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Visual Spheres of Expanding Thurston Maps: Their Weak Tangents and Porous Subsets

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ABSTRACT OF THE DISSERTATION

Visual Spheres of Expanding Thurston Maps:

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We study certain approximately self-similar metric spaces that arise from expanding Thurston maps $f : \mathbb{S}^2 \to \mathbb{S}^2$ called visual spheres. It is known[HP09, Theorem 4.2.11][BM17, Theorem 18.1(ii')] that the quasisymmetry class of a visual sphere of f is related to the rationality of f. We prove that a visual sphere is indeed approximately self-similar if f does not have periodic critical points. This is done by picking a nice visual metric of an expanding Thurston map. Using the nice metric, we study the solenoid of f. We put a specific metric on the leaves of f and show that the leaves and weak tangents are almost the same thing. We then study the visual spheres of expanding Thurston maps with an emphasis on a quasisymmetric invariant, called the Ahlfors regular conformal dimension. We show that the Ahlfors regular conformal dimension of any weak tangent of a visual sphere is the same as the Ahlfors regular conformal dimension of the visual sphere itself, and that the Ahlfors regular conformal dimension of any weak tangent is attainable if and only if Ahlfors regular conformal dimension of the visual sphere is attainable. We show by an example that the same does not hold for more general metric spaces. Finally, we show that a visual sphere has p-thick curve family supported on an f-invariant porous subset if f has a irreducible p-thick f-stable multicurve. We give an example to show that when $p = 2$, the condition of a p-thick f-stable multicurve is sharp.

The dissertation of Angela A. Wu is approved.

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CHAPTER 1

Introduction

This dissertation explores the dynamics on $\mathbb{S}^2$ induced by expanding Thurston maps and the geometries that they induce.

Expanding Thurston Maps: What Are They and Why Do They Matter?

Expanding Thurston Maps are maps that look like the following rational maps:

$$z \mapsto \frac{4z(z-1)(z+1)}{(z^2+1)^2}.$$

The Julia set of this rational map is $\widehat{\mathbb{C}}$. This map is an example of an Expanding Thurston map with no periodic critical point.

A *branched covering map* $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a map that looks like a power map everywhere, i.e., for every $x \in X$, there is a open neighborhood U of x, a positive integer m, and homeomorphisms $\varphi : U \to \mathbb{D}$, $\psi : f(U) \to \mathbb{D}$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & f(U) \arrow[d, "\psi"] \\ \mathbb{D} \arrow[r, "z \mapsto z^d"'] & \mathbb{D}. \end{tikzcd}$$

The best known branched covering maps on $\mathbb{S}^2$ are rational maps on the Riemann sphere. However, there are branched covering maps that are not rational maps. One example comes from the map

$$A : (x, y) \mapsto (2x, 3y) : \mathbb{T}^2 \to \mathbb{T}^2.$$

This map descends to a map $f : \mathbb{S}^2 \to \mathbb{S}^2$ in the following way:

$$\begin{tikzcd}
\mathbb{T}^2 \arrow[r, "A"] \arrow[d, "\sim"'] & \mathbb{T}^2 \arrow[d, "\sim"] \\
\mathbb{S}^2 \arrow[r, "f_{2,3}"'] & \mathbb{S}^2.
\end{tikzcd}$$

Here the map $\sim$ identifies (x, y) and $(-x, -y)$. The map $f_{2,3}$ is a branched covering map, but it does not come from a rational map.

Thurston studied a restricted collection of branched covering maps, called Thurston Maps. A branched covering map f is a *Thurston map* if its postcritical set of f $\operatorname{post}(f) = \{f^n(c) \in \mathbb{S}^2 : n \in \mathbb{N}, c \text{ a critical point}\}$ is finite. These maps are called *Thurston maps*. Given a Thurston map $f : \mathbb{S}^2 \to \mathbb{S}^2$, Thurston asked when f is *Thurston equivalent* to a rational map, i.e., when there exists a rational map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and homeomorphisms $\varphi, \psi : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ such that φ and ψ are isotopic rel $\operatorname{post}(f)$ and the following diagram commutes:

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathbb{S}^2 \arrow[d, "\varphi"] \\
\widehat{\mathbb{C}} \arrow[r, "R"'] & \widehat{\mathbb{C}}.
\end{tikzcd}$$

It turned out that, at least among generic Thurston maps f, the rational maps are characterized by the topological behavior of f.

In [BM17], Bonk and Meyer introduced the notion of expanding Thurston maps. Similar notions are studied by Haïssinsky and Pilgrim in [HP09] and their subsequent papers [HP08] and [HP14]. An expanding Thurston maps that behaves like a rational Thurston map whose Julia set is the whole Riemann sphere $\widehat{\mathbb{C}}$. Rigorously speaking, a Thurston map is *expanding* if there exists an open cover $\{U_i\}_{i \in I}$ of the 2-sphere such that

$$\lim_{n \to \infty} \sup\{\operatorname{diam}(V) : i \in I, V \text{ connected component of } f^{-1}(U_i)\} = 0.$$

Any rational Thurston map with $\mathcal{J}(f) = \widehat{\mathbb{C}}$ is an expanding Thurston map. The map $f_{2,3}$ in the commutative diagram (1), which is not Thurston equivalent to any rational maps, is also an expanding Thurston map.

When a Thurston map f is expanding, one can put a metric on the 2-sphere, called a *visual metric* with respect to f, such that f is lipschitz and locally bilipschitz away from the critical points. The 2-sphere $\mathbb{S}^2$, equipped with a visual metric, is called a *visual sphere*. See Chapter 2, Section 2.5 for the precise definition of visual spheres. In [HP09] and [BM17], rational maps are characterized by the geometries of visual spheres:

Theorem 1.0.1. *[HP09, Theorem 4.2.11][BM17, Theorem 18.1(ii')] An expanding Thurston map f with no periodic critical points is Thurston equivalent to a rational map if and only if one of its corresponding visual spheres is quasisymmetric to the standard 2-sphere.*

Here, quasisymmetries are maps that sends balls to sets that look like balls. A homeomorphism $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is a *quasisymmetry* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all $x, y, z \in X$, with $x \neq z$, we have

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right).$$

Quasisymmetries define an equivalence relation on metric spaces. Given an expanding Thurston map f, and two visual metrics $\rho, \widetilde{\rho}$ with respect to f, the identity map $\mathrm{id} : (\mathbb{S}^2, \rho) \to (\mathbb{S}^2, \widetilde{\rho})$ is a quasisymmetry. Thus the quasisymmetric equivalence of a visual sphere with respect to f to the standard 2-sphere in Theorem 1.0.1 depends only on f and not on the choice of the visual sphere.

The study of expanding Thurston maps is related to Cannon's conjecture in geometric group theory. One of the equivalent formulations of Cannon's conjecture concerns quasisymmetric classes of metric spaces that are topologically 2-spheres:

Conjecture 1.0.2 (Cannon's conjecture)**.** *Let G be a Gromov hyperbolic group. If $\partial_\infty G$ is homeomorphic to the 2-sphere, then $\partial_\infty G$ is quasisymmetrically equivalent to the standard 2-sphere.*

Just as rational Thurston maps can be considered as nice Thurston maps, Gromov hyperbolic groups whose boundaries at infinity are quasisymmetrically equivalent to the standard 2-sphere can be considered as nice Gromov hyperbolic groups:

Theorem 1.0.3. *[KB02] A Gromov hyperbolic group G admits a hyperbolic action on $\mathbb{H}^3$ if and only if $\partial_\infty G$ is quasisymmetrically equivalent to $\mathbb{S}^2$.*

As we can see, the study of expanding Thurston maps has motivation stemming from complex dynamics and geometric group theory. In fact, the two fields are often said to be connected by Sullivan's dictionary[Sul85]. In Sullivan's dictionary, the dynamics of rational maps are compared to the actions of by Kleinian groups G on the boundary at infinity of their Cayley graphs $\partial_\infty G$. However, it is not clear what should be the complex dynamical analogue of the Cayley graphs of a Gromov hyperbolic group. One possible analogue is called a *tile graph* of an expanding Thurston map, studied by Yin[Yin11] in her dissertation. The definition of a tile graph of f is given in Chapter 6, Section 6.2. Yin showed that visual spheres are boundaries at infinity of these tile graphs:

Theorem 1.0.4. *[Yin11, Proposition IV.7] Let f be an expanding Thurston map. There exists a Gromov hyperbolic space Γ, with a preferred based point, called the tile graph of f, such that ρ is a visual metric on $\mathbb{S}^2$ with respect to f if and only if ρ is a visual metric on $\partial_\infty\Gamma$ under the preferred based point.*

See also [HP09, Chapter 3], where Haïssinsky and Pilgirm made a similar construction and proved the same result.

In [HP09], Haïssinsky and Pilgrim proposed several additions to Sullivan's dictionary. See also [Yin16] for more possible entries in Sullivan's dictionary. In particular, Theorem 1.0.1 and Theorem 1.0.3 could form one entry in Sullivan's dictionary.

What Have We Done?

In this work study visual spheres expanding Thurston maps with an emphasis on quasisymmetric invariant called the Ahlfors regular conformal dimension.

Let $Q > 0$ be a constant. A metric space (X, d) is said to be Q-Ahlfors regular if there exists $C > 1$ such that for any $x \in X$ and $r \leq \operatorname{diam}(X)$,

$$C^{-1} r^Q \leq \mathcal{H}^Q(B(x, r)) \leq C r^Q.$$

When our metric spaces are Ahlfors regular, as they are for visual spheres of expanding

Thurston maps, we can define the notion of Ahlfors regular conformal dimension:

$$\dim_{\mathrm{AR}}([X]) = \inf\{\dim_H(Y) : \text{Y Ahlfors regular, there exists a quasisymmetry } f : X \to Y\}.$$

Visual spheres of expanding Thurston maps without periodic critical points are Ahlfors regular, but little is known of their Ahlfors regular conformal dimensions, except for those whose Ahlfors regular conformal dimensions are guaranteed to be 2 by Theorem 1.0.1 and visual spheres of Lattès type maps[Yin11, Corollary III.17]. In [HP14], Haïssinsky and Pilgrim proved that the Ahlfors regular conformal dimension of a visual sphere of a generic expanding Thurston map without periodic critical points is not attainable.

In Chapter 2, we make some important definitions that will be used throughout the dissertation.

Chapter 3 serves two purposes. The first is to clarify certain folklore beliefs about visual metrics of expanding Thurston maps without periodic critical points that, unfortunately, have not been well-documented in the literature.

Theorem 1.0.5. *Let f be an expanding Thurston map without periodic critical points. Then for any visual metric ρ of f, the visual sphere $(\mathbb{S}^2, \rho)$ is approximately self-similar.*

Here we take the definition of approximate self-similarity from [Kle06, Section 3]. Approximate self-similarity of these visual spheres has been helpful the study of their weak tangents. See, for instance, [HP14].

We proved the proposition by considering a particularly nice visual metric. Since we are interested in quasisymmetric invariants, and any pair of visual metrics of an expanding Thurston map f are quasisymmetrically equivalent, we will base the rest of the dissertation of this particular visual metric.

The second goal of Chapter 3 is to study the solenoid $\mathcal{S}(f)$ of an expanding Thurston map f, especially the path-components of $\mathcal{S}(f)$, also known as *leaves* of $\mathcal{S}(f)$. The *solenoid* $\mathcal{S}(f)$ of f is the inverse limit of the system

$$\cdots \xrightarrow{f} \mathbb{S}^2 \xrightarrow{f} \mathbb{S}^2 \xrightarrow{f} \mathbb{S}^2.$$

Thus we have natural projection maps $\pi_n : \mathcal{S}(f) \to \mathbb{S}^2$. Having fixed a visual metric ρ of f on $\mathbb{S}^2$, we introduce a metric d_L on each leaf L of $\mathcal{S}(f)$. Leaves are related to the weak tangents of the visual sphere. Roughly speaking, a weak tangent of a metric space (X, d) is a limit of $(X, x_n, \lambda_n d)$, where x_n is a sequence of points in X and λ_n is a sequence of positive real numbers tending to $+\infty$. We will define the definition of a weak tangent of a metric space in Chapter 2.

The following theorems, roughly speaking, say that every leaf is a weak tangent and every weak tangent is a branched covering of a leaf. These results suggest that solenoids are unions of weak tangents of visual spheres.

Theorem 1.0.6. *Let $x = \{x_n\}_{n\in\mathbb{N}_0}$ be a point in $\mathcal{S}(f)$, and let L be the leaf in $\mathcal{S}(f)$ containing x. Then the sequence $\{(\mathbb{S}^2, x_n, \Lambda^n\rho)\}_{n\in\mathbb{N}_0}$ converges in pointed-Gromov-Hausdorff sense to (L, x, d_L).*

Theorem 1.0.7. *Let (T, a, d) be the weak tangent of $(\mathbb{S}^2, \rho)$ with associated data (a_n, r_n), and suppose $\{x_n\}_{n\in\mathbb{N}_0} \in \mathcal{S}(f)$ represents (a_n, r_n). Let L be the leaf of $\mathcal{S}(f)$ containing $\{x_n\}_{n\in\mathbb{N}_0}$. Then the following statements hold.*

(i) There exists a branched covering $\pi : T \to L$.

(ii) If $\sup_{n\in\mathbb{N}_0} \Lambda^n\rho(x_n, \mathrm{post}(f)) = \infty$, then the map π is an isometry.

(iii) If $p = \{p_n\}_{n\in\mathbb{N}_0} \in L$ is a periodic sequence of postcritical points, then p is the only possible branched locus of π, and $\pi^{-1}(p)$ has exactly one point.

(iv) Let $b = \pi^{-1}(p)$. Then there exists $n_0 \in \mathbb{N}$ such that $\deg(\pi, b) = \deg(f^{n_0})$.

(v) There exists $q_0 \in f^{-n_0}(p_0)$ such that for all $R > 0$ there exists $k \in \mathbb{N}_0$ and an isometry $\imath : B_T(q, R) \to B_{\rho\Lambda^{n_0+k}}(q_0, R)$ such that the following diagram commutes:

$$\begin{tikzcd} B_{(T,d_T)}(b, R) \arrow[r, "\pi"] \arrow[d, "\imath"] & B_{(L,d_L)}(p, R) \arrow[d, "\pi_k"] \\ B_{(\mathbb{S}^2,\Lambda^{k+n_0}\rho)}(q_0, R) \arrow[r, "f^{n_0}"'] & B_{(\mathbb{S}^2,\Lambda^k\rho)}(p_0, R). \end{tikzcd}$$

In Chapter 4, we compare the Ahlfors regular conformal dimension of visual spheres with that of their weak tangents.

Theorem 1.0.8. *Let f be an expanding Thurston map f without periodic critical points. Let $(\mathbb{S}^2, \rho)$ be a visual sphere of f. Let (T, x) be a weak tangent of $(\mathbb{S}^2, \rho)$. Then*

$$\dim_{\mathrm{AR}}(\mathbb{S}^2, \rho) = \dim_{\mathrm{AR}}(T),$$

and $\dim_{\mathrm{AR}}(\mathbb{S}^2, \rho)$ is attainable if and only if $\dim_{\mathrm{AR}}(T)$ is attainable.

When the Ahlfors regular conformal dimension in the above theorem is 2 and attainable, we obtain the following corollary:

Theorem 1.0.9. *Let $(\mathbb{S}^2, \rho)$ be a visual sphere of an expanding Thurston map f without periodic critical points. The following are equivalent:*

(i) *$(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to the standard 2-sphere.*

(ii) *Every weak tangent of $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to the Euclidean plane $\mathbb{R}^2$.*

(iii) *There exists a weak tangent of $(\mathbb{S}^2, \rho)$ that is quasisymmetrically equivalent to the Euclidean plane $\mathbb{R}^2$.*

See also [Li18].

Theorem 1.0.8 and Theorem 1.0.9 roughly say that one can understand the geometry of visual spheres by looking at their weak tangents. The same assertion does not hold for arbitrary metric spaces. Chapter 5 offers the following complementary result.

Theorem 1.0.10. *For every $n \geq 2$, there exists a doubling, linearly locally contractible metric space X, topologically an n-sphere, such that every weak tangent of X is isometric to $(\mathbb{R}^n, 0)$ but X is not quasisymmetrically equivalent to the standard n-sphere.*

In Chapter 6 we return to our investigation on expanding Thurston maps and their visual spheres. Let f be an expanding Thurston map without periodic critical points. Chapter 6 concerns f-invariant porous subsets of f.

Let (X, d) be a metric space, and K a compact subset of X. We say that K is *porous* in X if there exists $a \in (0, 1)$ such that for all $x \in K$, and for every $r > 0$, there exists $y \in B_X(x, r)$ such that $B(y, ar) \cap K = \emptyset$. A porous subset $K \subset \mathbb{S}^2$ is said to be f-invariant if $f(K) = K$.

Definition 1.0.11. *A family Γ of curves in $(\mathbb{S}^2, \rho)$ is said to be p-thick if*

$$\inf_{\gamma \in \Gamma} \operatorname{diam}(\gamma) > 0$$

and

$$\limsup_{n \to \infty} \operatorname{mod}_p(\Gamma, \varepsilon) > 0.$$

For any $p \geq 2$, we ask when there can be a a p-thick curve family. The existence and nonexistence of p-thick curve family is related to Ahlfors regular conformal dimension:

Ahlfors regular conformal dimension of visual spheres of expanding Thurston maps with no periodic critical points are known to be related to Thurston matrices. Thurston matrices encode how expanding Thurston maps interact with simple closed curves γ in $\mathbb{S}^2$. A simple closed curve γ in $\mathbb{S}^2 \setminus \operatorname{post}(f)$ is *peripheral* if one of the components of $\mathbb{S}^2 \setminus \gamma$ contains at most one point in P, and *non-peripheral* otherwise. A *multicurve* is a finite collection of simple, closed disjoint non-homotopic, non-peripheral curves in $\mathbb{S}^2 \setminus \operatorname{post}(f)$. A multicurve Γ is said to be f-*stable* if, for all $\gamma \in \Gamma$, each non-peripheral simple closed curve in $f^{-1}(\gamma)$ is homotopic to a curve in Γ in $\mathbb{S}^2 \setminus \operatorname{post}(f)$.

Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be an f-stable multicurve Γ. For each $i, j \in \{1, \dots, n\}$, let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbb{S}^2 \setminus \operatorname{post}(f)$, and let $d_{i,j,\alpha} > 0$ be the degree of the restriction map $f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j$. Let $p > 0$ be arbitrary. Define

$$f_{\Gamma,p} : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$$

by

$$f_{\Gamma,p}(\gamma_j) = \sum_{\gamma_i \in \Gamma} \sum_{\alpha} d_{i,j,\alpha}^{1-p} \gamma_i.$$

Since $f_{\Gamma,p}$ is represented by a non-negative square matrix, Perron-Frobenius Theorem implies that the matrix $f_{\Gamma,p}$ has a real non-negative eigenvalue $\lambda(f_{\Gamma,p})$ equal to its spectral radius.

A multicurve Γ is called *irreducible* if there exists an iterate $f_{\Gamma,p}^n$ of $f_{\Gamma,p}$ such that every entry of $f_{\Gamma,p}^n$ is positive. If Γ contains an irreducible multicurve, then the function

$$p \mapsto \Lambda(\Gamma, p)$$

is strictly decreasing on $[1, \infty)$ ([HP08, Lemma A2]), and there exists unique $Q(\Gamma) \geq 1$ such that $\Lambda(\Gamma, Q(\Gamma)) = 1$. If Γ contains an irreducible multicurve, we define $Q(\Gamma) = 0$. Define

$$Q(f) = \sup\{Q(\Gamma) : \Gamma \text{ multicurve}\} \vee 2.$$

Definition 1.0.12. *A family Γ of curves in $(\mathbb{S}^2, \rho)$ is said to be p-thick if*

$$\inf_{\gamma \in \Gamma} \operatorname{diam}(\gamma) > 0$$

and

$$\limsup_{n \to \infty} \operatorname{mod}_p(\Gamma, \Lambda^{-n}) > 0.$$

Definition 1.0.13. *An irreducible f-stable multicurve Γ is said to be p-thick if either one of the following condition holds:*

1. $\lambda(\Gamma, p) > 1$*; or*
2. $\lambda(\Gamma, p) = 1$ *and there exists $\gamma \in \Gamma$ such that $f^{-1}(\gamma)$ contains a peripheral component.*

We prove the following theorem:

Theorem 1.0.14. *Let f be an expanding Thurston map with no periodic critical point, and let $p \geq 2$. If there exists a p-thick f-stable multicurve, then there exists a curve family Γ supported on a f-invariant porous subset such that Γ is p-thick.*

The above results are anticipated by Haïssinsky and Pilgrim, who proved the following inequality.

Theorem 1.0.15 (Haïssinski, Pilgrim)**.** *If f is an expanding Thurston map with no periodic critical point, then*

$$p_f \leq \dim_{\mathrm{AR}}(f). \tag{1.0.1}$$

Ultimately, all our work is related to the following conjecture:

Conjecture 1.0.16.

$$p_f = \dim_{\mathrm{AR}}(f).$$

CHAPTER 2

Preliminaries

2.1 A note on inequalities

Let $A(x)$ and $B(x)$ be two positive valued functions in x. We write

$$A(x) \lesssim B(x)$$

if there exists a constant C, independent of x, such that for all x,

$$A(x) \leq CB(x).$$

Similarly, we write

$$A(x) \gtrsim B(x)$$

if there exists a constant C, independent of x, such that for all x,

$$A(x) \geq CB(x).$$

Lastly, we write

$$A(x) \sim B(x)$$

if $A(x) \lesssim B(x)$ and $A(x) \gtrsim B(x)$.

2.2 Notations on metric spaces

Let (X, d) be a metric space. For any $x \in X$ and $r > 0$, we write

$$B_{(X,d)}(x, r) = \{y \in X : d(x, y) < r\}.$$

We also shorthand $B_{(X,d)}(x,r)$ as $B(x,r)$ when the underlying metric space is understood, and write $B_{(X,d)}(x,r)$ as $B_X(x,r)$ when there is a need to emphasize the underlying metric space. Occasionally we need to deal with multiple metrics on the same set X. In this case we write $B_{X,d}(x,r)$ as $B_d(x,r)$ when the underlying set X is understood.

Let $A, B \subset X$ be two subsets of a metric space (X,d). We define the *distance between A and B* by

$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}.$$

If $A = \{a\}$ is a singleton, we also write $d(a,B)$ for $d(A,B)$.

Let $A \subset X$ be a subset of X. We define the *diameter* of A by

$$\mathrm{diam}_d(A) = \sup\{d(a,b) : a, b \in A\}.$$

When there is no ambigurity, we write $\mathrm{diam}_d(A)$ as $\mathrm{diam}(A)$.

2.3 Quasisymmetry

In this section we introduce the notion of quasisymmetry. We mention several related notions in metric geometry that will appear frequently in the later chapters.

Definition 2.3.1. *Let (X,d_X) and (Y,d_Y) be two metric spaces. A homeomorphism $f : X \to Y$ is* bilipschitz *if there exists a positive constant $L > 1$ such that for all $x, y \in X$, we have*

$$L^{-1} d_X(x,y) \le d_Y(f(x), f(y)) \le L d_X(x,y).$$

If there exists a bilipschitz homeomorphisms between two metric spaces (X,d_X) and (Y,d_Y), then we say that (X,d_X) and (Y,d_Y) are bilipschitz equivalent.

Definition 2.3.2. *Let (X,d_X) and (Y,d_Y) be two metric spaces. Let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism. A homeomorphism $f : X \to Y$ is an η-*quasisymmetry *if for all $x, y, z \in X$, with $x \ne z$, we have*

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \le \eta\left(\frac{d_X(x,y)}{d_X(x,z)}\right).$$

A homeomorphism is a quasisymmetry *if it is an η-quasisymmetric for some η. The function η is called the* distortion function *of f.*

Note that if $f : X \to Y$ is a quasisymmetry, then $f^{-1} : Y \to X$ is a quasisymmetry.

Definition 2.3.3. *We say that X and Y are* quasisymmetrically equivalent *if there exists a quasisymmetry f between X and Y.*

Definition 2.3.4. *Let d_1, d_2 be two metrics on X. We say that d_1 and d_2 are quasisymmetrically equivalent if the identity map* $\mathrm{id} : (X, d_1) \to (X, d_2)$ *is a quasisymmetry.*

Below we list several properties of metric spaces that play crucial roles in our discussion.

Definition 2.3.5. *Let $N > 0$. A metric space is N-*doubling *if for all $R > 0$, every open ball of radius $2R$ can be covered by N balls of radius R. A metric space is* doubling *if it is N-doubling for some $N > 0$.*

Definition 2.3.6. *A metric space X is* uniformly perfect *if there exists constant $C \geq 1$ such that for each $x \in X$ and for each $r > 0$, the set $B(x, r)\backslash B(x, r/C)$ is nonempty whenever the set $X\backslash B(x, r)$ is nonempty.*

Definition 2.3.7. *Let $C \geq 1$. A metric space is C-bounded turning if every pair of points x, y in the space can be joined by a curve whose diameter dow not exceed $C\,|x - y|$. A metric space is bounded turning if it is C-bounded turning for some $C \geq 1$.*

Recall that a continuum is a non-empty compact connected set.

Definition 2.3.8. *Let $C \geq 1$ be a constant. A metric space X is said to be C-*linearly locally connected *if, for every open ball $B(x, R)$, every pair of points in $B(x, R)$ can be joined in $B(x, CR)$ by a continuum and every pair of points in $X\backslash B(x, R)$ can be joined in $X\backslash B(x, R/C)$ by a continuum.*

Doubling, uniform perfectness, bounded-turningness, and linear local connectivity are all preserved under quasisymmetries([Hei01]).

Finally, we introduce the notion of Ahlfors regularity and Ahlfors regular conformal dimension.

Let Q be a positive real number. A metric space (X,d) is *Q-Ahlfors regular*, or *Q-AR* if there exists $C>1$ such that for all $x\in X$ and $0<r<\operatorname{diam}_d(X)$, we have

$$C^{-1}r^Q \leq \mathcal{H}^Q(B(x,r)) \leq Cr^Q,$$

where $\mathcal{H}^Q$ is the Hausdorff Q-measure of (X,r).

We say that (X,d) is *Ahlfors regular* if it is Q-AR for some $Q>0$. In this case, we have $Q=\dim_H(X)$, where $\dim_H(X)$ denotes the Hausdorff dimension of X.

Proposition 2.3.9. *[Hei01, Corollary 14.15] A complete metric space is quasisymmetrically equivalent to an Ahlfors regular space if and only if it is uniformly perfect and doubling.*

Let (X,d) be an Ahlfors regular metric space. Define the *Ahlfors regular conformal dimension*

$$\dim_{\mathrm{AR}}(X) = \inf\{\dim_H(\varphi(X)) : \varphi \text{ quasisymmetry}, \varphi(X) \text{ Ahlfors regular}\}.$$

We say that the Ahlfors regular conformal dimension of X is *attainable* if the infimum in the above definition can be attained. We say that the Ahlfors regular conformal dimension of X is *attained* if $\dim_{\mathrm{AR}}(X)=\dim_H(X)$.

2.4 Gromov-Hausdorff limits and Weak Tangents

In this section, we review some basic concepts about Gromov-Hausdorff convergence of pointed metric spaces. The notion of Gromov-Hausdorff limit allows us to make sense of convergence of metric spaces that are not necessarily bounded. For example, we can "blow up" a metric space at a point to get what we call the weak tangent of the space. Most of the material in this section is standard. See Chapter 7 of [BBI01] or Chapter 11 of [HKST15] for reference.

Let (X,d) be a metric space and A be a subset of X. For any $\varepsilon>0$, we write

$$N_\varepsilon(A) = \{x\in X : \operatorname{diam}(x,A) < \varepsilon\}.$$

Let A and B be two nonempty subsets of X. The *Hausdorff distance* between A and B is

$$d_H^X(A, B) := \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}.$$

The function d_H^X defines a metric on the collection of all nonempty compact subsets of X.

Definition 2.4.1. *A* pointed metric space *is a triple (X, a, d), where X is a set, a is a point in X, and d is a metric on X.*

Let X, Y bet two sets, a be a point in X, and b be a point in Y. A function $f : (X, a) \to (Y, b)$ is a function $f : X \to Y$ such that $f(a) = b$.

Definition 2.4.2. *Let X be a metric space and let $\varepsilon > 0$. We say that $Y \subset X$ is* ε-dense *in X if for all $x \in X$, $d(x, Y) < \varepsilon$.*

Definition 2.4.3. *Let (X, a, d_X) and (Y, b, d_Y) be pointed metric spaces and let $\varepsilon > 0$. We say that $f\colon (X, a) \to (Y, b)$ is an* ε-rough embedding *of (X, a, d_X) and (Y, b, d_Y) if for all $x, y \in X$,*

$$d_X(x, y) - \varepsilon \le d_Y(f(x), f(y)) \le d_X(x, y) + \varepsilon.$$

The map f is an ε-rough isometry *if, in addition, $f(X)$ is ε-dense in X.*

Note that ε-rough embeddings and ε-rough isometries need not be continuous or injective.

We are now ready to define a notion of distance between two pointed metric spaces that generalizes Hausdorff distance.

Definition 2.4.4. *Let (X, a) and (Y, b) be two bounded pointed metric spaces. The* pointed-Gromov-Hausdorff distance *between (X, a) and (Y, b) is defined as*

$$d_{GH}((X, a), (Y, b)) = \inf\{\varepsilon > 0 : \exists \varepsilon\text{-rough isometry } f : (X, a) \to (Y, b)\}.$$

We are ready to define the notion of pointed-Gromov-Hausdorff convergence of pointed metric spaces.

Definition 2.4.5. *Let (X_n, a_n, d_n), $n = 1, 2, \ldots$, and (X, a, d) be a complete pointed metric spaces. We say that the sequence of pointed metric spaces (X_n, a_n, d_n)* converges *to (X, a, d)*

in the pointed-Gromov-Hausdorff sense, *and write* $(X_n, a_n, d_n) \xrightarrow{GH} (X, a, d)$, *if for all* $r > 0$ *and* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$, *there exists a map* $f \colon B_{X_n}(a_n, r) \to X$ *such that*

1. $f(a_n) = a$,

2. *f is a ε-rough embedding of $B_{X_n}(a_n, r)$ and $B_X(a, r)$.*

3. *$f(B_{X_n}(a_n, r))$ is ε-dense of $B_X(a, r)$.*

The following propositions are related to existence and uniqueness of pointed-Gromov-Hausdorff limits.

Proposition 2.4.6. *[Don11, Section 3.2] Let N be a natural number. Let $\mathcal{M}$ be the set of all isometry classes of N-doubling pointed metric spaces. Then the pointed-Gromov-Hausdorff convergence on $\mathcal{M}$ induces a metrizable topology on $\mathcal{M}$.*

Proposition 2.4.7 (Gromov's compactness theorem)**.** *[HKST15, Theorem 11.3.16] Let N be a positive integer, and let (X_n, a_n, d_n) be a sequence of pointed metric spaces such that (X_n, d_n) is N-doubling. Then there exists a subsequence of (X_n, a_n, d_n) that converges in pointed-Gromov-Hausdorff sense to a proper pointed metric space.*

Some properties of metric spaces are retained by the Gromov-Hausdorff limits.

Proposition 2.4.8. *Let $\{(X_n, a_n, d_n)\}_{n\in\mathbb{N}}$ be a sequence of pointed metric spaces and suppose $(X_n, a_n, d_n) \xrightarrow{GH} (X, a, d)$.*

1. *[HKST15, Proposition 11.3.14] If each X_n is proper, then X is proper.*

2. *[HKST15, Proposition 11.3.17] If there is an integer N such that each X_n is N-doubling, then X is N^2-doubling.*

We now define the notion of weak tangent.

Definition 2.4.9. *Let (X, d) be a metric space. We say that (T, b, d_T) is a* weak tangent *of (X, d) if there exists a sequence $\{a_n\}$ of points in X and a sequence $\{\lambda_n\}$ of positive numbers such that $\lambda_n \to \infty$ and that $(X, a_n, \lambda_n d)$ converges to (T, b, d_T) in Gromov-Hausdorff sense.*

In particular, we have $(X, x_n, \lambda_n d_X) \to (T, p, d_T)$ if for all $R > 0$, and for all $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$,

$$d_{GH}\left(\overline{B}_{\lambda_n d_X}(x_n, R), \overline{B}_{d_T}(p, R)\right) \leq \varepsilon.$$

The following proposition is a consequence of Proposition 2.4.7

Proposition 2.4.10. *If (X, d) is a doubling metric space, then for any $a_n \in X$ and any sequence $\{\lambda_n\}$ of positive numbers such that $\lambda_n \to \infty$, the spaces $(X, a_n, \lambda_n d)$ contains a subsequence that converges to some pointed metric space (T, b, d_T).*

We end this section by stating the theorem that quasisymmetries pass over to weak tangents.

Proposition 2.4.11. *[KL04, Lemma 2.4.7] Let $\{(X_n, p_n, d_n)\}$ and $\{Y_n, q_n, l_n)\}$ be sequences of proper pointed metric spaces that converge to (X, p, d) and (Y, q, l) respectively. Let $f_n : X_n \to Y_n$ be η-quasisymmetric homeomorphism for each $n \in \mathbb{N}$, where η is fixed. Further assume that $f_n(p_n) = q_n$, and that there exists $C > 0$, and that there exists a sequence $\{x_n\}$ of points in X_n such that for every $n \in \mathbb{N}$,*

$$C^{-1} \leq d_n(p_n, x_n) \leq C \quad \text{and} \quad C^{-1} \leq l_n(q_n, f(x_n)) \leq C.$$

Then, after passing to a subsequence, the functions $\{f_n\}$ converges to some η-quasisymmetric homeomorphism between X and Y.

As a particular instance of Proposition 2.4.11, we have the following statement about weak tangents.

Proposition 2.4.12. *Let X, Y be doubling metric spaces, let $f : X \to Y$ be a quasisymmetry, and let (T, x) be a weak tangent of X. Then there exists a weak tangent (S, y) of Y such that T and Y are quasisymmetrically equivalent.*

2.5 Expanding Thurston maps and Visual Metrics

Let X and Y be topological spaces. A map $f : X \to Y$ is a branched covering map if f is continuous, open (i.e. image of open sets are open), and discrete (i.e. preimage of a point is a discrete set). If X and Y are topological 2-manifolds, then by Stoilow's Theorem [Sto28, p. 372][LP19], for every $x \in X$, there is a open neighborhood U of x, a positive integer m, and homeomorphisms $\varphi : U \to \mathbb{D}$, $\psi : f(U) \to \mathbb{D}$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"] & f(U) \arrow[d, "\psi"] \\ \mathbb{D} \arrow[r, "z \mapsto z^d"] & \mathbb{D}. \end{tikzcd}$$

The number d is the degree of f at x. The point x is a *critical point* of f if $m \neq 1$. Let $\mathrm{crit}(f)$ be the set of critical points of f and let $\mathrm{post}(f) = \{f^n(c) : c \in \mathrm{crit}(f), n \geq 1\}$ be the set of postcritical points of f.

A *Thurston map* is a postcritically finite branched covering map $f : S^2 \to S^2$. For any $n \geq 1$, the point $f^n(x)$ is said to be a postcritical point of f. Here, and in what follows, f^n means the n-th iterate of f.

A Thurston map is an *expanding Thurston map* if there exists an open cover $\{U_\alpha\}$ of $\mathbb{S}^2$ such that

$$\lim_{n\to\infty} \sup\{\mathrm{diam}_\rho V : V \text{ connected component of } f^{-n}(U_\alpha)\} = 0.$$

Here ρ is any metric on $\mathbb{S}^2$ that generates the topology.

We have the following fact:

Proposition 2.5.1. *[BM17, Theorem 15.1] Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be an expanding Thurston map with postcritical set* $\mathrm{post}(f)$*. Then there exist a natural number $N \in \mathbb{N}$ and a Jordan curve $\mathcal{C} \subset \mathbb{S}^2$ such that* $\mathrm{post}(f) \subset \mathcal{C}$ *and $f^N(\mathcal{C}) \subset \mathcal{C}$.*

We say that $\mathcal{C}$ is f-invariant if $f(\mathcal{C}) \subset \mathcal{C}$. Replacing f by a higher iterate f^N, we may assume that there exists f-invariant Jordan curve $\mathcal{C}$ containing the postcritical set.

Suppose $\mathcal{C} \subset \mathbb{S}^2$ is an f-invariant Jordan curve such that $\mathrm{post}(f) \subset \mathcal{C}$. The Jordan curve $\mathcal{C}$ divides $\mathbb{S}^2$ into two connected components. The closure of each of the two complement components of $\mathcal{C}$ is called a 0-tile. In general, the preimages of $\mathcal{C}$ under f^n divide $\mathbb{S}^2$ into several connected components. The closure of each such connected component is called an n-tile. Each n-tile is contained in a unique $(n-1)$-tile. We let $\mathcal{D}_n(\mathbb{S}^2)$ denote the collection of all n-tiles, and let $\mathcal{D}(\mathbb{S}^2) = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n(\mathbb{S}^2)$ be the collection of all tiles. If $\tau \in \mathcal{D}_n(\mathbb{S}^2)$, we call n the *level* of τ.

Let W, B be the two 0-tiles. The tile W will be called the *white* 0-*tile* and B will be called the *black* 0-*tile*. For all $n \in \mathbb{N}$, a tile $\tau \in \mathcal{D}_n(\mathbb{S}^2)$ is a *white* n-*tile* if $f^n(\tau) = W$ and a *black* n-*tile* if $f^n(\tau) = B$. A tile is *white* if it is a white n-tile for some $n \in \mathbb{N}_0$, and it is *black* if it is a black n-tile for some $n \in \mathbb{N}_0$.

A 0-*edge* is a subarc of $\mathcal{C}$ with endpoints in $\mathrm{post}(f)$ and contains no other postcritical point. For all $n \in \mathbb{N}_0$, an n-*edge* is a subset e of a n-tile such that $f^n(e)$ is a 0-edge.

