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UNIVERSITY OF CALIFORNIA SANTA CRUZ

STRING HOMOLOGY AND LIE ALGEBRA STRUCTURES

A dissertation submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

 in

MATHEMATICS

by

Felicia Y. Tabing

June 2015

The Dissertation of Felicia Y. Tabing is approved:

Professor Hirotaka Tamanoi

Professor Geoffrey Mason

Professor Richard Montgomery

Tyrus Miller Vice Provost and Dean of Graduate Studies Copyright © by

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2015

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Abstract

String Homology and Lie Algebra Structures

by

Felicia Y. Tabing

Chas and Sullivan introduced string homology in [CS], which is the equivariant homology of the loop space with the S^1 action on loops by rotation. Craig Westerland computed the string homology for spheres with coefficients in $\mathbb{Z}/2\mathbb{Z}$ [We] and in Somnath Basu's dissertation [Ba], he computes the string homology and string bracket for spheres over rational coefficients, and he finds that the bracket is trivial. In this paper, we compute string homology and the string bracket for spheres with integer coefficients, treating the odd- and even-dimensional cases separately. We use the Gysin sequence and Leray-Serre spectral sequence to aid in our computations. We find that over the integers, the string Lie algebra bracket structure is more interesting, and not always zero as in [Ba]. The string bracket turns out to be non-zero on torsion coming from string homology.

We also make some computations of the Goldman Lie algebra structure, and more generally, the string Lie algebra structure of closed, orientable surfaces. To Michael Kusuda.

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Chapter 0

Introduction

The term *String Topology* came from the paper of the same name by Moira Chas and Dennis Sullivan in 1999. This paper discussed the various algebraic structures that arose from the homology of the free loop space. This paper came out of trying to generalize the Lie algebra structure that William M. Goldman described by the intersection and concatenation of loops on surfaces [Go].

Chapter 1 is an introduction to the Goldman Lie algebra, and we explore its structure. In particular we consider the structure of the Lie algebra for the closed torus, including computations showing it is finitely generated.

Chapter 2 introduces string topology background needed for the rest of this paper, and the various algebra structures of loop homology and string homology.

Chapter 3 contains the computations of the integral string homology and

string bracket structure for spheres, where some torsion phenomena appear. In our computations, we use the Leray-Serre spectral sequence, and the Gysin exact sequence.

Chapter 4 explores the string homology and bracket structure of surfaces.

Chapter 1

The Goldman Lie Algebra

The Goldman Lie algebra was introduced by William M. Goldman in 1986 [Go].

Throughout, let $\Sigma_{g,n}$ denote an oriented, genus g surface with $n \geq 0$ boundary components. Denote $\hat{\pi}(\Sigma_{g,n})$ to be the set of free homotopy classes of loops on $\Sigma_{g,n}$, where the surface is not mentioned in the notation of $\hat{\pi}$ when it is clear from the context that we are talking about some fixed surface. Recall the following.

Lemma 1.0.1. The set of free homotopy classes of loops on a surface $\Sigma_{g,n}$ is in one-to-one correspondence with conjugacy classes of $\pi_1(\Sigma_{g,n})$.

Remark 1.0.2. We can represent homotopy classes of loops by cyclically reduced words with letters the generators of the fundamental group.

Definition 1.0.3. Fix a surface $\Sigma_{g,n}$ and an orientation of $\Sigma_{g,n}$. Let $\alpha, \beta \in \hat{\pi}$.



Figure 1.1: Loops *aab* and *b* on the torus with one boundary component.

The **Goldman bracket** of α and β is defined to be

$$[\alpha,\beta] = \sum_{p \in \alpha \cap \beta} \epsilon(p) \alpha *_p \beta \tag{1.1}$$

where α and β intersect in transverse double points p, and $\epsilon(p)$ is the sign of the intersection, or $\epsilon(p) = 1$ if the ordered vectors in the tangent space to $\Sigma_{g,n}$ tangent to loop α and β match the orientation of the surface, and $\epsilon(p) = -1$ otherwise.

