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TOPOLOGICAL EXPANSION AND CUTKOSKY RULES

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ABSTRACT

An unambiguous way of cutting duality diagrams according to the Cutkosky rules is established which permits a topological classification of the terms appearing on the right-hand side of the unitarity equation.

The topological expansion recently proposed by Veneziano[(1)] has been used successfully to clarify various aspects of hadron dynamics, in particular those relevant to Regge theory[(1-6)]. However, it has not been possible, so far, to define the amplitudes appearing in the expansion in pure S-matrix language without reference to underlying field theoretical, or dual, models. One way of attempting such a formulation would be to provide an expansion of both sides of the unitarity equation in terms of the boundaries and handles of the amplitudes involved, so that these amplitudes can be defined through their discontinuities.

The purpose of the present paper is to show the duality diagrams representing the amplitudes have to be cut according to the Cutkosky rules to generate the discontinuities. We shall establish an unambiguous way of cutting diagrams according to their topological structure that will enable us to determine which terms have to be included in the unitarity sum at each level of the expansion. Our method does not take the boundary structures of the diagrams into account which will have to be analysed before the precise unitarity equations can be written down. Work along these lines is in progress.

* Participating guest; Lawrence Berkeley Laboratory.

The dual n-point function is represented by a sum of diagrams which are classified[(1,3)] in terms of their boundaries b (lines to which the external legs are attached) and their handles h. The parameter h is defined as the minimum genus (or number of handles) among all closed two-dimensional orientable surfaces in which the diagram can be embedded[(3,7)]. The topological structure of a diagram is uniquely defined by h and all different ways in which the diagram can be embedded in a surface are topologically equivalent[(8)].

To write down the unitarity equations, one can fix h on the left-hand side and then use the Cutkosky rules to write down the right-hand side[(3)]. This corresponds to cutting the diagrams into two connected pieces in such a way that one piece contains the incoming state and the other piece the outgoing state. Each cut through a particle line has to divide this line into two parts belonging to the two different pieces of the cut diagram. In other words, it must be possible to draw the cut through the diagram as a single line, e.g. as shown in Fig. 1a. Cuts, such as the one shown in Fig. 1b, which separate the diagram into two pieces and then mutilate one of the pieces further, are not allowed.

Topologically, there is no clear distribution between cutting off a cylinder and cutting off a plane from a larger surface[(9)]. In order to classify cuts unambiguously, we have to adopt certain conventions about how to draw diagrams on surfaces. During the following discussion, the reader should keep in mind that we are cutting diagrams; the surfaces merely serve to classify these diagrams in a convenient way.

To draw duality diagrams, we shall only draw particle lines and not quark lines, and we shall indicate a twist by a cross on the particle line. The diagram shown in Fig. 2b, for example, is drawn in this way. It is a shorthand notation for the quark-line diagram shown in Fig. 2a. Planar diagrams need only one side of the embedding surface which we shall identify with the front or back. Nonplanar diagrams need both sides and shall be drawn in such a way that all twists are on the "edges" of the surfaces. In Fig. 3, for example, we have drawn the diagram of Fig. 2 on a cylinder with horizontally oriented axis. We see that one of the diagram's planar parts (drawn in solid lines) occupies the front of the cylinder, whereas the other planar part (drawn in broken lines) occupies the back, the two twists lying on opposite points on the cylinder's "edges". Another kind of planar diagram is one with twists on the same edge of the surface such as the diagram shown in Fig. 4. Diagrams of this kind are considered to be planar because their twists can always be undone by changing the positions of some external legs.

Having established the conventions for drawing diagrams on surfaces, we can now study the various ways in which these surfaces can be cut so as to produce properly cut diagrams. For the cylinder, there are two basic cuts, shown in Fig. 5. In the first case (I), the cutting plane is perpendicular to the cylinder axis; in the second case (II)

it contains the cylinder axis. Cut II will, in general, cut the diagram into more than two pieces and will therefore be forbidden. Only diagrams with just one pair of twists, like the one shown in Fig. 3, will be cut into two pieces by cut II, but in that case the cut will be topologically equivalent to a cut of type I drawn through the two twists. Consequently, only cuts of type I will have to be considered in the following. Furthermore, we shall establish the rule that whenever a twisted line is cut, it shall be cut at the twist which is, obviously, always possible and will simplify the discussion.

