 0$. We see from Fig. E.1(b) that the contour Γ_ρ encloses a k-th order pole at $z = -j/2\lambda_n$. Using Cauchy's Residue Theorem [51, Sec. 6.1, Theorem 2], we have

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} h_{n,k}(z)\, dz = 2\pi j \cdot \mathrm{Res}\left(h_{n,k}; -\frac{j}{2\lambda_n}\right), \quad \lambda_n < 0, \tag{E.13}$$

where $\mathrm{Res}(f; z_0)$ is defined as the residue of the function f at $z = z_0$. Since, by Jordan's Lemma,

$$\lim_{\rho\to\infty} \int_{C_\rho^+} h_{n,k}(z)\, dz = 0, \quad \lambda_n < 0, \tag{E.14}$$

we obtain (see (E.8))

$$\begin{aligned} \text{p.v.} \int_{-\infty}^{\infty} h_{n,k}(\upsilon)\, d\upsilon &= \lim_{\rho\to\infty} \int_{\gamma_\rho} h_{n,k}(z)\, dz \\ &= 2\pi j \cdot \mathrm{Res}\left(h_{n,k}; -\frac{j}{2\lambda_n}\right), \quad \lambda_n < 0. \end{aligned} \tag{E.15}$$

Applying the general formula for a residue [51, Sec. 6.1, Theorem 1], we have

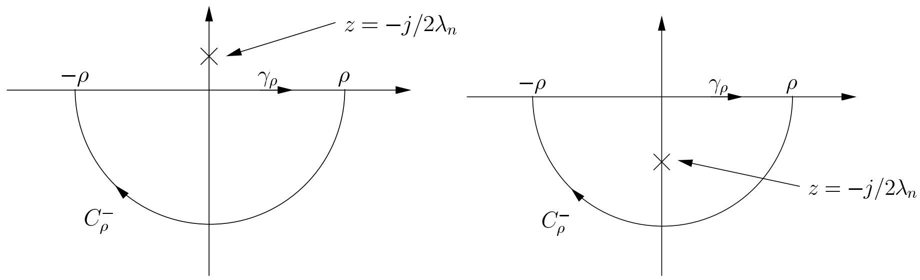
$$\begin{aligned} \mathrm{Res}\left(h_{n,k}; -\frac{j}{2\lambda_n}\right) &= \lim_{z\to -j/2\lambda_n} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[\left(z + \frac{j}{2\lambda_n}\right)^k h_{n,k}(z)\right] \\ &= \lim_{z\to -j/2\lambda_n} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(e^{-jxz}\right) \\ &= \lim_{z\to -j/2\lambda_n} \frac{1}{(k-1)!} (-jx)^{k-1} e^{-jxz} \\ &= \frac{1}{(k-1)!} (-jx)^{k-1} e^{-x/2\lambda_n}. \end{aligned} \tag{E.16}$$

To arrive at the above result, we used the formula

$$\frac{d^n}{dz^n}\left(e^{-jxz}\right) = (-jx)^n e^{-jxz}, \tag{E.17}$$

which is easily proved by induction. Substituting the results of (E.15) and (E.16) into (E.12), we obtain

$$\begin{aligned} f_Z(x) &= \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{j}{2\lambda_n}\right)^k \cdot \frac{1}{2\pi} \cdot 2\pi j \frac{1}{(k-1)!} (-jx)^{k-1} e^{-x/2\lambda_n} \\ &= \sum_{\substack{n=1 \\ \lambda_n < 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(-\frac{1}{2\lambda_n}\right)^k \frac{1}{(k-1)!} (-x)^{k-1} e^{-x/2\lambda_n}, \quad x < 0. \end{aligned} \tag{E.18}$$



(a) $\lambda_n < 0$ (b) $\lambda_n > 0$

Figure E.2: For $x > 0$, we close the contour with a semi-circle in the lower-half complex plane.

E.1.2 $x > 0$

For $x > 0$, we close the contour with expanding semi-circles in the *lower-half* complex plane. Consider the contour $\Gamma_\rho = \gamma_\rho + C_\rho^-$, where C_ρ^- is the negatively oriented (clockwise) semi-circle of radius ρ in the lower-half complex plane. We now examine the integral

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} h_{n,k}(z)\,dz = \lim_{\rho\to\infty} \int_{\gamma_\rho} h_{n,k}(z)\,dz + \lim_{\rho\to\infty} \int_{C_\rho^-} h_{n,k}(z)\,dz. \tag{E.19}$$

When $\lambda_n < 0$, we see from Fig. E.2(a) that the contour Γ_ρ does not enclose any poles, and $h_{n,k}(z)$ is analytic inside the contour. Using Cauchy's Integral Theorem and Jordan's Lemma, we have

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} h_{n,k}(z)\,dz = 0, \quad \lambda_n < 0, \tag{E.20}$$

and

$$\lim_{\rho\to\infty} \int_{C_\rho^-} h_{n,k}(z)\,dz = 0, \quad \lambda_n < 0, \tag{E.21}$$

which imply that (see (E.19))

$$\text{p.v.} \int_{-\infty}^{\infty} h_{n,k}(\upsilon)\,d\upsilon = \lim_{\rho\to\infty} \int_{\gamma_\rho} h_{n,k}(z)\,dz = 0, \quad \lambda_n < 0. \tag{E.22}$$

As a result, $f_Z(x)$ in (E.3) reduces to

$$f_Z(x) = \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{j}{2\lambda_n}\right)^k \cdot \frac{1}{2\pi} \text{ p.v.} \int_{-\infty}^{\infty} h_{n,k}(\upsilon)\, d\upsilon. \tag{E.23}$$

Now, when $\lambda_n > 0$, the contour Γ_ρ encloses a k-th order pole at $z = -j/2\lambda_n$ (see Fig. E.2(b)). Again, we use Cauchy's Residue Theorem and Jordan's Lemma to show that

$$\begin{aligned} \text{p.v.} \int_{-\infty}^{\infty} h_{n,k}(\upsilon)\, d\upsilon &= \lim_{\rho\to\infty} \int_{\gamma_\rho} h_{n,k}(z)\, dz \\ &= \lim_{\rho\to\infty} \left(\int_{\Gamma_\rho} h_{n,k}(z)\, dz - \int_{C_\rho^-} h_{n,k}(z)\, dz \right) \\ &= -2\pi j \cdot \text{Res}\left(h_{n,k}; -\frac{j}{2\lambda_n}\right), \quad \lambda_n > 0. \end{aligned} \tag{E.24}$$

The negative sign in (E.24) is required because the contour Γ_ρ is negatively oriented. Using the results of (E.16) and (E.24) in (E.23), we arrive at

$$\begin{aligned} f_Z(x) &= \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{j}{2\lambda_n}\right)^k \cdot \frac{1}{2\pi} \cdot (-2\pi j) \frac{1}{(k-1)!} (-jx)^{k-1} e^{-x/2\lambda_n} \\ &= \sum_{\substack{n=1 \\ \lambda_n > 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{1}{2\lambda_n}\right)^k \frac{1}{(k-1)!} x^{k-1} e^{-x/2\lambda_n}, \quad x > 0. \end{aligned} \tag{E.25}$$

E.1.3 Summary

In summary, $f_Z(x)$ is given by (see (E.18) and (E.25))

$$f_Z(x) = \begin{cases} \displaystyle\sum_{\substack{n=1 \\ \lambda_n < 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(-\frac{1}{2\lambda_n}\right)^k \frac{1}{(k-1)!} (-x)^{k-1} e^{-x/2\lambda_n}, & x < 0, \\ \displaystyle\sum_{\substack{n=1 \\ \lambda_n > 0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{1}{2\lambda_n}\right)^k \frac{1}{(k-1)!} x^{k-1} e^{-x/2\lambda_n}, & x > 0. \end{cases} \tag{E.26}$$

When the eigenvalues of **RF** are distinct (i.e., when $\Lambda = N$ and $m_n = 1$ for $n = 1, \ldots, \Lambda$), this result reduces to [36, Eq. (9.40)].

E.1.4 Sanity Check

We want to confirm that $\int_{-\infty}^{\infty} f_Z(x)\,dx = 1$. First, we have

$$P(Z<0) = \sum_{\substack{n=1 \\ \lambda_n<0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(-\frac{1}{2\lambda_n}\right)^k \frac{1}{(k-1)!} \int_{-\infty}^{0} (-x)^{k-1} e^{-x/2\lambda_n}\,dx. \tag{E.27}$$

Making the substitution $r_n = -\lambda_n$ in (E.27) and applying the relation for the Gamma function [33, Sec. 6.11.2],

$$\Gamma(k) = \int_0^{\infty} u^{k-1} e^{-u}\,du = (k-1)!\,, \quad k = 1, 2, \ldots, \tag{E.28}$$

we obtain

$$\begin{aligned} P(Z<0) &= \sum_{\substack{n=1 \\ r_n>0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{1}{2r_n}\right)^k \frac{1}{(k-1)!} \int_{-\infty}^{0} (-x)^{k-1} e^{x/2r_n}\,dx \\ &= \sum_{\substack{n=1 \\ r_n>0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \frac{1}{(k-1)!} \int_0^{\infty} u^{k-1} e^{-u}\,du \\ &= \sum_{\substack{n=1 \\ r_n>0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \\ &= \sum_{\substack{n=1 \\ \lambda_n<0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \\ &= \sum_{\substack{n=1 \\ \lambda_n<0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,k}. \end{aligned} \tag{E.29}$$

Similarly,

$$\begin{aligned} P(Z>0) &= \sum_{\substack{n=1 \\ \lambda_n>0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \left(\frac{1}{2\lambda_n}\right)^k \frac{1}{(k-1)!} \int_0^{\infty} x^{k-1} e^{-x/2\lambda_n}\,dx \\ &= \sum_{\substack{n=1 \\ \lambda_n>0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,(m_n-k+1)} \cdot \frac{1}{(k-1)!} \int_0^{\infty} u^{k-1} e^{-u}\,du \\ &= \sum_{\substack{n=1 \\ \lambda_n>0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,k}. \end{aligned} \tag{E.30}$$

Thus, we have

$$\int_{-\infty}^{\infty} f_Z(x)\,dx = \sum_{\substack{n=1\\ \lambda_n<0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,k} + \sum_{\substack{n=1\\ \lambda_n>0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,k}$$
$$= \sum_{n=1}^{\Lambda} \sum_{k=1}^{m_n} c_{n,k}$$
$$= 1, \tag{E.31}$$

where (A.20) was used in the last step.

E.2 Probability of Error

If we assume binary signaling, where a $+1$ and -1 are equally likely to be transmitted, then the probability of error is given by (see (E.29))

$$P_e = P(Z < 0) = \sum_{\substack{n=1\\ \lambda_n<0}}^{\Lambda} \sum_{k=1}^{m_n} c_{n,k}. \tag{E.32}$$

Appendix F

MVU and MMSE Estimators of Zero-Mean Complex Gaussian Random Variables

Given a known complex $N \times p$ matrix $\mathbf{H}$ with full-rank and $N > p$, and a complex $N \times 1$ random vector $\tilde{\mathbf{w}}$ with PDF $\tilde{\mathbf{w}} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\tilde{w}\tilde{w}})$, the complex extension of the *classical* linear model is given by [29, p. 529]

$$\tilde{\mathbf{x}} = \mathbf{H}\boldsymbol{\theta} + \tilde{\mathbf{w}}, \tag{F.1}$$

where $\boldsymbol{\theta}$ is a complex $p \times 1$ parameter vector to be estimated. It is shown that

$$\hat{\boldsymbol{\theta}}_{\mathrm{MVU}} = (\mathbf{H}^H \mathbf{C}_{\tilde{w}\tilde{w}}^{-1} \mathbf{H})^{-1} \mathbf{H}^H \mathbf{C}_{\tilde{w}\tilde{w}}^{-1} \tilde{\mathbf{x}} \tag{F.2}$$

is the minimum variance unbiased (MVU) estimator of $\boldsymbol{\theta}$ (see [29, Eq. (15.58)]). The complex *Bayesian* linear model, on the other hand, is also defined as

$$\tilde{\mathbf{x}} = \mathbf{H}\boldsymbol{\theta} + \tilde{\mathbf{w}}, \tag{F.3}$$

where $\mathbf{H}$, $\boldsymbol{\theta}$, and $\tilde{\mathbf{w}}$ are of the same dimensions as previously given. However, $\boldsymbol{\theta}$ is now a random vector with $\boldsymbol{\theta} \sim \mathcal{CN}(\mu_\theta, \mathbf{C}_{\theta\theta})$, $\tilde{\mathbf{w}}$ is independent of $\boldsymbol{\theta}$, and we possibly have $N \leq p$. From [29, Eq. (15.65)], the minimum mean square error (MMSE) estimator of $\boldsymbol{\theta}$ is given by

$$\hat{\boldsymbol{\theta}}_{\mathrm{MMSE}} = E(\boldsymbol{\theta}|\tilde{\mathbf{x}}) = \mu_\theta + (\mathbf{C}_{\theta\theta}^{-1} + \mathbf{H}^H \mathbf{C}_{\tilde{w}\tilde{w}}^{-1} \mathbf{H})^{-1} \mathbf{H}^H \mathbf{C}_{\tilde{w}\tilde{w}}^{-1} (\tilde{\mathbf{x}} - \mathbf{H}\mu_\theta). \tag{F.4}$$

When only one parameter is being estimated (i.e., $p = 1$), $\boldsymbol{\theta}$ becomes the scalar θ, and $\mathbf{H}$ reduces to a column vector $\mathbf{h}$:

$$\tilde{\mathbf{x}} = \mathbf{h}\theta + \tilde{\mathbf{w}}. \tag{F.5}$$

Assume we have white Gaussian noise, such that $\mathbf{C}_{\tilde{w}\tilde{w}} = \sigma_{\tilde{w}}^2 \mathbf{I}_N$. Under these assumptions, the MVU estimator given by (F.2) reduces to

$$\hat{\theta}_{\mathrm{MVU}} = (\mathbf{h}^H \mathbf{h})^{-1} \mathbf{h}^H \tilde{\mathbf{x}}. \tag{F.6}$$

Likewise, if we assume a prior pdf of $\theta \sim \mathcal{CN}(0, \sigma_\theta^2)$, the MMSE estimator becomes (see (F.4))

$$\hat{\theta}_{\mathrm{MMSE}} = \left(\frac{\sigma_{\tilde{w}}^2}{\sigma_\theta^2} + \mathbf{h}^H \mathbf{h} \right)^{-1} \mathbf{h}^H \tilde{\mathbf{x}} = \frac{\mathbf{h}^H \mathbf{h}}{\frac{\sigma_{\tilde{w}}^2}{\sigma_\theta^2} + \mathbf{h}^H \mathbf{h}} \, \hat{\theta}_{\mathrm{MVU}}. \tag{F.7}$$

Thus, when θ is zero-mean, the MVU and MMSE estimators for the complex linear model differ by only a multiplicative constant. Furthermore, if $\mathbf{h} = c\mathbf{1}_N$, where c is a constant and $\mathbf{1}_N$ is the $N \times 1$ all-ones vector, then

$$\hat{\theta}_{\mathrm{MVU}} = \frac{1}{c} \, \bar{x}, \tag{F.8}$$

$$\hat{\theta}_{\mathrm{MMSE}} = \frac{c^2 \sigma_\theta^2}{c^2 \sigma_\theta^2 + \frac{\sigma_{\tilde{w}}^2}{N}} \, \hat{\theta}_{\mathrm{MVU}}, \tag{F.9}$$

where $\bar{x} = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i$ is the sample average of the $\{\tilde{x}_i\}$.

F.1 Example

For the MC-CDMA system in Chapter 2, the demodulator outputs during the estimation phase are given in (2.26) as

$$W_{m,i,q} = A_1\, g_{m,i}^{(1)} + \Gamma_{W_{m,i,q}}, \quad q = 1, \ldots, Q_P, \tag{F.10}$$

where $\Gamma_{W_{m,i,q}}$ is defined as the sum of the MAI and AWGN terms. The $\{\Gamma_{W_{m,i,q}}\}$ are easily shown to be i.i.d. Since (F.10) is just a special case of (F.5) with $N = Q_P$ and $\mathbf{h} = A_1 \mathbf{1}_{Q_P}$, we use (F.8) to obtain the MVU estimate of $g_{m,i}^{(1)}$:

$$\hat{W}_{m,i,\text{MVU}} = \frac{1}{A_1}\left(\frac{1}{Q_P}\sum_{q=1}^{Q_P} W_{m,i,q}\right). \tag{F.11}$$

The MMSE estimate of $g_{m,i}^{(1)}$ is of the form

$$\hat{W}_{m,i,\text{MMSE}} = k_{m,i} \cdot \hat{W}_{m,i,\text{MVU}}, \tag{F.12}$$

where $k_{m,i}$ corresponds to the constant in (F.9). When these estimates are used as the weights in the maximal ratio combiner (see (2.35)),

$$Z_{m,q} = \sum_{i=1}^{R_1} \text{Re}\left\{\hat{W}_{m,i}^* Y_{m,i,q}\right\} = \sum_{i=1}^{R_1}\left\{\frac{1}{2}\hat{W}_{m,i}^* Y_{m,i,q} + \frac{1}{2}\hat{W}_{m,i} Y_{m,i,q}^*\right\}, \tag{F.13}$$

we observe that $k_{m,i}$ multiplies both the signal and interference/noise terms of $Y_{m,i,q}$, such that the SNR or SINR of the i-th term in (F.13) is the same for both estimators. Furthermore, since the fading and noise processes for the R_1 channels are statistically identical, we have $k_{m,i} \equiv k$ for all i. As a result, the total SNR or SINR of $Z_{m,q}$ is the same for both estimators, and this implies that both estimators yield the same probability of error.

Appendix G

Probability of Error: Case I

Consider the final test statistic

$$Z_{m,q} = \sum_{i=1}^{R} \left\{ \frac{1}{2} \hat{W}^*_{m,i} Y_{m,i,q} + \frac{1}{2} \hat{W}_{m,i} Y^*_{m,i,q} \right\}, \tag{G.1}$$

where $\hat{W}_{m,i}$ and $Y_{m,i,q}$ are a pair of correlated, zero-mean complex Gaussian random variables, and the R pairs $\{\hat{W}_{m,i}, Y_{m,i,q}\}_{i=1}^{R}$ are mutually statistically independent and identically distributed (i.i.d.). Since (G.1) is just a special case of the general quadratic form [27, Eq. (B-1)] with $A = 0$, $B = 0$, and $C = \frac{1}{2}$, the probability of error is obtained as [27, Eq. (B-21)]

$$P_e = P[Z_{m,q} < 0] = \frac{1}{\left(1 + \frac{\upsilon_2}{\upsilon_1}\right)^{2R-1}} \sum_{k=0}^{R-1} \binom{2R-1}{k} \left(\frac{\upsilon_2}{\upsilon_1}\right)^k, \tag{G.2}$$

where

$$\upsilon_1 = \frac{\sqrt{\mu_{WW}\mu_{YY}} - \mu_{WY}}{\mu_{WW}\mu_{YY} - \mu_{WY}^2}, \tag{G.3}$$

$$\upsilon_2 = \frac{\sqrt{\mu_{WW}\mu_{YY}} + \mu_{WY}}{\mu_{WW}\mu_{YY} - \mu_{WY}^2}, \tag{G.4}$$

and

$$\mu_{WW} \triangleq \frac{1}{2} E[|\hat{W}_{m,i}|^2], \tag{G.5}$$

$$\mu_{YY} \triangleq \frac{1}{2}E[|Y_{m,i,q}|^2], \tag{G.6}$$

$$\mu_{WY} \triangleq \frac{1}{2}E[\hat{W}_{m,i}Y^*_{m,i,q}]. \tag{G.7}$$

It is assumed in (G.3) and (G.4) that μ_{WY} is real and that $\mu_{WW}\mu_{YY} > \mu^2_{WY}$ (i.e., $|\mu| < 1$, where μ is given in (G.11) below). By applying [36, Eqs. (9.17),(9.18)], we can show that (G.2) is equal to

$$P_e = \frac{1}{\left(1+\frac{\upsilon_2}{\upsilon_1}\right)^R}\sum_{k=0}^{R-1}\binom{R-1+k}{k}\frac{1}{\left(1+\frac{\upsilon_1}{\upsilon_2}\right)^k}. \tag{G.8}$$

Furthermore, if we define

$$\mu = \frac{\upsilon_2 - \upsilon_1}{\upsilon_2 + \upsilon_1}, \tag{G.9}$$

such that

$$\left(1+\frac{\upsilon_2}{\upsilon_1}\right)^{-1} = \frac{1-\mu}{2},$$
$$\left(1+\frac{\upsilon_1}{\upsilon_2}\right)^{-1} = \frac{1+\mu}{2},$$

then we can re-write P_e as

$$P_e = \left(\frac{1-\mu}{2}\right)^R\sum_{k=0}^{R-1}\binom{R-1+k}{k}\left(\frac{1+\mu}{2}\right)^k. \tag{G.10}$$

Substituting the expressions for υ_1 and υ_2 into (G.9), we have

$$\mu = \frac{\mu_{WY}}{\sqrt{\mu_{WW}\mu_{YY}}}. \tag{G.11}$$

Note that μ is the correlation coefficient between $\hat{W}_{m,i}$ and $Y_{m,i,q}$.

G.1 MC-CDMA: SISO–Scenario #1

For the MC-CDMA system in Scenario #1, the final test statistic is given by (G.1) with $R = R_1$ (see (2.35)). The expressions for μ_{WW}, μ_{YY}, and μ_{WY} are given in Appendix T as

$$\mu_{WW} = \frac{1}{2}\left[1 + \frac{1}{Q_P}c_1\left(\frac{E_{b1}}{N_0}\right)\right], \tag{G.12}$$

$$\mu_{YY} = \frac{E_{b1}}{R_1}\left[1 + c_1\left(\frac{E_{b1}}{N_0}\right)\right], \tag{G.13}$$

$$\mu_{WY} = \frac{1}{2}\sqrt{\frac{2E_{b1}}{R_1}}. \tag{G.14}$$

Substituting (G.12)–(G.14) into (G.11), we get

$$\mu = \frac{\sqrt{Q_P}\,\bar{\gamma}}{\sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{G.15}$$

where

$$\bar{\gamma} = \left(\frac{1}{R_1}\frac{E_{b1}}{N_0}\right) \cdot \frac{1}{1 + (K-1)\left(\frac{1}{R_1}\frac{E_{b1}}{N_0}\right)}. \tag{G.16}$$

G.2 MC-DS-CDMA: SISO–Scenario #2

The final test statistic for the MC-DS-CDMA system in Scenario #2 is given by (G.1) with $R = R_2$ (see (2.49)). From Appendix Q, we have the following expressions for μ_{WW}, μ_{YY}, and μ_{WY}:

$$\mu_{WW} = \frac{1}{2}\left[1 + \frac{1}{Q_P}c_2\left(\frac{E_{b2}}{N_0}\right)\right], \tag{G.17}$$

$$\mu_{YY} = \frac{E_{b2}}{R_2}\left[1 + c_2\left(\frac{E_{b2}}{N_0}\right)\right], \tag{G.18}$$

$$\mu_{WY} = \frac{1}{2}\sqrt{\frac{2E_{b2}}{R_2}}. \tag{G.19}$$

We substitute (G.17)–(G.19) into (G.11) to get

$$\mu = \frac{\sqrt{Q_P}\,\bar{\gamma}}{\sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{G.20}$$

where

$$\bar{\gamma} = \left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right) \cdot \frac{1}{1 + \frac{K-1}{N}\left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right)}. \tag{G.21}$$

G.3 MC-DS-CDMA: SISO–Scenario #1 (Uniform MIP)

For the MC-DS-CDMA system in Scenario #1, the final test statistic is given by (2.43). When the MIP in each sub-band is uniform, $Z_{m,q}$ is just a sum of R_2L i.i.d. terms. Thus, we can think of $Z_{m,q}$ as having the same form as (G.1) with $R = R_2L$. By making the substitution $\Omega_{m,i,l}^{(1)} = 1/L$ in (R.33), (R.54), and (R.60), we obtain

$$\mu_{W_lW_l} = \frac{1}{2}\left[\frac{1}{L} + \frac{1}{Q_P}c_2\left(\frac{E_{b2}}{N_0}\right)\right] \equiv \mu_{WW}, \tag{G.22}$$

$$\mu_{Y_lY_l} = \frac{E_{b2}}{R_2}\left[\frac{1}{L} + c_2\left(\frac{E_{b2}}{N_0}\right)\right] \equiv \mu_{YY}, \tag{G.23}$$

$$\mu_{W_lY_l} = \frac{1}{2}\sqrt{\frac{2E_{b2}}{R_2}}\frac{1}{L} \equiv \mu_{WY}. \tag{G.24}$$

As a result, we have

$$\mu = \frac{\sqrt{Q_P}\,\bar{\gamma}}{\sqrt{(1+Q_P\bar{\gamma})(1+\bar{\gamma})}}, \tag{G.25}$$

where

$$\bar{\gamma} = \frac{1}{L}\left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right) \cdot \frac{1}{1 + \frac{K-1}{N}\left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right)}. \tag{G.26}$$

Appendix H

Probability of Error: Case II

From (2.43), the final test statistic for the MC-DS-CDMA system in Scenario #1 is given by

$$Z_{m,q} = \sum_{i=1}^{R_2} \sum_{l=1}^{L} \left\{ \frac{1}{2} \hat{W}^*_{m,i,l} Y_{m,i,q,l} + \frac{1}{2} \hat{W}_{m,i,l} Y^*_{m,i,q,l} \right\}, \tag{H.1}$$

where $\hat{W}_{m,i,l}$ and $Y_{m,i,q,l}$ are a pair of correlated, zero-mean complex Gaussian random variables. In this derivation, we assume the MIP is exponentially-decaying. For a given l, the R_2 pairs $\{\hat{W}_{m,i,l}, Y_{m,i,q,l}\}_{i=1,\ldots,R_2}$ are i.i.d., but for a given i, the L pairs $\{\hat{W}_{m,i,l}, Y_{m,i,q,l}\}_{l=1,\ldots,L}$ are assumed to be independent but not identically distributed. Thus, for a given l, the sum over i in (H.1) is just a special case of the quadratic form [27, Eq. (B-1)] with $A = 0$, $B = 0$, and $C = \frac{1}{2}$. We can use [27, Eq. (B-7)] to write the characteristic function of $Z_{m,q}$ as

$$\Phi_{Z_{m,q}}(j\upsilon) = \prod_{l=1}^{L} \frac{(\upsilon_{1l}\upsilon_{2l})^{R_2}}{(\upsilon + j\upsilon_{1l})^{R_2} (\upsilon - j\upsilon_{2l})^{R_2}} \equiv \Phi_1(j\upsilon)\Phi_2(j\upsilon), \tag{H.2}$$

where

$$\Phi_1(j\upsilon) \triangleq \prod_{l=1}^{L} \frac{\upsilon_{1l}^{R_2}}{(\upsilon + j\upsilon_{1l})^{R_2}}, \tag{H.3}$$

$$\Phi_2(j\upsilon) \triangleq \prod_{l=1}^{L} \frac{\upsilon_{2l}^{R_2}}{(\upsilon - j\upsilon_{2l})^{R_2}}, \tag{H.4}$$

and $j = \sqrt{-1}$. The expressions for υ_{1l} and υ_{2l} are identical to the ones given in (G.3) and (G.4), except that the moments μ_{WW}, μ_{YY}, and μ_{WY} are now a function of the path index l; that is, we have

$$\upsilon_{1l} = \frac{\sqrt{\mu_{W_l W_l}\mu_{Y_l Y_l}} - \mu_{W_l Y_l}}{\mu_{W_l W_l}\mu_{Y_l Y_l} - \mu^2_{W_l Y_l}}, \tag{H.5}$$

$$\upsilon_{2l} = \frac{\sqrt{\mu_{W_l W_l}\mu_{Y_l Y_l}} + \mu_{W_l Y_l}}{\mu_{W_l W_l}\mu_{Y_l Y_l} - \mu^2_{W_l Y_l}}, \tag{H.6}$$

where

$$\mu_{W_l W_l} \triangleq \frac{1}{2} E[|\hat{W}_{m,i,l}|^2], \tag{H.7}$$

$$\mu_{Y_l Y_l} \triangleq \frac{1}{2} E[|Y_{m,i,q,l}|^2], \tag{H.8}$$

$$\mu_{W_l Y_l} \triangleq \frac{1}{2} E[\hat{W}_{m,i,l} Y^*_{m,i,q,l}]. \tag{H.9}$$

It is assumed in (H.5) and (H.6) that $\mu_{W_l Y_l}$ is real. From (R.33), (R.54), and (R.60), we have

$$\mu_{W_l W_l} = \frac{1}{2}\left[\Omega^{(1)}_{m,i,l} + \frac{1}{Q_P} c_2\left(\frac{E_{b2}}{N_0}\right)\right], \tag{H.10}$$

$$\mu_{Y_l Y_l} = \frac{E_{b2}}{R_2}\left[\Omega^{(1)}_{m,i,l} + c_2\left(\frac{E_{b2}}{N_0}\right)\right], \tag{H.11}$$

$$\mu_{W_l Y_l} = \frac{1}{2}\sqrt{\frac{2E_{b2}}{R_2}}\,\Omega^{(1)}_{m,i,l}. \tag{H.12}$$

For the case of an exponential MIP, we simply substitute

$$\Omega^{(1)}_{m,i,l} = \Omega^{(1)}_{m,i,1} e^{-r_d(l-1)} \tag{H.13}$$

into the above equations. We then use (H.10)–(H.12) in (H.5) and (H.6) to obtain

$$\upsilon_{1l} = \frac{1}{\frac{1}{2}\sqrt{\frac{2E_{b2}}{R_2}}} \frac{1}{\sqrt{\left(\Omega^{(1)}_{m,i,l} + \frac{1}{Q_P} c_2\left(\frac{E_{b2}}{N_0}\right)\right)\left(\Omega^{(1)}_{m,i,l} + c_2\left(\frac{E_{b2}}{N_0}\right)\right)} + \Omega^{(1)}_{m,i,l}}, \tag{H.14}$$

$$\upsilon_{2l} = \frac{1}{\frac{1}{2}\sqrt{\frac{2E_{b2}}{R_2}}} \frac{1}{\sqrt{\left(\Omega^{(1)}_{m,i,l} + \frac{1}{Q_P} c_2\left(\frac{E_{b2}}{N_0}\right)\right)\left(\Omega^{(1)}_{m,i,l} + c_2\left(\frac{E_{b2}}{N_0}\right)\right)} - \Omega^{(1)}_{m,i,l}}. \tag{H.15}$$

Since $c_2\left(\frac{E_{b2}}{N_0}\right) > 0$, we observe that $\upsilon_{1l} > 0$ and $\upsilon_{2l} > 0$ for all l.

The inverse Fourier transform of $\Phi_{Z_{m,q}}(j\upsilon)$ yields the pdf of $Z_{m,q}$,

$$f_{Z_{m,q}}(x) = f_1(x) * f_2(x), \tag{H.16}$$

where

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_1(j\upsilon)e^{-j\upsilon x}\, d\upsilon, \tag{H.17}$$

$$f_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_2(j\upsilon)e^{-j\upsilon x}\, d\upsilon, \tag{H.18}$$

and $*$ denotes convolution. To evaluate these integrals, we will need to use contour integration and residue theory.

Finally, the probability of error is obtained by evaluating

$$P_e = \int_{-\infty}^{0} f_{Z_{m,q}}(x)\, dx. \tag{H.19}$$

H.1 $Q_P = 1$

For the case $Q_P = 1$, we have

$$\upsilon_{1l} = \frac{1}{\frac{1}{2}\sqrt{\frac{2E_{b2}}{R_2}}} \frac{1}{2\Omega_{m,i,l}^{(1)} + c_2\left(\frac{E_{b2}}{N_0}\right)}, \tag{H.20}$$

$$\upsilon_{2l} = \frac{1}{\frac{1}{2}\sqrt{\frac{2E_{b2}}{R_2}}} \frac{1}{c_2\left(\frac{E_{b2}}{N_0}\right)} \equiv \upsilon_2. \tag{H.21}$$

Since υ_{2l} is not a function of l, this allows us to re-write $\Phi_2(j\upsilon)$ in (H.4) as

$$\Phi_2(j\upsilon) = \frac{\upsilon_2^{R_2 L}}{(\upsilon - j\upsilon_2)^{R_2 L}}. \tag{H.22}$$

We also expand $\Phi_1(j\upsilon)$ in (H.3) in terms of partial fractions:

$$\Phi_1(j\upsilon) = \prod_{l=1}^{L}\left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{-R_2} = \sum_{l=1}^{L}\sum_{k=1}^{R_2} b_{l,(R_2-k+1)}\left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{-k}. \tag{H.23}$$

Using contour integration and residue theory to obtain $f_1(x)$ and $f_2(x)$, and then convolving the two functions to get $f_{Z_{m,q}}(x)$, we can show that (see Appendix I.1)

$$\begin{aligned} f_{Z_{m,q}}(x) = &\sum_{l=1}^{L}\sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot (-j\upsilon_{1l})^k (j\upsilon_2)^{R_2L} \sum_{n=0}^{R_2L-1} \binom{k-1+n}{n} \\ &\cdot \frac{1}{(R_2L-1-n)!} \frac{1}{(\upsilon_{1l}+\upsilon_2)^{k+n}} \cdot (-x)^{R_2L-1-n} e^{\upsilon_2 x}, \quad x<0. \end{aligned} \tag{H.24}$$

From (I.32), the probability of error is then given by

$$P_e = j^{R_2L} \sum_{l=1}^{L}\sum_{k=1}^{R_2} (-j)^k \, b_{l,(R_2-k+1)} \cdot \frac{1}{\left(1+\frac{\upsilon_2}{\upsilon_{1l}}\right)^k} \sum_{n=0}^{R_2L-1} \binom{k-1+n}{n} \frac{1}{\left(1+\frac{\upsilon_{1l}}{\upsilon_2}\right)^n}. \tag{H.25}$$

As we proved in Appendix A.2, we can express $b_{l,k}$ as

$$b_{l,k} = (-j)^{R_2(L-1)+k-1} \cdot \tilde{b}_{l,k}, \tag{H.26}$$

where the $\{\tilde{b}_{l,k}\}$ are the coefficients in the PFE

$$\prod_{l=1}^{L} \left(\frac{\upsilon}{\upsilon_{1l}}+1\right)^{-R_2} = \sum_{l=1}^{L}\sum_{k=1}^{R_2} \tilde{b}_{l,(R_2-k+1)} \left(\frac{\upsilon}{\upsilon_{1l}}+1\right)^{-k}. \tag{H.27}$$

Using (H.26) in (H.25), we obtain

$$P_e = \sum_{l=1}^{L}\sum_{k=1}^{R_2} \tilde{b}_{l,(R_2-k+1)} \cdot \frac{1}{\left(1+\frac{\upsilon_2}{\upsilon_{1l}}\right)^k} \sum_{n=0}^{R_2L-1} \binom{k-1+n}{n} \frac{1}{\left(1+\frac{\upsilon_{1l}}{\upsilon_2}\right)^n}. \tag{H.28}$$

Finally, by defining

$$\mu_l = \frac{\upsilon_2 - \upsilon_{1l}}{\upsilon_2 + \upsilon_{1l}}, \tag{H.29}$$

such that

$$\begin{aligned} \left(1+\frac{\upsilon_2}{\upsilon_{1l}}\right)^{-1} &= \frac{1-\mu_l}{2}, \\ \left(1+\frac{\upsilon_{1l}}{\upsilon_2}\right)^{-1} &= \frac{1+\mu_l}{2}, \end{aligned}$$

we can express P_e as

$$P_e = \sum_{l=1}^{L}\sum_{k=1}^{R_2} \tilde{b}_{l,(R_2-k+1)} \cdot \left(\frac{1-\mu_l}{2}\right)^k \sum_{n=0}^{R_2L-1} \binom{k-1+n}{n} \left(\frac{1+\mu_l}{2}\right)^n. \tag{H.30}$$

Substituting (H.20)–(H.21) into (H.29) and using (H.13), we show that

$$\mu_l = \frac{\mu_{W_l Y_l}}{\sqrt{\mu_{W_l W_l}\mu_{Y_l Y_l}}} = \frac{\bar{\gamma}_l}{1+\bar{\gamma}_l}, \tag{H.31}$$

where

$$\bar{\gamma}_l = \left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right) \cdot \frac{1}{1+\frac{K-1}{N}\left(\frac{1}{R_2}\frac{E_{b2}}{N_0}\right)}\, \Omega^{(1)}_{m,i,1}\, e^{-r_d(l-1)}. \tag{H.32}$$

H.2 $Q_P > 1$

For $Q_P > 1$, both υ_{1l} and υ_{2l} depend on l (see (H.14) and (H.15)). We expand both $\Phi_1(j\upsilon)$ and $\Phi_2(j\upsilon)$ in terms of partial fractions:

$$\Phi_1(j\upsilon) = \sum_{l=1}^{L}\sum_{k=1}^{R_2} c_{l,(R_2-k+1)} \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{-k}, \tag{H.33}$$

$$\Phi_2(j\upsilon) = \sum_{l=1}^{L}\sum_{k=1}^{R_2} d_{l,(R_2-k+1)} \left(\frac{\upsilon}{\upsilon_{2l}} - j\right)^{-k}. \tag{H.34}$$

Following the same steps as in the $Q_P = 1$ case, we obtain the following expression for the probability of error (see Appendix I.2):

$$\begin{aligned} P_e = &\sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2} (-j)^{k_1} j^{k_2} \cdot c_{l_1,(R_2-k_1+1)}\, d_{l_2,(R_2-k_2+1)} \\ &\cdot \frac{1}{\left(1+\frac{\upsilon_{2l_2}}{\upsilon_{1l_1}}\right)^{k_1}} \sum_{n=0}^{k_2-1} \binom{k_1-1+n}{n} \frac{1}{\left(1+\frac{\upsilon_{1l_1}}{\upsilon_{2l_2}}\right)^{n}}. \end{aligned} \tag{H.35}$$

Given the work in Appendices A.2 and A.3, we can express $c_{l,k}$ and $d_{l,k}$ as

$$c_{l,k} = (-j)^{R(L-1)+k-1} \cdot \tilde{c}_{l,k}, \tag{H.36}$$

$$d_{l,k} = j^{R(L-1)+k-1} \cdot \tilde{d}_{l,k}, \tag{H.37}$$

where the $\{\tilde{c}_{l,k}\}$ and the $\{\tilde{d}_{l,k}\}$ are the coefficients in the PFEs

$$\prod_{l=1}^{L}\left(\frac{\upsilon}{\upsilon_{1l}}+1\right)^{-R_2} = \sum_{l=1}^{L}\sum_{k=1}^{R_2} \tilde{c}_{l,(R_2-k+1)} \left(\frac{\upsilon}{\upsilon_{1l}}+1\right)^{-k}, \tag{H.38}$$

$$\prod_{l=1}^{L}\left(\frac{\upsilon}{\upsilon_{2l}}+1\right)^{-R_2} = \sum_{l=1}^{L}\sum_{k=1}^{R_2} \tilde{d}_{l,(R_2-k+1)} \left(\frac{\upsilon}{\upsilon_{2l}}+1\right)^{-k}. \tag{H.39}$$

By defining

$$\xi_{l_1,l_2} = \frac{\upsilon_{2l_2} - \upsilon_{1l_1}}{\upsilon_{2l_2} + \upsilon_{1l_1}}, \tag{H.40}$$

such that

$$\left(1 + \frac{\upsilon_{2l_2}}{\upsilon_{1l_1}}\right)^{-1} = \frac{1 - \xi_{l_1,l_2}}{2},$$
$$\left(1 + \frac{\upsilon_{1l_1}}{\upsilon_{2l_2}}\right)^{-1} = \frac{1 + \xi_{l_1,l_2}}{2},$$

and by using (H.36)–(H.37), we can show that (H.35) is re-written as

$$\begin{aligned} P_e = &\sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2} \tilde{c}_{l_1,(R_2-k_1+1)}\tilde{d}_{l_2,(R_2-k_2+1)} \\ &\cdot \left(\frac{1-\xi_{l_1,l_2}}{2}\right)^{k_1} \sum_{n=0}^{k_2-1} \binom{k_1-1+n}{n} \left(\frac{1+\xi_{l_1,l_2}}{2}\right)^{n}. \end{aligned} \tag{H.41}$$

Finally, substituting (H.14)–(H.15) into (H.40) and using (H.13), we have

$$\xi_{l_1,l_2} = \frac{e^{-r_d l_1}\left(1+\frac{1}{\mu_{l_1}}\right) + e^{-r_d l_2}\left(1-\frac{1}{\mu_{l_2}}\right)}{e^{-r_d l_1}\left(1+\frac{1}{\mu_{l_1}}\right) - e^{-r_d l_2}\left(1-\frac{1}{\mu_{l_2}}\right)}, \tag{H.42}$$

$$\mu_l = \frac{\sqrt{Q_P}\,\bar{\gamma}_l}{\sqrt{(1+Q_P\bar{\gamma}_l)(1+\bar{\gamma}_l)}}, \tag{H.43}$$

where $\bar{\gamma}_l$ is as defined in (H.32).

Appendix I

Derivations of $f_{Z_{m,q}}(x)$ and P_e

Given the expressions for $\Phi_1(j\upsilon)$ and $\Phi_2(j\upsilon)$ in Appendix H, we derive the PDF of $Z_{m,q}$ and the probability of error for the two cases $Q_P = 1$ and $Q_P > 1$.

I.1 $Q_P = 1$

From (H.22) and (H.23), we have

$$\Phi_1(j\upsilon) = \sum_{l=1}^{L}\sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{-k}, \tag{I.1}$$

$$\Phi_2(j\upsilon) = \frac{\upsilon_2^{R_2 L}}{(\upsilon - j\upsilon_2)^{R_2 L}} = \left(\frac{\upsilon}{\upsilon_2} - j\right)^{-R_2 L}. \tag{I.2}$$

We follow the same steps as in Appendix E and use contour integration and residue theory to derive expressions for

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_1(j\upsilon) e^{-j\upsilon x}\, d\upsilon, \tag{I.3}$$

$$f_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_2(j\upsilon) e^{-j\upsilon x}\, d\upsilon. \tag{I.4}$$

We consider $f_2(x)$ first and then $f_1(x)$.

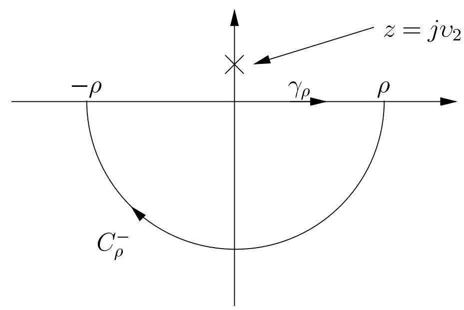


Figure I.1: For $x > 0$, we close the contour with a semi-circle in the lower-half complex plane.

I.1.1 $f_2(x)$

As in Appendix E, we treat the improper integral in (I.4) as a *principal value*,

$$f_2(x) = \frac{1}{2\pi} \text{ p.v.} \int_{-\infty}^{\infty} g(\upsilon)\, d\upsilon$$
$$= \frac{1}{2\pi} \lim_{\rho\to\infty} \int_{-\rho}^{\rho} g(\upsilon)\, d\upsilon, \tag{I.5}$$

where

$$g(\upsilon) \triangleq \frac{\upsilon_2^{R_2 L}}{(\upsilon - j\upsilon_2)^{R_2 L}}\, e^{-j\upsilon x}. \tag{I.6}$$

To solve (I.5), we consider the integral of the complex-valued function $g(z)$ along a simple closed contour Γ_ρ, such that γ_ρ, the directed line segment along the real axis from $-\rho$ to ρ, is one of its components. We will treat the cases $x < 0$ and $x > 0$ separately. Note that $g(z)$ has a pole at $z = j\upsilon_2$ of order $R_2 L$, where, as we mentioned in Appendix H, $\upsilon_2 > 0$.

For $x > 0$, we consider the contour $\Gamma_\rho = \gamma_\rho + C_\rho^-$, where C_ρ^- is the negatively oriented semi-circle of radius ρ in the *lower-half* complex plane (see Fig. I.1). Since $g(z)$ is analytic inside the contour (i.e., Γ_ρ does not enclose any poles), we have

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} g(z)\, dz = 0, \tag{I.7}$$

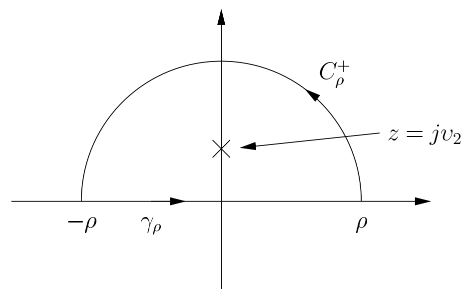


Figure I.2: For $x < 0$, we close the contour with a semi-circle in the upper-half complex plane.

after applying Cauchy's Integral Theorem. Using Jordan's Lemma, we also have

$$\lim_{\rho\to\infty} \int_{C_\rho^-} g(z)\, dz = 0. \tag{I.8}$$

Therefore,

$$\begin{aligned} \text{p.v.} \int_{-\infty}^{\infty} g(\upsilon)\, d\upsilon &= \lim_{\rho\to\infty} \int_{\gamma_\rho} g(z)\, dz \\ &= \lim_{\rho\to\infty} \int_{\Gamma_\rho} g(z)\, dz - \lim_{\rho\to\infty} \int_{C_\rho^-} g(z)\, dz \\ &= 0, \end{aligned} \tag{I.9}$$

which, when substituted into (I.5), yields

$$f_2(x) = 0, \quad x > 0. \tag{I.10}$$

For $x < 0$, we consider the contour $\Gamma_\rho = \gamma_\rho + C_\rho^+$, where C_ρ^+ is the positively oriented semi-circle of radius ρ in the *upper-half* complex plane (see Fig. I.2). This contour encloses a R_2L-th order pole at $z = j\upsilon_2$, since $\upsilon_2 > 0$. First, we apply Cauchy's Residue Theorem to get

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} g(z)\, dz = 2\pi j \cdot \text{Res}\,(g; j\upsilon_2)\,. \tag{I.11}$$

Since the application of Jordan's Lemma gives

$$\lim_{\rho\to\infty} \int_{C_\rho^+} g(z)\, dz = 0, \tag{I.12}$$

we obtain

$$\begin{aligned}
\text{p.v.} \int_{-\infty}^{\infty} g(\upsilon)\, d\upsilon &= \lim_{\rho \to \infty} \int_{\gamma_\rho} g(z)\, dz \\
&= \lim_{\rho \to \infty} \int_{\Gamma_\rho} g(z)\, dz - \lim_{\rho \to \infty} \int_{C_\rho^+} g(z)\, dz \\
&= 2\pi j \cdot \operatorname{Res}\left(g; j\upsilon_2\right). \qquad \text{(I.13)}
\end{aligned}$$

Using the formula given in (E.17), we evaluate the residue as

$$\begin{aligned}
\operatorname{Res}\left(g; j\upsilon_2\right) &= \lim_{z \to j\upsilon_2} \frac{1}{(R_2L-1)!} \frac{d^{R_2L-1}}{dz^{R_2L-1}} \left[\left(z - j\upsilon_2\right)^{R_2L} g(z) \right] \\
&= \lim_{z \to j\upsilon_2} \frac{\upsilon_2^{R_2L}}{(R_2L-1)!} \frac{d^{R_2L-1}}{dz^{R_2L-1}} \left(e^{-jxz}\right) \\
&= \lim_{z \to j\upsilon_2} \frac{\upsilon_2^{R_2L}}{(R_2L-1)!} \left(-jx\right)^{R_2L-1} e^{-jxz} \\
&= \frac{\upsilon_2^{R_2L}}{(R_2L-1)!} \left(-jx\right)^{R_2L-1} e^{\upsilon_2 x}. \qquad \text{(I.14)}
\end{aligned}$$

Substituting the results of (I.13) and (I.14) into (I.5), we obtain

$$f_2(x) = \frac{\left(j\upsilon_2\right)^{R_2L}}{(R_2L-1)!} \left(-x\right)^{R_2L-1} e^{\upsilon_2 x}, \quad x < 0. \qquad \text{(I.15)}$$

Thus, from (I.10) and (I.15), we have

$$f_2(x) = \begin{cases} \dfrac{\left(j\upsilon_2\right)^{R_2L}}{(R_2L-1)!} \left(-x\right)^{R_2L-1} e^{\upsilon_2 x}, & x < 0, \\ 0, & x > 0. \end{cases} \qquad \text{(I.16)}$$

I.1.2 $f_1(x)$

Treating the improper integral in (I.3) as a *principal value*, we have

$$\begin{aligned}
f_1(x) &= \sum_{l=1}^{L} \sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot \frac{1}{2\pi} \, \text{p.v.} \int_{-\infty}^{\infty} h_{l,k}(\upsilon)\, d\upsilon \\
&= \sum_{l=1}^{L} \sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot \frac{1}{2\pi} \lim_{\rho \to \infty} \int_{-\rho}^{\rho} h_{l,k}(\upsilon)\, d\upsilon, \qquad \text{(I.17)}
\end{aligned}$$

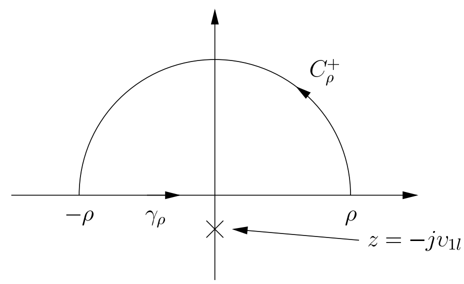


Figure I.3: For $x < 0$, we close the contour with a semi-circle in the upper-half complex plane.

where

$$h_{l,k}(\upsilon) \triangleq \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{-k} e^{-j\upsilon x} = \frac{\upsilon_{1l}^k}{(\upsilon + j\upsilon_{1l})^k}\, e^{-j\upsilon x}. \tag{I.18}$$

We will examine the integral of the complex-valued function $h_{l,k}(z)$ along a simple closed contour. Since $\upsilon_{1l} > 0$ for all l, the pole at $z = -j\upsilon_{1l}$ occurs in the lower-half complex plane.

For $x < 0$, we consider the contour $\Gamma_\rho = \gamma_\rho + C_\rho^+$ (see Fig. I.3). Note that $h_{l,k}$ is analytic inside the contour. Using Cauchy's Integral Theorem and Jordan's Lemma, we have

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} h_{l,k}(z)\, dz = 0, \tag{I.19}$$

and

$$\lim_{\rho\to\infty} \int_{C_\rho^+} h_{l,k}(z)\, dz = 0, \tag{I.20}$$

respectively. Thus,

$$\begin{aligned} \text{p.v.} \int_{-\infty}^{\infty} h_{l,k}(\upsilon)\, d\upsilon &= \lim_{\rho\to\infty} \int_{\gamma_\rho} h_{l,k}(z)\, dz \\ &= \lim_{\rho\to\infty} \int_{\Gamma_\rho} h_{l,k}(z)\, dz - \lim_{\rho\to\infty} \int_{C_\rho^-} h_{l,k}(z)\, dz \\ &= 0, \end{aligned} \tag{I.21}$$

which implies

$$f_1(x) = 0, \quad x < 0. \tag{I.22}$$

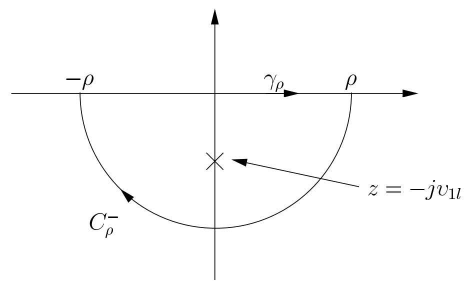


Figure I.4: For $x > 0$, we close the contour with a semi-circle in the lower-half complex plane.

For $x > 0$, we consider the contour $\Gamma_\rho = \gamma_\rho + C_\rho^-$ (see Fig. I.4). Since Γ_ρ encloses a pole at $z = -j\upsilon_{1l}$ of order k, we use Cauchy's Residue Theorem to obtain

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} h_{l,k}(z)\, dz = -2\pi j \cdot \mathrm{Res}\left(h_{l,k}; -j\upsilon_{1l}\right), \tag{I.23}$$

where the negative sign is required because Γ_ρ is negatively oriented. The residue at $z = -j\upsilon_{1l}$ is equal to

$$\begin{aligned}
\mathrm{Res}\left(h_{l,k}; -j\upsilon_{1l}\right) &= \lim_{z\to -j\upsilon_{1l}} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z + j\upsilon_{1l})^k\, h_{l,k}(z) \right] \\
&= \lim_{z\to -j\upsilon_{1l}} \frac{\upsilon_{1l}^k}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(e^{-jxz} \right) \\
&= \lim_{z\to -j\upsilon_{1l}} \frac{\upsilon_{1l}^k}{(k-1)!} (-jx)^{k-1} e^{-jxz} \\
&= \frac{\upsilon_{1l}^k}{(k-1)!} (-jx)^{k-1} e^{-\upsilon_{1l} x}.
\end{aligned} \tag{I.24}$$

Given that

$$\begin{aligned}
\mathrm{p.v.} \int_{-\infty}^{\infty} h_{l,k}(\upsilon)\, d\upsilon &= \lim_{\rho\to\infty} \int_{\gamma_\rho} h_{l,k}(z)\, dz \\
&= \lim_{\rho\to\infty} \int_{\Gamma_\rho} h_{l,k}(z)\, dz - \lim_{\rho\to\infty} \int_{C_\rho^-} h_{l,k}(z)\, dz \\
&= \lim_{\rho\to\infty} \int_{\Gamma_\rho} h_{l,k}(z)\, dz,
\end{aligned} \tag{I.25}$$

where the last equality follows after we apply Jordan's Lemma, we use (I.23)–(I.25) in (I.17) to get

$$f_1(x) = \sum_{l=1}^{L} \sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot \frac{(-j\upsilon_{1l})^k}{(k-1)!} \, x^{k-1} e^{-\upsilon_{1l} x}, \quad x > 0. \tag{I.26}$$

Combining the results of (I.22) and (I.26), we have

$$f_1(x) = \begin{cases} 0, & x < 0, \\ \displaystyle\sum_{l=1}^{L} \sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot \frac{(-j\upsilon_{1l})^k}{(k-1)!} \, x^{k-1} e^{-\upsilon_{1l} x}, & x > 0. \end{cases} \tag{I.27}$$

I.1.3 $f_{Z_{m,q}}(x)$

To obtain the probability of error, we only need an expression for $f_{Z_{m,q}}(x)$ over the range $x < 0$. From (H.16), $f_{Z_{m,q}}(x)$ is given by

$$f_{Z_{m,q}}(x) = f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(y) f_2(x-y) \, dy. \tag{I.28}$$

Substituting (I.16) and (I.27) into this integral, we get

$$\begin{aligned} f_{Z_{m,q}}(x) &= \sum_{l=1}^{L} \sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot \frac{(-j\upsilon_{1l})^k (j\upsilon_2)^{R_2 L}}{(k-1)! \, (R_2 L-1)!} \, e^{\upsilon_2 x} \\ &\quad \cdot \int_0^{\infty} y^{k-1} (y-x)^{R_2 L-1} e^{-(\upsilon_{1l}+\upsilon_2) y} \, dy \\ &= \sum_{l=1}^{L} \sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot \frac{(-j\upsilon_{1l})^k (j\upsilon_2)^{R_2 L}}{(k-1)! \, (R_2 L-1)!} \, e^{\upsilon_2 x} \\ &\quad \cdot \sum_{n=0}^{R_2 L-1} \binom{R_2 L-1}{n} \cdot (-x)^{R_2 L-1-n} \int_0^{\infty} y^{k-1+n} e^{-(\upsilon_{1l}+\upsilon_2) y} \, dy, \end{aligned} \tag{I.29}$$

where we applied a binomial expansion to $(y-x)^{R_2 L-1}$. Finally, we make the substitution $u = (\upsilon_{1l} + \upsilon_2) y$ and apply (E.28) to obtain

$$\begin{aligned} f_{Z_{m,q}}(x) &= \sum_{l=1}^{L} \sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot \frac{(-j\upsilon_{1l})^k (j\upsilon_2)^{R_2 L}}{(k-1)! \, (R_2 L-1)!} \, e^{\upsilon_2 x} \\ &\quad \cdot \sum_{n=0}^{R_2 L-1} \binom{R_2 L-1}{n} \cdot (-x)^{R_2 L-1-n} \frac{1}{(\upsilon_{1l}+\upsilon_2)^{k+n}} \int_0^{\infty} u^{k-1+n} e^{-u} \, du \\ &= \sum_{l=1}^{L} \sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot (-j\upsilon_{1l})^k (j\upsilon_2)^{R_2 L} \sum_{n=0}^{R_2 L-1} \binom{k-1+n}{n} \\ &\quad \cdot \frac{1}{(R_2 L-1-n)!} \frac{1}{(\upsilon_{1l}+\upsilon_2)^{k+n}} \cdot (-x)^{R_2 L-1-n} e^{\upsilon_2 x}, \quad x < 0. \end{aligned} \tag{I.30}$$

I.1.4 Probability of Error

Assuming a +1 is transmitted, we integrate over the negative values of $Z_{m,q}$ to obtain the probability of error:

$$\begin{aligned}
P_e &= \int_{-\infty}^{0} f_{Z_{m,q}}(x)\,dx \\
&= \sum_{l=1}^{L}\sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot (-j\upsilon_{1l})^k (j\upsilon_2)^{R_2L} \sum_{n=0}^{R_2L-1} \binom{k-1+n}{n} \\
&\quad \cdot \frac{1}{(R_2L-1-n)!}\frac{1}{(\upsilon_{1l}+\upsilon_2)^{k+n}} \int_{-\infty}^{0} (-x)^{R_2L-1-n} e^{\upsilon_2 x}\,dx \\
&= \sum_{l=1}^{L}\sum_{k=1}^{R_2} b_{l,(R_2-k+1)} \cdot (-j\upsilon_{1l})^k (j\upsilon_2)^{R_2L} \sum_{n=0}^{R_2L-1} \binom{k-1+n}{n} \\
&\quad \cdot \frac{1}{(R_2L-1-n)!}\frac{1}{(\upsilon_{1l}+\upsilon_2)^{k+n}}\frac{1}{\upsilon_2^{R_2L-n}} \int_{0}^{\infty} u^{R_2L-1-n} e^{-u}\,du.
\end{aligned} \tag{I.31}$$

We apply (E.28) and simplify the resulting expression to finally get

$$P_e = j^{R_2L} \sum_{l=1}^{L}\sum_{k=1}^{R_2} (-j)^k b_{l,(R_2-k+1)} \cdot \frac{1}{\left(1+\frac{\upsilon_2}{\upsilon_{1l}}\right)^k} \sum_{n=0}^{R_2L-1} \binom{k-1+n}{n} \frac{1}{\left(1+\frac{\upsilon_{1l}}{\upsilon_2}\right)^n}. \tag{I.32}$$

I.2 $Q_P > 1$

For $Q_P > 1$, we have (see (H.33) and (H.34))

$$\Phi_1(j\upsilon) = \sum_{l=1}^{L}\sum_{k=1}^{R_2} c_{l,(R_2-k+1)} \left(\frac{\upsilon}{\upsilon_{1l}} + j\right)^{-k}, \tag{I.33}$$

$$\Phi_2(j\upsilon) = \sum_{l=1}^{L}\sum_{k=1}^{R_2} d_{l,(R_2-k+1)} \left(\frac{\upsilon}{\upsilon_{2l}} - j\right)^{-k}, \tag{I.34}$$

where $f_1(x)$ and $f_2(x)$ are still given by (I.3) and (I.4), respectively.