Example 1.0.4. We compute [aab, b] on the surface $\Sigma_{1,1}$. The loops represented by words *aab* and *b* are shown in Figure 1.1, with intersection points

 p_1 and p_2 . At the intersection point p_1 , we smooth the intersection by creating a new loop, *aabb*, by following the red loop *aab* in the direction of its orientation at p_1 , and when returning to p_1 , we now follow the blue loop b in the direction of its orientation. When we return back to p_1 , we close the loop. At the intersection p_2 , we do the same, and create the loop *abab*. We get that $[aab, b] = \pm (aabb + abab)$ where the sign depends on the chosen orientation of $\Sigma_{1,1}$.

Theorem 1.0.5. (Goldman) The Goldman bracket is well defined, skewsymmetric, and satisfies the Jacobi identity

Proof. [Go]

We can extend the bracket linearly to $\mathbb{Z}[\hat{\pi}]$ (or $\mathbb{Q}[\hat{\pi}]$), the free module over \mathbb{Z} (or \mathbb{Q}) with basis $\hat{\pi}$, to get a bilinear map

$$[-,-]:\mathbb{Z}\hat{\pi}\times\mathbb{Z}\hat{\pi}\to\mathbb{Z}\hat{\pi}$$

. Thus, $\mathbb{Z}\hat{\pi}$ is a Lie algebra with bracket [-, -], which we call the *Goldman* Lie Algebra, denoted by \mathfrak{G} throughout the rest of this chapter. When it is unclear what the surface we are referring to, we use $\mathfrak{G}_{\Sigma_{g,n}}$

1.1 Goldman Lie Algebra Structure

What is the Lie Algebra structure of the Goldman Lie algebra? So far, the center of the Goldman Lie algebra is known, but much of the structure is still a mystery.

Theorem 1.1.1. (Etingof) The center of $\mathfrak{G}_{\Sigma_{g,0}}$ is spanned by the contractible loop [Et].

Theorem 1.1.2. (Kabiraj) The center of $\mathfrak{G}_{\Sigma_{g,n}}$ is generated by peripheral loops [Ka]

A question posed by Chas [Ch] is whether or not \mathfrak{G} is finitely generated. In Goldman's paper [Go], he also introduces what is called the homological Goldman Lie algebra. This Lie algebra is defined on intersection form on the first homology group of a surface. It is known that this Lie algebra is indeed finitely generated [KKT], but of course, the homological Goldman Lie algebra is simpler.

The closed torus is a special case. $\mathfrak{G}_{\Sigma_{1,0}}$ is finitely generated. Recall that we can represent free homotopy classes of loops on $\Sigma_{1,0}$ by cyclically reduced words in two letters, a and b, and we can represent all homotopy classes of loops on the torus by the word $a^l b^k$ for $k, l \in \mathbb{Z}$.

Proposition 1.1.3. The Goldman bracket structure of $\mathfrak{G}_{\Sigma_{0,1}}$ is given by

$$[a^i b^j, a^k b^l] = (il - jk)a^{i+k}b^{j+l}$$

Theorem 1.1.4. $\mathfrak{G}_{\Sigma_{1,0}}$ is finitely generated when considered as a Lie algebra over $\mathbb{Q}_{..}$

Proof. We denote a contractible loop by 1. We claim that $\mathfrak{G}_{\Sigma_{1,0}}$ is generated

by $\{a, b, a^{-1}, b^{-1}\}$. This will take many steps. We will first show that we can generate certain homotopy classes of loops. Below, we assume $n \neq 0$.

- 1. $a^n b^1 = [a, a^{n-1}b]$, which we get inductively,
- 2. $a^n = [b^{-1}, -\frac{1}{n}a^n b]$
- 3. $ab^n = [b, -ab^{n-1}]$
- 4. $b^n = [a^{-1}, -\frac{1}{n}ab^n]$
- 5. $a^n b^n = [a^n, \frac{1}{n^2} b^n]$
- 6. $a^{-n}b = [a^{-1}, -a^{-n+1}b]$
- 7. $a^{-n} = [b^{-1}, \frac{1}{n}a^{-n}b]$
- 8. $a^{-1}b^n [a^{-1}, -\frac{1}{n}b^n]$
- 9. $a^{-n}b^n = [a^{-n}, -\frac{1}{n^2}b^n]$
- 10. $ab^{-n} = [b^{-1}, ab^{n+1}]$ which we get inductively,
- 11. $b^{-1} = [a^{-1}, \frac{1}{n}ab^{-n}]$
- 12. $a^{-n}b^{-n} = [a^{-n}, \frac{1}{n^2}b^{-n}]$
- 13. $a^n b^{-n} = [a^n, -\frac{1}{n^2}b^{-n}].$
- 14. From 13. and 9. for n = 1, we get $a^0 b^0 = 1 = [ab^{-1}, \frac{1}{2}a^{-1}b]$.