The two pieces into which a cylinder diagram is separated by the cut can either be cylinder or planar diagrams, the difference being that the latter ones occupy only one side (front or back) of the surface, whereas the former need both sides and have lines going around the edges. To distinguish these two cases, we shall denote planar parts by drawing "demarcation lines" along the edges of the surface which cannot be crossed by any particle line (see Fig. 6). With this notation, the various combinations of cylinder (C) and planar (P) parts can be drawn as shown in Fig. 7.

Finally, we also have to include in the planar parts diagrams with twists on one edge. To do so, it is sufficient to draw demarcation lines along two edges of the cylinder pieces, so that particle lines can go around the third edge. The combination C + P, for example, will then include the terms shown in Fig. 8. In the following, we shall include all these terms, together with the untwisted planar parts, in the single symbolic notation shown in Fig. 9. With this notation, we can now list all cylinder cuts systematically, as shown in Fig. 10.

To discuss diagrams with $h \neq 0$, it will be convenient to draw them on cylinders with holes in them, which can always be done since a cylinder with h holes is topologically equivalent to a surface with h handles. Figure 11 shows an example of a diagram with $h = 2$ drawn on a cylinder with two holes. This will be our standard way of representing the embedding surfaces; not as spheres with handles, but as flattened cylinders with holes in them that allow particle lines to pass from the front to the back of the surface.

In cutting the diagrams drawn on these surfaces, we have to know how to classify cuts that go through a hole. Taking the torus $(h = 1)$ as an example, we see that a cut through the hole may or may not cut particle lines going through that hole. If it does, the only way of cutting these lines properly is to draw the diagram in such a way that the lines go through the hole exactly where it is touched by the cut. All other ways will produce forbidden types of cuts. Since any hole can accommodate at most two twists, there are three possibilities of cutting through it: (a) both twists are cut, (b) one twist is cut, and (c) neither of the twists is touched by the cut. In Fig. 12 we have drawn the diagram representing pomeron-pomeron exchange on a torus to illustrate these three possibilities.

The cuts shown in Fig. 12 can now be combined with our demarcation lines to define the combinations of cylinder and planar parts shown in Fig. 13. In the case where no demarcation lines are present, so that the result is $C + C$, the lines going through the hole may be cut or not, as shown in Fig. 14. When demarcation lines are present, turning one or both cylinders into a plane it shall be understood that the relevant lines going through the hole have been cut.

With this understanding, we can now list all torus cuts systematically, as shown in Fig. 15. Notice that only cylinder parts can be turned into planar parts by drawing demarcation lines. Cuts like the one shown in Fig. 16, for example, do not generate planes because of the presence of the hole. In this case, the left-hand side of the diagram will be either a cylinder or a torus and the cut will be among the ones listed in Fig. 15.

Because of the fact that all cylinder parts, and only cylinder parts can be turned into planar parts, we can incorporate the notion of the "full cylinder", defined by Veneziano[6] as the sum of cylinder and planar parts, into our notation. The two parts $T + C$ and $T + P$, for example, can be combined into $T + \bar{C}$, as shown in Fig. 17. With this notation, the torus cuts reduce to the three terms shown in Fig. 18.

It is now easy to generalize this analysis to cuts of diagrams with an arbitraty number of handles. To do so, we shall denote a cut dividing a surface with h handles into two pieces with h_- and h_+ handles, respectively, by $c^h_{h_-,h_+}$, with the understanding that all pieces with $h = 0$ are full cylinders including planar parts. Denoting by j the number of holes touched by such a cut, we have the relation

$$h_- + h_+ = h - j \qquad j=0,1\ldots h$$

and we can list the cuts systematically as shown in Table 1.

$j=h$	$c^h_{0,0}$
$j=h-1$	$c^h_{1,0}$ $c^h_{0,1}$
$j=h-2$	$c^h_{2,0}$ $c^h_{1,1}$ $c^h_{0,2}$
⋮	....
$j=0$	$c^h_{h,0}$ $c^h_{h-1,1}$ .... $c^h_{0,h}$

Table 1, Systematic list of cuts $c^h_{h_-,h_+}$.

The total number of cuts will be

$$n = \frac{1}{2}(h + 1)(h + 2).$$

Figure 19 shows a list of these cuts for diagrams with $h = 3$.