I.2.1 $f_1(x)$

Since $\Phi_1(j\upsilon)$ is identical in form to the $\Phi_1(j\upsilon)$ in the $Q_P = 1$ case (see (I.1)), we simply replace $b_{l,(R_2-k+1)}$ in (I.27) with $c_{l,(R_2-k+1)}$ to obtain

$$f_1(x) = \begin{cases} 0, & x < 0, \\ \displaystyle\sum_{l=1}^{L}\sum_{k=1}^{R_2} c_{l,(R_2-k+1)} \cdot \frac{(-j\upsilon_{1l})^k}{(k-1)!}\, x^{k-1} e^{-\upsilon_{1l} x}, & x > 0. \end{cases} \tag{I.35}$$

I.2.2 $f_2(x)$

Likewise, $\Phi_2(j\upsilon)$ is also similar in form to the $\Phi_2(j\upsilon)$ in the $Q_P = 1$ case (see (I.2)). Therefore, by replacing υ_2 with υ_{2l} and $R_2 L$ with k in (I.16), we can use the resulting expression to write

$$f_2(x) = \begin{cases} \displaystyle\sum_{l=1}^{L}\sum_{k=1}^{R_2} d_{l,(R_2-k+1)} \cdot \frac{(j\upsilon_{2l})^k}{(k-1)!}\, (-x)^{k-1} e^{\upsilon_{2l} x}, & x < 0, \\ 0, & x > 0. \end{cases} \tag{I.36}$$

I.2.3 $f_{Z_{m,q}}(x)$

Again, we only need to derive $f_{Z_{m,q}}(x)$ for $x < 0$. Substituting (I.35) and (I.36) into (I.28) and applying a binomial expansion, we get

$$\begin{aligned} f_{Z_{m,q}}(x) &= \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2} c_{l_1,(R_2-k_1+1)} d_{l_2,(R_2-k_2+1)} \cdot \frac{(-j\upsilon_{1l_1})^{k_1}(j\upsilon_{2l_2})^{k_2}}{(k_1-1)!\,(k_2-1)!}\, e^{\upsilon_{2l_2} x} \\ &\quad \cdot \int_0^{\infty} y^{k_1-1}(y-x)^{k_2-1} e^{-(\upsilon_{1l_1}+\upsilon_{2l_2})y}\, dy \\ &= \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2} c_{l_1,(R_2-k_1+1)} d_{l_2,(R_2-k_2+1)} \cdot \frac{(-j\upsilon_{1l_1})^{k_1}(j\upsilon_{2l_2})^{k_2}}{(k_1-1)!\,(k_2-1)!}\, e^{\upsilon_{2l_2} x} \\ &\quad \cdot \sum_{n=0}^{k_2-1} \binom{k_2-1}{n} \cdot (-x)^{k_2-1-n} \int_0^{\infty} y^{k_1-1+n} e^{-(\upsilon_{1l_1}+\upsilon_{2l_2})y}\, dy. \end{aligned} \tag{I.37}$$

We make the substitution $u = (\upsilon_{1l_1} + \upsilon_{2l_2})y$ and apply (E.28) to obtain

$$\begin{aligned} f_{Z_{m,q}}(x) = & \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2} c_{l_1,(R_2-k_1+1)} d_{l_2,(R_2-k_2+1)} \cdot \frac{(-j\upsilon_{1l_1})^{k_1}(j\upsilon_{2l_2})^{k_2}}{(k_1-1)!\,(k_2-1)!} e^{\upsilon_{2l_2}x} \\ & \cdot \sum_{n=0}^{k_2-1}\binom{k_2-1}{n} \cdot (-x)^{k_2-1-n} \frac{1}{(\upsilon_{1l_1}+\upsilon_{2l_2})^{k_1+n}} \int_0^\infty u^{k_1-1+n} e^{-u}\, du \\ = & \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2} c_{l_1,(R_2-k_1+1)} d_{l_2,(R_2-k_2+1)} \cdot (-j\upsilon_{1l_1})^{k_1}(j\upsilon_{2l_2})^{k_2} \sum_{n=0}^{k_2-1}\binom{k_1-1+n}{n} \\ & \cdot \frac{1}{(k_2-1-n)!}\frac{1}{(\upsilon_{1l_1}+\upsilon_{2l_2})^{k_1+n}} \cdot (-x)^{k_2-1-n} e^{\upsilon_{2l_2}x}, \quad x<0. \end{aligned} \tag{I.38}$$

I.2.4 Probability of Error

Again, we integrate over the negative values of $Z_{m,q}$ to obtain the probability of error:

$$\begin{aligned} P_e = & \int_{-\infty}^{0} f_{Z_{m,q}}(x)\, dx \\ = & \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2} c_{l_1,(R_2-k_1+1)} d_{l_2,(R_2-k_2+1)} \cdot (-j\upsilon_{1l_1})^{k_1}(j\upsilon_{2l_2})^{k_2} \sum_{n=0}^{k_2-1}\binom{k_1-1+n}{n} \\ & \cdot \frac{1}{(k_2-1-n)!}\frac{1}{(\upsilon_{1l_1}+\upsilon_{2l_2})^{k_1+n}} \cdot \int_{-\infty}^{0} (-x)^{k_2-1-n} e^{\upsilon_{2l_2}x}\, dx \\ = & \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2} c_{l_1,(R_2-k_1+1)} d_{l_2,(R_2-k_2+1)} \cdot (-j\upsilon_{1l_1})^{k_1}(j\upsilon_{2l_2})^{k_2} \sum_{n=0}^{k_2-1}\binom{k_1-1+n}{n} \\ & \cdot \frac{1}{(k_2-1-n)!}\frac{1}{(\upsilon_{1l_1}+\upsilon_{2l_2})^{k_1+n}} \frac{1}{\upsilon_{2l_2}^{k_2-n}} \int_0^\infty u^{k_2-1-n} e^{-u}\, du. \end{aligned} \tag{I.39}$$

Finally, we apply (E.28) and simplify the resulting expression to get

$$\begin{aligned} P_e = & \sum_{l_1=1}^{L}\sum_{l_2=1}^{L}\sum_{k_1=1}^{R_2}\sum_{k_2=1}^{R_2} (-j)^{k_1} j^{k_2} \cdot c_{l_1,(R_2-k_1+1)} d_{l_2,(R_2-k_2+1)} \\ & \cdot \frac{1}{\left(1+\frac{\upsilon_{2l_2}}{\upsilon_{1l_1}}\right)^{k_1}} \sum_{n=0}^{k_2-1}\binom{k_1-1+n}{n} \frac{1}{\left(1+\frac{\upsilon_{1l_1}}{\upsilon_{2l_2}}\right)^{n}}. \end{aligned} \tag{I.40}$$

Appendix J

Probability of Error: Case III

The final test statistic for the MC-CDMA system in Scenario #2 can be written as (see (2.35))

$$Z_{m,q} = \sum_{i=1}^{R_1} \left\{ \frac{1}{2}\hat{W}^*_{m,i} Y_{m,i,q} + \frac{1}{2}\hat{W}_{m,i} Y^*_{m,i,q} \right\} \equiv \sum_{u=1}^{MR_1/N} s_{m,u}, \tag{J.1}$$

where

$$s_{m,u} \triangleq \sum_{i=(u-1)R_b+1}^{uR_b} \left\{ \frac{1}{2}\hat{W}^*_{m,i} Y_{m,i,q} + \frac{1}{2}\hat{W}_{m,i} Y^*_{m,i,q} \right\} \tag{J.2}$$

represents the sum over the R_b correlated terms within the u-th frequency block that are associated with the m-th symbol. Once again, $\hat{W}_{m,i}$ and $Y_{m,i,q}$ are a pair of correlated, zero-mean complex Gaussian random variables. Given the assumptions made in the correlated block fading model (see Section 2.2.4), the $\{s_{m,u}\}_{u=1}^{MR_1/N}$ are i.i.d. Thus, the characteristic function of $Z_{m,q}$ can be written as

$$\Phi_{Z_{m,q}}(j\upsilon) = \prod_{u=1}^{MR_1/N} \Phi_{s_{m,u}}(j\upsilon) = \left[\Phi_{s_{m,1}}(j\upsilon)\right]^{MR_1/N}. \tag{J.3}$$

Define the $2R_b \times 1$ vector

$$\mathbf{v} = \begin{bmatrix} \hat{\mathbf{w}} \\ \mathbf{y} \end{bmatrix} \tag{J.4}$$

and the $2R_b \times 2R_b$ matrix

$$\mathbf{F} = \frac{1}{2}\begin{bmatrix} \mathbf{0}_{R_b} & \mathbf{I}_{R_b} \\ \mathbf{I}_{R_b} & \mathbf{0}_{R_b} \end{bmatrix}, \tag{J.5}$$

where

$$\hat{\mathbf{w}} = [\hat{W}_{m,1}\ \hat{W}_{m,2}\ \cdots\ \hat{W}_{m,R_b}]^T, \tag{J.6}$$

$$\mathbf{y} = [Y_{m,1,q}\ Y_{m,2,q}\ \cdots\ Y_{m,R_b,q}]^T. \tag{J.7}$$

We can then express $s_{m,1}$ as a Hermitian quadratic form in zero-mean complex Gaussian random variables:

$$s_{m,1} = \mathbf{v^H F v}. \tag{J.8}$$

If we denote the $2R_b \times 2R_b$ covariance matrix of $\mathbf{v}$ by

$$\mathbf{R} = \frac{1}{2}E[\mathbf{v v^H}], \tag{J.9}$$

then the characteristic function of $s_{m,1}$ is equal to [35, Eq. (B-3-16)]

$$\Phi_{s_{m,1}}(j\upsilon) = \frac{1}{\det(\mathbf{I} - 2j\upsilon\mathbf{RF})} = \prod_{n=1}^{\Lambda}(1 - 2j\upsilon\lambda_n)^{-m_n}, \tag{J.10}$$

where the second equality in (J.10) follows by assuming the matrix $\mathbf{RF}$ has $\Lambda \leq 2R_b$ distinct (non-zero) eigenvalues, $\{\lambda_i\}_{i=1}^{\Lambda}$, with corresponding multiplicities $\{m_i\}_{i=1}^{\Lambda}$, such that $\sum_{n=1}^{\Lambda} m_n = 2R_b$. We substitute (J.10) into (J.3), apply a partial fraction expansion (PFE), and define

$$\xi_n \triangleq m_n \cdot \frac{MR_1}{N} \tag{J.11}$$

to obtain

$$\Phi_{Z_{m,q}}(j\upsilon) = \prod_{n=1}^{\Lambda}(1 - 2j\upsilon\lambda_n)^{-\xi_n} = \sum_{n=1}^{\Lambda}\sum_{k=1}^{\xi_n} \frac{c_{n,(\xi_n-k+1)}}{(1 - 2j\upsilon\lambda_n)^k}. \tag{J.12}$$

The PFE coefficients in (J.12) are derived in closed form as (see Appendix A)

$$c_{n,k} = \prod_{\substack{r=1\\r\neq n}}^{\Lambda} \left(1 - \frac{\lambda_r}{\lambda_n}\right)^{-\xi_r} \sum_{\Xi_n} \prod_{\substack{p=1\\p\neq n}}^{\Lambda} \binom{\xi_p - 1 + i_p}{i_p} \left(1 - \frac{\lambda_n}{\lambda_p}\right)^{-i_p}$$
$$= \prod_{\substack{r=1\\r\neq n}}^{\Lambda} \left(\frac{1 - \mu_{rn}}{2}\right)^{\xi_r} \sum_{\Xi_n} \prod_{\substack{p=1\\p\neq n}}^{\Lambda} \binom{\xi_p - 1 + i_p}{i_p} \left(\frac{1 + \mu_{pn}}{2}\right)^{i_p}. \tag{J.13}$$

where

$$\mu_{mn} \triangleq \frac{\lambda_m + \lambda_n}{\lambda_m - \lambda_n}, \quad m \neq n, \tag{J.14}$$

and Ξ_n denotes the set of integers $\{i_1, \ldots, i_{n-1}, i_{n+1}, \ldots, i_\Lambda\}$ such that $\sum_{p=1,p\neq n}^{\Lambda} i_p = k - 1$. As shown in Appendix D.1, the characteristic function in (J.12) leads to a probability of error given by[1]

$$P_e = \sum_{\substack{n=1\\\lambda_n<0}}^{\Lambda} \sum_{k=1}^{\xi_n} c_{n,(\xi_n - k+1)}. \tag{J.15}$$

Since the PFE coefficients are a function of the eigenvalues of $\mathbf{RF}$, the first step is to obtain an expression for $\mathbf{RF}$. From (J.4), (J.5), and (J.9), we have

$$\mathbf{RF} = \frac{1}{2} \begin{bmatrix} \frac{1}{2}E[\hat{\mathbf{w}}\mathbf{y}^{\mathbf{H}}] & \frac{1}{2}E[\hat{\mathbf{w}}\hat{\mathbf{w}}^{\mathbf{H}}] \\ \frac{1}{2}E[\mathbf{y}\mathbf{y}^{\mathbf{H}}] & \frac{1}{2}E[\mathbf{y}\hat{\mathbf{w}}^{\mathbf{H}}] \end{bmatrix}. \tag{J.16}$$

We prove in Appendix S that

$$\frac{1}{2}E[\hat{W}_{m,i_1}\hat{W}^*_{m,i_2}] = \frac{1}{2}\left[1 + \frac{1}{Q_P}c_1\left(\frac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right], \tag{J.17}$$

$$\frac{1}{2}E[Y_{m,i_1,q}Y^*_{m,i_2,q}] = \frac{E_{b1}}{R_1}\left[1 + c_1\left(\frac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right], \tag{J.18}$$

$$\frac{1}{2}E[\hat{W}_{m,i_1}Y^*_{m,i_2,q}] = \frac{1}{2}\sqrt{\frac{2E_{b1}}{R_1}}, \tag{J.19}$$

when the indices i_1 and i_2 correspond to repetitions of the m-th symbol from the *same* block.

[1] An alternative derivation of (J.15) is given in Appendix E. It involves evaluating the inverse Fourier transform of $\Phi_{Z_{m,q}}(jv)$ to get the PDF of $Z_{m,q}$, $f_{Z_{m,q}}(x)$, and then solving $\int_{-\infty}^{0} f_{Z_{m,q}}(x)\,dx$.

As a result, we have

$$\frac{1}{2}E[\hat{\mathbf{w}}\hat{\mathbf{w}}^{\mathbf{H}}] = \frac{1}{2}\left[\mathbf{A}_{R_b} + \frac{1}{Q_P}c_1\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right], \tag{J.20}$$

$$\frac{1}{2}E[\mathbf{y}\mathbf{y}^{\mathbf{H}}] = \frac{E_{b1}}{R_1}\left[\mathbf{A}_{R_b} + c_1\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right], \tag{J.21}$$

$$\frac{1}{2}E[\hat{\mathbf{w}}\mathbf{y}^{\mathbf{H}}] = \frac{1}{2}\sqrt{\frac{2E_{b1}}{R_1}}\mathbf{A}_{R_b}, \tag{J.22}$$

where $\mathbf{A}_{R_b}$ is the $R_b \times R_b$ all-ones matrix. $\mathbf{RF}$ is then equal to

$$\mathbf{RF} = \frac{1}{4}\begin{bmatrix} \sqrt{\frac{2E_{b1}}{R_1}}\mathbf{A}_{R_b} & \mathbf{A}_{R_b} + \frac{1}{Q_P}c_1\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b} \\ \frac{2E_{b1}}{R_1}\left[\mathbf{A}_{R_b} + c_1\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right] & \sqrt{\frac{2E_{b1}}{R_1}}\mathbf{A}_{R_b} \end{bmatrix}, \tag{J.23}$$

which is a special case of (D.78) if we make the following substitutions:

$$A = 0, \quad B = 0, \quad C = \frac{1}{2},$$

$$N = 1, \quad L = R_b,$$

$$s_{xx} = \frac{1}{2}, \quad t_{xx} = \frac{1}{2}\frac{1}{Q_P}c_1\left(\frac{E_{b1}}{N_0}\right),$$

$$s_{yy} = \frac{E_{b1}}{R_1}, \quad t_{yy} = \frac{E_{b1}}{R_1}c_1\left(\frac{E_{b1}}{N_0}\right),$$

$$s_{xy} = \frac{1}{2}\sqrt{\frac{2E_{b1}}{R_1}}.$$

Using these substitutions in (D.105), we obtain the characteristic polynomial

$$\det(\lambda\mathbf{I}_{2R_b} - \mathbf{RF}) = \left\{\lambda^2 - \left[\frac{1}{\sqrt{Q_P}}\,c_1\left(\frac{E_{b1}}{N_0}\right)\cdot\frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}}\right]^2\right\}^{R_b-1}\cdot(\lambda^2 - d\lambda - e), \tag{J.24}$$

where

$$d = 2R_b\cdot\frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}}, \tag{J.25}$$

$$e = c_1\left(\frac{E_{b1}}{N_0}\right)\cdot\left[\left(1 + \frac{1}{Q_P}\right)R_b + \frac{1}{Q_P}c_1\left(\frac{E_{b1}}{N_0}\right)\right]\left(\frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}}\right)^2. \tag{J.26}$$

J.1 $Q_P = 1$

First, we consider the case $Q_P = 1$. Solving for the roots of $\det(\lambda \mathbf{I}_{2R_b} - \mathbf{RF})$ in (J.24), we can show that $\mathbf{RF}$ has only $\Lambda = 3$ distinct eigenvalues:

$$\lambda_1 = -\frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}} \cdot c_1\left(\frac{E_{b1}}{N_0}\right), \quad m_1 = R_b, \tag{J.27}$$

$$\lambda_2 = \frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}} \cdot c_1\left(\frac{E_{b1}}{N_0}\right), \quad m_2 = R_b - 1, \tag{J.28}$$

$$\lambda_3 = \frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}} \cdot \left[2R_b + c_1\left(\frac{E_{b1}}{N_0}\right)\right], \quad m_3 = 1. \tag{J.29}$$

Since $c_1\left(\frac{E_{b1}}{N_0}\right) > 0$, such that λ_1 is the only negative eigenvalue, (J.15) reduces to

$$P_e = \sum_{m=1}^{\xi_1} c_{1,(\xi_1 - m + 1)} = \sum_{m=1}^{\xi_1} c_{1,m}. \tag{J.30}$$

Using (J.13) in (J.30), we have

$$\begin{aligned} P_e = &\left(\frac{1-\mu_{21}}{2}\right)^{R_1 - MR_1/N} \left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N} \sum_{m=1}^{R_1} \sum_{n_2=0}^{m-1} \binom{(R_1 - MR_1/N) - 1 + n_2}{n_2} \\ &\cdot \binom{MR_1/N - 1 + (m-1-n_2)}{m-1-n_2} \left(\frac{1+\mu_{21}}{2}\right)^{n_2} \left(\frac{1+\mu_{31}}{2}\right)^{m-1-n_2}. \end{aligned} \tag{J.31}$$

We re-write the double summation above as

$$\sum_{m=1}^{R_1} \sum_{n_2=0}^{m-1} \longrightarrow \sum_{n_2=0}^{R_1-1} \sum_{m=n_2+1}^{R_1} \tag{J.32}$$

and make the substitution $p = m - 1 - n_2$ to get

$$\begin{aligned} P_e = &\left(\frac{1-\mu_{21}}{2}\right)^{R_1 - MR_1/N} \sum_{n_2=0}^{R_1-1} \binom{(R_1 - MR_1/N) - 1 + n_2}{n_2} \left(\frac{1+\mu_{21}}{2}\right)^{n_2} \\ &\cdot \left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N} \sum_{p=0}^{R_1-1-n_2} \binom{MR_1/N - 1 + p}{p} \left(\frac{1+\mu_{31}}{2}\right)^{p}. \end{aligned} \tag{J.33}$$

The expressions for μ_{21} and μ_{31} follow by substituting (J.27)–(J.29) into (J.14):

$$\mu_{21} = 0, \tag{J.34}$$

$$\mu_{31} = \frac{\bar{\gamma}}{1+\bar{\gamma}}, \tag{J.35}$$

where

$$\bar{\gamma} = \frac{R_b}{c_1\left(\frac{E_{b1}}{N_0}\right)} = \left(\frac{1}{MR_1/N}\frac{E_{b1}}{N_0}\right)\frac{1}{1+(K-1)\left(\frac{1}{R_1}\frac{E_{b1}}{N_0}\right)}. \tag{J.36}$$

Finally, since $\mu_{21} = 0$, (J.31) reduces to

$$\begin{aligned} P_e = &\left(\frac{1}{2}\right)^{R_1 - MR_1/N} \sum_{n_2=0}^{R_1-1} \binom{(R_1 - MR_1/N) - 1 + n_2}{n_2} \left(\frac{1}{2}\right)^{n_2} \\ &\cdot \left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N} \sum_{p=0}^{R_1-1-n_2} \binom{MR_1/N - 1 + p}{p} \left(\frac{1+\mu_{31}}{2}\right)^{p}. \end{aligned} \tag{J.37}$$

J.2 $Q_P > 1$

When $Q_P > 1$, we obtain $\Lambda = 4$ distinct solutions to the equation $\det(\lambda \mathbf{I}_{2R_b} - \mathbf{RF}) = 0$:

$$\lambda_1 = -\frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}} \cdot \frac{1}{\sqrt{Q_P}}\, c_1\!\left(\frac{E_{b1}}{N_0}\right), \quad m_1 = R_b - 1, \tag{J.38}$$

$$\lambda_2 = \frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}} \cdot \frac{1}{\sqrt{Q_P}}\, c_1\!\left(\frac{E_{b1}}{N_0}\right), \quad m_2 = R_b - 1, \tag{J.39}$$

$$\lambda_3 = \frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}} \cdot \left\{R_b - \sqrt{\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right] \cdot \left[R_b + \frac{1}{Q_P}\, c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]}\right\}, \quad m_3 = 1, \tag{J.40}$$

$$\lambda_4 = \frac{1}{4}\sqrt{\frac{2E_{b1}}{R_1}} \cdot \left\{R_b + \sqrt{\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right] \cdot \left[R_b + \frac{1}{Q_P}\, c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]}\right\}, \quad m_4 = 1. \tag{J.41}$$

There are now two negative eigenvalues (i.e., λ_1 and λ_3), so (J.15) reduces to

$$P_e = \sum_{m=1}^{\xi_1} c_{1,m} + \sum_{m=1}^{\xi_3} c_{3,m}. \tag{J.42}$$

Using (J.13) in (J.42), we get

$$\begin{aligned}P_e = &\left(\frac{1-\mu_{21}}{2}\right)^{R_1-MR_1/N}\left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N}\left(\frac{1-\mu_{41}}{2}\right)^{MR_1/N}\\ &\cdot\sum_{m=1}^{R_1-MR_1/N}\sum_{n_2=0}^{m-1}\sum_{n_3=0}^{m-1-n_2}\binom{(R_1-MR_1/N)-1+n_2}{n_2}\\ &\cdot\binom{MR_1/N-1+n_3}{n_3}\binom{MR_1/N-1+(m-1-n_2-n_3)}{m-1-n_2-n_3}\\ &\cdot\left(\frac{1+\mu_{21}}{2}\right)^{n_2}\left(\frac{1+\mu_{31}}{2}\right)^{n_3}\left(\frac{1+\mu_{41}}{2}\right)^{m-1-n_2-n_3}\\ &+\left(\frac{1-\mu_{13}}{2}\right)^{R_1-MR_1/N}\left(\frac{1-\mu_{23}}{2}\right)^{R_1-MR_1/N}\left(\frac{1-\mu_{43}}{2}\right)^{MR_1/N}\\ &\cdot\sum_{m=1}^{MR_1/N}\sum_{n_1=0}^{m-1}\sum_{n_2=0}^{m-1-n_1}\binom{(R_1-MR_1/N)-1+n_1}{n_1}\\ &\cdot\binom{R_1-MR_1/N-1+n_2}{n_2}\binom{MR_1/N-1+(m-1-n_1-n_2)}{m-1-n_1-n_2}\\ &\cdot\left(\frac{1+\mu_{13}}{2}\right)^{n_1}\left(\frac{1+\mu_{23}}{2}\right)^{n_2}\left(\frac{1+\mu_{43}}{2}\right)^{m-1-n_1-n_2}.\end{aligned} \tag{J.43}$$

We show in Appendix K that (J.43) can be re-written as

$$\begin{aligned}P_e = &\left(\frac{1-\mu_{21}}{2}\right)^{R_1-MR_1/N}\sum_{n_2=0}^{R_1-MR_1/N-1}\binom{(R_1-MR_1/N)-1+n_2}{n_2}\left(\frac{1+\mu_{21}}{2}\right)^{n_2}\\ &\cdot\left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N}\sum_{n_3=0}^{R_1-MR_1/N-1-n_2}\binom{MR_1/N-1+n_3}{n_3}\left(\frac{1+\mu_{31}}{2}\right)^{n_3}\\ &\cdot\left(\frac{1-\mu_{41}}{2}\right)^{MR_1/N}\sum_{r=0}^{R_1-MR_1/N-1-n_2-n_3}\binom{MR_1/N-1+r}{r}\left(\frac{1+\mu_{41}}{2}\right)^{r}\\ &+\left(\frac{1-\mu_{13}}{2}\right)^{R_1-MR_1/N}\sum_{n_1=0}^{MR_1/N-1}\binom{(R_1-MR_1/N)-1+n_1}{n_1}\left(\frac{1+\mu_{13}}{2}\right)^{n_1}\\ &\cdot\left(\frac{1-\mu_{23}}{2}\right)^{R_1-MR_1/N}\sum_{n_2=0}^{MR_1/N-1-n_1}\binom{(R_1-MR_1/N)-1+n_2}{n_2}\left(\frac{1+\mu_{23}}{2}\right)^{n_2}\\ &\cdot\left(\frac{1-\mu_{43}}{2}\right)^{MR_1/N}\sum_{r=0}^{MR_1/N-1-n_1-n_2}\binom{MR_1/N-1+r}{r}\left(\frac{1+\mu_{43}}{2}\right)^{r}.\end{aligned} \tag{J.44}$$

To obtain expressions for μ_{21}, μ_{31}, μ_{41}, μ_{13}, μ_{23} and μ_{43}, we substitute (J.38)–(J.41) into (J.14):

$$\mu_{21} = 0, \tag{J.45}$$

$$\mu_{31} = \frac{(\sqrt{Q_P}\,\bar{\gamma} - 1) + \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}{(\sqrt{Q_P}\,\bar{\gamma} + 1) + \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{J.46}$$

$$\mu_{41} = \frac{(\sqrt{Q_P}\,\bar{\gamma} - 1) - \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}{(\sqrt{Q_P}\,\bar{\gamma} + 1) - \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{J.47}$$

$$\mu_{13} = \frac{(1 - \sqrt{Q_P}\,\bar{\gamma}) + \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}{(1 + \sqrt{Q_P}\,\bar{\gamma}) - \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{J.48}$$

$$\mu_{23} = \frac{(1 + \sqrt{Q_P}\,\bar{\gamma}) - \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}{(1 - \sqrt{Q_P}\,\bar{\gamma}) + \sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{J.49}$$

$$\mu_{43} = \frac{\sqrt{Q_P}\,\bar{\gamma}}{\sqrt{(1 + Q_P\bar{\gamma})(1 + \bar{\gamma})}}, \tag{J.50}$$

where $\bar{\gamma}$ is as defined in (J.36). Finally, since $\mu_{21} = 0$, (J.44) reduces to

$$\begin{aligned} P_e = {} & \left(\frac{1}{2}\right)^{R_1 - MR_1/N} \sum_{n_2=0}^{R_1 - MR_1/N - 1} \binom{(R_1 - MR_1/N) - 1 + n_2}{n_2} \left(\frac{1}{2}\right)^{n_2} \\ & \cdot \left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N} \sum_{n_3=0}^{R_1 - MR_1/N - 1 - n_2} \binom{MR_1/N - 1 + n_3}{n_3} \left(\frac{1+\mu_{31}}{2}\right)^{n_3} \\ & \cdot \left(\frac{1-\mu_{41}}{2}\right)^{MR_1/N} \sum_{r=0}^{R_1 - MR_1/N - 1 - n_2 - n_3} \binom{MR_1/N - 1 + r}{r} \left(\frac{1+\mu_{41}}{2}\right)^{r} \\ & + \left(\frac{1-\mu_{13}}{2}\right)^{R_1 - MR_1/N} \sum_{n_1=0}^{MR_1/N - 1} \binom{(R_1 - MR_1/N) - 1 + n_1}{n_1} \left(\frac{1+\mu_{13}}{2}\right)^{n_1} \\ & \cdot \left(\frac{1-\mu_{23}}{2}\right)^{R_1 - MR_1/N} \sum_{n_2=0}^{MR_1/N - 1 - n_1} \binom{(R_1 - MR_1/N) - 1 + n_2}{n_2} \left(\frac{1+\mu_{23}}{2}\right)^{n_2} \\ & \cdot \left(\frac{1-\mu_{43}}{2}\right)^{MR_1/N} \sum_{r=0}^{MR_1/N - 1 - n_1 - n_2} \binom{MR_1/N - 1 + r}{r} \left(\frac{1+\mu_{43}}{2}\right)^{r}. \end{aligned} \tag{J.51}$$

Appendix K

Re-Writing P_e for MC-CDMA

From (J.43), the probability of error for the MC-CDMA (SISO) system in Scenario #2 when $Q_P > 1$ is given by

$$\begin{aligned}
P_e = &\left(\frac{1-\mu_{21}}{2}\right)^{R_1 - MR_1/N} \left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N} \left(\frac{1-\mu_{41}}{2}\right)^{MR_1/N} \\
&\cdot \sum_{m=1}^{R_1 - MR_1/N} \sum_{n_2=0}^{m-1} \sum_{n_3=0}^{m-1-n_2} \binom{(R_1 - MR_1/N) - 1 + n_2}{n_2} \\
&\cdot \binom{MR_1/N - 1 + n_3}{n_3} \binom{MR_1/N - 1 + (m-1-n_2-n_3)}{m-1-n_2-n_3} \\
&\cdot \left(\frac{1+\mu_{21}}{2}\right)^{n_2} \left(\frac{1+\mu_{31}}{2}\right)^{n_3} \left(\frac{1+\mu_{41}}{2}\right)^{m-1-n_2-n_3} \\
+ &\left(\frac{1-\mu_{13}}{2}\right)^{R_1 - MR_1/N} \left(\frac{1-\mu_{23}}{2}\right)^{R_1 - MR_1/N} \left(\frac{1-\mu_{43}}{2}\right)^{MR_1/N} \\
&\cdot \sum_{m=1}^{MR_1/N} \sum_{n_1=0}^{m-1} \sum_{n_2=0}^{m-1-n_1} \binom{(R_1 - MR_1/N) - 1 + n_1}{n_1} \\
&\cdot \binom{R_1 - MR_1/N - 1 + n_2}{n_2} \binom{MR_1/N - 1 + (m-1-n_1-n_2)}{m-1-n_1-n_2} \\
&\cdot \left(\frac{1+\mu_{13}}{2}\right)^{n_1} \left(\frac{1+\mu_{23}}{2}\right)^{n_2} \left(\frac{1+\mu_{43}}{2}\right)^{m-1-n_1-n_2} .
\end{aligned} \tag{K.1}$$

For both triple summations, we re-write the outer two summations as follows:

$$\sum_{m=1}^{R_1-MR_1/N}\sum_{n_2=0}^{m-1}\sum_{n_3=0}^{m-1-n_2} \longrightarrow \sum_{n_2=0}^{R_1-MR_1/N-1}\sum_{m=n_2+1}^{R_1-MR_1/N}\sum_{n_3=0}^{m-1-n_2}, \tag{K.2}$$

$$\sum_{m=1}^{MR_1/N}\sum_{n_1=0}^{m-1}\sum_{n_2=0}^{m-1-n_1} \longrightarrow \sum_{n_1=0}^{MR_1/N-1}\sum_{m=n_1+1}^{MR_1/N}\sum_{n_2=0}^{m-1-n_1}. \tag{K.3}$$

Next, we make the substitutions $p = m-1-n_2$ and $p = m-1-n_1$ in the first and second triple summations, respectively, to get

$$\begin{aligned}
P_e = {} & \left(\frac{1-\mu_{21}}{2}\right)^{R_1-MR_1/N} \sum_{n_2=0}^{R_1-MR_1/N-1} \binom{(R_1-MR_1/N)-1+n_2}{n_2} \left(\frac{1+\mu_{21}}{2}\right)^{n_2} \\
& \cdot \left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N} \left(\frac{1-\mu_{41}}{2}\right)^{MR_1/N} \sum_{p=0}^{R_1-MR_1/N-1-n_2} \sum_{n_3=0}^{p} \\
& \cdot \binom{MR_1/N-1+n_3}{n_3} \binom{MR_1/N-1+(p-n_3)}{p-n_3} \\
& \cdot \left(\frac{1+\mu_{31}}{2}\right)^{n_3} \left(\frac{1+\mu_{41}}{2}\right)^{p-n_3} \\
& + \left(\frac{1-\mu_{13}}{2}\right)^{R_1-MR_1/N} \sum_{n_1=0}^{MR_1/N-1} \binom{(R_1-MR_1/N)-1+n_1}{n_1} \left(\frac{1+\mu_{13}}{2}\right)^{n_1} \\
& \cdot \left(\frac{1-\mu_{23}}{2}\right)^{R_1-MR_1/N} \left(\frac{1-\mu_{43}}{2}\right)^{MR_1/N} \sum_{p=0}^{MR_1/N-1-n_1} \sum_{n_2=0}^{p} \\
& \cdot \binom{R_1-MR_1/N-1+n_2}{n_2} \binom{MR_1/N-1+(p-n_2)}{p-n_2} \\
& \cdot \left(\frac{1+\mu_{23}}{2}\right)^{n_2} \left(\frac{1+\mu_{43}}{2}\right)^{p-n_2}.
\end{aligned} \tag{K.4}$$

The inner two summations in both groups of terms can be re-written as

$$\sum_{p=0}^{R_1-MR_1/N-1-n_2}\sum_{n_3=0}^{p} \longrightarrow \sum_{n_3=0}^{R_1-MR_1/N-1-n_2}\sum_{p=n_3}^{R_1-MR_1/N-1-n_2}, \tag{K.5}$$

$$\sum_{p=0}^{MR_1/N-1-n_1}\sum_{n_2=0}^{p} \longrightarrow \sum_{n_2=0}^{MR_1/N-1-n_1}\sum_{p=n_2}^{MR_1/N-1-n_1}. \tag{K.6}$$

Finally, we make the substitutions $r = p - n_3$ and $r = p - n_2$ in the first and second triple summations, respectively, to obtain

$$\begin{aligned}P_e = &\left(\frac{1-\mu_{21}}{2}\right)^{R_1-MR_1/N} \sum_{n_2=0}^{R_1-MR_1/N-1} \binom{(R_1-MR_1/N)-1+n_2}{n_2} \left(\frac{1+\mu_{21}}{2}\right)^{n_2} \\ &\cdot \left(\frac{1-\mu_{31}}{2}\right)^{MR_1/N} \sum_{n_3=0}^{R_1-MR_1/N-1-n_2} \binom{MR_1/N-1+n_3}{n_3} \left(\frac{1+\mu_{31}}{2}\right)^{n_3} \\ &\cdot \left(\frac{1-\mu_{41}}{2}\right)^{MR_1/N} \sum_{r=0}^{R_1-MR_1/N-1-n_2-n_3} \binom{MR_1/N-1+r}{r} \left(\frac{1+\mu_{41}}{2}\right)^{r} \\ &+ \left(\frac{1-\mu_{13}}{2}\right)^{R_1-MR_1/N} \sum_{n_1=0}^{MR_1/N-1} \binom{(R_1-MR_1/N)-1+n_1}{n_1} \left(\frac{1+\mu_{13}}{2}\right)^{n_1} \\ &\cdot \left(\frac{1-\mu_{23}}{2}\right)^{R_1-MR_1/N} \sum_{n_2=0}^{MR_1/N-1-n_1} \binom{(R_1-MR_1/N)-1+n_2}{n_2} \left(\frac{1+\mu_{23}}{2}\right)^{n_2} \\ &\cdot \left(\frac{1-\mu_{43}}{2}\right)^{MR_1/N} \sum_{r=0}^{MR_1/N-1-n_1-n_2} \binom{MR_1/N-1+r}{r} \left(\frac{1+\mu_{43}}{2}\right)^{r}. \end{aligned} \tag{K.7}$$

Appendix L

Probability of Error: MC-DS-CDMA (MIMO)

We use the approach given in Appendix D to derive the probability of error for the MC-DS-CDMA system in Chapter 3. From (3.16), the final test statistic is given by

$$Z_{q,m} = \mathbf{v}_{q,m}^H \mathbf{F} \mathbf{v}_{q,m}, \quad q = Q_P + 1, \ldots, Q;\ q \text{ odd}, \tag{L.1}$$

where

$$\mathbf{v}_{q,m} = \begin{bmatrix} \mathbf{w}_m \\ \mathbf{y}_{q,m} \end{bmatrix} \tag{L.2}$$

is an $8R_2 \times 1$ vector, and

$$\mathbf{F} = \frac{1}{2} \begin{bmatrix} \mathbf{0}_{4R_2} & \mathbf{I}_{4R_2} \\ \mathbf{I}_{4R_2} & \mathbf{0}_{4R_2} \end{bmatrix} \tag{L.3}$$

is an $8R_2 \times 8R_2$ matrix. The $4R_2 \times 1$ vectors $\mathbf{w}_m$ and $\mathbf{y}_{q,m}$ are defined as

$$\mathbf{w}_m = \begin{bmatrix} \tilde{\mathbf{w}}_{m,1}^T & \ldots & \tilde{\mathbf{w}}_{m,R_2}^T \end{bmatrix}^T, \tag{L.4}$$

$$\mathbf{y}_{q,m} = \begin{bmatrix} \tilde{\mathbf{y}}_{q,m,1}^T & \dots & \tilde{\mathbf{y}}_{q,m,R_2}^T \end{bmatrix}^T, \tag{L.5}$$

where

$$\tilde{\mathbf{w}}_{m,i} = \begin{bmatrix} \hat{W}_{1,1,m,i} & \hat{W}_{2,1,m,i} & \hat{W}_{1,2,m,i} & \hat{W}_{2,2,m,i} \end{bmatrix}^T, \tag{L.6}$$

$$\tilde{\mathbf{y}}_{q,m,i} = \begin{bmatrix} Y_{q,1,m,i} & Y_{q,2,m,i} & Y_{q+1,1,m,i} & Y_{q+1,2,m,i} \end{bmatrix}^T. \tag{L.7}$$

The covariance matrix of $\mathbf{v}_{q,m}$ is given by

$$\mathbf{R} = \frac{1}{2}E[\mathbf{v}_{q,m}\mathbf{v}_{q,m}^H] = \begin{bmatrix} \boldsymbol{\Sigma}_{ww} & \boldsymbol{\Sigma}_{wy} \\ \boldsymbol{\Sigma}_{wy}^H & \boldsymbol{\Sigma}_{yy} \end{bmatrix}, \tag{L.8}$$

where

$$\boldsymbol{\Sigma}_{ww} = \frac{1}{2}E[\mathbf{w}_m\mathbf{w}_m^H], \tag{L.9}$$

$$\boldsymbol{\Sigma}_{yy} = \frac{1}{2}E[\mathbf{y}_{q,m}\mathbf{y}_{q,m}^H], \tag{L.10}$$

$$\boldsymbol{\Sigma}_{wy} = \frac{1}{2}E[\mathbf{w}_m\mathbf{y}_{q,m}^H]. \tag{L.11}$$

The matrix $\mathbf{RF}$ is then equal to

$$\mathbf{RF} = \frac{1}{2}\begin{bmatrix} \boldsymbol{\Sigma}_{wy} & \boldsymbol{\Sigma}_{ww} \\ \boldsymbol{\Sigma}_{yy} & \boldsymbol{\Sigma}_{wy}^H \end{bmatrix}. \tag{L.12}$$

Thus, to obtain $\mathbf{RF}$, we need expressions for $\boldsymbol{\Sigma}_{ww}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{wy}$.

L.1 Expressions for $\boldsymbol{\Sigma}_{ww}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{wy}$

In this section, we derive expressions for $\boldsymbol{\Sigma}_{ww}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{wy}$. We assume a +1 is transmitted (i.e., $a_{p,q,m}^{(1)} = +1$), and we condition on $a_{p,q+1,m}^{(1)}$. First, from Appendix U, we have (see (U.51), (U.74), (U.84), (U.97), (U.111), (U.120))

$$\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}] = \frac{1}{2}\left[1+\frac{1}{Q_P/2}\,c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}, \tag{L.13}$$

$$\begin{aligned}\frac{1}{2}E[Y_{q,v_1,m,i_1}Y^*_{q,v_2,m,i_2}] &= \frac{1}{2}E[Y_{q+1,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}]\\ &= \frac{E_{b2}}{R_2}\left[1+c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\delta_{v_1,v_2}\delta_{i_1,i_2},\end{aligned} \tag{L.14}$$

$$\frac{1}{2}E[Y_{q,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}] = 0, \tag{L.15}$$

$$\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}Y^*_{q,v_2,m,i_2}] = \begin{cases}\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\;a^{(1)}_{p,q,m}\delta_{v_1,v_2}\delta_{i_1,i_2}, & u_1=1,\\ \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\;a^{(1)}_{p,q+1,m}\delta_{v_1,v_2}\delta_{i_1,i_2}, & u_1=2,\end{cases} \tag{L.16}$$

$$\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}Y^*_{q+1,v_2,m,i_2}] = \begin{cases}-\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\;a^{(1)}_{p,q+1,m}\delta_{v_1,v_2}\delta_{i_1,i_2}, & u_1=1,\\ \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\;a^{(1)}_{p,q,m}\delta_{v_1,v_2}\delta_{i_1,i_2}, & u_1=2,\end{cases} \tag{L.17}$$

where

$$c_2\!\left(\frac{E_{b2}}{N_0}\right) \triangleq \frac{K-1}{N}+\frac{R_2}{E_{b2}/N_0}. \tag{L.18}$$

We then use these results, along with (L.6) and (L.7), to obtain

$$\frac{1}{2}E[\tilde{\mathbf{w}}_{m,i_1}\tilde{\mathbf{w}}^H_{m,i_2}] = \frac{1}{2}\left[1+\frac{1}{Q_P/2}\,c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\delta_{i_1,i_2}\mathbf{I}_4, \tag{L.19}$$

$$\frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,i_1}\tilde{\mathbf{y}}^H_{q,m,i_2}] = \frac{E_{b2}}{R_2}\left[1+c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\delta_{i_1,i_2}\mathbf{I}_4, \tag{L.20}$$

$$\begin{aligned}\frac{1}{2}E[\tilde{\mathbf{w}}_{m,i_1}\tilde{\mathbf{y}}^H_{q,m,i_2}] &= \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\,\delta_{i_1,i_2}\begin{bmatrix}\mathbf{I}_2 & -a^{(1)}_{p,q+1,m}\mathbf{I}_2\\ a^{(1)}_{p,q+1,m}\mathbf{I}_2 & \mathbf{I}_2\end{bmatrix}\\ &= \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\,\delta_{i_1,i_2}\left[\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right)\otimes\mathbf{I}_2\right],\end{aligned} \tag{L.21}$$

where the 2×2 matrix $\mathbf{B}_2(\epsilon)$ is defined as

$$\mathbf{B}_2(\epsilon) \triangleq \begin{bmatrix}1 & -\epsilon\\ \epsilon^* & 1\end{bmatrix}, \tag{L.22}$$

and $\otimes$ denotes the Kronecker product. Next, using (L.4), (L.5), and (L.9)–(L.11), the matrices $\boldsymbol{\Sigma}_{ww}$, $\boldsymbol{\Sigma}_{yy}$, and $\boldsymbol{\Sigma}_{wy}$ are expanded as

$$\boldsymbol{\Sigma}_{ww} = \begin{bmatrix} \frac{1}{2}E[\tilde{\mathbf{w}}_{m,1}\tilde{\mathbf{w}}_{m,1}^H] & \cdots & \frac{1}{2}E[\tilde{\mathbf{w}}_{m,1}\tilde{\mathbf{w}}_{m,R_2}^H] \\ \vdots & \ddots & \vdots \\ \frac{1}{2}E[\tilde{\mathbf{w}}_{m,R_2}\tilde{\mathbf{w}}_{m,1}^H] & \cdots & \frac{1}{2}E[\tilde{\mathbf{w}}_{m,R_2}\tilde{\mathbf{w}}_{m,R_2}^H] \end{bmatrix}, \quad \text{(L.23)}$$

$$\boldsymbol{\Sigma}_{yy} = \begin{bmatrix} \frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,1}\tilde{\mathbf{y}}_{q,m,1}^H] & \cdots & \frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,1}\tilde{\mathbf{y}}_{q,m,R_2}^H] \\ \vdots & \ddots & \vdots \\ \frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,R_2}\tilde{\mathbf{y}}_{q,m,1}^H] & \cdots & \frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,R_2}\tilde{\mathbf{y}}_{q,m,R_2}^H] \end{bmatrix}, \quad \text{(L.24)}$$

$$\boldsymbol{\Sigma}_{wy} = \begin{bmatrix} \frac{1}{2}E[\tilde{\mathbf{w}}_{m,1}\tilde{\mathbf{y}}_{q,m,1}^H] & \cdots & \frac{1}{2}E[\tilde{\mathbf{w}}_{m,1}\tilde{\mathbf{y}}_{q,m,R_2}^H] \\ \vdots & \ddots & \vdots \\ \frac{1}{2}E[\tilde{\mathbf{w}}_{m,R_2}\tilde{\mathbf{y}}_{q,m,1}^H] & \cdots & \frac{1}{2}E[\tilde{\mathbf{w}}_{m,R_2}\tilde{\mathbf{y}}_{q,m,R_2}^H] \end{bmatrix}. \quad \text{(L.25)}$$

We apply the results of (L.19)–(L.21) in (L.23)–(L.25) to finally get

$$\begin{aligned} \boldsymbol{\Sigma}_{ww} &= \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\, c_2\left(\frac{E_{b2}}{N_0}\right)\right] \begin{bmatrix} \mathbf{I}_4 & & \\ & \ddots & \\ & & \mathbf{I}_4 \end{bmatrix} \\ &= \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\, c_2\left(\frac{E_{b2}}{N_0}\right)\right] \mathbf{I}_{4R_2}, \end{aligned} \quad \text{(L.26)}$$

$$\begin{aligned} \boldsymbol{\Sigma}_{yy} &= \frac{E_{b2}}{R_2}\left[1 + c_2\left(\frac{E_{b2}}{N_0}\right)\right] \begin{bmatrix} \mathbf{I}_4 & & \\ & \ddots & \\ & & \mathbf{I}_4 \end{bmatrix} \\ &= \frac{E_{b2}}{R_2}\left[1 + c_2\left(\frac{E_{b2}}{N_0}\right)\right] \mathbf{I}_{4R_2}, \end{aligned} \quad \text{(L.27)}$$

$$\boldsymbol{\Sigma}_{wy} = \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}} \begin{bmatrix} \mathbf{B}_2\left(a_{p,q+1,m}^{(1)}\right) \otimes \mathbf{I}_2 & & \\ & \ddots & \\ & & \mathbf{B}_2\left(a_{p,q+1,m}^{(1)}\right) \otimes \mathbf{I}_2 \end{bmatrix}$$

$$= \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}} \left[\mathbf{I}_{R_2} \otimes \mathbf{B}_2\left(a^{(1)}_{p,q+1,m}\right) \otimes \mathbf{I}_2\right]. \tag{L.28}$$

Note that

$$\boldsymbol{\Sigma}^H_{wy} = \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}} \left[\mathbf{I}_{R_2} \otimes \mathbf{B}^H_2\left(a^{(1)}_{p,q+1,m}\right) \otimes \mathbf{I}_2\right], \tag{L.29}$$

which follows from Corollary 5 in Appendix B.2.

L.2 Characteristic Polynomial of RF

The characteristic polynomial of **RF** is given by (see (L.12))

$$\begin{aligned} &\det(\lambda\mathbf{I}_{8R_2} - \mathbf{RF}) \\ &= \begin{vmatrix} \lambda\mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}_{wy} & -\frac{1}{2}\boldsymbol{\Sigma}_{ww} \\ -\frac{1}{2}\boldsymbol{\Sigma}_{yy} & \lambda\mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}^H_{wy} \end{vmatrix} \\ &= \det\left(\lambda\mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right) \cdot \det\left\{\left(\lambda\mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}^H_{wy}\right) - \frac{1}{4}\boldsymbol{\Sigma}_{yy}\left(\lambda\mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right)^{-1}\boldsymbol{\Sigma}_{ww}\right\}, \end{aligned} \tag{L.30}$$

where we applied the expansion formula for the determinant of a partitioned matrix (see (D.52)). We now derive expressions for the two determinants in (L.30).

L.2.1 First Determinant of (L.30)

First, we use the relation $\mathbf{I}_{mn} = \mathbf{I}_m \otimes \mathbf{I}_n$, as well as Corollary 1 from Appendix B.2, to write $\lambda\mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}$ as

$$\begin{aligned} \lambda\mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}_{wy} &= \lambda(\mathbf{I}_{R_2} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2) - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}} \left[\mathbf{I}_{R_2} \otimes \mathbf{B}_2\left(a^{(1)}_{p,q+1,m}\right) \otimes \mathbf{I}_2\right] \\ &= \mathbf{I}_{R_2} \otimes \left[\lambda\mathbf{I}_2 - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\mathbf{B}_2\left(a^{(1)}_{p,q+1,m}\right)\right] \otimes \mathbf{I}_2. \end{aligned} \tag{L.31}$$

Similarly, we also have

$$\lambda \mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}^H = \mathbf{I}_{R_2} \otimes \left[\lambda \mathbf{I}_2 - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\mathbf{B}_2^H\Big(a_{p,q+1,m}^{(1)}\Big)\right] \otimes \mathbf{I}_2. \tag{L.32}$$

By making the substitutions $c = \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}$ and $\epsilon = a_{p,q+1,m}^{(1)}$ in both (C.25) and (C.26), and by using the fact that $|a_{p,q+1,m}^{(1)}|^2 = 1$, we get

$$\begin{aligned}\det\left[\lambda \mathbf{I}_2 - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\mathbf{B}_2\Big(a_{p,q+1,m}^{(1)}\Big)\right] &= \det\left[\lambda \mathbf{I}_2 - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\mathbf{B}_2^H\Big(a_{p,q+1,m}^{(1)}\Big)\right] \\ &= \lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda + \frac{E_{b2}}{8R_2}. \end{aligned}\tag{L.33}$$

The determinant of (L.31) is then evaluated by applying Corollary 2 in Appendix B.2 and by using the result in (L.33):

$$\begin{aligned}\det\left(\lambda \mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right) &= \left\{\det\left[\lambda \mathbf{I}_2 - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\mathbf{B}_2\Big(a_{p,q+1,m}^{(1)}\Big)\right]\right\}^{2R_2}. \\ &= \left(\lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda + \frac{E_{b2}}{8R_2}\right)^{2R_2}. \end{aligned}\tag{L.34}$$

Thus, we have the first determinant in (L.30).

L.2.2 Second Determinant of (L.30)

First, we apply Corollary 3 from Appendix B.2 to write the inverse of (L.31) as

$$\left(\lambda \mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right)^{-1} = \mathbf{I}_{R_2} \otimes \left[\lambda \mathbf{I}_2 - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\;\mathbf{B}_2\Big(a_{p,q+1,m}^{(1)}\Big)\right]^{-1} \otimes \mathbf{I}_2. \tag{L.35}$$

By making the substitutions $c = \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}$ and $\epsilon = a_{p,q+1,m}^{(1)}$ in (C.30), we get

$$\left[\lambda \mathbf{I}_2 - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\mathbf{B}_2\Big(a_{p,q+1,m}^{(1)}\Big)\right]^{-1} = \frac{1}{\lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda + \frac{E_{b2}}{8R_2}}\left[\lambda \mathbf{I}_2 - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\;\mathbf{B}_2^H\Big(a_{p,q+1,m}^{(1)}\Big)\right], \tag{L.36}$$

which, when substituted into (L.35), yields

$$\left(\lambda \mathbf{I}_{4R_2} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right)^{-1} = \frac{1}{\lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda + \frac{E_{b2}}{8R_2}}\left\{\mathbf{I}_{R_2} \otimes \left[\lambda \mathbf{I}_2 - \frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\;\mathbf{B}_2^H\Big(a_{p,q+1,m}^{(1)}\Big)\right] \otimes \mathbf{I}_2\right\}. \tag{L.37}$$

Next, using (L.37), along with the expressions for $\mathbf{\Sigma}_{ww}$ and $\mathbf{\Sigma}_{yy}$ in (L.26) and (L.27), we have

$$\begin{aligned}
&\frac{1}{4}\mathbf{\Sigma}_{yy}\left(\lambda\mathbf{I}_{4R_2}-\frac{1}{2}\mathbf{\Sigma}_{wy}\right)^{-1}\mathbf{\Sigma}_{ww} \\
&= \frac{E_{b2}}{8R_2}\left[1+c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\left[1+\frac{1}{Q_P/2}\,c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\frac{1}{\lambda^2-\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda+\frac{E_{b2}}{8R_2}} \\
&\quad\cdot\left\{\mathbf{I}_{R_2}\otimes\left[\lambda\mathbf{I}_2-\frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\,\mathbf{B}_2^H\!\left(a^{(1)}_{p,q+1,m}\right)\right]\otimes\mathbf{I}_2\right\}.
\end{aligned} \tag{L.38}$$

We combine the results of (L.32) and (L.38) and apply Corollary 1 from Appendix B.2 to get

$$\begin{aligned}
&\left(\lambda\mathbf{I}_{4R_2}-\frac{1}{2}\mathbf{\Sigma}_{wy}^H\right)-\frac{1}{4}\mathbf{\Sigma}_{yy}\left(\lambda\mathbf{I}_{4R_2}-\frac{1}{2}\mathbf{\Sigma}_{wy}\right)^{-1}\mathbf{\Sigma}_{ww} \\
&= \left(1-\frac{E_{b2}}{8R_2}\left[1+c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\left[1+\frac{1}{Q_P/2}\,c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\frac{1}{\lambda^2-\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda+\frac{E_{b2}}{8R_2}}\right) \\
&\quad\cdot\left\{\mathbf{I}_{R_2}\otimes\left[\lambda\mathbf{I}_2-\frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\,\mathbf{B}_2^H\!\left(a^{(1)}_{p,q+1,m}\right)\right]\otimes\mathbf{I}_2\right\} \\
&= \frac{\lambda^2-b\lambda-c}{\lambda^2-\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda+\frac{E_{b2}}{8R_2}}\left\{\mathbf{I}_{R_2}\otimes\left[\lambda\mathbf{I}_2-\frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\,\mathbf{B}_2^H\!\left(a^{(1)}_{p,q+1,m}\right)\right]\otimes\mathbf{I}_2\right\},
\end{aligned} \tag{L.39}$$

where

$$b = \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}, \tag{L.40}$$

$$c = \frac{E_{b2}}{8R_2}\left\{\left(1+\frac{1}{Q_P/2}\right)c_2\!\left(\frac{E_{b2}}{N_0}\right)+\frac{1}{Q_P/2}\left[c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]^2\right\}. \tag{L.41}$$

The determinant of (L.39) is then obtained by applying Corollary 2 from Appendix B.2 and using (L.33):

$$\begin{aligned}
&\det\left\{\left(\lambda\mathbf{I}_{4R_2}-\frac{1}{2}\mathbf{\Sigma}_{wy}^H\right)-\frac{1}{4}\mathbf{\Sigma}_{yy}\left(\lambda\mathbf{I}_{4R_2}-\frac{1}{2}\mathbf{\Sigma}_{wy}\right)^{-1}\mathbf{\Sigma}_{ww}\right\} \\
&= \left(\frac{\lambda^2-b\lambda-c}{\lambda^2-\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda+\frac{E_{b2}}{8R_2}}\right)^{4R_2}\det\left\{\mathbf{I}_{R_2}\otimes\left[\lambda\mathbf{I}_2-\frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\,\mathbf{B}_2^H\!\left(a^{(1)}_{p,q+1,m}\right)\right]\otimes\mathbf{I}_2\right\} \\
&= \left(\frac{\lambda^2-b\lambda-c}{\lambda^2-\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda+\frac{E_{b2}}{8R_2}}\right)^{4R_2}\left\{\det\left[\lambda\mathbf{I}_2-\frac{1}{4}\sqrt{\frac{E_{b2}}{R_2}}\,\mathbf{B}_2^H\!\left(a^{(1)}_{p,q+1,m}\right)\right]\right\}^{2R_2} \\
&= \left(\frac{\lambda^2-b\lambda-c}{\lambda^2-\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda+\frac{E_{b2}}{8R_2}}\right)^{4R_2}\left(\lambda^2-\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda+\frac{E_{b2}}{8R_2}\right)^{2R_2}
\end{aligned}$$

$$= \frac{\left(\lambda^2 - b\lambda - c\right)^{4R_2}}{\left(\lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\lambda + \frac{E_{b2}}{8R_2}\right)^{2R_2}}. \tag{L.42}$$

L.2.3 Combining the Results

We substitute the results of (L.34) and (L.42) into (L.30) to show that the characteristic polynomial of $\mathbf{RF}$ reduces to

$$\det(\lambda \mathbf{I}_{8R_2} - \mathbf{RF}) = \left(\lambda^2 - b\lambda - c\right)^{4R_2}. \tag{L.43}$$

L.3 Eigenvalues of RF

Using the quadratic formula to solve for the roots of (L.43), we see that $\mathbf{RF}$ has $\Lambda = 2$ distinct eigenvalues, both of multiplicity $4R_2$:

$$\lambda_1 = \frac{b - \sqrt{b^2 + 4c}}{2}, \quad m_1 = 4R_2, \tag{L.44}$$

$$\lambda_2 = \frac{b + \sqrt{b^2 + 4c}}{2}, \quad m_2 = 4R_2. \tag{L.45}$$

Since $c_2\left(\frac{E_{b2}}{N_0}\right) > 0$ (see (L.18)), we have $b^2 + 4c > b^2$, such that $\lambda_1 < 0$ and $\lambda_2 > 0$. Therefore, λ_1 is the only negative eigenvalue.