Given an expanding Thurston map f, we define the notion of *visual metric*. Following [BM17, Chapter 8], for any pair of distinct points $x, y \in S^2$, $x \neq y$, we define

$$m(x,y) = m_{f,\mathcal{C}}(x,y) = \max\{n \in \mathbb{N}_0 : \text{ there exist non-disjoint } n\text{-tiles } X, Y \\ \text{for } (f,\mathcal{C}) \text{ with } x \in X, y \in Y\}$$

and

$$m'(x,y) = m'_{f,\mathcal{C}}(x,y) = \min\{n \in \mathbb{N}_0 : \text{ there exist -disjoint } n\text{-tiles } X, Y \\ \text{for } (f,\mathcal{C}) \text{ with } x \in X, y \in Y\}$$

We also define, for all $x \in \mathcal{S}^2$,

$$m(x,x) = m'(x,x) = \infty.$$

There exists constant $k \in \mathbb{N}$ such that $|m(x,y) - m'(x,y)| \leq k$ for all $x, y \in \mathbb{S}^2$ with $x \neq y$ [BM17, Lemma 8.7(v)].

A metric ρ on $\mathbb{S}^2$ is a *visual metric with respect to* f, or a *visual metric of* f, if there exists $\Lambda > 1$ and $A > 1$ such that

$$A^{-1}\Lambda^{-m(x,y)} \leq \rho(x,y) \leq A\Lambda^{-m(x,y)}.$$

The number Λ is then called the *expansion factor* of the visual metric ρ, and the metric space $(\mathbb{S}^2, \rho)$ is called a *visual sphere* of f.

One can show ([BM17, Proposition 8.3]) that a visual metric exists. The collection of tiles $\mathcal{D}$ and the value of the function m depend on the choice of $\mathcal{C}$; but whether a metric is a visual metric or not is independent of $\mathcal{C}$([BM17, Proposition 8.3]).

In general, the notion of visual metric can be defined without a metric. See [BM17, Chapter 7] for more detail. We have the following proposition:

Proposition 2.5.2. *[BM17, Proposition 8.3] Let f be an expanding Thurston map, and let n be a positive integer. A metric ρ on $\mathbb{S}^2$ is a visual metric for f with expansion factor Λ if and only if ρ is a visual metric for f^n with expansion factor Λ^n.*

We list two facts about visual spheres that we will need:

Proposition 2.5.3. *[BM17, Proposition 18.5(iii)] Any visual sphere is linearly locally connected.*

Proposition 2.5.4. *[BM17, Proposition 18.1(i) and Lemma 18.6] Let f be an expanding Thurston map and ρ a visual metric of f. Then the following are equivalent.*

1. *$(\mathbb{S}^2, \rho)$ is doubling.*

2. *f does not have any periodic critical points.*

3. *There exists $N \in \mathbb{N}$ such that for all $p \in \mathbb{S}^2$ and $n \in \mathbb{N}$,*

$$\deg(f^n, p) \leq N.$$

Finally, we state one fact about weak tangents of visual spheres of expanding Thurston maps without periodic critical points, which we will need later on:

Proposition 2.5.5. *Let f be an expanding Thurston map without periodic critical points. Then every weak tangent of a visual sphere of f is doubling and linearly locally connected.*

Proof. Let ρ be a visual metric with respect to f. By Proposition 2.5.4, $(\mathbb{S}^2, \rho)$ is doubling. By Proposition 2.4.8, for every weak tangent (T, x) of $(\mathbb{S}^2, \rho)$, T is doubling.

To see that T is linearly locally connected, we need a stronger condition on $(\mathbb{S}^2, \rho)$ called annular linear local connectedness, defined in [BM17, Section 18.1]. It can be shown[Mac10] that weak tangents of annular linear local connectedness metric spaces are annularly linearly locally connected. Since every visual sphere is annularly linearly locally connected, every weak tangent of the visual sphere $(\mathbb{S}^2, \rho)$ is annularly linearly locally connected. This implies[Mac10] that every weak tangent of $(\mathbb{S}^2, \rho)$ is linearly locally connected. $\square$

2.6 A Note on Inverse Limits

In one of the later chapters we will use the language of inverse limits from category theory. For the sake of completeness, we include a short section on those terms. Readers can also consult any standard reference on category theory.

We are concerned with the category of topological spaces and continuous maps only. Let I be a set and $\leq$ be a relation on a subset of $I \times I$ such that

1. for all $i \in I$, $i \leq i$;
2. for $i, j \in I$, if $i \leq j$ and $j \leq i$, then $i = j$;
3. for all $i, j, k \in I$, if $i \leq j$ and $j \leq k$, then $i \leq k$; and
4. for all $i, j \in I$, there exists $k \in I$ such that $k \leq i$ and $k \leq j$.

An *inverse system* of topological spaces is a family $\{X_i\}_{i \in I}$ of topological spaces together with continuous functions $f_{i,j} : X_i \to X_j$ whenever $i \leq j$ such that $f_{j,k} \circ f_{i,j} = f_{i,k}$ for all $i \leq j \leq k$. An *inverse limit* of the inverse system $\{X_i\}_{i \in I}$ is the space

$$X = \{\{a_i\}_{i \in I} : f_{i,j}(a_i) = a_j \forall i \leq j\}.$$

From the definition of inverse limit, we see that there exists continuous maps $\pi_i : X \to X_i$, defined by

$$\pi_i(\{a_j\}_{j \in I}) = a_i.$$

Proposition 2.6.1 (The universal property of inverse limit). *Let $\{X_i\}_{i \in I}$ be an inverse system of topological spaces and let X be the inverse limit of $\{X_i\}_{i \in I}$. Let Y be another topological space and $\{\varphi_i : Y \to X_i\}_{i \in I}$ be continuous maps such that for each $i \leq j$, $f_{i,j} \circ \varphi_i = \varphi_j$. Then there exists a continuous map $u : Y \to X$ such that for each $i \leq j$, we have the following commutative diagram:*

$$\begin{tikzcd}
& Y \arrow[d, "u"] \arrow[ddl, "\varphi_i"'] \arrow[ddr, "\varphi_j"] & \\
& X \arrow[dl, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
X_i \arrow[rr, "f_{i,j}"] & & X_j.
\end{tikzcd}$$

CHAPTER 3

Visual Spheres and Their Weak Tangents

3.1 Introduction

Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be an expanding Thurston map. Let $\mathrm{crit}(f)$ be the set of critical points of f and let $\mathrm{post}(f) = \{f^n(c) : c \in \mathrm{crit}(f), n \geq 1\}$ be the set of postcritical points of f. In [BM17, Chapter 16] Bonk and Meyer obtained the following result:

Theorem 3.1.1. *[BM17, Theorem 16.1] Given an expanding Thurston map f, there exists $\Lambda_0(f) > 1$, called the combinatorial expansion factor of f, such that whenever $1 < \Lambda < \Lambda_0(f)$, there exists a visual metric ρ on $\mathbb{S}^2$ with expansion factor Λ, such that for every $x \in \mathbb{S}^2$, there exists $r_x > 0$ such that for every $y \in B(x, r_x)$,*

$$\rho(f(x), f(y)) = \Lambda\rho(x, y).$$

A stronger result can be found in [HP14]:

Theorem 3.1.2. *[HP14, Lemma 2.1] Let f be an expanding Thurston map without any periodic critical points. There exists $\lambda > 1$, a visual metric ρ of f with expansion factor Λ, and $r_0 > 0$ such that for all $x \in X$, for all $0 < r < r_0$, and $k \in \mathbb{N}_0$, we have*

$$f^k(B(x, \Lambda^{-k}r)) = B(f^k(x), r),$$

and if f^k is injective on $B(x, 4r\Lambda^{-k})$, then

$$\left|f^k(a) - f^k(b)\right| = \Lambda^{-k}\left|a - b\right|,$$

for all $a, b \in B(x, \Lambda^{-k}r)$.

In Section 3.2, we study this special visual metric ρ in greater detail. We are particularly interested in the possible value for r_x in Theorem 3.1.1 and when f^k in Theorem 3.1.2 is injective. When the point $f(x)$ is away from the postcritical point, we establish an estimate for the value r_x:

Proposition 3.1.3. *Given an expanding Thurston map f and a visual sphere ρ from the above theorem with expansion factor Λ. There exists $c \in (0,1)$ such that for all $x \in \mathbb{S}^2$ with $f(x) \notin \operatorname{post}(f)$, we can take $r_x = c\rho(f(x), \operatorname{post}(f))$.*

Suppose f does not have any periodic critical point. When the point $f(x)$ is in $\operatorname{post}(f)$, we obtain the following statement:

Proposition 3.1.4. *Let f be an expanding Thurston map without periodic critical points, and let ρ be the visual sphere from the above theorem with expansion factor Λ . Let $p \in \mathbb{S}^2$ be a postcritical point of f such that $f(p) = p$. Let $q \in \mathbb{S}^2$ be another point such that $f(q) = p$. Then there exists $r > 0$ and a scaling map $f' : B(q,r) \to B(q, \Lambda r)$ such that the following diagram commutes:*

$$\begin{tikzcd} B(q,r) \arrow[r, "f"] \arrow[d, "f'"] & B(p,\Lambda r) \arrow[d, "f"] \\ B(q,\Lambda r) \arrow[r, "f"] & B(p,\Lambda^2 r) \end{tikzcd}$$

Moreover, both $f' : B(q,r) \to B(q,\Lambda r)$ and $f : B(p,\Lambda r) \to B(p,\Lambda^2 r)$ are invertible.

Towards the end of the section, we prove that the visual sphere is self-similar in some sense.

Theorem (Theorem 1.0.5)**.** Let f be an expanding Thurston map without periodic critical points, for any visual metric ρ of f, the visual sphere $(\mathbb{S}^2, \rho)$ is approximately self-similar.

The definition of approximate self-similarity will be defined in Section 3.2.

Having clarified some basic properties of the visual metric ρ, we move on to study the *solenoids* of an expanding Thurston map f without periodic critical points. The solenoid $\mathcal{S}(f)$ of f is the inverse limit of the system

$$\cdots \xrightarrow{f} \mathbb{S}^2 \xrightarrow{f} \mathbb{S}^2 \xrightarrow{f} \mathbb{S}^2.$$

Thus we have natural projection maps $\pi_n : \mathcal{S}(f) \to \mathbb{S}^2$. When restricted to a path-connected component, also called a *leaf*, of $\mathcal{S}(f)$, these maps π_n are branched coverings (Theorem 3.3.8). Detailed discussion on the solenoid can be found in Section 3.

Leaves of the solenoid are closely related to weak tangents of the visual sphere. In Sections 4 and 5, we will see that a leaf of a solenoid is a weak tangent of the visual sphere:

Theorem (Theorem 1.0.6)**.** Let $x = \{x_n\}_{n\in\mathbb{N}_0}$ be a point in $\mathcal{S}(f)$, and let L be the leaf in $\mathcal{S}(f)$ containing x. Then the sequence $\{(\mathbb{S}^2, x_n, \Lambda^n\rho)\}_{n\in\mathbb{N}_0}$ converges in pointed-Gromov-Hausdorff sense to (L, x, d_L).

Conversely, Theorem 1.0.7 states, among other conclusions, that every weak tangent T of a visual sphere is a branched covering of a leaf of the solenoid:

Theorem (Theorem 1.0.7)**.** Let (T, a, d) be the weak tangent of $(\mathbb{S}^2, \rho)$ with associated data (a_n, r_n), and suppose $\{x_n\}_{n\in\mathbb{N}_0} \in \mathcal{S}(f)$ represents (a_n, r_n). Let L be the leaf of $\mathcal{S}(f)$ containing $\{x_n\}_{n\in\mathbb{N}_0}$. Then the following statements hold.

(i) There exists a branched covering $\pi : T \to L$.

(ii) If $\sup_{n\in\mathbb{N}_0} \Lambda^n\rho(x_n, \mathrm{post}(f)) = \infty$, then the map π is an isometry.

(iii) If $p = \{p_n\}_{n\in\mathbb{N}_0} \in L$ is a periodic sequence of postcritical points, then p is the only possible branched locus of π, and $\pi^{-1}(p)$ has exactly one point.

(iv) Let $b = \pi^{-1}(p)$. Then there exists $n_0 \in \mathbb{N}$ such that $\deg(\pi, b) = \deg(f^{n_0})$.

(v) There exists $q_0 \in f^{-n_0}(p_0)$ such that for all $R > 0$ there exists $k \in \mathbb{N}_0$ and an isometry $\imath : B_T(q, R) \to B_{\rho\Lambda^{n_0+k}}(q_0, R)$ such that the following diagram commutes:

$$\begin{tikzcd}
B_{(T,d_T)}(b,R) \arrow[r, "\pi"] \arrow[d, "\imath"] & B_{(L,d_L)}(p,R) \arrow[d, "\pi_k"] \\
B_{(\mathbb{S}^2,\Lambda^{k+n_0}\rho)}(q_0,R) \arrow[r, "f^{n_0}"'] & B_{(\mathbb{S}^2,\Lambda^{k}\rho)}(p_0,R).
\end{tikzcd}$$

3.2 A visual metric

In this section we construct a visual metric on $\mathbb{S}^2$. This construction first appears in [BM17, Chapter 16]. We reexamine the metric ρ. Towards the end of the section, we generalize Theorem 15.1 of [BM17] and Lemma 2.1 of [HP14].

A *tile chain joining x and y* is a sequence of tiles $X_1, X_2, \dots, X_n$, such that $x \in X_1$, $y \in X_n$, and for all $i = 1, 2, \dots, n-1$, $X_i \cap X_{i+1} \neq \emptyset$. Given a tile chain $\gamma = \{X_1, X_2, \dots, X_n\}$ joining two points in a sphere, we define the length of γ by

$$\ell(\gamma) := \sum_{i=1}^{n} \Lambda^{-m_i},$$

where m_i is the level of X_i. Let $\Lambda > 1$. For $x, y \in \mathbb{S}^2$, define

$$\rho(x, y) = \ell(\gamma),$$

where the infimum is taken over all possible tile chains γ joining x and y.

Lemma 3.2.1. *[BM17, Lemma 16.6 and Lemma 16.7] There exists $\Lambda_0(f) > 1$ such that for all $1 < \Lambda < \Lambda_0$, ρ is a visual metric. Moreover, for each $x \in \mathbb{S}^2$, there exists $r(x) > 0$ such that for all $y \in B(x, r(x))$, we have*

$$\rho(f(x), f(y)) = \Lambda \rho(x, y).$$

Lemma 3.2.2. *Let $x, y \in \mathbb{S}^2$. If $\rho(x, y) \leq 1$, then*

$$\rho(f(x), f(y)) \leq \Lambda \rho(x, y).$$

Proof. First assume that $\rho(x, y) < 1$. Given $\varepsilon \in (0, 1 - \rho(x, y))$, there exists a tile chain $\gamma = \{X_1, X_2, \dots X_n\}$ joining x and y such that

$$\rho(x, y) + \varepsilon \geq \ell(\gamma) = \sum_{i=1}^{n} \Lambda^{-m_i}.$$

where m_i is the level of X_i. If $\rho(x, y) < 1$, then $m_i \geq 1$ for all $i = 1, \dots, n$, and $f(X_i)$ is a $(m_i - 1)$-tile. The tile chain $f(\gamma) := \{f(X_1), f(X_2), \dots, f(X_n)\}$ joins $f(x)$ and $f(y)$, and hence

$$\rho(f(x), f(y)) \leq \ell(f(\gamma)) = \Lambda^{-m_1+1} + \cdots + \Lambda^{-m_n+1} \leq \Lambda \left(\rho(x, y) + \varepsilon\right).$$

Letting $\varepsilon \to 0$, we get $\rho(f(x), f(y)) \leq \Lambda \rho(x, y)$. $\square$

Lemma 3.2.3. *There exists $c > 0$ such that for all $x, x' \in \mathbb{S}^2$, if $f(x) = f(x')$ and $\rho(x, x') \leq c\Lambda^{-1}\rho(f(x), \mathrm{crit}(f)\backslash\{f(x)\})$, then $x = x'$.*

Proof. Suppose the lemma is false. Then for all $n \in \mathbb{N}$, there exists $x_n, x'_n \in \mathbb{S}^2$ such that $x_n \neq x'_n$, $f(x_n) = f(x'_n)$ and

$$\rho(x_n, x'_n) < \Lambda^{-n}\rho(f(x_n), \mathrm{post}(f)\backslash\{f(x_n)\}).$$

Hence $\rho(x_n, x'_n) \to 0$. Since $\mathbb{S}^2$ is compact, we may, by taking a subsequence of (x_n, x'_n) if necessary, assume that x_n converges to some $x^* \in \mathbb{S}^2$. Then x'_n converges to x^* as well.

If $f(x_n) = f(x'_n) = f(x^*)$, then because $f^{-1}(f(x^*))$ is finite, $x_n = x'_n$ for all sufficiently large n, a contradiction to our assumption that $x_n \neq x'_n$. Thus $f(x_n) = f(x'_n) \neq f(x^*)$ for large n. We claim that x^* is a critical point. Suppose not. Then there exists an open neighborhood U of x^* on which f is injective. For all large n, $x_n, x'_n \in U$. Therefore it is impossible to have $x_n \neq x'_n$ and $f(x_n) = f(x'_n)$ simultaneously. Thus x^* must be a critical point. Without loss of generality we assume that $x_n, x'_n \neq x^*$ and that x_n, x'_n are not critical points for all $n \in \mathbb{N}$. Moreover, for all $n \in \mathbb{N}$, we will assume that $f(x_n)$ is close enough to $f(x^*) \in \mathrm{post}(f)$ that $f(x_n) \notin \mathrm{post}(f)$.

For each $n \in \mathbb{N}$, choose a tile chain $\{X_1^n, \ldots, X_{k(n)}^n\}$ joining x_n and x'_n such that $X_i^n \in \mathcal{D}_{m_i^n}(\mathbb{S}^2)$ for all $i = 1, 2, \ldots, k(n)$, such that

$$\sum_{i=1}^{k(n)} \Lambda^{-m_i^n} < \Lambda^{-n}\rho(f(x_n), \mathrm{post}(f)).$$

Let δ_0 be the minimal distance between any pair of distinct postcritical points. Since f is postcritically finite, by Lemma 3.2.1, there exists $0 < \delta < \delta_0/3$ such that for all $r \in \mathbb{N}_0$, and for all $y \in \mathbb{S}^2$, if

$$\rho(f^r(y), f^r(x^*)) < \delta,$$

then

$$\rho(f^{r+1}(y), f^{r+1}(x^*)) = \Lambda\rho(f^r(y), f^r(x^*)).$$

For each $n \in \mathbb{N}$, choose r_n such that

$$\Lambda^{-1}\delta \leq \Lambda^r\rho(x, x^*) < \delta.$$

Applying f^{r_n} to each X_i^n we get a tile chain $\{f(X_1^n), \dots, f(X_{k(n)}^n))\}$ joining $f^{r_n}(x_n)$ and $f^{r_n}(x_n') = f^{r_n}(x_n)$. By Lemma 3.2.2, we have

$$\bigcup_{i=1}^{k(n)} f^{r_n}(X_i^n) \subset B(f^{r_n}(x^*), \delta).$$

Therefore the set $\bigcup_{i=1}^{k(n)} f^{r_n}(X_i^n)$ does not contain any postcritical point other than possibly $f^r(x^*)$. Since $x_n \neq x_n'$, there exists a Jordan curve $\gamma_n \subset \bigcup_{i=1}^{k(n)} f^{r_n}(X_i^n)$ passsing through $f^{r_n}(x_n)$ such that either γ is not null-homotopic in $\mathbb{S}^2 \setminus (\mathrm{post}(f) \setminus \{f^{r_n}(x^*)\})$, or γ contains $f^{r_n}(x^*)$. Let K_n be the closure of one component of $\mathbb{S}^2 \setminus post(f)$, and L_n be the closure of another component of $\mathbb{S}^2 \setminus post(f)$. Then

$$\inf_{n\in\mathbb{N}} \mathrm{diam}(K_n), \inf_{n\in\mathbb{N}} \mathrm{diam}(L_n) \geq \rho(f^{r_n}(x_n), \mathrm{post}(f)) > 0,$$

and

$$\begin{aligned}
\limsup_{n\to\infty} \mathrm{diam}(\gamma_n) &\leq \limsup_{n\to\infty} \Lambda^{r_n} \sum_{i=1}^{k(n)} \Lambda^{-m_i^n} \\
&\leq \limsup_{n\to\infty} \Lambda^{-n} \Lambda^{r_n} \rho(f(x_n), \mathrm{post}(f) \setminus \{f(x_n)\}) \\
&\leq C \limsup_{n\to\infty} \Lambda^{-n} \\
&= 0.
\end{aligned}$$

Thus along a subsequence of n, $\{K_n\}$ converges in Hausdorff sense to some compact set K, $\{L_n\}$ converges in Hausdorff sense to some compact set L, and $\gamma_n = \{K_n \cap L_n\}$ converges in Hausdorff sense to some compact set γ, such that $\mathrm{diam}(\gamma) = 0$, but $\mathrm{diam}(K_n), \mathrm{diam}(L_n) > 0$. But this is impossible since $(\mathbb{S}^2, \rho)$ is topologically a 2-sphere. $\square$

Lemma 3.2.4. *Take $c' = \frac{1}{2}\min\{c, 1\}$, where c is the constant in Lemma 3.2.3. If $\rho(x, y) \leq c'\Lambda^{-1}\rho(f(x), \mathrm{post}(f) \setminus \{f(x)\})$, then $\rho(f(x), f(y)) \geq \Lambda\rho(x, y)$.*

Proof. Let $x, y \in \mathbb{S}^2$ be fixed such that

$$\rho(x, y) \leq c'\Lambda^{-1}\rho(f(x), \mathrm{post}(f) \setminus \{f(x)\}).$$

By Lemma 3.2.2, we have

$$\rho(f(x), f(y)) \leq c' \rho(x, y) = \frac{1}{2} \min\{c, 1\} \rho(f(x), \mathrm{post}(f) \backslash \{f(x)\}).$$

For any $\varepsilon > 0$, we can find a tile chain $\{Y_1, Y_2, \ldots, Y_n\}$ joining $f(x)$ and $f(y)$ such that each Y_i is an m_i-tile and

$$\sum_{i=1}^{n} \Lambda^{-m_i} - \varepsilon < \rho(f(x), f(y)) \leq \sum_{i=1}^{n} \Lambda^{-m_i}.$$

For each $z \in \bigcup_{i=1}^{n} Y_i$, we have

$$\begin{aligned}
\rho(z, \mathrm{post}(f) \backslash \{f(x)\}) &\geq \rho(f(x), \mathrm{post}(f) \backslash \{f(x)\}) - \rho(f(x), z) \\
&\geq \rho(f(x), \mathrm{post}(f) \backslash \{f(x)\}) - \sum_{i=1}^{n} \Lambda^{-m_i} \\
&\geq \rho(f(x), \mathrm{post}(f) \backslash \{f(x)\}) - (\rho(f(x), f(y)) + \varepsilon) \\
&\geq \frac{1}{2} \rho(f(x), \mathrm{post}(f) \backslash \{f(x)\}) - \varepsilon.
\end{aligned}$$

If ε is sufficiently small, then $Y_2, \ldots, Y_n$ do not contain any postcritical points except possibly $f(x)$. There exists a lift of the tile chain $\{Y_1, \ldots, Y_n\}$ to a tile chain $\{X_1, \ldots, X_n\}$ such that $y \in X_n$. Let $x' \in X_1$ be the point such that $f(x') = f(x)$. Then

$$\rho(x', y) \leq \sum_{i=1}^{n} \Lambda^{-m_i - 1} \leq \Lambda^{-1} (\rho(f(x), f(y)) + \varepsilon) \leq \rho(x, y) + \Lambda^{-1} \varepsilon.$$

Therefore

$$\rho(x, x') \leq \rho(x, y) + \rho(x', y) \leq 2\rho(x, y) + \Lambda^{-1} \varepsilon.$$

Letting $\varepsilon \to 0$, we have

$$\rho(x, x') \leq 2\rho(x, y) \leq c\rho(f(x), \mathrm{post}(f) \backslash \{f(x)\}).$$

By Lemma 3.2.3, we have $x = x'$. Thus $X_1, \ldots, X_n$ is a tile chain that joins x and y, and

$$\rho(x, y) \leq \sum_{i=1}^{n} \Lambda^{-m_i - 1} \leq \Lambda^{-1} (\rho(f(x), f(y)) + \varepsilon).$$

Letting $\varepsilon \to 0$, we conclude that

$$\rho(x, y) \leq \Lambda^{-1} \rho(f(x), f(y)).$$

$\square$

Combining Lemma 3.2.2 and Lemma 3.2.4, we obtain the following propositions:

Proposition 3.2.5. *There exists $c' \in (0,1]$ such that for all $x, y \in \mathbb{S}^2$, if*

$$\rho(x,y) \leq c'\Lambda^{-1}\rho(f(x), \operatorname{post}(f)\backslash\{f(x)\}),$$

then

$$\rho(f(x), f(y)) = \Lambda\rho(x,y).$$

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ from X onto Y is a *scaling map* with *scaling factor* $\Lambda > 0$, or a Λ-scaling map, if for all $x_1, x_2 \in X$, we have

$$d_Y(f(x_1), f(x_2)) = \Lambda d_X(x_1, x_2).$$

Proposition (Proposition 3.1.3)**.** Let $c'' = \frac{c'}{2+c'}$, where c' is the constant in Proposition 3.2.5. Let x' be a point in $\mathbb{S}^2$, and suppose that $x = f(x') \notin \operatorname{post}(f)$. Let $R = \rho(x, \operatorname{post}(f)\backslash\{x\})$. Then the map

$$f : B(x', \Lambda^{-1}c''R) \to B(x, c''R)$$

is a Λ-scaling map.

Proof. Since $c'' < 1$ and $R \leq 1$, Lemma 3.2.2 implies $f(B(x', \Lambda^{-1}c''R)) \subset B(x, c''R)$.

For any $y \in B(x, c''R)$, let $Y_1, \ldots, Y_k$ be a tile chain joining x and y, and let $m_1, \ldots, m_k$ be the respective levels of $Y_1, \ldots, Y_k$. Assume

$$\sum_{i=1}^{k} \Lambda^{-m_i} < c''R.$$

This is possible since $\rho(x,y) < c''R$. Then $Y_i \subset B(x, c''R)$ for all $i = 1, \ldots, k$. Since $B(x, c''R)$ contains no postcritical points other than possibly x, By [BM17, Lemma A.19], we can lift $Y_1, \ldots, Y_k$ to a tile chain $X_1, \ldots, X_k$ such that $f(X_i) = Y_i$ for all i and $x' \in X_1$. This tile chain has length

$$\sum_{i=1}^{k} \Lambda^{-m_i-1} < \Lambda^{-1}c''R.$$

Therefore $X_1, \ldots, X_k$ is a tile chain in $B(x', \Lambda^{-1}c''R)$. Since $y \in Y_k = f(X_i)$, $y = f(y')$ for some $y' \in X_i \subset B(x', \Lambda^{-1}c''R)$. This proves that the map

$$f : B(x', \Lambda^{-1}c''R) \to B(x, c''R)$$

is surjective.

Let $y', z' \in B(x', \Lambda^{-1}c''R)$, and let $y = f(y')$. Then

$$\rho(y', z') < 2\Lambda^{-1}c''R.$$

Since $x, y \notin \operatorname{post}(f)$, we have $\operatorname{post}(f)\backslash\{y\} = \operatorname{post}(f)\backslash\{x\}$. Therefore

$$\begin{aligned} \rho(y, \operatorname{post}(f)\backslash\{y\}) &\geq \rho(y, \operatorname{post}(f)\backslash\{x\}) \\ &\geq \rho(x, \operatorname{post}(f)\backslash\{x\}) - \rho(x, y) \\ &\geq R - c''R \end{aligned}$$

For our choice of c'', we have

$$\rho(y', z') \leq 2\Lambda^{-1}c''R \leq c'(R - c''R) \leq c'\rho(y, \operatorname{post}(f)\backslash\{y\}).$$

By Proposition 3.2.5, we have

$$\rho(f(y'), f(z')) = \Lambda\rho(y', z').$$

$\square$

In other words, away from $f^{-1}(\operatorname{post}(f))$, f is locally a scaling map with scaling factor Λ. The next proposition describes the behavior of f near a preimage of a postcritical point.

Proposition (Proposition 3.1.4)**.** Suppose f is an expanding Thurston map without periodic critical points. Let $p \in \mathbb{S}^2$ be a postcritical point of f such that $f(p) = p$. Let $q \in \mathbb{S}^2$ be another point such that $f(q) = p$. Then there exists $r > 0$ and a scaling map $f' : B(q, r) \to B(q, \Lambda r)$ such that the following diagram commutes:

$$\begin{tikzcd} B(q, r) \arrow[r, "f"] \arrow[d, "f'"] & B(p, \Lambda r) \arrow[d, "f"] \\ B(q, \Lambda r) \arrow[r, "f"] & B(p, \Lambda^2 r) \end{tikzcd}$$

Moreover, both $f' : B(q,r) \to B(q, \Lambda r)$ and $f : B(p, \Lambda r) \to B(p, \Lambda^2 r)$ are invertible.

Proof. The point p is a periodic postcritical point, so it cannot be a critical point. The same argument as the proof of the previous proposition shows that the map

$$f : B(p, 2\Lambda r) \to B(p, 2\Lambda^2 r)$$

is surjective for sufficiently small r. By choose an even smaller r, we may assume that $2\Lambda r$ and the above map is also injective. We claim that by the map

$$f : B(p, \Lambda r) \to B(p, \Lambda^2 r)$$

is a scaling map with scaling factor Λ. Let $y, z \in (p, \Lambda r)$. Then $\rho(f(y), f(z)) < 2\Lambda^2 r$. Let $\varepsilon \in (0, \Lambda^2 - \max\{\rho(f(y), f(p)), \rho(f(z), f(p))\})$. We can therefore find a tile chain $\gamma = \{Y_1, \ldots, Y_n\}$ joining $f(y)$ and $f(z)$ such that

$$\ell(\gamma) \leq \rho(f(y), f(z)) + \varepsilon < 2\Lambda^2 r.$$

For any $x \in \bigcup_{i=1}^n Y_i$, we have

$$\rho(p, x) \leq \min\{\rho(p, y) + \rho(y, x), \rho(p, z) + \rho(z, x)\} < 2\Lambda r.$$

Therefore $\bigcup_{i=1}^n Y_i \subset B(p, 2\Lambda^2 r)$. The tile chain $\{Y_1, \ldots, Y_n\}$ has a unique lift to a tile chain $\{X_1, \ldots, X_n\}$ in $B(p, 2\Lambda r)$. The tile chain $\{X_1, \ldots, X_n\}$ joins y and z by injectivity of $f : B(p, \Lambda r) \to B(p, \Lambda^2 r)$. Thus

$$\rho(y, z) \leq \Lambda^{-1} \ell(\gamma) < \Lambda^{-1} \left(\rho(f(y), f(z)) + \varepsilon\right).$$

Letting $\varepsilon \to 0$, we get

$$\rho(y, z) \leq \Lambda^{-1} \rho(f(y), f(z)).$$

Combining this inequality with Lemma 3.2.2, we get the desired conclusion.

Since the maps

$$f : B(q, r) \backslash \{q\} \to B(p, \Lambda r) \backslash \{p\}$$

and

$$f : B(q, \Lambda r) \backslash \{q\} \to B(p, \Lambda^2 r) \backslash \{p\}$$

are covering maps of the same degree at q, we can lift f to a bijective map

$$f' : B(q,r)\backslash\{q\} \to B(q,\Lambda r)\backslash\{q\}$$

such that the following diagram commutes:

$$\begin{array}{ccc} B(q,r)\backslash\{q\} & \xrightarrow{f} & B(p,\Lambda r)\backslash\{p\} \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ B(q,\Lambda r)\backslash\{q\} & \xrightarrow{f} & B(p,\Lambda^2 r)\backslash\{p\}. \end{array}$$

By continuity of f, we can extend f' continuously to a bijection

$$f' : B(q,r) \to B(q,\Lambda r)$$

such that the following diagram commutes:

$$\begin{array}{ccc} B(q,r) & \xrightarrow{f} & B(p,\Lambda r) \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ B(q,\Lambda r) & \xrightarrow{f} & B(p,\Lambda^2 r). \end{array}$$

Moreover, f' maps an n-tile in $B(q,r)$ to an $(n-1)$-tile in $B(q,\Lambda r)$.

Let $r' = r/2$. For any $x, y \in B(q,r')$, we can find a tile chain $Y_1,\ldots,Y_k$ with levels $m_1,\ldots,m_k$ joining $f'(x)$ and $f'(y)$ such that

$$\sum_{i=1}^{k} \Lambda^{-m_i} < 2\Lambda r'.$$

For all $i = 1, 2, \ldots, k$, we have

$$\begin{aligned} \rho(q, Y_i) &\le \min\{\rho(q, f'(y)) + \rho(f'(y), Y_i), \rho(q, f'(x)) + \rho(f'(x), Y_i)\} \\ &\le \Lambda r' + \frac{1}{2}\sum_{i=1}^{k} \Lambda^{-m_i} < 2\Lambda r' = \Lambda r. \end{aligned}$$

Therefore $Y_i \subset B(q,\Lambda^2 r)$ for all i. The tile chain $Y_1,\ldots,Y_k$ can be lifted to a tile chain $X_1,\ldots,X_k$ joining x and y, and hence

$$\rho(x,y) \le \sum_{i=1}^{k} \Lambda^{-m_i-1}.$$

Infimizing over all tile chains joining $f'(x)$ and $f'(y)$, we get

$$\rho(x,y) \le \Lambda^{-1}\rho(f'(x), f'(y)).$$

Thus the map

$$f : B(q, \Lambda r') \to B(q, \Lambda^2 r')$$

is a scaling map with scaling factor Λ. $\square$

Lastly, we prove that the visual sphere is self-similar in some sense.

Definition 3.2.6. *A bounded metric space (X,d) is called approximately self-similar if there exists $R > 0$ and $L > 1$ such that for all $x \in X$ and $r < R$, there exists a injective map $f : B(x,r) \to X$ such that for all $x_1, x_2 \in B(x,r)$, we have*

$$L^{-1}d(f(x_1), f(x_2)) \le \frac{R}{r}d(x_1,x_2) \le Ld(f(x_1), f(x_2)). \tag{3.2.1}$$

See [Kle06, Section 3] for more discussion on approximate self-similarity.

Theorem (Theorem 1.0.5)**.** If f is an expanding Thurston map without periodic critical points, then the visual sphere $(\mathbb{S}^2, \rho)$ is approximately self-similar.

Proof. There exists $n \in \mathbb{N}$ such that every periodic postcritical point of f is a fixed point of f^n. For instance, we can take n to be a common multiple of the periods of all periodic postcritical points. By taking a sufficiently large multiple of n, we may assume that for every critical point c of f^n, $f^{2n}(c)$ is a postcritical fixed point. Let us assume $n = 1$.

By Proposition 3.1.4, for each $x \in f^{-1}(\mathrm{post}(f))$, there exists $r_x > 0$ and scaling maps

$$f' : B(x, r_x) \to B(x, \Lambda^2 r_x).$$

of scaling factor Λ^2. By Proposition 3.1.3, for each $x \in \mathbb{S}^2 \backslash f^{-1}(\mathrm{post}(f))$, there exists $r_x > 0$ such that the map

$$f^2 : B(x, r_x) \to B(f^2(x), \Lambda^2 r_x)$$

is a scaling map by a factor Λ^2. The collection $\{B(x, r_x)\}_{x\in\mathbb{S}^2}$ is an open cover of $\mathbb{S}^2$, therefore there exists $x_1, x_2, \ldots, x_n \in \mathbb{S}^2$ such that $\{B(x_i, r_{x_i})\}_{i=1}^n$ covers $\mathbb{S}^2$. Let $r_i = r_{x_i}$

and let $B_i = B(x_i, r_i)$. Then there exists $R > 0$ such that every open ball of radius R in $\mathbb{S}^2$ is contained in one of B_i. We say that an open ball $B(x, r)$ in $\mathbb{S}^2$ is good if there exists a scaling map $g : B(x, r) \to X$ with scaling factor λ^2 such that $\Lambda^{-2}R \leq \lambda r \leq R$.

Note that every ball of radius r in the range $[\Lambda^{-2}R, R]$ is good, because the identity map $\mathrm{id} : B(x, r) \to X$ is a scaling map with scaling factor 1, and $\Lambda^{-2}R \leq r \leq R$. Suppose not every ball of radius less than R is good. Let r_0 be the supremum of the radii of the balls $B(x, r)$, with $r < R$, that are not good. Then every ball of radius $r > r_0$ is good but there exists a ball $B(x, r)$ of radius $r \in (\Lambda^{-2}r_0, r_0]$ that is not good.

Since $r < R$, the ball $B(x, r)$ is in one of the B_i's, and there exists a scaling map $f' : B_i \to X$ with scaling factor Λ^2. The same map f', restricted to $B(x, r)$, gives a scaling map with scaling factor Λ^2 from $B(x, r)$ to another ball $B(y, \Lambda^2 r)$ of $\mathbb{S}^2$. By our hypothesis on r, $B(y, \Lambda^2 r)$ is good, i.e., there exists a map $g : B(y, \Lambda^2 r) \to X$ with scaling factor λ such that $\Lambda^{-2}R \leq \lambda\Lambda^2 r \leq \lambda R$. The composition $g \circ f' : B(x, r) \to X$ is then a scaling map with scaling factor $\Lambda^2\lambda$. This implies that $B(x, r)$ is good, a contradiction to our choice of $B(x, r)$. Thus every open ball is good. In particular, the visual sphere is approximately self-similar. $\square$

3.3 Solenoids and Leaves: Definitions

Given an expanding Thurston map f without periodic critical points, we define the *solenoid*

$$\mathcal{S}(f) = \mathcal{S}(\mathbb{S}^2, f) = \Big\{ \{a_n\}_{n\in\mathbb{N}_0} \in (\mathbb{S}^2)^{\mathbb{N}_0} : f(a_{n+1}) = a_n \forall n \in \mathbb{N}_0 \Big\}.$$

The solenoid $\mathcal{S}(f)$, as a subset of the product space $(\mathbb{S}^2)^{\mathbb{N}_0}$, inherits the product topology of $(\mathbb{S}^2)^{\mathbb{N}_0}$. It is the inverse limit of

$$\cdots \xrightarrow{f} \mathbb{S}^2 \xrightarrow{f} \mathbb{S}^2 \xrightarrow{f} \mathbb{S}^2. \tag{3.3.1}$$

Let ρ be the visual metric constructed in the previous section, and let Λ be the expansion factor of ρ. We define a relation $\sim$ on $\mathcal{S}(f)$ by $\{x_n\}_{n\in\mathbb{N}_0} \sim \{y_n\}_{n\in\mathbb{N}_0}$ if and only if

$$\sup_{n\to\mathbb{N}_0} \Lambda^n \rho(x_n, y_n) < \infty.$$

An equivalence class in $\mathcal{S}(f)$ is called a *leaf* of $\mathcal{S}(f)$.