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REFERENCES AND FOOTNOTES

1. G. Veneziano, Nucl. Phys. B74, 365 (1974); Phys. Letters 52B, 220 (1974).
2. G. F. Chew and C. Rosenzweig, Phys. Letters 58B, 93 (1975); Phys. Rev. D12, 3907 (1975); LBL-4603 preprint (1975).
3. M. Ciafaloni, G. Marchesini and G. Veneziano, Nucl. Phys. B98, 472, 493 (1975).
4. C. Schmid and C. Sorensen, Nucl. Phys. B96, 209 (1975).
5. J. W. Dash, Phys. Letters 61B, 199 (1976).
6. G. Veneziano, Kyoto University preprint (1976).
7. F. J. Capra, LBL-4295 preprint (1975).
8. See, for example, W. S. Massey, Algebraic Topology, Harcout, Brace and World, New York (1967).
9. We wish to make it clear that we really mean a sphere whenever we say "cylinder". However, we shall adhere to the term "cylinder", as it is the one commonly used in the literature.

FIGURE CAPTIONS

Fig. 1. Examples of a proper cut (a) and of a forbidden cut (b).

Fig. 2. A duality diagram drawn in terms of quark lines (a) and in terms of particle lines (b).

Fig. 3. Diagram of Fig. 2 drawn on a cylinder.

Fig. 4. Planar diagram with twists that can be undone by rearrangement of external legs.

Fig. 5. The two basic cylinder cuts.

Fig. 6. Cutting a cylinder diagram into a cylinder (C) plus planar (P) part, with "demarcation lines" denoting the planar part.

Fig. 7. Combinations of cylinder (C) and planar (P) parts generated by cutting the cylinders; shown with examples of cut diagrams

Fig. 8. $C + P$ terms with twisted planar parts, shown with examples of cut diagrams.

Fig. 9. Unified notation for all $C + P$ terms.

Fig. 10. Systematic list of cylinder cuts.

Fig. 11. A diagram with $h = 2$ drawn on a cylinder with two holes.

Fig. 12. Examples of cuts through the pomeron-pomeron diagram showing the three possibilities of cutting through the hole of the torus: (a) both twists are cut, (b) one twist is cut, (c) neither of the twists is cut.

Fig. 13. Combinations of cylinder (c) and planar (P) parts generated by cutting through the hole of the torus and applying demarcation lines.

Fig. 14. Examples of cutting a diagram with $h = 1$ into two cylinder parts with and without cutting lines going through the hole.

Fig. 15. Systematic list of torus cuts, giving all possible combinations of torus (T), cylinder (C), and planar (P) parts.

Fig. 16. A redundant torus cut.

Fig. 17. Definition of the full cylinder in terms of cut surfaces.

Fig. 18. Torus cuts in terms of torus (T) and full cylinder ($\bar{C}$) parts.

Fig. 19. Systematic list of cuts for diagrams with $h = 3$.

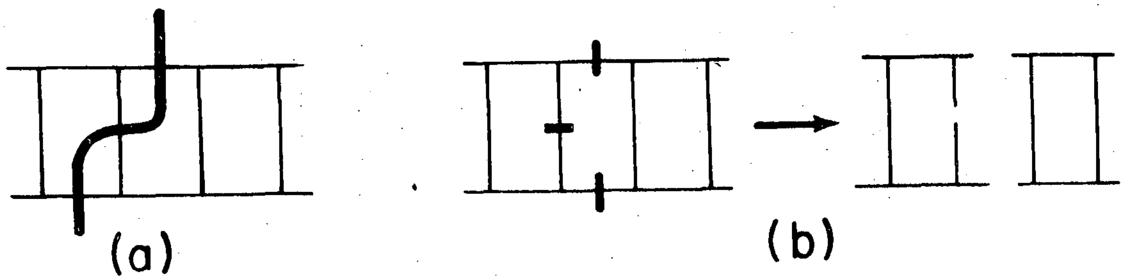


Fig. 1

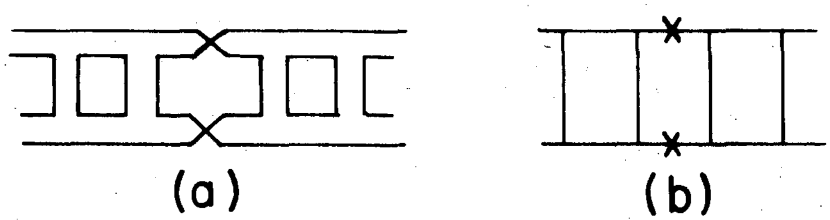
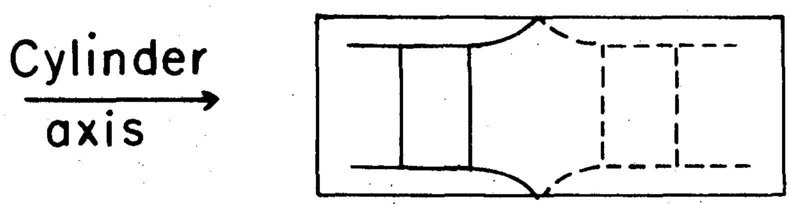