L.4 Probability of Error

We make the substitutions $\Lambda = 2$ and $m_1 = m_2 = 4R_2$ in (D.8), and we use the fact that λ_1 is the only negative eigenvalue to obtain the following expression for the probability of error:

$$P_e = \sum_{k=1}^{4R_2} c_{1,k}. \tag{L.46}$$

We also make the same substitutions in (D.5) to show that $c_{1,k}$ reduces to

$$c_{1,k} = \left(\frac{1 - \mu_{21}}{2}\right)^{4R_2} \binom{4R_2 - 1 + (k-1)}{k-1} \left(\frac{1 + \mu_{21}}{2}\right)^{k-1}, \tag{L.47}$$

where

$$\mu_{21} = \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}. \tag{L.48}$$

Thus,

$$\begin{aligned} P_e &= \left(\frac{1-\mu_{21}}{2}\right)^{4R_2} \sum_{k=1}^{4R_2} \binom{4R_2 - 1 + (k-1)}{k-1} \left(\frac{1+\mu_{21}}{2}\right)^{k-1} \\ &= \left(\frac{1-\mu_{21}}{2}\right)^{4R_2} \sum_{i=0}^{4R_2-1} \binom{4R_2 - 1 + i}{i} \left(\frac{1+\mu_{21}}{2}\right)^{i}. \end{aligned} \tag{L.49}$$

Finally, we substitute the expressions for λ_1 and λ_2 into (L.48) to get

$$\mu_{21} = \frac{b}{\sqrt{b^2 + 4c}} = \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}}, \tag{L.50}$$

where

$$\bar{\gamma} = \frac{b^2}{4c} = \left\{ 2\left(1 + \frac{1}{Q_P/2}\right) c_2\!\left(\frac{E_{b2}}{N_0}\right) + 2\frac{1}{Q_P/2}\left[c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]^2 \right\}^{-1}. \tag{L.51}$$

Appendix M

Probability of Error: MC-CDMA (MIMO)

We derive the probability of error for the MC-CDMA system in Chapter 3 using the approach outlined in Appendix D. As in the derivation of the probability of error for the MC-DS-CDMA system (see Appendix L), the final test statistic is given by

$$Z_{q,m} = \mathbf{v}_{q,m}^H \mathbf{F} \mathbf{v}_{q,m}, \quad q = Q_P + 1, \ldots, Q;\ q \text{ odd}, \tag{M.1}$$

where

$$\mathbf{v}_{q,m} = \begin{bmatrix} \mathbf{w}_m \\ \mathbf{y}_{q,m} \end{bmatrix} \tag{M.2}$$

is an $8R_1 \times 1$ vector, and

$$\mathbf{F} = \frac{1}{2} \begin{bmatrix} \mathbf{0}_{4R_1} & \mathbf{I}_{4R_1} \\ \mathbf{I}_{4R_1} & \mathbf{0}_{4R_1} \end{bmatrix} \tag{M.3}$$

is an $8R_1 \times 8R_1$ matrix. We define the $4R_1 \times 1$ vectors $\mathbf{w}_m$ and $\mathbf{y}_{q,m}$ as

$$\mathbf{w}_m = \begin{bmatrix} \tilde{\mathbf{w}}_{m,1}^T & \cdots & \tilde{\mathbf{w}}_{m,R_1}^T \end{bmatrix}^T, \tag{M.4}$$

$$\mathbf{y}_{q,m} = \begin{bmatrix} \tilde{\mathbf{y}}_{q,m,1}^T & \cdots & \tilde{\mathbf{y}}_{q,m,R_1}^T \end{bmatrix}^T, \tag{M.5}$$

where

$$\tilde{\mathbf{w}}_{m,i} = \begin{bmatrix} \hat{W}_{1,1,m,i} & \hat{W}_{2,1,m,i} & \hat{W}_{1,2,m,i} & \hat{W}_{2,2,m,i} \end{bmatrix}^T, \tag{M.6}$$

$$\tilde{\mathbf{y}}_{q,m,i} = \begin{bmatrix} Y_{q,1,m,i} & Y_{q,2,m,i} & Y_{q+1,1,m,i} & Y_{q+1,2,m,i} \end{bmatrix}^T. \tag{M.7}$$

The covariance matrix of $\mathbf{v}_{q,m}$ is given by

$$\mathbf{R} = \frac{1}{2} E[\mathbf{v}_{q,m} \mathbf{v}_{q,m}^H] = \begin{bmatrix} \mathbf{\Sigma}_{ww} & \mathbf{\Sigma}_{wy} \\ \mathbf{\Sigma}_{wy}^H & \mathbf{\Sigma}_{yy} \end{bmatrix}, \tag{M.8}$$

where

$$\mathbf{\Sigma}_{ww} = \frac{1}{2} E[\mathbf{w}_m \mathbf{w}_m^H], \tag{M.9}$$

$$\mathbf{\Sigma}_{yy} = \frac{1}{2} E[\mathbf{y}_{q,m} \mathbf{y}_{q,m}^H], \tag{M.10}$$

$$\mathbf{\Sigma}_{wy} = \frac{1}{2} E[\mathbf{w}_m \mathbf{y}_{q,m}^H]. \tag{M.11}$$

Thus, we have

$$\mathbf{RF} = \frac{1}{2} \begin{bmatrix} \mathbf{\Sigma}_{wy} & \mathbf{\Sigma}_{ww} \\ \mathbf{\Sigma}_{yy} & \mathbf{\Sigma}_{wy}^H \end{bmatrix}. \tag{M.12}$$

M.1 Expressions for $\mathbf{\Sigma}_{ww}$, $\mathbf{\Sigma}_{yy}$, and $\mathbf{\Sigma}_{wy}$

Again, we assume $a_{p,q,m}^{(1)} = +1$, and we condition on $a_{p,q+1,m}^{(1)}$. First, when the indices i_1 and i_2 are associated with the *same* block, we show in Appendix W that (see (W.49), (W.72), (W.82), (W.95), (W.110), (W.111), (W.121), and (W.122))

$$\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}] = \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right]\delta_{v_1,v_2}\delta_{u_1,u_2}, \tag{M.13}$$

$$\begin{aligned}\frac{1}{2}E[Y_{q,v_1,m,i_1}Y^*_{q,v_2,m,i_2}] &= \frac{1}{2}E[Y_{q+1,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}] \\ &= \frac{E_{b1}}{R_1}\left[1 + c_1\!\left(\frac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right]\delta_{v_1,v_2},\end{aligned} \tag{M.14}$$

$$\frac{1}{2}E[Y_{q,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}] = 0, \tag{M.15}$$

$$\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}Y^*_{q,v_2,m,i_2}] = \begin{cases}\frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\,a^{(1)}_{p,q,m}\delta_{v_1,v_2}, & u_1 = 1,\\ \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\,a^{(1)}_{p,q+1,m}\delta_{v_1,v_2}, & u_1 = 2,\end{cases} \tag{M.16}$$

$$\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}Y^*_{q+1,v_2,m,i_2}] = \begin{cases}-\frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\,a^{(1)}_{p,q+1,m}\delta_{v_1,v_2}, & u_1 = 1,\\ \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\,a^{(1)}_{p,q,m}\delta_{v_1,v_2}, & u_1 = 2,\end{cases} \tag{M.17}$$

where

$$c_1\!\left(\frac{E_{b1}}{N_0}\right) \triangleq (K-1) + \frac{R_1}{E_{b1}/N_0}. \tag{M.18}$$

We use these results, along with (M.6) and (M.7), to get

$$\frac{1}{2}E[\tilde{\mathbf{w}}_{m,i_1}\tilde{\mathbf{w}}^H_{m,i_2}] = \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right]\mathbf{I}_4, \tag{M.19}$$

$$\frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,i_1}\tilde{\mathbf{y}}^H_{q,m,i_2}] = \frac{E_{b1}}{R_1}\left[1 + c_1\!\left(\frac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right]\mathbf{I}_4, \tag{M.20}$$

$$\frac{1}{2}E[\tilde{\mathbf{w}}_{m,i_1}\tilde{\mathbf{y}}^H_{q,m,i_2}] = \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right)\otimes\mathbf{I}_2\right], \tag{M.21}$$

where

$$\mathbf{B}_2(\epsilon) = \begin{bmatrix}1 & -\epsilon\\ \epsilon^* & 1\end{bmatrix}. \tag{M.22}$$

We also prove in Appendix W that when i_1 and i_2 are associated with *different* blocks, we have (see (W.49), (W.72), (W.82), (W.95), (W.110), (W.111), (W.121), and (W.122))

$$\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}] = 0, \tag{M.23}$$

$$\frac{1}{2}E[Y_{q,v_1,m,i_1}Y^*_{q,v_2,m,i_2}] = 0, \tag{M.24}$$

$$\frac{1}{2}E[Y_{q+1,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}] = 0, \tag{M.25}$$

$$\frac{1}{2}E[Y_{q,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}] = 0, \tag{M.26}$$

$$\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}Y^*_{q,v_2,m,i_2}] = 0, \tag{M.27}$$

$$\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}Y^*_{q+1,v_2,m,i_2}] = 0, \tag{M.28}$$

from which we obtain

$$\frac{1}{2}E[\tilde{\mathbf{w}}_{m,i_1}\tilde{\mathbf{w}}^H_{m,i_2}] = \mathbf{0}_4, \tag{M.29}$$

$$\frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,i_1}\tilde{\mathbf{y}}^H_{q,m,i_2}] = \mathbf{0}_4, \tag{M.30}$$

$$\frac{1}{2}E[\tilde{\mathbf{w}}_{m,i_1}\tilde{\mathbf{y}}^H_{q,m,i_2}] = \mathbf{0}_4. \tag{M.31}$$

Next, using (M.4), (M.5), and (M.9)–(M.11), the matrices $\mathbf{\Sigma}_{ww}$, $\mathbf{\Sigma}_{yy}$, and $\mathbf{\Sigma}_{wy}$ are given by

$$\mathbf{\Sigma}_{ww} = \begin{bmatrix} \frac{1}{2}E[\tilde{\mathbf{w}}_{m,1}\tilde{\mathbf{w}}^H_{m,1}] & \cdots & \frac{1}{2}E[\tilde{\mathbf{w}}_{m,1}\tilde{\mathbf{w}}^H_{m,R_1}] \\ \vdots & \ddots & \vdots \\ \frac{1}{2}E[\tilde{\mathbf{w}}_{m,R_1}\tilde{\mathbf{w}}^H_{m,1}] & \cdots & \frac{1}{2}E[\tilde{\mathbf{w}}_{m,R_1}\tilde{\mathbf{w}}^H_{m,R_1}] \end{bmatrix}, \tag{M.32}$$

$$\mathbf{\Sigma}_{yy} = \begin{bmatrix} \frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,1}\tilde{\mathbf{y}}^H_{q,m,1}] & \cdots & \frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,1}\tilde{\mathbf{y}}^H_{q,m,R_1}] \\ \vdots & \ddots & \vdots \\ \frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,R_1}\tilde{\mathbf{y}}^H_{q,m,1}] & \cdots & \frac{1}{2}E[\tilde{\mathbf{y}}_{q,m,R_1}\tilde{\mathbf{y}}^H_{q,m,R_1}] \end{bmatrix}, \tag{M.33}$$

$$\mathbf{\Sigma}_{wy} = \begin{bmatrix} \frac{1}{2}E[\tilde{\mathbf{w}}_{m,1}\tilde{\mathbf{y}}^H_{q,m,1}] & \cdots & \frac{1}{2}E[\tilde{\mathbf{w}}_{m,1}\tilde{\mathbf{y}}^H_{q,m,R_1}] \\ \vdots & \ddots & \vdots \\ \frac{1}{2}E[\tilde{\mathbf{w}}_{m,R_1}\tilde{\mathbf{y}}^H_{q,m,1}] & \cdots & \frac{1}{2}E[\tilde{\mathbf{w}}_{m,R_1}\tilde{\mathbf{y}}^H_{q,m,R_1}] \end{bmatrix}. \tag{M.34}$$

We use (M.19) and (M.29) to show that

$$\boldsymbol{\Sigma}_{ww} = \begin{bmatrix} \mathbf{B}_{ww} & & \\ & \ddots & \\ & & \mathbf{B}_{ww} \end{bmatrix} = \mathbf{I}_{MR_1/N} \otimes \mathbf{B}_{ww}, \tag{M.35}$$

where the $4R_b \times 4R_b$ matrix $\mathbf{B}_{ww}$ is given by

$$\begin{aligned} \mathbf{B}_{ww} &= \begin{bmatrix} \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\, c_1\!\left(\frac{E_{b1}}{N_0}\right)\delta_{1,1}\right]\mathbf{I}_4 & \cdots & \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\, c_1\!\left(\frac{E_{b1}}{N_0}\right)\delta_{1,R_b}\right]\mathbf{I}_4 \\ \vdots & \ddots & \vdots \\ \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\, c_1\!\left(\frac{E_{b1}}{N_0}\right)\delta_{R_b,1}\right]\mathbf{I}_4 & \cdots & \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\, c_1\!\left(\frac{E_{b1}}{N_0}\right)\delta_{R_b,R_b}\right]\mathbf{I}_4 \end{bmatrix} \\ &= \left(\begin{bmatrix} \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\frac{1}{Q_P/2}\, c_1\!\left(\frac{E_{b1}}{N_0}\right) & & \\ & \ddots & \\ & & \frac{1}{2}\frac{1}{Q_P/2}\, c_1\!\left(\frac{E_{b1}}{N_0}\right) \end{bmatrix} \right) \otimes \mathbf{I}_4 \\ &= \frac{1}{2}\left[\mathbf{A}_{R_b} + \frac{1}{Q_P/2}\, c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right] \otimes \mathbf{I}_4, \end{aligned} \tag{M.36}$$

with $\mathbf{A}_{R_b}$ defined as the $R_b \times R_b$ all-ones matrix. Thus, we have

$$\boldsymbol{\Sigma}_{ww} = \mathbf{I}_{MR_1/N} \otimes \frac{1}{2}\left[\mathbf{A}_{R_b} + \frac{1}{Q_P/2}\, c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right] \otimes \mathbf{I}_4. \tag{M.37}$$

Similarly, by substituting (M.20) and (M.30) into (M.33), and (M.21) and (M.31) into (M.34), we can show that

$$\boldsymbol{\Sigma}_{yy} = \mathbf{I}_{MR_1/N} \otimes \frac{E_{b1}}{R_1}\left[\mathbf{A}_{R_b} + c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right] \otimes \mathbf{I}_4, \tag{M.38}$$

$$\boldsymbol{\Sigma}_{wy} = \mathbf{I}_{MR_1/N} \otimes \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right) \otimes \mathbf{I}_2\right]. \tag{M.39}$$

Note that

$$\boldsymbol{\Sigma}_{wy}^H = \mathbf{I}_{MR_1/N} \otimes \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2^H\!\left(a^{(1)}_{p,q+1,m}\right) \otimes \mathbf{I}_2\right], \tag{M.40}$$

which follows from Corollary 5 of Appendix B.2.

M.2 Characteristic Polynomial of RF

By applying (D.52), we can write the characteristic polynomial of **RF** as (see (M.12))

$$\begin{aligned}
&\det(\lambda \mathbf{I}_{8R_1} - \mathbf{RF}) \\
&\quad = \begin{vmatrix} \lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy} & -\frac{1}{2}\boldsymbol{\Sigma}_{ww} \\ -\frac{1}{2}\boldsymbol{\Sigma}_{yy} & \lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}^H \end{vmatrix} \\
&\quad = \det\left(\lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right) \cdot \det\left\{\left(\lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}^H\right) - \frac{1}{4}\boldsymbol{\Sigma}_{yy}\left(\lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right)^{-1}\boldsymbol{\Sigma}_{ww}\right\}.
\end{aligned} \tag{M.41}$$

We now derive expressions for the two determinants in (M.41).

M.2.1 First Determinant of (M.41)

First, we use the relation $\mathbf{I}_{mn} = \mathbf{I}_m \otimes \mathbf{I}_n$ and Corollary 1 from Appendix B.2 to write

$$\begin{aligned}
&\lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy} \\
&\quad = \lambda(\mathbf{I}_{MR_1/N} \otimes \mathbf{I}_{2R_b} \otimes \mathbf{I}_2) - \frac{1}{2}\left\{\mathbf{I}_{MR_1/N} \otimes \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2\left(a_{p,q+1,m}^{(1)}\right) \otimes \mathbf{I}_2\right]\right\} \\
&\quad = \mathbf{I}_{MR_1/N} \otimes \left\{\lambda \mathbf{I}_{2R_b} - \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2\left(a_{p,q+1,m}^{(1)}\right)\right]\right\} \otimes \mathbf{I}_2.
\end{aligned} \tag{M.42}$$

Similarly, we also have

$$\lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}^H = \mathbf{I}_{MR_1/N} \otimes \left\{\lambda \mathbf{I}_{2R_b} - \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2^H\left(a_{p,q+1,m}^{(1)}\right)\right]\right\} \otimes \mathbf{I}_2. \tag{M.43}$$

We make the substitutions $n = R_b$, $c = \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}$, and $\epsilon = a_{p,q+1,m}^{(1)}$ in both (C.43) and (C.44), and we note that $|a_{p,q+1,m}^{(1)}|^2 = 1$ to obtain

$$\begin{aligned}
&\det\left\{\lambda \mathbf{I}_{2R_b} - \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2\left(a_{p,q+1,m}^{(1)}\right)\right]\right\} \\
&\quad = \det\left\{\lambda \mathbf{I}_{2R_b} - \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2^H\left(a_{p,q+1,m}^{(1)}\right)\right]\right\} \\
&\quad = \lambda^{2R_b-2}X(\lambda),
\end{aligned} \tag{M.44}$$

where

$$X(\lambda) \triangleq \lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \lambda + \frac{E_{b1}}{8R_1} R_b^2. \tag{M.45}$$

The determinant of (M.42) is then evaluated by applying Corollary 2 in Appendix B.2 and (M.44):

$$\begin{aligned} \det\left(\lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right) &= \left(\det\left\{\lambda \mathbf{I}_{2R_b} - \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2\left(a^{(1)}_{p,q+1,m}\right)\right]\right\}\right)^{2MR_1/N} \\ &= \left[\lambda^{2R_b-2} X(\lambda)\right]^{2MR_1/N}. \end{aligned} \tag{M.46}$$

M.2.2 Second Determinant of (M.41)

First, we obtain the inverse of (M.42) by applying Corollary 3 from Appendix B.2:

$$\left(\lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right)^{-1} = \mathbf{I}_{MR_1/N} \otimes \left\{\lambda \mathbf{I}_{2R_b} - \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2\left(a^{(1)}_{p,q+1,m}\right)\right]\right\}^{-1} \otimes \mathbf{I}_2. \tag{M.47}$$

We make the substitutions $n = R_b$, $c = \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}$, and $\epsilon = a^{(1)}_{p,q+1,m}$ in (C.51) to get

$$\begin{aligned} &\left\{\lambda \mathbf{I}_{2R_b} - \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \mathbf{B}_2\left(a^{(1)}_{p,q+1,m}\right)\right]\right\}^{-1} \\ &\quad = \frac{1}{\lambda}\left\{\mathbf{I}_{2R_b} + \frac{1}{X(\lambda)}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \left(\lambda \mathbf{B}_2\left(a^{(1)}_{p,q+1,m}\right) - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \mathbf{I}_2\right)\right]\right\}, \end{aligned} \tag{M.48}$$

which implies

$$\begin{aligned} &\left(\lambda \mathbf{I}_{4R_1} - \frac{1}{2}\boldsymbol{\Sigma}_{wy}\right)^{-1} \\ &\quad = \mathbf{I}_{MR_1/N} \otimes \frac{1}{\lambda}\left\{\mathbf{I}_{2R_b} + \frac{1}{X(\lambda)}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \left(\lambda \mathbf{B}_2\left(a^{(1)}_{p,q+1,m}\right) - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \mathbf{I}_2\right)\right]\right\} \otimes \mathbf{I}_2. \end{aligned} \tag{M.49}$$

Next, using the relation $\mathbf{I}_{mn} = \mathbf{I}_m \otimes \mathbf{I}_n$, we re-write $\boldsymbol{\Sigma}_{ww}$ and $\boldsymbol{\Sigma}_{yy}$ in (M.37) and (M.38) as

$$\boldsymbol{\Sigma}_{ww} = \mathbf{I}_{MR_1/N} \otimes \left\{\frac{1}{2}\left[\mathbf{A}_{R_b} + \frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right] \otimes \mathbf{I}_2\right\} \otimes \mathbf{I}_2, \tag{M.50}$$

$$\boldsymbol{\Sigma}_{yy} = \mathbf{I}_{MR_1/N} \otimes \left\{\frac{E_{b1}}{R_1}\left[\mathbf{A}_{R_b} + c_1\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right] \otimes \mathbf{I}_2\right\} \otimes \mathbf{I}_2. \tag{M.51}$$

We apply (M.49)–(M.51), as well as Corollary 4 from Appendix B.2, to obtain

$$\begin{aligned}
&\frac{1}{4}\boldsymbol{\Sigma}_{yy}\left(\lambda\mathbf{I}_{4R_1}-\frac{1}{2}\boldsymbol{\Sigma}_{wy}\right)^{-1}\boldsymbol{\Sigma}_{ww}\\
&=\frac{1}{4}\left\{\mathbf{I}_{MR_1/N}\otimes\left(\frac{E_{b1}}{R_1}\left[\mathbf{A}_{R_b}+c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\otimes\mathbf{I}_2\right)\otimes\mathbf{I}_2\right\}\\
&\quad\cdot\left\{\mathbf{I}_{MR_1/N}\otimes\frac{1}{\lambda}\left\{\mathbf{I}_{2R_b}+\frac{1}{X(\lambda)}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b}\otimes\left(\lambda\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right)-\frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\mathbf{I}_2\right)\right]\right\}\otimes\mathbf{I}_2\right\}\\
&\quad\cdot\left\{\mathbf{I}_{MR_1/N}\otimes\left(\frac{1}{2}\left[\mathbf{A}_{R_b}+\frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\otimes\mathbf{I}_2\right)\otimes\mathbf{I}_2\right\}\\
&=\mathbf{I}_{MR_1/N}\otimes\mathbf{C}(\lambda)\otimes\mathbf{I}_2, \qquad\text{(M.52)}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{C}(\lambda)=\frac{1}{\lambda}\frac{E_{b1}}{8R_1}&\left(\left[\mathbf{A}_{R_b}+c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\otimes\mathbf{I}_2\right)\\
&\cdot\left\{\mathbf{I}_{2R_b}+\frac{1}{X(\lambda)}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b}\otimes\left(\lambda\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right)-\frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\mathbf{I}_2\right)\right]\right\} \qquad\text{(M.53)}\\
&\cdot\left(\left[\mathbf{A}_{R_b}+\frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\otimes\mathbf{I}_2\right)
\end{aligned}$$

is a $2R_b\times 2R_b$ matrix. Finally, using (M.43) and (M.52), we can write

$$\left(\lambda\mathbf{I}_{4R_1}-\frac{1}{2}\boldsymbol{\Sigma}^H_{wy}\right)-\frac{1}{4}\boldsymbol{\Sigma}_{yy}\left(\lambda\mathbf{I}_{4R_1}-\frac{1}{2}\boldsymbol{\Sigma}_{wy}\right)^{-1}\boldsymbol{\Sigma}_{ww}=\mathbf{I}_{MR_1/N}\otimes\mathbf{D}(\lambda)\otimes\mathbf{I}_2, \qquad\text{(M.54)}$$

where

$$\mathbf{D}(\lambda)=\lambda\mathbf{I}_{2R_b}-\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b}\otimes\mathbf{B}^H_2\!\left(a^{(1)}_{p,q+1,m}\right)\right]-\mathbf{C}(\lambda). \qquad\text{(M.55)}$$

The determinant of (M.54) then follows by applying Corollary 2 in Appendix B.2:

$$\det\left[\left(\lambda\mathbf{I}_{4R_1}-\frac{1}{2}\boldsymbol{\Sigma}^H_{wy}\right)-\frac{1}{4}\boldsymbol{\Sigma}_{yy}\left(\lambda\mathbf{I}_{4R_1}-\frac{1}{2}\boldsymbol{\Sigma}_{wy}\right)^{-1}\boldsymbol{\Sigma}_{ww}\right]=\left[\det\mathbf{D}(\lambda)\right]^{2MR_1/N}. \qquad\text{(M.56)}$$

M.2.3 Combining the Results

We substitute (M.46) and (M.56) into (M.41) to get

$$\det(\lambda\mathbf{I}_{8R_1}-\mathbf{RF})=\left[\lambda^{2R_b-2}X(\lambda)\det\mathbf{D}(\lambda)\right]^{2MR_1/N}. \qquad\text{(M.57)}$$

As shown in Appendix N.1, the matrix $\mathbf{D}(\lambda)$ in (M.55) can be re-written as

$$\mathbf{D}(\lambda) = \frac{1}{\lambda}\left(W(\lambda)\mathbf{I}_{2R_b} - \left[\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\frac{Y(\lambda)}{X(\lambda)}\mathbf{A}_{R_b} \otimes \mathbf{B}_2^H\left(\frac{Z(\lambda)}{Y(\lambda)}a_{p,q+1,m}^{(1)}\right)\right]\right), \tag{M.58}$$

where, from (N.12), (N.21), and (N.22), we have

$$W(\lambda) \triangleq \lambda^2 - \frac{E_{b1}}{8R_1}\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2, \tag{M.59}$$

$$\begin{aligned} Y(\lambda) \triangleq\ & \lambda^3 + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left(1 + \frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\lambda^2 \\ & - \frac{E_{b1}}{8R_1}c_1\left(\frac{E_{b1}}{N_0}\right)\left[R_b\left(1+\frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]\lambda \\ & - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}R_b\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2, \end{aligned} \tag{M.60}$$

$$Z(\lambda) \triangleq \lambda\left\{\lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\lambda - \frac{E_{b1}}{8R_1}c_1\left(\frac{E_{b1}}{N_0}\right)\left[R_b\left(1+\frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]\right\}. \tag{M.61}$$

Also, $X(\lambda)$ is given by (M.45). To derive an expression for $\det \mathbf{D}(\lambda)$, we first replace λ with $W(\lambda)$, n with R_b, c with $\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\frac{Y(\lambda)}{X(\lambda)}$, and ϵ with $\frac{Z(\lambda)}{Y(\lambda)}a_{p,q+1,m}^{(1)}$ in (C.44) to obtain

$$\begin{aligned} & \det\left(W(\lambda)\mathbf{I}_{2R_b} - \left[\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\frac{Y(\lambda)}{X(\lambda)}\mathbf{A}_{R_b} \otimes \mathbf{B}_2^H\left(\frac{Z(\lambda)}{Y(\lambda)}a_{p,q+1,m}^{(1)}\right)\right]\right) \\ & = [W(\lambda)]^{2R_b-2}\left\{[W(\lambda)]^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\frac{Y(\lambda)}{X(\lambda)}W(\lambda)\right. \\ & \quad \left. + \frac{E_{b1}}{16R_1}R_b^2\left(\frac{Y(\lambda)}{X(\lambda)}\right)^2\left[1 + \left(\frac{Z(\lambda)}{Y(\lambda)}\right)^2\right]\right\}. \end{aligned} \tag{M.62}$$

The determinant of $\mathbf{D}(\lambda)$ then follows from (M.58) and (M.62) as

$$\begin{aligned} \det \mathbf{D}(\lambda) &= \left(\frac{1}{\lambda}\right)^{2R_b}\det\left(W(\lambda)\mathbf{I}_{2R_b} - \left[\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\frac{Y(\lambda)}{X(\lambda)}\mathbf{A}_{R_b} \otimes \mathbf{B}_2^H\left(\frac{Z(\lambda)}{Y(\lambda)}a_{p,q+1,m}^{(1)}\right)\right]\right) \\ &= \left[\frac{W(\lambda)}{\lambda}\right]^{2R_b-2}\frac{1}{\lambda^2}\left\{[W(\lambda)]^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\frac{Y(\lambda)}{X(\lambda)}W(\lambda)\right. \\ & \quad \left. + \frac{E_{b1}}{16R_1}R_b^2\left[\left(\frac{Y(\lambda)}{X(\lambda)}\right)^2 + \left(\frac{Z(\lambda)}{X(\lambda)}\right)^2\right]\right\}. \end{aligned} \tag{M.63}$$

Using this result in (M.57), and noting that $R_b = \frac{N}{M}$ (see (3.25)), we get

$$\det(\lambda\mathbf{I}_{8R_1} - \mathbf{RF}) = [W(\lambda)]^{4(R_1 - MR_1/N)} \cdot [T(\lambda)]^{2MR_1/N}, \tag{M.64}$$

where

$$T(\lambda) \triangleq \frac{1}{\lambda^2}\left\{X(\lambda)\,[W(\lambda)]^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\,Y(\lambda)W(\lambda) + \frac{E_{b1}}{16R_1}R_b^2\frac{[Y(\lambda)]^2+[Z(\lambda)]^2}{X(\lambda)}\right\}. \tag{M.65}$$

Finally, we show in Appendix N.2 that $T(\lambda)$ reduces to

$$T(\lambda) = \lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e, \tag{M.66}$$

where (see (N.47)–(N.50))

$$b = -\sqrt{\frac{E_{b1}}{R_1}}R_b, \tag{M.67}$$

$$c = \frac{E_{b1}}{4R_1}\left\{R_b^2 - R_b\left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right) - \frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\}, \tag{M.68}$$

$$d = \sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}R_b c_1\!\left(\frac{E_{b1}}{N_0}\right)\left[R_b\left(1+\frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right], \tag{M.69}$$

$$e = \left(\frac{E_{b1}}{8R_1}\right)^2\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\left[R_b\left(1+\frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2. \tag{M.70}$$

M.3 Eigenvalues of RF

The eigenvalues of **RF** are given by the roots of its characteristic polynomial (see (M.64)). First, we solve the equation

$$[W(\lambda)]^{4(R_1 - MR_1/N)} = 0. \tag{M.71}$$

We substitute into (M.71) the expression for $W(\lambda)$ from (M.59) to get

$$\left\{\lambda^2 - \frac{E_{b1}}{8R_1}\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\}^{4(R_1 - MR_1/N)} = 0, \tag{M.72}$$

from which we obtain the solutions

$$\lambda_1 = -\frac{1}{2}\sqrt{\frac{E_{b1}}{2R_1}}\frac{1}{\sqrt{Q_P/2}}\,c_1\!\left(\frac{E_{b1}}{N_0}\right), \quad m_1 = 4(R_1 - MR_1/N), \tag{M.73}$$

$$\lambda_2 = \frac{1}{2}\sqrt{\frac{E_{b1}}{2R_1}}\frac{1}{\sqrt{Q_P/2}}\,c_1\!\left(\frac{E_{b1}}{N_0}\right), \quad m_2 = 4(R_1 - MR_1/N). \tag{M.74}$$

Next, we solve the fourth-order equation

$$\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e = 0. \tag{M.75}$$

Following the general method given in [33, Sec. 2.3], we begin by making the substitution

$$\lambda = y - \frac{b}{4} \tag{M.76}$$

in (M.75) to get

$$y^4 + py^2 + qy + r = 0, \tag{M.77}$$

where

$$p = \frac{8c - 3b^2}{8}, \tag{M.78}$$

$$q = \frac{b^3 - 4bc + 8d}{8}, \tag{M.79}$$

$$r = \frac{16b^2c + 256e - 3b^4 - 64bd}{256}. \tag{M.80}$$

Given the expressions in (M.67)–(M.70) for b, c, d, and e, we can show that

$$p = -\frac{E_{b1}}{8R_1}\left\{R_b^2 + 2R_b\left(1 + \frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right) + 2\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\}, \tag{M.81}$$

$$q = 0, \tag{M.82}$$

$$r = \frac{1}{4}\left(\frac{E_{b1}}{8R_1}\right)^2\left\{R_b^2 + 2R_b\left(1 + \frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right) + 2\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\}^2. \tag{M.83}$$

As a result, (M.77) reduces to a quadratic in y^2,

$$y^4 + py^2 + r = 0, \tag{M.84}$$

which leads to the following solutions:

$$y^2 = \frac{-p \pm \sqrt{p^2 - 4r}}{2}. \tag{M.85}$$

It is easily verified that

$$p^2 - 4r = 0, \tag{M.86}$$

so (M.85) further reduces to

$$y^2 = -\frac{p}{2}. \tag{M.87}$$

Hence, the two solutions to (M.84) each have multiplicity two and are given by

$$y = \pm\sqrt{-\frac{p}{2}} = \pm\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\sqrt{d\left(\frac{E_{b1}}{N_0}\right)}, \tag{M.88}$$

where

$$d\left(\frac{E_{b1}}{N_0}\right) \triangleq R_b^2 + 2R_b\left(1 + \frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right) + 2\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2. \tag{M.89}$$

Substituting (M.88) into (M.76), we get

$$\lambda = \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left(R_b \pm \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right). \tag{M.90}$$

Finally, since $T(\lambda) = \lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e$, the solutions to the equation

$$[T(\lambda)]^{2MR_1/N} = 0 \tag{M.91}$$

are equal to

$$\lambda_3 = \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left(R_b - \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right), \quad m_3 = 4MR_1/N, \tag{M.92}$$

$$\lambda_4 = \frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left(R_b + \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right), \quad m_4 = 4MR_1/N. \tag{M.93}$$

Thus, the matrix **RF** has $\Lambda = 4$ distinct eigenvalues. Since $c_1\left(\frac{E_{b1}}{N_0}\right) > 0$ (see (M.18)), this implies that $d\left(\frac{E_{b1}}{N_0}\right) > R_b^2$, and we have $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, and $\lambda_4 > 0$.

M.4 Probability of Error

Substituting $\Lambda = 4$, $m_1 = 4(R_1 - MR_1/N)$, and $m_3 = 4MR_1/N$ into (D.8), and using the fact that λ_1 and λ_3 are the only negative eigenvalues, we show that the probability of error reduces to

$$P_e = \sum_{k=1}^{4(R_1 - MR_1/N)} c_{1,k} + \sum_{k=1}^{4MR_1/N} c_{3,k}. \tag{M.94}$$

Next, we apply (D.5) to (M.94) and make the same substitutions for Λ, m_1, and m_3 to get

$$\begin{aligned}
P_e = &\left(\frac{1-\mu_{21}}{2}\right)^{4(R_1 - MR_1/N)} \left(\frac{1-\mu_{31}}{2}\right)^{4MR_1/N} \left(\frac{1-\mu_{41}}{2}\right)^{4MR_1/N} \\
&\cdot \sum_{m=1}^{4(R_1 - MR_1/N)} \sum_{n_2=0}^{m-1} \sum_{n_3=0}^{m-1-n_2} \binom{(4(R_1 - MR_1/N)) - 1 + n_2}{n_2} \\
&\cdot \binom{4MR_1/N - 1 + n_3}{n_3} \binom{4MR_1/N - 1 + (m-1-n_2-n_3)}{m-1-n_2-n_3} \\
&\cdot \left(\frac{1+\mu_{21}}{2}\right)^{n_2} \left(\frac{1+\mu_{31}}{2}\right)^{n_3} \left(\frac{1+\mu_{41}}{2}\right)^{m-1-n_2-n_3} \\
&+ \left(\frac{1-\mu_{13}}{2}\right)^{4(R_1 - MR_1/N)} \left(\frac{1-\mu_{23}}{2}\right)^{4(R_1 - MR_1/N)} \left(\frac{1-\mu_{43}}{2}\right)^{4MR_1/N} \\
&\cdot \sum_{m=1}^{4MR_1/N} \sum_{n_1=0}^{m-1} \sum_{n_2=0}^{m-1-n_1} \binom{(4(R_1 - MR_1/N)) - 1 + n_1}{n_1} \\
&\cdot \binom{4(R_1 - MR_1/N) - 1 + n_2}{n_2} \binom{4MR_1/N - 1 + (m-1-n_1-n_2)}{m-1-n_1-n_2} \\
&\cdot \left(\frac{1+\mu_{13}}{2}\right)^{n_1} \left(\frac{1+\mu_{23}}{2}\right)^{n_2} \left(\frac{1+\mu_{43}}{2}\right)^{m-1-n_1-n_2}.
\end{aligned} \tag{M.95}$$

By following the same steps given in Appendix K, we can re-write (M.95) as

$$\begin{aligned}
P_e = &\left(\frac{1-\mu_{21}}{2}\right)^{4(R_1 - MR_1/N)} \sum_{i_2=0}^{4(R_1 - MR_1/N)-1} \binom{4(R_1 - MR_1/N) - 1 + i_2}{i_2} \left(\frac{1+\mu_{21}}{2}\right)^{i_2} \\
&\cdot \left(\frac{1-\mu_{31}}{2}\right)^{4MR_1/N} \sum_{i_3=0}^{4(R_1 - MR_1/N)-1-i_2} \binom{4MR_1/N - 1 + i_3}{i_3} \left(\frac{1+\mu_{31}}{2}\right)^{i_3} \\
&\cdot \left(\frac{1-\mu_{41}}{2}\right)^{4MR_1/N} \sum_{r=0}^{4(R_1 - MR_1/N)-1-i_2-i_3} \binom{4MR_1/N - 1 + r}{r} \left(\frac{1+\mu_{41}}{2}\right)^{r} \\
&+ \left(\frac{1-\mu_{13}}{2}\right)^{4(R_1 - MR_1/N)} \sum_{i_1=0}^{4MR_1/N-1} \binom{4(R_1 - MR_1/N) - 1 + i_1}{i_1} \left(\frac{1+\mu_{13}}{2}\right)^{i_1} \\
&\cdot \left(\frac{1-\mu_{23}}{2}\right)^{4(R_1 - MR_1/N)} \sum_{i_2=0}^{4MR_1/N-1-i_1} \binom{4(R_1 - MR_1/N) - 1 + i_2}{i_2} \left(\frac{1+\mu_{23}}{2}\right)^{i_2} \\
&\cdot \left(\frac{1-\mu_{43}}{2}\right)^{4MR_1/N} \sum_{r=0}^{4MR_1/N-1-i_1-i_2} \binom{4MR_1/N - 1 + r}{r} \left(\frac{1+\mu_{43}}{2}\right)^{r}.
\end{aligned} \tag{M.96}$$

The expressions for μ_{21}, μ_{31}, μ_{41}, μ_{13}, μ_{23}, and μ_{43} are obtained by using (M.73)–(M.74) and (M.92)–(M.93) in (D.6):

$$\mu_{21} = 0, \tag{M.97}$$

$$\mu_{31} = \frac{R_b - \sqrt{d\left(\frac{E_{b1}}{N_0}\right)} - \sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right)}{R_b - \sqrt{d\left(\frac{E_{b1}}{N_0}\right)} + \sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right)}, \tag{M.98}$$

$$\mu_{41} = \frac{R_b + \sqrt{d\left(\frac{E_{b1}}{N_0}\right)} - \sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right)}{R_b + \sqrt{d\left(\frac{E_{b1}}{N_0}\right)} + \sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right)}, \tag{M.99}$$

$$\mu_{13} = \frac{\sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right) - \left(R_b - \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right)}{\sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right) + \left(R_b + \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right)}, \tag{M.100}$$

$$\mu_{23} = \frac{\sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right) + \left(R_b - \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right)}{\sqrt{2}\frac{1}{\sqrt{Q_P/2}}c_1\left(\frac{E_{b1}}{N_0}\right) - \left(R_b + \sqrt{d\left(\frac{E_{b1}}{N_0}\right)}\right)}, \tag{M.101}$$

$$\mu_{43} = \frac{R_b}{\sqrt{d\left(\frac{E_{b1}}{N_0}\right)}}. \tag{M.102}$$

Finally, since $\mu_{21} = 0$, (M.96) reduces to

$$
\begin{aligned}
P_e = &\left(\frac{1}{2}\right)^{4(R_1-MR_1/N)} \sum_{i_2=0}^{4(R_1-MR_1/N)-1} \binom{4(R_1-MR_1/N)-1+i_2}{i_2} \left(\frac{1}{2}\right)^{i_2} \\
&\cdot \left(\frac{1-\mu_{31}}{2}\right)^{4MR_1/N} \sum_{i_3=0}^{4(R_1-MR_1/N)-1-i_2} \binom{4MR_1/N-1+i_3}{i_3} \left(\frac{1+\mu_{31}}{2}\right)^{i_3} \\
&\cdot \left(\frac{1-\mu_{41}}{2}\right)^{4MR_1/N} \sum_{r=0}^{4(R_1-MR_1/N)-1-i_2-i_3} \binom{4MR_1/N-1+r}{r} \left(\frac{1+\mu_{41}}{2}\right)^{r} \\
&+ \left(\frac{1-\mu_{13}}{2}\right)^{4(R_1-MR_1/N)} \sum_{i_1=0}^{4MR_1/N-1} \binom{4(R_1-MR_1/N)-1+i_1}{i_1} \left(\frac{1+\mu_{13}}{2}\right)^{i_1} \\
&\cdot \left(\frac{1-\mu_{23}}{2}\right)^{4(R_1-MR_1/N)} \sum_{i_2=0}^{4MR_1/N-1-i_1} \binom{4(R_1-MR_1/N)-1+i_2}{i_2} \left(\frac{1+\mu_{23}}{2}\right)^{i_2} \\
&\cdot \left(\frac{1-\mu_{43}}{2}\right)^{4MR_1/N} \sum_{r=0}^{4MR_1/N-1-i_1-i_2} \binom{4MR_1/N-1+r}{r} \left(\frac{1+\mu_{43}}{2}\right)^{r}.
\end{aligned}
\tag{M.103}
$$

Appendix N

Derivations of (M.58) and (M.66)

In this appendix, we show that the expression for $\mathbf{D}(\lambda)$ in (M.55) can be re-written as (M.58). We also show that $T(\lambda)$, as given by (M.65), reduces to the expression in (M.66).

N.1 $\mathbf{D}(\lambda)$

First, we expand the expression for $\mathbf{C}(\lambda)$. From (M.53), $\mathbf{C}(\lambda)$ is equal to

$$\mathbf{C}(\lambda) = \mathbf{\Upsilon}_1(\lambda) \cdot \mathbf{\Upsilon}_2(\lambda) \cdot \mathbf{\Upsilon}_3(\lambda), \tag{N.1}$$

where

$$\mathbf{\Upsilon}_1(\lambda) \triangleq \frac{1}{\lambda}\frac{E_{b1}}{8R_1}\left\{\left[\mathbf{A}_{R_b} + c_1\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right] \otimes \mathbf{I}_2\right\}, \tag{N.2}$$

$$\mathbf{\Upsilon}_2(\lambda) \triangleq \mathbf{I}_{2R_b} + \frac{1}{X(\lambda)}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b} \otimes \left(\lambda\mathbf{B}_2\left(a^{(1)}_{p,q+1,m}\right) - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\mathbf{I}_2\right)\right], \tag{N.3}$$

$$\mathbf{\Upsilon}_3(\lambda) \triangleq \left[\mathbf{A}_{R_b} + \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right] \otimes \mathbf{I}_2. \tag{N.4}$$

By applying both the *mixed product rule* for Kronecker products (see Appendix B.2) and the relation (see (C.14))

$$\mathbf{A}_{R_b}\mathbf{A}_{R_b} = R_b\mathbf{A}_{R_b}, \tag{N.5}$$

we evaluate the product $\boldsymbol{\Upsilon}_2(\lambda)\cdot\boldsymbol{\Upsilon}_3(\lambda)$ as

$$\begin{aligned}
&\boldsymbol{\Upsilon}_2(\lambda)\cdot\boldsymbol{\Upsilon}_3(\lambda)\\
&\quad= \left[\mathbf{A}_{R_b} + \frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\otimes\mathbf{I}_2\\
&\qquad + \frac{1}{X(\lambda)}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left\{\mathbf{A}_{R_b}\left[\mathbf{A}_{R_b} + \frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\otimes\left[\lambda\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right) - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\mathbf{I}_2\right]\right\}\\
&\quad= \left[\mathbf{A}_{R_b} + \frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\otimes\mathbf{I}_2\\
&\qquad + \frac{1}{X(\lambda)}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left\{\left[R_b + \frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\mathbf{A}_{R_b}\otimes\left[\lambda\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right) - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\mathbf{I}_2\right]\right\}.
\end{aligned} \tag{N.6}$$

We then multiply this expression by $\boldsymbol{\Upsilon}_1(\lambda)$ and apply both the *mixed product rule* and (N.5) again to obtain

$$\begin{aligned}
\mathbf{C}(\lambda) &= \frac{1}{\lambda}\frac{E_{b1}}{8R_1}\left[\mathbf{A}_{R_b} + c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\left[\mathbf{A}_{R_b} + \frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\otimes\mathbf{I}_2\\
&\quad + \frac{1}{\lambda}\frac{1}{X(\lambda)}\frac{E_{b1}}{8R_1}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left\{\left[\mathbf{A}_{R_b} + c_1\!\left(\frac{E_{b1}}{N_0}\right)\mathbf{I}_{R_b}\right]\left[R_b + \frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\mathbf{A}_{R_b}\right.\\
&\qquad\qquad \left.\otimes\left[\lambda\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right) - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\mathbf{I}_2\right]\right\}\\
&= \frac{1}{\lambda}\frac{E_{b1}}{8R_1}\left\{\left[R_b + \left(1 + \frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\mathbf{A}_{R_b} + \frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\mathbf{I}_{R_b}\right\}\otimes\mathbf{I}_2\\
&\quad + \frac{1}{\lambda}\frac{1}{X(\lambda)}\frac{E_{b1}}{8R_1}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[R_b + \frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\\
&\qquad\qquad \cdot\left\{\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\mathbf{A}_{R_b}\otimes\left[\lambda\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right) - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\mathbf{I}_2\right]\right\}.
\end{aligned} \tag{N.7}$$

Using Property 3 from Appendix B.2, as well as the relation $\mathbf{I}_{mn} = \mathbf{I}_m \otimes \mathbf{I}_n$, we expand (N.7) as

$$\begin{aligned}
\mathbf{C}(\lambda) &= \frac{1}{\lambda}\frac{E_{b1}}{8R_1}\left[R_b + \left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right](\mathbf{A}_{R_b}\otimes\mathbf{I}_2) \\
&\quad + \frac{1}{\lambda}\frac{E_{b1}}{8R_1}\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\mathbf{I}_{2R_b} \\
&\quad + \frac{1}{\lambda}\frac{1}{X(\lambda)}\frac{E_{b1}}{8R_1}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b + \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right] \\
&\qquad \cdot\lambda\left[\mathbf{A}_{R_b}\otimes\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right)\right] \\
&\quad - \frac{1}{\lambda}\frac{1}{X(\lambda)}\left(\frac{E_{b1}}{8R_1}\right)^2 R_b\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b + \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right](\mathbf{A}_{R_b}\otimes\mathbf{I}_2) \\
&= \frac{1}{\lambda}\frac{E_{b1}}{8R_1}\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\mathbf{I}_{2R_b} \\
&\quad + \frac{1}{\lambda}\frac{1}{X(\lambda)}\frac{E_{b1}}{8R_1}\Bigg\{\left[R_b + \left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]X(\lambda)\,(\mathbf{A}_{R_b}\otimes\mathbf{I}_2) \\
&\qquad + \lambda\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b + \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[\mathbf{A}_{R_b}\otimes\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right)\right] \\
&\qquad - \frac{E_{b1}}{8R_1}R_b\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b + \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right](\mathbf{A}_{R_b}\otimes\mathbf{I}_2)\Bigg\}.
\end{aligned} \tag{N.8}$$

Given the expression for $X(\lambda)$ in (M.45), the two terms involving $\mathbf{A}_{R_b}\otimes\mathbf{I}_2$ can be combined as

$$\begin{aligned}
&\left[R_b + \left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]X(\lambda)\,(\mathbf{A}_{R_b}\otimes\mathbf{I}_2) \\
&\qquad - \frac{E_{b1}}{8R_1}R_b\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b + \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right](\mathbf{A}_{R_b}\otimes\mathbf{I}_2) \\
&= \left\{\left[R_b + \left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\lambda\left(\lambda - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\right) - \frac{E_{b1}}{8R_1}R_b\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\} \\
&\qquad \cdot(\mathbf{A}_{R_b}\otimes\mathbf{I}_2)\,.
\end{aligned} \tag{N.9}$$

Using this result in (N.8) and applying Property 3 from Appendix B.2, we finally get

$$\begin{aligned}\mathbf{C}(\lambda) = &\frac{1}{\lambda}\frac{E_{b1}}{8R_1}\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\mathbf{I}_{2R_b}\\ &+\frac{1}{\lambda}\frac{1}{X(\lambda)}\frac{E_{b1}}{8R_1}\Bigg\{\mathbf{A}_{R_b}\otimes\lambda\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[R_b+c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b+\frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right)\\ &\qquad+\mathbf{A}_{R_b}\otimes\Bigg(\left[R_b+\left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\lambda\left(\lambda-\frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\right)\\ &\qquad\qquad-\frac{E_{b1}}{8R_1}R_b\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\Bigg)\mathbf{I}_2\Bigg\}.\end{aligned} \tag{N.10}$$

We now expand $\mathbf{D}(\lambda)$ by substituting (N.10) into (M.55). This yields

$$\begin{aligned}\mathbf{D}(\lambda) = &\lambda\mathbf{I}_{2R_b}-\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[\mathbf{A}_{R_b}\otimes\mathbf{B}_2^H\!\left(a^{(1)}_{p,q+1,m}\right)\right]-\frac{1}{\lambda}\frac{E_{b1}}{8R_1}\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\mathbf{I}_{2R_b}\\ &-\frac{1}{\lambda}\frac{1}{X(\lambda)}\frac{E_{b1}}{8R_1}\Bigg\{\mathbf{A}_{R_b}\otimes\lambda\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\left[R_b+c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b+\frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right)\\ &\qquad+\mathbf{A}_{R_b}\otimes\Bigg(\left[R_b+\left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\lambda\left(\lambda-\frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\right)\\ &\qquad\qquad-\frac{E_{b1}}{8R_1}R_b\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\Bigg)\mathbf{I}_2\Bigg\}\\ =&\frac{1}{\lambda}W(\lambda)\mathbf{I}_{2R_b}-\frac{1}{\lambda}\frac{1}{X(\lambda)}\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}(\mathbf{A}_{R_b}\otimes\mathbf{\Phi}),\end{aligned} \tag{N.11}$$

where

$$W(\lambda) \triangleq \lambda^2-\frac{E_{b1}}{8R_1}\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2, \tag{N.12}$$

and

$$\mathbf{\Phi} \triangleq \mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3 \tag{N.13}$$

is a 2×2 matrix with

$$\mathbf{\Phi}_1 \triangleq \lambda X(\lambda)\mathbf{B}_2^H\!\left(a^{(1)}_{p,q+1,m}\right), \tag{N.14}$$

$$\mathbf{\Phi}_2 \triangleq \lambda\frac{E_{b1}}{8R_1}\left[R_b+c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b+\frac{1}{Q_P/2}\,c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right), \tag{N.15}$$

$$\mathbf{\Phi}_3 \triangleq \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\Bigg\{\left[R_b + \left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\lambda\left(\lambda - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\right) - \frac{E_{b1}}{8R_1}R_b\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\Bigg\}\mathbf{I}_2. \tag{N.16}$$

Let $[\mathbf{\Phi}]_{mn}$ denote the mn-th entry of $\mathbf{\Phi}$. Using the expressions for $X(\lambda)$ and $\mathbf{B}_2(\epsilon)$ (see (M.45) and (M.22), respectively), we have

$$\begin{aligned}
[\mathbf{\Phi}]_{11} &= [\mathbf{\Phi}_1]_{11} + [\mathbf{\Phi}_2]_{11} + [\mathbf{\Phi}_3]_{11} \\
&= \lambda X(\lambda)\left[\mathbf{B}_2^H\!\left(a^{(1)}_{p,q+1,m}\right)\right]_{11} \\
&\quad + \lambda\frac{E_{b1}}{8R_1}\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b + \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[\mathbf{B}_2\!\left(a^{(1)}_{p,q+1,m}\right)\right]_{11} \\
&\quad + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\Bigg\{\left[R_b + \left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\lambda\left(\lambda - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\right) \\
&\qquad - \frac{E_{b1}}{8R_1}R_b\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\Bigg\}[\mathbf{I}_2]_{11} \\
&= \lambda\left(\lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\lambda + \frac{E_{b1}}{8R_1}R_b^2\right) \\
&\quad + \lambda\frac{E_{b1}}{8R_1}\left[R_b + c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b + \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right] \\
&\quad + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\Bigg\{\left[R_b + \left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\lambda\left(\lambda - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\right) \\
&\qquad - \frac{E_{b1}}{8R_1}R_b\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2\Bigg\} \\
&= \lambda^3 + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left(1+\frac{1}{Q_P/2}\right)c_1\!\left(\frac{E_{b1}}{N_0}\right)\lambda^2 \\
&\quad - \frac{E_{b1}}{8R_1}c_1\!\left(\frac{E_{b1}}{N_0}\right)\left[R_b\left(1+\frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2}c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]\lambda \\
&\quad - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}R_b\frac{1}{Q_P/2}\left[c_1\!\left(\frac{E_{b1}}{N_0}\right)\right]^2
\end{aligned} \tag{N.17}$$

and

$$[\mathbf{\Phi}]_{12} = [\mathbf{\Phi}_1]_{12} + [\mathbf{\Phi}_2]_{12} + [\mathbf{\Phi}_3]_{12}$$

$$
\begin{aligned}
&= \lambda X(\lambda) \left[\mathbf{B}_2^H\left(a_{p,q+1,m}^{(1)}\right) \right]_{12} \\
&\quad + \lambda \frac{E_{b1}}{8R_1} \left[R_b + c_1\left(\frac{E_{b1}}{N_0}\right) \right] \left[R_b + \frac{1}{Q_P/2}\, c_1\left(\frac{E_{b1}}{N_0}\right) \right] \left[\mathbf{B}_2\left(a_{p,q+1,m}^{(1)}\right) \right]_{12} \\
&\quad + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} \left\{ \left[R_b + \left(1 + \frac{1}{Q_P/2}\right) c_1\left(\frac{E_{b1}}{N_0}\right) \right] \lambda \left(\lambda - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \right) \right. \\
&\qquad\qquad \left. - \frac{E_{b1}}{8R_1} R_b \frac{1}{Q_P/2} \left[c_1\left(\frac{E_{b1}}{N_0}\right) \right]^2 \right\} [\mathbf{I}_2]_{12} \\
&= \lambda \left(\lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \lambda + \frac{E_{b1}}{8R_1} R_b^2 \right) a_{p,q+1,m}^{(1)} \\
&\quad + \lambda \frac{E_{b1}}{8R_1} \left[R_b + c_1\left(\frac{E_{b1}}{N_0}\right) \right] \left[R_b + \frac{1}{Q_P/2}\, c_1\left(\frac{E_{b1}}{N_0}\right) \right] \left(-a_{p,q+1,m}^{(1)} \right) \\
&= \lambda \left\{ \lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \lambda - \frac{E_{b1}}{8R_1}\, c_1\left(\frac{E_{b1}}{N_0}\right) \left[R_b \left(1 + \frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2}\, c_1\left(\frac{E_{b1}}{N_0}\right) \right] \right\} a_{p,q+1,m}^{(1)}.
\end{aligned}
\tag{N.18}
$$

We can easily verify that

$$
[\mathbf{\Phi}]_{22} = [\mathbf{\Phi}]_{11}, \tag{N.19}
$$

$$
[\mathbf{\Phi}]_{21} = -[\mathbf{\Phi}]_{12}. \tag{N.20}
$$

Define

$$
\begin{aligned}
Y(\lambda) \triangleq \lambda^3 &+ \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} \left(1 + \frac{1}{Q_P/2}\right) c_1\left(\frac{E_{b1}}{N_0}\right) \lambda^2 \\
&- \frac{E_{b1}}{8R_1} c_1\left(\frac{E_{b1}}{N_0}\right) \left[R_b \left(1 + \frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right) \right] \lambda \\
&- \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} \frac{E_{b1}}{8R_1} R_b \frac{1}{Q_P/2} \left[c_1\left(\frac{E_{b1}}{N_0}\right) \right]^2,
\end{aligned}
\tag{N.21}
$$

$$
Z(\lambda) \triangleq \lambda \left\{ \lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \lambda - \frac{E_{b1}}{8R_1}\, c_1\left(\frac{E_{b1}}{N_0}\right) \left[R_b \left(1 + \frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2}\, c_1\left(\frac{E_{b1}}{N_0}\right) \right] \right\}, \tag{N.22}
$$

such that

$$
[\mathbf{\Phi}]_{11} = Y(\lambda), \tag{N.23}
$$

$$
[\mathbf{\Phi}]_{12} = Z(\lambda) a_{p,q+1,m}^{(1)}. \tag{N.24}
$$

We can then write $\mathbf{\Phi}$ as

$$\begin{aligned}\mathbf{\Phi} &= \begin{bmatrix} [\mathbf{\Phi}]_{11} & [\mathbf{\Phi}]_{12} \\ [\mathbf{\Phi}]_{21} & [\mathbf{\Phi}]_{22} \end{bmatrix} \\ &= \begin{bmatrix} Y(\lambda) & Z(\lambda)a^{(1)}_{p,q+1,m} \\ -Z(\lambda)a^{(1)}_{p,q+1,m} & Y(\lambda) \end{bmatrix} \\ &= Y(\lambda)\begin{bmatrix} 1 & \frac{Z(\lambda)}{Y(\lambda)}a^{(1)}_{p,q+1,m} \\ -\frac{Z(\lambda)}{Y(\lambda)}a^{(1)}_{p,q+1,m} & 1 \end{bmatrix} \\ &= Y(\lambda)\mathbf{B}_2^H\left(\frac{Z(\lambda)}{Y(\lambda)}a^{(1)}_{p,q+1,m}\right). \end{aligned} \tag{N.25}$$

Finally, we substitute (N.25) into (N.11) to get

$$\mathbf{D}(\lambda) = \frac{1}{\lambda}\left\{W(\lambda)\mathbf{I}_{2R_b} - \left[\frac{1}{4}\sqrt{\frac{E_{b1}}{R_1}}\frac{Y(\lambda)}{X(\lambda)}\mathbf{A}_{R_b} \otimes \mathbf{B}_2^H\left(\frac{Z(\lambda)}{Y(\lambda)}a^{(1)}_{p,q+1,m}\right)\right]\right\}. \tag{N.26}$$

N.2 $T(\lambda)$

From (M.65), $T(\lambda)$ is defined as

$$T(\lambda) = \frac{1}{\lambda^2}\left\{X(\lambda)\left[W(\lambda)\right]^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}R_b\,Y(\lambda)W(\lambda) + \frac{E_{b1}}{16R_1}R_b^2\frac{[Y(\lambda)]^2 + [Z(\lambda)]^2}{X(\lambda)}\right\}. \tag{N.27}$$

The terms in this expression will be evaluated in steps. First, we square the expression for $W(\lambda)$ in (N.12):

$$[W(\lambda)]^2 = \lambda^4 - \frac{E_{b1}}{4R_1}\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\lambda^2 + \left(\frac{E_{b1}}{8R_1}\right)^2\left(\frac{1}{Q_P/2}\right)^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4. \tag{N.28}$$

Using this result, along with the expression for $X(\lambda)$ in (M.45), we have

$$X(\lambda)[W(\lambda)]^2 = \lambda^6 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \lambda^5 + \frac{E_{b1}}{8R_1}\left\{R_b^2 - 2\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\}\lambda^4$$
$$+ \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{4R_1} R_b \frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2 \lambda^3$$
$$- \left(\frac{E_{b1}}{8R_1}\right)^2 \frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2 \left\{2R_b^2 - \frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\}\lambda^2 \tag{N.29}$$
$$- \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left(\frac{E_{b1}}{8R_1}\right)^2\left(\frac{1}{Q_P/2}\right)^2 R_b \left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4 \lambda$$
$$+ \left(\frac{E_{b1}}{8R_1}\right)^3\left(\frac{1}{Q_P/2}\right)^2 R_b^2 \left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4.$$

Given the expressions for $W(\lambda)$ and $Y(\lambda)$ in (N.12) and (N.21), respectively, we also have

$$\frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b Y(\lambda) W(\lambda) = \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \left\{\lambda^5 + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left(1 + \frac{1}{Q_P/2}\right) c_1\left(\frac{E_{b1}}{N_0}\right)\lambda^4\right.$$
$$- \frac{E_{b1}}{8R_1} R_b \left(1 + \frac{1}{Q_P/2}\right) c_1\left(\frac{E_{b1}}{N_0}\right)\lambda^3$$
$$- \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\left[R_b + \left(1 + \frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\right]\lambda^2$$
$$+ \left(\frac{E_{b1}}{8R_1}\right)^2\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^3\left[R_b\left(1 + \frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right)\right]\lambda$$
$$\left. + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left(\frac{E_{b1}}{8R_1}\right)^2\left(\frac{1}{Q_P/2}\right)^2 R_b\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4\right\}. \tag{N.30}$$

Thus, from (N.29) and (N.30), we obtain

$$X(\lambda)[W(\lambda)]^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b Y(\lambda) W(\lambda)$$
$$= \lambda^6 - \sqrt{\frac{E_{b1}}{R_1}} R_b \lambda^5 + \frac{E_{b1}}{8R_1}\left\{R_b^2 - 2R_b\left(1 + \frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right) - 2\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\}\lambda^4$$
$$+ \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1} R_b c_1\left(\frac{E_{b1}}{N_0}\right)\left\{R_b\left(1 + \frac{1}{Q_P/2}\right) + 2\frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right)\right\}\lambda^3$$
$$+ \left(\frac{E_{b1}}{8R_1}\right)^2\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^3\left\{2R_b\left(1 + \frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right)\right\}\lambda^2$$
$$- \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\left(\frac{E_{b1}}{8R_1}\right)^2\frac{1}{Q_P/2}\left(1 + \frac{1}{Q_P/2}\right) R_b^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^3\lambda \tag{N.31}$$
$$- \left(\frac{E_{b1}}{8R_1}\right)^3\left(\frac{1}{Q_P/2}\right)^2 R_b^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4.$$