We still have a few more cases to show, namely how to generate the homotopy class of the loop $a^i b^j$ in the following cases.

Case 1: Suppose i, j > 0.

- (a) Suppose i < j, then j = i + r for some $r \in \mathbb{Z} \{0\}$. Then $\frac{1}{ar}[a^i b^i, b^j] = a^i b^j$.
- (b) Suppose i > j, then i = j + r for $r \in \mathbb{Z} \{0\}$. Then $-\frac{1}{br}[a^j b^j, a^r] = a^i b^j$.

Case 2: Suppose i < 0 < j.

- (a) Suppose |i| < |j|, then j = -i + r for $r \in \mathbb{Z} \{0\}$. Then $\frac{1}{ar}[a^i b^{-i}, b^r] = a^i b^j$.
- (b) Suppose |a| > |b|, then i = -j + r for $r \in \mathbb{Z} \{0\}$. Then $-\frac{1}{br}[a^{-j}b^j, a^r] = a^i b^j$.

Case 3: The case i, j < 0, and $i \neq j$ is similar to Case 1.

Case 4: The case b < 0 < a is similar to Case 2.

Thus, everything in $\mathfrak{G}_{\Sigma_{1,0}}$ can be generated as a Lie algebra over \mathbb{Q} .

Corollary 1.1.5. We can refine the generators of $\mathfrak{G}_{\Sigma_{0,1}}$ to a smaller basis, namely $\{a, a^{-1}b^{-1} + b + 1, b\}$

Proof. We show that we generate the basis elements mentioned in the proof

of Theorem 1.1.4.

$$\begin{split} & [a^{-1}b^{-1} + b + 1, b] = -a^{-1}, \\ & [a^{-1}b^{-1} + b + 1, a] = b^{-1} - ab, \\ & [a, b] = ab. \end{split}$$

Corollary 1.1.6. $\mathfrak{G}_{\Sigma_{1,0}}$ as a Lie algebra over \mathbb{Q} is not nilpotent, nor solvable, since $[\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}] = \mathfrak{G}_{\Sigma_{1,0}}$.

Remark 1.1.7. $\mathfrak{G}_{\Sigma_{1,0}}$ is not finitely generated as a Lie algebra over \mathbb{Z} .

Proof. We will show that the set $\{(n-1)a^n\}_{n>2,n\in\mathbb{Z}}$ cannot be generated. Suppose to the contrary that we can generate $(n-1)a^n$, so there exists $(i_s, j_s), (k_s, l_s) \in \mathbb{Z}^2$ such that

$$\sum_{s=1}^{t} \pm [a^{i_s} b^{j_s}, a^{k_s}, b^{l_s}] = (n-1)a^n.$$

Then, as in Proposition 1.1.3,

$$\sum_{s=1}^{t} \pm [a^{i_s} b^{j_s}, a^{k_s}, b^{l_s}] = \sum_{s=1}^{t} \pm (i_s l_s - j_s k_s) a^{i_s + k_s} b^{j_s + l_s}.$$

We need that $i_s + k_s = n$ and $j_s + l_s = 0$, so $i_s l_s - j_s k_s = -i_s j_s - j_s n + j_s i_s = -j_s n$. So $n \mid i_s l_s - j_s k_s$, and $n \mid \sum_{s=1}^t \pm (i_s l_s - j_s k_s)$, so $n \mid (n-1)$, which is a contradiction.