Fig. 2



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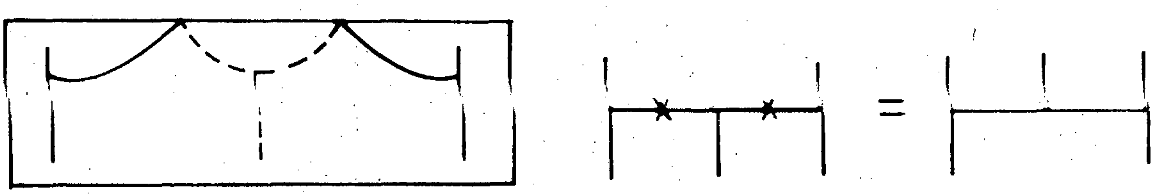

Fig. 4

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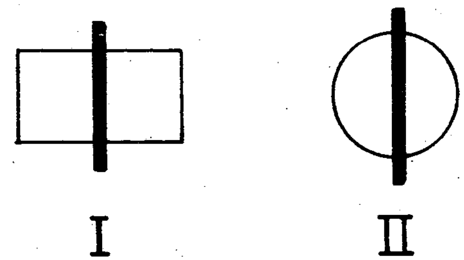

Fig. 5

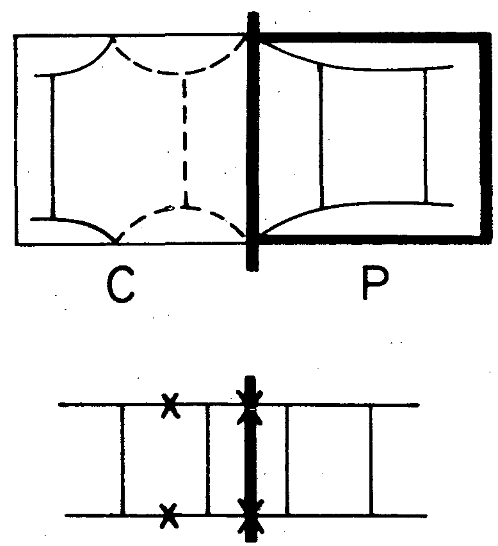

Fig. 6

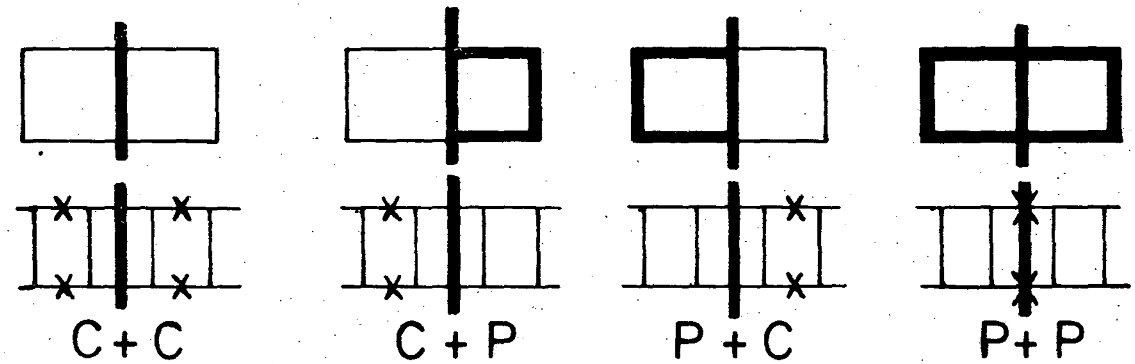

Fig. 7

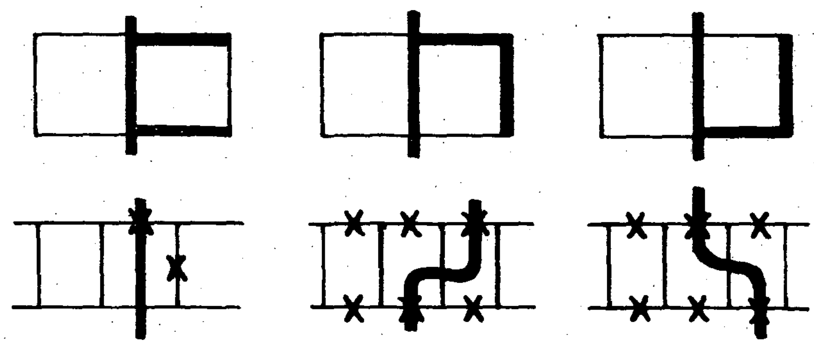

Fig. 8

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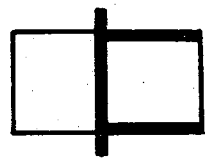
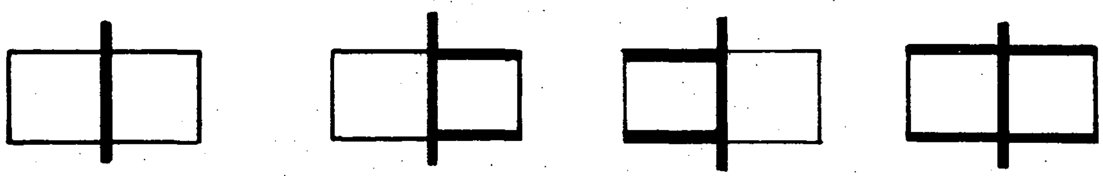


Fig. 9



C + C   C + P   P + C   P + P

Fig. 10

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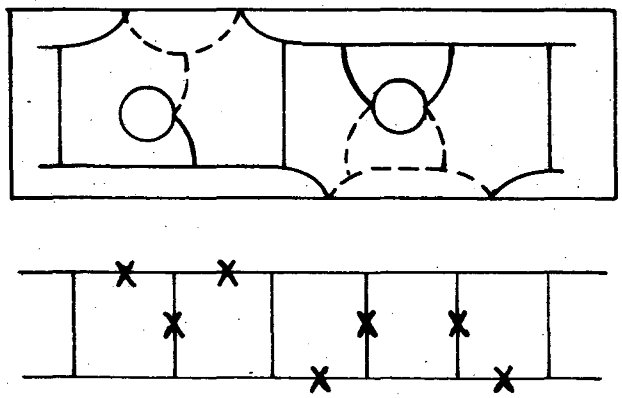