Next, we square the expressions for $Y(\lambda)$ and $Z(\lambda)$ in (N.21) and (N.22), respectively, and combine like terms to show that

$$\begin{aligned}
[Y(\lambda)]^2 = \lambda^6 &+ \sqrt{\frac{E_{b1}}{R_1}}\left(1+\frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\lambda^5 \\
&- \frac{E_{b1}}{4R_1}c_1\left(\frac{E_{b1}}{N_0}\right)\left\{\left(1+\frac{1}{Q_P/2}\right)\left[R_b-\left(1+\frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\right]-\frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right\}\lambda^4 \\
&- \sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\left\{R_b\frac{1}{Q_P/2}+\left(1+\frac{1}{Q_P/2}\right)\right. \\
&\qquad \left.\cdot\left[R_b\left(1+\frac{1}{Q_P/2}\right)-\frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]\right\}\lambda^3 \\
&+ \left(\frac{E_{b1}}{8R_1}\right)^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\left\{\left[R_b\left(1+\frac{1}{Q_P/2}\right)-\frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\right. \\
&\qquad \left. - 4\frac{1}{Q_P/2}\left(1+\frac{1}{Q_P/2}\right)R_bc_1\left(\frac{E_{b1}}{N_0}\right)\right\}\lambda^2 \\
&+ \sqrt{\frac{E_{b1}}{R_1}}\left(\frac{E_{b1}}{8R_1}\right)^2\frac{1}{Q_P/2}R_b\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^3\left[R_b\left(1+\frac{1}{Q_P/2}\right)-\frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]\lambda \\
&+ 2\left(\frac{E_{b1}}{8R_1}\right)^3\left(\frac{1}{Q_P/2}\right)^2R_b^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4
\end{aligned} \tag{N.32}$$

and

$$\begin{aligned}
[Z(\lambda)]^2 = \lambda^6 &- \sqrt{\frac{E_{b1}}{R_1}}R_b\lambda^5 + \frac{E_{b1}}{4R_1}\left\{R_b^2-R_b\left(1+\frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)-\frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\}\lambda^4 \\
&+ \sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}R_bc_1\left(\frac{E_{b1}}{N_0}\right)\left[R_b\left(1+\frac{1}{Q_P/2}\right)+\frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]\lambda^3 \\
&+ \left(\frac{E_{b1}}{8R_1}\right)^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\left[R_b\left(1+\frac{1}{Q_P/2}\right)+\frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\lambda^2.
\end{aligned} \tag{N.33}$$

Adding the results of (N.32) and (N.33), we get

$$\begin{aligned}
[Y(\lambda)]^2+[Z(\lambda)]^2 = 2\lambda^6 &- P_5\left(\frac{E_{b1}}{N_0}\right)\lambda^5 + P_4\left(\frac{E_{b1}}{N_0}\right)\lambda^4 + P_3\left(\frac{E_{b1}}{N_0}\right)\lambda^3 \\
&+ P_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0\left(\frac{E_{b1}}{N_0}\right),
\end{aligned} \tag{N.34}$$

where

$$P_5\left(\frac{E_{b1}}{N_0}\right) \triangleq \sqrt{\frac{E_{b1}}{R_1}}\left[R_b-\left(1+\frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\right], \tag{N.35}$$

$$P_4\left(\frac{E_{b1}}{N_0}\right) \triangleq \frac{E_{b1}}{4R_1}\left[R_b - \left(1+\frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2, \tag{N.36}$$

$$P_3\left(\frac{E_{b1}}{N_0}\right) \triangleq \sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}\left(1+\frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\left[R_b - c_1\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b - \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right], \tag{N.37}$$

$$P_2\left(\frac{E_{b1}}{N_0}\right) \triangleq 2\left(\frac{E_{b1}}{8R_1}\right)^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\left[R_b\left(1+\frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2, \tag{N.38}$$

$$P_1\left(\frac{E_{b1}}{N_0}\right) \triangleq \sqrt{\frac{E_{b1}}{R_1}}\left(\frac{E_{b1}}{8R_1}\right)^2\frac{1}{Q_P/2}R_b\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^3\left[R_b\left(1+\frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right], \tag{N.39}$$

$$P_0\left(\frac{E_{b1}}{N_0}\right) \triangleq 2\left(\frac{E_{b1}}{8R_1}\right)^3\left(\frac{1}{Q_P/2}\right)^2R_b^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4. \tag{N.40}$$

We show in Appendix O that $[Y(\lambda)]^2 + [Z(\lambda)]^2$, when divided by $X(\lambda)$ (see (M.45)), is equal to

$$\frac{[Y(\lambda)]^2 + [Z(\lambda)]^2}{X(\lambda)} = 2\lambda^4 + P_5'\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + \tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + \tilde{P}_3\left(\frac{E_{b1}}{N_0}\right)\lambda + \tilde{P}_2\left(\frac{E_{b1}}{N_0}\right), \tag{N.41}$$

where (see (O.10), (O.13), (O.16), and (O.19))

$$P_5'\left(\frac{E_{b1}}{N_0}\right) = \sqrt{\frac{E_{b1}}{R_1}}\left(1+\frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right), \tag{N.42}$$

$$\tilde{P}_4\left(\frac{E_{b1}}{N_0}\right) = \frac{E_{b1}}{4R_1}\left(1+\frac{1}{Q_P/2}\right)^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2, \tag{N.43}$$

$$\tilde{P}_3\left(\frac{E_{b1}}{N_0}\right) = \sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}\frac{1}{Q_P/2}\left(1+\frac{1}{Q_P/2}\right)\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^3, \tag{N.44}$$

$$\tilde{P}_2\left(\frac{E_{b1}}{N_0}\right) = 2\left(\frac{E_{b1}}{8R_1}\right)^2\left(\frac{1}{Q_P/2}\right)^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4. \tag{N.45}$$

Finally, we substitute the results of (N.31) and (N.41) into (N.27). After much simplification, we can show that $T(\lambda)$ equals

$$\begin{aligned}T(\lambda) = \frac{1}{\lambda^2}\Bigg\{&\lambda^6 - \sqrt{\frac{E_{b1}}{R_1}}R_b\lambda^5 \\ &+ \frac{E_{b1}}{4R_1}\left\{R_b^2 - R_b\left(1+\frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right) - \frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\right\}\lambda^4 \\ &+ \sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}R_bc_1\left(\frac{E_{b1}}{N_0}\right)\left[R_b\left(1+\frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]\lambda^3 \\ &+ \left(\frac{E_{b1}}{8R_1}\right)^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\left[R_b\left(1+\frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\lambda^2\Bigg\}\end{aligned}$$

$$= \lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e, \tag{N.46}$$

where

$$b = -\sqrt{\frac{E_{b1}}{R_1}} R_b, \tag{N.47}$$

$$c = \frac{E_{b1}}{4R_1} \left\{ R_b^2 - R_b \left(1 + \frac{1}{Q_P/2}\right) c_1\left(\frac{E_{b1}}{N_0}\right) - \frac{1}{Q_P/2} \left[c_1\left(\frac{E_{b1}}{N_0}\right) \right]^2 \right\}, \tag{N.48}$$

$$d = \sqrt{\frac{E_{b1}}{R_1}} \frac{E_{b1}}{8R_1} R_b c_1\left(\frac{E_{b1}}{N_0}\right) \left[R_b \left(1 + \frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right) \right], \tag{N.49}$$

$$e = \left(\frac{E_{b1}}{8R_1}\right)^2 \left[c_1\left(\frac{E_{b1}}{N_0}\right) \right]^2 \left[R_b \left(1 + \frac{1}{Q_P/2}\right) + \frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right) \right]^2. \tag{N.50}$$

Appendix O

Derivation of (N.41)

In this appendix, we perform step-by-step long division to evaluate

$$\frac{[Y(\lambda)]^2 + [Z(\lambda)]^2}{X(\lambda)}.$$

From (N.34) and (N.35)–(N.40), we have

$$\begin{aligned}[Y(\lambda)]^2 + [Z(\lambda)]^2 = 2\lambda^6 - P_5\left(\frac{E_{b1}}{N_0}\right)\lambda^5 + P_4\left(\frac{E_{b1}}{N_0}\right)\lambda^4 + P_3\left(\frac{E_{b1}}{N_0}\right)\lambda^3 \\ + P_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0\left(\frac{E_{b1}}{N_0}\right),\end{aligned} \tag{O.1}$$

where

$$P_5\left(\frac{E_{b1}}{N_0}\right) \triangleq \sqrt{\frac{E_{b1}}{R_1}}\left[R_b - \left(1 + \frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\right], \tag{O.2}$$

$$P_4\left(\frac{E_{b1}}{N_0}\right) \triangleq \frac{E_{b1}}{4R_1}\left[R_b - \left(1 + \frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2, \tag{O.3}$$

$$P_3\left(\frac{E_{b1}}{N_0}\right) \triangleq \sqrt{\frac{E_{b1}}{R_1}}\frac{E_{b1}}{8R_1}\left(1 + \frac{1}{Q_P/2}\right)c_1\left(\frac{E_{b1}}{N_0}\right)\left[R_b - c_1\left(\frac{E_{b1}}{N_0}\right)\right]\left[R_b - \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right], \tag{O.4}$$

$$P_2\left(\frac{E_{b1}}{N_0}\right) \triangleq 2\left(\frac{E_{b1}}{8R_1}\right)^2\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2\left[R_b\left(1 + \frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2, \tag{O.5}$$

$$P_1\left(\frac{E_{b1}}{N_0}\right) \triangleq \sqrt{\frac{E_{b1}}{R_1}} \left(\frac{E_{b1}}{8R_1}\right)^2 \frac{1}{Q_P/2} R_b \left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^3 \left[R_b\left(1+\frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right)\right], \tag{O.6}$$

$$P_0\left(\frac{E_{b1}}{N_0}\right) \triangleq 2\left(\frac{E_{b1}}{8R_1}\right)^3 \left(\frac{1}{Q_P/2}\right)^2 R_b^2 \left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4. \tag{O.7}$$

Also, the expression for $X(\lambda)$ is given by (M.45):

$$X(\lambda) \triangleq \lambda^2 - \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \lambda + \frac{E_{b1}}{8R_1} R_b^2. \tag{O.8}$$

Step #1:

$$\frac{2\lambda^6 - P_5\left(\frac{E_{b1}}{N_0}\right)\lambda^5 + P_4\left(\frac{E_{b1}}{N_0}\right)\lambda^4}{X(\lambda)} = 2\lambda^4 + \frac{P_5'\left(\frac{E_{b1}}{N_0}\right)\lambda^5 + P_4'\left(\frac{E_{b1}}{N_0}\right)\lambda^4}{X(\lambda)}, \tag{O.9}$$

where

$$\begin{aligned} P_5'\left(\frac{E_{b1}}{N_0}\right) &\triangleq -\left[P_5\left(\frac{E_{b1}}{N_0}\right) - \sqrt{\frac{E_{b1}}{R_1}} R_b\right] \\ &= \sqrt{\frac{E_{b1}}{R_1}}\left(1+\frac{1}{Q_P/2}\right) c_1\left(\frac{E_{b1}}{N_0}\right), \end{aligned} \tag{O.10}$$

and

$$\begin{aligned} P_4'\left(\frac{E_{b1}}{N_0}\right) &\triangleq P_4\left(\frac{E_{b1}}{N_0}\right) - \frac{E_{b1}}{4R_1} R_b^2 \\ &= -\frac{E_{b1}}{4R_1}\left(1+\frac{1}{Q_P/2}\right) c_1\left(\frac{E_{b1}}{N_0}\right)\left[2R_b - \left(1+\frac{1}{Q_P/2}\right) c_1\left(\frac{E_{b1}}{N_0}\right)\right]. \end{aligned} \tag{O.11}$$

Step #2:

$$\frac{P_5'\left(\frac{E_{b1}}{N_0}\right)\lambda^5 + P_4'\left(\frac{E_{b1}}{N_0}\right)\lambda^4 + P_3\left(\frac{E_{b1}}{N_0}\right)\lambda^3}{X(\lambda)} = P_5'\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + \frac{\tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)\lambda^4 + P_3'\left(\frac{E_{b1}}{N_0}\right)\lambda^3}{X(\lambda)}, \tag{O.12}$$

where

$$\tilde{P}_4\left(\frac{E_{b1}}{N_0}\right) \triangleq P_4'\left(\frac{E_{b1}}{N_0}\right) + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b P_5'\left(\frac{E_{b1}}{N_0}\right)$$
$$= \frac{E_{b1}}{4R_1}\left(1 + \frac{1}{Q_P/2}\right)^2 \left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2, \quad \text{(O.13)}$$

and

$$P_3'\left(\frac{E_{b1}}{N_0}\right) \triangleq P_3\left(\frac{E_{b1}}{N_0}\right) - \frac{E_{b1}}{8R_1} R_b^2 P_5'\left(\frac{E_{b1}}{N_0}\right)$$
$$= -\sqrt{\frac{E_{b1}}{R_1}} \frac{E_{b1}}{8R_1}\left(1 + \frac{1}{Q_P/2}\right)\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^2 \left[R_b\left(1 + \frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right)\right]. \quad \text{(O.14)}$$

Step #3:

$$\frac{\tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)\lambda^4 + P_3'\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + P_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2}{X(\lambda)} = \tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + \frac{\tilde{P}_3\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + P_2'\left(\frac{E_{b1}}{N_0}\right)\lambda^2}{X(\lambda)}, \quad \text{(O.15)}$$

where

$$\tilde{P}_3\left(\frac{E_{b1}}{N_0}\right) \triangleq P_3'\left(\frac{E_{b1}}{N_0}\right) + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)$$
$$= \sqrt{\frac{E_{b1}}{R_1}} \frac{E_{b1}}{8R_1} \frac{1}{Q_P/2}\left(1 + \frac{1}{Q_P/2}\right)\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^3, \quad \text{(O.16)}$$

and

$$P_2'\left(\frac{E_{b1}}{N_0}\right) \triangleq P_2\left(\frac{E_{b1}}{N_0}\right) - \frac{E_{b1}}{8R_1} R_b^2 \tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)$$
$$= -2\left(\frac{E_{b1}}{8R_1}\right)^2 \frac{1}{Q_P/2}\left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^3 \left[2R_b\left(1 + \frac{1}{Q_P/2}\right) - \frac{1}{Q_P/2} c_1\left(\frac{E_{b1}}{N_0}\right)\right]. \quad \text{(O.17)}$$

Step #4:

$$\frac{\tilde{P}_3\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + P_2'\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1\left(\frac{E_{b1}}{N_0}\right)\lambda}{X(\lambda)} = \tilde{P}_3\left(\frac{E_{b1}}{N_0}\right)\lambda + \frac{\tilde{P}_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1'\left(\frac{E_{b1}}{N_0}\right)\lambda}{X(\lambda)}, \quad \text{(O.18)}$$

where

$$\tilde{P}_2\left(\frac{E_{b1}}{N_0}\right) \triangleq P_2'\left(\frac{E_{b1}}{N_0}\right) + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \tilde{P}_3\left(\frac{E_{b1}}{N_0}\right)$$
$$= 2\left(\frac{E_{b1}}{8R_1}\right)^2 \left(\frac{1}{Q_P/2}\right)^2 \left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4, \quad \text{(O.19)}$$

and

$$P_1'\left(\frac{E_{b1}}{N_0}\right) \triangleq P_1\left(\frac{E_{b1}}{N_0}\right) - \frac{E_{b1}}{8R_1} R_b^2 \tilde{P}_3\left(\frac{E_{b1}}{N_0}\right)$$
$$= -\sqrt{\frac{E_{b1}}{R_1}} \left(\frac{E_{b1}}{8R_1}\right)^2 \left(\frac{1}{Q_P/2}\right)^2 R_b \left[c_1\left(\frac{E_{b1}}{N_0}\right)\right]^4. \quad \text{(O.20)}$$

Step #5:

$$\frac{\tilde{P}_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1'\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0\left(\frac{E_{b1}}{N_0}\right)}{X(\lambda)} = \tilde{P}_2\left(\frac{E_{b1}}{N_0}\right) + \frac{\tilde{P}_1\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0'\left(\frac{E_{b1}}{N_0}\right)}{X(\lambda)}, \quad \text{(O.21)}$$

where

$$\tilde{P}_1\left(\frac{E_{b1}}{N_0}\right) \triangleq P_1'\left(\frac{E_{b1}}{N_0}\right) + \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}} R_b \tilde{P}_2\left(\frac{E_{b1}}{N_0}\right) = 0, \quad \text{(O.22)}$$

$$P_0'\left(\frac{E_{b1}}{N_0}\right) \triangleq P_0\left(\frac{E_{b1}}{N_0}\right) - \frac{E_{b1}}{8R_1} R_b^2 \tilde{P}_2\left(\frac{E_{b1}}{N_0}\right) = 0. \quad \text{(O.23)}$$

Thus,

$$\frac{\tilde{P}_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1'\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0\left(\frac{E_{b1}}{N_0}\right)}{X(\lambda)} = \tilde{P}_2\left(\frac{E_{b1}}{N_0}\right). \quad \text{(O.24)}$$

Putting together the above results, we have

$$\frac{[Y(\lambda)]^2 + [Z(\lambda)]^2}{X(\lambda)}$$
$$= \frac{2\lambda^6 - P_5\left(\frac{E_{b1}}{N_0}\right)\lambda^5 + P_4\left(\frac{E_{b1}}{N_0}\right)\lambda^4 + P_3\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + P_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0\left(\frac{E_{b1}}{N_0}\right)}{X(\lambda)}$$
$$= 2\lambda^4 + \frac{P_5'\left(\frac{E_{b1}}{N_0}\right)\lambda^5 + P_4'\left(\frac{E_{b1}}{N_0}\right)\lambda^4 + P_3\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + P_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0\left(\frac{E_{b1}}{N_0}\right)}{X(\lambda)}$$

$$= 2\lambda^4 + P_5'\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + \frac{\tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)\lambda^4 + P_3'\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + P_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0\left(\frac{E_{b1}}{N_0}\right)}{X(\lambda)}$$
$$= 2\lambda^4 + P_5'\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + \tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + \frac{\tilde{P}_3\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + P_2'\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0\left(\frac{E_{b1}}{N_0}\right)}{X(\lambda)}$$
$$= 2\lambda^4 + P_5'\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + \tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + \tilde{P}_3\left(\frac{E_{b1}}{N_0}\right)\lambda + \frac{\tilde{P}_2\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + P_1'\left(\frac{E_{b1}}{N_0}\right)\lambda + P_0\left(\frac{E_{b1}}{N_0}\right)}{X(\lambda)}$$
$$= 2\lambda^4 + P_5'\left(\frac{E_{b1}}{N_0}\right)\lambda^3 + \tilde{P}_4\left(\frac{E_{b1}}{N_0}\right)\lambda^2 + \tilde{P}_3\left(\frac{E_{b1}}{N_0}\right)\lambda + \tilde{P}_2\left(\frac{E_{b1}}{N_0}\right). \quad \text{(O.25)}$$

Appendix P

Derivation of Energy-per-Bit

P.1 MC-DS-CDMA: SISO System

From (2.4), the complex lowpass equivalent signal transmitted at sub-carrier $f_{m,i}$ is given by

$$\tilde{s}_{m,i}^{(k)}(t) = A_2 \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} \sum_{n=1}^{N} a_{p,q,m}^{(k)} \, C_{p,q,n}^{(k)} \, h_2\big\{t - [(pQ + q - 1)N + n - 1]T_c\big\}. \tag{P.1}$$

We are only concerned with the m-th symbol during the q-th time slot of the p-th frame, so we ignore in (P.1) the contributions of the symbols from all other time slots. Thus, the signal due to the i-th repetition of the m-th symbol is given by

$$\tilde{\gamma}_{m,i}^{(k)}(t) \triangleq A_2 \sum_{n=1}^{N} a_{p,q,m}^{(k)} \, C_{p,q,n}^{(k)} \, h_2\big\{t - [(pQ + q - 1)N + n - 1]T_c\big\}. \tag{P.2}$$

Using the time shift property of the Fourier transform to obtain the transform pair

$$h_2\big\{t - [(pQ + q - 1)N + n - 1]T_c\big\} \stackrel{\mathcal{F}}{\longleftrightarrow} H_2(f) e^{-j2\pi f[(pQ+q-1)N+n-1]T_c}, \tag{P.3}$$

we derive the Fourier transform of $\tilde{\gamma}_{m,i}^{(k)}(t)$ as

$$\begin{aligned}\tilde{\Gamma}_{m,i}^{(k)}(f) &\triangleq \mathcal{F}\left\{\tilde{\gamma}_{m,i}^{(k)}(t)\right\} \\ &= A_2 \sum_{n=1}^{N} a_{p,q,m}^{(k)}\, C_{p,q,n}^{(k)}\, H_2(f) e^{-j2\pi f[(pQ+q-1)N+n-1]T_c}. \end{aligned} \tag{P.4}$$

From Parseval's Theorem, the energy of $\tilde{\gamma}_{m,i}^{(k)}(t)$ is defined as

$$E_{m,i} \triangleq \frac{1}{2}\int_{-\infty}^{\infty} \left|\tilde{\gamma}_{m,i}^{(k)}(t)\right|^2 dt = \frac{1}{2}\int_{-\infty}^{\infty} \left|\tilde{\Gamma}_{m,i}^{(k)}(f)\right|^2 df. \tag{P.5}$$

Substituting (P.4) into (P.5), we get

$$\begin{aligned} E_{m,i} &= \frac{A_2^2}{2} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \left(a_{p,q,m}^{(k)}\right)^2 C_{p,q,n_1}^{(k)} C_{p,q,n_2}^{(k)} \int_{-\infty}^{\infty} |H_2(f)|^2 e^{j2\pi f(n_2-n_1)T_c}\, df \\ &= \frac{A_2^2}{2} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C_{p,q,n_1}^{(k)} C_{p,q,n_2}^{(k)} \int_{-\infty}^{\infty} X_2(f) e^{j2\pi f(n_2-n_1)T_c}\, df \\ &= \frac{A_2^2}{2} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C_{p,q,n_1}^{(k)} C_{p,q,n_2}^{(k)} \cdot x_2[(n_2-n_1)T_c], \end{aligned} \tag{P.6}$$

where we used the fact that $(a_{p,q,m}^{(k)})^2 = 1$ and $x_2(t) \Leftrightarrow X_2(f) \triangleq |H_2(f)|^2$. From (2.25), we have

$$x_2(nT_c) = \delta_{n,0} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{P.7}$$

Thus, $E_{m,i}$ reduces to

$$E_{m,i} = \frac{NA_2^2}{2}. \tag{P.8}$$

Finally, since each symbol is repeated across R_2 sub-carriers, the energy of the m-th symbol (i.e., energy-per-bit) is equal to

$$E_{b2} = \sum_{i=1}^{R_2} E_{m,i} = \frac{NR_2A_2^2}{2}. \tag{P.9}$$

P.2 MC-CDMA: SISO System

From (2.3), the complex lowpass equivalent signal transmitted at sub-carrier $f_{m,i}$ is given by

$$\tilde{s}_{m,i}^{(k)}(t) = A_1 \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} a_{p,q,m}^{(k)} \, C_{p,q,m,i}^{(k)} \, h_1\big[t - (pQ + q - 1)T_s\big]. \tag{P.10}$$

If we ignore the contributions of the symbols from all other time slots, then the signal due to the i-th repetition of the m-th symbol is given by

$$\tilde{\gamma}_{m,i}^{(k)}(t) \triangleq A_1 \, a_{p,q,m}^{(k)} \, C_{p,q,m,i}^{(k)} \, h_1\big[t - (pQ + q - 1)T_s\big]. \tag{P.11}$$

Using the Fourier transform pair

$$h_1\big[t - (pQ + q - 1)T_s\big] \stackrel{\mathcal{F}}{\longleftrightarrow} H_1(f) e^{-j2\pi f (pQ+q-1)T_s}, \tag{P.12}$$

we derive the Fourier transform of $\tilde{\gamma}_{m,i}^{(k)}(t)$ as

$$\begin{aligned} \tilde{\Gamma}_{m,i}^{(k)}(f) &\triangleq \mathcal{F}\left\{\tilde{\gamma}_{m,i}^{(k)}(t)\right\} \\ &= A_1 a_{p,q,m}^{(k)} \, C_{p,q,m,i}^{(k)} \, H_1(f) e^{-j2\pi f (pQ+q-1)T_s}. \end{aligned} \tag{P.13}$$

The energy of $\tilde{\gamma}_{m,i}^{(k)}(t)$ is given by

$$\begin{aligned} E_{m,i} &= \frac{1}{2} \int_{-\infty}^{\infty} \left|\tilde{\Gamma}_{m,i}^{(k)}(f)\right|^2 df \\ &= \frac{A_1^2}{2} \left(C_{p,q,m,i}^{(k)}\right)^2 \left(a_{p,q,m}^{(k)}\right)^2 \int_{-\infty}^{\infty} |H_1(f)|^2 \, df \\ &= \frac{A_1^2}{2}, \end{aligned} \tag{P.14}$$

where we used the fact that $\int_{-\infty}^{\infty} |H_1(f)|^2 \, df = 1$ (see (2.20)). Therefore, the energy of the m-th symbol (i.e., energy-per-bit) is equal to

$$E_{b1} = \sum_{i=1}^{R_1} E_{m,i} = \frac{R_1 A_1^2}{2}. \tag{P.15}$$

P.3 MC-DS-CDMA: MIMO System

From (3.30), the complex lowpass equivalent signal transmitted at sub-carrier $f_{m,i}$ from the u-th antenna is given by

$$\tilde{s}^{(k)}_{u,m,i}(t) = \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} \sum_{n=1}^{N} A_{2,q}\, b^{(k)}_{p,q,u,m}\, C^{(k)}_{p,q,n}\, h_2\big\{t - [(pQ+q-1)N+n-1]T_c\big\}. \tag{P.16}$$

Following the same steps as in the SISO case, the signal from the u-th antenna due to the i-th repetition of the m-th symbol, along with its Fourier transform, are given by

$$\tilde{\gamma}^{(k)}_{u,m,i}(t) = A_{2,q} \sum_{n=1}^{N} b^{(k)}_{p,q,u,m}\, C^{(k)}_{p,q,n}\, h_2\big\{t - [(pQ+q-1)N+n-1]T_c\big\} \tag{P.17}$$

and

$$\begin{aligned}\tilde{\Gamma}^{(k)}_{u,m,i}(f) &\triangleq \mathcal{F}\left\{\tilde{\gamma}^{(k)}_{u,m,i}(t)\right\} \\ &= A_{2,q} \sum_{n=1}^{N} b^{(k)}_{p,q,u,m}\, C^{(k)}_{p,q,n}\, H_2(f) e^{-j2\pi f[(pQ+q-1)N+n-1]T_c},\end{aligned} \tag{P.18}$$

respectively. The energy of $\tilde{\gamma}^{(k)}_{u,m,i}(t)$ is obtained as

$$\begin{aligned}E_{u,m,i} &= \frac{1}{2}\int_{-\infty}^{\infty} \left|\tilde{\Gamma}^{(k)}_{u,m,i}(f)\right|^2 df \\ &= \frac{A^2_{2,q}}{2} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \left(b^{(k)}_{p,q,u,m}\right)^2 C^{(k)}_{p,q,n_1} C^{(k)}_{p,q,n_2} \int_{-\infty}^{\infty} |H_2(f)|^2 e^{j2\pi f(n_2-n_1)T_c}\, df \\ &= \frac{NA^2_{2,q}}{2}.\end{aligned} \tag{P.19}$$

In the above result, we assume q corresponds to a time slot where a signal is actually transmitted. During the estimation phase, we have $A_{2,q} = A_{2e}$ (see (3.31)). Since pilot symbols are transmitted on only one of the two antennas during each time slot, the energy of the m-th symbol (i.e., energy-per-bit) is derived as

$$E_{b2} = \sum_{i=1}^{R_2} E_{u,m,i} = \frac{NR_2A^2_{2e}}{2}. \tag{P.20}$$

On the other hand, during the data phase, we have $A_{2,q} = A_{2d}$. Since data symbols are transmitted on both antennas during each time slot, we obtain

$$E_{b2} = \sum_{u=1}^{2} \sum_{i=1}^{R_2} E_{u,m,i} = N R_2 A_{2d}^2. \tag{P.21}$$

P.4 MC-CDMA: MIMO System

From (3.21), the complex lowpass equivalent signal transmitted at sub-carrier $f_{m,i}$ is given by

$$\tilde{s}_{u,m,i}^{(k)}(t) = \sum_{p=-\infty}^{\infty} \sum_{q=1}^{Q} A_{1,q}\, b_{p,q,u,m}^{(k)}\, C_{p,q,m,i}^{(k)}\, h_1\big[t - (pQ + q - 1)T_s\big]. \tag{P.22}$$

Following the same steps as in the SISO case, the signal from the u-th antenna due to the i-th repetition of the m-th symbol, along with its Fourier transform, are given by

$$\tilde{\gamma}_{u,m,i}^{(k)}(t) \triangleq A_{1,q}\, b_{p,q,u,m}^{(k)}\, C_{p,q,m,i}^{(k)}\, h_1\big[t - (pQ + q - 1)T_s\big] \tag{P.23}$$

and

$$\begin{aligned} \tilde{\Gamma}_{u,m,i}^{(k)}(f) &\triangleq \mathcal{F}\left\{\tilde{\gamma}_{u,m,i}^{(k)}(t)\right\} \\ &= A_{1,q}\, b_{p,q,u,m}^{(k)}\, C_{p,q,m,i}^{(k)}\, H_1(f) e^{-j2\pi f (pQ+q-1)T_s}, \end{aligned} \tag{P.24}$$

respectively. The energy of $\tilde{\gamma}_{u,m,i}^{(k)}(t)$ is derived as

$$\begin{aligned} E_{u,m,i} &= \frac{1}{2} \int_{-\infty}^{\infty} \left|\tilde{\Gamma}_{u,m,i}^{(k)}(f)\right|^2 df \\ &= \frac{A_{1,q}^2}{2}. \end{aligned} \tag{P.25}$$

For the estimation phase, we use $A_{1,q} = A_{1e}$ (see (3.22)) to get

$$E_{b1} = \sum_{i=1}^{R_1} E_{u,m,i} = \frac{R_1 A_{1e}^2}{2}. \tag{P.26}$$

On the other hand, we use $A_{1,q} = A_{1d}$ for the data phase to get

$$E_{b1} = \sum_{u=1}^{2} \sum_{i=1}^{R_1} E_{u,m,i} = R_1 A_{1d}^2. \tag{P.27}$$

Appendix Q

Second Order Moments: MC-DS-CDMA (SISO–Scenario #2)

In this appendix, we derive the second moments μ_{WW}, μ_{YY}, and μ_{WY} for the MC-DS-CDMA (SISO) system in Scenario #2.

Q.1 Derivation of μ_{YY}

From (2.45), we have

$$Y_{m,i,q} = A_2\sqrt{N}\, g_{m,i}^{(1)} + I_{Y_{m,i,q}}^{(P)} + I_{Y_{m,i,q}}^{(D)} + N_{Y_{m,i,q}}, \quad q = Q_P + 1, \ldots, Q, \tag{Q.1}$$

where the expressions for $I^{(P)}_{Y_{m,i,q}}$, $I^{(D)}_{Y_{m,i,q}}$, and $N_{Y_{m,i,q}}$ are identical to the ones for $I^{(P)}_{W_{m,i,q}}$, $I^{(D)}_{W_{m,i,q}}$, and $N_{W_{m,i,q}}$ in (2.46)–(2.48):

$$\begin{aligned} I^{(P)}_{Y_{m,i,q}} &= \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q_P} \sum_{n'=1}^{N} C^{(k)}_{p',q',n'}\, g^{(k)}_{m,i} \\ &\quad \cdot x_2\big\{t - [(p'Q+q'-1)N+n'-1]T_c - \tau^{(k)}\big\}\Big|_{t=[(pQ+q-1)N+n-1]T_c}, \end{aligned} \tag{Q.2}$$

$$\begin{aligned} I^{(D)}_{Y_{m,i,q}} &= \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=Q_P+1}^{Q} \sum_{n'=1}^{N} a^{(k)}_{p',q',m}\, C^{(k)}_{p',q',n'}\, g^{(k)}_{m,i} \\ &\quad \cdot x_2\left\{t - [(p'Q+q'-1)N+n'-1]T_c - \tau^{(k)}\right\}\Big|_{t=[(pQ+q-1)N+n-1]T_c}, \end{aligned} \tag{Q.3}$$

$$N_{Y_{m,i,q}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \cdot \left[\tilde{n}_W(t) * \tilde{h}_{m,i,2}(t)\right]\Big|_{t=[(pQ+q-1)N+n-1]T_c}. \tag{Q.4}$$

We assume the channel gains, time delays, data sequences, and spreading sequences of the interfering users are mutually independent. These random quantities are also independent of the AWGN. We model the data and spreading sequences of the interfering users as independent random binary sequences of ±1's. As a result, we have

$$E\big[C^{(k)}_{p,q,n}\big] = 0, \tag{Q.5}$$

$$E\big[a^{(k)}_{p,q,m}\big] = 0, \tag{Q.6}$$

as well as the relations

$$E\big[C^{(k_1)}_{p_1,q_1,n_1} C^{(k_2)}_{p_2,q_2,n_2}\big] = \delta_{k_1,k_2}\delta_{p_1,p_2}\delta_{q_1,q_2}\delta_{n_1,n_2}, \tag{Q.7}$$

$$E\big[a^{(k_1)}_{p_1,q_1,m} a^{(k_2)}_{p_2,q_2,m}\big] = \delta_{k_1,k_2}\delta_{p_1,p_2}\delta_{q_1,q_2}. \tag{Q.8}$$

Since the fade amplitudes are Rayleigh distributed with unit second moment, we have

$$\frac{1}{2}E\big[|g^{(k)}_{m,i}|^2\big] = \frac{1}{2}E\big[(\alpha^{(k)}_{m,i})^2\big] = \frac{1}{2}. \tag{Q.9}$$

The channels gains for different users are also independent:

$$\frac{1}{2}E\big[g^{(k_1)}_{m,i}\big(g^{(k_2)}_{m,i}\big)^*\big] = 0, \quad k_1 \neq k_2. \tag{Q.10}$$

Combining the results of (Q.9) and (Q.10), we can write

$$\frac{1}{2}E\big[g_{m,i}^{(k_1)}\big(g_{m,i}^{(k_2)}\big)^*\big] = \frac{1}{2}\delta_{k_1,k_2}. \tag{Q.11}$$

The quantity μ_{YY} is defined in (G.6) as

$$\mu_{YY} = \frac{1}{2}E[|Y_{m,i,q}|^2] = \frac{1}{2}E[Y_{m,i,q}Y_{m,i,q}^*]. \tag{Q.12}$$

We substitute (Q.1) into (Q.12) and multiply out the terms to obtain

$$\mu_{YY} = \frac{1}{2}E\big[A_2^2 N \cdot \big|g_{m,i}^{(1)}\big|^2 + \big|I_{Y_{m,i,q}}^{(P)}\big|^2 + \big|I_{Y_{m,i,q}}^{(D)}\big|^2 + \big|N_{Y_{m,i,q}}\big|^2\big], \tag{Q.13}$$

where the expected values of all of the cross-terms are easily shown to equal zero.

Q.1.1 First Term in (Q.13)

Using (Q.9), we have

$$\frac{1}{2}E\big[A_2^2 N \cdot \big|g_{m,i}^{(1)}\big|^2\big] = \frac{A_2^2 N}{2}. \tag{Q.14}$$

Q.1.2 Second Term in (Q.13)

Define

$$\begin{aligned} I_{y_{m,i}}^{(P)}(t) = A_2 \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q_P} \sum_{n'=1}^{N} C_{p',q',n'}^{(k)}\, g_{m,i}^{(k)} \\ \cdot x_2\big\{t - [(p'Q + q' - 1)N + n' - 1]T_c - \tau^{(k)}\big\}, \end{aligned} \tag{Q.15}$$

such that $I_{Y_{m,i,q}}^{(P)}$ can be re-written as

$$I_{Y_{m,i,q}}^{(P)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C_{p,q,n}^{(1)} I_{y_{m,i}}^{(P)}\big\{[(pQ + q - 1)N + n - 1]T_c\big\}, \quad q = Q_P + 1, \ldots, Q. \tag{Q.16}$$

$I_{y_{m,i}}^{(P)}(t)$ represents the MAI due to the pilot symbols of the interfering users at the output of the matched filter (before sampling) at frequency $f_{m,i}$. We use (Q.16) to write

$$\frac{1}{2}E\big[\big|I_{Y_{m,i,q}}^{(P)}\big|^2\big] = \frac{1}{2}E\big[I_{Y_{m,i,q}}^{(P)}\big(I_{Y_{m,i,q}}^{(P)}\big)^*\big] = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)} \cdot R_{I_{y_{m,i}}^{(P)}}[(n_2-n_1)T_c], \tag{Q.17}$$

where

$$R_{I_{y_{m,i}}^{(P)}}(\tau) = \frac{1}{2}E\Big\{I_{y_{m,i}}^{(P)}(t)\big[I_{y_{m,i}}^{(P)}(t+\tau)\big]^*\Big\}. \tag{Q.18}$$

Substituting (Q.15) into (Q.18), we get

$$\begin{aligned} R_{I_{y_{m,i}}^{(P)}}(\tau) = A_2^2 \sum_{k_1=2}^{K}\sum_{k_2=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{p_2'=-\infty}^{\infty}\sum_{q_1'=1}^{Q_P}\sum_{q_2'=1}^{Q_P}\sum_{n_1'=1}^{N}\sum_{n_2'=1}^{N} E\big[C_{p_1',q_1',n_1'}^{(k_1)} C_{p_2',q_2',n_2'}^{(k_2)}\big] \\ \cdot \frac{1}{2}E\big[g_{m,i}^{(k_1)}\big(g_{m,i}^{(k_2)}\big)^*\big] \cdot E\Big[x_2\big\{t - [(p_1'Q+q_1'-1)N+n_1'-1]T_c - \tau^{(k_1)}\big\} \\ x_2\big\{t+\tau - [(p_2'Q+q_2'-1)N+n_2'-1]T_c - \tau^{(k_2)}\big\}\Big]. \end{aligned} \tag{Q.19}$$

Using the results of (Q.7) and (Q.11) in (Q.19), we obtain

$$\begin{aligned} R_{I_{y_{m,i}}^{(P)}}(\tau) = \frac{A_2^2}{2}\sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=1}^{Q_P}\sum_{n_1'=1}^{N} E\Big[x_2\big\{t - [(p_1'Q+q_1'-1)N+n_1'-1]T_c - \tau^{(k_1)}\big\} \\ x_2\big\{t+\tau - [(p_1'Q+q_1'-1)N+n_1'-1]T_c - \tau^{(k_1)}\big\}\Big], \end{aligned} \tag{Q.20}$$

which, when substituted into (Q.17), yields

$$\begin{aligned} \frac{1}{2}E\big[\big|I_{Y_{m,i,q}}^{(P)}\big|^2\big] = \frac{A_2^2}{2N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)} \sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=1}^{Q_P}\sum_{n_1'=1}^{N} \\ \cdot E\Big[x_2\big\{t - [(p_1'Q+q_1'-1)N+n_1'-1]T_c - \tau^{(k_1)}\big\} \\ x_2\big\{t+\tau - [(p_1'Q+q_1'-1)N+n_1'-1]T_c - \tau^{(k_1)}\big\}\Big]\Big|_{\tau=(n_2-n_1)T_c}. \end{aligned} \tag{Q.21}$$

Q.1.3 Third Term in (Q.13)

We can re-write $I_{Y_{m,i,q}}^{(D)}$ in (Q.3) as

$$I_{Y_{m,i,q}}^{(D)} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C_{p,q,n}^{(1)} I_{y_{m,i}}^{(D)}\big\{[(pQ+q-1)N+n-1]T_c\big\}, \quad q = Q_P+1,\ldots,Q, \tag{Q.22}$$

where

$$I_{y_{m,i}}^{(D)}(t) = A_2 \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=Q_P+1}^{Q} \sum_{n'=1}^{N} a_{p',q',m}^{(k)} C_{p',q',n'}^{(k)} g_{m,i}^{(k)} \cdot x_2\big\{t - [(p'Q + q' - 1)N + n' - 1]T_c - \tau^{(k)}\big\} \tag{Q.23}$$

represents the MAI due to the data symbols of the interfering users at the output of the matched filter (before sampling) at frequency $f_{m,i}$. Thus, we have

$$\begin{aligned} \frac{1}{2}E\big[\big|I_{Y_{m,i,q}}^{(D)}\big|^2\big] &= \frac{1}{2}E\big[I_{Y_{m,i,q}}^{(D)}\big(I_{Y_{m,i,q}}^{(D)}\big)^*\big] \\ &= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)} \cdot R_{I_{y_{m,i}}^{(D)}}[(n_2 - n_1)T_c], \end{aligned} \tag{Q.24}$$

where

$$R_{I_{y_{m,i}^{(D)}}}(\tau) = \frac{1}{2}E\Big\{I_{y_{m,i}}^{(D)}(t)\big[I_{y_{m,i}}^{(D)}(t+\tau)\big]^*\Big\}. \tag{Q.25}$$

We substitute (Q.23) into (Q.25) to get

$$\begin{aligned} R_{I_{y_{m,i}}^{(D)}}(\tau) = A_2^2 &\sum_{k_1=2}^{K}\sum_{k_2=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{p_2'=-\infty}^{\infty}\sum_{q_1'=Q_P+1}^{Q}\sum_{q_2'=Q_P+1}^{Q}\sum_{n_1'=1}^{N}\sum_{n_2'=1}^{N} E\big[a_{p_1',q_1',m}^{(k_1)} a_{p_2',q_2',m}^{(k_2)}\big] \\ &\cdot E\big[C_{p_1',q_1',n_1'}^{(k_1)} C_{p_2',q_2',n_2'}^{(k_2)}\big] \cdot \frac{1}{2}E\big[g_{m,i}^{(k_1)}\big(g_{m,i}^{(k_2)}\big)^*\big] \\ &\cdot E\Big[x_2\big\{t - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\} \\ &\quad x_2\big\{t + \tau - [(p_2'Q + q_2' - 1)N + n_2' - 1]T_c - \tau^{(k_2)}\big\}\Big]. \end{aligned} \tag{Q.26}$$

Applying the results of (Q.7), (Q.8), and (Q.11) to (Q.26), we obtain

$$\begin{aligned} R_{I_{y_{m,i}}^{(D)}}(\tau) = \frac{A_2^2}{2} &\sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=Q_P+1}^{Q}\sum_{n_1'=1}^{N} E\Big[x_2\big\{t - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\} \\ &x_2\big\{t + \tau - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\}\Big]. \end{aligned} \tag{Q.27}$$

Finally, we substitute this result back into (Q.24) to get

$$\begin{aligned}\frac{1}{2}E\big[\big|I^{(D)}_{Y_{m,i,q}}\big|^2\big] = \frac{A_2^2}{2N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1}C^{(1)}_{p,q,n_2}\sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=Q_P+1}^{Q}\sum_{n_1'=1}^{N}\\ \cdot E\Big[x_2\big\{t-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-\tau^{(k_1)}\big\}\\ x_2\big\{t+\tau-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\tau=(n_2-n_1)T_c}.\end{aligned} \tag{Q.28}$$

Q.1.4 Fourth Term in (Q.13)

Let

$$N_{y_{m,i}}(t) = \tilde{n}_W(t) * \tilde{h}_{m,i,2}(t) \tag{Q.29}$$

denote the AWGN at the output of the matched filter (before sampling) at frequency $f_{m,i}$. This allows us to re-write $N_{Y_{m,i,q}}$ as

$$N_{Y_{m,i,q}} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C^{(1)}_{p,q,n}N_{y_{m,i}}\big\{[(pQ+q-1)N+n-1]T_c\big\}, \quad q = Q_P+1,\ldots,Q. \tag{Q.30}$$

As a result, we have

$$\begin{aligned}\frac{1}{2}E\big[\big|N_{Y_{m,i,q}}\big|^2\big] &= \frac{1}{2}E\big[N_{Y_{m,i,q}}N^*_{Y_{m,i,q}}\big]\\ &= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1}C^{(1)}_{p,q,n_2}\cdot R_{N_{y_{m,i}}}[(n_2-n_1)T_c],\end{aligned} \tag{Q.31}$$

where

$$R_{N_{y_{m,i}}}(\tau) = \frac{1}{2}E\Big\{N_{y_{m,i}}(t)\big[N_{y_{m,i}}(t+\tau)\big]^*\Big\}. \tag{Q.32}$$

We substitute (Q.29) into (Q.32) and use

$$\frac{1}{2}E\big[\tilde{n}_W(\tau_1)\tilde{n}^*_W(\tau_2)\big] = N_0\,\delta(\tau_2-\tau_1) \tag{Q.33}$$

to obtain

$$\begin{aligned}R_{N_{y_{m,i}}}(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}E\big[\tilde{n}_W(\tau_1)\tilde{n}^*_W(\tau_2)\big]\tilde{h}_{m,i,2}(t-\tau_1)\tilde{h}^*_{m,i,2}(t+\tau-\tau_2)\,d\tau_2\,d\tau_1\\ &= N_0\int_{-\infty}^{\infty}\tilde{h}_{m,i,2}(t-\tau_1)\tilde{h}^*_{m,i,2}(t+\tau-\tau_1)\,d\tau_1.\end{aligned} \tag{Q.34}$$

From (2.19), we have

$$\tilde{h}_{m,i,2}(t) = h_2^*(-t) = h_2(-t), \tag{Q.35}$$

where the last equality follows because $h_2(t)$ is real (i.e., $H_2(f)$ is symmetric in the frequency domain). As a result, the integral in (Q.34) becomes

$$\begin{aligned} R_{N_{y_{m,i}}}(\tau) &= N_0 \int_{-\infty}^{\infty} h_2(-t+\tau_1)h_2(-t-\tau+\tau_1)\, d\tau_1 \\ &= N_0 \int_{-\infty}^{\infty} h_2(u)h_2[-(\tau-u)]\, du \\ &= N_0\, x_2(\tau), \end{aligned} \tag{Q.36}$$

since

$$x_2(t) = h_2(t) * h_2^*(-t) \Leftrightarrow |H_2(f)|^2 = X_2(f). \tag{Q.37}$$

Substituting (Q.36) into (Q.31), and using the fact that (see (2.25))

$$x_2(nT_c) = \delta_{n,0} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases} \tag{Q.38}$$

we finally have

$$\begin{aligned} \frac{1}{2}E\big[\big|N_{Y_{m,i,q}}\big|^2\big] &= \frac{N_0}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)} x_2[(n_2-n_1)T_c] \\ &= \frac{N_0}{N}\sum_{n_1=1}^{N}\big[C_{p,q,n_1}^{(1)}\big]^2 \\ &= N_0. \end{aligned} \tag{Q.39}$$

Q.1.5 Combining the Results

Given (Q.13), we add together the results of (Q.14), (Q.21), (Q.28), and (Q.39) to get

$$\mu_{YY} = \frac{A_2^2 N}{2} + \frac{A_2^2}{2N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)} \sum_{k_1=2}^{K} \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \sum_{n_1'=1}^{N}$$
$$\cdot E\Big[x_2\big\{t - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\}$$
$$x_2\big\{t + \tau - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\}\Big]\Big|_{\tau=(n_2-n_1)T_c} \tag{Q.40}$$
$$+ N_0.$$

Since $\tau^{(k_1)}$ is a uniform random variable over $[0, T_c)$, we have

$$\sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \sum_{n_1'=1}^{N} E\Big[x_2\big\{t - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\}$$
$$x_2\big\{t + \tau - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\}\Big]$$
$$= \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \sum_{n_1'=1}^{N} \frac{1}{T_c} \int_0^{T_c} x_2\big\{t - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\}$$
$$\cdot x_2\big\{t + \tau - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\}\, d\tau^{(k_1)}$$
$$= \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \sum_{n_1'=1}^{N} \frac{1}{T_c} \int_{-t+[(p_1'Q+q_1'-1)N+n_1'-1]T_c}^{-t+[(p_1'Q+q_1'-1)N+n_1']T_c} x_2(-u)x_2(\tau - u)\, du$$
$$= \frac{1}{T_c} \int_{-\infty}^{\infty} x_2(-u)x_2(\tau - u)\, du$$
$$= \frac{1}{T_c} [x_2(\tau) * x_2(-\tau)]$$
$$= \frac{1}{T_c} \int_{-\infty}^{\infty} |X_2(f)|^2 e^{j2\pi f \tau}\, df, \tag{Q.41}$$

where the last step follows from the fact that

$$x_2(t) * x_2^*(-t) \Leftrightarrow |X_2(f)|^2. \tag{Q.42}$$

Given that $X_2(f)$ is an even function (see (2.25)), we evaluate the inverse Fourier transform of $|X_2(f)|^2$ as

$$\begin{aligned}\mathcal{F}^{-1}\left\{|X_2(f)|^2\right\} &= \int_{-\infty}^{\infty} |X_2(f)|^2 e^{j2\pi f\tau}\, df \\ &= 2\int_{0}^{\infty} |X_2(f)|^2 \cos(2\pi f\tau)\, df \\ &= 2T_c^2 \int_{0}^{1/2T_c} \cos(2\pi f\tau)\, df \\ &= T_c \frac{\sin(\pi\tau/T_c)}{\pi\tau/T_c}. \end{aligned} \tag{Q.43}$$

Thus, we have

$$\int_{-\infty}^{\infty} |X_2(f)|^2 e^{j2\pi f n T_c}\, df = \begin{cases} T_c, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{Q.44}$$

Applying (Q.41) and (Q.44) to (Q.40), we obtain

$$\begin{aligned}\mu_{YY} &= \frac{A_2^2 N}{2} + \frac{A_2^2}{2NT_c} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)} \sum_{k_1=2}^{K} \int_{-\infty}^{\infty} |X_2(f)|^2 e^{j2\pi f\tau}\, df \Big|_{\tau=(n_2-n_1)T_c} + N_0 \\ &= \frac{A_2^2 N}{2} + \frac{A_2^2 (K-1)}{2NT_c} \cdot \sum_{n_1=1}^{N} \left[C_{p,q,n_1}^{(1)}\right]^2 \int_{-\infty}^{\infty} |X_2(f)|^2\, df \\ &\quad + \frac{A_2^2 (K-1)}{2NT_c} \sum_{n_1=1}^{N} \sum_{\substack{n_2=1 \\ n_2\neq n_1}}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)} \int_{-\infty}^{\infty} |X_2(f)|^2 e^{j2\pi f(n_2-n_1)T_c}\, df + N_0 \\ &= \frac{A_2^2 N}{2} + \frac{A_2^2 (K-1)}{2NT_c} \cdot NT_c + N_0 \\ &= \frac{A_2^2 N}{2} + \frac{A_2^2 (K-1)}{2} + N_0. \end{aligned} \tag{Q.45}$$

Finally, we use the expression

$$E_{b2} = \frac{NR_2 A_2^2}{2} \tag{Q.46}$$

from (P.9) to re-write (Q.45) as

$$\mu_{YY} = \frac{E_{b2}}{R_2}\left[1 + c_2\left(\frac{E_{b2}}{N_0}\right)\right], \tag{Q.47}$$

where

$$c_2\left(\frac{E_{b2}}{N_0}\right) = \frac{K-1}{N} + \frac{R_2}{E_{b2}/N_0}. \tag{Q.48}$$

Q.2 Derivation of μ_{WY}

From (2.44) and (2.50), we have

$$\hat{W}_{m,i} = \frac{1}{A_2\sqrt{N}}\left(\frac{1}{Q_P}\sum_{q=1}^{Q_P} W_{m,i,q}\right), \tag{Q.49}$$

where

$$W_{m,i,q} = A_2\sqrt{N}\,g_{m,i}^{(1)} + I_{W_{m,i,q}}^{(P)} + I_{W_{m,i,q}}^{(D)} + N_{W_{m,i,q}}, \quad q = 1,\ldots,Q_P. \tag{Q.50}$$

As we mentioned earlier, the expressions for $I_{Y_{m,i,q}}^{(P)}$, $I_{Y_{m,i,q}}^{(D)}$, and $N_{Y_{m,i,q}}$ are identical to the ones for $I_{W_{m,i,q}}^{(P)}$, $I_{W_{m,i,q}}^{(D)}$, and $N_{W_{m,i,q}}$.

The quantity μ_{WY} is defined in (G.7) as

$$\mu_{WY} = \frac{1}{2}E[\hat{W}_{m,i}Y_{m,i,q}^*]. \tag{Q.51}$$

Substituting (Q.49) into (Q.51), we obtain

$$\mu_{WY} = \frac{1}{A_2\sqrt{N}}\frac{1}{Q_P}\sum_{q'=1}^{Q_P}\frac{1}{2}E[W_{m,i,q'}Y_{m,i,q}^*]. \tag{Q.52}$$

We use (Q.1) and (Q.50) to write

$$\begin{aligned}\frac{1}{2}E[W_{m,i,q'}Y_{m,i,q}^*] = \frac{1}{2}E\big[A_2^2N\cdot\big|g_{m,i}^{(1)}\big|^2 + I_{W_{m,i,q'}}^{(P)}\big(I_{Y_{m,i,q}}^{(P)}\big)^* + I_{W_{m,i,q'}}^{(D)}\big(I_{Y_{m,i,q}}^{(D)}\big)^*\\ + N_{W_{m,i,q'}}\big(N_{Y_{m,i,q}}\big)^*\big],\end{aligned} \tag{Q.53}$$

where, again, the expected values of the cross terms are equal to zero.

Q.2.1 First Term in (Q.53)

Using (Q.9), we have

$$\frac{1}{2}E\big[A_2^2N\cdot\big|g_{m,i}^{(1)}\big|^2\big] = \frac{A_2^2N}{2}. \tag{Q.54}$$

Q.2.2 Second Term in (Q.53)

We express $I^{(P)}_{W_{m,i,q}}$ as

$$I^{(P)}_{W_{m,i,q}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} I^{(P)}_{y_{m,i}} \big\{[(pQ+q-1)N+n-1]T_c\big\}, \quad q = 1, \ldots, Q_P, \tag{Q.55}$$

where $I^{(P)}_{y_{m,i}}(t)$ was defined in (Q.15). Using (Q.16) and (Q.55), we can write

$$\frac{1}{2} E\big[I^{(P)}_{W_{m,i,q'}} \big(I^{(P)}_{Y_{m,i,q}}\big)^*\big] = \frac{1}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1} C^{(1)}_{p,q,n_2} \cdot R_{I^{(P)}_{y_{m,i}}} \big\{[(q-q')N + (n_2 - n_1)]T_c\big\}. \tag{Q.56}$$

Applying the expression for $R_{I^{(P)}_{y_{m,i}}}(\tau)$ in (Q.20) to (Q.56), we have

$$\begin{aligned} &\frac{1}{2} E\big[I^{(P)}_{W_{m,i,q'}} \big(I^{(P)}_{Y_{m,i,q}}\big)^*\big] \\ &\quad = \frac{A_2^2}{2N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1} C^{(1)}_{p,q,n_2} \sum_{k_1=2}^{K} \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q_P} \sum_{n_1'=1}^{N} \\ &\qquad \cdot E\Big[x_2\big\{t - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\} \\ &\qquad\qquad x_2\big\{t + \tau - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\}\Big]\Big|_{\tau = [(q-q')N + (n_2-n_1)]T_c}. \end{aligned} \tag{Q.57}$$

Q.2.3 Third Term in (Q.53)

Given the expression for $I^{(D)}_{y_{m,i}}(t)$ in (Q.23), we express $I^{(D)}_{W_{m,i,q}}$ as

$$I^{(D)}_{W_{m,i,q}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} I^{(D)}_{y_{m,i}} \big\{[(pQ+q-1)N+n-1]T_c\big\}, \quad q = 1, \ldots, Q_P. \tag{Q.58}$$

Using (Q.22) and (Q.58), we can write the third term in (Q.53) as

$$\frac{1}{2} E\big[I^{(D)}_{W_{m,i,q'}} \big(I^{(D)}_{Y_{m,i,q}}\big)^*\big] = \frac{1}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1} C^{(1)}_{p,q,n_2} \cdot R_{I^{(D)}_{y_{m,i}}} \big\{[(q-q')N + (n_2 - n_1)]T_c\big\}. \tag{Q.59}$$

We then apply the expression for $R_{I^{(D)}_{y_{m,i}}}(\tau)$ in (Q.27) to (Q.59) to obtain

$$\begin{aligned}
&\frac{1}{2}E\big[I^{(D)}_{W_{m,i,q'}}\big(I^{(D)}_{Y_{m,i,q}}\big)^*\big] \\
&\quad = \frac{A_2^2}{2N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1}C^{(1)}_{p,q,n_2}\sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=Q_P+1}^{Q}\sum_{n_1'=1}^{N} \\
&\qquad \cdot E\Big[x_2\big\{t-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-\tau^{(k_1)}\big\} \\
&\qquad\qquad x_2\big\{t+\tau-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\tau=[(q-q')N+(n_2-n_1)]T_c}.
\end{aligned} \tag{Q.60}$$

Q.2.4 Fourth Term in (Q.53)

With $N_{y_{m,i}}(t)$ given by (Q.29), we can write

$$N_{W_{m,i,q}} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C^{(1)}_{p,q,n}N_{y_{m,i}}\big\{[(pQ+q-1)N+n-1]T_c\big\}, \quad q=1,\ldots,Q_P. \tag{Q.61}$$

Using (Q.30) and (Q.61), we have

$$\frac{1}{2}E\big[N_{W_{m,i,q'}}N^*_{Y_{m,i,q}}\big] = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1}C^{(1)}_{p,q,n_2}\cdot R_{N_{y_{m,i}}}\big\{[(q-q')N+(n_2-n_1)]T_c\big\}. \tag{Q.62}$$

By applying the expressions for $R_{N_{y_{m,i}}}(\tau)$ and $x_2(nT_c)$ in (Q.36) and (Q.38), respectively, we get

$$\begin{aligned}
\frac{1}{2}E\big[N_{W_{m,i,q'}}N^*_{Y_{m,i,q}}\big] &= \frac{N_0}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1}C^{(1)}_{p,q,n_2}\cdot x_2\big\{[(q-q')N+(n_2-n_1)]T_c\big\} \\
&= 0,
\end{aligned} \tag{Q.63}$$

where the last equality follows because $q \neq q'$.