Proposition 3.3.1. *The relation $\sim$ is an equivalence relation.*

Proof. This is a consequence of the triangle inequality. $\square$

Lemma 3.3.2. *Let $\{x_n\}_{n\in\mathbb{N}_0} \in \mathcal{S}(\mathbb{S}^2, f)$ be arbitrary. If the condition*

$$\sup_{n\in\mathbb{N}_0} \Lambda^n \rho(x_n, \operatorname{post}(f)) < \infty$$

holds, then $\{x_n\}_{n\in\mathbb{N}_0} \sim \{p_n\}_{n\in\mathbb{N}_0}$ for one and only one periodic sequence $\{p_n\}_{n\in\mathbb{N}_0}$ of post-critical points in $\mathcal{S}(f)$.

Proof. Because $\operatorname{post}(f)$ is finite, there exists $\varepsilon > 0$ so that the distance between any two distinct points in $\operatorname{post}(f)$ is at least ε. If $\sup_{n\in\mathbb{N}_0} \Lambda^n \rho(x_n, \operatorname{post}(f)) < \infty$, then there exists $N \in \mathbb{N}_0$ such that

$$M = \sup_{n\in\mathbb{N}_0} \Lambda^n \rho(x_n, \operatorname{post}(f)) < \frac{\varepsilon}{2}\Lambda^N.$$

For each $n \in \mathbb{N}_0$, there exists $p_n \in \operatorname{post}(f)$ such that

$$\rho(x_n, p_n) \leq M\Lambda^{-n}.$$

Thus for all $n \geq N$,

$$\begin{aligned}
\rho(p_n, f(p_{n+1})) &\leq \rho(x_n, p_n) + \rho(f(x_{n+1}), f(p_{n+1})) \\
&\leq \rho(x_n, p_n) + \Lambda\rho(x_{n+1}, p_{n+1}) \\
&< M\Lambda^{-n} + M\Lambda^{-n} \\
&< \varepsilon.
\end{aligned}$$

Therefore $f(p_{n+1}) = p_n$. In other words, $\{p_{n+N}\}_{n\in\mathbb{N}_0}$, and therefore $\{f^N(p_{n+N})\}_{n\in\mathbb{N}_0}$, belongs to $\mathcal{S}(\mathbb{S}^2, f)$. As there are only finitely many postcritical points, $\{f^N(p_{n+N})\}_{n\in\mathbb{N}_0}$ is a periodic sequence of postcritical points, and

$$\begin{aligned}\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,f^N(p_{n+N})) &\le \sup_{n<N}\Lambda^n\rho(x_n,f^N(p_{n+N}))+\sup_{n\ge N}\Lambda^n\rho(x_n,f^N(p_{n+N}))\\ &=\sup_{n<N}\Lambda^n\rho(x_n,f^N(p_{n+N}))+\sup_{n\ge N}\Lambda^n\rho(x_n,p_n)\\ &<\sup_{n<N}\Lambda^n\rho(x_n,f^N(p_{n+N}))+M\\ &<+\infty.\end{aligned}$$

Suppose $\{q_n\}_{n\in\mathbb{N}_0}\in\mathcal{S}(f)$ is another periodic sequence of postcritical points such that $\{x_n\}_{n\in\mathbb{N}_0}\sim\{q_n\}_{n\in\mathbb{N}_0}$. Then $\{p_n\}_{n\in\mathbb{N}_0}\sim\{q_n\}_{n\in\mathbb{N}_0}$. In other words,

$$\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(p_n,q_n)=C<\infty.$$

Therefore, for all $n\in\mathbb{N}_0$, we have

$$\rho(p_n,q_n)\le C\Lambda^{-n}. \tag{3.3.2}$$

Since $\operatorname{post}(f)$ is a finite set, for all large n, (3.3.2) forces $p_n=q_n$. But if $p_n=q_n$, then $p_{n-1}=f(p_n)=f(q_n)=q_{n-1}$. Thus $\{p_n\}_{n\in\mathbb{N}_0}=\{q_n\}_{n\in\mathbb{N}_0}$ as points in $\mathcal{S}(f)$. □

The equivalence relation $\sim$ is the same as path-connectedness. To prove this we introduce a few lemmas.

Lemma 3.3.3. *For any $\{x_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{S}(f)$, there exists $N\in\mathbb{N}_0$ such that for all $n>N$, x_n is not a critical point.*

Proof. Suppose not. Then $\{x_n\}_{n\in\mathbb{N}_0}$ contains infinitely many critical points. Since the set $\operatorname{crit}(f)$ is finite, there exists $0\le n_1<n_2$ such that $x_{n_1}=x_{n_2}\in\operatorname{crit}(f)$. Since $f^{n_2-n_1}(x_{n_2})=x_{n_1}$, the point x_{n_2} is a periodic critical point, contradicting our assumption on f. □

Lemma 3.3.4. *Let $\{x_n\}_{n\in\mathbb{N}_0}$ be a point in $\mathcal{S}(f)$. Then for all $R>0$, there exists $N>0$ such that for all $n>N$, the map*

$$f:B_{\Lambda^n\rho}(x_n,R)\to B_{\Lambda^{n-1}\rho}(x_{n-1},R)$$

is an isometry.

Proof. **Case 1:** $\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,\mathrm{post}(f)) = +\infty$. Let $N_1 \in \mathbb{N}_0$ be an integer such that whenever $n > N_1$, the point x_n is not a critical point of f. Lemma 3.3.3 guarantees the existence of such N_1. Since $\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,\mathrm{post}(f)) = +\infty$, by Proposition 3.1.3, there exists $c > 0$ such that for all $n > N_1$ and for all $r < c\rho(x_{n-1},\mathrm{post}(f)\backslash\{x_{n-1}\})$, the map

$$f : B(x_n, \Lambda^{-1}r) \to B(x_{n-1}, r) \tag{3.3.3}$$

is a Λ-scaling map. For any $R > 0$, there exists $N \geq N_1$ such that

$$R < c\Lambda^N\rho(x_N,\mathrm{post}(f)\backslash\{x_N\}).$$

By induction on n and the fact that the map in (3.3.3) is a Λ-scaling map with scaling factor Λ, for every $n \geq N$, we have

$$R < c\Lambda^n\rho(x_n,\mathrm{post}(f)\backslash\{x_n\}).$$

Hence

$$f : B_{\Lambda^n\rho}(x_n, R) \to B_{\Lambda^{n-1}\rho}(x_{n-1}, R)$$

is an isometry.

Case 2: $\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,\mathrm{post}(f)) < +\infty$. Then by Lemma 3.3.2, there exists a periodic sequence of postcritical points $\{p_n\}_{n\in\mathbb{N}_0} \in \mathcal{S}(f)$ such that $\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,p_n) < \infty$. Let ε be the minimum distance between pairs of distinct points in $\mathrm{post}(f)$. We have

$$\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(p_n,\mathrm{post}(f)\backslash\{p_n\}) \geq \sup_{n\in\mathbb{N}_0}\Lambda^n\varepsilon = +\infty.$$

The same arguement as in the previous case concludes for every $R > 0$, there exists $N \in \mathbb{N}_0$ such that for all $n > N$, the map

$$f : B_{\Lambda^n\rho}(p_n, R + \sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,p_n)) \to B_{\Lambda^{n-1}\rho}(p_{n-1}, R + \sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,p_n))$$

is an isometry. Since

$$B_{\Lambda^n\rho}(x_n, R) \subset B_{\Lambda^n\rho}(p_n, R + \sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,p_n)),$$

the map

$$f : B_{\Lambda^n\rho}(p_n, R) \to B_{\Lambda^{n-1}\rho}(p_{n-1}, R)$$

is an isometry. $\square$

Proposition 3.3.5. *Two points $\{x_n\}$ and $\{y_n\}$ in $\mathcal{S}(\mathbb{S}^2, f)$ are connected by a path in $\mathcal{C}(f)$ if and only if $\{x_n\} \sim \{y_n\}$.*

Proof. Suppose $\{x_n\}$ and $\{y_n\}$ in $\mathcal{S}(\mathbb{S}^2, f)$ are connected by a path $\gamma : [0,1] \to \mathcal{S}(\mathbb{S}^2, f)$. We can write $\gamma = \{\gamma_0, \gamma_1, \ldots\}$, where each path $\gamma_n : [0,1] \to \mathbb{S}^2$ joins x_n and y_n and $f \circ \gamma_{n+1} = \gamma_n$. We have

$$\rho(x_n, y_n) \leq \operatorname{diam}(\gamma_n) \lesssim \Lambda^{-n},$$

where the implicit constant is independent of n. Here the last inequality follows from [BM17, Lemma 8.9]. Taking supremum over all $n \in \mathbb{N}$, we get

$$\sup_{n \in \mathbb{N}_0} \Lambda^n \rho(x_n, y_n) < \infty.$$

Conversely, suppose $\sup_{n \in \mathbb{N}_0} \Lambda^n \rho(x_n, y_n) < \infty$. Let $R = 2 \sup_{n \in \mathbb{N}_0} \Lambda^n \rho(x_n, y_n)$. By Lemma 3.3.4, there exists $N \in \mathbb{N}_0$ such that for all $n > N$, the map

$$f : B_{\Lambda^n \rho}(x_n, R) \to B_{\Lambda^{n-1} \rho}(x_{n-1}, R)$$

is an isometry.

Note that $y_N \in B_{\Lambda^N \rho}(x_N, R)$. Let γ_N be a curve in $B_{\Lambda^N \rho}(x_N, R)$ joining x_N and y_N. For all $n \geq N$, we can lift γ_N by f^{n-N} to a curve γ_n in $B_{\Lambda^n \rho}(x_n, R)$ that joins x_n and y_n. For all $n = 0, 1, \ldots, N-1$, let $\gamma_n = f^{N-n}(\gamma_N)$. This sequence $\{\gamma_n\}_{n \in \mathbb{N}_0}$ gives a path in $\mathcal{S}(f)$ joining $\{x_n\}_{n \in \mathbb{N}_0}$ and $\{y_n\}_{n \in \mathbb{N}_0}$. $\square$

By Proposition 3.3.5, a leaf of $\mathcal{S}(\mathbb{S}^2, f)$ is a path-connected component of $\mathcal{S}(\mathbb{S}^2, f)$.

Let $L \subset \mathcal{S}(\mathbb{S}^2, f)$ be a leaf. For $x = \{x_n\}, y = \{y_n\} \in L$, define

$$d_L(x, y) = \sup_{n \in \mathbb{N}_0} \Lambda^n \rho(x_n, y_n).$$

Proposition 3.3.6. *The function d_L is a metric on L.*

Proof. By the definition of $\sim$ and Proposition 3.3.5, we see that $0 \leq d_L(x, y) < \infty$ for all $x, y \in L$. Moreover $d_L(x, y) = 0$ if and only if $\rho(x_n, y_n) = 0$ for all $n \in \mathbb{N}_0$ if and only if $x = y$.

The symmetry of d_L is inherited from the symmetry of ρ.

Let $x = \{x_n\}, y = \{y_n\}, z = \{z_n\}$ be three elements in L. We have

$$\begin{aligned} d_L(x,y) + d_L(y,z) &= \sup_{n\in\mathbb{N}_0} \Lambda^n \rho(x_n, y_n) + \sup_{n\in\mathbb{N}_0} \Lambda^n \rho(y_n, z_n) \\ &\geq \sup_{n\in\mathbb{N}_0} \Lambda^n \left(\rho(x_n, y_n) + \rho(y_n, z_n)\right) \\ &\geq \sup_{n\in\mathbb{N}_0} \Lambda^n \left(\rho(x_n, z_n)\right) \\ &= d_L(x,z). \end{aligned}$$

□

Proposition 3.3.7. *The metric space (L, d_L) is complete.*

Proof. Suppose $x^k = \{x_n^k\}$ is a Cauchy sequence in (L, d_L). Then for all $n \in \mathbb{N}_0$,

$$\rho(x_n^k, x_n^l) \to 0$$

as $k, l \to \infty$, which implies x_n^k converges to some x_n as $k \to \infty$. As the space $\mathcal{S}(f)$ is closed under the product topology of $(\mathbb{S}^2)^{\mathbb{N}_0}$, $\{x_n\} \in \mathcal{S}(f)$. For all $n \in \mathbb{N}_0$,

$$\Lambda^n \rho(x_n^1, x_n) \leq \limsup_{k\in\mathbb{N}_0} \Lambda^n \rho(x_n^1, x_n^k) + \Lambda^n \rho(x_n^k, x_n) \leq \sup_{k\in\mathbb{N}_0} \Lambda^n \rho(x_n^1, x_n^k) \leq \sup_{k\in\mathbb{N}_0} d_L(x^1, x^k).$$

Therefore

$$\sup_{n\in\mathbb{N}_0} \Lambda^n \rho(x_n^1, x_n) \leq \sup_{n\in\mathbb{N}_0} \sup_{k\in\mathbb{N}_0} d_L(x^1, x^k) = \sup_{k\in\mathbb{N}_0} d_L(x^1, x^k) < \infty.$$

□

For all $k \in \mathbb{N}_0$, let $\pi_k : \mathcal{S}(f) \to \mathbb{S}^2$ denote the map

$$\{x_n\}_{n\in\mathbb{N}_0} \mapsto x_k.$$

Theorem 3.3.8. *Let L be a leaf of $\mathcal{S}(f)$. Regard L as a metric space equipped with the metric d_L. Then $n \in \mathbb{N}_0$, the map*

$$\pi_n|_L : L \to \mathbb{S}^2$$

defines a branched covering from (L, d_L) to $\mathbb{S}^2$. The only branched loci of π_n are points in post(f).

Proof. We prove the theorem for π_0. By Lemma 3.3.2, there exists $x = \{x_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{S}(f)$ that satisfies one of the two conditions in Lemma 3.3.4. Fix $r > 0$. By Lemma 3.3.4, there exists $N \in \mathbb{N}_0$ such that the map

$$\pi_N : B_{(L,d_L)}(x, r) \to B_{(\mathbb{S}^2,\Lambda^N\rho)}(x_N, r)$$

is an isometry. Postcomposing π_N with f^N, we conclude that on $B_{(L,d_L)}(x, r)$, the map π_0 is a continuous, open and discrete. This proves that $\pi_0|_{B_{(L,d_L)}(x,r)}$ is a branched covering. The argument works for all $r > 0$. Thus π_0 is a branched covering. The branched loci of $f^N \circ \pi_N$ are the branched loci of f^N, which have to be in $\operatorname{post}(f)$. $\square$

The maps π_n enable us to define the notion of tiles on leafs. The map

$$\pi_0 : T\backslash\pi_0^{-1}(\operatorname{post} P) \to \mathbb{S}^2\backslash \operatorname{post}(f)$$

is a covering map. For any tile σ in $\mathbb{S}^2$, and any point $x \in L$ such that $\pi_0(x) \in \sigma\backslash \operatorname{post}(f)$, the map π_0 lifts $\sigma\backslash \operatorname{post}(f)$ uniquely to a subset $\tau \subset L$ such that $x \in \tau$. The closure of τ in L is called a lift of σ, and if σ is an n-tile, we call τ an n-tile in L.

3.4 Weak Tangents and Leaves

In this section, we investigate the relation between weak tangents and leaves.

Theorem (Theorem 1.0.6)**.** Let $x = \{x_n\}_{n\in\mathbb{N}_0}$ be a point in $\mathcal{S}(f)$, and let L be the leaf in $\mathcal{S}(f)$ containing x. Then the sequence $\{(\mathbb{S}^2, x_n, \Lambda^n\rho)\}_{n\in\mathbb{N}_0}$ converges in pointed-Gromov-Hausdorff sense to (L, x, d_L).

Proof. Let us first suppose that $\{x_n\}_{n\in\mathbb{N}_0}$ satisfies either one of the two conditions in Lemma 3.3.4. Let $R > 0$ be arbitrary. By Lemma 3.3.4, there exists $N \in \mathbb{N}_0$ such that for all $n > N$, the map

$$f : B_{\Lambda^n\rho}(x_n, R) \to B_{\Lambda^{n-1}\rho}(x_{n-1}, R)$$

is an isometry. Hence the map

$$\pi_n : B_L(x, R) \to B_{\Lambda^n\rho}(x_n, R)$$

is an isometry. This proves that (L, x, d_L) is the pointed-Gromov-Hausdorff limit of the sequence $\{(\mathbb{S}^2, x_n, \Lambda^n\rho)\}_{n\in\mathbb{N}_0}$. $\square$

Theorem 1.0.6 implies that every leaf of the solenoid is a weak tangent of the visual sphere $(\mathbb{S}^2, \rho)$. In what follows let (T, a, d_T) be a weak tangent of $(\mathbb{S}^2, \rho)$. Then (T, a, d_T) is a pointed-Gromov-Hausdorff limit of a sequence $(\mathbb{S}^2, a_n, \Lambda^{r_n}\rho)$, where a_n is a sequence of points in $\mathbb{S}^2$, and r_n is a sequence of positive real numbers tending to $+\infty$. The pointed-Gromov-Hausdorff limit of the sequence $(\mathbb{S}^2, a_n, \Lambda^{r_n}\rho)$ is bilipschitz equivalent to the pointed-Gromov-Hausdorff limit of the sequence $(\mathbb{S}^2, a_n, \Lambda^{\lceil r_n\rceil}\rho)$. Since we are interested in studying the visual spheres up to quasisymmetric equivalence, we will assume that each r_n is a positive integer. The pointed metric space (T, a, d_T) is said to be a *weak tangent of* $(\mathbb{S}^2, \rho)$ *with associated data* (a_n, r_n).

Proposition 3.4.1. *Suppose T is a weak tangent of the visual sphere $(\mathbb{S}^2, \rho)$ with associated data (a_n, r_n). Then we can find a subsequence, indexed by n_k, such that for each $m \in \mathbb{N}_0$, the limit*

$$\lim_{k\to\infty} f^{r_{n_k}-m}(a_{n_k})$$

exists. Moreover, if $x_m := \lim_{k\to\infty} f^{r_{n_k}-m}(a_{n_k})$, then $\{x_m\}_{n\in\mathbb{N}_0} \in \mathcal{S}(f)$.

Proof. Since $(\mathbb{S}^2, \rho)$ is sequentially compact, for each $m \in \mathbb{N}_0$, the sequence $f^{r_n-m}(a_n)$ subconverges in $(\mathbb{S}^2, \rho)$. By diagonalization, we can replace (a_n, r_n) by a subsequence, indexed by n_k, such that for all m, the limit $\lim_{k\to\infty} f^{r_{n_k}-m}(a_{n_k})$ exists. By continuity of f,

$$\begin{aligned}
f(x_{m+1}) &= f(\lim_{k\to\infty} f^{r_{n_k}-(m+1)}(a_{n_k})) \\
&= \lim_{k\to\infty} f(f^{r_{n_k}-(m+1)}(a_{n_k})) \\
&= \lim_{k\to\infty} f^{r_{n_k}-m}(a_{n_k}) \\
&= x_m.
\end{aligned}$$

Therefore $\{x_m\}_{m\in\mathbb{N}_0} \in \mathcal{S}(f)$. $\square$

We say that the point $\{x_m\}_{m\in\mathbb{N}_0} \in \mathcal{S}(f)$ *represents* (a_n, r_n), or $\{x_m\}_{m\in\mathbb{N}_0}$ *is a representative of* (a_n, r_n), if for all $m \in \mathbb{N}_0$, we have

$$x_m = \lim_{n\to\infty} f^{r_n - m}(a_n).$$

Thus (a_n, r_n) may not always have a representative, but by Proposition 3.4.1, there exists a subsequence (a_{n_k}, r_{n_k}) of (a_n, r_n) such that (a_{n_k}, r_{n_k}) has a representative.

The following proposition shows that any point $\{x_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{S}(f)$ represents (x_n, n).

Proposition 3.4.2. *Let* $\{x_n\}_{n\in\mathbb{N}_0} \in \mathcal{S}(f)$. *Then* $x_n = \lim_{k\to\infty} f^{k-n}(x_k)$. *In other words,* $\{x_n\}_{n\in\mathbb{N}_0}$ *represents* (x_n, n).

Proof. For each $n \in \mathbb{N}_0$ and each $k \in \mathbb{N}_0$ such that $k > n$, we have

$$f^{k-n}(x_k) = x_{k-(k-n)} = x_n$$

$\square$

Theorem (Theorem 1.0.7)**.** Let (T, a, d) be the weak tangent of $(\mathbb{S}^2, \rho)$ with associated data (a_n, r_n), and suppose $\{x_n\}_{n\in\mathbb{N}_0} \in \mathcal{S}(f)$ represents (a_n, r_n). Let L be the leaf of $\mathcal{S}(f)$ containing $\{x_n\}_{n\in\mathbb{N}_0}$. Then the following statements hold.

(i) There exists a branched covering $\pi : T \to L$.

(ii) If $\sup_{n\in\mathbb{N}_0} \Lambda^n \rho(x_n, \mathrm{post}(f)) = \infty$, then the map π is an isometry.

(iii) If $p = \{p_n\}_{n\in\mathbb{N}_0} \in L$ is a periodic sequence of postcritical points, then p is the only possible branched locus of π, and $\pi^{-1}(p)$ has exactly one point.

(iv) Let $b = \pi^{-1}(p)$. Then there exists $n_0 \in \mathbb{N}$ such that $\deg(\pi, b) = \deg(f^{n_0})$.

(v) There exists $q_0 \in f^{-n_0}(p_0)$ such that for all $R > 0$ there exists $k \in \mathbb{N}_0$ and an isometry $\imath : B_T(q, R) \to B_{\rho\Lambda^{n_0+k}}(q_0, R)$ such that the following diagram commutes:

$$\begin{tikzcd}
B_{(T,d_T)}(b, R) \arrow[r, "\pi"] \arrow[d, "\imath"] & B_{(L,d_L)}(p, R) \arrow[d, "\pi_k"] \\
B_{(\mathbb{S}^2,\Lambda^{k+n_0}\rho)}(q_0, R) \arrow[r, "f^{n_0}"'] & B_{(\mathbb{S}^2,\Lambda^k\rho)}(p_0, R).
\end{tikzcd}$$

Theorem 1.0.7 is, in some sense, a converse of Theorem 1.0.6.

3.5 The Proof of Theorem 1.0.7

This section is devoted to the proof of Theorem 1.0.7.

Lemma 3.5.1. *Let (T, a, d) be a weak tangent of $(\mathbb{S}^2, \rho)$ with associated data (a_n, r_n). Let $\{b_n\}_{n\in\mathbb{N}_0}$ be a sequence of points in $\mathbb{S}^2$ such that*

$$\lim_{n\to\infty} \Lambda^{r_n}\rho(a_n, b_n) < \infty.$$

Then there exists $b \in T$ such that $d(a, b) = \lim_{n\to\infty} \Lambda^{r_n}\rho(a_n, b_n)$ and (T, b) is a weak tangent of the visual sphere $(\mathbb{S}^2, \rho)$ with associated data (b_{n_k}, r_{n_k}), where (b_{n_k}, r_{n_k}) is a subsequence of (b_n, r_n).

Proof. Without loss of generality we may assume that for all $n \in \mathbb{N}_0$ there exists a $\frac{1}{n}$-rough embedding $\varphi_n : B_{\Lambda^{r_n}\rho}(a_n, n) \to T$ such that $\varphi_n(a_n) = a$ and $\varphi_n(B_{\Lambda^{r_n}\rho}(a_n, n))$ is $\frac{1}{n}$-dense in $B_T(a, n)$.

For all large n, $\Lambda^{r_n}\rho(a_n, b_n) < n$, and hence $\varphi_n(b_n)$ is well-defined. Moreover, we have

$$\Lambda^{r_n}\rho(a_n, b_n) - \frac{1}{n} \leq d(a, \varphi_n(b_n)) \leq \Lambda^{r_n}\rho(a_n, b_n) + \frac{1}{n}.$$

As (T, d) is a complete doubling metric space, the sequence $\{\varphi_n(b_n)\}$ has a subsequence $\{\varphi_{n_k}(b_{n_k})\}$ such that $\{\varphi_{n_k}(b_{n_k})\}$ converges to some point $b \in T$ whose distance from a is

$$d(a, b) = \lim_{k\to\infty} \Lambda^{r_{n_k}}\rho(a_{n_k}, b_{n_k}).$$

We will assume that $n_k = n$ and that $d(\varphi_n(b_n), b) < \frac{1}{n}$.

For each $n \in \mathbb{N}_0$, define

$$\psi_n : B_{\Lambda^{r_n}\rho}(b_n, n - \Lambda^{r_n}\rho(a_n, b_n)) \to T$$

by $\psi_n(b_n) = b$ and $\psi_n = \varphi_n$ on $B_{\Lambda^{r_n}\rho}(b_n, n - \Lambda^{r_n}\rho(a_n, b_n))\setminus\{b\}$. Since each φ_n is a $\frac{1}{n}$-rough isometry, it follows that each ψ_n is a $\frac{2}{n}$-rough isometry. As $n \to \infty$, $n - \Lambda^{r_n}\rho(a_n, b_n) \to +\infty$. We conclude that (T, b) is the weak tangent with associated data (b_{n_k}, r_{n_k}), where (b_{n_k}, r_{n_k}) is a subsequence of (b_n, r_n). $\square$

Lemma 3.5.2. *Let (T, a, d) be a weak tangent of $(\mathbb{S}^2, \rho)$ with associated data (a_n, r_n). Suppose $\{x_n\}_{n\in\mathbb{N}_0} \in \mathcal{S}(f)$ represents (a_n, r_n). Then for each $k \in \mathbb{N}$ there exists $b_k \in f^{-r_k}(x_0)$ such that*

$$\lim_{k\to\infty} \Lambda^{r_k}\rho(b_k, a_k) = 0,$$

and for all $n \in \mathbb{N}_0$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $f^{r_k-n}(b_k) = x_n$.

Proof. For all k, we can choose a path γ_k that joins x_0 and $f^{r_k}(a_k)$. Since $x_0 = \lim_{k\to\infty} f^{r_k}(a_k)$, we may assume that γ_k lies within $B(x_0, \rho(x_0, 2f^{r_k}(a_k))$. Any lift of γ_k by f^{r_k} that lifts $f^{r_k}(a_k)$ to a_k will have another endpoint $b_k \in f^{-r_k}(x_0)$.

Let $r = c'\rho(x_0, \operatorname{post}(f)\backslash\{x_0\})$, where c' is the constant defined in Lemma 3.2.5. Since $x_0 = \lim_{k\to\infty} f^{r_k}(a_k)$, for all sufficiently large n, $\rho(x_0, f^{r_k}(a_k)) < c'r/2$, and the path γ_k lies inside $B(x_0, c'\rho(x_0, r))$. By Lemma 3.2.5,

$$\rho(b_k, a_k) = \Lambda^{-r_k}\rho(x_0, f^{r_k}(a_k)).$$

Therefore

$$\lim_{k\to\infty} \Lambda^{r_k}\rho(b_k, a_k) = \lim_{k\to\infty} \rho(x_0, f^{r_k}(a_k)) = 0.$$

Moreover, for each $n \in \mathbb{N}_0$, Lemma 3.2.5 gives

$$\lim_{k\to\infty} \rho(f^{r_k-n}(b_k), f^{r_k-n}(a_k)) = \Lambda^{-n}\rho(f^{r_k}(b_k), f^{r_k}(a_k)) = \Lambda^{-n}\rho(x_0, f^{r_k}(a_k)) = 0.$$

Therefore

$$\lim_{k\to\infty} f^{r_k-n}(b_k) = \lim_{k\to\infty} f^{r_k-n}(a_k) = x_n.$$

But for every $k \in \mathbb{N}_0$ with $r_k \geq n$, $f^{r_k-n}(b_k) \in f^{-n}(x_0)$. Since $|f^{-n}(x_0)| < \infty$, we must be able to find $K \in \mathbb{N}_0$ such that whenever $k \geq K$, $f^{r_k-n}(b_k) = x_n$. $\square$

Lemma 3.5.3. *Let (T, a) be the weak tangent of $(\mathbb{S}^2, \rho)$ with associated data (a_n, r_n), and let $\{x_n\}_{n\in\mathbb{N}_0} \in \mathcal{S}(f)$ be the representative of (a_n, r_n). Suppose $\{x_n\}_{n\in\mathbb{N}_0} \sim \{p\}_{n\in\mathbb{N}_0}$ for some $p \in \operatorname{post}(f)$ such that $f(p) = p$. Then there exists $b \in T$, $q \in \operatorname{crit}(f) \cup \operatorname{post}(f)$, and a sequence of natural numbers s_k tending to $+\infty$ such that $(\mathbb{S}^2, q, \Lambda^{s_k}\rho)$ converges in pointed-Gromov-Hausdorff sense to (T, b).*

Proof. Let $\{b_k\}_{k\in\mathbb{N}_0}$ be a sequence given by Lemma 3.5.2. Then $f^{r_k}(b_k) = x_0$ for all $k \in \mathbb{N}_0$. Let $\gamma = \{\gamma_n\}_{n\in\mathbb{N}_0} : [0,1] \to \mathcal{S}(f)$ be a path in $\mathcal{S}(f)$ such that $\gamma(0) = \{x_n\}_{n\in\mathbb{N}_0}$ and $\gamma(1) = \{p\}_{n\in\mathbb{N}_0}$. For each $k \in \mathbb{N}_0$, let n_k be the largest integer such that $0 \le n_k \le r_k$ and

$$f^{r_k - n_k}(b_k) = x_{n_k}.$$

Let γ^k be a lift of γ_{n_k} by $f^{r_k-n_k}$ such that $\gamma^k(0) = b_k$, and set $\gamma^k(1) = b'_k$. Then $f^{r_k}(b'_k) = p$. We verify that

$$\sup_{k\in\mathbb{N}_0} \Lambda^{r_k}\rho(a_k, b'_k) \le \sup_{k\in\mathbb{N}_0} \Lambda^{r_k}\rho(a_k, b_k) + \sup_{k\in\mathbb{N}_0} \Lambda^{r_k}\rho(b_k, b'_k) \le 0 + \sup_{k\in\mathbb{N}_0} \Lambda^{r_k}\operatorname{diam}(\gamma^k) < \infty.$$

Here we use Lemma 8.9 of [BM17] for the last inequality. By Lemma 3.5.1, there exists $b \in T$ such that a subsequence of $(\mathbb{S}^2, b'_k, \Lambda^{r_k}\rho)$ converges in pointed-Gromov-Hausdorff sense to (T, b). For the rest of the proof, we will assume that $(\mathbb{S}^2, b'_k, \Lambda^{r_k}\rho)$ converges in pointed-Gromov-Hausdorff sense to (T, b)

Let s_k be the largest possible integer such that $s_k \le r_k$ and $f^{r_k - s_k}(b'_k) \in \operatorname{crit}(f) \cup \operatorname{post}(f)$. Since $\operatorname{crit}(f) \cup \operatorname{post}(f)$ is a finite set, we may assume by taking a subsequence of $\{(a_k, r_k)\}$ that there exists $q \in \operatorname{crit}(f) \cup \operatorname{post}(f)$ such that $f^{r_k - s_k}(b'_k) = q$ for all $k \in \mathbb{N}_0$. We claim that $\lim_{k\to\infty} s_k = +\infty$. To prove out claim, let $N \in \mathbb{N}$ be arbitrary. By Lemma 3.5.2 there exists $K \in \mathbb{N}_0$ such that for all $k \ge K$,

$$f^{r_k - n}(b_k) = x_N.$$

Therefore $n_k \ge n$. We have

$$f^{r_k - N}(b'_k) = f^{n_k - N}(f^{r_k - n_k}(b'_k)) = f^{n_k - N}(p) = p \in \operatorname{crit}(f) \cup \operatorname{post}(f).$$

This implies that $r_k - N \le s_k$. Since $\lim_{r_k\to\infty}$, there exists $K_1 \ge K$ such that whenever $k \ge K_1$, we have $s_k \ge N$. This proves our claim.

It remains to prove that $(\mathbb{S}^2, q, \Lambda^{s_k}\rho)$ subconverges to (T, b). If $s_k < r_k$ for only finitely many $k \in \mathbb{N}$, then $b'_k = q$ and $(\mathbb{S}^2, q, \Lambda^{s_k}\rho) = (\mathbb{S}^2, b'_k, \Lambda^{r_k}\rho)$ along a subsequence so that they have the same pointed-Gromov-Hausdorff limit (T, b). If instead $s_k < r_k$ for infinitely many $k \in \mathbb{N}$, then by taking a subsequence we may assume that $s_k < r_k$ for all k, and that $\deg_{b_k}(f^{r_k - s_k}) = 1$.

By Proposition 3.1.3, there exists $R_0 > 0$ such that for all $r < R_0$ and $k \in \mathbb{N}_0$, the map

$$f^{r_k - s_k} : B_\rho(b'_k, \Lambda^{-(r_k - s_k)} r) \to B_\rho(q, r)$$

is a scaling map with scaling factor of $\Lambda^{r_k - s_k}$. Fix $R > 0$. Since $\lim_{k\to\infty} s_k = +\infty$, there exists $K \in \mathbb{N}_0$ such that for all $k \geq K$, we have $R < \Lambda^{s_k} \min\{R_0, R_1\}$, and therefore the map

$$f^{r_k - s_k} : B_{\Lambda^{r_k}\rho}(b'_k, R) \to B_{\Lambda^{s_k}\rho}(q, R)$$

is an isometry. This induces an isometry

$$B_T(b, R) \to B_{\Lambda^{s_k}\rho}(q, R).$$

Thus $(\mathbb{S}^2, q, \Lambda^{s_k}\rho)$ subconverges to (T, b). $\square$

We are ready to prove Theorem 1.0.7.

Proof of Theorem 1.0.7. We divide the proof into two cases:

Case 1: $\sup_{n\in\mathbb{N}_0} \Lambda^n \rho(x_n, \operatorname{post}(f)) = \infty$.

Fix $R > 0$. By Lemma 3.3.4, there exists $N \in \mathbb{N}_0$ such that for every $n \geq N$, the map

$$f : B_\rho(x_{n+1}, \Lambda^{-(n+1)} R) \to B_\rho(x_n, \Lambda^{-(n)} R)$$

is an isometry.

Take $\{b_k\}$ as defined in Lemma 3.5.2. Then there exists $K' > N$ such that for every $k \geq K'$, $f^{r_k - N}(b_k) = x_N$. The map

$$f^{r_k - N} : B_{\Lambda^{r_k}\rho}(b_k, R) \to B_{\Lambda^N\rho}(x_N, R)$$

is an isometry sending b_k to x_N. This induces an isometry

$$\pi : B_T(b, R) \to B_L(x, R)$$

where (T, b, d_T) is the limit of $(\mathbb{S}^2, b_k, \Lambda^{r_k}\rho)$. This holds for all $R > 0$. Thus we get an isometry

$$\pi : T \to L.$$

Case 2: $\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,\mathrm{post}(f))<\infty$.

By Lemma 3.3.2, there exists a periodic sequence $\{p_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{S}(f)$ such that $p_n\in\mathrm{post}(f)$ for each $n\in\mathbb{N}_0$ and $\sup_{n\in\mathbb{N}_0}\Lambda^n\rho(x_n,p_n)<\infty$. Let n_0 be the period of $\{p_n\}_{n\in\mathbb{N}_0}$, and let $g=f^{n_0}$. By Lemma 3.5.3, there exist $b\in T$, $q_0\in\mathrm{crit}(g)\cup\mathrm{post}(g)$ and a sequence s_k of positive integers tending to $+\infty$ such that $g(q_0)=p_0$ and (T,b,d_T) is the weak tangent with associated data (q_0,s_k). Let $R>0$. By Proposition 3.3.4, there exists $k\in\mathbb{N}_0$ such that the map

$$g: B_{\Lambda^k\rho}(p_0,R)\to B_{\Lambda^{k-1}\rho}(p_0,R)$$

is an isometry and there exists an isometry

$$g': B_{\Lambda^{n_0(k+1)}\rho}(q,R)\to B_{\Lambda^{n_0k}\rho}(q,R)$$

such that the following diagram commutes:

$$\begin{tikzcd}
B_{\Lambda^{n_0(k+1)}\rho}(q_0,R) \arrow[r, "g'"] \arrow[d, "g"'] & B_{\Lambda^{n_0k}\rho}(q_0,R) \arrow[d, "g"] \\
B_{\Lambda^{n_0k}\rho}(p_0,R) \arrow[r, "g"'] & B_{\Lambda^{n_0(k-1)}\rho}(p_0,R).
\end{tikzcd}$$

From the above commutative diagram, and by the definitions of (T,c) and (L,a'), we have

$$\begin{tikzcd}
 & & B_T(b,R) \arrow[dl, "\imath_{k+3}"'] \arrow[d, "\imath_{k+2}"'] \arrow[dr, "\imath_{k+1}"] & \\
\cdots \arrow[r, "g'"] & B_{\Lambda^{n_0(k+3)}\rho}(q_0,R) \arrow[r, "g'"] \arrow[d, "g"'] & B_{\Lambda^{n_0(k+2)}\rho}(q_0,R) \arrow[r, "g'"] \arrow[d, "g"] & B_{\Lambda^{n_0(k+1)}\rho}(q_0,R) \arrow[d, "g"] \\
\cdots \arrow[r, "g"] & B_{\Lambda^{n_0(k+2)}\rho}(p_0,R) \arrow[r, "g"] & B_{\Lambda^{n_0(k+1)}\rho}(p_0,R) \arrow[r, "g"] & B_{\Lambda^{n_0k}\rho}(p_0,R). \\
 & & B_L(p,R) \arrow[ul, "\pi_{n_0(k+2)}"] \arrow[u, "\pi_{n_0(k+1)}"] \arrow[ur, "\pi_{n_0k}"'] &
\end{tikzcd}$$

Here $p=\{p_n\}_{n\in\mathbb{N}_0}\in\mathcal{S}(f)$ is the constant sequence of postcritical points, and $\imath_{k+1},\imath_{k+2},\dots$ are isometries. The two open balls $B_T(b,R)$ and $B_L(p,R)$ are inverse limit of the two inverse systems as above, and the arrows pointing to $B_T(a,R)$ and $B_L(p,R)$ are all bijective maps. By the universal property of inverse limits, we obtain a branched covering map

$$\pi_R: B_T(b,R)\to B_L(p,R)$$

The only possible locus $\pi|_R$ is p, and $b \in T$ is the only point in $\pi^{-1}(p)$.