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Fig. 11

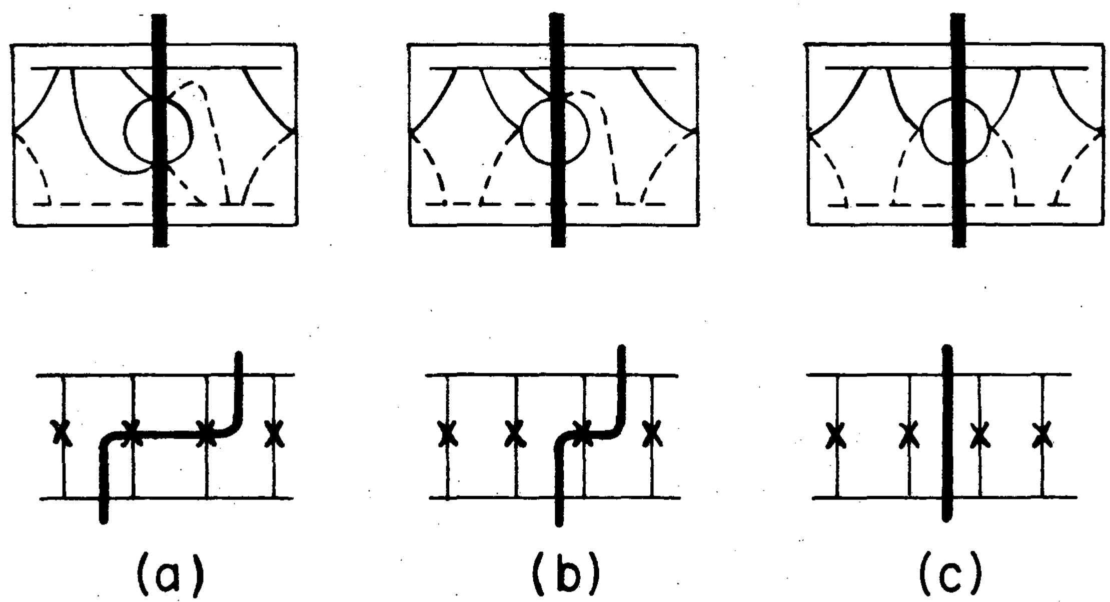

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Fig. 12