Q.2.5 Combining the Results

Substituting the results of (Q.54), (Q.57), (Q.60), and (Q.63) into (Q.53), and using (Q.41), we get

$$
\begin{aligned}
&\frac{1}{2}E[W_{m,i,q'}Y^*_{m,i,q}] \\
&\quad = \frac{A_2^2 N}{2} + \frac{A_2^2}{2N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=1}^{Q}\sum_{n_1'=1}^{N} \\
&\qquad\qquad \cdot E\Big[x_2\big\{t-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-\tau^{(k_1)}\big\} \\
&\qquad\qquad\quad x_2\big\{t+\tau-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\tau=[(q-q')N+(n_2-n_1)]T_c} \\
&\quad = \frac{A_2^2 N}{2} + \frac{A_2^2}{2NT_c}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \sum_{k_1=2}^{K}\int_{-\infty}^{\infty}|X_2(f)|^2 e^{j2\pi f\tau}\,df\Big|_{\tau=[(q-q')N+(n_2-n_1)]T_c}.
\end{aligned}
\tag{Q.64}
$$

The second term in the above equation is equal to zero since $q \neq q'$ (see (Q.44)). Thus, we have

$$\frac{1}{2}E[W_{m,i,q'}Y^*_{m,i,q}] = \frac{A_2^2 N}{2}. \tag{Q.65}$$

Finally, we substitute (Q.65) into (Q.52) and use (Q.46) to get

$$\mu_{WY} = \frac{A_2\sqrt{N}}{2} = \frac{1}{2}\sqrt{\frac{2E_{b2}}{R_2}}. \tag{Q.66}$$

Q.3 Derivation of μ_{WW}

From (G.5), the quantity μ_{WW} is defined as

$$\mu_{WW} = \frac{1}{2}E[|\hat{W}_{m,i}|^2] = \frac{1}{2}E[\hat{W}_{m,i}\hat{W}^*_{m,i}]. \tag{Q.67}$$

Substituting (Q.49) into (Q.67), we get

$$\mu_{WW} = \frac{1}{A_2^2 N}\frac{1}{Q_P^2}\sum_{q_1=1}^{Q_P}\sum_{q_2=1}^{Q_P}\frac{1}{2}E[W_{m,i,q_1}W^*_{m,i,q_2}]$$

$$= \frac{1}{A_2^2 N} \frac{1}{Q_P^2} \sum_{q_1=1}^{Q_P} \frac{1}{2} E[|W_{m,i,q_1}|^2] + \frac{1}{A_2^2 N} \frac{1}{Q_P^2} \sum_{q_1=1}^{Q_P} \sum_{\substack{q_2=1 \\ q_2 \neq q_1}}^{Q_P} \frac{1}{2} E[W_{m,i,q_1} W_{m,i,q_2}^*]$$

$$= \frac{1}{A_2^2 N} \frac{1}{Q_P^2} Q_P \cdot \frac{1}{2} E[|W_{m,i,q}|^2] + \frac{1}{A_2^2 N} \frac{1}{Q_P^2} Q_P (Q_P - 1) \cdot \frac{1}{2} E[W_{m,i,q_1} W_{m,i,q_2}^*]$$

$$= \frac{1}{A_2^2 N} \frac{1}{Q_P} \cdot \frac{1}{2} E[|W_{m,i,q}|^2] + \frac{1}{A_2^2 N} \frac{1}{Q_P} (Q_P - 1) \cdot \frac{1}{2} E[W_{m,i,q_1} W_{m,i,q_2}^*]. \quad \text{(Q.68)}$$

In (Q.68), we have

$$\frac{1}{2} E[|W_{m,i,q}|^2] = \frac{1}{2} E[|Y_{m,i,q'}|^2], \quad \text{(Q.69)}$$

and

$$\frac{1}{2} E[W_{m,i,q_1} W_{m,i,q_2}^*] = \frac{1}{2} E[W_{m,i,q} Y_{m,i,q'}^*], \quad q_1 \neq q_2, \quad \text{(Q.70)}$$

since $W_{m,i,q}$ and $Y_{m,i,q'}$ correspond to the same demodulator output, but are associated with different time slots. We apply both (Q.69) and (Q.70) to (Q.68):

$$\mu_{WW} = \frac{1}{A_2^2 N} \frac{1}{Q_P} \cdot \frac{1}{2} E[|Y_{m,i,q}|^2] + \frac{1}{A_2^2 N} \frac{1}{Q_P} (Q_P - 1) \cdot \frac{1}{2} E[W_{m,i,q_1} Y_{m,i,q_2}^*]$$

$$= \frac{1}{A_2^2 N} \frac{1}{Q_P} \cdot \mu_{YY} + \frac{1}{A_2^2 N} \frac{1}{Q_P} (Q_P - 1) \cdot \frac{1}{2} E[W_{m,i,q_1} Y_{m,i,q_2}^*]. \quad \text{(Q.71)}$$

Substituting (Q.45) and (Q.65) into this expression, we obtain

$$\mu_{WW} = \frac{1}{A_2^2 N} \frac{1}{Q_P} \cdot \left[\frac{A_2^2 N}{2} + \frac{A_2^2 (K-1)}{2} + N_0 \right] + \frac{1}{A_2^2 N} \frac{1}{Q_P} (Q_P - 1) \cdot \frac{A_2^2 N}{2}$$

$$= \frac{1}{A_2^2 N} \frac{1}{Q_P} \left[Q_P \frac{A_2^2 N}{2} + \frac{A_2^2 (K-1)}{2} + N_0 \right]$$

$$= \frac{1}{2} + \frac{1}{2} \frac{1}{Q_P} \left(\frac{K-1}{N} + \frac{2 N_0}{A_2^2 N} \right). \quad \text{(Q.72)}$$

Finally, by applying the expression for E_{b2} in (Q.46), we get

$$\mu_{WW} = \frac{1}{2} \left[1 + \frac{1}{Q_P} c_2\!\left(\frac{E_{b2}}{N_0} \right) \right], \quad \text{(Q.73)}$$

where $c_2\!\left(\frac{E_{b2}}{N_0}\right)$ is given in (Q.48).

Appendix R

Second Order Moments: MC-DS-CDMA (SISO–Scenario #1)

In this appendix, we derive the second moments $\mu_{W_lW_l}$, $\mu_{Y_lY_l}$, and $\mu_{W_lY_l}$ for the MC-DS-CDMA (SISO) system in Scenario #1.

R.1 Derivation of $\mu_{Y_lY_l}$

From (2.37), we have

$$Y_{m,i,q,l} = A_2\sqrt{N}\, g^{(1)}_{m,i,l} + K_{Y_{m,i,q,l}} + I^{(P)}_{Y_{m,i,q,l}} + I^{(D)}_{Y_{m,i,q,l}} + N_{Y_{m,i,q,l}}, \quad q = Q_P+1,\dots,Q, \quad \text{(R.1)}$$

where the expressions for $K_{Y_{m,i,q,l}}$, $I^{(P)}_{Y_{m,i,q,l}}$, $I^{(D)}_{Y_{m,i,q,l}}$, and $N_{Y_{m,i,q,l}}$ are identical to the ones for $K_{W_{m,i,q,l}}$, $I^{(P)}_{W_{m,i,q,l}}$, $I^{(D)}_{W_{m,i,q,l}}$, and $N_{W_{m,i,q,l}}$ in (2.38)–(2.41):

$$K_{Y_{m,i,q,l}} = \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \sum_{l'=1, l'\neq l}^{L} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q} \sum_{n'=1}^{N} a^{(1)}_{p',q',m} \, C^{(1)}_{p',q',n'} \, g^{(1)}_{m,i,l'} \cdot x_2\big\{t - [(p'Q + q' - 1)N + n' - 1]T_c - (l'-1)T_c\big\}\Big|_{t=\{[(pQ+q-1)N+n-1]+(l-1)\}T_c}, \tag{R.2}$$

$$I^{(P)}_{Y_{m,i,q,l}} = \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \sum_{k=2}^{K} \sum_{l'=1}^{L} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q_P} \sum_{n'=1}^{N} C^{(k)}_{p',q',n'} \, g^{(k)}_{m,i,l'} \cdot x_2\big\{t - [(p'Q + q' - 1)N + n' - 1]T_c - (l'-1)T_c - \tau^{(k)}\big\}\Big|_{t=\{[(pQ+q-1)N+n-1]+(l-1)\}T_c}, \tag{R.3}$$

$$I^{(D)}_{Y_{m,i,q,l}} = \frac{A_2}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \sum_{k=2}^{K} \sum_{l'=1}^{L} \sum_{p'=-\infty}^{\infty} \sum_{q'=Q_P+1}^{Q} \sum_{n'=1}^{N} a^{(k)}_{p',q',m} \, C^{(k)}_{p',q',n'} \, g^{(k)}_{m,i,l'} \cdot x_2\left\{t - [(p'Q + q' - 1)N + n' - 1]T_c - (l'-1)T_c - \tau^{(k)}\right\}\Big|_{t=\{[(pQ+q-1)N+n-1]+(l-1)\}T_c}, \tag{R.4}$$

$$N_{Y_{m,i,q,l}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} \cdot \left[\tilde{n}_W(t) * \tilde{h}_{m,i,2}(t)\right]\Big|_{t=\{[(pQ+q-1)N+n-1]+(l-1)\}T_c}. \tag{R.5}$$

As we argued in Chapter 2, the self-interference is negligible, so we can re-write $Y_{m,i,q,l}$ as

$$Y_{m,i,q,l} = A_2\sqrt{N}\, g^{(1)}_{m,i,l} + I^{(P)}_{Y_{m,i,q,l}} + I^{(D)}_{Y_{m,i,q,l}} + N_{Y_{m,i,q,l}}, \quad q = Q_P + 1, \ldots, Q. \tag{R.6}$$

We make the same assumptions regarding the channel gains, time delays, data sequences, and spreading sequences of the interfering users as in Appendix Q. Note, however, that since the fading in each sub-band is frequency-selective, the variance of the channel gain is now a function of the path index l:

$$\frac{1}{2}E\big[\big|g^{(k)}_{m,i,l}\big|^2\big] = \frac{1}{2}E\big[\big(\alpha^{(k)}_{m,i,l}\big)^2\big] = \frac{1}{2}\Omega^{(k)}_{m,i,l}. \tag{R.7}$$

We assume the channels gains for different users are independent. For a given user, the channel gains along different paths are also independent. Therefore, we have

$$\frac{1}{2}E\big[g_{m,i,l_1}^{(k_1)}\big(g_{m,i,l_2}^{(k_2)}\big)^*\big] = \frac{1}{2}\delta_{k_1,k_2}\delta_{l_1,l_2}\Omega_{m,i,l_1}^{(k_1)}. \tag{R.8}$$

From (H.8), $\mu_{Y_lY_l}$ is defined as

$$\mu_{Y_lY_l} = \frac{1}{2}E[|Y_{m,i,q,l}|^2] = \frac{1}{2}E[Y_{m,i,q,l}Y_{m,i,q,l}^*]. \tag{R.9}$$

We use (R.6) in (R.9) to obtain

$$\mu_{Y_lY_l} = \frac{1}{2}E\big[A_2^2N\cdot\big|g_{m,i,l}^{(1)}\big|^2 + \big|I_{Y_{m,i,q,l}}^{(P)}\big|^2 + \big|I_{Y_{m,i,q,l}}^{(D)}\big|^2 + \big|N_{Y_{m,i,q,l}}\big|^2\big], \tag{R.10}$$

where the expected values of all of the cross-terms are equal to zero.

R.1.1 First Term in (R.10)

Using (R.7), we have

$$\frac{1}{2}E\big[A_2^2N\cdot\big|g_{m,i,l}^{(1)}\big|^2\big] = \frac{A_2^2N}{2}\Omega_{m,i,l}^{(1)}. \tag{R.11}$$

R.1.2 Second Term in (R.10)

Define

$$\begin{aligned} I_{y_{m,i}}^{(P)}(t) = A_2\sum_{k=2}^{K}\sum_{l'=1}^{L}\sum_{p'=-\infty}^{\infty}\sum_{q'=1}^{Q_P}\sum_{n'=1}^{N} C_{p',q',n'}^{(k)}\, g_{m,i,l'}^{(k)} \\ \cdot\, x_2\big\{t-[(p'Q+q'-1)N+n'-1]T_c-(l'-1)T_c-\tau^{(k)}\big\}. \end{aligned} \tag{R.12}$$

This allows $I_{Y_{m,i,q,l}}^{(P)}$ to be re-written as

$$I_{Y_{m,i,q,l}}^{(P)} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C_{p,q,n}^{(1)} I_{y_{m,i}}^{(P)}\big(\{[(pQ+q-1)N+n-1]+(l-1)\}T_c\big), \quad q = Q_P+1,\ldots,Q. \tag{R.13}$$

We then have

$$\frac{1}{2}E\big[\big|I^{(P)}_{Y_{m,i,q,l}}\big|^2\big] = \frac{1}{2}E\big[I^{(P)}_{Y_{m,i,q,l}}\big(I^{(P)}_{Y_{m,i,q,l}}\big)^*\big]$$
$$= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1}C^{(1)}_{p,q,n_2}\cdot R_{I^{(P)}_{y_{m,i}}}[(n_2-n_1)T_c], \tag{R.14}$$

where

$$R_{I^{(P)}_{y_{m,i}}}(\tau) = \frac{1}{2}E\Big\{I^{(P)}_{y_{m,i}}(t)\big[I^{(P)}_{y_{m,i}}(t+\tau)\big]^*\Big\}. \tag{R.15}$$

We substitute (R.12) into (R.15) and apply (Q.7) and (R.8) to get

$$R_{I^{(P)}_{y_{m,i}}}(\tau) = \frac{A_2^2}{2}\sum_{k_1=2}^{K}\sum_{l_1'=1}^{L}\Omega^{(k_1)}_{m,i,l_1'}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=1}^{Q_P}\sum_{n_1'=1}^{N}$$
$$\cdot E\Big[x_2\big\{t-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-(l_1'-1)T_c-\tau^{(k_1)}\big\}$$
$$x_2\big\{t+\tau-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-(l_1'-1)T_c-\tau^{(k_1)}\big\}\Big]. \tag{R.16}$$

Finally, we use this result in (R.14) to obtain

$$\frac{1}{2}E\big[\big|I^{(P)}_{Y_{m,i,q,l}}\big|^2\big]$$
$$= \frac{A_2^2}{2N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}C^{(1)}_{p,q,n_1}C^{(1)}_{p,q,n_2}\sum_{k_1=2}^{K}\sum_{l_1'=1}^{L}\Omega^{(k_1)}_{m,i,l_1'}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=1}^{Q_P}\sum_{n_1'=1}^{N}$$
$$\cdot E\Big[x_2\big\{t-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-(l_1'-1)T_c-\tau^{(k_1)}\big\}$$
$$x_2\big\{t+\tau-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-(l_1'-1)T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\tau=(n_2-n_1)T_c}. \tag{R.17}$$

R.1.3 Third Term in (R.10)

$I^{(D)}_{Y_{m,i,q,l}}$ in (R.4) can be re-expressed as

$$I^{(D)}_{Y_{m,i,q,l}} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}C^{(1)}_{p,q,n}I^{(D)}_{y_{m,i}}\big\{[(pQ+q-1)N+n-1]T_c+(l-1)T_c\big\}, \quad q = Q_P+1,\ldots,Q, \tag{R.18}$$

where

$$I^{(D)}_{y_{m,i}}(t) = A_2 \sum_{k=2}^{K}\sum_{l'=1}^{L}\sum_{p'=-\infty}^{\infty}\sum_{q'=Q_P+1}^{Q}\sum_{n'=1}^{N} a^{(k)}_{p',q',m} C^{(k)}_{p',q',n'}\, g^{(k)}_{m,i,l'} \cdot x_2\big\{t - [(p'Q+q'-1)N + n' - 1]T_c - (l'-1)T_c - \tau^{(k)}\big\}. \tag{R.19}$$

Using (R.18), we have

$$\begin{aligned}\frac{1}{2}E\big[\big|I^{(D)}_{Y_{m,i,q,l}}\big|^2\big] &= \frac{1}{2}E\big[I^{(D)}_{Y_{m,i,q,l}}\big(I^{(D)}_{Y_{m,i,q,l}}\big)^*\big] \\ &= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \cdot R_{I^{(D)}_{y_{m,i}}}[(n_2-n_1)T_c],\end{aligned} \tag{R.20}$$

where

$$R_{I^{(D)}_{y_{m,i}}}(\tau) = \frac{1}{2}E\Big\{I^{(D)}_{y_{m,i}}(t)\big[I^{(D)}_{y_{m,i}}(t+\tau)\big]^*\Big\}. \tag{R.21}$$

We substitute (R.19) into (R.21) and apply (Q.7), (Q.8), and (R.8) to obtain

$$\begin{aligned}R_{I^{(D)}_{y_{m,i}}}(\tau) = \frac{A_2^2}{2}&\sum_{k_1=2}^{K}\sum_{l'_1=1}^{L}\Omega^{(k_1)}_{m,i,l'_1}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=Q_P+1}^{Q}\sum_{n'_1=1}^{N} \\ &\cdot E\Big[x_2\big\{t - [(p'_1Q+q'_1-1)N + n'_1 - 1]T_c - (l'_1-1)T_c - \tau^{(k_1)}\big\} \\ &\quad x_2\big\{t + \tau - [(p'_1Q+q'_1-1)N + n'_1 - 1]T_c - (l'_1-1)T_c - \tau^{(k_1)}\big\}\Big].\end{aligned} \tag{R.22}$$

Finally, we substitute this result back into (R.20) to get

$$\begin{aligned}&\frac{1}{2}E\big[\big|I^{(D)}_{Y_{m,i,q}}\big|^2\big] \\ &\quad= \frac{A_2^2}{2N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2}\sum_{k_1=2}^{K}\sum_{l'_1=1}^{L}\Omega^{(k_1)}_{m,i,l'_1}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=Q_P+1}^{Q}\sum_{n'_1=1}^{N} \\ &\quad\cdot E\Big[x_2\big\{t - [(p'_1Q+q'_1-1)N + n'_1 - 1]T_c - (l'_1-1)T_c - \tau^{(k_1)}\big\} \\ &\qquad x_2\big\{t + \tau - [(p'_1Q+q'_1-1)N + n'_1 - 1]T_c - (l'_1-1)T_c - \tau^{(k_1)}\big\}\Big]\Big|_{\tau=(n_2-n_1)T_c}.\end{aligned} \tag{R.23}$$

R.1.4 Fourth Term in (R.10)

By defining

$$N_{y_{m,i}}(t) = \tilde{n}_W(t) * \tilde{h}_{m,i,2}(t), \tag{R.24}$$

we can re-express $N_{Y_{m,i,q,l}}$ as

$$N_{Y_{m,i,q,l}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} N_{y_{m,i}}\big(\{[(pQ{+}q{-}1)N{+}n{-}1]{+}(l{-}1)\}T_c\big), \quad q = Q_P{+}1, \ldots, Q. \tag{R.25}$$

We use (R.25) to write

$$\begin{aligned}\frac{1}{2}E\big[\big|N_{Y_{m,i,q,l}}\big|^2\big] &= \frac{1}{2}E\big[N_{Y_{m,i,q,l}}\big(N_{Y_{m,i,q,l}}\big)^*\big] \\ &= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \cdot R_{N_{y_{m,i}}}[(n_2 - n_1)T_c],\end{aligned} \tag{R.26}$$

where

$$R_{N_{y_{m,i}}}(\tau) = \frac{1}{2}E\Big\{N_{y_{m,i}}(t)\big[N_{y_{m,i}}(t+\tau)\big]^*\Big\}. \tag{R.27}$$

Since (R.24) is identical to $N_{y_{m,i}}(t)$ in (Q.29), we follow the same steps as in Appendix Q.1.4 to get

$$R_{N_{y_{m,i}}}(\tau) = N_0\, x_2(\tau). \tag{R.28}$$

Using this result in (R.26) and applying (Q.38), we obtain

$$\begin{aligned}\frac{1}{2}E\big[\big|N_{Y_{m,i,q}}\big|^2\big] &= \frac{N_0}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} x_2[(n_2 - n_1)T_c] \\ &= N_0.\end{aligned} \tag{R.29}$$

R.1.5 Combining the Results

We substitute the results of (R.11), (R.17), (R.23), and (R.29) into (R.10) to get

$$\begin{aligned}\mu_{Y_l Y_l} = {} & \frac{A_2^2 N}{2}\Omega^{(1)}_{m,i,l} \\ & + \frac{A_2^2}{2N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \sum_{k_1=2}^{K}\sum_{l'_1=1}^{L}\Omega^{(k_1)}_{m,i,l'_1} \sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=1}^{Q}\sum_{n'_1=1}^{N} \\ & \quad \cdot E\Big[x_2\big\{t - [(p'_1 Q + q'_1 - 1)N + n'_1 - 1]T_c - (l'_1 - 1)T_c - \tau^{(k_1)}\big\} \\ & \qquad x_2\big\{t + \tau - [(p'_1 Q + q'_1 - 1)N + n'_1 - 1]T_c - (l'_1 - 1)T_c - \tau^{(k_1)}\big\}\Big]\Big|_{\tau=(n_2-n_1)T_c} \\ & + N_0.\end{aligned} \tag{R.30}$$

Given that $\tau^{(k_1)}$ is uniform over $[0, T_c)$, we have

$$\sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \sum_{n_1'=1}^{N} E\Big[x_2\big\{t - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - (l_1' - 1)T_c - \tau^{(k_1)}\big\}$$
$$x_2\big\{t + \tau - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - (l_1' - 1)T_c - \tau^{(k_1)}\big\}\Big]$$
$$= \frac{1}{T_c} \int_{-\infty}^{\infty} |X_2(f)|^2 e^{j2\pi f \tau}\, df, \tag{R.31}$$

which is proved by following the same steps as in Appendix Q.1.5. We substitute this result into (R.30) and use (Q.44), along with the fact that $\sum_{l=1}^{L} \Omega_{m,i,l}^{(k)} = 1$, to obtain

$$\mu_{Y_l Y_l} = \frac{A_2^2 N}{2} \Omega_{m,i,l}^{(1)} + \frac{A_2^2 (K-1)}{2} + N_0, \tag{R.32}$$
$$= \frac{E_{b2}}{R_2} \left[\Omega_{m,i,l}^{(1)} + c_2\!\left(\frac{E_{b2}}{N_0}\right) \right], \tag{R.33}$$

where

$$c_2\!\left(\frac{E_{b2}}{N_0}\right) = \frac{K-1}{N} + \frac{R_2}{E_{b2}/N_0}, \tag{R.34}$$

and

$$E_{b2} = \frac{N R_2 A_2^2}{2} \tag{R.35}$$

from (P.9). Note that for the case of a single path (i.e., $L = 1$), the substitution $\Omega_{m,i,l}^{(1)} = 1$ in (R.33) yields the same expression as μ_{YY} in Scenario #2 (see (Q.47)).

R.2 Derivation of $\mu_{W_l Y_l}$

From (2.36) and (2.42), we have

$$\hat{W}_{m,i,l} = \frac{1}{A_2 \sqrt{N}} \left(\frac{1}{Q_P} \sum_{q=1}^{Q_P} W_{m,i,q,l} \right), \tag{R.36}$$

where

$$W_{m,i,q,l} = A_2 \sqrt{N}\, g_{m,i,l}^{(1)} + K_{W_{m,i,q,l}} + I_{W_{m,i,q,l}}^{(P)} + I_{W_{m,i,q,l}}^{(D)} + N_{W_{m,i,q,l}}, \quad q = 1, \ldots, Q_P. \tag{R.37}$$

Once again, we ignore the self-interference, so $W_{m,i,q,l}$ reduces to

$$W_{m,i,q,l} = A_2\sqrt{N}\, g^{(1)}_{m,i,l} + I^{(P)}_{W_{m,i,q,l}} + I^{(D)}_{W_{m,i,q,l}} + N_{W_{m,i,q,l}}, \quad q = 1, \ldots, Q_P, \tag{R.38}$$

where the expressions for $I^{(P)}_{W_{m,i,q,l}}$, $I^{(D)}_{W_{m,i,q,l}}$, and $N_{W_{m,i,q,l}}$ are given by (2.39)–(2.41).

The quantity $\mu_{W_l Y_l}$ is defined in (H.9) as

$$\mu_{W_l Y_l} = \frac{1}{2} E[\hat{W}_{m,i,l} Y^*_{m,i,q,l}]. \tag{R.39}$$

Substituting (R.36) into (R.39), we obtain

$$\mu_{W_l Y_l} = \frac{1}{A_2\sqrt{N}} \frac{1}{Q_P} \sum_{q'=1}^{Q_P} \frac{1}{2} E[W_{m,i,q',l} Y^*_{m,i,q,l}]. \tag{R.40}$$

We use (R.6) and (R.38) to write

$$\begin{aligned} \frac{1}{2} E[W_{m,i,q',l} Y^*_{m,i,q,l}] = \frac{1}{2} E\big[A_2^2 N \cdot \big|g^{(1)}_{m,i,l}\big|^2 + I^{(P)}_{W_{m,i,q',l}} \big(I^{(P)}_{Y_{m,i,q,l}}\big)^* + I^{(D)}_{W_{m,i,q',l}} \big(I^{(D)}_{Y_{m,i,q,l}}\big)^* \\ + N_{W_{m,i,q',l}} \big(N_{Y_{m,i,q,l}}\big)^*\big], \end{aligned} \tag{R.41}$$

where, again, the expected values of the cross terms are equal to zero.

R.2.1 First Term in (R.41)

Using (R.7), we have

$$\frac{1}{2} E\big[A_2^2 N \cdot \big|g^{(1)}_{m,i,l}\big|^2\big] = \frac{A_2^2 N}{2} \Omega^{(1)}_{m,i,l}. \tag{R.42}$$

R.2.2 Second Term in (R.41)

With $I^{(P)}_{y_{m,i}}(t)$ defined in (R.12), we can express $I^{(P)}_{W_{m,i,q,l}}$ as

$$I^{(P)}_{W_{m,i,q,l}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} I^{(P)}_{y_{m,i}}\big(\{[(pQ+q-1)N+n-1]+(l-1)\}T_c\big), \quad q = 1, \ldots, Q_P. \tag{R.43}$$

Using (R.13) and (R.43), we can write

$$\frac{1}{2} E\big[I^{(P)}_{W_{m,i,q',l}} \big(I^{(P)}_{Y_{m,i,q,l}}\big)^*\big] = \frac{1}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1} C^{(1)}_{p,q,n_2} \cdot R_{I^{(P)}_{y_{m,i}}} \big\{[(q-q')N + (n_2-n_1)]T_c\big\}. \tag{R.44}$$

We apply to this result the expression for $R_{I^{(P)}_{y_{m,i}}}(\tau)$ from (R.16) to get

$$\begin{aligned}
&\frac{1}{2}E\big[I^{(P)}_{W_{m,i,q',l}}\big(I^{(P)}_{Y_{m,i,q,l}}\big)^*\big] \\
&= \frac{A_2^2}{2N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1}C^{(1)}_{p,q,n_2}\sum_{k_1=2}^{K}\sum_{l_1'=1}^{L}\Omega^{(k_1)}_{m,i,l_1'}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=1}^{Q_P}\sum_{n_1'=1}^{N} \\
&\quad \cdot E\Big[x_2\big\{t-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-(l_1'-1)T_c-\tau^{(k_1)}\big\} \\
&\qquad\quad x_2\big\{t+\tau-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-(l_1'-1)T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\tau=[(q-q')N+(n_2-n_1)]T_c}.
\end{aligned} \tag{R.45}$$

R.2.3 Third Term in (R.41)

We use the expression for $I^{(D)}_{y_{m,i}}(t)$ in (R.19) to write

$$I^{(D)}_{W_{m,i,q,l}} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C^{(1)}_{p,q,n}I^{(D)}_{y_{m,i}}\big(\{[(pQ+q-1)N+n-1]+(l-1)\}T_c\big), \quad q=1,\ldots,Q_P. \tag{R.46}$$

Given (R.18) and (R.46), we have

$$\frac{1}{2}E\big[I^{(D)}_{W_{m,i,q',l}}\big(I^{(D)}_{Y_{m,i,q,l}}\big)^*\big] = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1}C^{(1)}_{p,q,n_2}\cdot R_{I^{(D)}_{y_{m,i}}}\big\{[(q-q')N+(n_2-n_1)]T_c\big\}. \tag{R.47}$$

We then apply to (R.47) the expression for $R_{I^{(D)}_{y_{m,i}}}(\tau)$ in (R.22) to obtain

$$\begin{aligned}
&\frac{1}{2}E\big[I^{(D)}_{W_{m,i,q'}}\big(I^{(D)}_{Y_{m,i,q}}\big)^*\big] \\
&= \frac{A_2^2}{2N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1}C^{(1)}_{p,q,n_2}\sum_{k_1=2}^{K}\sum_{l_1'=1}^{L}\Omega^{(k_1)}_{m,i,l_1'}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=Q_P+1}^{Q}\sum_{n_1'=1}^{N} \\
&\quad \cdot E\Big[x_2\big\{t-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-(l_1'-1)T_c-\tau^{(k_1)}\big\} \\
&\qquad\quad x_2\big\{t+\tau-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-(l_1'-1)T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\tau=[(q-q')N+(n_2-n_1)]T_c}.
\end{aligned} \tag{R.48}$$

R.2.4 Fourth Term in (R.41)

We can write $N_{W_{m,i,q}}$ as

$$N_{W_{m,i,q}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C^{(1)}_{p,q,n} N_{y_{m,i}} \big\{ [(pQ + q - 1)N + n - 1]T_c \big\}, \quad q = 1, \ldots, Q_P, \tag{R.49}$$

where the expression for $N_{y_{m,i}}(t)$ is given in (R.24). Using (R.25) and (R.49), we have

$$\frac{1}{2} E\big[N_{W_{m,i,q'}} \big(N_{Y_{m,i,q}} \big)^* \big] = \frac{1}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1} C^{(1)}_{p,q,n_2} \cdot R_{N_{y_{m,i}}} \big\{ [(q-q')N + (n_2 - n_1)]T_c \big\}. \tag{R.50}$$

We apply the expression for $R_{N_{y_{m,i}}}(\tau)$ in (R.28) to get

$$\begin{aligned} \frac{1}{2} E\big[N_{W_{m,i,q'}} \big(N_{Y_{m,i,q}} \big)^* \big] &= \frac{N_0}{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C^{(1)}_{p,q',n_1} C^{(1)}_{p,q,n_2} \cdot x_2 \big\{ [(q - q')N + (n_2 - n_1)]T_c \big\} \\ &= 0, \end{aligned} \tag{R.51}$$

where the last equality follows from (Q.38) and the fact that $q \neq q'$.

R.2.5 Combining the Results

Substituting the results of (R.42), (R.45), (R.48), and (R.51) into (R.41), and using (R.31), we get

$$\begin{aligned} &\frac{1}{2} E[W_{m,i,q',l} Y^*_{m,i,q,l}] \\ &= \frac{A_2^2 N}{2} \Omega^{(1)}_{m,i,l} \\ &\quad + \frac{A_2^2}{2N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \sum_{k_1=2}^{K} \sum_{l_1'=1}^{L} \Omega^{(k_1)}_{m,i,l_1'} \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \sum_{n_1'=1}^{N} \\ &\quad\quad \cdot E\Big[x_2 \big\{ t - [(p_1' Q + q_1' - 1)N + n_1' - 1]T_c - (l_1' - 1)T_c - \tau^{(k_1)} \big\} \\ &\quad\quad\quad x_2 \big\{ t + \tau - [(p_1' Q + q_1' - 1)N + n_1' - 1]T_c - (l_1' - 1)T_c - \tau^{(k_1)} \big\} \Big] \Big|_{\tau = [(q-q')N + (n_2 - n_1)]T_c} \end{aligned}$$

$$= \frac{A_2^2 N}{2}\Omega_{m,i,l}^{(1)}$$
$$+ \frac{A_2^2}{2NT_c}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)} \sum_{k_1=2}^{K}\sum_{l_1'=1}^{L}\Omega_{m,i,l_1'}^{(k_1)}\int_{-\infty}^{\infty}|X_2(f)|^2 e^{j2\pi f\tau}\,df\Big|_{\tau=[(q-q')N+(n_2-n_1)]T_c}. \quad \text{(R.52)}$$

The second term in (R.52) is equal to zero since $q \neq q'$ (see (Q.44)). Thus, we have

$$\frac{1}{2}E[W_{m,i,q',l}Y_{m,i,q,l}^*] = \frac{A_2^2 N}{2}\Omega_{m,i,l}^{(1)}. \quad \text{(R.53)}$$

Finally, we substitute (R.53) into (R.40) and use (R.35) to get

$$\mu_{W_l Y_l} = \frac{A_2\sqrt{N}}{2}\Omega_{m,i,l}^{(1)} = \frac{1}{2}\sqrt{\frac{2E_{b2}}{R_2}}\Omega_{m,i,l}^{(1)}. \quad \text{(R.54)}$$

For the case of a single path ($L = 1$), such that $\Omega_{m,i,l}^{(1)} = 1$, (R.54) reduces to the same expression as μ_{WY} in Scenario #2 (see (Q.66)).

R.3 Derivation of $\mu_{W_l W_l}$

From (H.7), $\mu_{W_l W_l}$ is defined as

$$\mu_{W_l W_l} = \frac{1}{2}E[|\hat{W}_{m,i,l}|^2] = \frac{1}{2}E[\hat{W}_{m,i,l}\hat{W}_{m,i,l}^*]. \quad \text{(R.55)}$$

Substituting (R.36) into (R.55), and following the same steps as in the derivation for (Q.68), we get

$$\mu_{W_l W_l} = \frac{1}{A_2^2 N}\frac{1}{Q_P^2}\sum_{q_1=1}^{Q_P}\sum_{q_2=1}^{Q_P}\frac{1}{2}E[W_{m,i,q_{1,l}}W_{m,i,q_2,l}^*]$$
$$= \frac{1}{A_2^2 N}\frac{1}{Q_P}\cdot\frac{1}{2}E[|W_{m,i,q,l}|^2] + \frac{1}{A_2^2 N}\frac{1}{Q_P}(Q_P - 1)\cdot\frac{1}{2}E[W_{m,i,q_1,l}W_{m,i,q_2,l}^*]. \quad \text{(R.56)}$$

We note that

$$\frac{1}{2}E[|W_{m,i,q,l}|^2] = \frac{1}{2}E[|Y_{m,i,q,l}|^2] \quad \text{(R.57)}$$

and

$$\frac{1}{2}E[W_{m,i,q_1,l}W^*_{m,i,q_2,l}] = \frac{1}{2}E[W_{m,i,q,l}Y^*_{m,i,q',l}]. \tag{R.58}$$

and we use (R.32) and (R.53) to show that

$$\begin{aligned}
\mu_{W_lW_l} &= \frac{1}{A_2^2N}\frac{1}{Q_P}\cdot\left[\frac{A_2^2N}{2}\Omega^{(1)}_{m,i,l} + \frac{A_2^2(K-1)}{2} + N_0\right] + \frac{1}{A_2^2N}\frac{1}{Q_P}(Q_P-1)\cdot\frac{A_2^2N}{2}\Omega^{(1)}_{m,i,l} \\
&= \frac{1}{A_2^2N}\frac{1}{Q_P}\left[Q_P\frac{A_2^2N}{2}\Omega^{(1)}_{m,i,l} + \frac{A_2^2(K-1)}{2} + N_0\right] \\
&= \frac{1}{2}\Omega^{(1)}_{m,i,l} + \frac{1}{2}\frac{1}{Q_P}\left(\frac{K-1}{N} + \frac{2N_0}{A_2^2N}\right).
\end{aligned} \tag{R.59}$$

Finally, by applying the expression for E_{b2} in (R.35), we get

$$\mu_{W_lW_l} = \frac{1}{2}\left[\Omega^{(1)}_{m,i,l} + \frac{1}{Q_P}c_2\left(\frac{E_{b2}}{N_0}\right)\right], \tag{R.60}$$

where $c_2\left(\frac{E_{b2}}{N_0}\right)$ is given in (R.34). Again, when $L = 1$, such that $\Omega^{(1)}_{m,i,l} = 1$, $\mu_{W_lW_l}$ reduces to μ_{WW} in Scenario #2 (see (Q.73)).

Appendix S

Second Order Moments: MC-CDMA (SISO–Scenario #2)

We derive expressions for $\frac{1}{2}E[\hat{W}_{m,i_1}\hat{W}^*_{m,i_2}]$, $\frac{1}{2}E[Y_{m,i_1,q}Y^*_{m,i_2,q}]$, and $\frac{1}{2}E[\hat{W}_{m,i_1}Y^*_{m,i_2,q}]$ for the MC-CDMA (SISO) system in Scenario #2.

S.1 Derivation of $\frac{1}{2}E[Y_{m,i_1,q}Y^*_{m,i_2,q}]$

From (2.27), we have

$$Y_{m,i,q} = A_1\, g^{(1)}_{m,i} + I^{(P)}_{Y_{m,i,q}} + I^{(D)}_{Y_{m,i,q}} + N_{Y_{m,i,q}}, \quad q = Q_P + 1, \ldots, Q, \tag{S.1}$$

where the expressions for $I^{(P)}_{Y_{m,i,q}}$, $I^{(D)}_{Y_{m,i,q}}$, and $N_{Y_{m,i,q}}$ are identical to the ones for $I^{(P)}_{W_{m,i,q}}$, $I^{(D)}_{W_{m,i,q}}$, and $N_{W_{m,i,q}}$ in (2.28)–(2.30):

$$\begin{aligned} I^{(P)}_{Y_{m,i,q}} &= A_1\, C^{(1)}_{p,q,m,i} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q_P} C^{(k)}_{p',q',m,i}\, g^{(k)}_{m,i} \\ &\quad \cdot x_1 \left[t - (p'Q + q' - 1)T_s - \tau^{(k)}\right] \Big|_{t=[(pQ+q-1)T_s]}, \end{aligned} \tag{S.2}$$

$$I_{Y_{m,i,q}}^{(D)} = A_1\, C_{p,q,m,i}^{(1)} \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=Q_P+1}^{Q} a_{p',q',m}^{(k)}\, C_{p',q',m,i}^{(k)}\, g_{m,i}^{(k)} \cdot x_1 \left[t - (p'Q + q' - 1)T_s - \tau^{(k)}\right] \Big|_{t=[(pQ+q-1)T_s]}, \tag{S.3}$$

$$N_{Y_{m,i,q}} = C_{p,q,m,i}^{(1)} \cdot \left[\tilde{n}_W(t) * \tilde{h}_{m,i,1}(t)\right] \Big|_{t=[(pQ+q-1)T_s]}. \tag{S.4}$$

As in the derivations for the MC-DS-CDMA system in Scenario #2 (see Appendix Q), we assume

$$E\big[a_{p,q,m}^{(k)}\big] = 0, \tag{S.5}$$

$$E\big[a_{p_1,q_1,m}^{(k_1)} a_{p_2,q_2,m}^{(k_2)}\big] = \delta_{k_1,k_2}\delta_{p_1,p_2}\delta_{q_1,q_2}. \tag{S.6}$$

Once again, the spreading sequences of the interfering users are modeled as independent random binary sequences:

$$E\big[C_{p,q,m,i}^{(k)}\big] = 0, \tag{S.7}$$

$$E\big[C_{p_1,q_1,m,i_1}^{(k_1)} C_{p_2,q_2,m,i_2}^{(k_2)}\big] = \delta_{k_1,k_2}\delta_{p_1,p_2}\delta_{q_1,q_2}\delta_{i_1,i_2}. \tag{S.8}$$

We still have

$$\frac{1}{2}E\big[\big|g_{m,i}^{(k)}\big|^2\big] = \frac{1}{2}E\big[\big(\alpha_{m,i}^{(k)}\big)^2\big] = \frac{1}{2}, \tag{S.9}$$

and we still assume the channel gains between different users are independent:

$$\frac{1}{2}E\big[g_{m,i_1}^{(k_1)}\big(g_{m,i_2}^{(k_2)}\big)^*\big] = \frac{1}{2}E\big[g_{m,i_1}^{(k_1)}\big(g_{m,i_2}^{(k_1)}\big)^*\big]\delta_{k_1,k_2}. \tag{S.10}$$

Given the correlated block fading model, the channel gains from two different sub-bands are either *perfectly correlated* if the sub-bands are in the *same* frequency block, or are *independent* if the sub-bands are from *different* blocks. Thus, we have

$$\frac{1}{2}E\big[g_{m,i_1}^{(k_1)}\big(g_{m,i_2}^{(k_2)}\big)^*\big] = \begin{cases} \frac{1}{2}\delta_{k_1,k_2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks.} \end{cases} \tag{S.11}$$

Using these above assumptions, along with (S.1), we write

$$\frac{1}{2}E[Y_{m,i_1,q}Y^*_{m,i_2,q}] = \frac{1}{2}E\big[A_1^2 \cdot g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^* + I^{(P)}_{Y_{m,i_1,q}}\big(I^{(P)}_{Y_{m,i_2,q}}\big)^* + I^{(D)}_{Y_{m,i_1,q}}\big(I^{(D)}_{Y_{m,i_2,q}}\big)^* \\ + N^{(P)}_{Y_{m,i_1,q}}\big(N^{(P)}_{Y_{m,i_2,q}}\big)^*\big], \tag{S.12}$$

where the expected values of all of the cross-terms are easily shown to equal zero.

S.1.1 First Term in (S.12)

The first term in (S.12) is just

$$\frac{1}{2}E\big[A_1^2 \cdot g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big] = A_1^2 \cdot \frac{1}{2}E\big[g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big]. \tag{S.13}$$

S.1.2 Second Term in (S.12)

Define

$$I^{(P)}_{y_{m,i}}(t) = A_1 \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q_P} C^{(k)}_{p',q',m,i}\, g^{(k)}_{m,i} \cdot x_1\big[t - (p'Q + q' - 1)T_s - \tau^{(k)}\big], \tag{S.14}$$

such that $I^{(P)}_{Y_{m,i,q}}$ can be re-expressed as

$$I^{(P)}_{Y_{m,i,q}} = C^{(1)}_{p,q,m,i} I^{(P)}_{y_{m,i}}[(pQ + q - 1)T_s], \quad q = Q_P + 1, \ldots, Q. \tag{S.15}$$

This allows us to write

$$\frac{1}{2}E\big[I^{(P)}_{Y_{m,i_1,q}}\big(I^{(P)}_{Y_{m,i_2,q}}\big)^*\big] \\ = C^{(1)}_{p,q,m,i_1} C^{(1)}_{p,q,m,i_2} \cdot \frac{1}{2}E\Big\{I^{(P)}_{y_{m,i_1}}[(pQ + q - 1)T_s]\Big(I^{(P)}_{y_{m,i_2}}[(pQ + q - 1)T_s]\Big)^*\Big\}. \tag{S.16}$$

By using (S.14) and applying (S.8) and (S.10), we have

$$\frac{1}{2}E\Big\{I^{(P)}_{y_{m,i_1}}[(pQ + q - 1)T_s]\Big(I^{(P)}_{y_{m,i_2}}[(pQ + q - 1)T_s]\Big)^*\Big\} \\ = A_1^2 \sum_{k_1=2}^{K} \sum_{k_2=2}^{K} \sum_{p_1'=-\infty}^{\infty} \sum_{p_2'=-\infty}^{\infty} \sum_{q_1'=1}^{Q_P} \sum_{q_2'=1}^{Q_P} E\big[C^{(k_1)}_{p_1',q_1',m,i_1} C^{(k_2)}_{p_2',q_2',m,i_2}\big] \cdot \frac{1}{2}E\big[g^{(k_1)}_{m,i_1}\big(g^{(k_2)}_{m,i_2}\big)^*\big] \\ \cdot E\Big\{x_1\big[t - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big] x_1\big[t - (p_2'Q + q_2' - 1)T_s - \tau^{(k_2)}\big]\Big\}\Big|_{t=(pQ+q-1)T_s}$$

$$
\begin{aligned}
&= A_1^2 \sum_{k_1=2}^{K} \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q_P} \delta_{i_1,i_2} \cdot \frac{1}{2} E\big[g_{m,i_1}^{(k_1)} \big(g_{m,i_2}^{(k_1)}\big)^*\big] \\
&\quad \cdot E\Big\{x_1^2\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t=(pQ+q-1)T_s}.
\end{aligned} \tag{S.17}
$$

Note that for any function $f(i_1, i_2)$, we have

$$
f(i_1, i_2) \cdot \delta_{i_1,i_2} = f(i_1, i_1) \cdot \delta_{i_1,i_2}. \tag{S.18}
$$

We apply this relation and use the result in (S.9) to show that (S.17) reduces to

$$
\begin{aligned}
&\frac{1}{2} E\Big\{I_{y_{m,i_1}}^{(P)}[(pQ+q-1)T_s]\Big(I_{y_{m,i_2}}^{(P)}[(pQ+q-1)T_s]\Big)^*\Big\} \\
&\quad = \frac{A_1^2}{2}\, \delta_{i_1,i_2} \sum_{k_1=2}^{K} \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q_P} E\Big\{x_1^2\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t=(pQ+q-1)T_s}.
\end{aligned} \tag{S.19}
$$

Finally, we substitute (S.19) into (S.16) and apply (S.18) once again to get

$$
\begin{aligned}
&\frac{1}{2} E\big[I_{Y_{m,i_1,q}}^{(P)} \big(I_{Y_{m,i_2,q}}^{(P)}\big)^*\big] \\
&\quad = \Big(C_{p,q,m,i_1}^{(1)}\Big)^2 \frac{A_1^2}{2}\, \delta_{i_1,i_2} \sum_{k_1=2}^{K} \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q_P} E\Big\{x_1^2\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t=(pQ+q-1)T_s} \\
&\quad = \frac{A_1^2}{2}\, \delta_{i_1,i_2} \sum_{k_1=2}^{K} \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q_P} E\Big\{x_1^2\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t=(pQ+q-1)T_s}.
\end{aligned} \tag{S.20}
$$

S.1.3 Third Term in (S.12)

We can re-write $I_{Y_{m,i,q}}^{(D)}$ in (S.3) as

$$
I_{Y_{m,i,q}}^{(D)} = C_{p,q,m,i}^{(1)} I_{y_{m,i}}^{(D)}[(pQ+q-1)T_s], \quad q = Q_P+1, \ldots, Q, \tag{S.21}
$$

where

$$
I_{y_{m,i}}^{(D)}(t) = A_1 \sum_{k=2}^{K} \sum_{p'=-\infty}^{\infty} \sum_{q'=Q_P+1}^{Q} a_{p',q',m}^{(k)} C_{p',q',m,i}^{(k)}\, g_{m,i}^{(k)} \cdot x_1[t-(p'Q+q'-1)T_s-\tau^{(k)}]. \tag{S.22}
$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{2} E\big[I_{Y_{m,i_1,q}}^{(D)} \big(I_{Y_{m,i_2,q}}^{(D)}\big)^*\big] \\
&\quad = C_{p,q,m,i_1}^{(1)} C_{p,q,m,i_2}^{(1)} \cdot \frac{1}{2} E\Big\{I_{y_{m,i_1}}^{(D)}[(pQ+q-1)T_s]\Big(I_{y_{m,i_2}}^{(D)}[(pQ+q-1)T_s]\Big)^*\Big\}.
\end{aligned} \tag{S.23}
$$

We use (S.22) and apply (S.6) and (S.8)–(S.10) to get

$$\begin{aligned}
&\frac{1}{2}E\Big\{I^{(D)}_{y_{m,i_1}}[(pQ+q-1)T_s]\Big(I^{(D)}_{y_{m,i_2}}[(pQ+q-1)T_s]\Big)^*\Big\} \\
&\quad = A_1^2 \sum_{k_1=2}^{K}\sum_{k_2=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{p_2'=-\infty}^{\infty}\sum_{q_1'=Q_P+1}^{Q}\sum_{q_2'=Q_P+1}^{Q} E\big[a^{(k_1)}_{p_1',q_1',m}a^{(k_2)}_{p_2',q_2',m}\big] \\
&\qquad \cdot E\big[C^{(k_1)}_{p_1',q_1',m,i_1}C^{(k_2)}_{p_2',q_2',m,i_2}\big]\cdot\frac{1}{2}E\big[g^{(k_1)}_{m,i_1}\big(g^{(k_2)}_{m,i_2}\big)^*\big] \\
&\qquad \cdot E\Big\{x_1\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]x_1\big[t-(p_2'Q+q_2'-1)T_s-\tau^{(k_2)}\big]\Big\}\Big|_{t=(pQ+q-1)T_s} \\
&\quad = \frac{A_1^2}{2}\delta_{i_1,i_2}\sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=Q_P+1}^{Q} E\Big\{x_1^2\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t=(pQ+q-1)T_s}, \qquad \text{(S.24)}
\end{aligned}$$

which, when substituted into (S.23), yields

$$\begin{aligned}
&\frac{1}{2}E\big[I^{(D)}_{Y_{m,i_1,q}}\big(I^{(D)}_{Y_{m,i_2,q}}\big)^*\big] \\
&\quad = \frac{A_1^2}{2}\delta_{i_1,i_2}\sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=Q_P+1}^{Q} E\Big\{x_1^2\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t=(pQ+q-1)T_s}.
\end{aligned} \qquad \text{(S.25)}$$

S.1.4 Fourth Term in (S.12)

Define

$$N_{y_{m,i}}(t) = \tilde{n}_W(t) * \tilde{h}_{m,i,1}(t) \qquad \text{(S.26)}$$

as the AWGN at the output of the matched filter (before sampling) at frequency $f_{m,i}$. We can then re-write $N_{Y_{m,i,q}}$ in (S.4) as

$$N_{Y_{m,i,q}} = C^{(1)}_{p,q,m,i}N_{y_{m,i}}[(pQ+q-1)T_s], \quad q = Q_P+1,\ldots,Q, \qquad \text{(S.27)}$$

such that

$$\begin{aligned}
&\frac{1}{2}E\big[N_{Y_{m,i_1,q}}N^*_{Y_{m,i_2,q}}\big] \\
&\quad = C^{(1)}_{p,q,m,i_1}C^{(1)}_{p,q,m,i_2}\cdot\frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q-1)T_s]\Big(N^*_{y_{m,i_2}}[(pQ+q-1)T_s]\Big)\Big\}.
\end{aligned} \qquad \text{(S.28)}$$

For $i_1 \neq i_2$, $N_{y_{m,i_1}}(t)$ and $N_{y_{m,i_2}}(t)$ represent AWGN at the outputs of two non-overlapping bandpass matched filters centered at frequencies f_{m,i_1} and f_{m,i_2}, respectively. Hence, $N_{y_{m,i_1}}(t)$

and $N_{y_{m,i_2}}(t)$ are independent. Since $N_{y_{m,i}}(t)$ is zero-mean, this implies

$$\frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q-1)T_s]N^*_{y_{m,i_2}}[(pQ+q-1)T_s]\Big\} = 0, \quad i_1 \neq i_2. \tag{S.29}$$

For $i_1 = i_2$, we use (S.26) and apply (Q.33) to obtain

$$\begin{aligned}
&\frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q-1)T_s]N^*_{y_{m,i_1}}[(pQ+q-1)T_s]\Big\} \\
&\quad = \frac{1}{2}E\Big\{\big(\tilde{n}_W(t) * \tilde{h}_{m,i_1,1}(t)\big)\big(\tilde{n}^*_W(t) * \tilde{h}^*_{m,i_1,1}(t)\big)\Big\}\Big|_{t=(pQ+q-1)T_s} \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}E\big[\tilde{n}_W(\tau_1)\tilde{n}^*_W(\tau_2)\big]\tilde{h}_{m,i_1,1}(t-\tau_1)\tilde{h}^*_{m,i_1,1}(t-\tau_2)\,d\tau_2\,d\tau_1\Big|_{t=(pQ+q-1)T_s} \\
&\quad = N_0\int_{-\infty}^{\infty}\big|\tilde{h}_{m,i_1,1}(t-\tau_1)\big|^2\,d\tau_1\Big|_{t=(pQ+q-1)T_s}.
\end{aligned} \tag{S.30}$$

From (2.18), we have

$$\tilde{h}_{m,i,1}(t) = h^*_1(-t) = h_1(-t), \tag{S.31}$$

since $h_1(t)$ is real. As a result, (S.30) becomes

$$\begin{aligned}
&\frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q-1)T_s]N^*_{y_{m,i_1}}[(pQ+q-1)T_s]\Big\} \\
&\quad = N_0\int_{-\infty}^{\infty}\big|h_1(\tau_1)\big|^2\,d\tau_1 \\
&\quad = N_0\int_{-\infty}^{\infty}\big|H_1(f)\big|^2\,df \\
&\quad = N_0,
\end{aligned} \tag{S.32}$$

where the last equality follows from (2.20). Combining the results of (S.29) and (S.32), we have

$$\frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q-1)T_s]N^*_{y_{m,i_2}}[(pQ+q-1)T_s]\Big\} = N_0\,\delta_{i_1,i_2}. \tag{S.33}$$

Finally, we use (S.33) in (S.28) to get

$$\begin{aligned}
\frac{1}{2}E\big[N_{Y_{m,i_1,q}}N^*_{Y_{m,i_2,q}}\big] &= \Big(C^{(1)}_{p,q,m,i_1}\Big)^2 N_0\,\delta_{i_1,i_2} \\
&= N_0\,\delta_{i_1,i_2}.
\end{aligned} \tag{S.34}$$

S.1.5 Combining the Results

Given (S.12), we add together the results of (S.13), (S.20), (S.25), and (S.34) to get

$$
\begin{aligned}
\frac{1}{2}E[Y_{m,i_1,q}Y^*_{m,i_2,q}] = A_1^2 \cdot \frac{1}{2}E\big[g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big] \\
+ \frac{A_1^2}{2}\,\delta_{i_1,i_2} \sum_{k_1=2}^{K} \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \\
\cdot E\Big\{x_1^2\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t=(pQ+q-1)T_s} \\
+ N_0\,\delta_{i_1,i_2}.
\end{aligned}
\tag{S.35}
$$

Since $\tau^{(k_1)}$ is a uniform random variable over $[0,T_s)$, we have

$$
\begin{aligned}
&\sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} E\Big\{x_1^2\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\} \\
&= \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \frac{1}{T_s}\int_0^{T_s} x_1^2\big[t-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\,d\tau^{(k_1)} \\
&= \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \frac{1}{T_s}\int_{t-(p_1'Q+q_1')T_s}^{t-(p_1'Q+q_1'-1)T_s} x_1^2(u)\,du \\
&= \frac{1}{T_s}\int_{-\infty}^{\infty} x_1^2(u)\,du \\
&= \frac{1}{T_s}\int_{-\infty}^{\infty} |X_1(f)|^2\,df
\end{aligned}
\tag{S.36}
$$

By following the same steps as in the derivation for (Q.43), we can easily show that

$$
\mathcal{F}^{-1}\left\{|X_1(f)|^2\right\} = T_s\frac{\sin(\pi\tau/T_s)}{\pi\tau/T_s}, \tag{S.37}
$$

which leads to the relation

$$
\int_{-\infty}^{\infty} |X_1(f)|^2 e^{j2\pi f n T_s}\,df = \begin{cases} T_s, & n=0, \\ 0, & n\neq 0. \end{cases} \tag{S.38}
$$

Applying (S.36) and (S.38) to (S.35) and using (S.11), we obtain

$$
\frac{1}{2}E[Y_{m,i_1,q}Y^*_{m,i_2,q}]
$$

$$= A_1^2 \cdot \frac{1}{2} E\big[g_{m,i_1}^{(1)} \big(g_{m,i_2}^{(1)}\big)^*\big] + \frac{A_1^2}{2}(K-1)\,\delta_{i_1,i_2} + N_0\,\delta_{i_1,i_2} \tag{S.39}$$

$$= \begin{cases} \dfrac{A_1^2}{2} + \left(\dfrac{A_1^2(K-1)}{2} + N_0\right)\delta_{i_1,i_2}, & i_1 \text{ and } i_2 \text{ associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ associated with different blocks.} \end{cases} \tag{S.40}$$

Finally, we use the expression

$$E_{b1} = \frac{R_1 A_1^2}{2} \tag{S.41}$$

from (P.15) to re-write $\frac{1}{2}E[Y_{m,i_1,q} Y_{m,i_2,q}^*]$ as

$$\frac{1}{2}E[Y_{m,i_1,q} Y_{m,i_2,q}^*] = \begin{cases} \dfrac{E_{b1}}{R_1}\left[1 + c_1\!\left(\dfrac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right], & i_1 \text{ and } i_2 \text{ associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ associated with different blocks,} \end{cases} \tag{S.42}$$

where

$$c_1\!\left(\frac{E_{b1}}{N_0}\right) = (K-1) + \frac{R_1}{E_{b1}/N_0}. \tag{S.43}$$

S.2 Derivation of $\frac{1}{2}E[\hat{W}_{m,i_1} Y_{m,i_2,q}^*]$

From (2.26) and (2.34), we have

$$\hat{W}_{m,i} = \frac{1}{A_1}\left(\frac{1}{Q_P}\sum_{q=1}^{Q_P} W_{m,i,q}\right), \tag{S.44}$$

where

$$W_{m,i,q} = A_1\, g_{m,i}^{(1)} + I_{W_{m,i,q}}^{(P)} + I_{W_{m,i,q}}^{(D)} + N_{W_{m,i,q}}, \quad q = 1, \ldots, Q_P. \tag{S.45}$$

The expressions for $I_{W_{m,i,q}}^{(P)}$, $I_{W_{m,i,q}}^{(D)}$, and $N_{W_{m,i,q}}$ are given by (2.28)–(2.30).

We substitute the expression for $\hat{W}_{m,i}$ into $\frac{1}{2}E[\hat{W}_{m,i_1} Y_{m,i_2,q}^*]$ to write

$$\frac{1}{2}E[\hat{W}_{m,i_1} Y_{m,i_2,q}^*] = \frac{1}{A_1}\frac{1}{Q_P}\sum_{q'=1}^{Q_P}\frac{1}{2}E[W_{m,i_1,q'} Y_{m,i_2,q}^*]. \tag{S.46}$$

Using (S.1) and (S.45), we have

$$\begin{aligned}\frac{1}{2}E[W_{m,i_1,q'}Y^*_{m,i_2,q}] = \frac{1}{2}E\big[A_1^2 \cdot g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^* + I^{(P)}_{W_{m,i_1,q'}}\big(I^{(P)}_{Y_{m,i_2,q}}\big)^* + I^{(D)}_{W_{m,i_1,q'}}\big(I^{(D)}_{Y_{m,i_2,q}}\big)^* \\ + N^{(P)}_{W_{m,i_1,q'}}\big(N^{(P)}_{Y_{m,i_2,q}}\big)^*\big],\end{aligned} \tag{S.47}$$

where, again, the expected values of the cross terms are equal to zero.