Since $\pi|R_1 = \pi_{R_2}$ on $B_T(b, \min\{R_1, R_2\})$, we get a branched covering $\pi : T \to L$ as required in (i). The point $p \in L$ is the only possible branched locus, and $\pi^{-1}(b)$ has exactly 1 point, yielding (ii). The local degree of π on $\pi^{-1}(b)$ is the local degree of g at q_0, yielding (iv). Taking $\imath = \imath_{n_0(k+1)}$, we get the communicative diagram in (v). $\square$

3.6 Tiles on Leafs and Weak Tangents

In this section we will define the notion of tiles on leafs and weak tangents, using the branched covering maps we obtained in Theorem 3.3.8 and Theorem 1.0.7.

Let L be a leaf of $\mathcal{S}(f)$. We extend the inverse system (3.3.1) as follows:

$$\begin{tikzcd} & & \mathcal{S}(f) \arrow[dl, "\pi_1"'] \arrow[d, "\pi_0"] \arrow[dr, "\pi_{-1}"] & & \\ \cdots \arrow[r, "f"] & \mathbb{S}^2 \arrow[r, "f"] & \mathbb{S}^2 \arrow[r, "f"] & \mathbb{S}^2 \arrow[r, "f"] & \cdots, \end{tikzcd} \tag{3.6.1}$$

where the maps π_i for $i \in \mathbb{Z}$ are the natural projection maps from $\mathcal{S}(f)$ to $\mathbb{S}^2$. By Theorem 3.3.8, for every non-negative integer i, the maps $\pi_i : L \to \mathbb{S}^2$, restricted to L, is a branched covering map. When i is a negative integer, then $\pi_i : L \to \mathbb{S}^2$ is the composition of branched covering maps, hence it is a branched covering map. Moreover, for every $i \in \mathbb{N}_0$, all the branched loci of π_i are contained in $\operatorname{post}(f)$.

Let $k \in \mathbb{Z}$, and let $a \in L \backslash \pi_{-k}^{-1}(\operatorname{post}(f))$ be a point. Suppose $\pi_{-k}(a)$ is a point in the 0-tile $\tau \in \mathcal{D}_0(\mathbb{S}^2)$. Then there exists a unique lift σ of τ by π_{-k} such that $a \in \sigma$. More precisely, since the map $\pi_{-k} : L \backslash \pi_k^{-1}(\operatorname{post}(f)) \to \mathbb{S}^2 \backslash \operatorname{post}(f)$ is a covering map, there exists a unique lift σ' of $\tau \backslash \operatorname{post}(f)$ by π_{-k} such that $a \in \sigma'$, and σ is the closure of σ'. We call σ a *k-tile of L*, and we say that k is the *level of σ*. We also call σ a *tile of L* when we do not wish to specify the level of L.

For $k \in \mathbb{Z}$, let $\mathcal{D}_k(L)$ be the collection of k-tiles of L. Let $\mathcal{D}(L) = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k(L)$ be the collection of tiles in L.

Proposition 3.6.1. *Let $k, l \in \mathbb{Z}$, and suppose $k + l \geq 0$. Then for all $\sigma \in \mathcal{D}_k(L)$, $\pi_l(\sigma)$ is a*

$(k+l)$-tile in $\mathbb{S}^2$.

Proof. Let $\tau = \pi_l(\sigma)$. Then $f^{(k+l)}(\tau) = f^{(k+l)}(\pi_l(\sigma)) = \pi_{-k}(\sigma)$ is a 0-tile. The map $\pi_{-k} : \sigma \to \pi_{-k}(\sigma)$ is injective, and $\pi_{-k}(\partial\sigma) = \partial(\pi_{-k}(\sigma)) = \mathcal{C}$ is the f-invariant Jordan curve, therefore $\tau \cap f^{-k}(\mathcal{C}) = \partial\tau$. We conclude that τ is a $(k+l)$-tile. $\square$

Proposition 3.6.2. *For all $k \in \mathbb{Z}$ and $\sigma \in \mathcal{D}_k(L)$, we have*

$$\operatorname{diam}(\sigma) \sim \Lambda^{-k},$$

where the implicit constant of $\sim$ is independent of σ or k.

Proof. For each $k \in \mathbb{Z}$, it follows from Proposition 3.2.2 and the definition of the metric d_L on L that

$$\pi_k|_L : L \to \mathbb{S}^2.$$

is a Λ^{-k}-lipschits map. Let $\sigma \in \mathcal{D}_k(L)$ be arbitrary. We have

$$\operatorname{diam}(\pi_{-k}(\sigma)) \leq \Lambda^k \operatorname{diam}(\sigma).$$

But $\pi_{-k}(\sigma)$ is a 0-tile. This proves

$$\operatorname{diam}(\sigma) \geq \Lambda^{-k}.$$

To prove the other inequality, let $x = \{x_n\}_{n\in\mathbb{N}_0}$ and $y = \{y_n\}_{n\in\mathbb{N}_0}$ be two points in σ. Let $l \in \mathbb{N}_0$ be a non-negative integer such that $k + l \geq 0$. Then $\pi_l(\sigma)$ is a $(k+l)$-tile in $\mathbb{S}^2$, therefore

$$\rho(x_l, y_l) \sim \Lambda^{-(k+l)},$$

where the similarity constant does not depend on l, k, x, y. For $l' > l$, we have the same inequality

$$\rho(x_{l'}, y_{l'}) \sim \Lambda^{-(k+l')},$$

with the same implicit constants. For $l < l'$, we have

$$\rho(x_{l'}, y_{l'}) \leq \Lambda^{l-l'} \rho(x_l, y'_l) \sim \Lambda^{-(k+l')}.$$

Therefore

$$d_L(\{x_n\}_{n\in\mathbb{N}_0}, \{y_n\}_{n\in\mathbb{N}_0}) = \sup_{n\in\mathbb{N}_0} \Lambda^n \rho(x_n, y_n) \sim \Lambda^{-k}.$$

This completes our proof. □

Let (T, x) be a weak tangent of $\mathbb{S}^2$. We can similarly define the notion of tiles in T by considering the branched covering map $\pi : T \to L$, where L is a suitable leaf of $\mathcal{S}(f)$, and π is the branched covering map in Theorem 1.0.7. If $\tau \in \mathcal{D}_k(L)$ is a tile in L, and $a \in T$ is a point such that $\pi(a)$ is not a postcritical point, then there exists a unique lift σ of τ into T by π such that $a \in \sigma$. We call σ a *k-tile of T* if τ is a k-tile of $\mathbb{S}^2$. Let $\mathcal{D}_k(T)$ be the collection of k-tiles in T and $\mathcal{D}(T) = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k(T)$ be the collection of tiles in T.

Proposition 3.6.3. *For all $k \in \mathbb{Z}$ and $\tau \in \mathcal{D}_k(L)$, we have*

$$\operatorname{diam}(\tau) \sim \Lambda^{-k},$$

where the implicit constant of $\sim$ is independent of τ or k.

Proof. This follows from Proposition 3.6.2 and Theorem 1.0.7(ii) and (v). □

CHAPTER 4

Ahlfors Regular Conformal Dimension of Visual Spheres and Their Weak Tangents

4.1 Introduction

The goal of this chapter is to prove the following theorem:

Theorem (Theorem 1.0.8)**.** Let f be an expanding Thurston map f without periodic critical points. Let $(\mathbb{S}^2, \rho)$ be a visual sphere of f. Let (T, x) be a weak tangent of $(\mathbb{S}^2, \rho)$. Then

$$\dim_{\mathrm{AR}}(\mathbb{S}^2, \rho) = \dim_{\mathrm{AR}}(T),$$

and $\dim_{\mathrm{AR}}(\mathbb{S}^2, \rho)$ is attainable if and only if $\dim_{\mathrm{AR}}(T)$ is attainable.

We will offer two proofs for the if part. The second proof is shorter, but the first proof illustrates the use of gauge functions. Most of the chapter will be devoted to the first proof. In the last section, we discuss the second proof.

We restate Theorem 1.0.8 to provide a different perspective:

Theorem (Theorem 1.0.8, alternative formulation)**.** Let f be an expanding Thurston map f without periodic critical points. Let $(\mathbb{S}^2, \rho)$ be a visual sphere of f. Let (T, x) be a weak tangent of $(\mathbb{S}^2, \rho)$. For any $Q > 0$, $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to a Q-Ahlfors regular space if and only if T is quasisymmetrically equivalent to a Q-Ahlfors regular space.

One implication of Theorem 1.0.8 holds in greater generality.

Theorem 4.1.1. *Let X be a complete doubling metric space, and let T be a weak tangent of X. If X is quasisymmetrically equivalent to a Q-Ahlfors regular space, then T is quasisymmetrically equivalent to a Q-Ahlfors regular space.*

See [MT10, Section 6.1] for a proof of Theorem 4.1.1. The converse of Theorem 4.1.1 does not hold, as we will see in the next chapter.

To prove the "if" implication of Theorem 1.0.8, suppose T is quasisymmetrically equivalent to a Q-AR space T_Q. We first prove that every weak tangent T of a visual sphere $(\mathbb{S}^2, \rho)$ is a branched covering of $(\mathbb{S}^2, \rho)$. Then we make use of the branched covering $\pi : T \to \mathbb{S}^2$ and the quasisymmetry between T and T_Q to construct a Q-Ahlfors regular metric on $\mathbb{S}^2$.

Note that the above statement is independent of the choice of visual metric ρ since any two visual metrics are quasisymmetrically equivalent to each other, and if X and Y are two quasisymmetrically equivalent metric spaces, then any weak tangent T_X of X is quasisymmetrically equivalent to some weak tangent T_Y of Y. Thus we will only prove under a particular visual metric, which we constructed in Section 3.2. This visual metric simplifies our proofs because it gives rise to isometries instead of bilipschitz maps.

In Section 4.2, we define a notion of gauge functions on visual spheres. Gauge functions are used to construct an new metric on a metric space so that the new metric is quasisymmetrically equivalent to the old one. In Section 4.3, we show how we can construct a gauge function on $\mathbb{S}^2$ from a gauge function on a weak tangent T of the visual sphere. Section 4.4 contains the proof Theorem 1.0.8 based on the material built up in Section 4.2 and Section 4.3. We also mention an application of Theorem 1.0.8. In Section 4.5, we give an alternative proof of Theorem 1.0.8. The second proof made use of a quasisymmetric gluing result, proven in [Haï09], and has the potential to be generalized.

4.2 Gauge Functions on Visual Spheres

In this section, we introduce a tool to construct a Q-Ahlfors regular metric on the visual sphere $(\mathbb{S}^2, \rho)$. Our tool, Proposition 4.2.2, is a modification of Theorem 1.1 of [CP13]. See also [KL04, Proposition 5.1.1] and [Kig18] for uses of gauge functions in the study of Ahlfors regular conformal dimension. Roughly speaking, instead of constructing a new metric on $\mathbb{S}^2$ directly, we assign each tile a new diameter. If the new diameters of the tiles are well-selected, then we show that there exists a metric on $\mathbb{S}^2$ realizing those new diameters.

Definition 4.2.1. *Let $Q > 1$ be a constant. A function $g : \mathcal{D}(\mathbb{S}^2) \to [0, \infty)$ is called a* Q-gauge function *on $(\mathbb{S}^2, \rho)$ if g satisfies the following conditions:*

(i) There exists $0 < \eta_1 < \eta_2 < 1$ such that for all $n \in \mathbb{N}_0$, $\tau \in \mathcal{D}_n(\mathbb{S}^2)$ and $\tau' \in \mathcal{D}_{n+1}(\mathbb{S}^2)$ with $\tau' \subset \tau$.

$$\eta_1 \leq \frac{g(\tau')}{g(\tau)} \leq \eta_2,$$

(ii) There exists a constant $K_0 \geq 1$ such that for all $n \in \mathbb{N}_0$ and for all $\tau', \tau \in \mathcal{D}_n(\mathbb{S}^2)$ with $\tau \cap \tau' \neq \emptyset$,

$$\frac{g(\tau')}{g(\tau)} \leq K_0,$$

(iii) There exists $K_1 > 0$ such that for all $x, y \in \mathbb{S}^2$, for all $\tau \in \mathcal{D}_{m(x,y)}(\mathbb{S}^2)$ such that $x \in \tau$ or $y \in \tau$, and for all tile chains $\gamma = \{\tau_1, \tau_2, \ldots, \tau_n\}$ joining x and y, we have

$$\ell_g(\gamma) := \sum_{i=1}^{n} g(\tau_i) \geq K_1^{-1} g(\tau).$$

(iv) There exists a constant $K_2 \geq 1$ such that for all $n \in \mathbb{N}_0$, for all $\tau \in \mathcal{D}_n(\mathbb{S}^2)$, and for all $m \geq n$, we have

$$K_2^{-1} g(\tau)^Q \leq \sum_{\tau' \in \mathcal{D}_m(\mathbb{S}^2), \tau' \cap \tau \neq \emptyset} g(\tau')^Q \leq K_2 g(\tau)^Q.$$

(v) We have $g(\tau) \neq 0$ for all $\tau \in \mathcal{D}_0(\mathbb{S}^2)$.

Proposition 4.2.2. *Let $g : \mathcal{D}(\mathbb{S}^2) \to [0, \infty)$ be a Q-gauge function on $\mathbb{S}^2$. For $x, y \in \mathbb{S}^2$, define*

$$q(x, y) = \inf\{\ell_g(\gamma) : \gamma \text{ tile chain joining } x \text{ and } y\}.$$

Then q is a Q-Ahlfors regular metric on $\mathbb{S}^2$, and the identity map $\mathrm{id} : (\mathbb{S}^2, \rho) \to (\mathbb{S}^2, q)$ *is a quasisymmetry.*

The proof of Proposition 4.2.2 will be broken up into Proposition 4.2.3, Proposition 4.2.4, and Proposition 4.2.10.

Proposition 4.2.3. *The function q defined in Proposition 4.2.2 is a metric.*

Proof. We have $q(x,x) = 0$. If $x \neq y$, then condition (iii) implies that

$$q(x,y) \geq K_1^{-1} g(\tau),$$

where τ is any $m(x,y)$-tile containing x or y. By condition (i) and (v), we have

$$q(x,y) \geq K_1^{-1} g(\tau) \geq K_1^{-1} \eta_1^{m(x,y)} \min_{\tau \in \mathbb{D}_0} g(\tau) > 0. \tag{4.2.1}$$

Therefore q is positive definite.

The symmetry of q follows from the fact that any tile chain $\gamma = \{\tau_1, \tau_2, \ldots, \tau_n\}$ joining x and y corresponds to the tile chain $\gamma' = \{\tau_n, \tau_{n-1}, \ldots, \tau_1\}$ joining y and x, and $\ell_g(\gamma) = \ell_g(\gamma')$.

If $x, y, z \in \mathbb{S}^2$ are three points, γ is tile chain joining x and y, and γ' is a tile chain joining y and z, then $\gamma \cup \gamma'$ is a tile chain joining x and z. We have

$$q(x,z) \leq \ell_g(\gamma \cup \gamma') = \ell_g(\gamma) + \ell_g(\gamma').$$

Infimizing over all possible choice of γ and γ', we get

$$q(x,z) \leq q(x,y) + q(y,z).$$

Thus q is a metric on $\mathbb{S}^2$. $\square$

Proposition 4.2.4. *The identity map* $\mathrm{id} : (\mathbb{S}^2, \rho) \to (\mathbb{S}^2, q)$ *is a quasisymmetry.*

Proof. For all $x, y \in \mathbb{S}^2$, and all $\tau \in \mathcal{D}_{m(x,y)}(\mathbb{S}^2)$ such that $x \in \tau$, there exists $\tau' \in \mathcal{D}_{m(x,y)}(\mathbb{S}^2)$ such that $y \in \tau'$ and $\tau \cap \tau' \neq \emptyset$. The tile chain $\{\tau, \tau'\}$ joins x and y, therefore

$$q(x,y) \leq \ell_g(\{\tau, \tau'\}) = g(\tau) + g(\tau') \leq g(\tau) + K_0 g(\tau). \tag{4.2.2}$$

Combining (4.2.1) and (4.2.2), we get

$$K_1^{-1} g(\tau) \leq q(x,y) \leq (1 + K_0) g(\tau).$$

For all $x, y, z \in \mathbb{S}^2$ with $x \neq z$, let τ_1 be an $m(x,y)$-tile and τ_1 be an $m(x,z)$-tile such that $x \in \tau_1 \cap \tau_2$. Assume that $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$. If $m(x,y) \geq m(x,z)$, then $\tau_1 \subset \tau_2$, and

$$\begin{aligned}
\frac{q(x,y)}{q(x,z)} &\leq K_1(1 + K_0) \frac{g(\tau_1)}{g(\tau_2)} \\
&\leq K_1(1 + K_0) \eta_2^{(m(x,y) - m(x,z))} \\
&\leq A \frac{\rho(x,y)}{\rho(x,z)}^{\frac{-\log \eta_2}{\log \Lambda}},
\end{aligned}$$

where A is a constant. If $m(x,y) \le m(x,z)$, then $\tau_2 \subset \tau_1$, and

$$\begin{aligned}\frac{q(x,y)}{q(x,z)} &\le K_1(1+K_0)\frac{g(\tau_1)}{g(\tau_2)} \\ &= K_1(1+K_0)\eta_1^{-(m(x,z)-m(x,y))} \\ &\le B\frac{\rho(x,y)}{\rho(x,z)}^{\frac{-\log \eta_1}{\log \Lambda}},\end{aligned}$$

where B is a constant. Thus the map $\mathrm{id} : (\mathbb{S}^2, \rho) \to (\mathbb{S}^2, q)$ is an η-quasisymmetry with

$$\eta(t) = \max\{At^{\frac{-\log \eta_2}{\log \Lambda}}, Bt^{\frac{-\log \eta_1}{\log \Lambda}}\}.$$

$\square$

For each $k \in \mathbb{N}_0$ and $x \in \mathbb{S}^2$, let $T(x,k)$ be the union of all k-tiles τ in $\mathcal{D}(\mathbb{S}^2)$ such that $\tau \cap \tau' \neq \emptyset$ for another $\tau' \in \mathcal{D}_k(\mathbb{S}^2)$ with $x \in \tau'$.

Lemma 4.2.5 (Tiles are quasi-balls)**.** *There exists a constant $K_3 > 0$ such that for all $n \in \mathbb{N}_0$ and $\tau \in \mathcal{D}_n(\mathbb{S}^2)$, there exists $x \in \tau$ such that*

$$B(x, K_3 g(\tau)) \subset \tau \subset B(x, g(\tau)).$$

Proof. There exists $n_0 \in \mathbb{N}_0$ such that for all for all $n \in \mathbb{N}_0$ and $\tau \in \mathcal{D}_n(\mathbb{S}^2)$ there exists $x \in \tau$ such that $T(x, n+n_0) \subset \tau$. If $y \notin T(x, n+n_0)$, then

$$q(x,y) \ge K_1^{-1} g(\tau'),$$

where τ' is another $(n+n_0)$-tile such that $\tau' \subset T(x, n+n_0)$. By Property (ii) of the gauge function g, we have

$$g(\tau') \ge \eta_1^{n_0} g(\tau).$$

Therefore

$$B(x, K_1^{-1}\eta_1^{n_0} g(\tau)) \subset T(x, n+n_0) \subset \tau.$$

Conversely, if $y \in \tau$, then $\{\tau\}$ is a tile chain joining x and y, and we have

$$q(x,y) \le g(\tau).$$

This proves

$$B(x, K_3 g(\tau)) \subset \tau \subset B(x, g(\tau)),$$

where $K_3 = K_1^{-1}\eta_1^{n_0}$. □

Corollary 4.2.6. *For every $\tau \in \mathcal{D}_n(\mathbb{S}^2)$ we have*

$$K_3 g(\tau) \leq \operatorname{diam}(\tau) \leq 2g(\tau),$$

where K_3 is the constant in Lemma 4.2.5.

Lemma 4.2.7. *For any $x \in \mathbb{S}^2$, and $r < \frac{\eta_1 \min\{g(\tau):\tau\in\mathcal{D}_0(\mathbb{S}^2)\}}{K_1}$, there exists $k \in \mathbb{N}$ such that $B(x, r) \subset T(x, k)$ and if τ is a k-tile in $T(x, k)$, then*

$$\frac{K_1}{K_0} r \leq g(\tau) \leq K_4 r$$

where $K_4 = K_0^2 K_1 \eta_1^{-1}$.

Proof. Let k be the largest integer such that

$$g(\tau) \geq K_1 r$$

for all k-tile containing x. Note that for every 1-tile τ, we have

$$g(\tau) \geq \eta_1 \min\{g(\tau) : \tau \in \mathcal{D}_0(\mathbb{S}^2)\} \geq K_1 r.$$

Therefore $k \geq 1$.

For all $y \notin T(x, k)$, $m(x, y) \leq k - 1$, therefore

$$q(x, y) \geq K^{-1} g(\tau) \geq r$$

for any k-tile containing x. This implies $y \notin B(x, r)$. Therefore $B(x, r) \subset T(x, k)$.

IF τ is a k-tile containing x and τ' is another k-tile such that $\tau' \subset T(x, k)$, then

$$g(\tau') \geq K_0^{-1} g(\tau) \geq K_0^{-1} K_1 r.$$

Since k is the largest integer such that $g(\tau) \geq K_1 r$, if τ'' is a $(k+1)$-tile such that $\sigma \subset T(x, k+1)$, and τ' is a k-tile such that $\tau'' \subset \tau'$, and τ is a k-tile containing x, then $\tau \cap \tau' \neq \emptyset$, and we have

$$g(\tau'') \geq \eta_1 g(\tau') \geq \eta_1 K_0^{-1} g(\tau).$$

Therefore

$$g(\tau) \leq K_0 \eta_1^{-1} K_1 r.$$

Finally, if τ''' is another k-tile such that $\tau \subset T(x, k)$, then

$$g(\tau''') \leq K_0 g(\tau) \leq K_0^2 K_1 \eta_1^{-1} r.$$

□

Lemma 4.2.8. *For any $x \in \mathbb{S}^2$, and $r < \frac{\min\{\mathrm{diam}_q(\tau): \tau \in \mathcal{D}_0(\mathbb{S}^2)\}}{2}$, there exists $\tau \in \mathcal{D}(\mathbb{S}^2)$ such that $\tau \subset B(x, r)$ and*

$$\eta_1 r \leq \mathrm{diam}(\tau) \leq r.$$

Proof. If $\tau \in \mathcal{D}(\mathbb{S}^2)$ contains x and $\mathrm{diam}(\tau) < r$, then $\tau \subset B(x, r)$. Let k be the smallest integer such that there exists $\tau \in \mathcal{D}_k(\mathbb{S}^2)$ with $\tau \subset B(x, r)$. By our assumption on r, $k \geq 1$. Moreover, if τ' is a $(k-1)$-tile containing τ, then $\mathrm{diam}(\tau') \geq r$. We have

$$g(\tau) \geq \eta_1 g(\tau') \geq \eta_1 \,\mathrm{diam}(\tau') \geq \eta_1 r.$$

□

Proposition 4.2.9. *Let $\mathcal{H}^Q$ be the Hausdorff Q-measure of* $(\mathbb{S}^2, q)$. *Then*

$$\mathcal{H}^Q(\tau) \sim g(\tau)^Q$$

for all $\tau \in \mathcal{D}(\mathbb{S}^2)$.

Proof. For all $\tau \in \mathcal{D}(\mathbb{S}^2)$ and for all large n, $\{\tau' \in \mathcal{D}_n(\mathbb{S}^2), \tau' \subset \tau\}$ is a cover of τ, therefore

$$\begin{aligned}\mathcal{H}^Q(\tau) &\leq \limsup_{n \to +\infty} \sum_{\tau' \in \mathcal{D}_n(\mathbb{S}^2), \tau' \subset \tau} \mathrm{diam}(\tau')^Q \\ &\leq \limsup_{n \to +\infty} \sum_{\tau' \in . \mathcal{D}_n(\mathbb{S}^2), \tau' \subset \tau} g(\tau')^Q \leq K_2 g(\tau)^Q.\end{aligned}$$

Conversely, fix $\varepsilon > 0$ and suppose that $\{B(x_i, r_i)\}_{i=1}^n$ is an open cover of τ such that

$$\mathcal{H}^Q(\tau) \geq \sum_{i=1}^n r_i^Q - \varepsilon.$$

For each $i = 1, 2, \dots, n$, there exists $k_i \in \mathbb{N}_0$ such that $B(x_i, r_i) \subset T(x, k_i)$ and for each $\tau \in \mathcal{D}_{k_i}(\mathbb{S}^2)$, $g(\tau) \sim r_i$. Thus

$$r_i^Q \sim \sum_{\tau' \in \mathcal{D}_{k_i}(\mathbb{S}^2), \tau' \subset T(x, k_i)} g(\tau')^Q.$$

Let $k_{\max} = \max\{k_1, \dots, k_n\}$. By condition (iv) of the function g,

$$\sum_{\tau' \in \mathcal{D}_{k_i}(\mathbb{S}^2), \tau' \subset T(x, k_i)} g(\tau')^Q \sim \sum_{\tau' \in \mathcal{D}_{k_{\max}}(\mathbb{S}^2), \tau' \subset T(x, k_i)} g(\tau')^Q.$$

Therefore

$$\mathcal{H}^Q(\tau) + \varepsilon \geq \sum_{i=1}^n r_i^Q \gtrsim \sum_{i=1}^n \sum_{\tau' \in \mathcal{D}_{k_{\max}}(\mathbb{S}^2), \tau' \subset T(x, k_i)} g(\tau')^Q.$$

As $\{B(x_i, r_i)\}_{i=1}^n$ covers τ, $\{T(x, k_i)\}_{i=1}^n$ covers τ, and

$$\mathcal{H}^Q(\tau) + \varepsilon \gtrsim \sum_{i=1}^n \sum_{\tau' \in \mathcal{D}_{k_{\max}}(\mathbb{S}^2), \tau' \subset T(x, k_i)} g(\tau')^Q \gtrsim \sum_{\tau' \in \mathcal{D}_{k_{\max}}(\mathbb{S}^2), \tau' \subset \tau} g(\tau')^Q \gtrsim g(\tau)^Q.$$

□

Proposition 4.2.10. *The metric q defined in Proposition 4.2.2 is Q-Ahlfors regular.*

Proof. By Lemma 4.2.8, for every $r < \frac{\min\{\operatorname{diam}_q(\tau) : \tau \in \mathcal{D}_0(\mathbb{S}^2)\}}{2}$, and every $x \in \mathbb{S}^2$, there exists $\tau \in \mathcal{D}(\mathbb{S}^2)$ such that $\tau \subset B(x, r)$ and $\operatorname{diam}(\tau) \geq \eta_1 r$. Applying Proposition 4.2.9, we get

$$\mathcal{H}^Q(B(x, r)) \geq \mathcal{H}^Q(\tau) \sim \operatorname{diam}(\tau)^Q \sim r.$$

where the implicit constant is independent of x and r.

Conversely, by Lemma 4.2.7, for every $r < \frac{\eta_1 \min\{g(\tau) : \tau \in \mathcal{D}_0(\mathbb{S}^2)\}}{K_1}$, and for every $x \in \mathbb{S}^2$, there exists $k \in \mathbb{N}_0$ such that $B(x, r) \subset T(x, k)$, and $g(\tau) \sim r$. Applying Proposition 4.2.9, we get

$$\begin{aligned}
\mathcal{H}^Q(B(x, r)) &\leq \mathcal{H}^Q(T(x, k)) \\
&\leq \sum_{\tau \in \mathcal{D}_k(\mathbb{S}^2), \tau \subset T(x, k)} \mathcal{H}^Q(\tau) \\
&\lesssim |\{\tau \in \mathcal{D}_k(\mathbb{S}^2) : \tau \subset T(x, k)\}| \max\{g(\tau) : \tau \in \mathcal{D}_k(\mathbb{S}^2), \tau \subset T(x, k)\}^Q \\
&\lesssim r^Q.
\end{aligned}$$

Here $\sup_{x,k} |\{\tau \in \mathcal{D}_k(\mathbb{S}^2) : \tau \subset T(x,k)\}| < \infty$ since $\sup_{n\in\mathbb{N}_0} \sup_{x\in\mathbb{S}^2} \deg_x(f^n) < \infty$, and if $x \in f^{-n}(\text{post}(f))$, then the number of n-tiles containing x is equal to $2\deg_x(f^n)$. $\square$

We can similarly define the notion of gauge function on a tile τ of $\mathbb{S}^2$.

Definition 4.2.11. *Let $Q > 1$ be a constant. Let τ be a tile in $\mathbb{S}^2$ and let $\mathcal{D}(\tau) = \{\tau' \in \mathcal{D}(\tau) : \tau' \subset \tau\}$. A function $g : \mathcal{D}(\tau) \to [0,\infty)$ is called a Q-*gauge function* on (τ,ρ) if g satisfies the following conditions:*

(i) there exists $0 < \eta_1 < \eta_2 < 1$ such that for all $n \in \mathbb{N}_0$, $\tau \in \mathcal{D}_n(\tau)$ and $\tau' \in \mathcal{D}_{n+1}(\tau)$ such that $\tau' \subset \tau$, we have
$$\eta_1 \leq \frac{g(\tau')}{g(\tau)} \leq \eta_2,$$

(ii) there exists a constant $K_0 \geq 1$ such that for all $n \in \mathbb{N}_0$ and for all $\tau', \tau \in \mathcal{D}_n(\tau)$ such that $\tau \cap \tau' \neq \emptyset$, we have
$$\frac{g(\tau')}{g(\tau)} \leq K_0,$$

(iii) there exists $K_1 > 0$ such that for all $x, y \in \mathbb{S}^2$, for all $\tau \in \mathcal{D}_{m(x,y)}(\tau)$ such that $x \in \tau$ or $y \in \tau$, and for all tile chain $\gamma = \{\tau_1, \tau_2, \ldots, \tau_n\}$ joining x and y, we have
$$\ell_g(\gamma) := \sum_{i=1}^{n} g(\tau_i) \geq K_1^{-1} g(\tau).$$

(iv) there exists a constant $K_2 \geq 1$ such that for all $n \in \mathbb{N}_0$, for all $\tau \in \mathcal{D}_n(\tau)$, and for all $m \geq n$, we have
$$K_2^{-1} g(\tau)^Q \leq \sum_{\tau' \in \mathcal{D}_m(\tau), \tau' \cap \tau \neq \emptyset} g(\tau')^Q \leq K_2 g(\tau)^Q.$$

(v) $g(\tau) \neq 0$ for all $\tau \in \mathcal{D}_0(\tau)$

Proposition 4.2.12. *Let τ be a tile in $\mathbb{S}^2$. Let $g : \mathcal{D}(\tau) \to [0,\infty)$ be a Q-gauge function on $\mathbb{S}^2$. Then (τ,ρ) is qusisymmetrically equivalent to a Q-Ahlfors regular metric space.*

The proof for Proposition 4.2.2 also works for Proposition 4.2.12.

4.3 Construction of a Gauge Function

Let T be a weak tangent of the visual sphere $(\mathbb{S}^2, \rho)$, let T_Q be a Q-Ahlfors regular metric space, and let $\varphi : T \to T_Q$ be a quasisymmetry. We construct a gauge function on the set $\mathcal{D}(\mathbb{S}^2)$ of tiles on the visual sphere. Let $\pi : T \to \mathbb{S}^2$ be a composition of a branched covering map in Theorem 1.0.7, and π_0 restricted to an appropriate leaf. By Theorem 3.3.8, the map π is a branched covering map.

Fix $x \in \mathcal{C} \setminus \operatorname{post}(f)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be an enumeration of the preimages of x under π in T. Let W be the white 0-tile in $\mathbb{S}^2$, and let B be the black 0-tile in $\mathbb{S}^2$. For each $n \in \mathbb{N}$, let $j_n(W)$ be the unique lift of W by φ in T that contains x_n and let $j_n(B)$ be the unique lift of the B by φ in T that contains x_n. Let

$$K_n = \bigcup_{k\le n} j_n(W) \cup j_n(B).$$

let

$$\partial_n = \#\{\tau : \tau \text{ a 0-tile of } T, \tau \subset K_n, \tau \cap \partial K_n \neq \emptyset\}.$$

Here ∂K_n is the boundary of K_n as a subspace in T. By enumerating x_n appropriately we may assume there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such

$$\lim_{k\to\infty} \frac{\partial_{n_k}}{n_k} = 0.$$

Let $\tau \in (\mathbb{S}^2)$ be an m-tile. Then for each $n \in \mathbb{N}$ there exists a unique m-tile $\sigma \in \mathcal{D}(T)$ such that $\sigma \subset j_n(W) \cup j_n(B)$ and $\pi(\sigma) = \tau$. We write $j_n(\tau) = \sigma$.

For every tile σ in T, let

$$g(\sigma) = \frac{\operatorname{diam}(\varphi(\sigma))}{\operatorname{diam}(\varphi(\sigma_0))},$$

where σ_0 is a 0-tile in $\mathcal{T}$ that contains σ.

For all $\tau \in \mathcal{D}(\mathbb{S}^2)$, and for all $n \in \mathbb{N}$, let

$$g_n(\tau) = \left(\frac{1}{n}\left((g(j_1(\tau)))^Q + \cdots + (g(j_n(\tau)))^Q\right)\right)^{\frac{1}{Q}}.$$

Since $\mathcal{D}(\mathbb{S}^2) = \bigcup_{m\in\mathbb{N}_0} \mathcal{D}_m(\mathbb{S}^2)$ is countable, and $g_n(\tau) \in [0,1]$ for all $n \in \mathbb{N}$ and $\tau \in \mathcal{D}(\mathbb{S}^2)$, by an Arzela-Ascoli argument there exists a subsequence $\{n_r\}_{r\in\mathbb{N}}$ of $\{n_k\}_{k\to\infty}$ such that for all $\tau \in \mathcal{D}(\mathbb{S}^2)$,

$$g_\infty(\tau) = \lim_{r \to +\infty} g_{n_r}(\tau)$$

exists.

Proposition 4.3.1. *The function $g_\infty : \mathcal{D}(\mathbb{S}^2) \to [0, \infty)$ verifies all the conditions in Proposition 4.2.2.*

Proof. **Verification of condition (i):** If $\tau^{(1)} \in \mathcal{D}_m(\mathbb{S}^2)$ and $\tau^{(2)} \in \mathcal{D}_{m+k}(\mathbb{S}^2)$ are two tiles in $\mathbb{S}^2$ such that $\tau^{(2)} \subset \tau^{(1)}$, then for all $n \in \mathbb{N}$, $j_n(\tau^{(2)}) \subset j_n(\tau^{(1)})$, and

$$\frac{\operatorname{diam}(\varphi(j_n(\tau^{(2)})))}{\operatorname{diam}(\varphi(j_n(\tau^{(1)})))} \leq A\eta\left(c\frac{\operatorname{diam}(j_n(\tau^{(2)}))}{\operatorname{diam}(j_n(\tau^{(1)}))}\right) \leq A'\eta\left(c'\Lambda^{-k}\right) \leq K_0(k),$$

where A, A', c, c' are constant, and $K_0(k)$ constants depending only on k. Since $K_0(k) \to 0$ as $k \to \infty$, by taking a higher iterate of f if necessary, we may assume that $0 < \eta_2 = K_0(1) < 1$. Since φ^{-1} is also quasisymmetry, when $k = 1$, we have

$$\frac{\operatorname{diam}(\varphi(j_n(\tau^{(2)})))}{\operatorname{diam}(\varphi(j_n(\tau^{(1)})))} \geq \eta_1 > 0.$$

As this is true for all $n \in \mathbb{N}$, condition (i) is verified.

Verification of condition (ii): For all $m \in \mathbb{N}_0$, there exists $K_m > 0$ such that for all $\tau \in \mathcal{D}_m(\mathbb{S}^2)$ and $n \in \mathbb{N}$, we have

$$L(m) \leq g(\tau_n) \leq K(m),$$

where $0 < L(m) \leq K(m)$ are constants depending only on m. Also note that if $\sigma, \sigma' \in \mathcal{D}(T)$ are adjacent tiles in T at the same level, then

$$\frac{\operatorname{diam}(\varphi(\sigma))}{\operatorname{diam}(\varphi(\sigma'))} \leq A''\eta\left(c\frac{\operatorname{diam}(\sigma)}{\operatorname{diam}(\sigma')}\right) \leq K_0'$$

where K_0' is a constant. If σ_0, σ_0' are 0-tiles in T such that $\sigma \subset \sigma_0$ and $\sigma' \subset \sigma_0'$, then $\sigma_0 \cap \sigma_0' \neq \emptyset$, therefore

$$\frac{g(\sigma)}{g(\sigma')} \leq \frac{\operatorname{diam}(\varphi(\sigma))}{\operatorname{diam}(\varphi(\sigma'))}\frac{\operatorname{diam}(\varphi(\sigma_0))}{\operatorname{diam}(\varphi(\sigma_0'))} \leq K_0,$$

where $K_0 = (K_0')^2$ is a constant.