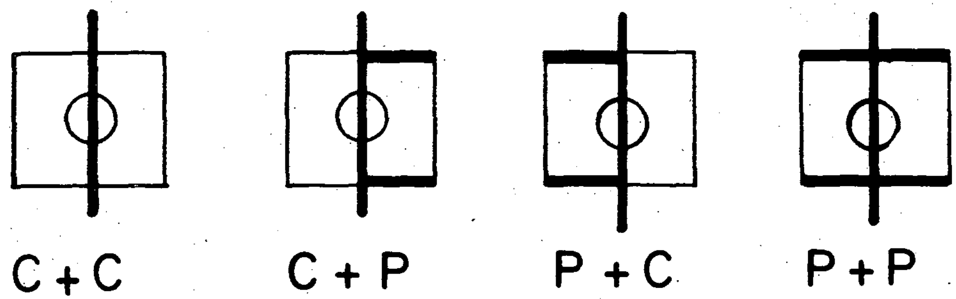

Fig. 13

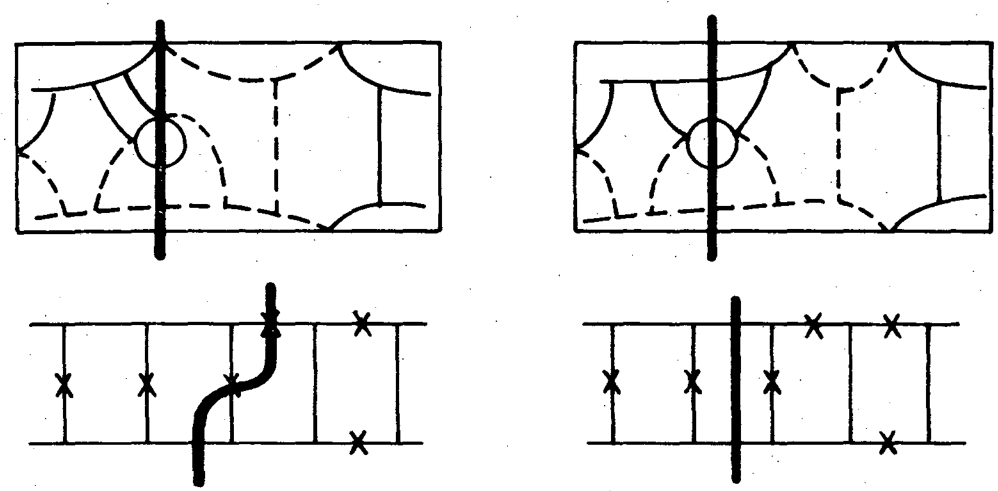

Fig. 14

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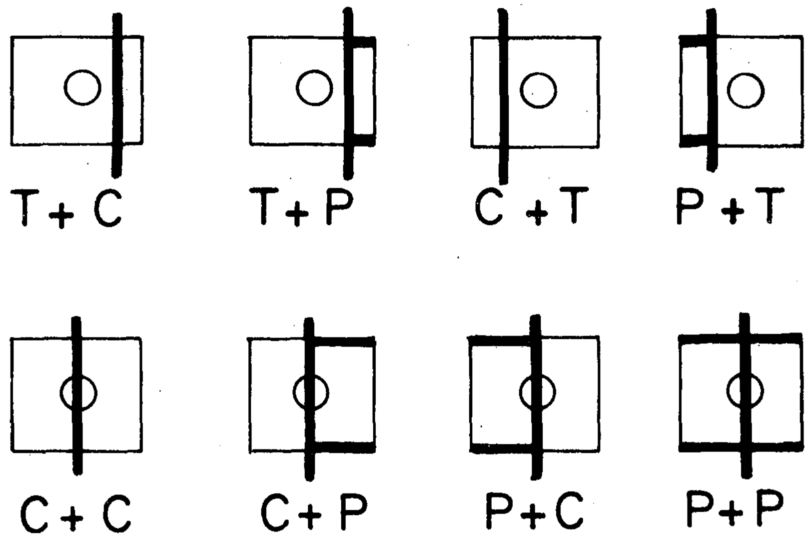