Conjecture 1.1.8. We conjecture that $\mathfrak{G}_{\Sigma_{g,n}}$ for $g \geq 1$ and n > 1 is not finitely generated. For the particular case for a punctured torus, the peripheral loop is given by a commutator word. We noticed in using Chas' program for computing the bracket seems to not generate a commutator word, nor products of commutators. This needs more work, but this would mean we have a set $\{(aba^{-1}b^{-1})^n\}_{n\in\mathbb{Z}}$ of infinitely many homotopy classes of loops that each cannot be generated by any other homotopy classes of loops.

Proposition 1.1.9. The derived Lie algebra for $\mathfrak{G}_{\Sigma_{1,0}}$ is given by

$$[\mathfrak{G}_{\Sigma_{1,0}},\mathfrak{G}_{\Sigma_{1,0}}] = \langle d(a^i b^j), na^n, nb^n \rangle$$

for d = gcd(i, j) and $n \in \mathbb{Z} - \{0\}$.

Proof. We first show $[\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}] \subset \langle d(a^i b^j), na^n, nb^n \rangle$

Case 1: Suppose $d = gcd(i, j), i, j \neq 0$ and $ma^i b^j \in [\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}]$ for some $m \in \mathbb{Z}$. Write xi + yj = d for some $x, y \in \mathbb{Z}$ and $ma^i b^j = [a^k b^l, a^p, b^q]$ for $k, l, p, q \in \mathbb{Z}$. But

$$[a^k b^l, a^p, b^q] = (kq - lp)a^{k+p}b^{l+q}$$
(1.2)

so we have that k + p = i, l + q = j, and $kq - lp = kj - li = d(k(\frac{j}{d}) - l(\frac{i}{d})) = m$. Thus $d \mid m$.

Case 2: Suppose that $n \neq 0$ and that $[a^i b^j, a^k b^l] = ma^n$ for $i, j, k, l, m \in \mathbb{Z}$.

Then i + k = n, j + l = 0, so

$$[a^i b^j, a^k b^l] = -jna^n \tag{1.3}$$

so $n \mid m$.

Case 3: Showing that for $n \neq 0$ and $n \mid m$ for $mb^n \in [\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}]$ is similar to Case 2.

To show the other containment, we can consider the equality 1.2 with k = y and l = -x for Case 1, we can consider the equality 1.3 with j = -1, and we can do something similar for Case 3.

Proposition 1.1.10. The lower central series for $\mathfrak{G}_{\Sigma_{1,0}}$ stabilizes, i.e.

$$[\mathfrak{G}_{\Sigma_{1,0}}, G_i] = \langle d(a^i b^j), na^n, nb^n \rangle$$

where d = gcd(i, j), $n \in \mathbb{Z} - \{0\}$, for all $i \ge 0$, and $G_i = [\mathfrak{G}_{\Sigma_{1,0}}, G_{i-1}]$ defined inductively, where $G_0 = \mathfrak{G}_{\Sigma_{1,0}}$.

Proof. For i = 1, this is just Proposition 1.1.9. For i = 2, we need to show that

$$[\mathfrak{G}_{\Sigma_{1,0}}, [\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}]] = \langle d(a^i b^j), na^n, nb^n \rangle.$$

The " \subset " containment is clear. First, consider $a^i b^{-1} \in \mathfrak{G}_{\Sigma_{1,0}}$ and $a^{n-i} b \in \mathfrak{G}_{\Sigma_{1,0}}$

 $\langle d(a^i b^j), na^n, nb^n \rangle$ (since gcd(n-i, 1) = 1). We have that

$$[a^{i}b^{-1}, a^{n-i}b] = na^{n}.$$

In a similar way, we can show that $nb^n \in [\mathfrak{G}_{\Sigma_{1,0}}, [\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}]]$. Now consider d = gcd(i.j), so we can write d = xi + yj. Consider $a^{i+y}b^{j-x} \in \mathfrak{G}_{\Sigma_{1,0}}$ and $a^{-y}b^x \in [\mathfrak{G}_{\Sigma_{1,0}}, \mathfrak{G}_{\Sigma_{1,0}}]$ (since $1 = \frac{i}{d}x + \frac{j}{d}y$ implies gcd(x, y) = 1. We have

$$[a^{i+y}b^{j-x}, a^{-y}b^x] = da^i b^j.$$

Thus, it follows that the lower central series stabilizes.