Now, suppose $\tau^{(1)}, \tau^{(2)} \in \mathcal{D}_m(\mathbb{S}^2)$ are two adjacent m-tiles in $\mathbb{S}^2$. For any $n \in \mathbb{N}$, at most δ_n-many m-tiles among $j_1(\tau^{(1)}), \ldots, j_n(\tau^{(1)})$ are not adjacent to a tile among $j_1(\tau^{(2)}), \ldots, j_n(\tau^{(2)})$. Moreover, there exist distinct numbers $a_1, \ldots, a_{n-\delta_n} \in \{1, 2, \ldots, n\}$ and distinct numbers $b_1, \ldots, b_{n-\delta_n} \in \{1, 2, \ldots, n\}$ such that $j_{a_i}(\tau^{(1)}) \cap j_{b_i}(\tau^{(2)}) \neq \emptyset$ for all $i = 1, 2, \ldots, n - \delta_n$. We have

$$\begin{aligned} g_n(\tau^{(1)}) &= \left(\frac{1}{n}\left(g(j_1(\tau^{(1)}))^Q + \cdots + g(j_n(\tau^{(1)}))^Q\right)\right)^{\frac{1}{Q}} \\ &\leq \left(\left(1 + \frac{\delta_n}{N - \delta_n}\left(\frac{K(m)}{L(m)}\right)^Q\right)\left(g(j_{a_1}(\tau^{(1)}))^Q + \cdots + g(j_{a_{\delta_n}}(\tau^{(1)}))^Q\right)\right)^{\frac{1}{Q}} \end{aligned}$$

and

$$\begin{aligned} g_n(\tau^{(2)}) &= \left(\frac{1}{n}\left(g(j_1(\tau^{(2)}))^Q + \cdots + g(j_n(\tau^{(2)})^Q\right)\right)^{\frac{1}{Q}} \\ &\leq \left(\left(1 + \frac{\delta_n}{N - \delta_n}\left(\frac{K(m)}{L(m)}\right)^Q\right)\left(g(j_{a_1}(\tau^{(2)}))^Q + \cdots + g(j_{a_{\delta_n}}(\tau^{(2)}))^Q\right)\right)^{\frac{1}{Q}} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{g_n(\tau^{(1)})}{g_n(\tau^{(2)})} &\leq \left(\frac{1 + \frac{\delta_n}{n-\delta_n}\left(\frac{K(m)}{L(m)}\right)^Q}{1 + \frac{\delta_n}{n-\delta_n}\left(\frac{L(m)}{K(m)}\right)^Q} \frac{g(j_{a_1}(\tau^{(1)}))^Q + \cdots + g(j_{a_{n-\delta_n}}(\tau^{(1)}))^Q}{g(j_{b_1}(\tau^{(2)}))^Q + \cdots + g(j_{b_{n-\delta_n}}(\tau^{(2)}))^Q}\right)^{\frac{1}{Q}} \\ &\leq \left(\frac{1 + \frac{\delta_n}{n-\delta_n}\left(\frac{K(m)}{L(m)}\right)^Q}{1 + \frac{\delta_n}{n-\delta_n}\left(\frac{L(m)}{K(m)}\right)^Q}\right)^{\frac{1}{Q}} K_0. \end{aligned}$$

By our assumption that $\lim_{r\to\infty} \frac{\delta_{n_r}}{n_r} = 0$, we obtain

$$\frac{g_\infty(\tau^{(1)})}{g_\infty(\tau^{(2)})} = \lim_{r\to\infty} \frac{g_{n_r}(\tau^{(1)})}{g_{n_r}(\tau^{(2)})} = K_0.$$

Thus we have prove condition (ii).

Verification of condition (iii): Let $x, y \in \mathbb{S}^2$ and $\tau \in \mathcal{D}_{m(x,y)}(\mathbb{S}^2)$ be a tile containing x. By enlarging K_1 if necessary, we may assume that $\rho(x, y) < d_0$, where $d_0 > 0$ is the shortest length of any tile chain that joins opposite 0-edges in $\mathbb{S}^2$. Let $\gamma = \{\tau^{(1)}, \ldots, \tau^{(k)}\}$ be a tile chain in $\mathbb{S}$ joining x and y, and suppose that $\tau^{(i)} \in \mathcal{D}_{m_i}(\mathbb{S}^2)$. Since q is a bounded metric, the inequality in (iii) always hold if $\ell_g(\gamma)$ is large. In particular, we may assume that γ does not join opposite edges of $\mathcal{C}$. We also assume that no tile appear in γ twice, for otherwise we may remove a portion of γ and form a tile chain γ' such that $\ell_g(\gamma') < \ell_g(\gamma)$.

By the same logic we employed to verify condition (ii), for any $n \in \mathbb{N}$, there exists integers $a_{i,j}$, where $i = 1, \ldots, k$, and $j = 1, \ldots, n - k\delta_n$, such that $1 \leq a_{i,j} \leq n$, $a_{i,j} \neq a_{i,j'}$ if $1 \leq j < j' \leq n - k\delta_n$, and for each $j = 1, 2 \ldots, n - k\delta_n$,

$$\gamma^j = \{j_{a_{1,j}}(\tau^{(1)}), \ldots, j_{a_{k,j}}(\tau^{(k)})\}$$

is a tile chain in T. Moreover, there exists distinct integer $b_1, b_2, \ldots, b_{n-\delta_n} \in \{1, \ldots, n\}$ such that γ^j contains a tile that intersect τ_{b_j} for each $j = 1, 2, \ldots, n - k\delta_n$.

Fix $j \in \{1, 2, \ldots, n - k\delta_n\}$. Let $\sigma_1, \ldots, \sigma_l$ be all the 0-tiles in T whose intersection with $\bigcup_{\sigma \in \gamma^j} \sigma$ is nonempty. Since γ does not join opposite sides of $\mathcal{C}$, $\sigma_1, \ldots, \sigma_l$ must intersect at a common point $p \in T$. Therefore for any $*, \dagger \in \{1, \ldots, \ell\}$,

$$\frac{1}{K_0'} \leq \frac{\operatorname{diam}(\varphi(\sigma_*))}{\operatorname{diam}(\varphi(\sigma_\dagger))} \leq K_0'.$$

By triangle inequality, we have

$$\operatorname{diam}(\varphi(j_{a_{1,j}}(\tau^{(1)})) + \cdots + \operatorname{diam}(\varphi(j_{a_{k,j}}(\tau^{(k)})) \geq |\varphi(x) - \varphi(y)|_{T_Q},$$

where $|\varphi(x) - \varphi(y)|_{T_Q}$ denotes the distance between $\varphi(x)$ and $\varphi(y)$ in T_Q. Since φ is quasisymmetry, and the distance between x and y in T is proportional to $\operatorname{diam}(j_{b_j}(\tau))$, we have

$$\frac{\operatorname{diam}(\varphi(j_{a_{1,j}}(\tau^{(1)}))) + \cdots + \operatorname{diam}(\varphi(j_{a_{k,j}}(\tau^{(k)})))}{\operatorname{diam}(\varphi(j_{b_j}(\tau)))} \geq \frac{|\varphi(x) - \varphi(y)|_{T_Q}}{\operatorname{diam}(\varphi(j_{b_j}(\tau)))} \geq K_1', \tag{4.3.1}$$

where K_1' is a constant independent of $x, y, \gamma, j_{b_j}(\tau)$. We thus have

$$\frac{g(j_{a_{1,j}}(\tau^{(1)})) + \cdots + g(j_{a_{1,j}}(\tau^{(1)}))}{g(j_{b_j}(\tau))} \geq \frac{K_1'}{K_0'} \geq K_1^{-1}.$$

Here $K_1 \geq 1$ is a constant independent of $x, y, \gamma, j_{b_j}(\tau)$. We have

$$\begin{aligned} n^{\frac{1}{Q}} \sum_{i=1}^{k} g_n(\tau^{(i)}) &\geq \sum_{i=1}^{k} \left(\left(1 + \frac{\delta_n}{n - k\delta_n} \left(\frac{L(m_i)}{K(m_i)} \right)^Q \right) \left(\sum_{j=1}^{n-k\delta_n} g\left(j_{a_{i,j}}(\tau^{(i)}) \right)^Q \right) \right)^{\frac{1}{Q}} \\ &\geq \min_{i=1,2,\ldots,k} \left(1 + \frac{\delta_n}{n - k\delta_n} \left(\frac{L(m_i)}{K(m_i)} \right)^Q \right)^{\frac{1}{Q}} \sum_{i=1}^{k} \left(\sum_{j=1}^{n-k\delta_n} g\left(j_{a_{i,j}}(\tau^{(i)}) \right)^Q \right)^{\frac{1}{Q}}. \end{aligned}$$

By Minkowski inequaltiy and (4.3.1), we have

$$\sum_{i=1}^{k}\left(\sum_{j=1}^{n-k\delta_n} g\left(j_{a_{i,j}}(\tau^{(i)})\right)^Q\right)^{\frac{1}{Q}} \geq \left(\sum_{j=1}^{n-k\delta_n}\left(\sum_{i=1}^{k} g\left(j_{a_{i,j}}(\tau^{(i)})\right)\right)^Q\right)^{\frac{1}{Q}} \geq K_1^{-1}\left(\sum_{j=1}^{n-k\delta_n}\left(g\left(j_{b_j}(\tau)\right)\right)^Q\right)^{\frac{1}{Q}}.$$

Thus

$$\begin{aligned} n^{\frac{1}{Q}}\sum_{i=1}^{k} g_n(\tau^{(i)}) &\geq K_1^{-1}\min_{i=1,2,\ldots,k}\left(1+\frac{\delta_n}{n-k\delta_n}\left(\frac{L(m_i)}{K(m_i)}\right)^Q\right)^{\frac{1}{Q}}\left(\sum_{j=1}^{n-k\delta_n}\left(g\left(j_{b_j}(\tau)\right)\right)^Q\right)^{\frac{1}{Q}} \\ &\geq K_1^{-1}E(n)n^{\frac{1}{Q}}g_n(\tau), \end{aligned}$$

where $m = m(x,y)$ is the level of τ, and

$$E(n) = \min_{i=1,2,\ldots,k}\left(1+\frac{\delta_n}{n-k\delta_n}\left(\frac{L(m_i)}{K(m_i)}\right)^Q\right)^{\frac{1}{Q}}\left(1+\frac{\delta_n}{n-k\delta_n}\left(\frac{L(m)}{K(m)}\right)^Q\right)^{-Q}.$$

Again using $\lim_{r\to\infty}\frac{\delta_{n_r}}{n} = 0$, we get

$$\lim_{r\to\infty} E(n_r) = 1.$$

Therefore

$$\sum_{i=1}^{k} g_\infty(\tau^{(i)}) \geq K_1^{-1}g_n(\tau).$$

Verification of condition (iv): Condition (iv) holds for all g_n,. Therefore it holds for g_∞. □

4.4 The First Proof of Theorem 1.0.8

Theorem (Theorem 1.0.8)**.** Let $(\mathbb{S}^2,\rho)$ be a visual sphere of an expanding Thurston map f with no periodic critical point. Let (T,x) be a weak tangent of $(\mathbb{S}^2,\rho)$. For any $Q>0$, $(\mathbb{S}^2,\rho)$ is quasisymmetrically equivalent to a Q-Ahlfors regular space if and only if T is quasisymmetrically equivalent to a Q-Ahlfors regular space.

Proof. As mentioned in the introduction of this chapter, the "only if" implication holds in greater generality. To prove the "if" implication, suppose T is quasisymmetrically equivalent to a Q-Ahlfors regular space T_Q. By Theorem 4.3.1, there exists a gauge function g on the collection $\mathcal{D}(\mathbb{S}^2)$ of tiles on $\mathbb{S}^2$ satisfying all the conditions of Theorem 4.2.2. By Theorem 4.2.2, there exists a Q-Ahlfors regular metric q on $\mathbb{S}^2$ such that the identity map $\mathrm{id} : (\mathbb{S}^2, \rho) \to (\mathbb{S}^2, q)$ is a quasisymmetry. $\square$

The case when the Ahlfors regular conformal dimension is attainable as 2 is of particular interest.

Theorem 4.4.1. *Let $(\mathbb{S}^2, \rho)$ be a visual sphere of an expanding Thurston map f with no periodic critical point. The followings are equivalent:*

(i) $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to the standard 2-sphere.

(ii) Every weak tangent of $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to the Euclidean plane $\mathbb{R}^2$.

(iii) There exists a weak tangent of $(\mathbb{S}^2, \rho)$ that is quasisymmetrically equivalent to the Euclidean plane $\mathbb{R}^2$.

Proof. (i) $\implies$ (ii): The standard 2-sphere is 2-Ahlfors regular. If $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to a 2-Ahlfors regular space, then by Theorem 1.0.8, every weak tangent T of $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to a 2-Ahlfors regular space. But T is LLC. By [Wil08], the T is quasisymmetrically equivalent to the Euclidean plane.

(ii) $\implies$ (iii) is clear.

(iii) $\implies$ (i): If a weak tangent of $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to the Euclidean plane $\mathbb{R}^2$, then by Theorem 1.0.8, $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to a 2-Ahlfors regular space. Since $(\mathbb{S}^2, \rho)$ is LLC, by [BK02], $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to the standard 2-sphere. $\square$

4.5 An Alternative Proof of Theorem 1.0.8

As promised, we offer an alternative proof of the "if" implication of Theorem 1.0.8.

Lemma 4.5.1. *For every tile $\tau \in \mathcal{D}(\mathbb{S}^2)$, $\partial\tau$ is a porous subset of σ.*

Proof. Suppose τ is a k-tile. Let $x \in \partial\tau$ be arbitrary, and suppose $\Lambda^{-(n+1)} \leq r < \Lambda^{-n}$ for some $n \geq k$. Regarding τ as a metric space, every open ball in τ with radius r contains a $(n+k)$-tile (cf [BM17, Lemma 8.11]). This tile contains a ball B of radius cr, where $c > 0$ is a constant independent of x, r, and n, such that $B \in \tau$ and $B \cap \partial\tau = \emptyset$. □

Lemma 4.5.2. *Let (T, x) be a weak tangent of $\mathbb{S}^2$. Every tile $\sigma \in \mathcal{D}(T)$ is quasisymmetrically equivalent to a Q-AR metric space.*

Proof. This is a consequence of Theorem 4.2.12. □

Theorem 4.5.3. *[Haï09, Proposition 4.1] Let X_1, X_2 be proper metric spaces containing at least two points. Let us assume that $Y_1 \subset X_1$, $Y_2 \subset X_2$ are two closed uniformly perfect subsets, and $f : Y_1 \to Y_2$ is a quasisymmetry. Suppose that X_1 is bounded if Y_1 is. Suppose further that Y_1 and Y_2 are porous, and X_1 and X_2 are both Q-AR. Then there exists a metric $\widehat{d}$ on $\widehat{X} = X_1 \cup X_2/(f)$ and a constant $c > 0$ such that*

(1) For any $(x_1, x_2) \in X_1 \times X_2$, $\widehat{d}(x_1, x_2) \geq c \cdot \inf_{y \in Y_1} \widehat{d}(x_1, y) + \widehat{d}(f(y), x_2)$.

(2) For $j = 1, 2$, the map $\mathrm{id} : X_j \to \widehat{X}$ is quasisymmetric.

(3) $\widehat{X}$ is Q-AR.

Theorem (Theorem 1.0.9)**.** Let $(\mathbb{S}^2, \rho)$ be a visual sphere of an expanding Thurston map f with no periodic critical point. Let (T, x) be a weak tangent of $(\mathbb{S}^2, \rho)$. If T is quasisymmetrically equivalent to a Q-Ahlfors regular space, then $(\mathbb{S}^2, \rho)$ is quasisymmetrically equivalent to a Q-Ahlfors regular space.

Proof. Let (T, x) be a weak tangent of $\mathbb{S}^2$. By Theorem 1.0.7(ii) and (v), and by how we defined tiles on T, there exists a white tile τ_W in $\mathbb{S}^2$ that is quasisymmetrically equivalent to a white tile in T. By Lemma 4.5.2, τ_W is quasisymmetrically equivalent to a Q-Ahlfors regular space. Since τ_W is bilipschitz equivalent, therefore quasisymmetrically equivalent, to the white 0-tile W, the white 0-tile W in $\mathbb{S}^2$ is also quasisymmetrically equivalent to a Q-AR space. Similarly, the black 0-tile in $\mathbb{S}^2$ is quasisymmetrically equivalent to a Q-AR space. By Lemma 4.5.1, $W \cap B$ is porous in W as well as B. Therefore we can apply Theorem 4.5.3 to conclude that $(\mathbb{S}^2, \rho)$ is quasisymmetrically to a Q-Ahlfors regular space. $\square$

CHAPTER 5

A Metric Sphere Not a Quasisphere but Whose Weak Tangents are Euclidean

5.1 Introduction

In [Kin17] Kinneberg characterized metric circles that are quasisymmetric to the standard circle in terms of weak tangents:

Theorem 5.1.1 (Kinneberg, [Kin17])**.** *A doubling metric circle $\mathcal{C}$ is quasisymmetrically equivalent to the standard circle $\mathbb{S}^1$ if and only if every weak tangent of $\mathcal{C}$ quasisymmetrically equivalent to the real line $\mathbb{R}$ based at* 0*.*

In this paper we prove that Kinneberg's result cannot be extended to higher dimensions:

Theorem (Proposition 1.0.10)**.** For every $n \geq 2$, there exists a doubling, linearly locally contractible metric space X, topologically an n-sphere, such that every weak tangent of X is isometric to $(\mathbb{R}^n, 0)$ but X is not quasisymmetrically equivalent to the standard n-sphere.

When $n = 2$, one can compare our result with the following Theorem:

Theorem 5.1.2 (Bonk and Kleiner, [BK02])**.** *Let Z be a 2-Ahlfors regular metric space homeomorphic to $\mathbb{S}^2$. Then Z is quasisymmetric to $\mathbb{S}^2$ if and only if Z is linearly locally contractible.*

Theorem 1.0.10 shows that the conclusion of Theorem 5.1.2 is false if we replace 2-Ahlfors regularity with Q-Ahlfors regularity for $Q > 2$.

Our study is also related to the following theorem, proven in [GW18]:

Theorem 5.1.3. *Let (Z, d) be a doubling metric space homeomorphic to $\mathbb{S}^2$. The following are equivalent:*

(i) (Z, d) is quasisymmetrically equivalent to the standard 2-sphere.

(ii) For every $x \in Z$, there exists an open neighborhood U of x in Z such that U is quasisymmetrically equivalent to $\mathbb{D}$.

Roughly speaking, Theorem 5.1.3 says local properties promote to global properties. Since weak tangents are local, one could ask the following question:

Question 5.1.4. Suppose (Z, d) is doubling and linearly locally connected. Are the following two statements equivalent?

(i) Z is quasisymmetrically equivalent to the standard d-sphere $\mathbb{S}^d$

(ii) Every weak tangent of Z is quasisymmetrically equivalent to $\mathbb{R}^d$.

When Z is a doubling and linearly locally connected metric sphere, statement (i) implies statement (ii). However, our construction shows that statement (ii) does not imply statement (i).

5.2 The First Construction

In this section, we will outline the construction of a doubling, linearly locally contractible metric space X', homeomorphic to the 2-sphere, such that every weak tangent of X' is bilipschitz equivalent to $(\mathbb{R}^2, 0)$ but X' itself is not quasisymmetrically equivalent to the standard 2-sphere. The construction of X' is not enough to prove Theorem 1.0.10, because the weak tangents of X' are only bilipschitz equivalent to $\mathbb{R}^2$. However, the construction of X' motivates the less intuitive second construction that proves Theorem 1.0.10. It is therefore helpful to understand the first construction.

Our first construction is based on the idea of discretizing von-Koch snowflakes.

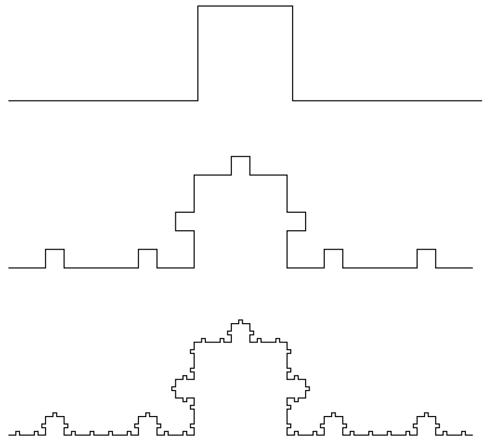

Figure 5.1: L_2^1,L_2^2, and L_2^3.

For each $n \in \mathbb{N}$, let L_n^0 be the unit interval $[0,1]$. Let L_n^1 be the the curve obtained by dividing L_n^0 into $2n+1$ intervals of equal length, and replacing the middle interval with three sides of a square. Thus each L_n^1 has $2n+3$ segments of length $1/(2n+1)$. For each $k \in \mathbb{N}_0$, we divide each segment in L_n^k into $2n+1$ smaller intervals of equal length and replace the middle interval with three sides of a square to obtain L_n^{k+1}. Figure 5.1 shows L_2^1, L_2^2, and L_2^3. The sequence L_n^k converges in Hausdorff sense in $\mathbb{R}^2$ to a curve L_n, whose Hausdorff dimension is

$$\dim_H(L_n) = \frac{\log(2n+3)}{\log(2n+1)}.$$

The curves L_n are called snowflake curves. It is well-known that L_n is bilipschitz equivalent to the metric $([0,1], d^{\alpha_n})$, where d is the Euclidean metric and

$$\alpha_n = \frac{1}{\dim_H(L_n)}.$$

L_n^k is called the k-step iteration of the curve L_n.

We need the following two lemmas:

Lemma 5.2.1. *Let σ be the chordal metric on the 2-sphere $\mathbb{S}^2$. Let $A, B \subset \mathbb{S}^2$ be two subsets such that $\sigma(A,B)$, the distance between A and B, is positive. Let $\Gamma(A,B)$ be the family of curves joining A and B. Then*

$$\operatorname{mod}_2(\Gamma(A,B)) \lesssim \pi \left(1 + \frac{\min\{\operatorname{diam}(A), \operatorname{diam}(B)\}}{\sigma(A,B)}\right)^2$$

Lemma 5.2.2. *Let L be a rectifiable curve of length ℓ, $X = L \times [0,1]$, $A = L \times \{0\}$, $B = L \times \{1\}$. Then*

$$\mathrm{mod}_2(\Gamma(A,B)) \geq \ell.$$

Motivated by these two lemmas, and the belief that 2-moduli should behave well under quasisymmetry, we choose a sequence k_n of positive integers such that

$$\lim_{n\to\infty} \ell(L_n^{k_n}) = \infty.$$

Here $\ell(L)$ denotes the length of a curve L. We choose a sequence s_n of positive real numbers such that $s_n < \frac{1}{n} - \frac{1}{n+1}$. Let $a_n = \frac{1}{n} - s_n$ and $b_n = \frac{1}{n}$, so that $[a_n, b_n]$ is a segment in $[\frac{1}{n+1}, \frac{1}{n}]$ of length s_n. For each $n \in \mathbb{N}$, we construct a new metric space on $\mathbb{R}$ by replacing the segments $[a_n, b_n]$ with the snowflake curves $L_n^{k_n}$ scaled by a factor s_n. Call this topologically 1-dimensional space Γ. Then we form $\Gamma \times [0,1]$ to produce a metric space of topological dimension 2. FInally, we smooth the boundary of $\Gamma \times [0,1]$ and embed the construction into the standard 2-sphere. We call the fimal metric sphere X'.

The two lemmas above imply that X' is not quasisymmetric to the standard 2-sphere. We now give a heruistic argument that every weak tangent of X' is bilipschitz equivalent to $(\mathbb{R}^2, 0)$. The metric space X' is obtained by modifying part of the standard 2-sphere into $\Gamma \times [0,1]$. Every weak tangnet of the standard 2-sphere is isometric to $(\mathbb{R}^2, 0)$. Every weak tangent of $L_n^{k_n}$ is bilipschitz equivalent to $(\mathbb{R}, 0)$, thus every weak tangent of $L_n^{k_n} \times [0,1]$ is bilipschitz equivalent $(\mathbb{R}^2, 0)$. Finally, if we zoom in near the point 0, then the curves $L_n^{k_n}$ becomes milder in the sense that the curves $L_n^{k_n}$ become closer to a line segment. Thus every the weak tangents of our metric space is bilipschitz equivalent to $\mathbb{R}^2$.

One can verify that X' is doubling and linearly locally contractible. This completes our discussion of the first construction.

5.3 The Second Construction

In this section, we replace $L_n^{k_n}$ with another "discretized snowflake curve". To begin the construction let $\alpha \in (0,1)$ and $c \in (0,1)$ and define

$$\varphi_{\alpha,c}(x) = \begin{cases} L(\alpha,c)x, & x \in [0,c] \\ \left(\frac{x-c(1-\alpha)}{1-c(1-\alpha)}\right)^{\alpha}, & x \in [c,1], \end{cases}$$

where

$$L(\alpha,c) = \frac{1}{c}\left(\frac{c\alpha}{1-c(1-\alpha)}\right)^{\alpha}.$$

Lemma 5.3.1. *The function $\varphi_{\alpha,c} : [0,1] \to +\infty$ is the unique function on $[0,1]$ that has the following properties:*

(3.1) $\varphi_{\alpha,c}(0) = 0$,

(3.2) $\varphi_{\alpha,c}(1) = 1$,

(3.3) $\varphi_{\alpha,c}$ *is linear on* $[0,c]$,

(3.4) There exists $a, b \in \mathbb{R}$, with $a \neq 0$, such that $\varphi_{\alpha,c}(x) = (ax-b)^{\alpha}$ when $x \in [c,1]$,

(3.5) $\varphi_{\alpha,c}$ is continuous and differentiable at c.

In addition to properties (3.1) - (3.5), $\varphi_{\alpha,c}$ has the following extra properties:

(3.6) $\varphi_{\alpha,c} : [0,1] \to [0,1]$ is a homeomorphism,

(3.7) $\varphi_{\alpha,c}$ is concave on $[0,1]$.

Lemma 5.3.2. *For any $\alpha \in (0,1)$, $c \in (0,1]$, and $a, b \in [0,1]$, we have*

$$\varphi_{\alpha,c}(ab) \leq \varphi_{\alpha,c}(a)\varphi_{\alpha,c}(b).$$

Proof. For fixed $\alpha \in (0,1)$, the function $c \mapsto L(\alpha,c)$ is decreasing on $(0,1)$. Therefore if $0 < c_1 \leq c_2 \leq 1$, then

$$\varphi_{\alpha,c_1} \geq \varphi_{\alpha,c_2}.$$

Let
$$\varphi_*(x) = \frac{\varphi_{\alpha,c}(ax)}{\varphi_{\alpha,c}(a)}.$$
The function φ_* satisfies conditions (1)-(5) listed above for the same α and $c_* = ca^{-1} \in [c, 1]$, therefore $\varphi_* = \varphi_{\alpha,c_*}$. This implies that $\varphi_* \leq \varphi_{\alpha,c}(x)$ for all $x \in [0, 1]$. Taking $x = b$, we have
$$\varphi_{\alpha,c}(ab) \leq \varphi_{\alpha,c}(a)\varphi_{\alpha,c}(b).$$
□

Lemma 5.3.3. *For any $\alpha \in (0, 1)$, $c \in (0, 1)$, and $0 \leq t \leq x \leq 1$, we have*
$$0 \leq \varphi_{\alpha,c}(t) + \varphi_{\alpha,c}(x - t) - \varphi_{\alpha,c}(x) \leq (2 - 2^{\alpha})\, \varphi_{\alpha,c}(x/2).$$

Proof. When $x \leq c$, we have
$$\varphi_{\alpha,c}(t) + \varphi_{\alpha,c}(x - t) - \varphi_{\alpha,c}(x) = 0$$
for all $t \in [0, x]$. When $x \geq 2c$ is fixed, $\varphi_{\alpha,c}(t) + \varphi_{\alpha,c}(x - t) - \varphi_{\alpha,c}(x)$ is maximized when
$$\varphi'_{\alpha,c}(t) - \varphi'_{\alpha,c}(x - t) = 0,$$
which is possible only if $x - t = t$ i.e. $2t = x$. Thus we have
$$\begin{aligned}\varphi_{\alpha,c}(t)+&\varphi_{\alpha,c}(x - t) - \varphi_{\alpha,c}(x) \leq 2\varphi_{\alpha,c}(x/2) - \varphi_{\alpha,c}(x)\\ &\leq 2\varphi_{\alpha,c}(x/2) - 2^{\alpha}\varphi_{\alpha,c}(x/2) = (2 - 2^{\alpha})\varphi_{\alpha,c}(x/2).\end{aligned}$$

When $c < x < 2c$, we have
$$\varphi_{\alpha,c}(t) + \varphi_{\alpha,c}(x - t) \leq L(\alpha, c)x = \frac{x}{c}\varphi_{\alpha,c}(c).$$
By the concavity of $\varphi_{\alpha,c}$, we also have
$$\frac{\varphi_{\alpha,c}(x)}{x} \geq \frac{\varphi_{\alpha,c}(2c)}{2c}.$$
Therefore
$$\varphi_{\alpha,c}(t) + \varphi_{\alpha,c}(x - t) - \varphi_{\alpha,c}(x) \leq \frac{x}{c}\varphi_{\alpha,c}(c) - \frac{x\varphi_{\alpha,c}(2c)}{2c} = \frac{x}{2c}\left(2\varphi_{\alpha,c}(c) - \varphi_{\alpha,c}(2c)\right).$$

But since

$$2\varphi_{\alpha,c}(c) - \varphi_{\alpha,c}(2c) \leq (2-2^{\alpha})\varphi_{\alpha,c}(c) = (2-2^{\alpha})cL(\alpha, c),$$

we obtain

$$\varphi_{\alpha,c}(t) + \varphi_{\alpha,c}(x-t) - \varphi_{\alpha,c}(x) \leq \frac{x}{2c}(2-2^{\alpha})cL(\alpha, c) = (2-2^{\alpha})\varphi_{\alpha,c}(x/2).$$

$\square$

For any $\alpha \in (0,1)$, we have

$$\lim_{c\to 0^+} L(\alpha, c) = +\infty. \tag{5.3.1}$$

Let α_n be an increasing sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 1$. By (5.3.1), we can choose $c_n \in (0,1)$ such that $c_n \to 0$, and

$$\lim_{n\to\infty} L(\alpha_n, c_n) = +\infty.$$

Let $\varphi_n = \varphi_{\alpha_n,c_n}$. Choose a sequence s_n such that $s_n < 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$, $s_n L(\alpha_n, c_n)$ is decreasing in n, and $\sum_{n\in\mathbb{N}} s_n L(\alpha_n, c_n) < \infty$.

For all $n \in \mathbb{N}$, let $a_n = \frac{1}{n} - s_n$ and $b_n = \frac{1}{n}$. Write $I_n = [a_n, b_n]$, and equip I_n with the metric $\delta_n = s_n\varphi_n \circ (s_n^{-1}d)$, where d is the usual Euclidean metric on I_n. Note that

(i) The distance between the two endpoints of I_n is $\delta_n(a_n, b_n) = s_n$.

(ii) I_n is rectifiable and the length of I_n is $\ell(I_n) = s_n L(\alpha_n, c_n)$.

For $x \leq y$, define

$$\delta(x,y) = \begin{cases} d(x,y), & \text{if } x, y \in \mathbb{R}\setminus\bigcup_{i\geq} I_i, \\ \delta_n(x,y), & \text{if } x,y \in I_n, \\ d(x,a_n) + \delta_n(a_n, y), & \text{if } x \in \mathbb{R}\setminus\bigcup_{i\in\mathbb{N}} I_i, y \in I_n, \\ \delta_n(x, b_n) + d(b_n, y), & \text{if } x \in I_n, y \in \mathbb{R}\setminus\bigcup_{i\in\mathbb{N}} I_i, \\ \delta_n(x,b_n) + d(b_n, a_m) + d_m(a_m, y), & \text{if } x \in I_n, y \in I_m. \end{cases}$$

For $x > y$, define $\delta(x,y) = \delta(y,x)$. Then δ is a metric on $\mathbb{R}$.

5.4 The Weak Tangents of $(\mathbb{R}, \delta)$

The goal of this section is to prove the following proposition that describes all the weak tangents of $(\mathbb{R}, \delta)$.

Proposition 5.4.1. *For all $a_n \in \mathbb{R}$, and for all positive integers $\lambda_n \to +\infty$, $(\mathbb{R}, a_n, \lambda_n \delta)$ converges in pointed Gromov-Hausdorff sense to $(\mathbb{R}, 0, d)$.*

Note that Proposition 5.4.1 guarantees the existence of weak tangents of $(\mathbb{R}, \delta)$. To prove the above proposition we will use the following three lemmas:

Lemma 5.4.2. *Suppose $x, y, z \in \mathbb{R}$ are three points so that $x \leq y \leq z$. Suppose $N = \inf\{n \in \mathbb{N} : \{x, y, z\} \cap I_n \neq \emptyset\} < \infty$. Then*

$$0 \leq \delta(x, y) + \delta(y, z) - \delta(z, x) \leq (2 - 2^{\alpha_N}) \min\{s_N, \delta(x, y)\}$$

Proof. If $N = +\infty$, then $\delta(x, y) + \delta(y, z) - \delta(z, x) = 0$. Otherwise, let

$$a = \sup\{a_n \leq y : n \in \mathbb{N}\} \vee \sup\{b_n \leq y : n \in \mathbb{N}\} \vee x$$
$$b = \inf\{a_n \geq y : n \in \mathbb{N}\} \wedge \inf\{b_n \geq y : n \in \mathbb{N}\} \wedge z$$

We have $x \leq a \leq y \leq b \leq z$. By definition of δ, we have

$$\begin{aligned} \delta(x, y) + \delta(y, z) - \delta(x, z) &= (\delta(x, a) + \delta(a, y)) + (\delta(y, b) + \delta(b, z)) \\ &\quad - (\delta(x, a) + \delta(a, b) + \delta(b, z)) \\ &= \delta(a, y) + \delta(y, b) - \delta(a, b). \end{aligned}$$

Bu our choice of a and b, either $a, y, b \in I_n$ for some $n \geq N$, or $(a, b) \cap \bigcup_{n \in \mathbb{N}} I_n = \emptyset$. In the latter case, we have

$$\delta(x, y) + \delta(y, z) - \delta(z, x) = \delta(a, y) + \delta(y, b) - \delta(a, b) = 0.$$

In the former case, Lemma 5.3.3 gives

$$\begin{aligned} 0 \leq \delta(x, y) + \delta(y, z) - \delta(z, x) &= \delta(a, y) + \delta(y, b) - \delta(a, b) \\ \leq (2 - 2^{\alpha_n}) \, \delta(a, b) &= (2 - 2^{\alpha_N}) \min\{s_N, \delta(x, y)\}. \end{aligned}$$

Since $\alpha_N \leq \alpha_n \leq 1$, we have our desired conclusion. ∎

Lemma 5.4.3. *Let $p \in \mathbb{R}$ and $r > 0$ be arbitrary. Suppose $N = \inf\{n \in \mathbb{N} : \overline{B}_\delta(p,r) \cap I_n \neq \emptyset\} < \infty$. Let $a = \inf\{x \in \mathbb{R} : \delta(x,p) \le r\}$ and $b = \sup\{x \in \mathbb{R} : \delta(x,p) \le r\}$. Then the map*

$$\psi : ([a,b], p, \delta) \to ([-r,r], 0, d)$$
$$\psi(x) = \begin{cases} -\delta(p,x), & x \le p \\ \delta(p,x), & x > p. \end{cases}$$

is an ε-rough isometry, where $\varepsilon = (2 - 2^{\alpha_N}) \min\{s_N, 2r\}$.

Proof. Since ψ is surjective and fixes p, it remains to check that for all $x, y \in [a,b]$, we have

$$(d(\psi(x), \psi(y)) - \delta(x,y)) \le \varepsilon.$$

Suppose $x \le y$. If $x \le y \le p$, then

$$\begin{aligned} |d(\psi(x), \psi(y)) - \delta(x,y)| &= ||\delta(p,x) - \delta(p,y)| - \delta(x,y)| \\ &= |\delta(p,y) - \delta(p,x) - \delta(x,y)| \\ &\le (2 - 2^{\alpha_N}) \min\{s_N, \delta(p,x)\} \\ &= (2 - 2^{\alpha_N}) \min\{s_N, 2r\}. \end{aligned}$$

The second to last inequality is a consequence of Lemma 5.4.2. If $x \le p \le y$, then

$$\begin{aligned} |d(\psi(x), \psi(y)) - \delta(x,y)| &= ||\delta(p,x) - \delta(p,y)| - \delta(x,y)| \\ &= |\delta(p,y) + \delta(p,x) - \delta(x,y)| \\ &\le (2 - 2^{\alpha_N}) \min\{s_N, \delta(x,y)\} \\ &= (2 - 2^{\alpha_N}) \min\{s_N, 2r\}. \end{aligned}$$

If $p \le x \le y$, then following a similar argument as when $x \le y \le p$, we get

$$|d(\psi(x), \psi(y)) - \delta(x,y)| \le (2 - 2^{\alpha_N}) \min\{s_N, 2r\}$$

This verifies that ψ is an ε-rough isometry. $\square$

Lemma 5.4.4. *For $r \in (0,1)$, we have*

$$\sup_{p \in \mathbb{R}} d_{GH}((\overline{B}_\delta(p,r), p, \delta), (\overline{B}_d(0,r), 0, d)) = o(r)$$

as $r \to 0$.