S.2.1 First Term in (S.47)

Again, we have

$$\frac{1}{2}E\big[A_1^2 \cdot g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big] = A_1^2 \cdot \frac{1}{2}E\big[g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big]. \tag{S.48}$$

S.2.2 Second Term in (S.47)

We express $I^{(P)}_{W_{m,i,q}}$ as

$$I^{(P)}_{W_{m,i,q}} = C^{(1)}_{p,q,m,i} I^{(P)}_{y_{m,i}}[(pQ+q-1)T_s], \quad q = 1, \ldots, Q_P, \tag{S.49}$$

where $I^{(P)}_{y_{m,i}}(t)$ was defined in (S.14). Using (S.15) and (S.49), we can write

$$\begin{aligned}&\frac{1}{2}E\big[I^{(P)}_{W_{m,i_1,q'}}\big(I^{(P)}_{Y_{m,i_2,q}}\big)^*\big] \\ &= C^{(1)}_{p,q',m,i_1} C^{(1)}_{p,q,m,i_2} \cdot \frac{1}{2}E\Big\{I^{(P)}_{y_{m,i_1}}[(pQ+q'-1)T_s]\Big(I^{(P)}_{y_{m,i_2}}[(pQ+q-1)T_s]\Big)^*\Big\}.\end{aligned} \tag{S.50}$$

By using (S.14) and applying (S.8)–(S.10), we have

$$\begin{aligned}&\frac{1}{2}E\Big\{I^{(P)}_{y_{m,i_1}}[(pQ+q'-1)T_s]\Big(I^{(P)}_{y_{m,i_2}}[(pQ+q-1)T_s]\Big)^*\Big\} \\ &= A_1^2 \sum_{k_1=2}^{K}\sum_{k_2=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{p'_2=-\infty}^{\infty}\sum_{q'_1=1}^{Q_P}\sum_{q'_2=1}^{Q_P} E\big[C^{(k_1)}_{p'_1,q'_1,m,i_1} C^{(k_2)}_{p'_2,q'_2,m,i_2}\big] \cdot \frac{1}{2}E\big[g^{(k_1)}_{m,i_1}\big(g^{(k_2)}_{m,i_2}\big)^*\big] \\ &\quad \cdot E\Big\{x_1\big[t_1 - (p'_1Q+q'_1-1)T_s - \tau^{(k_1)}\big] \\ &\qquad x_1\big[t_2 - (p'_2Q+q'_2-1)T_s - \tau^{(k_2)}\big]\Big\}\Big|_{t_1=(pQ+q'-1)T_s,\, t_2=(pQ+q-1)T_s} \\ &= \frac{A_1^2}{2}\,\delta_{i_1,i_2}\sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=1}^{Q_P} E\Big\{x_1\big[t_1 - (p'_1Q+q'_1-1)T_s - \tau^{(k_1)}\big] \\ &\qquad x_1\big[t_2 - (p'_1Q+q'_1-1)T_s - \tau^{(k_1)}\big]\Big\}\Big|_{t_1=(pQ+q'-1)T_s,\, t_2=(pQ+q-1)T_s}.\end{aligned} \tag{S.51}$$

We then substitute this result into (S.50) and apply (S.18) to get

$$\frac{1}{2}E\big[I^{(P)}_{W_{m,i_1,q'}}\big(I^{(P)}_{Y_{m,i_2,q}}\big)^*\big]$$
$$= \frac{A_1^2}{2}\,\delta_{i_1,i_2}C^{(1)}_{p,q',m,i_1}C^{(1)}_{p,q,m,i_1}\sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=1}^{Q_P}E\Big\{x_1\big[t_1-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big] \quad \text{(S.52)}$$
$$x_1\big[t_2-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t_1=(pQ+q'-1)T_s,\ t_2=(pQ+q-1)T_s}.$$

S.2.3 Third Term in (S.47)

Given the expression for $I^{(D)}_{y_{m,i}}(t)$ in (S.22), we express $I^{(D)}_{W_{m,i,q}}$ as

$$I^{(D)}_{W_{m,i,q}} = C^{(1)}_{p,q,m,i}I^{(D)}_{y_{m,i}}[(pQ+q-1)T_s],\quad q=1,\ldots,Q_P. \quad \text{(S.53)}$$

We substitute (S.21) and (S.53) into $\frac{1}{2}E\big[I^{(D)}_{W_{m,i_1,q'}}\big(I^{(D)}_{Y_{m,i_2,q}}\big)^*\big]$ to get

$$\frac{1}{2}E\big[I^{(D)}_{W_{m,i_1,q'}}\big(I^{(D)}_{Y_{m,i_2,q}}\big)^*\big]$$
$$= C^{(1)}_{p,q',m,i_1}C^{(1)}_{p,q,m,i_2}\cdot\frac{1}{2}E\Big\{I^{(D)}_{y_{m,i_1}}[(pQ+q'-1)T_s]\Big(I^{(D)}_{y_{m,i_2}}[(pQ+q-1)T_s]\Big)^*\Big\}. \quad \text{(S.54)}$$

We then use (S.22) and apply (S.6) and (S.8)–(S.10) to obtain

$$\frac{1}{2}E\Big\{I^{(D)}_{y_{m,i_1}}[(pQ+q'-1)T_s]\Big(I^{(D)}_{y_{m,i_2}}[(pQ+q-1)T_s]\Big)^*\Big\}$$
$$= \frac{A_1^2}{2}\,\delta_{i_1,i_2}\sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=Q_P+1}^{Q}E\Big\{x_1\big[t_1-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big]$$
$$x_1\big[t_2-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t_1=(pQ+q'-1)T_s,\ t_2=(pQ+q-1)T_s},$$

which, when substituted into (S.54), yields

$$\frac{1}{2}E\big[I^{(P)}_{W_{m,i_1,q'}}\big(I^{(P)}_{Y_{m,i_2,q}}\big)^*\big]$$
$$= \frac{A_1^2}{2}\,\delta_{i_1,i_2}C^{(1)}_{p,q',m,i_1}C^{(1)}_{p,q,m,i_1}\sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=Q_P+1}^{Q}E\Big\{x_1\big[t_1-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big]$$
$$x_1\big[t_2-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t_1=(pQ+q'-1)T_s,\ t_2=(pQ+q-1)T_s}.$$
$$\text{(S.55)}$$

S.2.4 Fourth Term in (S.47)

With $N_{y_{m,i}}(t)$ given by (S.26), we can write

$$N_{W_{m,i,q}} = C^{(1)}_{p,q,m,i} N_{y_{m,i}}[(pQ+q-1)T_s], \quad q = 1, \ldots, Q_P. \tag{S.56}$$

Using (S.27) and (S.56), we have

$$\begin{aligned} &\frac{1}{2}E\big[N_{W_{m,i_1,q'}} N^*_{Y_{m,i_2,q}}\big] \\ &\quad = C^{(1)}_{p,q',m,i_1} C^{(1)}_{p,q,m,i_2} \cdot \frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q'-1)T_s] N^*_{y_{m,i_2}}[(pQ+q-1)T_s]\Big\}. \end{aligned} \tag{S.57}$$

For $i_1 \neq i_2$, $N_{y_{m,i_1}}(t_1)$ and $N_{y_{m,i_2}}(t_2)$ are independent because they represent the outputs from two non-overlapping bandpass matched filters, so we have

$$\frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q'-1)T_s] N^*_{y_{m,i_2}}[(pQ+q-1)T_s]\Big\} = 0, \quad i_1 \neq i_2. \tag{S.58}$$

For $i_1 = i_2$, we use (S.26) and apply (Q.33) to obtain

$$\begin{aligned} &\frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q'-1)T_s] N^*_{y_{m,i_1}}[(pQ+q-1)T_s]\Big\} \\ &\quad = \frac{1}{2}E\Big\{\big(\tilde{n}_W(t_1) * \tilde{h}_{m,i_1,1}(t_1)\big)\big(\tilde{n}^*_W(t_2) * \tilde{h}^*_{m,i_1,1}(t_2)\big)\Big\}\Big|_{t_1=(pQ+q'-1)T_s,\, t_2=(pQ+q-1)T_s} \\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}E\big[\tilde{n}_W(\tau_1)\tilde{n}^*_W(\tau_2)\big] \\ &\qquad\qquad \cdot \tilde{h}_{m,i_1,1}(t_1-\tau_1)\tilde{h}^*_{m,i_1,1}(t_2-\tau_2)\, d\tau_2\, d\tau_1\Big|_{t_1=(pQ+q'-1)T_s,\, t_2=(pQ+q-1)T_s} \\ &\quad = N_0 \int_{-\infty}^{\infty} \tilde{h}_{m,i_1,1}(t_1-\tau_1)\tilde{h}^*_{m,i_1,1}(t_2-\tau_1)\, d\tau_1\Big|_{t_1=(pQ+q'-1)T_s,\, t_2=(pQ+q-1)T_s}. \end{aligned} \tag{S.59}$$

By using the expression $\tilde{h}_{m,i,1}(t) = h_1(-t)$ from (S.31), we can show that

$$\begin{aligned} \int_{-\infty}^{\infty} \tilde{h}_{m,i_1,1}(t_1-\tau_1)\tilde{h}^*_{m,i_1,1}(t_2-\tau_1)\, d\tau_1 &= \int_{-\infty}^{\infty} h_1(-t_1+\tau_1)h_1(-t_2+\tau_1)\, d\tau_1 \\ &= \int_{-\infty}^{\infty} h_1(\tau-u)h_1(-u)\, du\Big|_{\tau=t_2-t_1} \\ &= x_1(\tau)\Big|_{\tau=t_2-t_1}, \end{aligned} \tag{S.60}$$

since

$$x_1(t) = h_1(t) * h_1^*(-t) \Leftrightarrow |H_1(f)|^2 = X_1(f). \tag{S.61}$$

Thus, (S.59) becomes

$$\begin{aligned}\frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q'-1)T_s]N^*_{y_{m,i_1}}[(pQ+q-1)T_s]\Big\} &= N_0\, x_1(\tau)\Big|_{\tau=t_2-t_1=(q-q')T_s} \\ &= 0, \end{aligned} \tag{S.62}$$

where the last equality follows from the fact that (see (2.24))

$$x_1(nT_s) = \delta_{n,0} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{S.63}$$

Combining the results of (S.58) and (S.62), we have

$$\frac{1}{2}E\Big\{N_{y_{m,i_1}}[(pQ+q'-1)T_s]N^*_{y_{m,i_2}}[(pQ+q-1)T_s]\Big\} = 0. \tag{S.64}$$

Finally, we use (S.64) in (S.57) to get

$$\frac{1}{2}E\big[N_{W_{m,i_1,q'}}N^*_{Y_{m,i_2,q}}\big] = 0. \tag{S.65}$$

S.2.5 Combining the Results

Substituting the results of (S.48), (S.52), (S.55), and (S.65) into (S.47), we get

$$\begin{aligned}&\frac{1}{2}E[W_{m,i_1,q'}Y^*_{m,i_2,q}] \\ &\quad = A_1^2 \cdot \frac{1}{2}E\big[g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big] \\ &\qquad + \frac{A_1^2}{2}\,\delta_{i_1,i_2}C^{(1)}_{p,q',m,i_1}C^{(1)}_{p,q,m,i_1}\sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=1}^{Q}E\Big\{x_1\big[t_1-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big] \\ &\qquad\qquad x_1\big[t_2-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t_1=(pQ+q'-1)T_s,\ t_2=(pQ+q-1)T_s}.\end{aligned} \tag{S.66}$$

Since $\tau^{(k_1)}$ is uniform over $[0, T_s)$, we have

$$\begin{aligned}
&\sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} E\Big\{x_1\big[t_1 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big]x_1\big[t_2 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big]\Big\} \\
&= \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \frac{1}{T_s}\int_0^{T_s} x_1\big[t_1 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big] \\
&\qquad \cdot x_1\big[t_2 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big]\, d\tau^{(k_1)} \\
&= \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \frac{1}{T_s}\int_{-t_1+(p_1'Q+q_1'-1)T_s}^{-t_1+(p_1'Q+q_1')T_s} x_1(-u)x_1(\tau - u)\, du\Big|_{\tau=t_2-t_1} \\
&= \frac{1}{T_s}\int_{-\infty}^{\infty} x_1(-u)x_1(\tau-u)\, du\Big|_{\tau=t_2-t_1} \\
&= \frac{1}{T_s}\big[x_1(\tau) * x_1^*(-\tau)\big]\Big|_{\tau=t_2-t_1} \\
&= \frac{1}{T_s}\int_{-\infty}^{\infty} |X_1(f)|^2 e^{j2\pi f\tau}\, df\Big|_{\tau=t_2-t_1}. \qquad \text{(S.67)}
\end{aligned}$$

We use (S.67) and (S.38) to show that the second term in (S.66) is equal to zero. Thus,

$$\frac{1}{2}E[W_{m,i_1,q'}Y^*_{m,i_2,q}] = A_1^2 \cdot \frac{1}{2}E\big[g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big]. \qquad \text{(S.68)}$$

Finally, we substitute (S.68) into (S.46) to show that

$$\frac{1}{2}E[\hat{W}_{m,i_1}Y^*_{m,i_2,q}] = A_1 \cdot \frac{1}{2}E\big[g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big]. \qquad \text{(S.69)}$$

Using (S.11) and (S.41), we can re-write (S.69) as

$$\begin{aligned}
\frac{1}{2}E[W_{m,i_1,q'}Y^*_{m,i_2,q}] &= \begin{cases} \dfrac{A_1}{2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\[2ex] 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks,} \end{cases} \\
&= \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{2E_{b1}}{R_1}}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\[2ex] 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks.} \end{cases} \qquad \text{(S.70)}
\end{aligned}$$

S.3 Derivation of $\frac{1}{2}E[\hat{W}_{m,i_1}\hat{W}^*_{m,i_2}]$

Using (S.44), we have

$$\begin{aligned}
\frac{1}{2}E[\hat{W}_{m,i_1}\hat{W}^*_{m,i_2}] &= \frac{1}{A_1^2}\frac{1}{Q_P^2}\sum_{q_1=1}^{Q_P}\sum_{q_2=1}^{Q_P}\frac{1}{2}E[W_{m,i_1,q_1}W^*_{m,i_2,q_2}] \\
&= \frac{1}{A_1^2}\frac{1}{Q_P^2}\sum_{q_1=1}^{Q_P}\frac{1}{2}E[W_{m,i_1,q_1}W^*_{m,i_2,q_1}] + \frac{1}{A_1^2}\frac{1}{Q_P^2}\sum_{q_1=1}^{Q_P}\sum_{\substack{q_2=1\\ q_2\neq q_1}}^{Q_P}\frac{1}{2}E[W_{m,i_1,q_1}W^*_{m,i_2,q_2}] \\
&= \frac{1}{A_1^2}\frac{1}{Q_P^2}\cdot Q_P\cdot\frac{1}{2}E[W_{m,i_1,q}W^*_{m,i_2,q}] \\
&\quad + \frac{1}{A_1^2}\frac{1}{Q_P^2}\cdot Q_P(Q_P-1)\cdot\frac{1}{2}E[W_{m,i_1,q_1}W^*_{m,i_2,q_2}], \quad q_1\neq q_2.
\end{aligned} \tag{S.71}$$

Since $W_{m,i,q}$ and $Y_{m,i,q'}$ are both outputs of the same demodulator but from different time slots, we have

$$\frac{1}{2}E[W_{m,i_1,q_1}W^*_{m,i_2,q_1}] = \frac{1}{2}E[Y_{m,i_1,q_2}Y^*_{m,i_2,q_2}] \tag{S.72}$$

and

$$\frac{1}{2}E[W_{m,i_1,q_1}W^*_{m,i_2,q_2}] = \frac{1}{2}E[W_{m,i_1,q}Y^*_{m,i_2,q'}], \quad q_1\neq q_2. \tag{S.73}$$

We apply both (S.72) and (S.73) to (S.71) to get

$$\frac{1}{2}E[\hat{W}_{m,i_1}\hat{W}^*_{m,i_2}] = \frac{1}{A_1^2}\frac{1}{Q_P}\cdot\frac{1}{2}E[Y_{m,i_1,q}Y^*_{m,i_2,q}] + \frac{1}{A_1^2}\frac{1}{Q_P}(Q_P-1)\cdot\frac{1}{2}E[W_{m,i_1,q_1}Y^*_{m,i_2,q_2}]. \tag{S.74}$$

Substituting (S.39) and (S.68) into this expression, we obtain

$$\begin{aligned}
\frac{1}{2}E[\hat{W}_{m,i_1}\hat{W}^*_{m,i_2}] &= \frac{1}{A_1^2}\frac{1}{Q_P}\cdot\Big\{A_1^2\cdot\frac{1}{2}E\big[g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big] + \frac{A_1^2}{2}(K-1)\,\delta_{i_1,i_2} + N_0\,\delta_{i_1,i_2}\Big\} \\
&\quad + \frac{1}{A_1^2}\frac{1}{Q_P}(Q_P-1)\cdot A_1^2\cdot\frac{1}{2}E\big[g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big] \\
&= \frac{1}{A_1^2}\frac{1}{Q_P}\Big\{Q_P\cdot A_1^2\cdot\frac{1}{2}E\big[g^{(1)}_{m,i_1}\big(g^{(1)}_{m,i_2}\big)^*\big] + \frac{A_1^2}{2}(K-1)\,\delta_{i_1,i_2} + N_0\,\delta_{i_1,i_2}\Big\} \\
&= \frac{1}{2}E\big[g^{(k_1)}_{m,i_1}\big(g^{(k_1)}_{m,i_2}\big)^*\big] + \frac{1}{2}\frac{1}{Q_P}\left((K-1) + \frac{2N_0}{A_1^2N}\right)\delta_{i_1,i_2}.
\end{aligned} \tag{S.75}$$

Finally, by applying the expressions for $\frac{1}{2}E\big[g_{m,i_1}^{(k_1)}\big(g_{m,i_2}^{(k_1)}\big)^*\big]$ and E_{b2} in (S.11) and (S.41), respectively, we get

$$\frac{1}{2}E[\hat{W}_{m,i_1}\hat{W}_{m,i_2}^*] = \begin{cases} \frac{1}{2}\left[1+\frac{1}{Q_P}c_1\left(\frac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right], & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks,} \end{cases} \tag{S.76}$$

where $c_1\Big(\frac{E_{b1}}{N_0}\Big)$ is given in (S.43).

Appendix T

Second Order Moments: MC-CDMA (SISO–Scenario #1)

Since the corresponding expressions for $W_{m,i,q}$ and $Y_{m,i,q}$ in the MC-CDMA (SISO) system are the same in both Scenarios #1 and #2 (see (2.26) and (2.27)), the expressions for μ_{WW}, μ_{YY}, and μ_{WY} in Scenario #1 can be derived from the results for $\frac{1}{2}E[\hat{W}_{m,i_1}\hat{W}^*_{m,i_2}]$, $\frac{1}{2}E[Y_{m,i_1,q}Y^*_{m,i_2,q}]$, and $\frac{1}{2}E[\hat{W}_{m,i_1}Y^*_{m,i_2,q}]$ in Scenario #2 (see Appendix S).

T.1 Derivation of μ_{YY}

The quantity μ_{YY} is defined in (G.6) as

$$\mu_{YY} = \frac{1}{2}E[|Y_{m,i,q}|^2] = \frac{1}{2}E[Y_{m,i,q}Y^*_{m,i,q}]. \tag{T.1}$$

The expression $\frac{1}{2}E[Y_{m,i,q}Y^*_{m,i,q}]$ can be obtained by setting $i_1 = i_2 = i$ in (S.42):

$$\mu_{YY} = \frac{E_{b1}}{R_1}\left[1 + c_1\left(\frac{E_{b1}}{N_0}\right)\right], \tag{T.2}$$

where

$$c_1\left(\frac{E_{b1}}{N_0}\right) = (K-1) + \frac{R_1}{E_{b1}/N_0} \tag{T.3}$$

and

$$E_{b1} = \frac{R_1 A_1^2}{2} \tag{T.4}$$

from (P.15). This result can be verified by substituting

$$Y_{m,i,q} = A_1\, g_{m,i}^{(1)} + I_{Y_{m,i,q}}^{(P)} + I_{Y_{m,i,q}}^{(D)} + N_{Y_{m,i,q}}, \quad q = Q_P+1, \ldots, Q, \tag{T.5}$$

into (T.1) and by following the same steps as in Appendix S.1.

T.2 Derivation of μ_{WY}

μ_{WY} is defined in (G.7) as

$$\mu_{WY} = \frac{1}{2} E[\hat{W}_{m,i} Y_{m,i,q}^*]. \tag{T.6}$$

Substituting $i_1 = i$ and $i_2 = i$ into (S.70), we get

$$\mu_{WY} = \frac{1}{2}\sqrt{\frac{2E_{b1}}{R_1}}. \tag{T.7}$$

To verify this result, we first substitute

$$\hat{W}_{m,i} = \frac{1}{A_1}\left(\frac{1}{Q_P}\sum_{q=1}^{Q_P} W_{m,i,q}\right) \tag{T.8}$$

into (T.6) to obtain

$$\mu_{WY} = \frac{1}{A_1}\frac{1}{Q_P}\sum_{q'=1}^{Q_P}\frac{1}{2}E[W_{m,i,q'} Y_{m,i,q}^*]. \tag{T.9}$$

Then, we expand $\frac{1}{2}E[W_{m,i,q'} Y_{m,i,q}^*]$ by using (T.5) and

$$W_{m,i,q} = A_1\, g_{m,i}^{(1)} + I_{W_{m,i,q}}^{(P)} + I_{W_{m,i,q}}^{(D)} + N_{W_{m,i,q}}, \quad q = 1, \ldots, Q_P, \tag{T.10}$$

and by following the same steps as in Appendix S.2.

T.3 Derivation of μ_{WW}

From (G.5), the quantity μ_{WW} is defined as

$$\mu_{WW} = \frac{1}{2}E[|\hat{W}_{m,i}|^2] = \frac{1}{2}E[\hat{W}_{m,i}\hat{W}^*_{m,i}]. \tag{T.11}$$

The substitutions $i_1 = i$ and $i_2 = i$ in (S.76) yield

$$\mu_{WW} = \frac{1}{2}\left[1 + \frac{1}{Q_P}c_1\left(\frac{E_{b1}}{N_0}\right)\right]. \tag{T.12}$$

Again, we can verify this result by substituting (T.8) into (T.11) and by following the same steps as in Appendix S.3.

Appendix U

Second Order Moments: MC-DS-CDMA (MIMO)

In this appendix, we provide derivations for the expressions in (L.13)–(L.17). Again, we assume $a^{(1)}_{p,q,m} = +1$ and we condition on $a^{(1)}_{p,q+1,m}$.

U.1 Derivation of $\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}]$

From (3.63), we have

$$\hat{W}_{v,u,m,i} = \begin{cases} g^{(1)}_{v,u,m,i} + \dfrac{1}{A_{2e}\sqrt{N}}\dfrac{1}{Q_P/2}\displaystyle\sum_{\substack{q=1\\ q \text{ odd}}}^{Q_P}\big(I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}}\big), & u = 1, \\ g^{(1)}_{v,u,m,i} + \dfrac{1}{A_{2e}\sqrt{N}}\dfrac{1}{Q_P/2}\displaystyle\sum_{\substack{q=1\\ q \text{ even}}}^{Q_P}\big(I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}}\big), & u = 2. \end{cases} \tag{U.1}$$

The expressions for $I_{W_{q,v,m,i}}$ and $N_{W_{q,v,m,i}}$ are given in (3.61) and (3.62) as

$$I_{W_{q,v,m,i}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C_{p,q,n}^{(1)} \sum_{k=2}^{K} \sum_{u=1}^{2} g_{v,u,m,i}^{(k)} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q} \sum_{n'=1}^{N} A_{2,q'} b_{p',q',u,m}^{(k)} C_{p',q',n'}^{(k)} \cdot x_2\{t - [(p'Q + q' - 1)N + n' - 1]T_c - \tau^{(k)}\}\Big|_{t=[(pQ+q-1)N+n-1]T_c}, \tag{U.2}$$

$$N_{W_{q,v,m,i}} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} C_{p,q,n}^{(1)} \left(\tilde{n}_v(t) * \tilde{h}_{m,i,2}(t)\right)\Big|_{t=[(pQ+q-1)N+n-1]T_c}. \tag{U.3}$$

As in the analysis for the SISO system, we assume that the channel gains, time delays, data sequences, and spreading sequences of the interfering users are not only mutually independent, but are also independent of the AWGN. We model the data sequences and spreading sequences of the interfering users as independent random binary sequences of ± 1's. As a result, we have

$$E\big[C_{p,q,n}^{(k)}\big] = 0, \tag{U.4}$$

$$E\big[a_{p,q,m}^{(k)}\big] = 0, \quad q = Q_P + 1, \ldots, Q, \tag{U.5}$$

and

$$E\big[C_{p_1,q_1,n_1}^{(k_1)} C_{p_2,q_2,n_2}^{(k_2)}\big] = \delta_{k_1,k_2}\delta_{p_1,p_2}\delta_{q_1,q_2}\delta_{n_1,n_2}, \tag{U.6}$$

$$E\big[a_{p_1,q_1,m}^{(k_1)} a_{p_2,q_2,m}^{(k_2)}\big] = \delta_{k_1,k_2}\delta_{p_1,p_2}\delta_{q_1,q_2}, \quad q_i = Q_P + 1, \ldots, Q;\ i = 1, 2. \tag{U.7}$$

Finally, since the channel gains between different users, different transmit-receive antenna pairs, and different repetitions of the same data symbol are independent, we have

$$\frac{1}{2} E\big[g_{v_1,u_1,m,i_1}^{(k_1)} \big(g_{v_2,u_2,m,i_2}^{(k_2)}\big)^*\big] = \frac{1}{2}\,\delta_{k_1,k_2}\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}. \tag{U.8}$$

Let us define

$$\Gamma_{W_{q,v,m,i}} \triangleq I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}} \tag{U.9}$$

and

$$q(u) \triangleq q + (u - 1), \tag{U.10}$$

such that $\hat{W}_{v,u,m,i}$ in (U.1) can be written as

$$\hat{W}_{v,u,m,i} = g^{(1)}_{v,u,m,i} + \frac{1}{A_{2e}\sqrt{N}} \frac{1}{Q_P/2} \sum_{\substack{q=1 \\ q \text{ odd}}}^{Q_P} \Gamma_{W_{q(u),v,m,i}}. \tag{U.11}$$

We use this expression to write

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\Bigg[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,u_2,m,i_2}\big)^* + \frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\, g^{(1)}_{v_1,u_1,m,i_1} \sum_{\substack{q=1 \\ q \text{ odd}}}^{Q_P} \Gamma^*_{W_{q(u_2),v_2,m,i_2}} \\
&\qquad + \frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\big(g^{(1)}_{v_2,u_2,m,i_2}\big)^* \sum_{\substack{q=1 \\ q \text{ odd}}}^{Q_P} \Gamma_{W_{q(u_1),v_1,m,i_2}} \\
&\qquad + \left(\frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\right)^2 \sum_{\substack{q_1=1 \\ q_1 \text{ odd}}}^{Q_P} \sum_{\substack{q_2=1 \\ q_2 \text{ odd}}}^{Q_P} \Gamma_{W_{q_1(u_1),v_1,m,i_1}}\Gamma^*_{W_{q_2(u_2),v_2,m,i_2}}\Bigg].
\end{aligned} \tag{U.12}$$

where $q_1(u_1) \triangleq q_1 + (u_1 - 1)$ and $q_2(u_2) \triangleq q_2 + (u_2 - 1)$. Using the assumptions in (U.4)–(U.8), we can show that the expected values of the two cross terms are equal to zero, so that

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\Bigg[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,u_2,m,i_2}\big)^* \\
&\qquad + \left(\frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\right)^2 \sum_{\substack{q_1=1 \\ q_1 \text{ odd}}}^{Q_P} \sum_{\substack{q_2=1 \\ q_2 \text{ odd}}}^{Q_P} \Gamma_{W_{q_1(u_1),v_1,m,i_1}}\Gamma^*_{W_{q_2(u_2),v_2,m,i_2}}\Bigg] \\
&\quad = \frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,u_2,m,i_2}\big)^*\big] \\
&\qquad + \left(\frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\right)^2 \sum_{\substack{q_1=1 \\ q_1 \text{ odd}}}^{Q_P} \frac{1}{2}E\big[\Gamma_{W_{q_1(u_1),v_1,m,i_1}}\Gamma^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\qquad + \left(\frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\right)^2 \sum_{\substack{q_1=1 \\ q_1 \text{ odd}}}^{Q_P} \sum_{\substack{q_2=1 \\ q_2 \text{ odd} \\ q_2 \neq q_1}}^{Q_P} \frac{1}{2}E\big[\Gamma_{W_{q_1(u_1),v_1,m,i_1}}\Gamma^*_{W_{q_2(u_2),v_2,m,i_2}}\big].
\end{aligned} \tag{U.13}$$

U.1.1 First Term in (U.13)

The first term in (U.13) is equal to (see (U.8))

$$\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,u_2,m,i_2}\big)^*\big] = \frac{1}{2}\,\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}. \tag{U.14}$$

U.1.2 Second Term in (U.13)

We use (U.9) to expand the term $\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1}\Gamma^*_{W_{q_1(u_2)},v_2,m,i_2}\big]$ as

$$\begin{aligned}
&\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1}\Gamma^*_{W_{q_1(u_2)},v_2,m,i_2}\big]\\
&\quad= \frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1}I^*_{W_{q_1(u_2)},v_2,m,i_2} + I_{W_{q_1(u_1)},v_1,m,i_1}N^*_{W_{q_1(u_2)},v_2,m,i_2}\\
&\qquad + N_{W_{q_1(u_1)},v_1,m,i_1}I^*_{W_{q_1(u_2)},v_2,m,i_2} + N_{W_{q_1(u_1)},v_1,m,i_1}N^*_{W_{q_1(u_2)},v_2,m,i_2}\big].
\end{aligned} \tag{U.15}$$

The expected values of the cross terms are equal to zero, so

$$\begin{aligned}
&\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1}\Gamma^*_{W_{q_1(u_2)},v_2,m,i_2}\big]\\
&\quad= \frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1}I^*_{W_{q_1(u_2)},v_2,m,i_2} + N_{W_{q_1(u_1)},v_1,m,i_1}N^*_{W_{q_1(u_2)},v_2,m,i_2}\big].
\end{aligned} \tag{U.16}$$

First Term in (U.16)

Define

$$\begin{aligned}
I_{y_{v,m,i}}(t) = \sum_{k=2}^{K}\sum_{u'=1}^{2} g^{(k)}_{v,u',m,i}\sum_{p'=-\infty}^{\infty}\sum_{q'=1}^{Q}\sum_{n'=1}^{N} A_{2,q'}b^{(k)}_{p',q',u',m}C^{(k)}_{p',q',n'}\\
\cdot\, x_2\big\{t-[(p'Q+q'-1)N+n'-1]T_c-\tau^{(k)}\big\},
\end{aligned} \tag{U.17}$$

such that $I_{W_{q(u)},v,m,i}$ can be re-written as (see (U.2))

$$I_{W_{q(u)},v,m,i} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C^{(1)}_{p,q(u),n}I_{y_{v,m,i}}\big\{[(pQ+q(u)-1)N+n-1]T_c\big\}, \quad q=1,\ldots,Q_P. \tag{U.18}$$

Note that $I_{y_{v,m,i}}(t)$ represents the MAI at the output of the matched filter (before sampling) at frequency $f_{m,i}$ of the v-th receive antenna. We use (U.17) and (U.18) to write

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_1(u_2)},v_2,m,i_2}\big] \\
&\quad = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2} \cdot \frac{1}{2}E\Big[I_{y_{v_1,m,i_1}}\big\{[(pQ+q_1(u_1)-1)N+n_1-1]T_c\big\} \\
&\qquad\qquad I^*_{y_{v_2,m,i_2}}\big\{[(pQ+q_1(u_2)-1)N+n_2-1]T_c\big\}\Big] \\
&\quad = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2} \sum_{k_1=2}^{K}\sum_{k_2=2}^{K}\sum_{u_1'=1}^{2}\sum_{u_2'=1}^{2} \\
&\qquad \cdot \frac{1}{2}E\big[g^{(k_1)}_{v_1,u_1',m,i_1}\big(g^{(k_2)}_{v_2,u_2',m,i_2}\big)^*\big] \sum_{p_1'=-\infty}^{\infty}\sum_{p_2'=-\infty}^{\infty}\sum_{q_1'=1}^{Q}\sum_{q_2'=1}^{Q}\sum_{n'=1}^{N}\sum_{n_2'=1}^{N} A_{2,q_1'}A_{2,q_2'} \\
&\qquad \cdot E\big[b^{(k_1)}_{p_1',q_1',u_1',m} b^{(k_2)}_{p_2',q_2',u_2',m}\big] E\big[C^{(k_1)}_{p_1',q_1',n_1'} C^{(k_2)}_{p_2',q_2',n_2'}\big] \\
&\qquad \cdot E\Big[x_2\big\{t_1 - [(p_1'Q+q_1'-1)N+n_1'-1]T_c - \tau^{(k_1)}\big\} \\
&\qquad\qquad x_2\big\{t_2 - [(p_2'Q+q_2'-1)N+n_2'-1]T_c - \tau^{(k_2)}\big\}\Big]\Big|_{\substack{t_1=[(pQ+q_1(u_1)-1)N+n_1-1]T_c \\ t_2=[(pQ+q_1(u_2)-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.19}$$

To further evaluate (U.19), we first expand the summations over q_1' and q_2' as

$$\begin{aligned}
\sum_{q_1'=1}^{Q}\sum_{q_2'=1}^{Q} &= \Bigg(\sum_{\substack{q_1'=1\\ q_1' \text{ odd}}}^{Q_P} + \sum_{\substack{q_1'=1\\ q_1' \text{ even}}}^{Q_P} + \sum_{q_1'=Q_P+1}^{Q}\Bigg)\Bigg(\sum_{\substack{q_2'=1\\ q_2' \text{ odd}}}^{Q_P} + \sum_{\substack{q_2'=1\\ q_2' \text{ even}}}^{Q_P} + \sum_{q_2'=Q_P+1}^{Q}\Bigg) \\
&= \sum_{\substack{q_1'=1\\ q_1' \text{ odd}}}^{Q_P}\sum_{\substack{q_2'=1\\ q_2' \text{ odd}}}^{Q_P} + \sum_{\substack{q_1'=1\\ q_1' \text{ odd}}}^{Q_P}\sum_{\substack{q_2'=1\\ q_2' \text{ even}}}^{Q_P} + \sum_{\substack{q_1'=1\\ q_1' \text{ odd}}}^{Q_P}\sum_{q_2'=Q_P+1}^{Q} \\
&\quad + \sum_{\substack{q_1'=1\\ q_1' \text{ even}}}^{Q_P}\sum_{\substack{q_2'=1\\ q_2' \text{ odd}}}^{Q_P} + \sum_{\substack{q_1'=1\\ q_1' \text{ even}}}^{Q_P}\sum_{\substack{q_2'=1\\ q_2' \text{ even}}}^{Q_P} + \sum_{\substack{q_1'=1\\ q_1' \text{ even}}}^{Q_P}\sum_{q_2'=Q_P+1}^{Q} \\
&\quad + \sum_{q_1'=Q_P+1}^{Q}\sum_{\substack{q_2'=1\\ q_2' \text{ odd}}}^{Q_P} + \sum_{q_1'=Q_P+1}^{Q}\sum_{\substack{q_2'=1\\ q_2' \text{ even}}}^{Q_P} + \sum_{q_1'=Q_P+1}^{Q}\sum_{q_2'=Q_P+1}^{Q}.
\end{aligned} \tag{U.20}$$

Then, we substitute both

$$A_{2,q} = \begin{cases} A_{2e}, & q = 1,\ldots,Q_P, \\ A_{2d}, & q = Q_P+1,\ldots,Q, \end{cases} \tag{U.21}$$

Table U.1: Definition of $b^{(k)}_{p,q,u,m}$

Estimation phase ($q = 1, \ldots, Q_P$):	$b^{(k)}_{p,q,u,m} = \begin{cases} \delta_{u,1}, & q \text{ odd} \\ \delta_{u,2}, & q \text{ even} \end{cases}$
Data phase ($q = Q_P + 1, \ldots, Q$):	$b^{(k)}_{p,q,u,m} = \begin{cases} a^{(k)}_{p,q,m}, & u = 1, q \text{ odd} \\ a^{(k)}_{p,q+1,m}, & u = 2, q \text{ odd} \\ -a^{(k)}_{p,q,m}, & u = 1, q \text{ even} \\ a^{(k)}_{p,q-1,m}, & u = 2, q \text{ even} \end{cases}$

from (3.31) and $b^{(k)}_{p,q,u,m}$ from Table U.1 (same as Table 3.1) into the resulting expression, and we apply (U.5) to get

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_1(u_2)},v_2,m,i_2}\big] \\
&\quad= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2} \sum_{k_1=2}^{K}\sum_{k_2=2}^{K}\sum_{u_1'=1}^{2}\sum_{u_2'=1}^{2} \\
&\qquad \cdot \frac{1}{2}E\big[g^{(k_1)}_{v_1,u_1',m,i_1}\big(g^{(k_2)}_{v_2,u_2',m,i_2}\big)^*\big] \sum_{p_1'=-\infty}^{\infty}\sum_{p_2'=-\infty}^{\infty} \\
&\qquad \cdot \Bigg(\sum_{\substack{q_1'=1 \\ q_1' \text{ odd}}}^{Q_P} \sum_{\substack{q_2'=1 \\ q_2' \text{ odd}}}^{Q_P} A_{2e}^2 \delta_{u_1',1}\delta_{u_2',1} + \sum_{\substack{q_1'=1 \\ q_1' \text{ odd}}}^{Q_P} \sum_{\substack{q_2'=1 \\ q_2' \text{ even}}}^{Q_P} A_{2e}^2 \delta_{u_1',1}\delta_{u_2',2} + \sum_{\substack{q_1'=1 \\ q_1' \text{ even}}}^{Q_P} \sum_{\substack{q_2'=1 \\ q_2' \text{ odd}}}^{Q_P} A_{2e}^2 \delta_{u_1',2}\delta_{u_2',1} \\
&\qquad\quad + \sum_{\substack{q_1'=1 \\ q_1' \text{ even}}}^{Q_P} \sum_{\substack{q_2'=1 \\ q_2' \text{ even}}}^{Q_P} A_{2e}^2 \delta_{u_1',2}\delta_{u_2',2} + \sum_{q_1'=Q_P+1}^{Q} \sum_{q_2'=Q_P+1}^{Q} A_{2d}^2\, E\big[b^{(k_1)}_{p_1',q_1',u_1',m} b^{(k_2)}_{p_2',q_2',u_2',m}\big] \Bigg) \\
&\qquad \cdot \sum_{n'=1}^{N}\sum_{n_2'=1}^{N} E\big[C^{(k_1)}_{p_1',q_1',n_1'} C^{(k_2)}_{p_2',q_2',n_2'}\big] \\
&\qquad \cdot E\Big[x_2\big\{t_1 - [(p_1'Q + q_1' - 1)N + n_1' - 1]T_c - \tau^{(k_1)}\big\} \\
&\qquad\qquad x_2\big\{t_2 - [(p_2'Q + q_2' - 1)N + n_2' - 1]T_c - \tau^{(k_2)}\big\}\Big]\Bigg|_{\substack{t_1=[(pQ+q_1(u_1)-1)N+n_1-1]T_c \\ t_2=[(pQ+q_1(u_2)-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.22}$$

In Appendix V, we show that

$$\sum_{u_1'=1}^{2}\sum_{u_2'=1}^{2}\sum_{q_1'=Q_P+1}^{Q}\sum_{q_2'=Q_P+1}^{Q} A_{2d}^2 E\big[b^{(k_1)}_{p_1',q_1',u_1',m} b^{(k_2)}_{p_2',q_2',u_2',m}\big] = \sum_{q_1'=Q_P+1}^{Q} 2A_{2d}^2\, \delta_{p_1',p_2'}\delta_{k_1,k_2}. \tag{U.23}$$

We use this result, along with (U.6) and (U.8), to show that (U.22) reduces to

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\quad = \frac{1}{2}\frac{1}{N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2}\sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty} \\
&\qquad \cdot\Bigg(\sum_{\substack{q'_1=1\\ q'_1\text{ odd}}}^{Q_P} A_{2e}^2 + \sum_{\substack{q'_1=1\\ q'_1\text{ even}}}^{Q_P} A_{2e}^2 + \sum_{q'_1=Q_P+1}^{Q} 2A_{2d}^2\Bigg) \\
&\qquad \cdot\sum_{n'=1}^{N} E\Big[x_2\big\{t_1-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\} \\
&\qquad\qquad x_2\big\{t_2-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\substack{t_1=[(pQ+q_1(u_1)-1)N+n_1-1]T_c\\ t_2=[(pQ+q_1(u_2)-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.24}$$

Given the relation

$$A_{2d} = \frac{A_{2e}}{\sqrt{2}} \tag{U.25}$$

from (3.32), we can write

$$\sum_{\substack{q'_1=1\\ q'_1\text{ odd}}}^{Q_P} A_{2e}^2 + \sum_{\substack{q'_1=1\\ q'_1\text{ even}}}^{Q_P} A_{2e}^2 + \sum_{q'_1=Q_P+1}^{Q} 2A_{2d}^2 = \sum_{q'_1=1}^{Q} A_{2e}^2. \tag{U.26}$$

Thus, (U.24) equals

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\quad = \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2}\sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=1}^{Q}\sum_{n'=1}^{N} \\
&\qquad \cdot E\Big[x_2\big\{t_1-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\} \\
&\qquad\qquad x_2\big\{t_2-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\substack{t_1=[(pQ+q_1(u_1)-1)N+n_1-1]T_c\\ t_2=[(pQ+q_1(u_2)-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.27}$$

Finally, we apply (Q.41) to the above equation and use (Q.44) to get

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\quad = \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2} \\
&\qquad \cdot\sum_{k_1=2}^{K}\frac{1}{T_c}\int_{-\infty}^{\infty}|X_2(f)|^2 e^{j2\pi f\tau}\,df\Big|_{\tau=t_2-t_1=[(u_2-u_1)N+(n_2-n_1)]T_c}
\end{aligned}$$

$$= \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2}\sum_{k_1=2}^{K}\frac{1}{T_c}\cdot T_c\,\delta_{u_1,u_2}\delta_{n_1,n_2}$$

$$= \frac{A_{2e}^2(K-1)}{2N}\,\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\left[C^{(1)}_{p,q_1(u_1),n_1}\right]^2$$

$$= \frac{A_{2e}^2(K-1)}{2}\,\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}. \tag{U.28}$$

Second Term in (U.16)

Let

$$N_{y_{v,m,i}}(t) = \tilde{n}_v(t) * \tilde{h}_{m,i,2}(t) \tag{U.29}$$

denote the AWGN at the output of the matched filter (before sampling) at frequency $f_{m,i}$ of the v-th receive antenna. This allows us to re-write $N_{W_{q(u),v,m,i}}$ as (see (U.3))

$$N_{W_{q(u),v,m,i}} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C^{(1)}_{p,q(u),n} N_{y_{v,m,i}}\big\{[(pQ+q-1)N+n-1]T_c\big\}, \quad q=1,\ldots,Q_P, \tag{U.30}$$

such that

$$\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$$
$$= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2}\cdot\frac{1}{2}E\Big[N_{y_{v_1,m,i_1}}\big\{[(pQ+q_1(u_1)-1)N+n_1-1]T_c\big\}$$
$$N^*_{y_{v_2,m,i_2}}\big\{[(pQ+q_1(u_2)-1)N+n_2-1]T_c\big\}\Big]. \tag{U.31}$$

When $i_1 \neq i_2$, the AWGN terms $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are from two different frequencies. Thus, $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are independent, and since $N_{y_{v_1,m,i_1}}(t)$ is zero-mean, we have

$$\frac{1}{2}E\Big[N_{y_{v_1,m,i_1}}\big\{[(pQ+q_1(u_1)-1)N+n_1-1]T_c\big\}$$
$$N^*_{y_{v_2,m,i_2}}\big\{[(pQ+q_1(u_2)-1)N+n_2-1]T_c\big\}\Big] = 0, \quad i_1 \neq i_2. \tag{U.32}$$

This, in turn, implies that

$$\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_1(u_2),v_2,m,i_2}}\big] = 0, \quad i_1 \neq i_2. \tag{U.33}$$

When $i_1 = i_2$, we use (U.29) and (U.31) to write

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_1(u_2),v_2,m,i_1}}\big] \\
&= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2} \cdot \frac{1}{2}E\Big[N_{y_{v_1,m,i_1}}\big\{[(pQ+q_1(u_1)-1)N+n_1-1]T_c\big\} \\
&\qquad N^*_{y_{v_2,m,i_1}}\big\{[(pQ+q_1(u_2)-1)N+n_2-1]T_c\big\}\Big]. \\
&= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}E\big[\tilde{n}_{v_1}(\tau_1)\tilde{n}^*_{v_2}(\tau_2)\big] \\
&\quad \cdot \tilde{h}_{m,i_1,2}(t_1-\tau_1)\tilde{h}^*_{m,i_1,2}(t_2-\tau_2)\,d\tau_2\,d\tau_1\Big|_{\substack{t_1=[(pQ+q_1(u_1)-1)N+n_1-1]T_c \\ t_2=[(pQ+q_1(u_2)-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.34}$$

The AWGN processes at the two receive antennas are assumed to be independent, so we have

$$\frac{1}{2}E\big[\tilde{n}_{v_1}(\tau_1)\tilde{n}^*_{v_2}(\tau_2)\big] = N_0\,\delta(\tau_1-\tau_2)\,\delta_{v_1,v_2}. \tag{U.35}$$

Substituting this result into (U.34), we get

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_1(u_2),v_2,m,i_1}}\big] \\
&\quad = \frac{N_0}{N}\,\delta_{v_1,v_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2} \\
&\qquad \cdot \int_{-\infty}^{\infty} \tilde{h}_{m,i_1,2}(t_1-\tau_1)\tilde{h}^*_{m,i_1,2}(t_2-\tau_1)\,d\tau_1\Big|_{\substack{t_1=[(pQ+q_1(u_1)-1)N+n_1-1]T_c \\ t_2=[(pQ+q_1(u_2)-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.36}$$

From (Q.34) and (Q.36), we have

$$\int_{-\infty}^{\infty} \tilde{h}_{m,i,2}(t-\tau_1)\tilde{h}^*_{m,i,2}(t+\tau-\tau_1)\,d\tau_1 = x_2(\tau). \tag{U.37}$$

Applying (U.37) to (U.36) and using (Q.38), we obtain

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_1(u_2),v_2,m,i_1}}\big] \\
&\quad = \frac{N_0}{N}\,\delta_{v_1,v_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2} \cdot x_2(\tau)\Big|_{\tau=t_2-t_1=[(u_2-u_1)N+(n_2-n_1)]T_c} \\
&\quad = \frac{N_0}{N}\,\delta_{v_1,v_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_1(u_2),n_2} \cdot \delta_{u_1,u_2}\delta_{n_1,n_2} \\
&\quad = \frac{N_0}{N}\,\delta_{v_1,v_2}\delta_{u_1,u_2}\sum_{n_1=1}^{N}\Big[C^{(1)}_{p,q_1(u_1),n_1}\Big]^2
\end{aligned}$$

$$= N_0\, \delta_{v_1,v_2}\delta_{u_1,u_2}. \tag{U.38}$$

Finally, we combine the results of (U.33) and (U.38) to write

$$\frac{1}{2}E\big[N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_1(u_2)},v_2,m,i_2}\big] = N_0\, \delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}. \tag{U.39}$$

Combining the Results

Substituting the results of (U.28) and (U.39) into (U.16), we get

$$\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1} \Gamma^*_{W_{q_1(u_2)},v_2,m,i_2}\big] = \left(\frac{A_{2e}^2(K-1)}{2} + N_0\right) \delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}. \tag{U.40}$$

U.1.3 Third Term in (U.13)

We use (U.9) to expand the term $\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1} \Gamma^*_{W_{q_2(u_2)},v_2,m,i_2}\big]$, where $q_1 \neq q_2$, as

$$\begin{aligned}
&\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1} \Gamma^*_{W_{q_2(u_2)},v_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_2(u_2)},v_2,m,i_2} + I_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_2(u_2)},v_2,m,i_2} \\
&\qquad + N_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_2(u_2)},v_2,m,i_2} + N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_2(u_2)},v_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_2(u_2)},v_2,m,i_2} + N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_2(u_2)},v_2,m,i_2}\big].
\end{aligned} \tag{U.41}$$

Again, the expected values of the cross terms are easily shown to equal zero.

First Term in (U.41)

To derive an expression for $\frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_2(u_2)},v_2,m,i_2}\big]$, we simply replace $q_1(u_2)$ with $q_2(u_2)$ in the derivation just presented for $\frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_1(u_2)},v_2,m,i_2}\big]$. Thus, given

(U.27), we have

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_2(u_2),v_2,m,i_2}}\big] \\
&\quad = \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_2(u_2),n_2}\sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=1}^{Q}\sum_{n'=1}^{N} \\
&\qquad\cdot E\Big[x_2\big\{t_1-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-\tau^{(k_1)}\big\} \\
&\qquad\qquad x_2\big\{t_2-[(p_1'Q+q_1'-1)N+n_1'-1]T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\substack{t_1=[(pQ+q_1(u_1)-1)N+n_1-1]T_c \\ t_2=[(pQ+q_2(u_2)-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.42}$$

We apply (Q.41) to the above equation and use (Q.44), along with the fact that $q_1 \neq q_2$, to get

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_2(u_2),v_2,m,i_2}}\big] \\
&\quad = \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_2(u_2),n_2}\sum_{k_1=2}^{K} \\
&\qquad\cdot \frac{1}{T_c}\int_{-\infty}^{\infty}|X_2(f)|^2 e^{j2\pi f\tau}\,df\Big|_{\tau=t_2-t_1=\big\{[(q_2-q_1)+(u_2-u_1)]N+(n_2-n_1)\big\}T_c} \\
&\quad = \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_2(u_2),n_2}\sum_{k_1=2}^{K}\frac{1}{T_c}\cdot T_c\,\delta_{q_1,q_2}\delta_{u_1,u_2}\delta_{n_1,n_2} \\
&\quad = 0.
\end{aligned} \tag{U.43}$$

Second Term in (U.41)

Likewise, to derive an expression for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_2(u_2),v_2,m,i_2}}\big]$, we also replace $q_1(u_2)$ with $q_2(u_2)$ in the derivation for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$. Again, we have

$$\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_2(u_2),v_2,m,i_2}}\big] = 0, \quad i_1 \neq i_2, \tag{U.44}$$

since $N_{W_{q_1(u_1),v_1,m,i_1}}$ and $N_{W_{q_2(u_2),v_2,m,i_2}}$ correspond to AWGN terms from two different frequencies. For $i_1 = i_2$, we use (U.36) to write

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_2(u_2),v_2,m,i_1}}\big] \\
&\quad = \frac{N_0}{N}\,\delta_{v_1,v_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_2(u_2),n_2} \\
&\qquad\cdot\int_{-\infty}^{\infty}\tilde{h}_{m,i_1,2}(t_1-\tau_1)\tilde{h}^*_{m,i_1,2}(t_2-\tau_1)\,d\tau_1\Big|_{\substack{t_1=[(pQ+q_1(u_1)-1)N+n_1-1]T_c \\ t_2=[(pQ+q_1(u_2)-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.45}$$

We then apply (U.37) and use (Q.38), along with the fact that $q_1 \neq q_2$, to obtain

$$\frac{1}{2}E\big[N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_2(u_2)},v_2,m,i_1}\big]$$
$$= \frac{N_0}{N}\,\delta_{v_1,v_2} \sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q_1(u_1),n_1} C^{(1)}_{p,q_2(u_2),n_2} \cdot x_2(\tau)\Big|_{\tau=t_2-t_1=\left\{[(q_2-q_1)+(u_2-u_1)]N+(n_2-n_1)\right\}T_c}$$
$$= 0. \tag{U.46}$$

Therefore, from (U.44) and (U.46), we have

$$\frac{1}{2}E\big[N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_2(u_2)},v_2,m,i_2}\big] = 0. \tag{U.47}$$

Combining the Results

Substituting (U.43) and (U.47) into (U.41), we obtain

$$\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1} \Gamma^*_{W_{q_2(u_2)},v_2,m,i_2}\big] = 0, \quad q_1 \neq q_2. \tag{U.48}$$

U.1.4 Combining the Results

Finally, we use the results of (U.14), (U.40), and (U.48) in (U.13) to get

$$\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1} \hat{W}^*_{v_2,u_2,m,i_2}\big]$$
$$= \frac{1}{2}\,\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2} + \left(\frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\right)^2 \sum_{\substack{q_1=1\\ q_1\ \text{odd}}}^{Q_P} \left(\frac{A_{2e}^2(K-1)}{2} + N_0\right)\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}$$
$$= \frac{1}{2}\,\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2} + \left(\frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\right)^2 \cdot \frac{Q_P}{2}\left(\frac{A_{2e}^2(K-1)}{2} + N_0\right)\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}.$$
$$= \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\left(\frac{K-1}{N} + \frac{2N_0}{A_{2e}^2 N}\right)\right]\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}. \tag{U.49}$$

Since

$$E_{b2} = \frac{NR_2A_{2e}^2}{2} = NR_2A_{2d}^2 \tag{U.50}$$

from (P.20), we can re-write (U.49) as

$$\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1} \hat{W}^*_{v_2,u_2,m,i_2}\big] = \frac{1}{2}\left[1 + \frac{1}{Q_P/2}\,c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}, \tag{U.51}$$

where

$$c_2\left(\frac{E_{b2}}{N_0}\right) = \frac{K-1}{N} + \frac{R_2}{E_{b2}/N_0}. \tag{U.52}$$

U.2 Derivation of $\frac{1}{2}E[Y_{q,v_1,m,i_1} Y^*_{q,v_2,m,i_2}]$

From (3.60), we have

$$Y_{q,v,m,i} = A_{2d}\sqrt{N}\sum_{u=1}^{2} b^{(1)}_{p,q,u,m}\, g^{(1)}_{v,u,m,i} + I_{Y_{q,v,m,i}} + N_{Y_{q,v,m,i}}, \quad q = Q_P+1,\ldots,Q, \tag{U.53}$$

where the expressions for $I_{Y_{q,v,m,i}}$ and $N_{Y_{q,v,m,i}}$ are identical to the ones for $I_{W_{q,v,m,i}}$ and $N_{W_{q,v,m,i}}$ in (3.61)–(3.62):

$$\begin{aligned} I_{Y_{q,v,m,i}} &= \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C^{(1)}_{p,q,n}\sum_{k=2}^{K}\sum_{u=1}^{2} g^{(k)}_{v,u,m,i}\sum_{p'=-\infty}^{\infty}\sum_{q'=1}^{Q}\sum_{n'=1}^{N} A_{2,q'}\, b^{(k)}_{p',q',u,m} C^{(k)}_{p',q',n'} \\ &\quad \cdot x_2\{t - [(p'Q+q'-1)N+n'-1]T_c - \tau^{(k)}\}\Big|_{t=[(pQ+q-1)N+n-1]T_c}, \end{aligned} \tag{U.54}$$

$$N_{Y_{q,v,m,i}} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C^{(1)}_{p,q,n}\left(\tilde{n}_v(t) * \tilde{h}_{m,i,2}(t)\right)\Big|_{t=[(pQ+q-1)N+n-1]T_c}. \tag{U.55}$$

Assume q is *odd*. We define

$$\Gamma_{Y_{q,v,m,i}} \triangleq I_{Y_{q,v,m,i}} + N_{Y_{q,v,m,i}} \tag{U.56}$$

and make the substitution (see Table U.1 on p. 246)

$$b^{(k)}_{p,q,u,m} = \begin{cases} a^{(k)}_{p,q,m}, & u=1, \\ a^{(k)}_{p,q+1,m}, & u=2, \end{cases} \quad (q = Q_P+1,\ldots,Q;\ q \text{ odd}) \tag{U.57}$$

in (U.53) to get

$$\begin{aligned} Y_{q,v,m,i} &= A_{2d}\sqrt{N}\, a^{(1)}_{p,q,m}\, g^{(1)}_{v,1,m,i} + A_{2d}\sqrt{N}\, a^{(1)}_{p,q+1,m}\, g^{(1)}_{v,2,m,i} \\ &\quad + \Gamma_{Y_{q,v,m,i}}, \quad q = Q_P+1,\ldots,Q;\ q \text{ odd}. \end{aligned} \tag{U.58}$$

Using this expression, we can show that

$$\frac{1}{2}E\big[Y_{q,v_1,m,i_1}Y^*_{q,v_2,m,i_2}\big] = A_{2d}^2 N\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A_{2d}^2 N\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] + \frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q,v_2,m,i_2}}\big], \quad q = Q_P+1,\ldots,Q;\ q \text{ odd}, \tag{U.59}$$

where the expected values of the cross terms are easily shown to equal zero.

U.2.1 Sum of First and Second Terms in (U.59)

The sum of the first and second terms in (U.59) is equal to (see (U.8))

$$\begin{aligned} &A_{2d}^2 N\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A_{2d}^2 N\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\ &\quad = A_{2d}^2 N\cdot\frac{1}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2} + A_{2d}^2 N\cdot\frac{1}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2} \\ &\quad = A_{2d}^2 N\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \end{aligned} \tag{U.60}$$

U.2.2 Third Term in (U.59)

Using (U.56) to expand the third term in (U.59), we have

$$\frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q,v_2,m,i_2}}\big] = \frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}}I^*_{Y_{q,v_2,m,i_2}} + N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q,v_2,m,i_2}}\big], \tag{U.61}$$

where, again, the expected values of the cross terms are equal to zero.

First Term in (U.61)

We can re-write $I_{Y_{q,v,m,i}}$ in (U.54) as

$$I_{Y_{q,v,m,i}} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C^{(1)}_{p,q,n} I_{y_{v,m,i}}\big\{[(pQ+q-1)N+n-1]T_c\big\}, \quad q = Q_P+1,\ldots,Q;\ q \text{ odd}, \tag{U.62}$$

where $I_{y_{v,m,i}}(t)$ was defined in (U.17). Thus, we have

$$\begin{aligned}
&\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \\
&\qquad \cdot \frac{1}{2}E\Big[I_{y_{v_1,m,i_1}}\big\{[(pQ+q-1)N+n_1-1]T_c\big\} I^*_{y_{v_2,m,i_2}}\big\{[(pQ+q-1)N+n_2-1]T_c\big\}\Big] \\
&\quad = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \sum_{k_1=2}^{K}\sum_{k_2=2}^{K}\sum_{u'_1=1}^{2}\sum_{u'_2=1}^{2} \\
&\qquad \cdot \frac{1}{2}E\big[g^{(k_1)}_{v_1,u'_1,m,i_1}\big(g^{(k_2)}_{v_2,u'_2,m,i_2}\big)^*\big] \sum_{p'_1=-\infty}^{\infty}\sum_{p'_2=-\infty}^{\infty}\sum_{q'_1=1}^{Q}\sum_{q'_2=1}^{Q}\sum_{n'_1=1}^{N}\sum_{n'_2=1}^{N} A_{2,q'_1}A_{2,q'_2} \\
&\qquad \cdot E\big[b^{(k_1)}_{p'_1,q'_1,u'_1,m} b^{(k_2)}_{p'_2,q'_2,u'_2,m}\big] E\big[C^{(k_1)}_{p'_1,q'_1,n'_1} C^{(k_2)}_{p'_2,q'_2,n'_2}\big] \\
&\qquad \cdot E\Big[x_2\big\{t_1-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\} \\
&\qquad\qquad x_2\big\{t_2-[(p'_2Q+q'_2-1)N+n'_2-1]T_c-\tau^{(k_2)}\big\}\Big]\Big|_{\substack{t_1=[(pQ+q-1)N+n_1-1]T_c \\ t_2=[(pQ+q-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.63}$$

So far, the derivation is identical to the derivation of $\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ in Section U.1.2, except that $q_1(u_1)$ and $q_1(u_2)$ have been replaced by q. Therefore, we follow the same steps as outlined between (U.20) and (U.27) to obtain

$$\begin{aligned}
&\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=1}^{Q}\sum_{n'=1}^{N} \\
&\qquad \cdot E\Big[x_2\big\{t_1-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\} \\
&\qquad\qquad x_2\big\{t_2-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\substack{t_1=[(pQ+q-1)N+n_1-1]T_c \\ t_2=[(pQ+q-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.64}$$

We then apply (Q.41) to the above equation and use (Q.44) to get

$$\begin{aligned}
&\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q,n_2} \sum_{k_1=2}^{K}\frac{1}{T_c}\int_{-\infty}^{\infty}|X_2(f)|^2 e^{j2\pi f\tau}\,df\Big|_{\tau=t_2-t_1=(n_2-n_1)T_c}
\end{aligned} \tag{U.65}$$

$$= \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)} \sum_{k_1=2}^{K}\frac{1}{T_c}\cdot T_c\,\delta_{n_1,n_2}$$
$$= \frac{A_{2e}^2(K-1)}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \tag{U.66}$$

Second Term in (U.61)

We use $N_{y_{v,m,i}}(t)$ in (U.29) to re-write $N_{Y_{q,v,m,i}}$ as (see (U.55))

$$N_{Y_{q,v,m,i}} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} C_{p,q,n}^{(1)} N_{y_{v,m,i}}\big\{[(pQ+q-1)N+n-1]T_c\big\}, \quad q = Q_P+1,\ldots,Q;\ q \text{ odd.} \tag{U.67}$$

Thus, we have

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N_{Y_{q,v_2,m,i_2}}^*\big]$$
$$= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)}\cdot\frac{1}{2}E\Big[N_{y_{v_1,m,i_1}}\big\{[(pQ+q-1)N+n_1-1]T_c\big\} \tag{U.68}$$
$$N_{y_{v_2,m,i_2}}^*\big\{[(pQ+q-1)N+n_2-1]T_c\big\}\Big].$$

When $i_1 \neq i_2$, the AWGN terms $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are once again independent, and this implies that $N_{Y_{q,v_1,m,i_1}}$ and $N_{Y_{q,v_2,m,i_2}}$ are independent as well:

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N_{Y_{q,v_2,m,i_2}}^*\big] = 0, \quad i_1 \neq i_2. \tag{U.69}$$

When $i_1 = i_2$, we use (U.29) in (U.68) to write

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N_{Y_{q,v_2,m,i_1}}^*\big]$$
$$= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)}\cdot\frac{1}{2}E\Big[N_{y_{v_1,m,i_1}}\big\{[(pQ+q-1)N+n_1-1]T_c\big\}$$
$$N_{y_{v_2,m,i_1}}^*\big\{[(pQ+q-1)N+n_2-1]T_c\big\}\Big]$$
$$= \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C_{p,q,n_1}^{(1)} C_{p,q,n_2}^{(1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}E\big[\tilde{n}_{v_1}(\tau_1)\tilde{n}_{v_2}^*(\tau_2)\big]$$
$$\cdot\,\tilde{h}_{m,i_1,2}(t_1-\tau_1)\tilde{h}_{m,i_1,2}^*(t_2-\tau_2)\,d\tau_2\,d\tau_1\Big|_{\substack{t_1=[(pQ+q-1)N+n_1-1]T_c\\ t_2=[(pQ+q-1)N+n_2-1]T_c}}. \tag{U.70}$$

This derivation is identical to the one for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}}N^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ in Section U.1.2, except that $q_1(u_1)$ and $q_1(u_2)$ have been replaced by q. Therefore, we follow the same steps as in that derivation to show that (U.70) is equal to

$$\begin{aligned}\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q,v_2,m,i_1}}\big] &= \frac{N_0}{N}\,\delta_{v_1,v_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}C^{(1)}_{p,q,n_1}C^{(1)}_{p,q,n_2}\cdot x_2(\tau)\Big|_{\tau=t_2-t_1=(n_2-n_1)T_c} \\ &= \frac{N_0}{N}\,\delta_{v_1,v_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}C^{(1)}_{p,q,n_1}C^{(1)}_{p,q,n_2}\cdot\delta_{n_1,n_2} \\ &= N_0\,\delta_{v_1,v_2}. \end{aligned} \tag{U.71}$$

Combining the results of (U.69) and (U.71), we get

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q,v_2,m,i_2}}\big] = N_0\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \tag{U.72}$$

Combining the Results

Substituting (U.66) and (U.72) into (U.61), we obtain

$$\frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q,v_2,m,i_2}}\big] = \left(\frac{A_{2e}^2(K-1)}{2}+N_0\right)\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \tag{U.73}$$

U.2.3 Combining the Results

Finally, we substitute the results of (U.60) and (U.73) into (U.59), and we apply (U.50) to get

$$\begin{aligned}\frac{1}{2}E\big[Y_{q,v_1,m,i_1}Y^*_{q,v_2,m,i_2}\big] &= \left(A_{2d}^2N+\frac{A_{2e}^2(K-1)}{2}+N_0\right)\,\delta_{v_1,v_2}\delta_{i_1,i_2} \\ &= \left(\frac{E_{b2}}{R_2}+\frac{E_{b2}}{R_2}\frac{K-1}{N}+N_0\right)\,\delta_{v_1,v_2}\delta_{i_1,i_2} \\ &= \frac{E_{b2}}{R_2}\left[1+c_2\!\left(\frac{E_{b2}}{N_0}\right)\right]\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \end{aligned} \tag{U.74}$$

U.3 Derivation of $\frac{1}{2}E[Y_{q+1,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}]$

For q odd, we use (U.53) and

$$b^{(k)}_{p,q,u,m} = \begin{cases} -a^{(k)}_{p,q,m}, & u = 1, \\ a^{(k)}_{p,q-1,m}, & u = 2, \end{cases} \quad (q = Q_P + 1, \ldots, Q;\ q \text{ even}) \tag{U.75}$$

from Table U.1 (see p. 246) to write

$$Y_{q+1,v,m,i} = -A_{2d}\sqrt{N}\, a^{(1)}_{p,q+1,m}\, g^{(1)}_{v,1,m,i} + A_{2d}\sqrt{N}\, a^{(1)}_{p,q,m}\, g^{(1)}_{v,2,m,i} + \Gamma_{Y_{q+1,v,m,i}}, \quad q = Q_P + 1, \ldots, Q;\ q \text{ odd}, \tag{U.76}$$

where

$$\Gamma_{Y_{q+1,v,m,i}} \triangleq I_{Y_{q+1,v,m,i}} + N_{Y_{q+1,v,m,i}}. \tag{U.77}$$

Using this expression, we have

$$\begin{aligned} &\frac{1}{2}E\big[Y_{q+1,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}\big] \\ &\quad = A^2_{2d}N\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A^2_{2d}N\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\ &\quad + \frac{1}{2}E\big[\Gamma_{Y_{q+1,v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big], \quad q = Q_P + 1, \ldots, Q;\ q \text{ odd}, \end{aligned} \tag{U.78}$$

where the expected values of the cross terms are easily shown to equal zero.