Corollary 1.1.11. $\mathfrak{G}_{\Sigma_{1,0}}$ as a Lie algebra over \mathbb{Z} is not nilpotent.

Chapter 2

String Topology Preliminaries

Here we describe the basic algebraic structures appearing in the homology and equivariant homology of the free loop space, as described by Chas and Sullivan in *String Topology*. Throughout this paper, let M be a manifold of dimension d, ΩM the based loop space of M, and the space of all continuous, piecewise smooth loops on M, $LM = Map(S^1, M)$, the free loop space of M. Note that LM can be considered to be an infinite-dimensional manifold, and it is topologised with the compact-open topology. We will consider homology and cohomology with integer coefficients, unless otherwise stated. We denote the usual homology of the free loop space of M as $H_*(LM)$ and equivariant homology will be denoted by $H_*^{S^1}(LM)$.

2.1 Loop Homology Algebra Preliminaries

We first describe the Chas-Sullivan loop product, which Chas and Sullivan defined on the chain level of LM, the space of all continuous, piecewise smooth loops on M.

The loop product is a combination of the intersection product an the product given by the concatenation of loops. It is defined transversally at the chain level.

Consider an *i*-chain of loops in LM. We can think of a simplex in this chain as a map $\sigma : \Delta_i \to LM$ or as a map $\sigma : \Delta_i \otimes S^1 \to M$. So we can think of an *i* chain of loops as a map from a simplex with loops above it into M. Intuitively, if we have an *i*-chain and a *j*-chain of loops where the marked points intersect transversally, then we get a new i + j - d-chain of loops consisting of the intersecting marked points, and at each marked point, the new loop is formed by going around the *i*-chain loops then around the *j*-chain of loops. This description at the chain level can pass to homology to form the *Chas-Sullivan loop product*.

Here we give a more precise description of the product given in *String* Topology and Cyclic Homology [CHV]. Let $Map(8, M) = Map(S^1 \vee S^1)$ be the space of continuous, piecewise smooth maps from the figure eight, or the wedge sum of two circles to M. This is topologised with the compact-open topology and can be considered as an infinite-dimensional manifold, but we need piecewise smooth in order for it to be some sort of manifold. It can also be viewed as a subspace of $LM \times LM$ where the loops agree at 0.

Consider the following diagram. The left square is a pullback diagram.:

$$LM \times LM \xleftarrow{\rho_{\text{in}}} Map(8, M) \xrightarrow{\rho_{\text{out}}} LM$$
$$\downarrow^{ev \times ev} \qquad \qquad \qquad \downarrow^{ev}$$
$$M \times M \xleftarrow{\Delta} M$$

where ρ_{in} is the restriction of the figure eight to the two different loops, and ρ_{out} is where the figure eight loop is considered as one loop. The maps ev are the evaluation of the loops at 0, and Δ is the diagonal map. ev is a locally trivial fibration so if η_{Δ} is a tubular neighborhood of the diagonal embedding, then a tubular neighborhood of the $\eta_{\rho_{in}} = (ev \times ev)^{-1}(\eta_{\Delta})$ is homeomorphic to $ev^*(TM) = ev^*(\Delta(M))$. We can have a tubular neighborhood since ρ_{in} is a codimension d embedding. Since $ev^*(\eta_{\Delta})$ is the pullback of η_{Δ} , which has fiber dimension d since it is the normal bundle, the pullback $ev^*(\eta_{\Delta})$ has fibers isomorphic to fibers of η_{Δ} , so $ev^*(\eta_{\Delta})$ also has fiber dimension d. This means that the normal bundle of map(8, M) in $LM \times LM$ has codimension d. Since in the above diagram, the left square is a pullback diagram of fiber bundles, we have that ρ_{in} is a codimension d embedding.