Proof. Let $p \in \mathbb{R}$ and $r > 0$ be arbitrary. Let $N = \inf\{n \in \mathbb{N} : \overline{B}_\delta(p,r) \cap I_n \neq \emptyset\}$. If $N = +\infty$, then $\delta = d$ on $\overline{B}_\delta(p,r)$. We have

$$d_{GH}((\overline{B}_\delta(p,r), p, \delta), (\overline{B}_d(0,r), 0, d)) = 0.$$

If $N < \infty$, we consider two cases:

Case 1: $r \geq \frac{s_N}{2}\varphi_N(c_N)$. As $r \to 0$, $N \to +\infty$, and therefore $(2 - 2^{\alpha_N}) \to 0$. In this case Lemma 5.4.3 implies

$$d_{GH}((\overline{B}_\delta(p,r), p, \delta), (\overline{B}_d(0,r), 0, d)) \leq 2\,(2 - 2^{\alpha_N})\,r = o(r).$$

Case 2: $r < \frac{s_N}{2}\varphi_N(c_N)$. In this case, δ is a length metric on $[a,b]$, and ψ is an isometry between two length spaces. Thus we have

$$d_{GH}((\overline{B}_\delta(p,r), p, \delta), (\overline{B}_d(0,r), 0, d)) = 0.$$

□

Proof of Proposition 5.4.1. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers that diverges to $+\infty$. By Lemma 5.4.4,

$$\begin{aligned} &d_{GH}\left((\overline{B}_{\lambda_n\delta}(a_n,R), a_n, \lambda_n\delta), (\overline{B}_{\lambda_n d}(0,R), 0, \lambda_n d)\right) \\ &\quad = \lambda_n d_{GH}\left((\overline{B}_\delta(a_n, \lambda_n^{-1}R), a_n, \delta), (\overline{B}_d(0, \lambda_n^{-1}R), 0, d)\right) = \lambda_n o(\lambda_n^{-1}R). \end{aligned}$$

But $(\overline{B}_{\lambda_n d}(0,R), 0, \lambda_n d) = \overline{B}_d(0,R), 0, d))$ by the symmetry of $\mathbb{R}$. As $n \to \infty$, $\lambda_n^{-1}R \to 0$, and we have

$$(\overline{B}_{\lambda_n\delta}(a_n,R), a_n, \delta) \to \overline{B}_d(0,R), 0, d).$$

This is true for all $R > 0$. We conclude that $(\mathbb{R}, a_n, \lambda_n\delta) \to (\mathbb{R}, 0, d)$. □

5.5 Linear Local Contractibility and Assouad Dimension of $(\mathbb{R}, \delta)$

In this section we establish two properties of the space $(\mathbb{R}, \delta)$. These properties often appear in the study of quasisymmetry classes of metric spheres. Both properties are discussed in detail in [Hei01].

Definition 5.5.1. *Let $C > 1$ be a constant. A metric space is C-linearly locally contractible if every small ball is contractible inside a ball whose radius is C times larger. A metric space is linearly locally contractible if it is C-linearly locally contractible for some $C > 0$.*

Definition 5.5.2. *Let $N > 0$. A metric space is N-doubling if for all $R > 0$, every open ball of $2R$ can be covered by N balls of radius R. A metric space is doubling if it is N-doubling for some $N > 0$.*

Both doubling and linear local contractibility are preserved under quasisymmetry. The Euclidean spaces $\mathbb{R}^n$ or $\mathbb{S}^n$ are doubling and linearly locally contractible. The doubling property also ensures the existence of weak tangents.

Proposition 5.5.3. *The space $(\mathbb{R}, \delta)$ is 1-linearly locally contractible.*

Proof. Any open ball B in $(\mathbb{R}, \delta)$ is an open interval (a, b). Denote p the center of B (in $(\mathbb{R}, \delta)$.) Note that the map $x \mapsto \delta(p, x)$ is increasing on $\{x \in \mathbb{R} : x \geq p$, and decreasing on $\{x \in \mathbb{R} : x \leq p\}$. Therefore the map $H(x, t) = tx + (1 - t)p$ is a homotopy of (a, b) to $\{p\}$ in B. This proves that $(\mathbb{R}, \delta)$ is 1-linearly locally contractible. □

Proposition 5.5.4. *The space $(\mathbb{R}, \delta)$ is doubling.*

If a metric space X is doubling, then there exists $\beta > 0$ and $C > 0$ such that for all $\varepsilon \in (0, 1/2)$ and $r > 0$, any set of diameter r in X can be covered by at most $C\varepsilon^{-\beta}$ subsets of diameter at most εr. The function $\varepsilon \mapsto C\varepsilon^{-\beta}$ is called the *covering function* of X. The *Assouad dimension* of X is defined to be the infimum of all β so that a covering function of the form $\varepsilon \mapsto C\varepsilon^{-\beta}$ of X exists. Conversely, any metric space of finite Assouad dimension is doubling. Proposition 5.5.4 will follow from the stronger proposition below.

Lemma 5.5.5. *For each $n \in \mathbb{N}$, the function $f_n(\varepsilon) = 2\varepsilon^{-\alpha_n^{-1}}$ is a covering function of (I_n, δ).*

Proof. Every subinterval of I_n of δ-diameter $r \in [0, s_n]$ has d-diameter $s_n\varphi_n^{-1}(s_n^{-1}r)$. Thus our goal is to show that for every $r \in [0, s_n]$, and every $\varepsilon \in (0, 1/2)$, every subset of I_n of d-diameter $s_n\varphi_n^{-1}(s_n^{-1}r)$ can be covered by no more than $(\varepsilon^{-\alpha_n} + 1)$ subintervals of I_n of

d-diameter at most $s_n\varphi_n^{-1}(s_n^{-1}\varepsilon r)$. The number of subintervals we need can be bounded from above by

$$\frac{s_n\varphi_n^{-1}(s_n^{-1}r)}{s_n\varphi_n^{-1}(s_n^{-1}\varepsilon r)}+1\leq\sup_{y\in(0,1]}\frac{\varphi_n^{-1}(y)}{\varphi_n^{-1}(\varepsilon y)}+1=\sup_{y\in[\varphi_n(c_n),1]}\frac{\varphi_n^{-1}(y)}{\varphi_n^{-1}(\varepsilon y)}+1.$$

We claim that the last supremum is attained when $y=1$. This is equivalent to

$$\frac{\varphi_n(\varphi_n^{-1}(\varepsilon y))}{\varphi_n(\varphi_n^{-1}(y))}\leq\varphi_n\left(\frac{\varphi_n^{-1}(\varepsilon y)}{\varphi_n^{-1}(y)}\right),\tag{5.5.1}$$

which follows from Lemma 5.3.2.

Suppose $\varphi_n(x_0)=\varepsilon$. When $\varepsilon>\varphi_n(c_n)$, we have $x_0>c_n$, and

$$\varepsilon^{\alpha_n^{-1}}=\frac{x_0-c(1-\alpha_n)}{1-c(1-\alpha_n)}\leq x_0.$$

When $0<\varepsilon<\varphi_n(c_n)$, we have $x_0<c_n$ and $\varepsilon=\varphi_n(x_0)=\frac{x_0}{c}\varphi_n(c_n)$. As $\varphi_n(c_n)^{\alpha_n^{-1}}\leq c$, we have

$$\frac{1}{x_0}=\frac{\varphi(c)}{c\varepsilon}=\varepsilon^{-\alpha_n^{-1}}\left(\frac{\varepsilon}{\varphi(c)}\right)^{\alpha_n^{-1}-1}\frac{\varphi(c)^{\alpha_n^{-1}}}{c}\leq\varepsilon^{-\alpha_n^{-1}}.$$

In any case, we can take the covering function of I_n to be

$$\varepsilon^{-\alpha_n^{-1}}+1\leq 2\varepsilon^{-\alpha_n^{-1}}.$$

□

Proposition 5.5.6. *The Assouad dimension of* $(\mathbb{R},\delta)$ *is* 1.

Proof. Let $\beta>1$ be arbitrary. There exists $N\in\mathbb{N}$ such that when $n\geq N$, $\alpha^{-1}<\beta$. Let $C=\max_{n<N}\{2^{\alpha_n^{-1}-\beta}\}\geq 1$. Then the function $\varepsilon\mapsto 2C\varepsilon^{-\beta}$, where $\varepsilon\in(0,1/2]$, is a covering function of (I_n,δ) for all $n\in\mathbb{N}$. Thus $\varepsilon\mapsto 4C\varepsilon^{-\beta}$ is a covering function of $(\mathbb{R},\delta)$. □

5.6 The Proof of Theorem 1.0.10

Let $d\geq 2$. We will denote by d_{Euclid} the Euclidean metric on $\mathbb{R}^d$. Let

$$X_d=\left(\mathbb{R}\times\mathbb{R}^{d-1},\sqrt{\delta^2+d_{Euclid}^2}\right)$$

be the product of $(\mathbb{R},\delta)$ and $(\mathbb{R}^{d-1},d_{Euclid})$. Write $\rho_n=\sqrt{\delta^2+d_{Euclid}^2}$. Here are some facts about X_d.

Proposition 5.6.1. *(a) Every weak tangent of X_d is isometric to $(\mathbb{R}^d, 0, d_{Euclid})$.*

(b) X_d is doubling and linearly locally contractible.

Proof. (a) Every weak tangent of X_d is of the form $(T \times \mathbb{R}^{n-1}, (x, 0), d_T \times d_{Euclid})$, where (T, x, d_T) is a weak tangent of $(\mathbb{R}, \delta)$.

(b) Recall that $(\mathbb{R}, \delta)$ is doubling (Proposition 5.5.4). X_d is the product of doubling metric spaces, hence doubling. By Proposition 5.5.3, $(\mathbb{R}, \delta)$ is C-linearly locally contractible for some $C > 1$. Let $x = (x_1, x_2)$ be any point in $\mathbb{R} \times \mathbb{R}^{d-1}$, and $r > 0$ be arbitrary. The ball $B(x, r)$ in X_d can first be contracted to $\{x_1\} \times B(x_2, r)$ within a $B(x, Cr)$, which can then be contracted to the point $\{x_1, x_2\}$. $\square$

Every finite segment in $(\mathbb{R}, \delta)$ is rectifiable. Let μ_1 be a the measure on $(\mathbb{R}, \delta)$ given by length. For $d \geq 2$, let μ_d be the product measure $\mu_1 \times \lambda_{d-1}$ on X_d, where λ_{d-1} is the $(d-1)$-dimensional Lebesgue measure on $\mathbb{R}^{d-1}$.

In the remaining of this section we show that X_d is not quasisymmetrically equivalent to $\mathbb{R}^d$. To do that we consider a geometric quantity, roughly preserved under quasisymmetry, called modulus. Given a family Γ of curves in a measured metric space (X, d_X, μ), we say that a Borel function $\rho : X \to [0, \infty)$ is admissible if for all locally rectifiable $\gamma \in \Gamma$,

$$\int_\gamma \rho(x)\, ds \geq 1.$$

Let $Q > 0$. We define the *Q-modulus* of Γ as

$$\operatorname{mod}_Q(\Gamma) = \inf\{\int_X \rho^Q\, d\mu : \rho \text{ admissible } \}.$$

Let $E, F \subset X$ be two disjoint nondegenerate continua in X. Let $\Gamma_{E,F}$ to be the collection of all rectifiable curves joining E and F. We write $\operatorname{mod}_Q(E, F) = \operatorname{mod}_Q(\Gamma_{E,F})$.

Moduli behave nicely under quasisymmetry, as illustrated by the following theorem.

Theorem 5.6.2 (Tyson, [Tys98])**.** *Let X, Y be locally compact, connected, Q-Ahlfors regular metric spaces, where $Q > 1$, and let $f : X \to Y$ be a quasisymmetric homeomorphism. Then there exists $C > 1$ such that for all curve family $\Gamma \subset X$, we have*

$$\frac{1}{C} \operatorname{mod}_Q(\Gamma) \leq \operatorname{mod}_Q(f(\Gamma)) \leq C \operatorname{mod}_Q(\Gamma).$$

See the next section for the definition of Q-Ahlfors regularity.

We can now prove that X_d is not quasisymmetrically equivalent to $\mathbb{R}^d$. The idea is that if the two spaces are quasisymmetrically equivalent, then Γ and $f(\Gamma)$ should have comparable moduli for any curve family Γ. We know that $\mathbb{R}^d$ has the property that any disjoint nondegenerate continua E, F in $\mathbb{R}^d$ satisfy

$$\operatorname{mod}_d(E, F) \leq \phi\left(\frac{d(E, F)}{\operatorname{diam}(E) \wedge \operatorname{diam}(F)}\right). \tag{5.6.1}$$

for some non-increasing function $\phi : [0, \infty) \to (0, \infty)$. However, Proposition 5.6.3 below shows that some sequence disjoint nondegenerate continua E_n, F_n in X_d do not satisfy (5.6.1). The only problem is that X_d is not d-Ahlfors regular, so we cannot apply Theorem 5.6.2 directly. In Proposition 5.6.4, however, we will show that the inequality

$$\operatorname{mod}_d(E_n, F_n) \leq C \operatorname{mod}_d(f(E_n), f(F_n))$$

holds for some C independent of C.

From now on, we will denote $\Delta(E, F) = \frac{d(E,F)}{\operatorname{diam}(E) \wedge \operatorname{diam}(F)}$.

Proposition 5.6.3. *There exists $E_n, F_n \subset X_d$ such that $\Delta(E_n, F_n) = \frac{d(E_n, F_n)}{\operatorname{diam} E_n \wedge \operatorname{diam} F_n} = 1$ and $\operatorname{mod}_d(E_n, F_n) \to +\infty$.*

Proof. Take $E_n = I_n \times \{0\} \times [0, s_n]^{d-2}, F_n = I_n \times \{s_n\} \times [0, s_n]^{d-2}$. Then $d(E_n, F_n) = s_n$, and $\operatorname{diam} E_n = \operatorname{diam} F_n = s_n$, therefore

$$\Delta(E_n, F_n) = \frac{d(E_n, F_n)}{\operatorname{diam} E_n \wedge \operatorname{diam} F_n} = 1.$$

For each $n \in \mathbb{N}$, and for $x \in I_n$ and $(v_2, \ldots, v_{d-1}) \in [0, s_n]^{d-2}$, let $\gamma_{x,v_2,\ldots,v_{d-1}}$ be the path

$$t \mapsto (x, t, v_2, \ldots, v_{d-1})\,, t \in [0, s_n].$$

Let

$$\Gamma_n = \{\gamma_{x,v_2,\ldots,v_{d-1}} : x \in I_n, (v_2, \ldots, v_{d-1}) \in [0, s_n]^{d-2}\}$$

be the family of straight lines joining E_n and F_n that meet E_n orthogonally. Then

$$\operatorname{mod}_d(E_n, F_n) \geq \operatorname{mod}_d(\Gamma_n).$$

Let $\rho : X_d \to \mathbb{R}_{\geq 0}$ be an admissible function for Γ_n. This means for all $\gamma_{x,v_2,v_3,\ldots,v_{d-1}} \in \Gamma_n$, we have

$$\int_{\gamma_{x,v_2,v_3,\ldots,v_{d-1}}} \rho(t)\,dt = \int_0^{s_n} \rho(x,t,v_2,\ldots,v_{d-1})\,dt \geq 1.$$

Integrating over $I_n \times [0,s_n]^{d-2}$ and applying Fubini's theorem, we get

$$\begin{aligned}\int_{I_n\times[0,s_n]^{d-2}} d\mu_1 \times \lambda_{d-2} &\leq \int_{I_n\times[0,s_n]^{d-2}} \int_{\gamma_{x,v_2,v_3,\ldots,v_{d-1}}} \rho(t)\,dt\,d(\mu_1\times\lambda_{d-2}) \\ &= \int_{I_n\times[0,s_n]^{d-1}} \rho\,d\mu_d.\end{aligned}$$

Applying Hölder's inequality, we get

$$\int_{I_n\times[0,s_n]^{d-1}} \rho\,d\mu_d \leq \left(\int_{I_n\times[0,s_n]^{d-1}} d\mu_d\right)^{\frac{1}{\delta}} \left(\int_{I_n\times[0,s_n]^{d-1}} \rho^d\,d\mu_d\right)^{\frac{1}{d}}$$

where δ is the conjugate exponent of d (so $\frac{1}{\delta}+\frac{1}{d}=1$). Thus we have

$$\begin{aligned}\int_{I_n\times[0,s_n]^{d-1}} \rho^d\,d\mu_d &\geq \left(\int_{I_n\times[0,s_n]^{d-1}} d\mu_d\right)^{-\frac{d}{\delta}} \left(\int_{I_n\times[0,s_n]^{d-2}} d\mu_1\times\lambda_{d-2}\right)^{d} \\ &= \left((s_n)^{(d-1)}\ell(I_n)\right)^{-\frac{d}{\delta}} \left((s_n)^{d-2}\ell(I_n)\right)^{d} \\ &= 2^n\ell(I_n).\end{aligned}$$

This is true for all admissible function ρ, therefore

$$\operatorname{mod}_d(\Gamma_n) \geq s_n^{-1}\ell(I_n).$$

As $n\to\infty$, $s_n^{-1}\ell(I_n) = L(\alpha_n, c_n) \to +\infty$. We have

$$\operatorname{mod}_d(E_n,F_n) \geq \operatorname{mod}_d(\Gamma_n) \to +\infty.$$

□

Proposition 5.6.4. *X_d is not quasisymmetrically equivalent to any subset of $\mathbb{R}^d$.*

Proof. For each $n\in\mathbb{N}$, and for $x\in I_n, (v_2,\ldots,v_{d-1}) \in [0,s_n]^{d-2}\}$, let $\gamma_{x,v_2,\ldots,v_{d-1}}$ be the path

$$t \mapsto (x,t,v_2,\ldots,v_{d-1})\,, t\in[0,s_n].$$

Let

$$\Gamma_n = \{\gamma_{x,v_2,\ldots,v_{d-1}} : x \in I_n, (v_2, \ldots, v_{d-1}) \in [0, s_n]^{d-2}\}$$

be the family of straight lines joining E_n and F_n that meet E_n orthogonally. Suppose $f : X_d \to \mathbb{R}^d$ is an η-quasisymmetric embedding. By abuse of notation, we also write γ for the image of the path γ. For a path family Γ, define $\Gamma^* = \bigcup_{\gamma\in\Gamma} \gamma$, and define $\operatorname{diam}(\Gamma) = \inf\{\operatorname{diam}\gamma : \gamma \in \Gamma\}$.

Choose E_n and F_n as in Proposition 5.6.3. We know from $\operatorname{diam}(E_n) \wedge \operatorname{diam}(F_n) = d(E_n, F_n)$ that

$$\frac{d(f(E_n), f(F_n))}{\operatorname{diam}(f(E_n)) \wedge \operatorname{diam}(f(F_n))} \sim 1,$$

where the implicit constant for $\sim$ depends only on η. For the same reason, there exists $\alpha > 1$, depending only on $\alpha, \beta > 1$, depending only on η, so that $f(\Gamma_n)^* \subset B_n$ for a ball B_n with diameter $r_n \leq \alpha\, d(E_n, F_n) \leq \beta \operatorname{diam}(f(\Gamma))$.

For each $n \in \mathbb{N}$, we can cover Γ^* by squares $\{R_i\}_{i\in I_n}$ of diameter $s_n\varphi_n(c_n)$ so that their sides are either parallel to the paths in Γ or orthogonal to the paths in Γ. We can choose $\{R_i\}$ so that these R_i's don't overlap and their union is precisely Γ^*. For each rectifiable path γ, denote by $\ell(\gamma)$ its length. Let

$$\rho_n = (\operatorname{diam} f(\Gamma_n))^{-1} \sum_{i\in I_n} \frac{\operatorname{diam}(f(R_i))}{\operatorname{diam}(R_i)} \mathbb{1}_{f^{-1}(B_n)\cap R_i}.$$

be a function on X_d. For all $\gamma \in \Gamma_n$,

We have

$$\int_\gamma \rho_n(s)\, ds = (\operatorname{diam} f(\Gamma_n))^{-1} \sum_{i\in I_n} \frac{\operatorname{diam}(f(R_i))}{\operatorname{diam}(R_i)} \ell(f^{-1}(B_n) \cap R_i \cap \gamma).$$

For each i, $f^{-1}(B_n) \cap R_i \cap \gamma = R_i \cap \gamma$. When $R_i \cap \gamma \neq \emptyset$, $\ell(R_i \cap \gamma) = \operatorname{diam}(R_i)$. As $\{R_i\}_{i\in I_n}$ covers Γ^*, we have

$$\int_\gamma \rho_n(s)\, ds \geq (\operatorname{diam} f(\Gamma_n))^{-1} \sum_{i\in I_n, R_i\cap\gamma\neq\emptyset} \operatorname{diam}(f(R_i)) \geq (\operatorname{diam} f(\Gamma_n))^{-1} \operatorname{diam}(f(\gamma)) \geq 1.$$

This means ρ_n is admissible for Γ_n. We have

$$\begin{aligned}\operatorname{mod}_d \Gamma_n \leq \int \rho_n^d \, d\mu_d &= (\operatorname{diam} f(\Gamma_n))^{-d} \sum_{i \in I_n} \left(\frac{\operatorname{diam}(f(R_i))}{\operatorname{diam}(R_i)} \right)^d \mu_d(f^{-1}(B_n) \cap R_i) \\ &= (\operatorname{diam} f(\Gamma_n))^{-d} \sum_{i \in I_n} \left(\frac{\operatorname{diam}(f(R_i))}{\operatorname{diam}(R_i)} \right)^d \mu_d(R_i) \\ &= (\operatorname{diam} f(\Gamma_n))^{-d} \sum_{i \in I_n} \left(\frac{\operatorname{diam}(f(R_i))}{\operatorname{diam}(R_i)} \right)^d \operatorname{diam}(R_i)^d.\end{aligned}$$

For the last equality, we make use of the fact that R_i are chosen so small that $\mu_d(R_i) = \operatorname{diam}(R_i)^d$. We get

$$\operatorname{mod}_d \Gamma_n \lesssim (\operatorname{diam} f(\Gamma_n))^{-d} \sum_{i \in I_n} (\operatorname{diam}(f(R_i)))^d \lesssim (\operatorname{diam} f(\Gamma_n))^{-d} \sum_{i \in I_n} \lambda_d(f(R_i)).$$

Here we use the fact that the Lebesgue λ_d on $\mathbb{R}^d$ is d-Ahlfors regular and that f is a quasisymmetric. Since $\{f(R_i)\}_{n \in \mathbb{N}}$ are disjoint subsets of B_n, we have

$$\operatorname{mod}_d \Gamma_n \lesssim (\operatorname{diam} f(\Gamma_n))^{-d} \lambda_d(B_n) \lesssim 1.$$

where all the implicit constants for $\lesssim$ depends only on η. But this is a contradiction to Proposition 5.6.3. □

From Proposition 5.6.1 and Proposition 5.6.4, X_d is homeomorphic to $\mathbb{R}^d$, doubling and linearly locally connected, it is not quasisymmetric to $\mathbb{R}^d$. Our proof shows that the any ball in X_d centered at 0 cannot be quasisymmetrically embedded into $\mathbb{R}^d$.

We conclude this section with a proof of Theorem 1.0.10.

Proof of Theorem 1.0.10. The d-dimensional unit ball $B(0,1)$ in $\mathbb{R}^d$, equipped with the metric

$$\widetilde{\rho}(x,y) = \frac{\rho\left(\frac{x}{1-|x|}, \frac{y}{1-|y|}\right)}{1 + \rho\left(\frac{x}{1-|x|}, \frac{y}{1-|y|}\right)}.$$

The completion of the space $(B(0,1), \widetilde{\rho})$ is $\overline{B}(0,1)$, and the metric on the boundary is same as the Euclidean metric. gluing the space $\overline{B}(0,1)$ with another hemisphere to form a topological d-sphere. This d-sphere is doubling, locally linearly contractible, and every weak tangent is isometric to $\mathbb{R}^d$, but it cannot be a quasisphere. □

5.7 Ahlfors Regularity

Let $Q > 0$. A measured metric space (X, d, μ) is said to be *Q-Ahlfors regular* if for all $x \in X$ and $r \leq \operatorname{diam} X$, we have

$$\mu(B(x,r)) \sim r^Q.$$

Ahlfors regularity and Assouad dimension are related concepts. We record a result from [Hei01]:

Theorem 5.7.1. *[Hei01, Theorem 14.6] Let X be a complete, connected metric space of finite Assouad dimension β. Then for each $Q > \beta$, there exists a quasisymmetric homeomorphism of X onto a closed Q-Ahlfors regular subset of some $\mathbb{R}^N$.*

With these facts our example gives:

Theorem 5.7.2. *For every $Q > 2$, there exists a Q-Ahlfors regular and linearly locally contractible metric space X that is topologically a 2-sphere such that every weak tangent is uniformly quasisymmetric to $\mathbb{R}^2$ but X is not quasisymmetric to the standard 2-sphere.*

Proof. Let $Q > 2$. By Proposition 5.5.6, the Assouad dimension of X_2 is 2. Proposition 5.7.1 says that there exist a distortion function $\eta : [0, \infty) \to [0, \infty)$ and an η-quasisymmetry $\varphi : X_2 \to X'$, where X' is a closed Q-Ahlfors regular subset of $\mathbb{R}^N$. By Proposition 5.6.4 and Proposition 5.6.1(b), X' is not quasisymmetric to the standard 2-sphere, but every weak tangent of X' is η-quasisymmetric to $(\mathbb{R}^2, 0)$. $\square$

CHAPTER 6

Porous Subsets of Visual Spheres

6.1 Introduction

In this chapter, we study the f-invariant porous subsets of the visual sphere. We ask whether there are porous subset that are large in some sense. This question turns out to be related to the topological data of f that can be described by multicurves.

A simple closed curve γ in $\mathbb{S}^2\setminus \operatorname{post}(f)$ is *peripheral* if one of the components of $\mathbb{S}^2\setminus\gamma$ contains at most one point in $\operatorname{post}(f)$, and *non-peripheral* otherwise. A *multicurve* is a finite, disjoint, non-isotopic collection of non-peripheral simple closed curves in $\mathbb{S}^2\setminus \operatorname{post}(f)$. A multicurve Γ is said to be *f-stable* if, for all $\gamma \in \Gamma$, each non-peripheral simple closed curve in $f^{-1}(\gamma)$ is isotopic relative $\operatorname{post}(f)$ to a curve in Γ.

Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be an f-stable multicurve Γ. For each $i, j \in \{1, \dots, n\}$, let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ isotopic to γ_i in $\mathbb{S}^2\setminus \operatorname{post}(f)$, and let $d_{i,j,\alpha} > 0$ be the degree of the restriction map $f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j$. Let $p > 0$ be arbitrary. Define

$$M(f, \Gamma, p) : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$$

by

$$M(f, \Gamma, p)(\gamma_j) = \sum_{\gamma_i\in\Gamma} \sum_{\alpha} d_{i,j,\alpha}^{1-p} \gamma_i.$$

Since $M(f, \Gamma, p)$ is represented by a non-negative square matrix, the Perron-Frobenius Theorem implies that the matrix $M(f, \Gamma, p)$ has a real non-negative eigenvalue $\Lambda(M(f, \Gamma, p))$ equal to its spectral radius.

When $p = 2$, the matrix $f_{\Gamma,2}$ is usually referred to as the Thurston matrix associated to f and Γ. Let $\Lambda(\Gamma, p)$ be the largest eigenvalue of $M(f, p, \Gamma)$. Thurston gave a characterization of the rationality of f in terms of these matrices if f has hyperbolic orbifold. An *orbifold* of f can be described by a function $\alpha : \mathbb{S}^2 \to \mathbb{N} \cup \{\infty\}$, such that $\alpha(x)$ is the lowest common divisor of $\{\deg_{x'}(f^n) : n \in \mathbb{N}, f^n(x') = x\}$. Notice that $\alpha(x) = 1$ for every $x \in \mathbb{S}^2 \setminus \operatorname{post}(f)$. The Thurston map f is said to have hyperbolic orbifold if

$$2 - \sum_{x \in \operatorname{post}(f)} 1 - \frac{1}{\alpha(x)} < 0.$$

Theorem 6.1.1 (Thurston's obstruction)**.** *A Thurston map f with hyperbolic orbifold is not equivalent to a rational map if and only if there exists an f-stable family Γ of non-isotopic non-peripheral simple closed curves such that $\Lambda(\Gamma, 2)$ greater than or equal to 1.*

The exact statement and the proof are contained in [DH93] and [Pil01].

A multicurve Γ is called *irreducible* if there exists an iterate f^n of f such that every entry of $M(f^n, \Gamma, p)$ is positive. If Γ contains an irreducible multicurve, then the function

$$p \mapsto \Lambda(\Gamma, p)$$

is strictly decreasing on $[1, \infty)$ ([HP08, Lemma A2]), and there exists unique $Q(\Gamma) \geq 1$ such that $\Lambda(\Gamma, Q(\Gamma)) = 1$. If Γ does not contain any irreducible multicurve, we define $Q(\Gamma) = 0$. Define

$$Q(f) = \sup\{Q(\Gamma) : \Gamma \text{ multicurve}\}.$$

When f is an expanding Thurston map without periodic critical points, we have the following inequality.

Theorem 6.1.2 (Haissinski-Pilgrim, [HP08])**.**

$$Q(f) \leq \dim_{\mathrm{AR}}(f).$$

We are interested in the porous subsets of visual spheres.

Definition 6.1.3. *Let (X, d) be a metric space, and K a compact subset of X. Let $a \in (0, 1)$. We say that K is* a-porous *in X if for all $x \in K$, and for every $0 < r < \operatorname{diam}_d(X)$, there exists $y \in B_X(x, r)$ such that $B(y, ar) \cap K = \emptyset$. We say that K is* porous *if it is a-porous for some $a \in (0, 1)$. A compact subset K of a visual sphere of f is said to be f-invariant if $f(K) \subset K$.*

We would like to know when f exists a "thick" family of curves.

Definition 6.1.4. *An f-stable multicurve Γ is said to be p-*thick *if either one of the following conditions holds:*

1. $\Lambda(\Gamma, p) > 1$*; or*

2. $\Lambda(\Gamma, p) = 1$ *and there exists $\gamma \in \Gamma$ such that $f^{-1}(\gamma)$ contains a peripheral component.*

The existence of a p-thick f-stable multicurve is related to the existence of a curve family having large discrete p-modulus. For each $n \in \mathbb{N}$, let G_n be the graph with vertex set $V_n = \mathcal{D}_n(\mathbb{S}^2)$, and join two tiles $\tau, \tau' \in \mathcal{D}_n$ by an edge if and only if $\tau \cap \tau' \neq \emptyset$. Given a curve family Γ on a graph $G_n = (V_n, E_n)$, we say that $\rho : V_n \to [0, \infty)$ is Γ-admissible if, for all $\gamma \in \Gamma$, we have

$$\sum_{\tau \in V_n : \tau \cap \gamma \neq \emptyset} \rho(\tau) \geq 1.$$

Given $p > 0$, we define

$$\mathrm{mod}_p(\Gamma, \Lambda^{-n}) = \inf\{\sum_{\tau \in V_n} \rho(\tau)^p : \rho \text{ is } \Gamma\text{-admissible}\}.$$

Definition 6.1.5. *A family Γ of curves in $(\mathbb{S}^2, \rho)$ is said to be p-*thick *if*

$$\inf_{\gamma \in \Gamma} \mathrm{diam}(\gamma) > 0$$

and

$$\limsup_{n \to \infty} \mathrm{mod}_p(\Gamma, \Lambda^{-n}) > 0.$$

Theorem 6.1.6. *Let f be an expanding Thurston map without periodic critical points, and let $p \geq 2$. If there exists a p-thick f-stable multicurve, then there exists a curve family Γ supported on an f-invariant porous subset such that Γ is p-thick.*

When $p = 2$, we give an example (Example 6.9.3) to show that if f does not have 2-thick f-stable multicurve, then every curve family Γ supported on a f-invariant porous subset is not 2-thick.

6.2 Discrete Moduli

Discrete moduli, also called combinatorial moduli, are used to study metric spaces that do not have many locally rectifiable curves. They are particularly useful in the study of Ahlfors regular conformal dimension. See [KL04], [HP08], and [CP13]. In this section we define discrete moduli of metric spaces and prove a few basic properties about discrete moduli.

Let $G = (V, E)$ be a connected graph. A sequence of points $\gamma = \{v_1, v_2, \dots, v_n\}$ is called a *curve* in G if v_i neighbors v_{i+1} for all $i = 1, 2, \dots, n-1$. A curve γ in G is *simple closed* if v_n neighbors v_1 and v_i's are distinct. For any function $\rho : V \to [0, \infty)$, we define the *ρ-length* of γ to be

$$\ell_\rho(\gamma) = \sum_{v \in \gamma} \gamma(v).$$

Note that γ may not be simple, but each vertex appears at most once in the above sum. Let Γ be a family of curves in V. We say that ρ is Γ-*admissible* if $\ell_\rho(\gamma) \geq 1$ for all $\gamma \in \Gamma$. Let $p > 0$. Define the p-modulus of Γ in G by

$$\mathrm{mod}_p(\Gamma, G) = \inf_\rho \{\sum_{v \in V} \rho(v)^p\},$$

where the infimum is taken over all Γ-admissible function ρ.

We list some basic properties of modulus. These propositions are related to the Serial Law and Parallel Law from the theory of electrical networks, hence their names.

Proposition 6.2.1 (The Parallel Law)**.** *Let $G = (V, E)$ be a connected graph, and let Γ be a family of disjoint curves. Then*

$$mod_p(\Gamma, G) = \sum_{\gamma \in \Gamma} \mathrm{mod}_p(\{\gamma\}, G).$$

Proof. If $\rho : V \to [0, \infty)$ is a Γ-admissible function, then for each $\gamma \in \Gamma$, the function

$$\rho_\gamma(v) = \begin{cases} \rho(v), & \text{if } v \in \gamma \\ 0, & \text{otherwise} \end{cases}$$

is $\{\gamma\}$admissible. We have

$$\sum_{v\in V}\rho(v)^p \geq \sum_{\gamma\in\Gamma}\sum_{v\in\Gamma}\rho(v)^p \geq \sum_{\gamma\in\Gamma}\operatorname{mod}_p(\{\gamma\},G).$$

Infimizing over all possible choice of ρ we get

$$\operatorname{mod}_p(\Gamma,G) \geq \sum_{\gamma\in\Gamma}\operatorname{mod}_p(\{\gamma\},G).$$

Conversely, for each $\gamma\in\Gamma$, let ρ_γ be an admissible function for $\{\gamma\}$ on G. Then the function

$$\rho'_\gamma(v) = \begin{cases}\rho_\gamma(v), \text{ if } v\in\gamma \\ 0, \text{ otherwise}\end{cases}$$

is $\{\gamma\}$-admissible. Let $\rho = \sum_{\gamma\in\Gamma}\rho'_\gamma$. Then ρ is Γ-admissible, and we have

$$\operatorname{mod}_p(\Gamma,G) \leq \sum_{v\in V}\rho(v)^p = \sum_{\gamma\in\Gamma}\sum_{v\in\gamma}\rho'_\gamma(v)^p \leq \sum_{\gamma\in\Gamma}\sum_{v\in\gamma}\rho_\gamma(v)^p$$

Infimizing over all possible choice of ρ_γ, we get

$$\operatorname{mod}_p(\Gamma,G) \leq \sum_{\gamma\in\Gamma}\operatorname{mod}_p(\{\gamma\},G).$$

□

We can view G as a topological space on the underlying set $V\cup E$, such that each edge in E is homeomorphic to the unit interval $[0,1]$ joining two vertices in V.

Proposition 6.2.2 (The Serial Law)**.** *Let $G=(V,E)$, $\widetilde{G}=(\widetilde{V},\widetilde{E})$ be two connected graphs and let $\pi:\widetilde{G}\to G$ be a covering map such that $\pi^{-1}(V)=\widetilde{V}$ and $\pi(\widetilde{V})=V$. Let $\Gamma=\{\gamma\}$ be a family of closed curves in G and $\widetilde{\Gamma}$ be a family of closed curves in G such that the map π lifts each curve γ in Γ to a curve γ' in Γ'. Let us further assume that there exists an integer $d\in\mathbb{N}_0$ such that for all $\gamma\in\Gamma$, $\ell(\gamma')=d\ell(\gamma)$. Then*

$$\operatorname{mod}_p(\widetilde{\Gamma},\widetilde{G}) \leq d^{1-p}\operatorname{mod}_p(\Gamma,G).$$

Proof. Given a Γ-admissible function ρ on G, we define $\widetilde{\rho}:\widetilde{V}\to[0,\infty)$ by the formula $\widetilde{\rho}=\frac{1}{d}\rho\circ\pi$. Then $\widetilde{\rho}$ is $\widetilde{\Gamma}$-admissible on $\widetilde{G}$, and we have

$$\sum_{v\in\widetilde{V}}\widetilde{\rho}(v)^p = d\sum_{w\in V}\frac{1}{d^p}\rho(w)^p = d^{1-p}\sum_{w\in V}\rho(w)^p.$$

Infimizing over all possible choices of ρ, we get the inequality

$$\operatorname{mod}_p(\widetilde{\Gamma}, \widetilde{G}) \leq d^{1-p} \operatorname{mod}_p(\Gamma, G).$$

$\square$

Following [CP13], we describe the notion of hyperbolic filling of a doubling and uniformly perfect metric space. Similar constructions can be found in [BP03], [Lin16], and [BS18], among other works. We do not need the full construction of a hyperbolic filling, but we describe the construction in full regardless, as we believe this gives the reader the right perspective.

Let (X, d) be a doubling and uniformly compact metric space. Let V be a cover of (X, d). Suppose that there exists $\varepsilon > 0$ such each $\tau \in V$ contains a point $x_\tau \in X$ such that $B(x_\tau, \varepsilon) \subset \tau$, and if $\tau, \tau' \in V$ are two different sets, then $B(x_\tau, \varepsilon) \cap B(x_{\tau'}, \varepsilon) = \emptyset$. We call V a *ε-thick ε-separated cover of* (X, d).

Let V be a ε-thick ε-separated cover of (X, d). We construct a graph G, called the *graph of* V, whose vertex set is V, and $\tau, \tau' \in V$ are connected by an edge if and only if $\tau \neq \tau'$ and $\tau \cap \tau' \neq \emptyset$.