Fig. 15

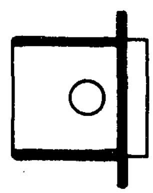

Fig. 16

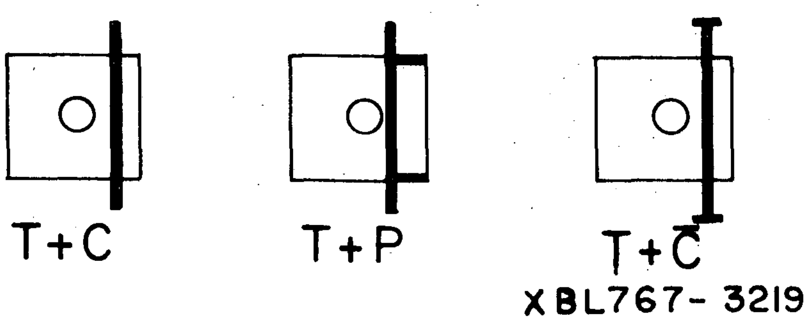

Fig. 17

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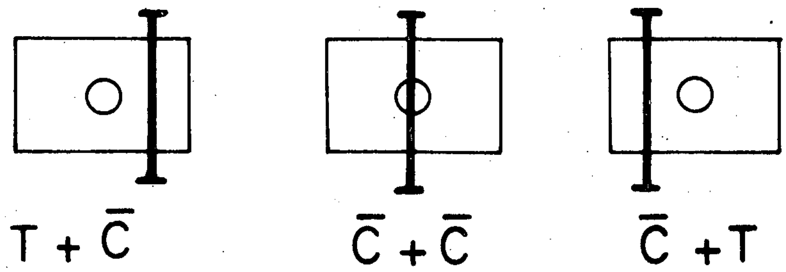

Fig. 18

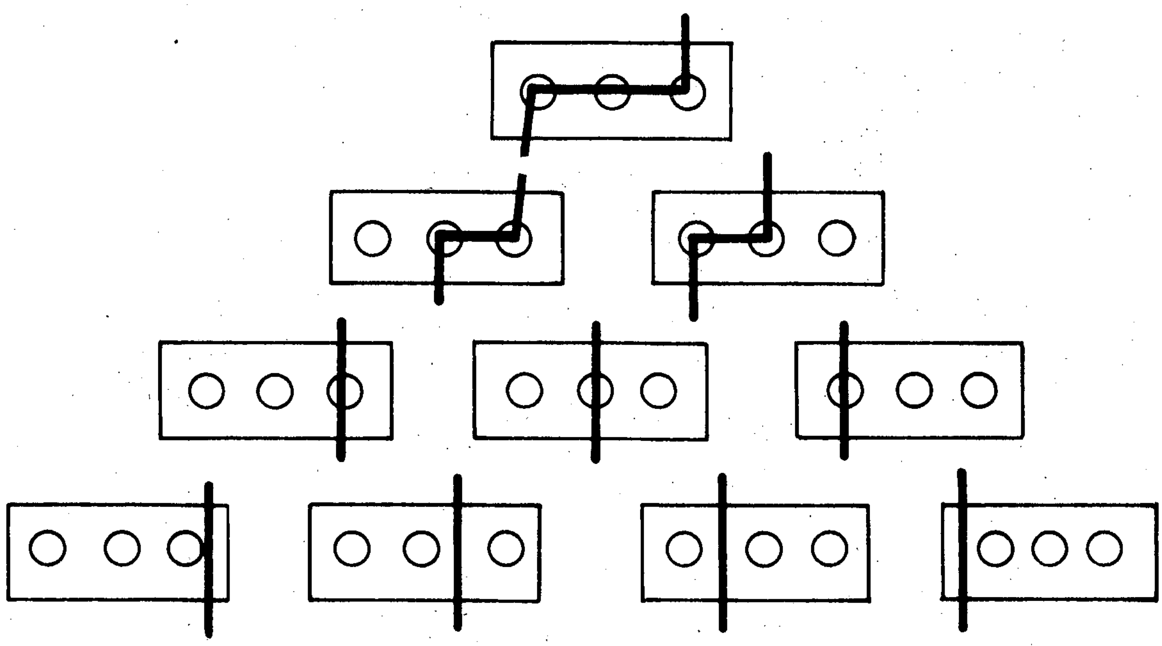

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Fig. 19

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