The induced maps on homology go in the wrong direction, in order to remedy this, we need to turn the map ρ_{in} around somehow. We do this by using the Pontrjagin-Thom collapse map:

$$LM \times LM \rightarrow LM \times LM/LM \times LM - ev^*(TM) \cong Thom(Map(8, M))$$

Define the umkehr map $(\rho_{in})_!$ containing the induced map on homology above as follows:

$$(\rho_{in})_! : H_*(LM) \otimes H_*(LM) \cong H_*(LM \times LM) \to H_*(Thom(Map(8, M)))$$
$$\cong H_{*-d}(Map(8, M))$$

The last isomorphism is given by the Thom Isomorphism by taking the cap product with $u \in H^d(Thom(Map(8, M)))$, the Thom class given by the orientation.

Definition 2.1.1. The following composition gives the *Chas-Sullivan loop* product (or just *loop product*):

$$- \bullet - = (\rho_{out})_* \circ (\rho_{in})_! : H_*(LM \times LM) \to H_{*-d}(Map(8, M)) \to H_{*-d}(LM)$$
(2.1)

This product can be extended to homology. It is convenient to regrade the loop homology as follows:

$$\mathbb{H}_*(LM) := H_{*+d}(LM)$$

we can rewrite the product:

$$-\bullet - = \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \to \mathbb{H}_*(LM)$$

We may drop the LM from the notation and denote loop homology by \mathbb{H}_* when it is clear which manifold we are referring to.

Theorem 2.1.2. (Chas-Sullivan) $(\mathbb{H}_*(LM), \bullet)$ is an associative, graded, commutative algebra.

Definition 2.1.3. There is a *Batalin-Vilkovisky operator* denoted by Δ , which comes from the natural action given by rotation of loops,

$$\rho: S^1 \times LM \to LM$$

given by $\rho(t,\gamma)(s) = \gamma(s+t)$. This action defines a degree one operator on loop homology: $\Delta : \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(LM)$ given by $\delta(\alpha) = \rho_*([S^1] \otimes \alpha)$ for $\alpha \in H_k(LM)$.

Theorem 2.1.4. (Chas-Sullivan) $(\mathbb{H}_*(LM), \Delta)$ is a Batalin-Vilkovisky algebra,

- 1. $(\mathbb{H}_*(LM), \bullet)$ is a graded, commutative, associative algebra
- 2. $\Delta \circ \Delta = 0$
- 3. $(-1)^{|\alpha|} \Delta(\alpha \bullet \beta) (-1)^{|\alpha|} \Delta(\alpha) \bullet \beta \alpha \bullet \Delta(\beta)$ is a derivation in each variable.

We can also define a Lie bracket with \bullet and Δ as in part 3 of Theorem 2.1.4.

Definition 2.1.5. The *loop bracket* is defined as

$$\{\alpha,\beta\} := (-1)^{|\alpha|} \Delta(\alpha \bullet \beta) - (-1)^{|\alpha|} \Delta(\alpha) \bullet \beta - \alpha \bullet \Delta(\beta)$$

which is the deviation of Δ from being a derivation of \bullet .

Theorem 2.1.6. (Chas-Sullivan) $(\mathbb{H}_*(LM), \bullet, \{-, -\})$ has the structure of a Gerstenhaber algebra,

- 1. $(\mathbb{H}_*(LM), \bullet)$ is a graded, commutative, associative algebra
- 2. $\{-, -\}$ is a degree 1 Lie bracket, (a) $\alpha, \beta = (-1)^{(|\alpha|+1)(|\beta|+1)+1} \{\beta, \alpha\},$ (b) $\{\alpha, \{\beta, \gamma\}\} = \{\{\alpha, \beta\}, \gamma\} + (-1)^{(|\alpha|+1)(|\beta|+1)} \{\beta, \{\alpha, \gamma\}\},$ 3. $\{\alpha, \beta \bullet \gamma\} = \{\alpha, \beta\} \bullet \gamma + (-1)^{(|\alpha|-1)|\beta|} \beta \bullet \{\alpha, \gamma\}.$

2.2 String Homology Algebra Preliminaries

Now we consider algebraic structures on the equivariant homology of the free loop space with respect to the action of rotation of loops, $H_*^{S^1}(LM)$. Consider the fibration

$$S^2 \to LM \times ES^1 \to LM \times_{S^1} ES^1.$$

This induces a long exact sequence on homology, the Gysin sequence from which we will use to describe a Lie bracket on $H_*^{S^1}(LM)$.