Let (X, d) be a metric space, let V be a cover of X, and let $G = (V, E)$ be a connected graph. For each curve $\gamma : [0, 1] \to X$, we say that the curve $\gamma' = \{v_1, \dots, v_n\}$ is a *projection of* γ *onto* G if there exists a increasing onto map $\varphi : [0, 1] \to \{1, 2, \dots, n\}$ such that $\gamma(t)) \in \gamma'(\varphi(t))$. Given a family Γ of curves in X, let Γ_G be the collection of projections of curves in Γ onto G, and call Γ_G the *projection of* Γ *onto* G. For any $p \geq 0$, we write

$$\operatorname{mod}_p(\Gamma, G) = \operatorname{mod}_p(\Gamma_G, G).$$

Proposition 6.2.3. *Let V, V' be two ε-thick ε-separated covers of (X, d). Let G, G' be the graphs of V and V' respectively. Assume*

$$\sup_{v \in V'} |\{w \in V : w \cap v \neq \emptyset\}| = C_1 < \infty,$$

and

$$\sup_{w \in V} |\{v \in V' : w \cap v \neq \emptyset\}| = C_2 < \infty.$$

Then there exists $C > 1$, depending only on C_1 and C_2, such that for all curve family Γ in (X, d), and for any $p \geq 1$, we have

$$C^{-1} \operatorname{mod}_p(\Gamma, G) \leq \operatorname{mod}_p(\Gamma, G') \leq C \operatorname{mod}_p(\Gamma, G).$$

Proof. Let ρ be an admissible function on G_n. Define $\widetilde{\rho} : V' \to [0, \infty)$ by

$$\widetilde{\rho}(v) = \sum_{w \in V : w \cap v \neq \emptyset} \rho(w).$$

Let $\gamma' \in \Gamma_{G'}$ be a curve. Then γ' is a projection of some $\widetilde{\gamma} \in \Gamma$. Let $\gamma \in \Gamma_G$ be a projection of $\widetilde{\gamma}$ onto G_n. Then for every $w \in \gamma$, there exists $v \in \gamma'$ such that $w \cap v \cap \gamma \neq \emptyset$. Thus

$$\sum_{v \in V' : \gamma' \cap v \neq \emptyset} \widetilde{\rho}(v) = \sum_{v \in \gamma'} \sum_{w \in V : w \cap v \neq \emptyset} \rho(w) \geq \sum_{w \in V : w \cap \gamma \neq \emptyset} \rho(w) \geq 1.$$

This shows that $\widetilde{\rho}$ is admissible in G.

Using the definitions of C_1 and C_2, we get

$$\begin{aligned}
\sum_{v \in V} \widetilde{\rho}(v)^p &= \sum_{v \in V} \left(\sum_{w \in V : w \cap v \neq \emptyset} \rho(w) \right)^p \\
&\leq \sum_{v \in V} C_2^{p-1} \sum_{w \in V : w \cap v \neq \emptyset} \rho(w)^p \\
&= C_2^{p-1} \sum_{w \in V} \sum_{v \in V : w \cap v \neq \emptyset} \rho(w)^p \\
&\leq C_2^{p-1} \sum_{w \in V} C_1 \rho(w)^p \\
&= C_1 C_2^{p-1} \sum_{w \in V} \rho(w)^p.
\end{aligned}$$

Thus

$$\operatorname{mod}_p(\Gamma_G, G) \leq C_1 C_2^{p-1} \sum_{w \in V} \rho(w)^p.$$

The above inequality holds for every function ρ that is admissible for Γ_{G_n} on G_n. We conclude that

$$\operatorname{mod}_p(\Gamma_G, G) \leq C \operatorname{mod}_p(\Gamma_G, G).$$

where $C = C_1 C_2^{p-1}$. The other symmetry is obtained by switching the role of G and G'. $\square$

We say that V is an $(\varepsilon, \varepsilon')$-*discretization of* (X,d) if V is a ε-thick ε-separated cover of (X,d), and there exists $\varepsilon' > \varepsilon$ such that for each $\tau \in V$, $\tau \subset B(x_\tau, \varepsilon')$.

Fix $a \in (0,1)$ and $\lambda > 1$, and suppose that for each $n \in \mathbb{N}$, V_n is a $(a^n, \lambda a^n)$-discretization of (X,d). Let $V = \coprod_{n\in\mathbb{N}} V_n$ be the disjoint union of the sets V_n. Let G whose vertex set is V such that $\tau, \tau' \in V$ are joined by an edge when $\tau \cap \tau' \neq \emptyset$ and $\tau \in V_n$ and $\tau' \in V_n$ for some $m, n \in \mathbb{N}$ with $|m-n| \leq 1$. The graph G is called a *hyperbolic filling of* (X,d) *with parameter* a, λ.

Let G_n be the graph of V_n. Let $p > 0$. For each $\tau \subset V_n$, and $k \geq 1$, let B be a ball with radius λa^n such that $\tau \in B$, and let $\Gamma_{\tau,n+k}$ be the set of paths in G_{n+k} that connect an element in $\{\tau \in V_{n+k} : \tau \cap B \neq \emptyset\}$ and an element in $\{\tau \in V_{n+k} : \tau \cap (X \backslash 2B) \neq \emptyset\}$. Here λB is the ball with the same center as B and λ times the radius of B. Define

$$M_{p,k} := \sup_{\tau \in V} \operatorname{mod}_p(\Gamma_{\tau,n+k}, G_{n+k}).$$

Let

$$Q_N = \inf\{p \in (0,\infty) : \liminf_{k\to\infty} M_{p,k} = 0\}.$$

Theorem 6.2.4. *[CP13, Theorem 1.2] Let (X,d) be a doubling, uniformly perfect, compact metric space. Then the* $\dim_{\mathrm{AR}}(X) = Q_N$.

Note that by Proposition 6.2.3, whether $Q_N = 0$ or not does not depend on the choice of the hyperbolic filling V.

We are interested in the case that (X,d) is a visual sphere of an expanding Thurston map f with no periodic critical point. Given such f, Yin[Yin11] constructed an object called the *tile graph of f*. In that construction, the set V_n is taken to be the collection $\mathcal{D}_n$ of tiles at level n. The graphs G_n are constructed by joining tiles $\tau, \tau' \in \mathcal{D}_n$ such that $\tau \cap \tau' \neq \emptyset$. We call G_n the level n tile graph.

Theorem 6.2.5. *[BM17, Lemma 8.11] Let $(\mathbb{S}^2, \rho)$ be the visual sphere constructed in Chapter 3. Then there exists a constant $\kappa \geq 1$ such that for every n-tile $\tau \in \mathbb{D}$, there exists a point $p \in \tau$ such that*

$$B_\rho(p, \kappa^{-1}\Lambda^{-n}) \subset \tau \subset B_\rho(p, \kappa\Lambda^{-n}).$$

Very similar mathematical objects were considered by Haïssinsky and Pilgrim's [HP09, Chapter 2], and similar result as the above one is obtained. In view of Proposition 6.2.3 and Theorem 6.2.5, for any curve family Γ in the visual sphere, and any $p > 1$, we write

$$\mathrm{mod}_p(\Gamma, \Lambda^{-n}) = \mathrm{mod}_p(\Gamma, G_n)$$

6.3 Symbolic Dynamics on Curves

Let Γ_0 be a f-stable multicurve in $\mathbb{S}^2 \backslash \operatorname{post}(f)$. For all $n \geq 1$, let Γ_n be the union of all non-peripheral components of $f^{-n}(\gamma)$ over all $\gamma \in \Gamma_0$.

We view curves in Γ_n as functions from $\mathbb{S}^1$ onto $\mathbb{S}^2 \backslash \operatorname{post}(f)$. To do that, for each $\gamma \in \Gamma_0$, we choose a parameterization of γ, which we also denote as γ. Thus γ can either be a injective function from $\mathbb{S}^1$ to $\mathbb{S}^2 \operatorname{post}(f)$ or the image of the function. Now suppose $\gamma_i \in \Gamma_1$ is a component in $f^{-1}(\gamma)$. Then there exists unique integer $d_i \in \mathbb{Z}\backslash\{0\}$ and a parameterization $\gamma_i : \mathbb{S}^1 \to \operatorname{post}(f)$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & \mathbb{S}^2\backslash f^{-1}(\operatorname{post}(f)) \arrow[d, "f"] \\
\mathbb{S}^1 \arrow[r, "d_j"'] \arrow[urr, "\gamma_i"] & \mathbb{S}^1 \arrow[r, "\gamma"'] & \mathbb{S}^2\backslash \operatorname{post}(f).
\end{tikzcd}$$

Thus each γ_i can also be regarded as a function $\gamma_i : \mathbb{S}^1 \to \operatorname{post}(f)$.

Let $\mathbb{A}^1 = \{1, 2, \ldots, |\Gamma_1|\}$. For each $j \in \mathbb{A}^1$, γ_j is isotopic to γ for exactly one $\gamma \in \Gamma_0$. Let

$$H_j : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^2\backslash \operatorname{post}(f)$$

be an isotopy such that $H_j(\cdot, 0) = \gamma_j$ and $H_j(\cdot, 1) = \gamma$. Suppose $\gamma_i \in \Gamma_1$ is a component in $f^{-1}(\gamma)$. Then there exists a unique isotopy

$$H_{ij} : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^2\backslash f^{-1}(\operatorname{post}(f))$$

such that $H_{ij}(\cdot, 1) = \gamma_i$ and

$$\begin{tikzcd}
 & & \mathbb{S}^2\backslash f^{-1}(\operatorname{post}(f)) \arrow[d, "f"] \\
\mathbb{S}^1 \times [0,1] \arrow[r, "d_i \times \mathrm{id}"'] \arrow[urr, "H_{ij}"] & \mathbb{S}^1 \times [0,1] \arrow[r, "H_j"'] & \mathbb{S}^2\backslash \operatorname{post}(f).
\end{tikzcd}$$

Let $\gamma_{ij}(\cdot) = H_{ij}(\cdot, 0)$.

Let $i_1, \ldots, i_n \in \{1, 2, \ldots |\Gamma_1|\}$. We say that $i_1 i_2 \ldots i_n$ is an *admissible word of length n for* (f, Γ_0) if for each $k = 1, 2, \ldots, k-1$, $\gamma_{i_k} \in \Gamma_1$ is isotopic to a component of the preimage of $\gamma_{i_{k+1}} \in \Gamma_1$ or the inverse of $\gamma_{i_{k+1}}$. Let $\mathbb{A}^n$ be the collection of all admissible words of length n for (f, Γ_0). We inductively define the parameterizations $\gamma_{i_1 \ldots, i_n} : \mathbb{S}^1 \to \mathbb{S}^2 \setminus \operatorname{post}(f)$ and the isotopies $H_{i_1 \ldots, i_n} : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^2 \setminus \operatorname{post}(f)$ in the following way.

Let $i_1 \ldots i_n$ be an admissible word for (f, Γ_0). Suppose $\gamma_{i_2 \ldots i_n} : \mathbb{S}^1 \to \mathbb{S}^2 \setminus \operatorname{post}(f)$ and $H_{i_2, \ldots, i_n} : \mathbb{S}^1 \times [0,1] \times \mathbb{S}^2 \setminus \operatorname{post}(f)$ are defined. Then there exists unique $H_{i_1 \ldots i_n}$ such that the following diagram commutes:

$$\begin{tikzcd} & & \mathbb{S}^2 \setminus f^{-1}(\operatorname{post}(f)) \arrow[d, "f"] \\ \mathbb{S}^1 \times [0,1] \arrow[r, "d_{i_1} \times \mathrm{id}"'] \arrow[urr, "H_{i_1 \ldots i_n}"] & \mathbb{S}^1 \times [0,1] \arrow[r, "H_{i_2 \ldots i_n}"'] & \mathbb{S}^2 \setminus \operatorname{post}(f). \end{tikzcd}$$

Now we set $\gamma_{i_1 \ldots i_n}(\cdot) = H_{i_1 \ldots i_n}(\cdot, 0)$.

The following estimate follows from [BM17, Lemma 8.9].

Lemma 6.3.1. *There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, $i_1 \ldots i_{n+1} \in \mathbb{A}^{n+1}$, and $(x,t) \in \mathbb{S}^1 \times [0,1]$, we have*

$$|H_{i_1 \ldots i_n}(x,t) - H_{i_1 \ldots i_{n+1}}(x,t)| \leq C \Lambda^{-n}.$$

Let $i_1, i_2, \ldots \in \{1, \ldots, |\Gamma_1|\}$ be a sequence of letters. We say that $i_1 i_2 \ldots$ is an *admissible infinte word for* (f, Γ_0) if and only if $i_1 \ldots i_n \in \mathbb{A}^n$ for all $n \in \mathbb{N}$. Let $\mathbb{A}^\infty$ be the collection of admissible infinte words for (f, Γ_0). As a corollary of Lemma 6.3.1, we have the following proposition:

Proposition 6.3.2. *Let $i_1 i_2 \ldots \in \mathbb{A}^\infty$ be an admissible infinite word for (f, Γ_0). Then the sequence $\gamma_{i_1 i_2 \ldots, i_n} : \mathbb{S}^1 \to \mathbb{S}^2$ converges uniformly as a sequence of functions to a function $\mathbb{S}^1 \to \mathbb{S}^2$, which we denote by $\gamma_{i_1 i_2 \ldots}$.*

Proposition 6.3.2 allows us to make sense of γ_ω for an admissible infinite word ω for (f, Γ_0). Let $\Gamma_\infty = \{\gamma_\omega : \omega \in \mathbb{A}^\infty\}$.

Proposition 6.3.3.

$$\inf_{\gamma \in \Gamma_\infty} \operatorname{diam}(\gamma) > 0.$$

Proof. First note that there exists $c > 0$ such that every simple closed curve γ in $\mathbb{S}^2 \setminus \operatorname{post}(f)$ that is isotopic to a curve in Γ_0 satisfies

$$\operatorname{diam}_\rho(\gamma) > c.$$

Thus for any $n \in \mathbb{N}$,

$$\inf_{\gamma \in \Gamma_n} \operatorname{diam}(\gamma) \geq c.$$

Taking the Hausdorff limit, we get

$$\inf_{\gamma \in \Gamma_\infty} \operatorname{diam}(\gamma) \geq c > 0.$$

□

6.4 Invariant Porous Subsets of Visual Spheres

In this section, we gave a few basic facts about existence and behavior of invariant porous subsets of visual spheres.

Lemma 6.4.1. *A compact subset K of $\mathbb{S}^2$ is f-invariant if and only if $K = \mathbb{S}^2 \setminus \bigcup_{n=0}^{\infty} f^{-n}(\mathbb{S}^2 \setminus K)$.*

Proof. Since $f(K) \subset K$, we have $K \subset f^{-1}(K) \subset f^{-2}(K) \subset \cdots$, therefore for all $n \in \mathbb{N}$,

$$\mathbb{S}^2 \setminus \bigcup_{n=0}^{\infty} f^{-n}(\mathbb{S}^2 \setminus K) = \bigcap_{n=0}^{\infty} f^{-n}(K) = K.$$

Conversely, if $K = \mathbb{S}^2 \setminus \bigcup_{n=0}^{\infty} f^{-n}(\mathbb{S}^2 \setminus K) = \bigcap_{n=0}^{\infty} f^{-n}(K)$, then

$$f(K) = f(\bigcap_{n=0}^{\infty} f^{-n}(K)) \subset \bigcap_{n=0}^{\infty} f(f^{-n}(K)) \subset \bigcap_{n=0}^{\infty} f^{-n}(f(K)) \subset \bigcap_{n=0}^{\infty} f^{-n}(K) = K.$$

□

Lemma 6.4.2. *An f-invariant subset K of the visual sphere is either $\mathbb{S}^2$ or porous.*

Proof. Suppose $K \neq \mathbb{S}^2$. Then there exists an open ball $B = B(x_0, \varepsilon)$ such that $B \cap K \neq \emptyset$. We may even assume, by choosing a smaller open ball if necessary, that B does not intersect that f-invariant Jordan curve $\mathcal{C}$. Without loss of generality, we may assume that B is in the white 0-tile.

Let $x \in K$ be a point. Then there exists $n_0 \in \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$, the open ball $B(x, r)$ contains a white $(n + n_0)$-tile τ such that $x \notin \tau$. By Theorem 3.2.2, $f^{-n}(B) \cap \tau$ contains a ball of radius $\Lambda^{-n}\varepsilon$. Thus K is porous. □

Lemma 6.4.3. *If K is an f^n-invariant compact porous subset, then $f(K)$ is compact, f^n-invariant, and porous.*

Proof. Since f is continuous, $f(K)$ is compact. The set K is f^n-invariant since

$$f^n(f(K)) = f(f^n(K)) \subset f(K).$$

Since $f^n(K) \subset K$ and $K \neq \mathbb{S}^2$, we have $f^{n-1}(f(K)) = f^n(K) \neq \mathbb{S}^2$. Therefore $f(K) \neq \mathbb{S}^2$. By Lemma 6.4.2, $f(K)$ is porous. □

Lemma 6.4.4. *Let $n \in \mathbb{N}$. Every f^n-invariant subset of the visual sphere $(\mathbb{S}^2, \rho)$ of f is contained in a f-invariant porous subset of the visual sphere.*

Proof. Let K be a f^n-invariant porous subset of the visual sphere. By Lemma 6.4.3, $f(K), f^2(K), \ldots, f^{n-1}(K)$ are compact and porous. We claim that

$$K' = K \cap f(K) \cap \ldots \cap f^{n-1}(K)$$

is compact, f-invariant and porous.

For each $r \in \mathbb{N}$, the map f^r is continuous, therefore $f^r(K)$ is compact. Union of compact subsets are compact, thus $K' = K \cap f(K) \cap \ldots \cap f^{n-1}(K)$ is compact. Using the fact that $f^n(K) \subset K$, we have

$$\begin{aligned} f(K') &= f(K \cap f(K) \cap \ldots \cap f^{n-1}(K)) \\ &\subset f(K) \cap f^2(K) \cap \ldots \cap f^n(K) \\ &\subset f(K) \cap f^2(K) \cap \ldots \cap K \\ &= K'. \end{aligned}$$

Thus K' is f-invariant.

Suppose $K, f(K), f^2(K), \ldots, f^{n-1}(K)$ are a-porous for some $a \in (0,1)$. Suppose $x \in K'$, and let $0 < r < \operatorname{diam}_\rho(\mathbb{S}^2)$. If $B(x, r/2) \cap K = \emptyset$, then the ball $B(x, ar/2)$ is a ball in $B(x, r)$ and $B(x, ar/2) \cap K = \emptyset$. If instead $B(x, r/2) \cap K \neq \emptyset$, then for any $y_1 \in B(x, r/2) \cap K$, there by porosity of K there exists a ball $B(x_1, ar/2) \subset B(y_1, r/2) \subset B(x, r)$ such that $B(x_1, ar/2) \cap K = \emptyset$. In any case, we can find $x_1 \in B(x, r)$ such that $B(x_1, ar/2) \subset B(x, r)$ and $B(x_1, ar/2) \cap K \neq \emptyset$. Similarly, there exists $x_2 \in B(x_1, ar/2)$ such that $B(x_2, (a/2)^2 r) \subset B(x_1, ar/2)$ and $B(x_2, (a/2)^2 r) \cap f(K) = \emptyset$. By induction, there exists $x_n \in B(x, r)$ such that $B(x_n, (a/2)^n r) \subset B(x, r)$ and $B(x_n, (a/2)^n r) \cap K \cap f(K) \cap \cdots \cap f^{n-1}(K) = \emptyset$. Thus K' is $(a/2)^n$-porous. $\square$

Recall from Section 6.4 that $\mathbb{A}^k$ is the space of all admissible words of length k, and $\mathbb{A}^\infty$ is the space of all admissible infinite words. We define a shift map on $\mathbb{A}^\infty$ and study subsets of $\mathbb{A}^\infty$ that are invariant under the shift map.

Lemma 6.4.5. *Let $n \geq 2$, and let $i_1, \ldots, i_n \in \{1, \ldots, |\Gamma_0|\}$. If $i_1 i_2 \ldots i_k \in \mathbb{A}^k$, then $i_2 \ldots i_k \in \mathbb{A}^{k-1}$.*

Proof. If $i_1 i_2 \ldots i_n \in \mathbb{A}^k$, then for each $k = 1, 2, \ldots, k-1$, $\gamma_{i_k} \in \Gamma_1$ is isotopic to a component of the preimage of $\gamma_{i_{k+1}} \in \Gamma_1$ or the inverse of $\gamma_{i_{k+1}}$. In particular, $\gamma_{i_k} \in \Gamma_1$ is isotopic to a component of the preimage of $\gamma_{i_{k+1}} \in \Gamma_1$ or the inverse of $\gamma_{i_{k+1}}$ for each $k = 2, 3, \ldots, k-1$. Therefore $i_2 \ldots i_k \in \mathbb{A}^{k-1}$. $\square$

Here is a corollary of Lemma 6.4.5.

Corollary 6.4.6. *Let $i_1, i_2, \ldots \in \{1, \ldots, |\Gamma_0|\}$ If $i_1 i_2 i_3 \ldots \in \mathbb{A}^\infty$, then $i_2 i_3 \ldots \in \mathbb{A}^\infty$.*

Let $\sigma : \mathbb{A}^\infty \to \mathbb{A}^\infty$ be the map $\sigma(i_1 i_2 i_3 \ldots) = i_2 i_3 \ldots$. We call σ the *shift map* on $\mathbb{A}^\infty$. We say that $E \subset \mathbb{A}^\infty$ is σ-invariant if $\sigma(E) \subset E$.

Lemma 6.4.7. *For any $\omega \in E$, $f(\gamma_\omega) \subset \gamma_{\sigma(\omega)}$.*

Proof. Write $\omega = i_1 i_2 i_3 \ldots$. For each $n \in \mathbb{N}$, we have

$$f(\gamma_{i_1 \ldots i_n}) = \gamma_{i_2 \ldots, i_n}$$

as sets. Taking Hausdorff limits on both sides, and noting that f is continuous, we have the desired conclusion. □

For any subset $E \subset \mathbb{A}^\infty$, let $\Gamma_E = \{\gamma_\omega \in \Gamma_\infty : \omega \in E\}$, and let $K_E = \bigcup_{\omega \in E} \gamma_\omega$.

Lemma 6.4.8. *If $E \subset \mathbb{A}^\infty$ is σ-invariant, then K_E is f-invariant.*

Proof.

$$f(K_E) = f\left(\bigcup_{\omega \in E} \gamma_\omega\right) = \bigcup_{\omega \in E} f(\gamma_\omega) = \bigcup_{\omega \in E} \gamma_{\sigma(\omega)} \subset \bigcup_{\omega \in E} \gamma_\omega.$$

□

6.5 Constructing the Porous Subsets

If A is a square matrix with positive entries, then by the Perron-Frobenius theorem, there exists $\lambda > 0$ such that λ is an eigenvalue of A, that the eigenspace of A with eigenvalue λ is 1-dimensional, and that λ is the spectral radius of A. We let $\rho(A) = \lambda$ and call $\rho(A)$ the *Perron-Frobenius root* of A.

The proof of the following lemma comes from [js2].

Lemma 6.5.1. *Let $n \geq 1$ be an integer. Let A, B be two $n \times n$ matrices with positive entries. Let v be an eigenvector of A with eigenvalue $\rho(A)$. Let $\kappa(A) = \|v\|_\infty^{-1} > 0$. Then*

$$|\rho(A) - \rho(B)| \leq \kappa(A) \max_{i,j} |A_{i,j} - B_{i,j}|.$$

Proof. Let B^t be the transpose of B, and let w be an eigenvector of B with eigenvalue $\rho(B)$. Let $(\cdot, \cdot)$ denote the usual dot product on $\mathbb{R}^n$. Then

$$((A - B)v, w) = (Av, w) - (v, B^t w) = (\rho(A) - \rho(B))(v, w).$$

Note that

$$(v, w) \geq \kappa(A) \|w\|_1 ,$$

and

$$((A-B)v, w) \leq \kappa(A) \max_{i,j} |A_{i,j} - B_{i,j}| \|w\|_1 .$$

Note that $\|w\|_1 > 0$. Therefore

$$|\rho(A) - \rho(B)| \leq \kappa(A) |A_{i,j} - B_{i,j}| .$$

□

Let p be a positive number such that $p \geq 2$. Assume Γ_0 is a p-thick f-invariant multicurve. Let us rewrite the matrix $M(f, \Gamma_0, p)$, as follows. Define functions

$$p : \coprod_{n \in \mathbb{N}} \mathbb{A}^n \to \Gamma_0$$
$$h : \coprod_{n \in \mathbb{N}} \mathbb{A}^n \to \Gamma_0$$
$$d : \coprod_{n \in \mathbb{N}} \mathbb{A}^n \to \mathbb{N},$$

so that for any $n \in \mathbb{N}$ and $\omega \in \mathbb{A}^n$, $\gamma_\omega \in \Gamma_n$ is a component of the preimage of $p(\omega) \in \Gamma_0$ under f^n, γ_ω is isotopic to $h(\omega) \in \Gamma_0$, and $\deg(f|_{\gamma_\omega}) = d(\omega)$. Then For each $\gamma \in \Gamma_0$, we have

$$M(f, \Gamma_0, p)(\gamma) = \sum_{i \in \mathbb{A}^1 : p(i) = \gamma} d(i)^{1-p} h(i).$$

One can verify that for all $n \geq 1$,

$$M(f, \Gamma_0, p)^n(\gamma) = \sum_{\omega \in \mathbb{A}^n : p(\omega) = \gamma} d(\omega)^{1-p} h(\omega) = (f^n)_{\Gamma_0, p}.$$

Recall that $\Lambda(\Gamma_0, p)$ is the Perron-Frobenius root of $M(f, \Gamma_0, p)$, which is also the spectral radius of $M(f, \Gamma_0, p)$.

Lemma 6.5.2. *Let Γ_0 be an irreducible f-stable multicurve. Let $M = M(f, \Gamma_0, p)$, and $\lambda = \Lambda(M(f, \Gamma_0, p))$. If $\lambda > 1$, then there exists $N \in \mathbb{N}$ and $\omega_0 \in \mathbb{A}^N$ such that the matrix defined by*

$$M_{\omega_0}(\gamma) = \sum_{\omega \in \mathbb{A}^N \setminus \{\omega_0\}: p(i) = \gamma} d(i)^{1-p} h(i),$$

has positive entries and $\rho(M') > 1$.

Proof. Let v be an eigenvector of M with eigenvalue λ. Then for each $n \in \mathbb{N}$, v is an eigenvector of M^n with eigenvalue λ^n. Since Γ_0 is irreducible, there exists $N' \in \mathbb{N}$ such that for every $n \geq N'$, every entry of M^n is positive. Suppose $\omega = i_1 i_2 \ldots i_n \in \mathbb{A}^n$. Then $d(\omega) = d(i_1) d(i_2) \ldots d(i_n) \geq 2^n$. Therefore for all $\omega_0 \in \mathbb{A}^n$, if we set

$$M_{\omega_0} = \sum_{\omega \in \mathbb{A}^n \setminus \{\omega_0\}: p(\omega) = \gamma} d(\omega)^{1-p} h(\omega) = (f^n)_{\Gamma_0, p},$$

then every entry of M' is positive, and the absolute value of any entry in $M^n - M_\omega$ is bounded from above by $2^{(1-p)n}$. By Lemma 6.5.1, we have

$$|\rho(M^n) - \rho(M_{\omega_0})| \leq \kappa(M) 2^{(1-p)n}.$$

Therefore

$$\rho(M_{\omega_0}) \geq \lambda^n - \kappa(M) 2^{(1-p)n}.$$

Since $\lambda > 1$ and $1 - p < 0$, by choosing sufficiently large $N \geq N'$, we nave

$$\rho(M_\omega) > 1$$

for arbitrary choice of ω in $\mathbb{A}^N$. □

6.6 Modulus Estimates

Throughout this section, we assume that Γ_0 is an irreducible p-thick curve. We also assume either one of the following:

Case 1: There exists $\gamma \in \Gamma_0$ with a peripheral component in $f^{-1}(\gamma)$. In this case, we let $E_\infty = \mathbb{A}^\infty$, and for every $n \in \mathbb{N}$, we let $E_n = \mathbb{A}^\infty$. We also let

$$M = M(f, \Gamma_0, p).$$

Case 2: $M(f,\Gamma_0,p)>1$, and there exists $i_0\in\mathbb{A}^1$ such that every entry of the matrix

$$M=\sum_{i\in\mathbb{A}^1\setminus\{i_0\}} d(i)^{1-p}h(i).$$

is positive, and $\rho(M)>1$. In this case, we let E_∞ be the set of admissible infinite words for (f,Γ_0) that onto i_0, and for every $n\in\mathbb{N}$, we let E_n be the set of admissible finite words of length n for (f,Γ_0) that do not contain the letter i.

In both cases, for every $n\in\mathbb{N}$, let $\Gamma'_n=\{\gamma_\omega:\omega\in E_n\}$, and let $K_n=\bigcup_{\omega\in E_n}\gamma_\omega$. Also let $\Gamma'_\infty=\Gamma_{E_\infty}$, and let $K_\infty=K_{E_n}$.

To complete proof, we replace Γ_0 by a slightly easier choice to work with, and we choose an integer $k\in\mathbb{N}$ such that G_n is a sufficiently fine discretization for each $n\geq k$.

Lemma 6.6.1. *There exists a collection Γ_0^* of f-stable multicurves and a bijection $j:\Gamma_0^*\to\Gamma_0$ such that for all $\gamma\in\Gamma_0^*$, $j(\gamma)$ and γ are isotopic in $\mathbb{S}^2\setminus\operatorname{post}(f)$, and there exists $k\in\mathbb{N}$ such that for all $n\geq k$, every simple closed curve in the subset $\bigcup_{\tau\in\mathcal{D}_n(\mathbb{S}^2),\gamma\cap\tau\neq\emptyset}\tau$ of $\mathbb{S}^2$ is either isotopic in $\mathbb{S}^2\setminus\operatorname{post}(f)$ to γ or null-isotopic in $\mathbb{S}^2\setminus\operatorname{post}(f)$, and for every $\gamma,\gamma'\in\Gamma_0^*$ with $\gamma\neq\gamma'$, we have $\bigcup_{\tau\in\mathcal{D}_n(\mathbb{S}^2),\gamma\cap\tau\neq\emptyset}\tau\cap\bigcup_{\tau\in\mathcal{D}_n(\mathbb{S}^2),\gamma'\cap\tau\neq\emptyset}\tau=\emptyset$.*

Proof. Since curves in Γ are disjoint and avoid $\operatorname{post}(f)$, there exists $k\in\mathbb{N}_0$ such that if $\gamma,\gamma'\in\Gamma_0$ are distinct, then for all $n\geq k$, we have

$$\bigcup_{\tau\in\mathcal{D}_n(\mathbb{S}^2),\gamma\cap\tau\neq\emptyset}\tau\cap\bigcup_{\tau\in\mathcal{D}_n(\mathbb{S}^2),\gamma'\cap\tau\neq\emptyset}\tau=\emptyset,$$

and

$$\operatorname{post}(f)\cap\bigcup_{\tau\in\mathcal{D}_n(\mathbb{S}^2),\gamma\cap\tau\neq\emptyset}\tau=\emptyset.$$

Each γ can be replaced by a simple closed curve $\widetilde{\gamma}$ in $f^{-k}(\mathcal{C})$ such that

$$\widetilde{\gamma}\subset\bigcup_{\tau\in\mathcal{D}_k(\mathbb{S}^2),\gamma\cap\tau\neq\emptyset}\tau.$$

such that $\widetilde{\gamma}$ is isotopic to γ. This gives us the f-stable multicurve Γ_0^*. Since $\widetilde{\gamma}\subset f^{-k}(\mathcal{C})$, for every large n, every simple closed curve in the subset $\bigcup_{\tau\in\mathcal{D}_n(\mathbb{S}^2),\gamma\cap\tau\neq\emptyset}\tau$ of $\mathbb{S}^2$ is either isotopic in $\mathbb{S}^2\setminus\operatorname{post}(f)$ to γ or null-isotopic in $\mathbb{S}^2\setminus\operatorname{post}(f)$. $\square$

For the rest of this section, we will assume that $\Gamma_0 = \Gamma_0^*$ and k be the positive integer as in Lemma 6.6.1.

Lemma 6.6.2. *There exists a vector $w \in \mathbb{R}^{|\Gamma_0|}$ with positive entries such that for all $n \in \mathbb{N}_0$, we have*

$$\operatorname{mod}_p(\Gamma'_n, \Lambda^{-(n+k)}) \sim v^t \left(M(f, \Gamma, p)\right)^n w.$$

where $v = (1, 1, \ldots, 1)^t$ is the column vector of $|\Gamma_0|$-entries, each entry being 1.

Proof. Note that by our choice of f, for every $n \geq 1$ and $\mathrm{I} \in \mathbb{A}^n$, the map

$$f^n : \left(\tau : \tau \in \mathcal{D}_{n+k}(\mathbb{S}^2), \gamma_{\mathrm{I}} \cap \tau \neq\right) \to \left(\tau : \tau \in \mathcal{D}_k(\mathbb{S}^2), f^n(\gamma_{\mathrm{I}}) \cap \tau \neq \emptyset\right)$$

is a covering map satisfying the condition in Proposition 6.2.2, with $d = \deg(f^n|_{\gamma_{\mathrm{I}}}) = d_{\mathrm{I}}$. Moreover, if $\mathrm{I} \in \mathbb{A}^n$, then every simple closed curve in the subset $\bigcup_{\tau \in \mathcal{D}_{n+k}(\mathbb{S}^2), \gamma_{\mathrm{I}} \cap \tau \neq \emptyset} \tau$ of $\mathbb{S}^2$ is either isotopic in $\mathbb{S}^2 \setminus \operatorname{post}(f)$ to γ_{I} or it is null-isotopic in $\mathbb{S}^2 \setminus \operatorname{post}(f)$, and that if $\mathrm{I}, \mathrm{J} \in \mathbb{A}^n$ are distinct, then

$$\left(\bigcup_{\tau \in \mathcal{D}_{n+k}(\mathbb{S}^2), \gamma_{\mathrm{I}} \cap \tau \neq \emptyset} \tau \right) \cap \left(\bigcup_{\tau \in \mathcal{D}_{n+k}(\mathbb{S}^2), \gamma_{\mathrm{J}} \cap \tau \neq \emptyset} \tau \right) = \emptyset.$$

By the Parallel Law (Proposition 6.2.1),

$$\operatorname{mod}_p(\Gamma'_n, \Lambda^{-(n+k)}) = \sum_{\mathrm{I} \in \mathbb{A}^n} \operatorname{mod}_p(\gamma_{\mathrm{I}}, \Lambda^{-(n+k)}). \tag{6.6.1}$$

For each $n \in \mathbb{N}$ and $\mathrm{I} \in \mathbb{A}^n$, the map

$$f^n : \left(\tau : \tau \in \mathcal{D}_{n+k}(\mathbb{S}^2), \gamma_{\mathrm{I}} \cap \tau \neq\right) \to \left(\tau : \tau \in \mathcal{D}_k(\mathbb{S}^2), f^n(\gamma_{\mathrm{I}}) \cap \tau \neq \emptyset\right)$$

is a normal covering map as described in the Serial Law (Proposition 6.2.2), hence the Serial Law applies and we get

$$\operatorname{mod}_p(\gamma_{\mathrm{I}}, \Lambda^{-(n+k)}) = \deg(f^n|_{\gamma_{\mathrm{I}}})^{1-p} \operatorname{mod}_p(f^n(\gamma_{\mathrm{I}}), \Lambda^{-k}). \tag{6.6.2}$$

We compute that if $\mathrm{I} = i_1 i_2 \ldots i_n$, then

$$\deg(f^n|_{\gamma_{\mathrm{I}}}) = \left| d_{i_1} d_{i_2} \ldots d_{i_n} \right|.$$

Combining (6.6.1) and (6.6.2), we get

$$\operatorname{mod}_p(\Gamma'_n, \Lambda^{-(n+k)}) = v^t M^n w,$$

where $v = (1, 1, \dots, 1)^t$ is the column vector of $|\Gamma_0|$-entries, each entry being 1, and

$$w = (\operatorname{mod}_p(\gamma, \Lambda^{-k}))_{\gamma \in \Gamma_0}$$

is a vector with positive entries. $\square$

Lemma 6.6.3. *For every $k \in \mathbb{N}_0$, we have*

$$\operatorname{mod}_p(\Gamma'_\infty, \Lambda^{-(n+k)}) \gtrsim \operatorname{mod}_p(\Gamma'_n, \Lambda^{-(n+k)})$$

where the implicit constants are independent of n.

Proof. By Lemma 6.3.1 and Lemma 6.2.3 we have

$$\operatorname{mod}_p(\Gamma'_\infty, \Lambda^{-(n+k)}) \sim \operatorname{mod}_p(\Gamma'_\infty, G^C_{(n+k)}),$$

where C is the constant in 6.3.1, so that for all $k \in \mathbb{N}$, we have the inclusion

$$(\Gamma'_\infty)_{G^C_{(n+k)}} \supset (\Gamma'_n)_{\Lambda^{-(n+k)}}.$$

The above inclusion implies that

$$\operatorname{mod}_p(\Gamma'_\infty, G^C_{(n+k)}) \gtrsim \operatorname{mod}_p(\Gamma'_n, \Lambda^{-(n+k)}).$$

$\square$

Lemma 6.6.4. *The set K_{E_∞} is compact, f-invariant and porous.*

Proof. Suppose x is a point in the closure of K_{E_∞}. Then there exists a sequence $\{\omega^n\}_{n\in\mathbb{N}}$ in E_∞ such that $\rho(x, \gamma_{\omega^n}) \to 0$ as $n \to +\infty$. Since the alphabet set $\mathbb{A}^1$ is finite, there exists a subsequence ω^{n_k} of ω^n and a sequence $\{i_k\}_{k\in\mathbb{N}}$ of letters such that for each $k \in \mathbb{N}$, the first k letters of ω^{n_k} is $i_1 i_2 \dots i_k$. Let $\omega = i_1 i_2 \dots$. By Proposition 6.3.1, $\gamma_{\omega^{n_k}} : [0, 1] \to \mathbb{S}^2$ converges uniformly as functions to γ_ω. Therefore

$$\rho(x, \gamma_{\gamma^n}) \leq \lim_{n\to\infty} \rho(x, \gamma_{\omega^{n_k}}) = 0.$$

This proves that K_{E_∞} is compact.

Since E is σ-invariant, by Lemma 6.4.8, K_{E_∞} is f-invariant. We claim that $K_{E_\infty} \neq \mathbb{S}^2$. By Lemma 6.4.2, K_{E_∞} is then porous.