U.3.1 Sum of First and Second Terms in (U.78)

The sum of the first and second terms in (U.78) is given by (U.60):

$$A^2_{2d}N\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A^2_{2d}N\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] = A^2_{2d}N\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \tag{U.79}$$

U.3.2 Third Term in (U.78)

Using (U.77), we have

$$\frac{1}{2}E\big[\Gamma_{Y_{q+1,v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{1}{2}E\big[I_{Y_{q+1,v_1,m,i_1}}I^*_{Y_{q+1,v_2,m,i_2}} + N_{Y_{q+1,v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_2}}\big]. \tag{U.80}$$

First Term in (U.80)

We replace q with $q+1$ in the derivation for $\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q,v_2,m,i_2}}\big]$ to obtain the same result as (U.66):

$$\frac{1}{2}E\big[I_{Y_{q+1,v_1,m,i_1}} I^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{A_{2e}^2(K-1)}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \tag{U.81}$$

Second Term in (U.80)

Similarly, we replace q with $q+1$ in the derivation for $\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big]$ to obtain (see (U.72))

$$\frac{1}{2}E\big[N_{Y_{q+1,v_1,m,i_1}} N^*_{Y_{q+1,v_2,m,i_1}}\big] = N_0\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \tag{U.82}$$

Combining the Results

We substitute (U.81) and (U.82) into (U.80) to get

$$\frac{1}{2}E\big[\Gamma_{Y_{q+1,v_1,m,i_1}} \Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = \left(\frac{A_{2e}^2(K-1)}{2} + N_0\right)\delta_{v_1,v_2}\delta_{i_1,i_2}. \tag{U.83}$$

U.3.3 Combining the Results

Finally, we substitute the results of (U.79) and (U.83) into (U.78) to get the same result as (U.74):

$$\begin{aligned}\frac{1}{2}E\big[Y_{q+1,v_1,m,i_1} Y^*_{q+1,v_2,m,i_2}\big] &= \left(A_{2d}^2 N + \frac{A_{2e}^2(K-1)}{2} + N_0\right)\delta_{v_1,v_2}\delta_{i_1,i_2} \\ &= \frac{E_{b2}}{R_2}\left[1 + c_2\left(\frac{E_{b2}}{N_0}\right)\right]\delta_{v_1,v_2}\delta_{i_1,i_2}. \end{aligned} \tag{U.84}$$

U.4 Derivation of $\frac{1}{2}E[Y_{q,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}]$

We use (U.58) and (U.76) to write

$$\begin{aligned}\frac{1}{2}E\big[Y_{q,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}\big] &= -A^2_{2d}Na^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] \\ &\quad + A^2_{2d}Na^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\ &\quad + \frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big], \quad q = Q_P+1,\ldots,Q;\ q \text{ odd},\end{aligned} \tag{U.85}$$

where the expected values of the cross terms are equal to zero.

U.4.1 Sum of First and Second Terms in (U.85)

Using (U.8), we evaluate the sum of the first and second terms in (U.85) as

$$\begin{aligned}&-A^2_{2d}Na^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] \\ &\qquad + A^2_{2d}Na^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\ &\quad = -A^2_{2d}Na^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2} + A^2_{2d}Na^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2} \\ &\quad = 0.\end{aligned} \tag{U.86}$$

U.4.2 Third Term in (U.85)

We use (U.56) and (U.77) to expand the third term in (U.85) as

$$\frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}}I^*_{Y_{q+1,v_2,m,i_2}} + N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_2}}\big], \tag{U.87}$$

where, again, the expected values of the cross terms are equal to zero.

First Term in (U.87)

We use the expression for $I_{Y_{q,v,m,i}}$ in (U.62) to write

$$\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q+1,n_2} \frac{1}{2}E\Big[I_{y_{v_1,m,i_1}}\big\{[(pQ+q-1)N+n_1-1]T_c\big\} I^*_{y_{v_2,m,i_2}}\big\{[(pQ+q)N+n_2-1]T_c\big\}\Big]. \tag{U.88}$$

By replacing $q_1(u_1)$ and $q_1(u_2)$ in the derivation for $\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ (see Section U.1.2) with q and $q+1$, respectively, we obtain (see (U.27))

$$\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{A^2_{2e}}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q+1,n_2}\sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=1}^{Q}\sum_{n'=1}^{N} \cdot E\Big[x_2\big\{t_1-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\} x_2\big\{t_2-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\}\Big]\Big|_{\substack{t_1=[(pQ+q-1)N+n_1-1]T_c\\ t_2=[(pQ+q)N+n_2-1]T_c}}. \tag{U.89}$$

We then apply (Q.41) to the above equation and use (Q.44) to get

$$\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{A^2_{2e}}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q+1,n_2} \cdot \sum_{k_1=2}^{K}\frac{1}{T_c}\int_{-\infty}^{\infty}|X_2(f)|^2 e^{j2\pi f\tau}\,df\Big|_{\tau=t_2-t_1=[N+(n_2-n_1)]T_c} = 0. \tag{U.90}$$

Second Term in (U.87)

We use the expression for $N_{Y_{q,v,m,i}}$ in (U.67) to write

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q+1,n_2}\cdot\frac{1}{2}E\Big[N_{y_{v_1,m,i_1}}\big\{[(pQ+q-1)N+n_1-1]T_c\big\} N^*_{y_{v_2,m,i_2}}\big\{[(pQ+q)N+n_2-1]T_c\big\}\Big]. \tag{U.91}$$

When $i_1 \neq i_2$, the AWGN terms $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are independent, so $N_{Y_{q,v_1,m,i_1}}$ and $N_{Y_{q+1,v_2,m,i_2}}$ are independent as well:

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q+1,v_2,m,i_2}}\big] = 0, \quad i_1 \neq i_2. \tag{U.92}$$

For $i_1 = i_2$, we use the derivation for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ in Section U.1.2 to evaluate (U.91). We replace $q_1(u_1)$ and $q_1(u_2)$ with q and $q+1$, respectively, in (U.36) to obtain

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q+1,v_2,m,i_1}}\big] = \frac{N_0}{N}\,\delta_{v_1,v_2} \sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q+1,n_2} \cdot \int_{-\infty}^{\infty} \tilde{h}_{m,i_1,2}(t_1-\tau_1)\tilde{h}^*_{m,i_1,2}(t_2-\tau_1)\,d\tau_1\Big|_{\substack{t_1=[(pQ+q-1)N+n_1-1]T_c \\ t_2=[(pQ+q)N+n_2-1]T_c}}. \tag{U.93}$$

To the above equation, we then apply (U.37) and (Q.38) to get

$$\begin{aligned}\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q+1,v_2,m,i_1}}\big] &= \frac{N_0}{N}\,\delta_{v_1,v_2} \sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q,n_1} C^{(1)}_{p,q+1,n_2} \cdot x_2(\tau)\Big|_{\tau=t_2-t_1=[N+(n_2-n_1)]T_c} \\ &= 0.\end{aligned} \tag{U.94}$$

Combining the results of (U.92) and (U.94), we have

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q+1,v_2,m,i_2}}\big] = 0. \tag{U.95}$$

Combining the Results

Substituting (U.90) and (U.95) into (U.87), we obtain

$$\frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}} \Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = 0. \tag{U.96}$$

U.4.3 Combining the Results

Finally, we substitute the results of (U.86) and (U.96) into (U.85) to get

$$\frac{1}{2}E\big[Y_{q,v_1,m,i_1} Y^*_{q+1,v_2,m,i_2}\big] = 0. \tag{U.97}$$

U.5 Derivation of $\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1} Y^*_{q,v_2,m,i_2}]$

Assuming $q = Q_P + 1, \ldots, Q$ and q odd, we use (U.11) and (U.58) to write

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1} Y^*_{q,v_2,m,i_2}\big] \\
&= \frac{1}{2}E\Bigg[\Bigg(g^{(1)}_{v_1,u_1,m,i_1} + \frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\sum_{\substack{q'=1\\ q' \text{ odd}}}^{Q_P} \Gamma_{W_{q'(u_1),v_1,m,i_1}}\Bigg) \\
&\quad \cdot \Bigg(A_{2d}\sqrt{N}\, a^{(1)}_{p,q,m}\, g^{(1)}_{v_2,1,m,i_2} + A_{2d}\sqrt{N}\, a^{(1)}_{p,q+1,m}\, g^{(1)}_{v_2,2,m,i_2} + \Gamma_{Y_{q,v_2,m,i_2}}\Bigg)^*\Bigg] \\
&= A_{2d}\sqrt{N}\, a^{(1)}_{p,q,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A_{2d}\sqrt{N}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\
&\quad + \frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\sum_{\substack{q'=1\\ q' \text{ odd}}}^{Q_P} \frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}}\Gamma^*_{Y_{q,v_2,m,i_2}}\big],
\end{aligned} \tag{U.98}$$

where $q'(u) \triangleq q' + (u-1)$, and we used the fact that

$$\frac{1}{2}E\Big[g^{(k)}_{v_1,u,m,i_1}\Gamma^*_{Y_{q,v_2,m,i_2}}\Big] = 0, \tag{U.99}$$

which is straightforward to prove.

U.5.1 First and Second Terms in (U.98)

Using (U.8), we have

$$A_{2d}\sqrt{N}\, a^{(1)}_{p,q,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] = A_{2d}\sqrt{N}a^{(1)}_{p,q,m} \cdot \frac{1}{2}\delta_{v_1,v_2}\delta_{u_1,1}\delta_{i_1,i_2}. \tag{U.100}$$

Similarly, we also have

$$A_{2d}\sqrt{N}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] = A_{2d}\sqrt{N}a^{(1)}_{p,q+1,m} \cdot \frac{1}{2}\delta_{v_1,v_2}\delta_{u_1,2}\delta_{i_1,i_2}. \tag{U.101}$$

U.5.2 Third Term in (U.98)

We use (U.9) and (U.56) to write

$$\frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}}\Gamma^*_{Y_{q,v_2,m,i_2}}\big] = \frac{1}{2}E\big[I_{W_{q'(u_1),v_1,m,i_1}}I^*_{Y_{q,v_2,m,i_2}} + N_{W_{q'(u_1),v_1,m,i_1}}N^*_{Y_{q,v_2,m,i_2}}\big], \tag{U.102}$$

where, again, the expected values of the cross terms are equal to zero.

First Term in (U.102)

We use the expressions for $I_{W_{q(u),v,m,i}}$ and $I_{Y_{q,v,m,i}}$ in (U.18) and (U.62), respectively, to write

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q'(u_1),v_1,m,i_1}}I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}C^{(1)}_{p,q'(u_1),n_1}C^{(1)}_{p,q,n_2}\frac{1}{2}E\Big[I_{y_{v_1,m,i_1}}\big\{[(pQ+q'(u_1)-1)N+n_1-1]T_c\big\} \\
&\qquad\qquad I^*_{y_{v_2,m,i_2}}\big\{[(pQ+q-1)N+n_2-1]T_c\big\}\Big].
\end{aligned} \tag{U.103}$$

By replacing $q_1(u_1)$ and $q_1(u_2)$ in the derivation for $\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}}I^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ (see Section U.1.2) with $q'(u_1)$ and q, respectively, we can show that

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q'(u_1),v_1,m,i_1}}I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}C^{(1)}_{p,q'(u_1),n_1}C^{(1)}_{p,q,n_2}\sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=1}^{Q}\sum_{n'=1}^{N} \\
&\qquad \cdot E\Big[x_2\big\{t_1-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\} \\
&\qquad\qquad x_2\big\{t_2-[(p'_1Q+q'_1-1)N+n'_1-1]T_c-\tau^{(k_1)}\big\}\Big]\bigg|_{\substack{t_1=[(pQ+q'(u_1)-1)N+n_1-1]T_c \\ t_2=[(pQ+q-1)N+n_2-1]T_c}} \\
&\quad = \frac{A_{2e}^2}{2N}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}C^{(1)}_{p,q'(u_1),n_1}C^{(1)}_{p,q,n_2} \\
&\qquad \cdot \sum_{k_1=2}^{K}\frac{1}{T_c}\int_{-\infty}^{\infty}|X_2(f)|^2e^{j2\pi f\tau}\,df\bigg|_{\tau=t_2-t_1=\big\{[q-q'-(u_1-1)]N+(n_2-n_1)\big\}T_c} \\
&\quad = 0.
\end{aligned} \tag{U.104}$$

The last two equalities follow from (Q.41) and (Q.44), along with the fact that $q - q' - (u_1 - 1)$ can never equal zero, since $q' = 1, \ldots, Q_P$ and $q = Q_P + 1, \ldots, Q$, and q' and q are both odd.

Second Term in (U.102)

We use the expressions for $N_{W_{q(u),v,m,i}}$ and $N_{Y_{q,v,m,i}}$ in (U.30) and (U.67), respectively, to write

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = \frac{1}{N}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q'(u_1),n_1} C^{(1)}_{p,q,n_2} \cdot \frac{1}{2}E\Big[N_{y_{v_1,m,i_1}}\big\{[(pQ + q'(u_1) - 1)N + n_1 - 1]T_c\big\} \\
&\qquad\qquad N^*_{y_{v_2,m,i_2}}\big\{[(pQ + q - 1)N + n_2 - 1]T_c\big\}\Big].
\end{aligned} \tag{U.105}$$

When $i_1 \neq i_2$, the AWGN terms $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are independent, so $N_{W_{q'(u_1),v_1,m,i_1}}$ and $N_{Y_{q,v_2,m,i_2}}$ are independent as well:

$$\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big] = 0, \quad i_1 \neq i_2. \tag{U.106}$$

For $i_1 = i_2$, we use the derivation for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ in Section U.1.2 to evaluate (U.105). That is, we replace $q_1(u_1)$ and $q_1(u_2)$ in (U.36) with $q'(u_1)$ and q, respectively, to obtain

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_1}}\big] \\
&\quad = \frac{N_0}{N}\,\delta_{v_1,v_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q'(u_1),n_1} C^{(1)}_{p,q,n_2} \\
&\qquad \cdot \int_{-\infty}^{\infty} \tilde{h}_{m,i_1,2}(t_1 - \tau_1)\tilde{h}^*_{m,i_1,2}(t_2 - \tau_1)\,d\tau_1\Big|_{\substack{t_1 = [(pQ+q'(u_1)-1)N+n_1-1]T_c \\ t_2 = [(pQ+q-1)N+n_2-1]T_c}}.
\end{aligned} \tag{U.107}$$

We then apply (U.37) and (Q.38) to the above equation to get

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_1}}\big] \\
&\quad = \frac{N_0}{N}\,\delta_{v_1,v_2}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N} C^{(1)}_{p,q'(u_1),n_1} C^{(1)}_{p,q,n_2} \cdot x_2(\tau)\Big|_{\tau = t_2 - t_1 = \big\{[q - q' - (u_1 - 1)]N + (n_2 - n_1)\big\}T_c} \\
&\quad = 0.
\end{aligned} \tag{U.108}$$

Again, the quantity $q - q' - (u_1 - 1)$ can never equal zero.

Combining the results of (U.106) and (U.108), we have

$$\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big] = 0. \tag{U.109}$$

Combining the Results

Substituting (U.104) and (U.109) into (U.102), we obtain

$$\frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}} \Gamma^*_{Y_{q,v_2,m,i_2}}\big] = 0. \tag{U.110}$$

U.5.3 Combining the Results

Finally, we substitute the results of (U.100), (U.101), and (U.110) into (U.98) to get

$$\begin{aligned}
\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1} Y^*_{q,v_2,m,i_2}\big] &= \frac{A_{2d}\sqrt{N}}{2}\left(a^{(1)}_{p,q,m}\delta_{u_1,1} + a^{(1)}_{p,q+1,m}\delta_{u_1,2}\right)\delta_{v_1,v_2}\delta_{i_1,i_2} \\
&= \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\left(a^{(1)}_{p,q,m}\delta_{u_1,1} + a^{(1)}_{p,q+1,m}\delta_{u_1,2}\right)\delta_{v_1,v_2}\delta_{i_1,i_2} \\
&= \begin{cases} \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\, a^{(1)}_{p,q,m}\delta_{v_1,v_2}\delta_{i_1,i_2}, & u_1 = 1, \\ \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\, a^{(1)}_{p,q+1,m}\delta_{v_1,v_2}\delta_{i_1,i_2}, & u_1 = 2, \end{cases}
\end{aligned} \tag{U.111}$$

where we used the expression for E_{b2} in (U.50).

U.6 Derivation of $\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1} Y^*_{q+1,v_2,m,i_2}]$

Again, we assume $q = Q_P+1, \ldots, Q$ and q odd, but this time, we use (U.11) and (U.76) to write

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1} Y^*_{q+1,v_2,m,i_2}\big] \\
&\quad= \frac{1}{2}E\Bigg[\Bigg(g^{(1)}_{v_1,u_1,m,i_1} + \frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\sum_{\substack{q'=1\\q'\text{ odd}}}^{Q_P} \Gamma_{W_{q'(u_1),v_1,m,i_1}}\Bigg) \\
&\qquad\cdot\Big(-A_{2d}\sqrt{N}\,a^{(1)}_{p,q+1,m}\,g^{(1)}_{v_2,1,m,i_2} + A_{2d}\sqrt{N}\,a^{(1)}_{p,q,m}\,g^{(1)}_{v_2,2,m,i_2} + \Gamma_{Y_{q+1,v_2,m,i_2}}\Big)^*\Bigg] \\
&\quad= -A_{2d}\sqrt{N}\,a^{(1)}_{p,q+1,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] \\
&\qquad+ A_{2d}\sqrt{N}\,a^{(1)}_{p,q,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\
&\qquad+ \frac{1}{A_{2e}\sqrt{N}}\frac{1}{Q_P/2}\sum_{\substack{q'=1\\q'\text{ odd}}}^{Q_P}\frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big],
\end{aligned} \tag{U.112}$$

where $q'(u) \triangleq q' + (u-1)$, and we used the fact that

$$\frac{1}{2}E\Big[g^{(k)}_{v_1,u,m,i_1}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\Big] = 0, \tag{U.113}$$

which is straightforward to prove.

U.6.1 First and Second Terms in (U.112)

Using (U.8), we have

$$-A_{2d}\sqrt{N}\,a^{(1)}_{p,q+1,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] = -A_{2d}\sqrt{N}\,a^{(1)}_{p,q+1,m}\cdot\frac{1}{2}\delta_{v_1,v_2}\,\delta_{u_1,1}\,\delta_{i_1,i_2}, \tag{U.114}$$

$$A_{2d}\sqrt{N}\,a^{(1)}_{p,q,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] = A_{2d}\sqrt{N}\,a^{(1)}_{p,q,m}\cdot\frac{1}{2}\delta_{v_1,v_2}\,\delta_{u_1,2}\,\delta_{i_1,i_2}. \tag{U.115}$$

U.6.2 Third Term in (U.112)

Using (U.9) and (U.77), we can write

$$\frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{1}{2}E\big[I_{W_{q'(u_1),v_1,m,i_1}}I^*_{Y_{q+1,v_2,m,i_2}} + N_{W_{q'(u_1),v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_2}}\big], \tag{U.116}$$

where, again, the expected values of the cross terms are equal to zero.

First Term in (U.116)

We simply replace q with $q+1$ in the derivation for $\frac{1}{2}E\big[I_{W_{q'(u_1),v_1,m,i_1}}I^*_{Y_{q,v_2,m,i_2}}\big]$ in Section U.5.2 to obtain (see (U.104))

$$\frac{1}{2}E\big[I_{W_{q'(u_1),v_1,m,i_1}}I^*_{Y_{q+1,v_2,m,i_2}}\big] = 0, \tag{U.117}$$

since $q - q' - (u_1 - 1) + 1$ can never equal zero.

Second Term in (U.116)

Similarly, we replace q with $q+1$ in the derivation for $\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}}N^*_{Y_{q,v_2,m,i_2}}\big]$ to obtain (see (U.109))

$$\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_2}}\big] = 0. \tag{U.118}$$

Combining the Results

Substituting (U.117) and (U.118) into (U.116), we get

$$\frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = 0. \tag{U.119}$$

U.6.3 Combining the Results

Finally, we substitute the results of (U.114), (U.115), and (U.119) into (U.112) to get

$$\begin{aligned}\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1}Y^*_{q+1,v_2,m,i_2}\big] &= -\frac{A_{2d}\sqrt{N}}{2}\left(a^{(1)}_{p,q+1,m}\delta_{u_1,1} - a^{(1)}_{p,q,m}\delta_{u_1,2}\right)\delta_{v_1,v_2}\delta_{i_1,i_2} \\ &= -\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\left(a^{(1)}_{p,q+1,m}\delta_{u_1,1} - a^{(1)}_{p,q,m}\delta_{u_1,2}\right)\delta_{v_1,v_2}\delta_{i_1,i_2} \\ &= \begin{cases} -\frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\,a^{(1)}_{p,q+1,m}\delta_{v_1,v_2}\delta_{i_1,i_2}, & u_1 = 1, \\ \frac{1}{2}\sqrt{\frac{E_{b2}}{R_2}}\,a^{(1)}_{p,q,m}\delta_{v_1,v_2}\delta_{i_1,i_2}, & u_1 = 2. \end{cases}\end{aligned} \tag{U.120}$$

Appendix V

Proof of (U.23)

First, we define

$$\beta(u_1', u_2') \triangleq A_{2d}^2 \sum_{q_1'=Q_P+1}^{Q} \sum_{q_2'=Q_P+1}^{Q} E\big[b_{p_1',q_1',u_1',m}^{(k_1)} b_{p_2',q_2',u_2',m}^{(k_2)}\big], \tag{V.1}$$

such that the left hand side of (U.23) can be re-written as

$$\sum_{u_1'=1}^{2} \sum_{u_2'=1}^{2} \sum_{q_1'=Q_P+1}^{Q} \sum_{q_2'=Q_P+1}^{Q} A_{2d}^2 E\big[b_{p_1',q_1',u_1',m}^{(k_1)} b_{p_2',q_2',u_2',m}^{(k_2)}\big] = \sum_{u_1'=1}^{2} \sum_{u_2'=1}^{2} \beta(u_1', u_2'). \tag{V.2}$$

We then expand $\beta(u_1', u_2')$ as

$$\beta(u_1', u_2') = A_{2d}^2 \Bigg(\sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ odd}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ odd}}}^{Q} + \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ odd}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ even}}}^{Q} + \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ even}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ odd}}}^{Q} + \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ even}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ even}}}^{Q} \Bigg) \cdot E\big[b_{p_1',q_1',u_1',m}^{(k_1)} b_{p_2',q_2',u_2',m}^{(k_2)}\big]. \tag{V.3}$$

The right hand side of (V.2) consists of a sum of four terms. We examine each of these terms separately.

V.1 $u_1' = 1$, $u_2' = 1$

For $q = Q_P + 1, \ldots, Q$, $b^{(k)}_{p,q,u,m}$ is defined as (see Table U.1 on p. 246)

$$b^{(k)}_{p,q,u,m} = \begin{cases} a^{(k)}_{p,q,m}, & u = 1,\, q \text{ odd}, \\ a^{(k)}_{p,q+1,m}, & u = 2,\, q \text{ odd}, \\ -a^{(k)}_{p,q,m}, & u = 1,\, q \text{ even}, \\ a^{(k)}_{p,q-1,m}, & u = 2,\, q \text{ even}, \end{cases} \quad (q = Q_P + 1, \ldots, Q). \tag{V.4}$$

We use (V.4) in (V.3) to write $\beta(1,1)$ as

$$\begin{aligned}\beta(1,1) = {} & A_{2d}^2 \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ odd}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ odd}}}^{Q} E\big[a^{(k_1)}_{p_1',q_1',m} a^{(k_1)}_{p_2',q_2',m}\big] - A_{2d}^2 \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ odd}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ even}}}^{Q} E\big[a^{(k_1)}_{p_1',q_1',m} a^{(k_1)}_{p_2',q_2',m}\big] \\ & - A_{2d}^2 \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ even}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ odd}}}^{Q} E\big[a^{(k_1)}_{p_1',q_1',m} a^{(k_1)}_{p_2',q_2',m}\big] + A_{2d}^2 \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ even}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ even}}}^{Q} E\big[a^{(k_1)}_{p_1',q_1',m} a^{(k_1)}_{p_2',q_2',m}\big].\end{aligned} \tag{V.5}$$

We then apply (see (U.7))

$$E\big[a^{(k_1)}_{p_1,q_1,m} a^{(k_2)}_{p_2,q_2,m}\big] = \delta_{k_1,k_2}\delta_{p_1,p_2}\delta_{q_1,q_2}, \quad q_i = Q_P + 1, \ldots, Q;\ i = 1, 2, \tag{V.6}$$

to this expression to get

$$\begin{aligned}\beta(1,1) = {} & A_{2d}^2 \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ odd}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ odd}}}^{Q} \delta_{p_1',p_2'}\delta_{q_1',q_2'}\delta_{k_1,k_2} - A_{2d}^2 \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ odd}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ even}}}^{Q} \delta_{p_1',p_2'}\delta_{q_1',q_2'}\delta_{k_1,k_2} \\ & - A_{2d}^2 \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ even}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ odd}}}^{Q} \delta_{p_1',p_2'}\delta_{q_1',q_2'}\delta_{k_1,k_2} + A_{2d}^2 \sum_{\substack{q_1'=Q_P+1 \\ q_1' \text{ even}}}^{Q} \sum_{\substack{q_2'=Q_P+1 \\ q_2' \text{ even}}}^{Q} \delta_{p_1',p_2'}\delta_{q_1',q_2'}\delta_{k_1,k_2}.\end{aligned} \tag{V.7}$$

The second term in (V.7) is equal to zero because the $\{q_1'\}$ are odd while the $\{q_2'\}$ are even, such that $q_1' \neq q_2'$ for all q_1' and q_2'. Likewise, the third term in (V.7) is also equal to zero because

$q'_1 \neq q'_2$ for all q'_1 and q'_2, since the $\{q'_1\}$ are even while the $\{q'_2\}$ are odd. Thus, $\beta(1,1)$ reduces to

$$\begin{aligned}\beta(1,1) &= A_{2d}^2 \sum_{\substack{q'_1=Q_P+1 \\ q'_1 \text{ odd}}}^{Q} \delta_{p'_1,p'_2}\delta_{k_1,k_2} + A_{2d}^2 \sum_{\substack{q'_1=Q_P+1 \\ q'_1 \text{ even}}}^{Q} \delta_{p'_1,p'_2}\delta_{k_1,k_2} \\ &= A_{2d}^2 \sum_{q'_1=Q_P+1}^{Q} \delta_{p'_1,p'_2}\delta_{k_1,k_2}. \end{aligned} \tag{V.8}$$

V.2 $u'_1 = 1$, $u'_2 = 2$

Similarly, $\beta(1,2)$ is derived as

$$\begin{aligned}\beta(1,2) = {} & A_{2d}^2 \sum_{\substack{q'_1=Q_P+1 \\ q'_1 \text{ odd}}}^{Q} \sum_{\substack{q'_2=Q_P+1 \\ q'_2 \text{ odd}}}^{Q} E\big[a_{p'_1,q'_1,m}^{(k_1)} a_{p'_2,q'_2+1,m}^{(k_1)}\big] + A_{2d}^2 \sum_{\substack{q'_1=Q_P+1 \\ q'_1 \text{ odd}}}^{Q} \sum_{\substack{q'_2=Q_P+1 \\ q'_2 \text{ even}}}^{Q} E\big[a_{p'_1,q'_1,m}^{(k_1)} a_{p'_2,q'_2-1,m}^{(k_1)}\big] \\ & - A_{2d}^2 \sum_{\substack{q'_1=Q_P+1 \\ q'_1 \text{ even}}}^{Q} \sum_{\substack{q'_2=Q_P+1 \\ q'_2 \text{ odd}}}^{Q} E\big[a_{p'_1,q'_1,m}^{(k_1)} a_{p'_2,q'_2+1,m}^{(k_1)}\big] \\ & - A_{2d}^2 \sum_{\substack{q'_1=Q_P+1 \\ q'_1 \text{ even}}}^{Q} \sum_{\substack{q'_2=Q_P+1 \\ q'_2 \text{ even}}}^{Q} E\big[a_{p'_1,q'_1,m}^{(k_1)} a_{p'_2,q'_2-1,m}^{(k_1)}\big]. \end{aligned} \tag{V.9}$$

This time, the first and fourth terms above are equal to zero, so

$$\beta(1,2) = A_{2d}^2 \sum_{\substack{q'_1=Q_P+1 \\ q'_1 \text{ odd}}}^{Q} \delta_{p'_1,p'_2}\delta_{k_1,k_2} - A_{2d}^2 \sum_{\substack{q'_1=Q_P+1 \\ q'_1 \text{ even}}}^{Q} \delta_{p'_1,p'_2}\delta_{k_1,k_2}. \tag{V.10}$$

V.3 $u'_1 = 2,\ u'_2 = 1$

Again, we use (V.4) in (V.3) to get

$$\begin{aligned}\beta(2,1) = A_{2d}^2 &\sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ odd}}}^{Q} \sum_{\substack{q'_2=Q_P+1\\ q'_2 \text{ odd}}}^{Q} E\big[a^{(k_1)}_{p'_1,q'_1+1,m} a^{(k_1)}_{p'_2,q'_2,m}\big] - A_{2d}^2 \sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ odd}}}^{Q} \sum_{\substack{q'_2=Q_P+1\\ q'_2 \text{ even}}}^{Q} E\big[a^{(k_1)}_{p'_1,q'_1+1,m} a^{(k_1)}_{p'_2,q'_2,m}\big] \\ &+ A_{2d}^2 \sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ even}}}^{Q} \sum_{\substack{q'_2=Q_P+1\\ q'_2 \text{ odd}}}^{Q} E\big[a^{(k_1)}_{p'_1,q'_1-1,m} a^{(k_1)}_{p'_2,q'_2,m}\big] \\ &- A_{2d}^2 \sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ even}}}^{Q} \sum_{\substack{q'_2=Q_P+1\\ q'_2 \text{ even}}}^{Q} E\big[a^{(k_1)}_{p'_1,q'_1-1,m} a^{(k_1)}_{p'_2,q'_2,m}\big].\end{aligned} \tag{V.11}$$

The first and fourth terms above are equal to zero, so

$$\beta(2,1) = -A_{2d}^2 \sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ odd}}}^{Q} \delta_{p'_1,p'_2}\delta_{k_1,k_2} + A_{2d}^2 \sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ even}}}^{Q} \delta_{p'_1,p'_2}\delta_{k_1,k_2}. \tag{V.12}$$

V.4 $u'_1 = 2,\ u'_2 = 2$

Finally, we have

$$\begin{aligned}\beta(2,2) = A_{2d}^2 &\sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ odd}}}^{Q} \sum_{\substack{q'_2=Q_P+1\\ q'_2 \text{ odd}}}^{Q} E\big[a^{(k_1)}_{p'_1,q'_1+1,m} a^{(k_1)}_{p'_2,q'_2+1,m}\big] \\ &+ A_{2d}^2 \sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ odd}}}^{Q} \sum_{\substack{q'_2=Q_P+1\\ q'_2 \text{ even}}}^{Q} E\big[a^{(k_1)}_{p'_1,q'_1+1,m} a^{(k_1)}_{p'_2,q'_2-1,m}\big] \\ &+ A_{2d}^2 \sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ even}}}^{Q} \sum_{\substack{q'_2=Q_P+1\\ q'_2 \text{ odd}}}^{Q} E\big[a^{(k_1)}_{p'_1,q'_1-1,m} a^{(k_1)}_{p'_2,q'_2+1,m}\big] \\ &+ A_{2d}^2 \sum_{\substack{q'_1=Q_P+1\\ q'_1 \text{ even}}}^{Q} \sum_{\substack{q'_2=Q_P+1\\ q'_2 \text{ even}}}^{Q} E\big[a^{(k_1)}_{p'_1,q'_1-1,m} a^{(k_1)}_{p'_2,q'_2-1,m}\big].\end{aligned} \tag{V.13}$$

Here, the second and third terms are equal to zero, so $\beta(2,2)$ reduces to

$$\begin{aligned}\beta(2,2) &= A_{2d}^2 \sum_{\substack{q_1'=Q_P+1\\ q_1' \text{ odd}}}^{Q} \delta_{p_1',p_2'}\delta_{k_1,k_2} + A_{2d}^2 \sum_{\substack{q_1'=Q_P+1\\ q_1' \text{ even}}}^{Q} \delta_{p_1',p_2'}\delta_{k_1,k_2} \\ &= A_{2d}^2 \sum_{q_1'=Q_P+1}^{Q} \delta_{p_1',p_2'}\delta_{k_1,k_2}. \end{aligned} \tag{V.14}$$

V.5 Combining the Results

Using the results of (V.8), (V.10), (V.12), and (V.14), we show that (V.2) is equal to

$$\begin{aligned}&\sum_{u_1'=1}^{2}\sum_{u_2'=1}^{2}\sum_{q_1'=Q_P+1}^{Q}\sum_{q_2'=Q_P+1}^{Q} A_{2d}^2 E\big[b_{p_1',q_1',u_1',m}^{(k_1)} b_{p_2',q_2',u_2',m}^{(k_2)}\big] \\ &= \beta(1,1)+\beta(1,2)+\beta(2,1)+\beta(2,2) \\ &= 2A_{2d}^2 \sum_{q_1'=Q_P+1}^{Q} \delta_{p_1',p_2'}\delta_{k_1,k_2}. \end{aligned} \tag{V.15}$$

Appendix W

Second Order Moments: MC-CDMA (MIMO)

In this appendix, we provide derivations for the expressions in (M.13)–(M.17) and (M.23)–(M.28). We assume $a^{(1)}_{p,q,m} = +1$ and condition on $a^{(1)}_{p,q+1,m}$.

W.1 Derivation of $\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}]$

From (3.41), we have

$$\hat{W}_{v,u,m,i} = \begin{cases} g^{(1)}_{v,u,m,i} + \dfrac{1}{A_{1e}}\dfrac{1}{Q_P/2}\displaystyle\sum_{\substack{q=1\\ q \text{ odd}}}^{Q_P}\big(I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}}\big), & u=1,\\ g^{(1)}_{v,u,m,i} + \dfrac{1}{A_{1e}}\dfrac{1}{Q_P/2}\displaystyle\sum_{\substack{q=1\\ q \text{ even}}}^{Q_P}\big(I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}}\big), & u=2. \end{cases} \tag{W.1}$$

The expressions for $I_{W_{q,v,m,i}}$ and $N_{W_{q,v,m,i}}$ are given in (3.39) and (3.40) as

$$I_{W_{q,v,m,i}} = C^{(1)}_{p,q,m,i} \sum_{k=2}^{K} \sum_{u=1}^{2} g^{(k)}_{v,u,m,i} \sum_{p'=-\infty}^{\infty} \sum_{q'=1}^{Q} A_{1,q'} b^{(k)}_{p',q',u,m} C^{(k)}_{p',q',m,i} \cdot x_1[t-(p'Q+q'-1)T_s - \tau^{(k)}]\Big|_{t=(pQ+q-1)T_s} \tag{W.2}$$

$$N_{W_{q,v,m,i}} = C^{(1)}_{p,q,m,i} \left(\tilde{n}_v(t) * \tilde{h}_{m,i,1}(t)\right)\Big|_{t=(pQ+q-1)T_s} \tag{W.3}$$

As in the MC-DS-CDMA (MIMO) analysis (see Appendix U), we make the following assumptions regarding the data sequences and the spreading sequences of the interfering users:

$$E\big[C^{(k)}_{p,q,m,i}\big] = 0, \tag{W.4}$$

$$E\big[a^{(k)}_{p,q,m}\big] = 0, \quad q = Q_P+1, \ldots, Q, \tag{W.5}$$

and

$$E\big[C^{(k_1)}_{p_1,q_1,m,i_1} C^{(k_2)}_{p_2,q_2,m,i_2}\big] = \delta_{k_1,k_2}\delta_{p_1,p_2}\delta_{q_1,q_2}\delta_{i_1,i_2}, \tag{W.6}$$

$$E\big[a^{(k_1)}_{p_1,q_1,m} a^{(k_2)}_{p_2,q_2,m}\big] = \delta_{k_1,k_2}\delta_{p_1,p_2}\delta_{q_1,q_2}, \quad q_i = Q_P+1, \ldots, Q;\ i=1,2. \tag{W.7}$$

Finally, since the channel gains between different users and different transmit-receive antenna pairs are independent, we have

$$\frac{1}{2}E\big[g^{(k_1)}_{v_1,u_1,m,i_1}\big(g^{(k_2)}_{v_2,u_2,m,i_2}\big)^*\big] = \frac{1}{2}E\big[g^{(k_1)}_{v_1,u_1,m,i_1}\big(g^{(k_1)}_{v_1,u_1,m,i_2}\big)^*\big]\,\delta_{k_1,k_2}\delta_{v_1,v_2}\delta_{u_1,u_2}. \tag{W.8}$$

By applying the assumptions from the correlated block fading model, we can re-write (W.8) as

$$\frac{1}{2}E\big[g^{(k_1)}_{v_1,u_1,m,i_1}\big(g^{(k_2)}_{v_2,u_2,m,i_2}\big)^*\big] = \begin{cases} \frac{1}{2}\,\delta_{k_1,k_2}\delta_{v_1,v_2}\delta_{u_1,u_2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks.} \end{cases} \tag{W.9}$$

We define

$$\Gamma_{W_{q,v,m,i}} \triangleq I_{W_{q,v,m,i}} + N_{W_{q,v,m,i}} \tag{W.10}$$

and

$$q(u) \triangleq q + (u-1), \tag{W.11}$$

such that $\hat{W}_{v,u,m,i}$ in (W.1) reduces to

$$\hat{W}_{v,u,m,i} = g^{(1)}_{v,u,m,i} + \frac{1}{A_{1e}} \frac{1}{Q_P/2} \sum_{\substack{q=1 \\ q \text{ odd}}}^{Q_P} \Gamma_{W_{q(u),v,m,i}}. \tag{W.12}$$

We use this expression to write

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\Bigg[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,u_2,m,i_2}\big)^* + \frac{1}{A_{1e}}\frac{1}{Q_P/2}\, g^{(1)}_{v_1,u_1,m,i_1} \sum_{\substack{q=1 \\ q \text{ odd}}}^{Q_P} \Gamma^*_{W_{q(u_2),v_2,m,i_2}} \\
&\qquad + \frac{1}{A_{1e}}\frac{1}{Q_P/2}\big(g^{(1)}_{v_2,u_2,m,i_2}\big)^* \sum_{\substack{q=1 \\ q \text{ odd}}}^{Q_P} \Gamma_{W_{q(u_1),v_1,m,i_2}} \\
&\qquad + \left(\frac{1}{A_{1e}}\frac{1}{Q_P/2}\right)^2 \sum_{\substack{q_1=1 \\ q_1 \text{ odd}}}^{Q_P} \sum_{\substack{q_2=1 \\ q_2 \text{ odd}}}^{Q_P} \Gamma_{W_{q_1(u_1),v_1,m,i_1}} \Gamma^*_{W_{q_2(u_2),v_2,m,i_2}}\Bigg],
\end{aligned} \tag{W.13}$$

where $q_1(u_1) \triangleq q_1+(u_1-1)$ and $q_2(u_2) \triangleq q_2+(u_2-1)$. By using the assumptions in (W.4)–(W.8), we can show that the expected values of the two cross terms are equal to zero, so that

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,u_2,m,i_2}\big)^*\big] + \left(\frac{1}{A_{1e}}\frac{1}{Q_P/2}\right)^2 \sum_{\substack{q_1=1 \\ q_1 \text{ odd}}}^{Q_P} \frac{1}{2}E\big[\Gamma_{W_{q_1(u_1),v_1,m,i_1}}\Gamma^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\qquad + \left(\frac{1}{A_{1e}}\frac{1}{Q_P/2}\right)^2 \sum_{\substack{q_1=1 \\ q_1 \text{ odd}}}^{Q_P} \sum_{\substack{q_2=1 \\ q_2 \text{ odd} \\ q_2\neq q_1}}^{Q_P} \frac{1}{2}E\big[\Gamma_{W_{q_1(u_1),v_1,m,i_1}}\Gamma^*_{W_{q_2(u_2),v_2,m,i_2}}\big].
\end{aligned} \tag{W.14}$$

W.1.1 First Term in (W.14)

The first term in (W.14) is equal to (see (W.8))

$$\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,u_2,m,i_2}\big)^*\big] = \frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_1,u_1,m,i_2}\big)^*\big]\delta_{v_1,v_2}\delta_{u_1,u_2}. \tag{W.15}$$

W.1.2 Second Term in (W.14)

We use (W.10) to expand the term $\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1),v_1,m,i_1}}\Gamma^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ as

$$\begin{aligned}
&\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1),v_1,m,i_1}}\Gamma^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\quad= \frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}}I^*_{W_{q_1(u_2),v_2,m,i_2}} + I_{W_{q_1(u_1),v_1,m,i_1}}N^*_{W_{q_1(u_2),v_2,m,i_2}} \\
&\qquad + N_{W_{q_1(u_1),v_1,m,i_1}}I^*_{W_{q_1(u_2),v_2,m,i_2}} + N_{W_{q_1(u_1),v_1,m,i_1}}N^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\quad= \frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}}I^*_{W_{q_1(u_2),v_2,m,i_2}} + N_{W_{q_1(u_1),v_1,m,i_1}}N^*_{W_{q_1(u_2),v_2,m,i_2}}\big],
\end{aligned} \tag{W.16}$$

where the expected values of the cross terms are easily shown to equal zero.

First Term in (W.16)

Define

$$\begin{aligned}
I_{y_{v,m,i}}(t) = \sum_{k=2}^{K}\sum_{u'=1}^{2} g^{(k)}_{v,u',m,i} \sum_{p'=-\infty}^{\infty}\sum_{q'=1}^{Q} A_{1,q'} b^{(k)}_{p',q',u',m} C^{(k)}_{p',q',m,i} \\
\cdot\, x_1\big[t-(p'Q+q'-1)T_s-\tau^{(k)}\big]\Big|_{t=(pQ+q-1)T_s},
\end{aligned} \tag{W.17}$$

such that $I_{W_{q(u),v,m,i}}$ can be re-written as (see (W.2))

$$I_{W_{q(u),v,m,i}} = C^{(1)}_{p,q(u),m,i} I_{y_{v,m,i}}[(pQ+q(u)-1)T_s], \quad q=1,\ldots,Q_P. \tag{W.18}$$

We use (W.17) and (W.18) to write

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}}I^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\quad= C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_2} \\
&\qquad\cdot \frac{1}{2}E\Big\{I_{y_{v_1,m,i_1}}[(pQ+q_1(u_1)-1)T_s] I^*_{y_{v_2,m,i_2}}[(pQ+q_1(u_2)-1)T_s]\Big\}
\end{aligned}$$

Table W.1: Definition of $b^{(k)}_{p,q,u,m}$

Estimation phase ($q = 1, \ldots, Q_P$):	$b^{(k)}_{p,q,u,m} = \begin{cases} \delta_{u,1}, & q \text{ odd} \\ \delta_{u,2}, & q \text{ even} \end{cases}$
Data phase ($q = Q_P + 1, \ldots, Q$):	$b^{(k)}_{p,q,u,m} = \begin{cases} a^{(k)}_{p,q,m}, & u = 1,\, q \text{ odd} \\ a^{(k)}_{p,q+1,m}, & u = 2,\, q \text{ odd} \\ -a^{(k)}_{p,q,m}, & u = 1,\, q \text{ even} \\ a^{(k)}_{p,q-1,m}, & u = 2,\, q \text{ even} \end{cases}$

$$\begin{aligned} &= C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_2} \sum_{k_1=2}^{K} \sum_{k_2=2}^{K} \sum_{u_1'=1}^{2} \sum_{u_2'=1}^{2} \frac{1}{2} E\big[g^{(k_1)}_{v_1,u_1',m,i_1} \big(g^{(k_2)}_{v_2,u_2',m,i_2}\big)^*\big] \\ &\cdot \sum_{p_1'=-\infty}^{\infty} \sum_{p_2'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \sum_{q_2'=1}^{Q} A_{1,q_1'} A_{1,q_2'} E\big[b^{(k_1)}_{p_1',q_1',u_1',m} b^{(k_2)}_{p_2',q_2',u_2',m}\big] E\big[C^{(k_1)}_{p_1',q_1',m,i_1} C^{(k_2)}_{p_2',q_2',m,i_2}\big] \\ &\cdot E\Big\{x_1\big[t_1 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big] \\ &\qquad x_1\big[t_2 - (p_2'Q + q_2' - 1)T_s - \tau^{(k_2)}\big]\Big\}\Big|_{t_1=[pQ+q_1(u_1)-1]T_s,\; t_2=[pQ+q_1(u_2)-1]T_s}. \end{aligned} \tag{W.19}$$

Substituting

$$A_{1,q} = \begin{cases} A_{1e}, & q = 1, \ldots, Q_P, \\ \\ A_{1d}, & q = Q_P + 1, \ldots, Q, \end{cases} \tag{W.20}$$

from (3.22) into (W.19) and applying both (U.20) and the expression for $b^{(k)}_{p,q,u,m}$ from Table W.1 (same as Table 3.1) into the resulting expression, we get

$$
\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\quad = C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_2} \sum_{k_1=2}^{K} \sum_{k_2=2}^{K} \sum_{u_1'=1}^{2} \sum_{u_2'=1}^{2} \\
&\qquad \cdot \frac{1}{2}E\big[g^{(k_1)}_{v_1,u_1',m,i_1} \big(g^{(k_2)}_{v_2,u_2',m,i_2}\big)^*\big] \sum_{p_1'=-\infty}^{\infty} \sum_{p_2'=-\infty}^{\infty} \\
&\qquad \cdot \Bigg(\sum_{\substack{q_1'=1 \\ q_1' \text{ odd}}}^{Q_P} \sum_{\substack{q_2'=1 \\ q_2' \text{ odd}}}^{Q_P} A_{1e}^2 \delta_{u_1',1}\delta_{u_2',1} + \sum_{\substack{q_1'=1 \\ q_1' \text{ odd}}}^{Q_P} \sum_{\substack{q_2'=1 \\ q_2' \text{ even}}}^{Q_P} A_{1e}^2 \delta_{u_1',1}\delta_{u_2',2} + \sum_{\substack{q_1'=1 \\ q_1' \text{ even}}}^{Q_P} \sum_{\substack{q_2'=1 \\ q_2' \text{ odd}}}^{Q_P} A_{1e}^2 \delta_{u_1',2}\delta_{u_2',1} \\
&\qquad\quad + \sum_{\substack{q_1'=1 \\ q_1' \text{ even}}}^{Q_P} \sum_{\substack{q_2'=1 \\ q_2' \text{ even}}}^{Q_P} A_{1e}^2 \delta_{u_1',2}\delta_{u_2',2} \sum_{q_1'=Q_P+1}^{Q} \sum_{q_2'=Q_P+1}^{Q} A_{1d}^2\, E\big[b^{(k_1)}_{p_1',q_1',u_1',m} b^{(k_2)}_{p_2',q_2',u_2',m}\big] \Bigg) \\
&\qquad \cdot E\big[C^{(k_1)}_{p_1',q_1',m,i_1} C^{(k_2)}_{p_2',q_2',m,i_2}\big] \\
&\qquad \cdot E\Big\{ x_1\big[t_1 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big] \\
&\qquad\qquad x_1\big[t_2 - (p_2'Q + q_2' - 1)T_s - \tau^{(k_2)}\big] \Big\}\Big|_{t_1=[pQ+q_1(u_1)-1]T_s,\ t_2=[pQ+q_1(u_2)-1]T_s}.
\end{aligned}
\tag{W.21}
$$

By following the same steps as in Appendix V, we can show that

$$
\sum_{u_1'=1}^{2} \sum_{u_2'=1}^{2} \sum_{q_1'=Q_P+1}^{Q} \sum_{q_2'=Q_P+1}^{Q} A_{1d}^2 E\big[b^{(k_1)}_{p_1',q_1',u_1',m} b^{(k_2)}_{p_2',q_2',u_2',m}\big] = \sum_{q_1'=Q_P+1}^{Q} 2A_{1d}^2\, \delta_{p_1',p_2'} \delta_{k_1,k_2}. \tag{W.22}
$$

We use this result, along with (W.6), (W.8), and the general relation in (S.18), to show that (W.21) reduces to

$$
\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\
&\quad = C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_1} \cdot \frac{1}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2} \sum_{k_1=2}^{K} \sum_{p_1'=-\infty}^{\infty} \\
&\qquad \cdot \Bigg(\sum_{\substack{q_1'=1 \\ q_1' \text{ odd}}}^{Q_P} A_{1e}^2 + \sum_{\substack{q_1'=1 \\ q_1' \text{ even}}}^{Q_P} A_{1e}^2 + \sum_{q_1'=Q_P+1}^{Q} 2A_{1d}^2 \Bigg) \\
&\qquad \cdot E\Big\{ x_1\big[t_1 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big] \\
&\qquad\qquad x_1\big[t_2 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big] \Big\}\Big|_{t_1=[pQ+q_1(u_1)-1]T_s,\ t_2=[pQ+q_1(u_2)-1]T_s}.
\end{aligned}
\tag{W.23}
$$

By using the relation

$$A_{1d} = \frac{A_{1e}}{\sqrt{2}} \tag{W.24}$$

from (3.23), we can write

$$\sum_{\substack{q_1'=1 \\ q_1' \text{ odd}}}^{Q_P} A_{1e}^2 + \sum_{\substack{q_1'=1 \\ q_1' \text{ even}}}^{Q_P} A_{1e}^2 + \sum_{q_1'=Q_P+1}^{Q} 2A_{1d}^2 \;=\; \sum_{q_1'=1}^{Q} A_{1e}^2. \tag{W.25}$$

Thus, (W.23) equals

$$\begin{aligned} &\frac{1}{2} E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\ &\quad = \frac{A_{1e}^2}{2}\, \delta_{v_1,v_2}\delta_{i_1,i_2} \cdot C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_1} \sum_{k_1=2}^{K} \sum_{p_1'=-\infty}^{\infty} \sum_{q_1'=1}^{Q} \\ &\qquad \cdot E\Big\{ x_1\big[t_1 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big] \\ &\qquad\qquad x_1\big[t_2 - (p_1'Q + q_1' - 1)T_s - \tau^{(k_1)}\big] \Big\}\Big|_{t_1=[pQ+q_1(u_1)-1]T_s,\; t_2=[pQ+q_1(u_2)-1]T_s}. \end{aligned} \tag{W.26}$$

Finally, we apply (S.67) to the above equation and use (S.38) to get

$$\begin{aligned} &\frac{1}{2} E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big] \\ &\quad = \frac{A_{1e}^2}{2}\, \delta_{v_1,v_2}\delta_{i_1,i_2} \cdot C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_1} \sum_{k_1=2}^{K} \frac{1}{T_s} \int_{-\infty}^{\infty} |X_1(f)|^2 e^{j2\pi f\tau}\, df \Big|_{\tau=t_2-t_1=(u_2-u_1)T_s} \\ &\quad = \frac{A_{1e}^2}{2}\, \delta_{v_1,v_2}\delta_{i_1,i_2} \cdot C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_1} \sum_{k_1=2}^{K} \frac{1}{T_s} \cdot T_s\, \delta_{u_1,u_2} \\ &\quad = \frac{A_{1e}^2(K-1)}{2}\, \delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2} \cdot \Big[C^{(1)}_{p,q_1(u_1),m,i_1}\Big]^2 \\ &\quad = \frac{A_{1e}^2(K-1)}{2}\, \delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}. \end{aligned} \tag{W.27}$$

Second Term in (W.16)

By defining

$$N_{y_{v,m,i}}(t) = \tilde{n}_v(t) * \tilde{h}_{m,i,1}(t), \tag{W.28}$$

we can re-write $N_{W_{q(u),v,m,i}}$ as (see (W.3))

$$N_{W_{q(u),v,m,i}} = C^{(1)}_{p,q(u),m,i} N_{y_{v,m,i}}[(pQ + q(u) - 1)T_s], \quad q = 1, \ldots, Q_P, \tag{W.29}$$

such that

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_1(u_2)},v_2,m,i_2}\big] \\
&\qquad = C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_2} \cdot \frac{1}{2}E\Big\{N_{y_{v_1,m,i_1}}[(pQ+q_1(u_1)-1)T_s] \\
&\qquad\qquad N^*_{y_{v_2,m,i_2}}[(pQ+q_1(u_2)-1)T_s]\Big\}.
\end{aligned} \tag{W.30}$$

When $i_1 \neq i_2$, the AWGN terms $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are from two different frequencies. Thus, $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are independent, so

$$\frac{1}{2}E\Big\{N_{y_{v_1,m,i_1}}[(pQ+q_1(u_1)-1)T_s]N^*_{y_{v_2,m,i_2}}[(pQ+q_1(u_2)-1)T_s]\Big\} = 0, \quad i_1 \neq i_2. \tag{W.31}$$

This, in turn, implies that

$$\frac{1}{2}E\big[N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_1(u_2)},v_2,m,i_2}\big] = 0, \quad i_1 \neq i_2. \tag{W.32}$$

When $i_1 = i_2$, we use (W.28) and (W.30) to write

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_1(u_2)},v_2,m,i_1}\big] \\
&\quad = C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_1} \cdot \frac{1}{2}E\Big\{N_{y_{v_1,m,i_1}}[(pQ+q_1(u_1)-1)T_s] \\
&\qquad\qquad N^*_{y_{v_2,m,i_1}}[(pQ+q_1(u_2)-1)T_s]\Big\} \\
&\quad = C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}E\big[\tilde{n}_{v_1}(\tau_1)\tilde{n}^*_{v_2}(\tau_2)\big] \\
&\qquad \cdot \tilde{h}_{m,i_1,1}(t_1-\tau_1)\tilde{h}^*_{m,i_1,1}(t_2-\tau_2)\, d\tau_2\, d\tau_1\Big|_{t_1=[pQ+q_1(u_1)-1]T_s,\ t_2=[pQ+q_1(u_2)-1]T_s}.
\end{aligned} \tag{W.33}$$

The AWGN processes at the two receive antennas are assumed to be independent, so we have

$$\frac{1}{2}E\big[\tilde{n}_{v_1}(\tau_1)\tilde{n}^*_{v_2}(\tau_2)\big] = N_0\,\delta(\tau_1-\tau_2)\,\delta_{v_1,v_2}. \tag{W.34}$$

Substituting this result into (W.33), we get

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_1(u_2)},v_2,m,i_1}\big] \\
&\quad = N_0\,\delta_{v_1,v_2} \cdot C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_1} \\
&\qquad \cdot \int_{-\infty}^{\infty} \tilde{h}_{m,i_1,1}(t_1-\tau_1)\tilde{h}^*_{m,i_1,1}(t_2-\tau_1)\, d\tau_1\Big|_{t_1=[pQ+q_1(u_1)-1]T_s,\ t_2=[pQ+q_1(u_2)-1]T_s}.
\end{aligned} \tag{W.35}$$

We apply (S.60) and (S.63) to (W.35) to obtain

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_1(u_2)},v_2,m,i_1}\big] \\
&\quad = N_0\,\delta_{v_1,v_2}\cdot C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_1}\cdot x_1(\tau)\Big|_{\tau=t_2-t_1=(u_2-u_1)T_s} \\
&\quad = N_0\,\delta_{v_1,v_2}\cdot C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_1(u_2),m,i_1}\cdot \delta_{u_1,u_2} \\
&\quad = N_0\,\delta_{v_1,v_2}\delta_{u_1,u_2}\Big[C^{(1)}_{p,q_1(u_1),m,i_1}\Big]^2 \\
&\quad = N_0\,\delta_{v_1,v_2}\delta_{u_1,u_2}. \qquad \text{(W.36)}
\end{aligned}$$

Finally, we combine the results of (W.32) and (W.36) to write

$$\frac{1}{2}E\big[N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_1(u_2)},v_2,m,i_2}\big] = N_0\,\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}. \qquad \text{(W.37)}$$

Combining the Results

Substituting the results of (W.27) and (W.37) into (W.16), we get

$$\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1} \Gamma^*_{W_{q_1(u_2)},v_2,m,i_2}\big] = \left(\frac{A_{1e}^2(K-1)}{2} + N_0\right)\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}. \qquad \text{(W.38)}$$

W.1.3 Third Term in (W.14)

We use (W.10) to expand the term $\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1} \Gamma^*_{W_{q_2(u_2)},v_2,m,i_2}\big]$, where $q_1 \neq q_2$, as

$$\begin{aligned}
&\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1)},v_1,m,i_1} \Gamma^*_{W_{q_2(u_2)},v_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_2(u_2)},v_2,m,i_2} + I_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_2(u_2)},v_2,m,i_2} \\
&\qquad + N_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_2(u_2)},v_2,m,i_2} + N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_2(u_2)},v_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_2(u_2)},v_2,m,i_2} + N_{W_{q_1(u_1)},v_1,m,i_1} N^*_{W_{q_2(u_2)},v_2,m,i_2}\big]. \qquad \text{(W.39)}
\end{aligned}$$

Again, the expected values of the cross terms are easily shown to equal zero.