To prove our claim, for each $n \in \mathbb{N}$, we count the number of n-tiles that intersect K_{E_∞}. Let $k \in \mathbb{N}$ be as in Lemma 6.6.1. Then Let A be a positive $|\Gamma_0| \times |\Gamma_0|$ matrix, defined by

$$A(\gamma) = \sum_{i \in E_1 : p(i) = \gamma} d(i) h(i)$$

for each $\gamma \in \Gamma_0$. Note that for each $\gamma \in \Gamma_0$,

$$\sum_{i \in E_1 : p(i) = \gamma} d(i) \leq \deg(f),$$

where $\deg(f)$ is the global degree of f. Moreover, by our choice of E_1, there exists $\gamma \in \Gamma_0$ such that equality does not hold. This implies $\rho(A) < \deg(f)$. For each $n \in \mathbb{N}$, we have

$$\#\{\tau \in \mathcal{D}_{n+k} : \tau \cap K_{E_n} \neq \emptyset\} = v^t A^n w.$$

Here $v = (1, 1, \ldots, 1)^t$ is the column vector of $|\Gamma_0|$-entries, each entry being 1, and

$$w = (\#\{\tau \in \mathcal{D}_{n+k} : \tau \cap \gamma \neq \emptyset\})_{\gamma \in \Gamma_0}$$

is a vector with non-negative entries. Since the graph G_{n+k} has uniformly bounded vertex degree, By Lemma 6.4.8, we have

$$\#\{\tau \in \mathcal{D}_{n+k} : \tau \cap K_{E_\infty} \neq \emptyset\} = C v^t A^n w,$$

where C is a constant. We have

$$\begin{aligned}
\lim_{n \to \infty} \frac{\#\{\tau \in \mathcal{D}_{n+k} : \tau \cap K_{E_\infty} \neq \emptyset\}}{\#\mathcal{D}_{n+k}} &= \lim_{n \to \infty} \frac{C v^t A^n w}{2 \deg(f)^{(n+k)}} \\
&= \lim_{n \to \infty} \left(\frac{\rho(A)}{\deg(f)} \right)^n \frac{C}{2 \deg(f)^k} v^t \frac{A^n}{\rho(A)^n} w \\
&= 0.
\end{aligned}$$

Thus there exists $n \in \mathbb{N}$ such that $\{\tau \in \mathcal{D}_{n+k} : \tau \cap K_{E_\infty} \neq \emptyset\} \neq \mathcal{D}_{n+k}$. Thus $K_{E_\infty} \neq \emptyset(\mathbb{S}^2)$. □

Proposition 6.6.5. *The curve family Γ_∞ is supported on K_{E_∞} with*

$$\liminf_{n\to\infty} \operatorname{mod}_p(\Gamma_\infty, G_n) > 0.$$

Proof. By the definition of K_{E_∞}, Γ_∞ is supported on K_{E_∞}. By Proposition 6.6.3, for all $n \in \mathbb{N}_0$, we have

$$\operatorname{mod}_p(\Gamma'_\infty, \Lambda^{-(n+k)}) \gtrsim \operatorname{mod}_p(\Gamma'_n, \Lambda^{-(n+k)}).$$

By Proposition 6.6.2, we have

$$\operatorname{mod}_p(\Gamma'_n, \Lambda^{-(n+k)}) \sim v^t M^n w.$$

where $v = (1, 1, \ldots, 1)^t$ is the column vector of $|\Gamma_0|$-entries, each entry being 1, and $w \in \mathbb{R}^{|\Gamma_0|}$ is a vector with positive entries. Thus

$$\operatorname{mod}_p(\Gamma'_\infty, \Lambda^{-(n+k)}) \gtrsim v^t M^n w.$$

By the Perron-Frobenius Theorem,

$$\lim_{n\to\infty} \left| v^t \frac{M^n}{\rho(M)^n} w \right| \in (0, 1).$$

Since $\rho(M) > 1$, we have

$$\liminf_{n\to\infty} \operatorname{mod}_p(\Gamma'_\infty, \Lambda^{-(n+k)}) \gtrsim v^t M^n w \gtrsim \lim_{n\to\infty} \left| v^t \frac{M^n}{\rho(M)^n} w \right| \rho(M)^n = \infty.$$

□

6.7 The Proof of Theorem 6.1.6

We are ready to put together the proof of Theorem 6.1.6.

Proof of Theorem 6.1.6. Let $p > 0$. Let Γ_0 be a p-thick irreducible f-stable multicurve. Then either one of the two cases hold:

Case 1: There exists $\gamma \in \Gamma_0$ with a peripheral component in $f^{-1}(\gamma)$. In this case we take $r = 1$.

Case 2: $M(f, \Gamma_0, p) > 1$. Then by Lemma 6.5.1, there exists $r \in \mathbb{N}$ and $\omega_0 \in \mathbb{A}^r$ such that every entry of the matrix

$$M = \sum_{\omega \in \mathbb{A}^r \setminus \{\omega_0\}} d(\omega)^{1-p} h(\omega).$$

is positive, and $\rho(M) > 1$.

In either case, we can apply Lemma 6.3.3, Proposition 6.6.4 and Proposition 6.6.5 to f^r and Γ_0 to conclude the exists a curve family Γ supported on an f^r-invariant porous subset K for some $r \in \mathbb{N}$ that is p-thick. By Proposition 6.4.4, there exists an f-invariant porous subset K' such that $K \subset K'$. This completes the proof of Theorem 6.1.6. $\square$

6.8 Flap construction

In this section we describe a procedure to construct a new expanding Thurston map from an old one. The procedure, which we call the *flap construction*, was studied in by Pilgrim and Tan in [PT98] where the construction took the name "blowing-up the arc". However, the existence of f-invariant Jordan curves gives us a straightforward combinatorial description of the procedure when f is an expanding Thurston map.

Our description of the procedure relies on the notion of two-tile subdivision rules, stated in [BM17, Chapter 12]. We start by recalling some language about two-tile subdivision rules. Our exposition for cell decomposition mainly follows [BM17, Sections 5.1 and 5.2].

Let $\mathcal{X}$ be a locally compact Hausdorff space. For each $n \in \mathbb{N}$, a *n-cell* in $\mathcal{X}$ is a set $c \subset \mathcal{X}$ that is homeomorphic to the closed unit ball $\overline{\mathbb{B}}^n$ in $\mathbb{R}^n$. We denote by ∂c the set of points corresponding to $\partial \mathbb{B}^n$, and we denote by $\mathrm{int}(c)$ the set of points corresponding to $\mathbb{B}^n$. We also write $\dim(c) = n$. A 0-*cell* in $\mathcal{X}$ is a singleton in $\mathcal{X}$. In this case we define $\partial c = \emptyset$, $\mathrm{int}(c) = c$, and $\dim(c) = 0$.

Definition 6.8.1 (Cell decompositions). *Let $\mathcal{X}$ be a locally compact Hausdorff space. Suppose that $\mathcal{D}$ is a collection of cells in a locally compact Hausdorff space $\mathcal{X}$. We say that $\mathcal{D}$ is a* cell decomposition of $\mathcal{X}$ *if the following conditions are satisfied:*

1. *the union of all cells in $\mathcal{D}$ is equal to $\mathcal{X}$,*

2. *if $\sigma, \tau \in \mathcal{D}$ are two distinct cells, then* $\mathrm{int}(\sigma) \cap \mathrm{int}(\tau) = \emptyset$.

3. *if $\tau \in \mathcal{D}$, then $\partial\tau$ is a union of cells in $\mathcal{D}$.*

4. *every point in $\mathcal{X}$ has a neighborhood that meets only finitely many cells in $\mathcal{D}$.*

Definition 6.8.2 (Refinement)**.** *Let $\mathcal{D}'$ and $\mathcal{D}$ be two cell decompositions of the space $\mathcal{X}$. We say that $\mathcal{D}'$ is a* refinement *of $\mathcal{D}$ if the following conditions are satisfied:*

1. *every cell $\sigma \in \mathcal{D}'$ is a subset of a cell $\tau \in \mathcal{D}$.*

2. *every cell $\tau \in \mathcal{D}$ is the union of the cells $\sigma \in \mathcal{D}'$ such that $\sigma \subset \tau$.*

Definition 6.8.3 (Cellular maps and cellular Markov partition)**.** *Let $\mathcal{D}$ and $\mathcal{D}'$ be two cell decompositions of $\mathcal{X}$, and let $f : \mathcal{X} \to \mathcal{X}$ be a continuous map. Then f is* cellular for $(\mathcal{D}', \mathcal{D})$ *if, for all $\sigma \in \mathcal{D}'$, $f(\sigma)$ is a cell in $\mathcal{D}$ and $f|_\sigma$ is a homeomorphism of σ onto $f(\sigma)$. If f is cellular with respect to $(\mathcal{D}', \mathcal{D})$ and $\mathcal{D}'$ is a refinement of $\mathcal{D}$, then the pair $(\mathcal{D}', \mathcal{D})$ is called a* cellular Markov partition for *f.*

If $\mathcal{X} = \mathbb{S}^2$, then any cell decomposition $\mathcal{D}$ of $\mathbb{S}^2$ has to have finite number of cells, and the dimension of each cell is at most 2. A 2-cell c in a cell decomposition $\mathcal{D}$ of $\mathbb{S}^2$ is an *n-gon* if c contains exactly n distinct 0-cells of $\mathcal{D}$.

An *orientation* on $\mathbb{S}^2$ is a triple (c_0, c_1, c_2), where $\mathrm{diam}(c_i) = i$ for $i = 0, 1, 2$, and $c_0 \subset c_1 \subset c_2$. Two triples (c_0, c_1, c_2) and (c_0', c_1', c_2') give the same orientation on $\mathbb{S}^2$ if and only if there exists an orientation-preserving homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\varphi(c_i) = c_i'$ for $i = 0, 1, 2$. One can check that there are two equivalence classes of orientations on $\mathbb{S}^2$.

Definition 6.8.4 (Labeling)**.** *Let $\mathcal{D}^1, \mathcal{D}^0$ be two cell decompositions of $\mathbb{S}^2$. A map $L : \mathcal{D}^1 \to \mathcal{D}^0$ is a* labeling *of $(\mathcal{D}^1, \mathcal{D}^0)$ if the following conditions hold:*

1. $\dim(L(\tau)) = \dim(\tau)$ *for all $\tau \in \mathcal{D}^1$,*

2. *for all $\sigma, \tau \in \mathcal{D}^1$, if $\sigma \subset \tau$, then $L(\sigma) \subset L(\tau)$.*

3. *if* $\sigma, \tau, c \in \mathcal{D}^1$, $\sigma, \tau \subset c$ *and* $L(\sigma) = L(\tau)$, *then* $\sigma = \tau$.

Definition 6.8.5. *A labeling* L *of* $(\mathcal{D}^1, \mathcal{D}^0)$ *is* orientation-preserving *if for all* (c_0, c_1, c_2) *in* $\mathcal{D}^1$, $(L(c_0), L(c_1), L(c_2))$ *and* (c_0, c_1, c_2) *have the same orientation.*

Definition 6.8.6 (Two-tile subdivision rules)**.** *A two-tile subdivision rule for* $\mathbb{S}^2$ *is a triple* $(\mathcal{D}^1, \mathcal{D}^0, L)$ *of cell decompositions* $\mathcal{D}^0$ *and* $\mathcal{D}^1$ *of* $\mathbb{S}^2$ *and an orientation preserving labeling* $L : \mathcal{D}^1 \to \mathcal{D}^0$ *such that*

1. $\mathcal{D}^0$ *has precisely two* 2*-cells.*

2. $\mathcal{D}^1$ *is a refinement of* $\mathcal{D}^0$, *and* $\mathcal{D}^1$ *contains more than two 2-cells.*

3. *If* k *is the number of* 0*-cells in* $\mathcal{D}^0$, *then* $k \geq 3$ *and every tile in* $\mathcal{D}^1$ *is a* k*-gon.*

4. *Every* 0*-tile in* $\mathcal{D}^1$ *is contained in an even number of* 2*-cells in* $\mathcal{D}^1$.

Given a Thurston map $f : \mathbb{S}^2 \to \mathbb{S}^2$ and an f-invariant Jordan curve $\mathcal{C}$, define $\mathcal{D}^0(f, \mathcal{C})$ to be the cell decomposition of $\mathbb{S}^2$ consisting of the following cells: The 0-cells for $\mathcal{D}$ will be $\{x\}$ for $x \in \operatorname{post}(f)$. The 1-cells are the segments c of $\mathcal{D}$ bounded by two points in $\operatorname{post}(f)$ such that $\operatorname{int}(c) \cap \operatorname{post}(f) = \emptyset$. The 2-cells of $\mathcal{D}$ are the closures of the two components of $\mathbb{S}^2 \backslash \mathcal{C}$.

Proposition 6.8.7. *[BM17, Lemma 5.12 and Proposition 12.2] If* $f : \mathbb{S}^2 \to \mathbb{S}^2$ *is a Thurston map with* $\operatorname{post}(f) \geq 3$ *and* $\mathcal{C}$ *is an* f*-invariant Jordan curve, then there exists unique cell-decomposition* $\mathcal{D}^1(f, \mathcal{C})$ *of* $\mathbb{S}^2$ *and a labeling* $L : \mathcal{D}^1 \to \mathcal{D}^0$ *such that for all* $\tau \in \mathcal{D}^1$, $L(\tau) = f(\tau)$. *Moreover,* $(\mathcal{D}^1(f, \mathcal{C}), \mathcal{D}^0(f, \mathcal{C}), L)$ *is a two-tile subdivision rule.*

We call $(\mathcal{D}^1(f, \mathcal{C}), \mathcal{D}^0(f, \mathcal{C}))$ in the above theorem the *two-tile subdivision rule of* $(f, \mathcal{C})$. In fact, every two-tile subdivision rule comes from an orientation preserving Thurston map.

Theorem 6.8.8. *[BM17, Proposition 12.2 and Proposition 12.3] Let* $(\mathcal{D}^1, \mathcal{D}^0, L)$ *be a two-tile subdivision rule on* $\mathbb{S}^2$. *Then there exists a Thurston map* $f : \mathbb{S}^2 \to \mathbb{S}^2$ *and an* f*-invariant Jordan curve* $\mathcal{C}$ *such that* $\mathcal{D}^0 = \mathcal{D}^0(f, \mathcal{C})$, $\mathcal{D}^1 = \mathcal{D}^1(f, \mathcal{C})$ *and for all* $\tau \in \mathcal{D}^1$, $f(\tau) = L(\tau)$.

Let $\mathcal{C}$ be a f-invariant Jordan curve in $\mathbb{S}^2$, and $(\mathcal{D}^1, \mathcal{D}^2)$ be the corresponding two tile-subdivision rule. A 2-cell c of $\mathcal{D}^1$ is said to *join two opposite sides of* $\mathcal{C}$ if $|\mathrm{post}(f)| = 3$ and there exists $e, e' \in \mathcal{D}^1$ such that $c \cap e, c \cap e' \not\emptyset$ and $c \cap e \cap e' = \emptyset$, or if $|\mathrm{post}(f)| \geq 4$ and there exists $e, e' \in \mathcal{D}^0$ such that $e \cap e' = \emptyset$ and $c \cap e, c \cap e' \neq \emptyset$. The above theorem gives a combinatorial characterization of expanding Thurston maps among Thurston maps.

Theorem 6.8.9. *Let f be a Thurston map, and let $\mathcal{C}$ be a f-invariant Jordan curve in $\mathbb{S}^2$. Let $(\mathcal{D}^1, \mathcal{D}^2, L)$ be the corresponding two-tile subdivision rule. Suppose that no 2-cell c in $\mathcal{D}^1$ joins two opposite sides. Then the map f is Thurston equivalent to an expanding Thurston map. Conversely, if f is an expanding Thurston map, then there exists $n \in \mathbb{N}$ and an f^n-invariant Jordan curve $\mathcal{C}$ such that in the corresponding two-tile subdivision rule $(\mathcal{D}^1, \mathcal{D}^0, L)$ for f^n, no 2-cells c in $\mathcal{D}^1$ join opposite sides of $\mathcal{C}$.*

We can now define the notion of flap constructions. Let f be an expanding Thurston map with no periodic critical points, and let $\mathcal{C}$ be an f-invariant Jordan curve. Let $(\mathcal{D}^1, \mathcal{D}^0, L)$ be a two-tile subdivision rule for $(f, \mathcal{C})$. Let α be a 1-cell in $\mathcal{D}^1$ such that $\alpha \cap \mathrm{post}(f) = \emptyset$. We will construction a new 2-tile subdivision rule $(\widetilde{\mathcal{D}}^1, \widetilde{\mathcal{D}}^0, \widetilde{L})$

Think of $\mathcal{D}^0$ and $\mathcal{D}^1$ as cell decompositions of $\widehat{\mathbb{C}}$, such that the union of 1-cells of $\mathcal{D}^0$ is $\partial\mathbb{D}$. Suppose α is a 1-cell of $\mathcal{D}^1$ in $\mathbb{D}$. Fix an orientation on $\mathcal{D}^0$. Let c_+ and c_- be the two 2-cells adjacent to α, and let p_1 and p_n be the two 0-cells in $\partial\alpha$ such that (p_1, α, c_+) is positively oriented. Then (p_1, α, c_-) is negatively oriented.

Let $\mathcal{X} = \widehat{\mathbb{C}} \backslash \alpha$. Let $\mathcal{Y} = \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$. Then there exists a conformal map $h : \mathcal{Y} \to \mathcal{X}$. This conformal map h can be extended continuously to a map $\widehat{\mathbb{C}} \backslash \mathbb{D} \to \widehat{\mathbb{C}}$, which we will also denote by h. The map h is surjective, $h^{-1}(\alpha) = \partial\mathbb{D}$, and for each $p \in \partial\alpha$, $|h^{-1}(p)| = 1$. By precomposing h with a suitable Möbvious transformation, we may assume that $h^{-1}(\partial(\alpha)) = \{-1, 1\}$.

We would like to build a new cell decomposition on $\widehat{\mathbb{C}}$ by gluing a cell decomposition on $\overline{\mathcal{Y}} = \widehat{\mathbb{C}} \backslash \mathbb{D}$ and cell decomposition on $\overline{\mathbb{D}}$. The cell decomposition on $\overline{\mathcal{Y}}$ is essentially the pullback of the cell decomposition $\mathcal{D}^1$ on $\mathcal{X}$. To be precise, let $n = |\mathrm{post}(f)|$. Let $p_1 = \{-1\}$ and $p_n = \{1\}$ be two 0-cells in $\overline{\mathcal{Y}}$ corresponding to the two points of $\widetilde{h}^{-1}(\partial(\alpha))$.

Let $\alpha_- = \{e^{i\theta} : \theta \in [0, \pi]\}$, and let $\alpha_+ = \{e^{i\theta} : \theta \in [\pi, 2\pi]\}$. For every $\tau \in \mathcal{D}^1 \backslash \{\alpha, \partial(\alpha)\}$, either $\tau \in \mathcal{X}$, therefore $h^{-1}(\tau)$ is a cell in $\mathcal{Y}$, or τ is not a 0-cell and $\mathrm{int}(\tau) \subset \mathcal{X}$, therefore $\overline{h^{-1}(\mathrm{int}(\tau))}$ is a cell in $\mathcal{Y}$. Let $h^{-1}(\mathcal{D}^1)'$ be the collection of cells in $\overline{\mathcal{Y}}$ obtained by one of the 2 ways described above. Let

$$h^{-1}(\mathbb{D}^1) = h^{-1}(\mathcal{D}^1)' \cup \{\alpha_+, \alpha_-, p_1, p_2\}.$$

Then $h^{-1}(\mathcal{D}^1)$ is a cell decomposition of $\overline{\mathcal{Y}}$. Moreover, the map

$$h : \overline{\mathcal{Y}} \to \overline{cX}$$

gives us a map

$$\overline{h}^1 : h^{-1}(\mathcal{D}^1) \to \mathcal{D}^1$$

sending τ to $h(\tau)$. Let us orient $h^{-1}(\mathcal{D}^1)$ such that $\overline{h}^1$ is orientation preserving.

We now construct a cell-decomposition on $\overline{\mathbb{D}}$, illustrated in Figure 6.1. For $j = 2, \dots, n-1$, let $p_j = \{\frac{2(j-1)}{n-1} - 1\}$ (this formula holds for $j = 1, n$ as well). For $j = 1, 2, \dots, n-1$, let $e_j = [\frac{2(j-1)}{n-1} - 1, \frac{2j}{n-1} - 1]$. Let $c_{n,+} = \overline{\mathbb{D}} \cap \{\mathrm{Im}(z) \leq 0\}$ and $c_{n,-} = \overline{\mathbb{D}} \cap \{\mathrm{Im}(z) \geq 0\}$. Then $\{p_1, \dots, p_n, e_1, \dots, e_{n-1}, \alpha_+, \alpha_-, c_{n,+}, c_{n,-}\}$ is a cell decomposition of $\overline{\mathbb{D}}$, which we call $\mathcal{D}^1_n$.

two 2-cells in such that $\partial(c_{n,+}) = \bigcup_{i=1}^{n-1} e_i \cup \alpha_-$, and $\partial(c_{n,-}) = \bigcup_{i=1}^{n-1} e_i \cup \alpha_+$. Then for each $i = 1, 2, \dots, n-1$, $(p_i, e_i, c_{n,-})$ is positively oriented, and $(p_i, e_i, c_{n,+})$ is negatively oriented.

Let

$$\widetilde{\mathcal{D}}^1 = h^{-1}(\mathcal{D}^1) \cup \mathcal{D}^1_n.$$

Define

$$\widetilde{\mathcal{D}}^0 = \{h{-}1(\tau) : \tau \in \mathcal{D}^0 \backslash \{\overline{\mathbb{D}}\}\} \cup \{\mathcal{D}\}.$$

We have a bijection

$$\widetilde{h}^0 : \widetilde{\mathcal{D}}^0 \to \mathcal{D}^0$$

given by

$$\widetilde{h}^0(\tau) = \begin{cases} h(\tau), & \text{if } \tau \neq \sigma_0 \\ \overline{\mathbb{S}^2 \backslash \sigma_0}, & \text{if } \tau = \sigma_0. \end{cases}$$

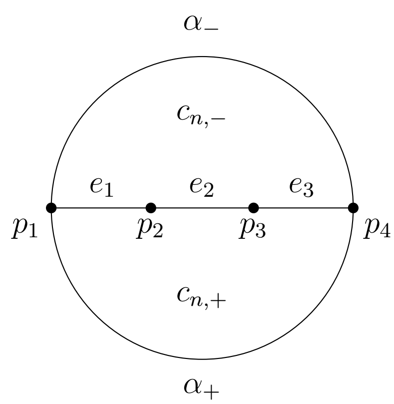

Figure 6.1: The cell decomposition $\overline{\mathbb{D}}$ with $|\operatorname{post}(f)| = 4$

Proposition 6.8.10. *The cell decomposition $\widetilde{\mathcal{D}}^1$ is a refinement of $\widetilde{\mathcal{D}}^0$, and there exists an orientation preserving labeling map $\widetilde{L} : \widetilde{\mathcal{D}}^1 \to \widetilde{\mathcal{D}}^0$ such that $(\widetilde{\mathcal{D}}^1, \widetilde{\mathcal{D}}^0, \widetilde{L})$ is a two-tile subdivision rule.*

Proof. On $h^{-1}(\mathcal{D}^1)$, we define

$$\widetilde{L} = \left(\widetilde{h}^0\right)^{-1} \circ L \circ \widetilde{h}^1.$$

Let $q_1, q_2, \ldots, q_n$ be the n 0-cells in $\mathcal{D}_0$, and $f_1, \ldots, f_n$ be the n 1-cells in $\mathcal{D}_0$ such that $\partial f_i = q_i \cup q_{i+1}$ for $i = 1, 2, \ldots, n-1$, $L(\widetilde{h}(p_1)) = q_1$, $L(\widetilde{h}(p_n)) = q_n$, and $L(\alpha) = f_n$. Define

$$\widetilde{L}(p_i) = \left(\widetilde{h}^0\right)^{-1} (q_i) \quad \forall i = 1, \ldots, n.$$

$$\widetilde{L}(e_i) = \left(\widetilde{h}^0\right)^{-1} (f_i) \quad \forall i = 1, \ldots, n-1.$$

$$\widetilde{L}(\alpha_+) = \widetilde{L}(\alpha_-) = \left(\widetilde{h}^0\right)^{-1} \circ L(\alpha).$$

$$\widetilde{L}(c_{n,+}) = \left(\widetilde{h}^0\right)^{-1} \circ L(c_+).$$

$$\widetilde{L}(c_{n,-}) = \left(\widetilde{h}^0\right)^{-1} \circ L(c_-).$$

Then $(\widetilde{\mathcal{D}}^1, \widetilde{\mathcal{D}}^0, \widetilde{L})$ is a two-tile subdivision rule on $\mathcal{Y} \cap c = \mathbb{S}^2$. □

Proposition 6.8.11. *The two tile subdivision rule $(\widetilde{\mathcal{D}}^1, \widetilde{\mathcal{D}}^0, \widetilde{L})$ is the two-tile subdivision rule of $(g, \widetilde{\mathcal{C}})$, where g is an expanding Thurston map without periodic critical points and $\widetilde{\mathcal{C}}$ is a g-invariant Jordan curve.*

Proof. By Theorem 6.8.8, two tile subdivision rule $(\widetilde{\mathcal{D}}^1, \widetilde{\mathcal{D}}^0, \widetilde{L})$ is the two-tile subdivision rule of $(g, \widetilde{\mathcal{C}})$, where g is a Thurston map and $\mathcal{C}$ is a g-invariant Jordan curve. By our assumption that no 2-cells in $\mathcal{D}^1$ join opposite sides of $\mathcal{C}$, and the fact that the new 2-cells added to $\widetilde{\mathcal{D}}^1$ do not join opposite sides of $\widetilde{\mathcal{C}}$, no 2-cells in $\widetilde{\mathcal{D}}^1$ join opposite sides of $\widetilde{\mathcal{C}}$. By Theorem 6.8.9, g is an expanding Thurston map. Finally, every 0-cell q in $\widetilde{\mathcal{D}}^0$ is still adjacent to only 2 2-cells in $\widetilde{\mathcal{D}}^1$, therefore g does not have periodic critical points. $\square$

6.9 Examples

In this section we mention some important examples in the study of expanding Thurston maps.

Example 6.9.1 ($m \times n$ subdivision rule)**.** The first example we are interested in comes from affine maps on $\mathbb{R}^2$. Let m and n be two integers greater than 1. Let

$$A = \begin{bmatrix} n & 0 \\ 0 & m \end{bmatrix}.$$

Then the matrix A can be viewed as an affine map on $\mathbb{R}^2$. There exists $f_A : \mathbb{S}^2 \to \mathbb{S}^2$ that makes the following diagram commute:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "\pi"] & \mathbb{R}^2 \arrow[d, "\pi"] \\ \mathbb{S}^2 \arrow[r, "f_A"] & \mathbb{S}^2. \end{tikzcd}$$

where π is the map we obtained by identifying $z \in \mathbb{R}^2$ with $z + w$ for all $w \in \mathbb{Z}^2$ and identifying z with $-z$. The map f_A leaves $\{k + \imath y : k \in \frac{1}{2}\mathbb{Z}, y \in \mathbb{R}\} \cup \{x + \imath k : x \in \mathbb{R}, k \in \frac{1}{2}\mathbb{Z}\}$ fixed, therefore there exists an f_A-invariant curve $\mathcal{C}$ in $\mathbb{S}^2$.

The visual sphere is two isometric copies of Rickman's rugs glued together along the boundaries. A *Rickman's rug* is a metric space of the form $([0,1], d) \times ([0,1], d^\alpha$, where d is the usual Euclidean metric on $[0,1]$ and $\alpha \in (0,1)$. For the visual sphere, each Rickman's rug is of the form $\alpha = \frac{\log(mn)}{\log(m \wedge n)}$. The f_A-invariant Jordan curve $\mathcal{C}$ corresponds to the boundary of the two Rickman's rug.

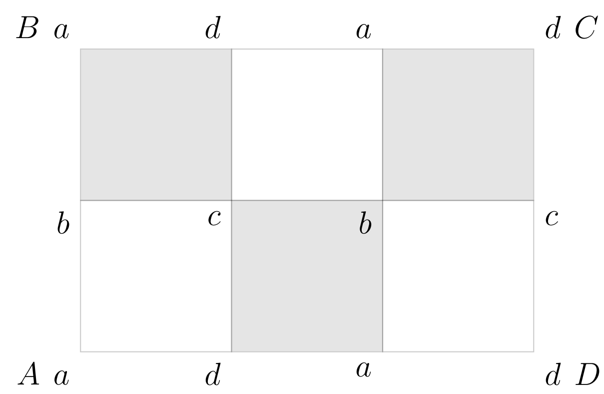

Figure 6.2: 3×2-subdivision rule.

By Theorem 6.8.7, $(f_A, \mathcal{C})$ gives us a two-tile subdivision subdivision rule. Figure 6.2 shows one side of the two-tile subdivision subdivision rule for $n = 3, m = 2$. The other side is at the "back" of Figure 6.2. When $n = 3, m = 2$, each 0-tile is a quadrilateral with vertices A, B, C, D. The 1-tiles subdivide each 0-tile into 2×3-many small quadrilaterals. The map f_A sends vertices marked by a small letter to the vertices with the corresponding capital letter. The shaded 1-tiles are sent to the large 0-tile at the back of the picture, whereas the white 1-tiles are sent to the large 0-tile at the front of the picture.

If $m = n$, then A is a conformal map on $\mathbb{R}^2$, hence the induced map $f_A : \mathbb{S}^2 \to \mathbb{S}^2$ is Thurston equivalent to a rational map. Conversely, if $m \neq n$, then f_A has a Thurston obstruction. In this case the Ahlfors regular conformal dimension of the visual sphere is $\frac{\log(mn)}{\log(m \wedge n)}$, and this Ahlfors regular conformal dimension is attainable.

Example 6.9.2 (2 × 2 subdivision rule with flaps)**.** Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

and let f_A be the map defined in Example 6.9.1. We add a flap to the subdivision to obtain a new subdivision rule, shown in Figure 6.3.

Let the new expanding Thurston map be g_1. Then g has hyperbolic orbifold, and a horizontal loop γ separating $\{A, D\}$ and $\{B, D\}$ gives a irreducible g_1-stable multicurve

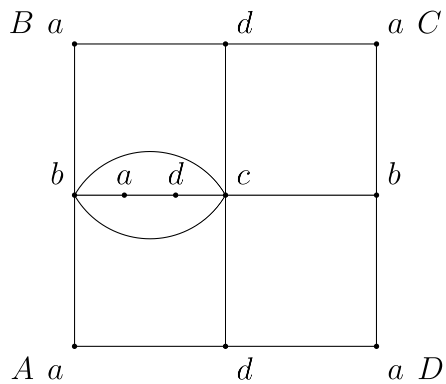

Figure 6.3: The Two-tile subdivision rule for 2×2 subdivision rule with a flap.

$\Gamma_0 = \{\gamma\}$ such that

$$\Lambda(\Gamma_0, 2) = 1.$$

Thus g_1 is not Thurston equivalent to a rational map. Moreover, since $g_1^{-1}(\Gamma_0)$ has a peripheral component, there exists a 2-thick curve family Γ_0 supported on a g_1-invariant porous subset. The porous subset can also be obtained by removing the 2 flaps added to the subdivision rule and their preimages under the map g_1^n for $n \in \mathbb{N}$. Yet another way to visualize the porous subset is to take a square, cut off certain open intervals with dyadic endpoints, then take the completion of the metric space equipped with the length metric, to form a Sierpiński carpets, and glue two Sierpiński carpets together. See Figure 6.4 for an illustration. Similar Sierpiński carpets have been studied in [HL19].

Example 6.9.3 (2×4 subdivision rule with flaps)**.** Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

and let f_A be the map defined in the previous example. We add eight flaps to the subdivision to obtain a new subdivision rule. Figure 6.5 shows the position of 4 new flaps. The other 4 new flaps are at the "back" of Figure 6.5, so that the front and the back of the subdivision rules are the same. Let the new expanding Thurston map be g_2. Then g_2 has hyperbolic orbifold, and a vertical loop γ separating $\{A, D\}$ and $\{B, D\}$ gives a g_2-stable multicurve

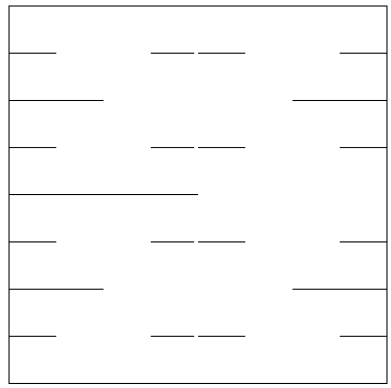

Figure 6.4: An invariant porous subset in a visual sphere of the 2×2 subdivision rule with flaps.

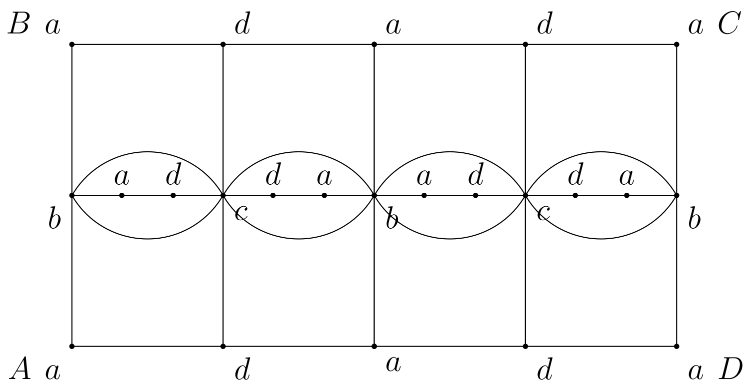

Figure 6.5: The Two-tile subdivision rule for 2×4 subdivision rule with 4 flaps on each side.

$\Gamma = \{\gamma\}$ such that

$$\Lambda(\Gamma, 2) = 1.$$

Thus g_2 is not Thurston equivalent to a rational map. In contrast with Example 6.9.2, there are no 2-thick curve families supported on any g_2-invariant porous subset of the visual sphere. This is because any g_2-invariant porous subset K of the visual sphere must omit a tile τ that avoids $\operatorname{post}(f)$. Let h be a new expanding Thurston map obtained by adding a flap at each edge of τ. Then h is a Thurston map with hyperbolic orbifold and h and every h-stable multicurve Γ_0 in $\mathbb{S}^2 \setminus \operatorname{post}(h)$ satisfies $\Lambda(\Gamma_0, p) < 1$. By Thurston's obstruction theorem, h is Thurston equivalent to a rational map. By Theorem 5.1.2, any visual sphere $(\mathbb{S}^2, \rho)$ of h

is quasisymmetrically equivalent to the standard 2-sphere. But the porous subset K can be embedded quasisymmetrically into $(\mathbb{S}^2, \rho)$, therefore K is quasisymmetrically equivalent to a porous subset of the standard 2-sphere. The non-existence of g_2-invariant porous subset supporting a 2-thick curve families follows from the following Lemma:

Lemma 6.9.4. *Let σ is the chordal metric on the standard 2-sphere, and let ρ be a visual metric of an expanding Thurston map f with no periodic critical points on $\mathbb{S}^2$ with expansion factor Λ. Assume that* $\mathrm{id} : (\mathbb{S}^2, \rho) \to (\mathbb{S}^2, \sigma)$ *is a quasisymmetry. Let K be a compact subset of $\mathbb{S}^2$ with Lebesgue measure 0. Let Γ be a curve family in $\mathbb{S}^2$ supported on K such that*

$$\inf_{\gamma \in \Gamma} \operatorname{diam}_\rho(\gamma) > 0.$$

Then

$$\limsup_{n \to \infty} \operatorname{mod}_2(\Gamma, \Lambda^{-n}) = 0.$$

Proof. First observe that

$$\inf_{\gamma \in \Gamma} \operatorname{diam}_\sigma(\gamma) > 0.$$

To see why, assume there exists $\gamma_n \in \Gamma$ such that $\operatorname{diam}_\sigma(\gamma_n) < \frac{1}{n}$. Since each γ_n is a compact subset of $\mathbb{S}^2$, a subsequence of γ_n converges in Hausdorff sense to some compact subset $\gamma \in \mathbb{S}^2$. We have

$$\operatorname{diam}_\rho(\gamma) \geq \inf_{n \in \mathbb{N}} \operatorname{diam}_\rho(\gamma_n) > 0$$

and

$$\operatorname{diam}_\sigma(\gamma) \leq \lim_{n \to \infty} \operatorname{diam}_\sigma(\gamma_n) = 0,$$

a contradiction.

Let

$$L = \inf_{\gamma \in \Gamma} \operatorname{diam}_\sigma(\gamma).$$

For each $n \in \mathbb{N}$, define $\rho_\varepsilon : \mathcal{D}_n(\mathbb{S}^2) \to [0, \infty)$ by

$$\rho_n(\tau) = \begin{cases} L^{-1} \operatorname{diam}_\sigma(\tau), & \text{if } \tau \cap K \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

For any curve $\gamma \in \Gamma_G$, the collection $\{\tau \in \mathcal{D}_n(\mathbb{S}^2) : \tau \cap \gamma \neq \emptyset\}$ forms a cover of γ, therefore

$$\sum_{\tau \in \mathcal{D}_n(\mathbb{S}^2) : \tau \cap \gamma \neq \emptyset} \rho_n(\tau) \geq L^{-1} \operatorname{diam}_\sigma(\gamma) \geq 1.$$

Thus ρ_n is Γ-admissible. Let $\tau_1, \ldots, \tau_k$ be all the n-tiles that intersect K. Using the fact that tiles in $\mathcal{D}(\mathbb{S}^2)$ are uniform quasiballs (Proposition 6.2.5) and that quasisymmetry sends quasiballs to quasiballs quantitatively, we get

$$|\tau| \sim \operatorname{diam}_\sigma(\tau)^2$$

where $|E|$ is the Lebesgue measure of E. We have

$$\sum_{\tau \in \mathcal{D}_n(\mathbb{S}^2)} \rho_n(\tau)^2 \lesssim \sum_{\tau \in \mathcal{D}_n(\mathbb{S}^2), \tau \cap K \neq \emptyset} |\tau| \lesssim |\bigcup_{\tau \in \mathcal{D}_n(\mathbb{S}^2), \tau \cap K \neq \emptyset} \tau| \leq |N_{\varepsilon(n)}(K)|,$$

where $\varepsilon(n) = \sup_{\tau \in \mathcal{D}_n(\mathbb{S}^2)} \operatorname{diam}_\sigma(\tau)$. As $n \to \infty$, $\varepsilon(n) \to 0$. Therefore

$$\limsup_{n \to \infty} \sum_{\tau \in \mathcal{D}_n(\mathbb{S}^2)} \rho_n(\tau)^2 \lesssim |K| = 0.$$

$\square$

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