First Term in (W.39)

To derive an expression for $\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_2(u_2),v_2,m,i_2}}\big]$, we simply replace $q_1(u_2)$ with $q_2(u_2)$ in the derivation for $\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ (see Section W.1.2). Thus, using (W.26), we have

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_2(u_2),v_2,m,i_2}}\big] \\
&\quad = \frac{A_{1e}^2}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\cdot C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_2(u_2),m,i_2} \sum_{k_1=2}^{K}\sum_{p_1'=-\infty}^{\infty}\sum_{q_1'=1}^{Q} \\
&\qquad \cdot E\Big\{x_1\big[t_1-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big] \\
&\qquad\qquad x_1\big[t_2-(p_1'Q+q_1'-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t_1=[pQ+q_1(u_1)-1]T_s,\ t_2=[pQ+q_2(u_2)-1]T_s}.
\end{aligned} \tag{W.40}$$

We apply (S.67) to the above equation and use (S.38), along with the fact that $q_1 \neq q_2$, to get

$$\begin{aligned}
&\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_2(u_2),v_2,m,i_2}}\big] \\
&\quad = \frac{A_{1e}^2}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2} C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_2(u_2),m,i_2} \sum_{k_1=2}^{K} \\
&\qquad \cdot \frac{1}{T_s}\int_{-\infty}^{\infty} |X_1(f)|^2 e^{j2\pi f\tau}\,df\Big|_{\tau=t_2-t_1=[(q_2-q_1)+(u_2-u_1)]T_s} \\
&\quad = \frac{A_{1e}^2}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2} C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_2(u_2),m,i_2} \sum_{k_1=2}^{K} \frac{1}{T_s}\cdot T_s\,\delta_{q_1,q_2}\delta_{u_1,u_2} \\
&\quad = 0.
\end{aligned} \tag{W.41}$$

Second Term in (W.39)

Likewise, to derive an expression for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_2(u_2),v_2,m,i_2}}\big]$, we also replace $q_1(u_2)$ with $q_2(u_2)$ in the derivation for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_2(u_2),v_2,m,i_2}}\big]$ in Section W.1.2. Again, we have

$$\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_2(u_2),v_2,m,i_2}}\big] = 0, \quad i_1 \neq i_2, \tag{W.42}$$

since $N_{W_{q_1(u_1),v_1,m,i_1}}$ and $N_{W_{q_2(u_2),v_2,m,i_2}}$ correspond to AWGN terms from two different frequencies. For $i_1 = i_2$, we use (W.35) to write

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_2(u_2),v_2,m,i_1}}\big] \\
&\quad = N_0\,\delta_{v_1,v_2} C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_2(u_2),m,i_1} \\
&\qquad \cdot \int_{-\infty}^{\infty} \tilde{h}_{m,i_1,1}(t_1-\tau_1)\tilde{h}^*_{m,i_1,1}(t_2-\tau_1)\,d\tau_1\Big|_{t_1=[pQ+q_1(u_1)-1]T_s,\ t_2=[pQ+q_2(u_2)-1]T_s}.
\end{aligned} \tag{W.43}$$

We then apply (S.60) and use (S.63), along with the fact that $q_1 \neq q_2$, to obtain

$$\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_2(u_2),v_2,m,i_1}}\big] \\
&\quad = N_0\,\delta_{v_1,v_2} C^{(1)}_{p,q_1(u_1),m,i_1} C^{(1)}_{p,q_2(u_2),m,i_1} \cdot x_1(\tau)\Big|_{\tau=t_2-t_1=[(q_2-q_1)+(u_2-u_1)]T_s} \\
&\quad = 0.
\end{aligned} \tag{W.44}$$

Therefore, from (W.42) and (W.44), we have

$$\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_2(u_2),v_2,m,i_2}}\big] = 0. \tag{W.45}$$

Combining the Results

Substituting (W.41) and (W.45) into (W.39), we obtain

$$\frac{1}{2}E\big[\Gamma_{W_{q_1(u_1),v_1,m,i_1}} \Gamma^*_{W_{q_2(u_2),v_2,m,i_2}}\big] = 0, \quad q_1 \neq q_2. \tag{W.46}$$

W.1.4 Combining the Results

We use the results of (W.15), (W.38), and (W.46) in (W.14) to get

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_1,u_1,m,i_2}\big)^*\big]\delta_{v_1,v_2}\delta_{u_1,u_2} \\
&\qquad + \left(\frac{1}{A_{1e}}\frac{1}{Q_P/2}\right)^2 \sum_{\substack{q_1=1\\ q_1\ \text{odd}}}^{Q_P} \left(\frac{A_{1e}^2(K-1)}{2} + N_0\right)\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}
\end{aligned}$$

$$= \frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_1,u_1,m,i_2}\big)^*\big]\delta_{v_1,v_2}\delta_{u_1,u_2}$$

$$+\left(\frac{1}{A_{1e}}\frac{1}{Q_P/2}\right)^2\cdot\frac{Q_P}{2}\left(\frac{A_{1e}^2(K-1)}{2}+N_0\right)\delta_{v_1,v_2}\delta_{u_1,u_2}\delta_{i_1,i_2}$$

$$= \left\{\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_1,u_1,m,i_2}\big)^*\big]+\frac{1}{2}\frac{1}{Q_P/2}\left((K-1)+\frac{2N_0}{A_{1e}^2}\right)\delta_{i_1,i_2}\right\}\delta_{v_1,v_2}\delta_{u_1,u_2}. \quad \text{(W.47)}$$

Finally, we use the expression

$$E_{b1}=\frac{R_1A_{1e}^2}{2}=R_1A_{1d}^2 \quad \text{(W.48)}$$

from (P.26), along with (W.9), to re-write (W.47) as

$$\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1}\hat{W}^*_{v_2,u_2,m,i_2}\big]$$

$$=\begin{cases}\frac{1}{2}\left[1+\frac{1}{Q_P/2}c_1\left(\frac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right]\delta_{v_1,v_2}\delta_{u_1,u_2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,}\\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks,}\end{cases}$$

$$\text{(W.49)}$$

where

$$c_1\left(\frac{E_{b1}}{N_0}\right)=(K-1)+\frac{R_1}{E_{b1}/N_0}. \quad \text{(W.50)}$$

W.2 Derivation of $\frac{1}{2}E[\hat{Y}_{q,v_1,m,i_1}\hat{Y}^*_{q,v_2,m,i_2}]$

From (3.38), we have

$$Y_{q,v,m,i}=A_{1d}\sum_{u=1}^{2}b^{(1)}_{p,q,u,m}\,g^{(1)}_{v,u,m,i}+I_{Y_{q,v,m,i}}+N_{Y_{q,v,m,i}}, \quad q=Q_P+1,\ldots,Q, \quad \text{(W.51)}$$

where the expressions for $I_{Y_{q,v,m,i}}$ and $N_{Y_{q,v,m,i}}$ are identical to the ones for $I_{W_{q,v,m,i}}$ and $N_{W_{q,v,m,i}}$ (see (3.39)–(3.40)):

$$I_{Y_{q,v,m,i}}=C^{(1)}_{p,q,m,i}\sum_{k=2}^{K}\sum_{u=1}^{2}g^{(k)}_{v,u,m,i}\sum_{p'=-\infty}^{\infty}\sum_{q'=1}^{Q}A_{1,q'}b^{(k)}_{p',q',u,m}C^{(k)}_{p',q',m,i}$$
$$\cdot\, x_1[t-(p'Q+q'-1)T_s-\tau^{(k)}]\Big|_{t=(pQ+q-1)T_s} \quad \text{(W.52)}$$

$$N_{Y_{q,v,m,i}}=C^{(1)}_{p,q,m,i}\left(\tilde{n}_v(t)*\tilde{h}_{m,i,1}(t)\right)\Big|_{t=(pQ+q-1)T_s} \quad \text{(W.53)}$$

Assume q is *odd*. We define

$$\Gamma_{Y_{q,v,m,i}} \triangleq I_{Y_{q,v,m,i}} + N_{Y_{q,v,m,i}} \tag{W.54}$$

and make the substitution (see Table W.1 on p. 279)

$$b^{(k)}_{p,q,u,m} = \begin{cases} a^{(k)}_{p,q,m}, & u = 1, \\ a^{(k)}_{p,q+1,m}, & u = 2, \end{cases} \quad (q = Q_P + 1, \ldots, Q;\ q \text{ odd}) \tag{W.55}$$

in (W.51) to get

$$Y_{q,v,m,i} = A_{1d}\, a^{(1)}_{p,q,m}\, g^{(1)}_{v,1,m,i} + A_{1d}\, a^{(1)}_{p,q+1,m}\, g^{(1)}_{v,2,m,i} + \Gamma_{Y_{q,v,m,i}}, \quad q = Q_P + 1, \ldots, Q. \tag{W.56}$$

Using this expression, we can show that

$$\begin{aligned} \frac{1}{2}E\big[Y_{q,v_1,m,i_1} Y^*_{q,v_2,m,i_2}\big] = A^2_{1d}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A^2_{1d}\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\ + \frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q,v_2,m,i_2}}\big], \quad q \text{ odd}, \end{aligned} \tag{W.57}$$

where the expected values of the cross terms are easily shown to equal zero.

W.2.1 Sum of First and Second Terms in (W.57)

The sum of the first and second terms in (W.57) are equal to (see (W.8))

$$\begin{aligned} &A^2_{1d}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A^2_{1d}\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\ &\quad = A^2_{1d}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{i_1,i_2} + A^2_{1d}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{i_1,i_2} \\ &\quad = 2A^2_{1d}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \end{aligned} \tag{W.58}$$

W.2.2 Third Term in (W.57)

Using (W.54) to expand the third term in (W.57), we have

$$\frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q,v_2,m,i_2}}\big] = \frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q,v_2,m,i_2}} + N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big], \tag{W.59}$$

where, again, the expected values of the cross terms are equal to zero.

First Term in (W.59)

We can re-write $I_{Y_{q,v,m,i}}$ in (W.52) as

$$I_{Y_{q,v,m,i}} = C^{(1)}_{p,q,m,i} I_{y_{v,m,i}}[(pQ+q-1)T_s], \quad q = Q_P+1,\ldots,Q, \tag{W.60}$$

where $I_{y_{v,m,i}}(t)$ was defined in (W.17). Thus, we have

$$\begin{aligned}
&\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = C^{(1)}_{p,q,m,i_1} C^{(1)}_{p,q,m,i_2} \cdot \frac{1}{2}E\Big\{I_{y_{v_1,m,i_1}}[(pQ+q-1)T_s] I^*_{y_{v_2,m,i_2}}[(pQ+q-1)T_s]\Big\} \\
&\quad = C^{(1)}_{p,q,m,i_1} C^{(1)}_{p,q,m,i_2} \sum_{k_1=2}^{K}\sum_{k_2=2}^{K}\sum_{u'_1=1}^{2}\sum_{u'_2=1}^{2} \\
&\qquad \cdot \frac{1}{2}E\big[g^{(k_1)}_{v_1,u'_1,m,i_1}\big(g^{(k_2)}_{v_2,u'_2,m,i_2}\big)^*\big] \sum_{p'_1=-\infty}^{\infty}\sum_{p'_2=-\infty}^{\infty}\sum_{q'_1=1}^{Q}\sum_{q'_2=1}^{Q} A_{1,q'_1}A_{1,q'_2} \\
&\qquad \cdot E\big[b^{(k_1)}_{p'_1,q'_1,u'_1,m} b^{(k_2)}_{p'_2,q'_2,u'_2,m}\big] E\big[C^{(k_1)}_{p'_1,q'_1,m,i_1} C^{(k_2)}_{p'_2,q'_2,m,i_2}\big] \\
&\qquad \cdot E\Big\{x_1\big[t_1-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big] \\
&\qquad\qquad x_1\big[t_2-(p'_2Q+q'_2-1)T_s-\tau^{(k_2)}\big]\Big\}\Big|_{t_1=(pQ+q-1)T_s,\ t_2=(pQ+q-1)T_s}.
\end{aligned} \tag{W.61}$$

So far, the derivation is identical to the derivation of $\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}} I^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ in Section W.1.2, except that $q_1(u_1)$ and $q_1(u_2)$ have been replaced by q. Therefore, we follow the same steps as outlined between (W.19) and (W.26) to obtain

$$\begin{aligned}
&\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = \frac{A_{1e}^2}{2}\delta_{v_1,v_2}\delta_{i_1,i_2}\Big[C^{(1)}_{p,q,m,i_1}\Big]^2 \sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=1}^{Q} \\
&\qquad \cdot E\Big\{x_1\big[t_1-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big] \\
&\qquad\qquad x_1\big[t_2-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t_1=(pQ+q-1)T_s,\ t_2=(pQ+q-1)T_s}.
\end{aligned} \tag{W.62}$$

We then apply (S.67) to the above equation and use (S.38) to get

$$\begin{aligned}
&\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad= \frac{A_{1e}^2}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\left[C^{(1)}_{p,q,m,i_1}\right]^2 \sum_{k_1=2}^{K} \frac{1}{T_s}\int_{-\infty}^{\infty} |X_1(f)|^2 e^{j2\pi f\tau}\,df\Big|_{\tau=t_2-t_1=0} \\
&\quad= \frac{A_{1e}^2}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2}\sum_{k_1=2}^{K}\frac{1}{T_s}\cdot T_s \\
&\quad= \frac{A_{1e}^2(K-1)}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \qquad \text{(W.63)}
\end{aligned}$$

Second Term in (W.59)

We use $N_{y_{v,m,i}}(t)$ in (W.28) to re-write $N_{Y_{q,v,m,i}}$ as (see (W.53))

$$N_{Y_{q,v,m,i}} = C^{(1)}_{p,q,m,i_1} N_{y_{v,m,i}}[(pQ+q-1)T_s], \quad q = Q_P+1,\ldots,Q. \qquad \text{(W.64)}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big] \\
&= C^{(1)}_{p,q,m,i_1}C^{(1)}_{p,q,m,i_2}\cdot\frac{1}{2}E\Big\{N_{y_{v_1,m,i_1}}[(pQ+q-1)T_s]N^*_{y_{v_2,m,i_2}}[(pQ+q-1)T_s]\Big\}.
\end{aligned} \qquad \text{(W.65)}$$

When $i_1 \neq i_2$, the AWGN terms $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are once again independent, and this implies that $N_{Y_{q,v_1,m,i_1}}$ and $N_{Y_{q,v_2,m,i_2}}$ are independent as well:

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big] = 0, \quad i_1 \neq i_2. \qquad \text{(W.66)}$$

When $i_1 = i_2$, we use (W.28) in (W.65) to write

$$\begin{aligned}
&\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_1}}\big] \\
&\quad= [C^{(1)}_{p,q,m,i_1}]^2\cdot\frac{1}{2}E\Big\{N_{y_{v_1,m,i_1}}[(pQ+q-1)T_s]N^*_{y_{v_2,m,i_1}}[(pQ+q-1)T_s]\Big\} \\
&\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}E\big[\tilde{n}_{v_1}(\tau_1)\tilde{n}^*_{v_2}(\tau_2)\big] \\
&\qquad\cdot\tilde{h}_{m,i_1,1}(t_1-\tau_1)\tilde{h}^*_{m,i_1,1}(t_2-\tau_2)\,d\tau_2\,d\tau_1\Big|_{t_1=(pQ+q-1)T_s,\ t_2=(pQ+q-1)T_s}.
\end{aligned} \qquad \text{(W.67)}$$

This derivation is identical to the one for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}} N^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ in Section W.1.3, except that $q_1(u_1)$ and $q_1(u_2)$ have been replaced by q. Therefore, we follow the same steps as in that derivation to show that

$$\begin{aligned}\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_1}}\big] &= N_0\,\delta_{v_1,v_2}\cdot\cdot x_1(\tau)\Big|_{\tau=t_2-t_1=0} \\ &= N_0\,\delta_{v_1,v_2}. \qquad \text{(W.68)}\end{aligned}$$

Combining the results of (W.66) and (W.68), we get

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_1}}\big] = N_0\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \qquad \text{(W.69)}$$

Combining the Results

Substituting (W.63) and (W.69) into (W.59), we obtain

$$\frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}} \Gamma^*_{Y_{q,v_2,m,i_2}}\big] = \left(\frac{A_{1e}^2(K-1)}{2} + N_0\right)\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \qquad \text{(W.70)}$$

W.2.3 Combining the Results

Finally, we substitute the results of (W.58) and (W.70) into (W.57) to get

$$\begin{aligned}&\frac{1}{2}E\big[Y_{q,v_1,m,i_1} Y^*_{q,v_2,m,i_2}\big] \\ &\quad= 2A_{1d}^2\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{i_1,i_2} + \left(\frac{A_{1e}^2(K-1)}{2} + N_0\right)\,\delta_{v_1,v_2}\delta_{i_1,i_2} \\ &\quad= \left\{\frac{2E_{b1}}{R_1}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big] + \frac{E_{b1}}{R_1}(K-1) + N_0\right\}\,\delta_{v_1,v_2}\delta_{i_1,i_2}. \qquad \text{(W.71)}\end{aligned}$$

We use (W.9) and apply both (W.48) and (W.50) to the resulting expression to re-write (W.71) as

$$\frac{1}{2}E\big[Y_{q,v_1,m,i_1}Y^*_{q,v_2,m,i_2}\big] = \begin{cases} \dfrac{E_{b1}}{R_1}\left[1+c_1\!\left(\dfrac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right]\delta_{v_1,v_2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks.} \end{cases} \tag{W.72}$$

W.3 Derivation of $\frac{1}{2}E[Y_{q+1,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}]$

For q odd, we use (W.51) and

$$b^{(k)}_{p,q,u,m} = \begin{cases} -a^{(k)}_{p,q,m}, & u=1, \\ a^{(k)}_{p,q-1,m}, & u=2, \end{cases} \quad (q = Q_P+1,\ldots,Q;\ q \text{ even}) \tag{W.73}$$

from Table W.1 (see p. 279) to write

$$\begin{aligned} Y_{q+1,v,m,i} &= -A_{1d}\,a^{(1)}_{p,q+1,m}\,g^{(1)}_{v,1,m,i} + A_{1d}\,a^{(1)}_{p,q,m}\,g^{(1)}_{v,2,m,i} \\ &\quad + \Gamma_{Y_{q+1,v,m,i}}, \quad q = Q_P+1,\ldots,Q;\ q \text{ odd}, \end{aligned} \tag{W.74}$$

where

$$\Gamma_{Y_{q+1,v,m,i}} \triangleq I_{Y_{q+1,v,m,i}} + N_{Y_{q+1,v,m,i}}. \tag{W.75}$$

Using this expression, we have

$$\begin{aligned} &\frac{1}{2}E\big[Y_{q+1,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}\big] \\ &= A_{1d}^2\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A_{1d}^2\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\ &\quad + \frac{1}{2}E\big[\Gamma_{Y_{q+1,v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big], \quad q = Q_P+1,\ldots,Q;\ q \text{ odd}, \end{aligned} \tag{W.76}$$

where the expected values of the cross terms are easily shown to equal zero.

W.3.1 Sum of First and Second Terms in (W.76)

The sum of the first and second terms in (W.76) is given by (W.58):

$$\begin{aligned} &A_{1d}^2 \frac{1}{2} E\big[g_{v_1,1,m,i_1}^{(1)} \big(g_{v_2,1,m,i_2}^{(1)}\big)^*\big] + A_{1d}^2 \frac{1}{2} E\big[g_{v_1,2,m,i_1}^{(1)} \big(g_{v_2,2,m,i_2}^{(1)}\big)^*\big] \\ &\quad = 2A_{1d}^2 \frac{1}{2} E\big[g_{v_1,1,m,i_1}^{(1)} \big(g_{v_1,1,m,i_2}^{(1)}\big)^*\big]\, \delta_{v_1,v_2} \delta_{i_1,i_2}. \end{aligned} \tag{W.77}$$

W.3.2 Third Term in (W.76)

Using (W.75), we have

$$\frac{1}{2} E\big[\Gamma_{Y_{q+1,v_1,m,i_1}} \Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{1}{2} E\big[I_{Y_{q+1,v_1,m,i_1}} I^*_{Y_{q+1,v_2,m,i_2}} + N_{Y_{q+1,v_1,m,i_1}} N^*_{Y_{q+1,v_2,m,i_2}}\big]. \tag{W.78}$$

First Term in (W.78)

We replace q with $q+1$ in the derivation for $\frac{1}{2} E\big[I_{Y_{q,v_1,m,i_1}} I^*_{Y_{q,v_2,m,i_2}}\big]$ (see Section W.2.2) to obtain the same result as (W.63):

$$\frac{1}{2} E\big[I_{Y_{q+1,v_1,m,i_1}} I^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{A_{1e}^2 (K-1)}{2}\, \delta_{v_1,v_2} \delta_{i_1,i_2}. \tag{W.79}$$

Second Term in (W.78)

Similarly, we replace q with $q+1$ in the derivation for $\frac{1}{2} E\big[N_{Y_{q,v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big]$ in Section W.2.2 to obtain (see (W.69))

$$\frac{1}{2} E\big[N_{Y_{q+1,v_1,m,i_1}} N^*_{Y_{q+1,v_2,m,i_1}}\big] = N_0\, \delta_{v_1,v_2} \delta_{i_1,i_2}. \tag{W.80}$$

Combining the Results

We substitute (W.79) and (W.80) into (W.78) to get

$$\frac{1}{2} E\big[\Gamma_{Y_{q+1,v_1,m,i_1}} \Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = \left(\frac{A_{1e}^2 (K-1)}{2} + N_0 \right) \delta_{v_1,v_2} \delta_{i_1,i_2}. \tag{W.81}$$

W.3.3 Combining the Results

Finally, we substitute the results of (W.77) and (W.81) into (W.76) to get the same result as (W.72):

$$
\begin{aligned}
&\frac{1}{2}E\big[Y_{q+1,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}\big] \\
&\quad= \left\{\frac{2E_{b1}}{R_1}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big] + \frac{E_{b1}}{R_1}(K-1)+N_0\right\}\delta_{v_1,v_2}\delta_{i_1,i_2} \\
&\quad= \begin{cases} \dfrac{E_{b1}}{R_1}\left[1+c_1\left(\dfrac{E_{b1}}{N_0}\right)\delta_{i_1,i_2}\right]\delta_{v_1,v_2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks.} \end{cases}
\end{aligned}
\tag{W.82}
$$

W.4 Derivation of $\frac{1}{2}E\big[Y_{q,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}\big]$

We use (W.56) and (W.74) to write

$$
\begin{aligned}
\frac{1}{2}E\big[Y_{q,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}\big] &= -A^2_{1d}a^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] \\
&\quad + A^2_{1d}a^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\
&\quad +\frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big], \quad q=Q_P+1,\ldots,Q;\ q \text{ odd,}
\end{aligned}
\tag{W.83}
$$

where the expected values of the cross terms are equal to zero.

W.4.1 Sum of First and Second Terms in (W.83)

Using (W.8), we evaluate the sum of the first and second terms in (W.83) as

$$
\begin{aligned}
&-A^2_{1d}a^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A^2_{1d}a^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\
&\quad= -A^2_{1d}a^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big]\delta_{v_1,v_2} \\
&\qquad + A^2_{1d}a^{(1)}_{p,q,m}a^{(1)}_{p,q+1,m}\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big]\delta_{v_1,v_2} \\
&\quad= 0.
\end{aligned}
\tag{W.84}
$$

W.4.2 Third Term in (W.83)

We use (W.54) and (W.75) to expand the third term in (W.83) as

$$\frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = \frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}}I^*_{Y_{q+1,v_2,m,i_2}} + N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_2}}\big], \tag{W.85}$$

where, again, the expected values of the cross terms are equal to zero.

First Term in (W.85)

We use the expression for $I_{Y_{q,v,m,i}}$ in (W.60) to write

$$\begin{aligned}&\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}}I^*_{Y_{q+1,v_2,m,i_2}}\big]\\&\quad= C^{(1)}_{p,q,m,i_1}C^{(1)}_{p,q+1,m,i_2}\frac{1}{2}E\Big\{I_{y_{v_1,m,i_1}}[(pQ+q-1)T_s]I^*_{y_{v_2,m,i_2}}[(pQ+q)T_s]\Big\}.\end{aligned} \tag{W.86}$$

By replacing $q_1(u_1)$ and $q_1(u_2)$ in the derivation for $\frac{1}{2}E\big[I_{W_{q_1(u_1),v_1,m,i_1}}I^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ (see Section W.1.2) with q and $q+1$, respectively, we obtain (see (W.26))

$$\begin{aligned}&\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}}I^*_{Y_{q+1,v_2,m,i_2}}\big]\\&\quad= \frac{A^2_{1e}}{2}\delta_{v_1,v_2}\delta_{i_1,i_2}C^{(1)}_{p,q,m,i_1}C^{(1)}_{p,q+1,m,i_1}\sum_{k_1=2}^{K}\sum_{p'_1=-\infty}^{\infty}\sum_{q'_1=1}^{Q}\\&\qquad\cdot E\Big\{x_1\big[t_1-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big]\\&\qquad\qquad x_1\big[t_2-(p'_1Q+q'_1-1)T_s-\tau^{(k_1)}\big]\Big\}\Big|_{t_1=(pQ+q-1)T_s,\ t_2=(pQ+q)T_s}.\end{aligned} \tag{W.87}$$

We then apply (S.67) to the above equation and use (S.38) to get

$$\begin{aligned}\frac{1}{2}E\big[I_{Y_{q,v_1,m,i_1}}I^*_{Y_{q+1,v_2,m,i_2}}\big] &= \frac{A^2_{1e}}{2}\delta_{v_1,v_2}\delta_{i_1,i_2}C^{(1)}_{p,q,m,i_1}C^{(1)}_{p,q+1,m,i_1}\\&\qquad\cdot\sum_{k_1=2}^{K}\frac{1}{T_s}\int_{-\infty}^{\infty}|X_1(f)|^2e^{j2\pi f\tau}\,df\Big|_{\tau=t_2-t_1=T_s}\\&= 0.\end{aligned} \tag{W.88}$$

Second Term in (W.85)

We use the expression for $N_{Y_{q,v,m,i}}$ in (W.64) to write

$$\begin{aligned}&\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_2}}\big]\\&\quad=C^{(1)}_{p,q,m,i_1}C^{(1)}_{p,q+1,m,i_2}\cdot\frac{1}{2}E\Big\{N_{y_{v_1,m,i_1}}[(pQ+q-1)T_s]N^*_{y_{v_2,m,i_2}}[(pQ+q)T_s]\Big\}.\end{aligned}\tag{W.89}$$

When $i_1 \neq i_2$, the AWGN terms $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are independent, so $N_{Y_{q,v_1,m,i_1}}$ and $N_{Y_{q+1,v_2,m,i_2}}$ are independent as well:

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_2}}\big]=0,\quad i_1\neq i_2.\tag{W.90}$$

For $i_1 = i_2$, we use the derivation for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}}N^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ (see Section W.1.2) to evaluate (W.89). We replace $q_1(u_1)$ and $q_1(u_2)$ in (W.35) with q and $q+1$, respectively, to obtain

$$\begin{aligned}\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_1}}\big]&=N_0\,\delta_{v_1,v_2}C^{(1)}_{p,q,m,i_1}C^{(1)}_{p,q+1,m,i_1}\\&\quad\cdot\int_{-\infty}^{\infty}\tilde{h}_{m,i_1,1}(t_1-\tau_1)\tilde{h}^*_{m,i_1,1}(t_2-\tau_1)\,d\tau_1\Big|_{\substack{t_1=(pQ+q-1)T_s\\t_2=(pQ+q)T_s}}.\end{aligned}\tag{W.91}$$

To the above equation, we then apply (S.60) and (S.63) to get

$$\begin{aligned}\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_1}}\big]&=N_0\,\delta_{v_1,v_2}C^{(1)}_{p,q,m,i_1}C^{(1)}_{p,q+1,m,i_1}\cdot x_1(\tau)\Big|_{\tau=t_2-t_1=T_s}\\&=0.\end{aligned}\tag{W.92}$$

Combining the results of (W.90) and (W.92), we have

$$\frac{1}{2}E\big[N_{Y_{q,v_1,m,i_1}}N^*_{Y_{q+1,v_2,m,i_2}}\big]=0.\tag{W.93}$$

Combining the Results

Substituting (W.88) and (W.93) into (W.85), we obtain

$$\frac{1}{2}E\big[\Gamma_{Y_{q,v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big]=0.\tag{W.94}$$

W.4.3 Combining the Results

Finally, we substitute the results of (W.84) and (W.94) into (W.83) to get

$$\frac{1}{2}E\big[Y_{q,v_1,m,i_1}Y^*_{q+1,v_2,m,i_2}\big] = 0. \tag{W.95}$$

W.5 Derivation of $\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1}Y^*_{q,v_2,m,i_2}]$

Assuming $q = Q_P + 1, \ldots, Q$ and q odd, we use (W.12) and (W.56) to write

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1}Y^*_{q,v_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\Bigg[\Bigg(g^{(1)}_{v_1,u_1,m,i_1} + \frac{1}{A_{1e}}\frac{1}{Q_P/2}\sum_{\substack{q'=1 \\ q' \text{ odd}}}^{Q_P}\Gamma_{W_{q'(u_1),v_1,m,i_1}}\Bigg) \\
&\qquad \cdot\Bigg(A_{1d}\,a^{(1)}_{p,q,m}\,g^{(1)}_{v_2,1,m,i_2} + A_{1d}\,a^{(1)}_{p,q+1,m}\,g^{(1)}_{v_2,2,m,i_2} + \Gamma_{Y_{q,v_2,m,i_2}}\Bigg)^*\Bigg] \\
&\quad = A_{1d}\,a^{(1)}_{p,q,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A_{1d}\,a^{(1)}_{p,q+1,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\
&\qquad + \frac{1}{A_{1e}}\frac{1}{Q_P/2}\sum_{\substack{q'=1 \\ q' \text{ odd}}}^{Q_P}\frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}}\Gamma^*_{Y_{q,v_2,m,i_2}}\big],
\end{aligned} \tag{W.96}$$

where $q'(u) \triangleq q' + (u-1)$, and we used the fact that

$$\frac{1}{2}E\big[g^{(k)}_{v_1,u,m,i_1}\Gamma^*_{Y_{q,v_2,m,i_2}}\big] = 0, \tag{W.97}$$

which is straightforward to prove.

W.5.1 First and Second Terms in (W.96)

Using (W.8), we have

$$A_{1d}\,a^{(1)}_{p,q,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] = A_{1d}\,a^{(1)}_{p,q,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\delta_{v_1,v_2}\delta_{u_1,1}. \tag{W.98}$$

Similarly, we also have

$$A_{1d}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2} E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] = A_{1d}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2} E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_1,2,m,i_2}\big)^*\big] \delta_{v_1,v_2}\delta_{u_1,2}. \tag{W.99}$$

W.5.2 Third Term in (W.96)

We use (W.10) and (W.54) to write

$$\frac{1}{2} E\big[\Gamma_{W_{q'(u_1)},v_1,m,i_1} \Gamma^*_{Y_{q,v_2,m,i_2}}\big] = \frac{1}{2} E\big[I_{W_{q'(u_1)},v_1,m,i_1} I^*_{Y_{q,v_2,m,i_2}} + N_{W_{q'(u_1)},v_1,m,i_1} N^*_{Y_{q,v_2,m,i_2}}\big], \tag{W.100}$$

where, again, the expected values of the cross terms are equal to zero.

First Term in (W.100)

We use the expressions for $I_{W_{q(u),v,m,i}}$ and $I_{Y_{q,v,m,i}}$ in (W.18) and (W.60), respectively, to write

$$\begin{aligned}
&\frac{1}{2} E\big[I_{W_{q'(u_1)},v_1,m,i_1} I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = C^{(1)}_{p,q'(u_1),m,i_1} C^{(1)}_{p,q,m,i_2} \frac{1}{2} E\Big\{ I_{y_{v_1,m,i_1}}[(pQ+q'(u_1)-1)T_s] I^*_{y_{v_2,m,i_2}}[(pQ+q-1)T_s] \Big\}.
\end{aligned} \tag{W.101}$$

By replacing $q_1(u_1)$ and $q_1(u_2)$ in the derivation for $\frac{1}{2}E\big[I_{W_{q_1(u_1)},v_1,m,i_1} I^*_{W_{q_1(u_2)},v_2,m,i_2}\big]$ (see Section W.1.2) with $q'(u_1)$ and q, respectively, we can show that

$$\begin{aligned}
&\frac{1}{2} E\big[I_{W_{q'(u_1)},v_1,m,i_1} I^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = \frac{A^2_{1e}}{2}\, \delta_{v_1,v_2}\delta_{i_1,i_2} C^{(1)}_{p,q'(u_1),m,i_1} C^{(1)}_{p,q,m,i_1} \sum_{k_1=2}^{K} \sum_{p'_1=-\infty}^{\infty} \sum_{q'_1=1}^{Q} \\
&\qquad \cdot E\Big\{ x_1\big[t_1 - (p'_1 Q + q'_1 - 1)T_s - \tau^{(k_1)}\big] \\
&\qquad\qquad x_1\big[t_2 - (p'_1 Q + q'_1 - 1)T_s - \tau^{(k_1)}\big] \Big\}\Big|_{t_1=[pQ+q'(u_1)-1]T_s,\ t_2=(pQ+q-1)T_s}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A_{1e}^2}{2}\,\delta_{v_1,v_2}\delta_{i_1,i_2}C^{(1)}_{p,q'(u_1),m,i_1}C^{(1)}_{p,q,m,i_1} \\
&\quad \cdot \sum_{k_1=2}^{K}\frac{1}{T_s}\int_{-\infty}^{\infty}|X_1(f)|^2 e^{j2\pi f\tau}\,df\Big|_{\tau=t_2-t_1=[q-q'-(u_1-1)]T_s} \\
&= 0. \qquad\qquad \text{(W.102)}
\end{aligned}
$$

The last equality follows from (S.38), along with the fact that $q-q'-(u_1-1)$ can never equal zero, since $q'=1,\ldots,Q_P$ and $q=Q_P+1,\ldots,Q$, and q' and q are both odd.

Second Term in (W.100)

We use the expressions for $N_{W_{q(u),v,m,i}}$ and $N_{Y_{q,v,m,i}}$ in (W.29) and (W.64), respectively, to write

$$
\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}}N^*_{Y_{q,v_2,m,i_2}}\big] \\
&\quad = C^{(1)}_{p,q'(u_1),m,i_1}C^{(1)}_{p,q,m,i_2}\cdot\frac{1}{2}E\Big\{N_{y_{v_1,m,i_1}}[(pQ+q'(u_1)-1)T_s]N^*_{y_{v_2,m,i_2}}[(pQ+q-1)T_s]\Big\}.
\end{aligned}
\qquad \text{(W.103)}
$$

When $i_1\neq i_2$, the AWGN terms $N_{y_{v_1,m,i_1}}(t)$ and $N_{y_{v_2,m,i_2}}(t)$ are independent, so $N_{W_{q'(u_1),v_1,m,i_1}}$ and $N_{Y_{q,v_2,m,i_2}}$ are independent as well:

$$
\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}}N^*_{Y_{q,v_2,m,i_2}}\big]=0,\quad i_1\neq i_2. \qquad \text{(W.104)}
$$

For $i_1=i_2$, we use the derivation for $\frac{1}{2}E\big[N_{W_{q_1(u_1),v_1,m,i_1}}N^*_{W_{q_1(u_2),v_2,m,i_2}}\big]$ to evaluate (W.103) (see Section W.1.2). That is, we replace $q_1(u_1)$ and $q_1(u_2)$ in (W.35) with $q'(u_1)$ and q, respectively, to obtain

$$
\begin{aligned}
&\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}}N^*_{Y_{q,v_2,m,i_1}}\big] \\
&\quad = N_0\,\delta_{v_1,v_2}C^{(1)}_{p,q'(u_1),m,i_1}C^{(1)}_{p,q,m,i_2} \qquad\qquad \text{(W.105)}\\
&\qquad \cdot\int_{-\infty}^{\infty}\tilde{h}_{m,i_1,1}(t_1-\tau_1)\tilde{h}^*_{m,i_1,1}(t_2-\tau_1)\,d\tau_1\Big|_{t_1=[pQ+q'(u_1)-1]T_s,\ t_2=(pQ+q-1)T_s}.
\end{aligned}
$$

We then apply (S.60) and (S.63) to the above equation to get

$$\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_1}}\big] = N_0\,\delta_{v_1,v_2} C^{(1)}_{p,q'(u_1),m,i_1} C^{(1)}_{p,q,m,i_2} \cdot x_2(\tau)\Big|_{\tau=t_2-t_1=[q-q'-(u_1-1)]T_s}$$
$$= 0. \quad \text{(W.106)}$$

Again, the quantity $q - q' - (u_1 - 1)$ can never equal zero.

Combining the results of (W.104) and (W.106), we have

$$\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big] = 0. \quad \text{(W.107)}$$

Combining the Results

Substituting (W.102) and (W.107) into (W.100), we obtain

$$\frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}} \Gamma^*_{Y_{q,v_2,m,i_2}}\big] = 0. \quad \text{(W.108)}$$

W.5.3 Combining the Results

Finally, we substitute the results of (W.98), (W.99), and (W.108) into (W.96) to get

$$\begin{aligned}
\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1} Y^*_{q,v_2,m,i_2}\big] &= A_{1d}\, a^{(1)}_{p,q,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{u_1,1} \\
&\quad + A_{1d}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_1,2,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{u_1,2} \\
&= \sqrt{\frac{E_{b1}}{R_1}}\, a^{(1)}_{p,q,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{u_1,1} \\
&\quad + \sqrt{\frac{E_{b1}}{R_1}}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_1,2,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{u_1,2},
\end{aligned} \quad \text{(W.109)}$$

where we used the expression for E_{b1} in (W.48). The expression for $\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_1,u_1,m,i_2}\big)^*\big]$ is given by (W.9). When $u_1 = 1$, (W.109) yields

$$\frac{1}{2}E\big[\hat{W}_{v_1,1,m,i_1} Y^*_{q,v_2,m,i_2}\big]$$
$$= \begin{cases} \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\, a^{(1)}_{p,q,m}\,\delta_{v_1,v_2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks.} \end{cases} \quad \text{(W.110)}$$

Similarly, when $u_1 = 2$, we have

$$\frac{1}{2}E\big[\hat{W}_{v_1,2,m,i_1} Y^*_{q,v_2,m,i_2}\big]$$
$$= \begin{cases} \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\, a^{(1)}_{p,q+1,m}\, \delta_{v_1,v_2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks.} \end{cases} \quad \text{(W.111)}$$

W.6 Derivation of $\frac{1}{2}E[\hat{W}_{v_1,u_1,m,i_1} Y^*_{q+1,v_2,m,i_2}]$

Again, we assume $q = Q_P + 1, \ldots, Q$ and q odd, but this time, we use (W.12) and (W.74) to write

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1} Y^*_{q+1,v_2,m,i_2}\big] \\
&\quad = \frac{1}{2}E\Bigg[\Bigg(g^{(1)}_{v_1,u_1,m,i_1} + \frac{1}{A_{1e}}\frac{1}{Q_P/2}\sum_{\substack{q'=1 \\ q' \text{ odd}}}^{Q_P} \Gamma_{W_{q'(u_1),v_1,m,i_1}}\Bigg) \\
&\qquad \cdot\Bigg(-A_{1d}\, a^{(1)}_{p,q+1,m}\, g^{(1)}_{v_2,1,m,i_2} + A_{1d}\, a^{(1)}_{p,q,m}\, g^{(1)}_{v_2,2,m,i_2} + \Gamma_{Y_{q+1,v_2,m,i_2}}\Bigg)^*\Bigg] \\
&\quad = -A_{1d}\, a^{(1)}_{p,q+1,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] + A_{1d}\, a^{(1)}_{p,q,m}\cdot\frac{1}{2}E\big[g^{(1)}_{v_1,u_1,m,i_1}\big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] \\
&\qquad + \frac{1}{A_{1e}}\frac{1}{Q_P/2}\sum_{\substack{q'=1 \\ q' \text{ odd}}}^{Q_P} \frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big],
\end{aligned} \quad \text{(W.112)}$$

where $q'(u) \triangleq q' + (u-1)$, and we used the fact that

$$\frac{1}{2}E\big[g^{(k)}_{v_1,u,m,i_1}\Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = 0, \quad \text{(W.113)}$$

which is straightforward to prove.

W.6.1 First and Second Terms in (W.112)

Using (W.8), we have

$$\begin{aligned} & - A_{1d}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2} E\big[g^{(1)}_{v_1,u_1,m,i_1} \big(g^{(1)}_{v_2,1,m,i_2}\big)^*\big] \\ & \quad = -A_{1d}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2} E\big[g^{(1)}_{v_1,u_1,m,i_1} \big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\, \delta_{v_1,v_2}\, \delta_{u_1,1}, \end{aligned} \tag{W.114}$$

and

$$A_{1d}\, a^{(1)}_{p,q,m} \cdot \frac{1}{2} E\big[g^{(1)}_{v_1,u_1,m,i_1} \big(g^{(1)}_{v_2,2,m,i_2}\big)^*\big] = A_{1d}\, a^{(1)}_{p,q,m} \cdot \frac{1}{2} E\big[g^{(1)}_{v_1,u_1,m,i_1} \big(g^{(1)}_{v_1,2,m,i_2}\big)^*\big]\, \delta_{v_1,v_2}\, \delta_{u_1,2}. \tag{W.115}$$

W.6.2 Third Term in (W.112)

Using (W.10) and (W.75), we can expand the third term in (W.112) as

$$\frac{1}{2} E\big[\Gamma_{W_{q'(u_1)},v_1,m,i_1} \Gamma^*_{Y_{q+1},v_2,m,i_2}\big] = \frac{1}{2} E\big[I_{W_{q'(u_1)},v_1,m,i_1} I^*_{Y_{q+1},v_2,m,i_2} + N_{W_{q'(u_1)},v_1,m,i_1} N^*_{Y_{q+1},v_2,m,i_2}\big], \tag{W.116}$$

where, again, the expected values of the cross terms are equal to zero.

First Term in (W.116)

We simply replace q with $q+1$ in the derivation for $\frac{1}{2} E\big[I_{W_{q'(u_1)},v_1,m,i_1} I^*_{Y_q,v_2,m,i_2}\big]$ in Section W.5.2 to obtain (see (W.102))

$$\frac{1}{2} E\big[I_{W_{q'(u_1)},v_1,m,i_1} I^*_{Y_{q+1},v_2,m,i_2}\big] = 0, \tag{W.117}$$

since $q - q' - (u_1 - 1) + 1$ can never equal zero.

Second Term in (W.116)

Similarly, we replace q with $q+1$ in the derivation for $\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}} N^*_{Y_{q,v_2,m,i_2}}\big]$ to obtain (see Section W.5.2)

$$\frac{1}{2}E\big[N_{W_{q'(u_1),v_1,m,i_1}} N^*_{Y_{q+1,v_2,m,i_2}}\big] = 0. \tag{W.118}$$

Combining the Results

Substituting (W.117) and (W.118) into (W.116), we get

$$\frac{1}{2}E\big[\Gamma_{W_{q'(u_1),v_1,m,i_1}} \Gamma^*_{Y_{q+1,v_2,m,i_2}}\big] = 0. \tag{W.119}$$

W.6.3 Combining the Results

Finally, we substitute (W.114), (W.115), and (W.119) into (W.112) to get

$$\begin{aligned}
\frac{1}{2}E\big[\hat{W}_{v_1,u_1,m,i_1} Y^*_{q,v_2,m,i_2}\big] &= -A_{1d}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{u_1,1} \\
&\quad + A_{1d}\, a^{(1)}_{p,q,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_1,2,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{u_1,2} \\
&= -\sqrt{\frac{E_{b1}}{R_1}}\, a^{(1)}_{p,q+1,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,1,m,i_1}\big(g^{(1)}_{v_1,1,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{u_1,1} \\
&\quad + \sqrt{\frac{E_{b1}}{R_1}}\, a^{(1)}_{p,q,m} \cdot \frac{1}{2}E\big[g^{(1)}_{v_1,2,m,i_1}\big(g^{(1)}_{v_1,2,m,i_2}\big)^*\big]\,\delta_{v_1,v_2}\delta_{u_1,2},
\end{aligned} \tag{W.120}$$

where we used the expression for E_{b1} in (W.48). By using (W.9) in (W.120), we have

$$\begin{aligned}
&\frac{1}{2}E\big[\hat{W}_{v_1,1,m,i_1} Y^*_{q,v_2,m,i_2}\big] \\
&\quad = \begin{cases} -\dfrac{1}{2}\sqrt{\dfrac{E_{b1}}{R_1}}\, a^{(1)}_{p,q+1,m}\,\delta_{v_1,v_2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks,} \end{cases}
\end{aligned} \tag{W.121}$$

when $u_1 = 1$, and

$$\frac{1}{2}E\big[\hat{W}_{v_1,2,m,i_1}Y^*_{q,v_2,m,i_2}\big] = \begin{cases} \frac{1}{2}\sqrt{\frac{E_{b1}}{R_1}}\, a^{(1)}_{p,q,m}\,\delta_{v_1,v_2}, & i_1 \text{ and } i_2 \text{ are associated with the same block,} \\ 0, & i_1 \text{ and } i_2 \text{ are associated with different blocks,} \end{cases} \tag{W.122}$$

when $u_1 = 2$.

Bibliography

[1] E. A. Sourour and M. Nakagawa, “Performance of Orthogonal Multicarrier CDMA in a Multipath Fading Channel,” *IEEE Trans. Commun.*, vol. 44, no. 3, pp. 356–367, Mar. 1996.

[2] S. Kondo and L. B. Milstein, “Performance of Multicarrier DS CDMA Systems,” *IEEE Trans. Commun.*, vol. 44, no. 2, pp. 238–246, Feb. 1996.

[3] D. Lee and L. B. Milstein, “Comparison of Multicarrier DS-CDMA Broadcast Systems in a Multipath Fading Channel,” *IEEE Trans. Commun.*, vol. 47, no. 12, pp. 1897–1904, Dec. 1999.

[4] W. Xu and L. B. Milstein, “On the Performance of Multicarrier RAKE Systems,” *IEEE Trans. Commun.*, vol. 49, no. 10, pp. 1812–1823, Oct. 2001.

[5] L. L. Chong and L. B. Milstein, “Error Rate of a Multicarrier CDMA System with Imperfect Channel Estimates,” in *Proc. 2000 IEEE Int. Conf. Commun. (ICC’00)*, vol. 2, New Orleans, LA, Jun. 18–22, 2000, pp. 934–938.

[6] D. N. Rowitch and L. B. Milstein, “Convolutionally Coded Multicarrier DS-CDMA Systems in a Multipath Fading Channel—Part I: Performance Analysis,” *IEEE Trans. Commun.*, vol. 47, no. 10, pp. 1570–1582, Oct. 1999.

[7] L.-L. Yang and L. Hanzo, “Performance of Generalized Multicarrier DS-CDMA over Nakagami-m Fading Channels,” *IEEE Trans. Commun.*, vol. 50, no. 6, pp. 956–966, Jun. 2002.

[8] N. Yee, J. M. G. Linnartz, and G. Fettweis, “Multicarrier CDMA in Indoor Wireless Radio Networks,” *IEICE Trans. Commun.*, vol. E77-B, no. 7, pp. 900–904, Jul. 1994.

[9] X. Gui and T. S. Ng, “Performance of Asynchronous Orthogonal Multicarrier CDMA System in Frequency Selective Fading Channel,” *IEEE Trans. Commun.*, vol. 47, no. 7, pp. 1084–1091, Jul. 1999.

[10] Z. Hou and V. K. Dubey, “Bit Error Probability of MC-CDMA System over Rayleigh Fading Channels,” in *Proc. 2004 IEEE 8th Int. Symp. Spread Spectrum Techniques Appl. (ISSSTA’04)*, Sydney, Australia, Aug. 30–Sep. 2, 2004, pp. 320–324.

[11] Z. Kang and K. Yao, “Performance Comparison of MC-CDMA over Frequency-Selective Nakagami-m and Rayleigh Fading Channels,” in *Proc. 2004 IEEE 60th Veh. Technol. Conf. (VTC’04–Fall)*, vol. 6, Los Angeles, CA, Sep. 26–29, 2004, pp. 4228–4232.

[12] Q. Shi and M. Latva-Aho, "Accurate Bit-Error Rate Evaluation for Synchronous MC-CDMA over Nakagami-m Fading Channels Using Moment Generating Functions," *IEEE Trans. Wireless Commun.*, vol. 4, no. 2, pp. 422–433, Mar. 2005.

[13] Z. Li and M. Latva-Aho, "Analysis of MRC Receivers for Asynchronous MC-CDMA with Channel Estimation Errors," in *Proc. 2002 IEEE 7th Int. Symp. Spread Spectrum Techniques Appl. (ISSSTA'02)*, vol. 2, Prague, Czech Republic, Sep. 2–5, 2002, pp. 343–347.

[14] S. Kaiser and P. Hoeher, "Performance of Multicarrier CDMA Systems with Channel Estimation in Two Dimensions," in *Proc. 8th IEEE Int. Symp. Personal, Indoor, Mobile Radio Commun. (PIMRC'97)*, vol. 1, Helsinki, Finland, Sep. 1–4, 1997, pp. 115–119.

[15] K. Zheng, G. Zeng, and W. Wang, "Analysis and Optimization of Pilot-Symbol-Assisted MC-CDMA Systems," in *Proc. 2004 IEEE Wireless Commun. Networking Conf. (WCNC'04)*, vol. 2, Atlanta, GA, Mar. 21–25, 2004, pp. 706–710.

[16] C. S. Patel, G. L. Stuber, and T. G. Pratt, "Analysis of OFDM/MC-CDMA under Channel Estimation and Jamming," in *Proc. 2004 IEEE Wireless Commun. Networking Conf. (WCNC'04)*, vol. 2, Atlanta, GA, Mar. 21–25, 2004, pp. 954–958.

[17] R. Prasad and S. Hara, "An Overview of Multicarrier CDMA," in *Proc. 1996 IEEE 4th Int. Symp. Spread Spectrum Techniques Appl. (ISSSTA'96)*, vol. 1, Mainz, Germany, Sep. 22–25, 1996, pp. 107–114.

[18] S. Abeta, H. Atarashi, M. Sawahashi, and F. Adachi, "Coherent Multicarrier/DS-CDMA and MC-CDMA for Broadband Packet Wireless Access," in *Proc. 2000 IEEE 51st Veh. Technol. Conf. (VTC'00–Spring)*, vol. 3, Tokyo, Japan, May 15–18, 2000, pp. 1918–1922.

[19] S. Abeta, H. Atarashi, and M. Sawahashi, "Forward Link Capacity of Coherent DS-CDMA and MC-CDMA Broadband Packet Wireless Access in a Multi-Cell Environment," in *Proc. 2000 IEEE 52nd Veh. Technol. Conf. (VTC'00–Fall)*, vol. 5, Boston, MA, Sep. 24–28, 2000, pp. 2213–2218.

[20] H. Atarashi, S. Abeta, and M. Sawahashi, "Broadband Packet Wireless Access Appropriate for High-Speed and High-Capacity Throughput," in *Proc. 2001 IEEE 53rd Veh. Technol. Conf. (VTC'01–Spring)*, vol. 1, Rhodes, Greece, May 6–9, 2001, pp. 566–570.

[21] S. Suwa, H. Atarashi, and M. Sawahashi, "Performance Comparison Between MC/DS-CDMA and MC-CDMA for Reverse Link Broadband Packet Wireless Access," in *Proc. 2002 IEEE 56th Veh. Technol. Conf. (VTC'02–Fall)*, vol. 4, Vancouver, BC, Canada, Sep. 24–28, 2002, pp. 2076–2080.

[22] R. S. El-Khamy, S. E. Shaaban, I. A. Ghaleb, and H. N. Kheirallah, "The Enhanced Performance of Combined Multicarrier and CDMA Techniques in Multipath Fading Channels," in *Proc. Twenty-First National Radio Science Conference*, Cairo, Egypt, Mar. 16–18, 2004, pp. C34-1–C34-10.

[23] L. Liu, M. Lei, P. Zhang, K. Hamaguchi, and H. Wakana, "Performance of the Uplink Multicarrier CDMA Systems for WPAN Indoor Applications on MMW Band," in *Proc. 2004 IEEE 59th Veh. Technol. Conf. (VTC'04–Spring)*, vol. 4, Milan, Italy, May 17–19, 2004, pp. 2027–2031.

[24] S. Verdu, *Multiuser Detection.* Cambridge: Cambridge University Press, 1998.

[25] S. M. Alamouti, "A Simple Transmit Diversity Technique for Wireless Communications," *IEEE J. Sel. Areas Commun.*, vol. 16, no. 10, pp. 1451–1458, Oct. 1998.

[26] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time Block Codes from Orthogonal Designs," *IEEE Trans. Inf. Theory*, vol. 45, no. 5, pp. 1456–1467, Jul. 1999.

[27] J. G. Proakis, *Digital Communications*, 4th ed. New York: McGraw-Hill, 2001.

[28] A. Richardson, *WCDMA Design Handbook.* Cambridge: Cambridge University Press, 2005.

[29] S. M. Kay, *Fundamentals of Statistical Signal Processing, Vol. 1: Estimation Theory.* Upper Saddle River: Prentice Hall, 1993.

[30] I.-M. Kim, "Exact BER Analysis of OSTBCs in Spatially Correlated MIMO Channels," *IEEE Trans. Commun.*, vol. 54, no. 8, pp. 1365–1373, Aug. 2006.

[31] Q. T. Zhang, "Maximal-Ratio Combining over Nakagami Fading Channels with an Arbitrary Branch Covariance Matrix," *IEEE Trans. Veh. Technol.*, vol. 48, no. 4, pp. 1141–1150, Jul. 1999.

[32] S. V. Amari and R. B. Misra, "Closed-Form Expressions for Distribution of Sum of Exponential Random Variables," *IEEE Trans. Rel.*, vol. 46, no. 4, pp. 519–522, Dec. 1997.

[33] D. Zwillinger, *CRC Standard Mathematical Tables and Formulae*, 31st ed. Boca Raton: Chapman & Hall/CRC Press, 2003.

[34] R. A. Horn and C. R. Johnson, *Matrix Analysis.* New York: Cambridge University Press, 1985.

[35] M. Schwartz, W. Bennett, and S. Stein, *Communication Systems and Techniques.* New York: McGraw-Hill, 1966.

[36] M. K. Simon, *Probability Distributions Involving Gaussian Random Variables: A Handbook for Engineers and Scientists.* Boston: Kluwer Academic Publishers, 2002.

[37] G. L. Turin, "The Characteristic Function of Hermitian Quadratic Forms in Complex Normal Variables," *Biometrika*, vol. 47, pp. 199–201, 1960.

[38] P. Bello and B. D. Nelin, "Predetection Diversity Combining with Selectively Fading Channels," *IRE Trans. on Commun. Syst.*, vol. CS-10, no. 1, pp. 32–42, Mar. 1962.

[39] J. G. Proakis, "Probabilities of Error for Adaptive Reception of M-Phase Signals," *IEEE Trans. Commun. Technol.*, vol. COM-16, no. 1, pp. 71–81, Feb. 1968.

[40] K. H. Biyari and W. C. Lindsey, "Statistical Distributions of Hermitian Quadratic Forms in Complex Gaussian Variables," *IEEE Trans. Inf. Theory*, vol. 39, no. 3, pp. 1076–1082, May 1993.

[41] F. A. Dietrich and W. Utschick, "Maximal Ratio Combining of Correlated Rayleigh Fading Channels with Imperfect Channel Knowledge," *IEEE Commun. Lett.*, vol. 7, no. 9, pp. 419–421, Sep. 2003.

[42] L. Schmitt, T. Grundler, C. Schreyoegg, I. Viering, and H. Meyr, "Maximum Ratio Combining of Correlated Diversity Branches with Imperfect Channel State Information and Colored Noise," in *Proc. 2004 IEEE 8th Int. Symp. Spread Spectrum Techniques Appl. (ISSSTA'04)*, Sydney, Australia, Aug. 2004, pp. 859–863.

[43] Y. Ma, R. Schober, and S. Pasupathy, "Effect of Imperfect Channel Estimation on MRC Diversity in Fading Channels," in *Proc. 2004 IEEE Int. Conf. Commun. (ICC'04)*, vol. 6, Paris, France, Jun. 2004, pp. 3163–3167.

[44] Y. Ma and T. J. Lim, "Bit Error Probability for MDPSK and NCFSK over Arbitrary Rician Fading Channels," *IEEE J. Select. Areas Commun.*, vol. 18, no. 11, pp. 2179–2189, Nov. 2000.

[45] R. M. Buehrer and N. A. Kumar, "The Impact of Channel Estimation Error on Space-Time Block Codes," in *Proc. 2002 IEEE 56th Veh. Technol. Conf. (VTC'02–Fall)*, vol. 3, Vancouver, BC, Canada, Sep. 2002, pp. 1921–1925.

[46] J. Jootar, J. R. Zeidler, and J. G. Proakis, "Performance of Alamouti Space-Time Code in Time-Varying Channels with Noisy Channel Estimates," in *Proc. 2005 IEEE Wireless Commun. Networking Conf. (WCNC'05)*, vol. 1, New Orleans, LA, Mar. 2005, pp. 498–503.

[47] P. Garg, R. K. Mallik, and H. M. Gupta, "Performance Analysis of Space-Time Coding with Imperfect Channel Estimation," *IEEE Trans. Wireless Commun.*, vol. 4, no. 1, pp. 257–265, Jan. 2005.

[48] F. A. Dietrich and W. Utschick, "Impact of Imperfect Channel Knowledge on Transmit Processing Concepts," in *Proc. 2005 IEEE Int. Conf. Commun. (ICC'05)*, vol. 3, Seoul, South Korea, May 2005, pp. 1916–1921.

[49] S. Manohar, V. Tikiya, R. Annavajjala, and A. Chockalingam, "BER-Optimal Linear Parallel Interference Cancellation for Multicarrier DS-CDMA in Rayleigh Fading," *IEEE Trans. Commun.*, vol. 55, no. 6, pp. 1253–1265, Jun. 2007.

[50] R. Annavajjala and L. B. Milstein, "Performance Analysis of Linear Diversity-Combining Schemes on Rayleigh Fading Channels with Binary Signaling and Gaussian Weighting Errors," *IEEE Trans. Wireless Commun.*, vol. 4, no. 5, pp. 2267–2278, Sep. 2005.

[51] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering.* Upper Saddle River: Prentice Hall, 1993.

[52] S. Haykin, *Adaptive Filter Theory.* Upper Saddle River: Prentice Hall, 1996.