$$\cdots \to \mathbb{H}_{*-d}(LM) \xrightarrow{e} H^{S^1}_*(LM) \xrightarrow{\cap} H^{S^1}_{*-2}(LM) \xrightarrow{M} \mathbb{H}_{*-d-1}(LM) \to \cdots$$

where e and M are informally called the "erasing map" and "marking map," respectively. The map e forgets the marked points on the loops, and the map M puts markings back on the loops in all possible places. We have that M is a homomorphism of graded Lie algebras, it preserves the brackets, going from the string bracket to the loop bracket. The map e is the induced fibration map. For the rest of this paper, it will be clear from the context which space we are referring to, so we often drop the LM from the homology notation.

Remark 2.2.1. Note that $e \circ M = 0$ by exactness, and $\Delta = M \circ e$.

Definition 2.2.2. For two classes $\alpha, \beta \in H^{S^1}_*(LM)$, we can define the *string* bracket by

$$[\alpha,\beta] = (-1)^{|\alpha|-d} e(M(\alpha) \bullet M(\beta))$$

where \bullet was the loop product mentioned in Definition 2.1.1.

Theorem 2.2.3. (Chas-Sullivan) $(H_*^{S^1}(LM), [-, -])$ is a graded Lie algebra, with Lie bracket of degree 2 - d.

More precisely, our bracket is a map:

$$[-,-]: H_i^{S^1}(LM) \times H_j^{S^1}(LM) \to H_{i+j+2-d}^{S^1}(LM).$$

In the following chapters, we compute the $H^{S^1}_*(LS^n)$ for all $n \in \mathbb{N}$ and we compute the structure of the string bracket.

Chapter 3

Computations of String Homology and the String Bracket

In this chapter, we compute explicitly the integral string homology and the string bracket for spheres. Somnath Basu made some computations of rational string homology for spheres in his Ph.D. thesis [Ba] using rational homotopy theory and minimal models. Craig Westerland also made computations of string homology over \mathbb{Z}_2 for spheres in *String Homology of Spheres* and Projective Spaces [We] using a spectral sequence. We separate the computations for the even- and odd-dimensional spheres. First, we compute particular examples, S^1 , S^3 , and S^2 , to get a better hold on the computation, then generalize to the higher-dimensional spheres. We use primarily the Gysin exact sequence, and the Leray-Serre spectral sequence to aid in computations of string homology. We find that there is a lot of interesting torsion in integral string homology, and the bracket structure is not always zero.

3.1 String Homology and String Bracket of S^1

We compute the string homology of S^1 using the Gysin sequence for the circle bundle

$$S^1 \to LS^1 \times ES^1 \to LS^1 \times_{S^1} ES^1$$

Basu computed this in his thesis, but here we use elementary techniques.

Recall that the non-equivariant homology of LS^1 is given as follows [CJY], [He],

$$\mathbb{H}_*(LS^1) = \Lambda_{\mathbb{Z}}[a] \otimes \mathbb{Z}[x, x^{-1}], \ |a| = -1, |x| = 0.$$

where $\mathbb{H}_*(LS^1) = H_{*+1}(LS^1)$ and *a* corresponds to the dual of $[S^1]$ under the geometric grading [Se], [CJY].

The BV-operator $(\Delta = M \circ e)$ acts on generators of $\mathbb{H}_*(LS^1)$ as follows,

[Me]:

$$\begin{split} \Delta(a\otimes x^i) &= i(1\otimes x^i)\\ \Delta(1\otimes x^i) &= 0. \end{split}$$

Consider the Gysin sequence for the above circle bundle:

$$\begin{array}{c} & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & &$$

The end of the Gysin sequence gives us that $H_0^{S^1}(LS^1) \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}(e(a \otimes x^n))$. Using the information from the BV-operator, $M \circ e(a \otimes x^n) = \Delta(a \otimes x^n) = n(1 \otimes x^n)$. Since e is surjective and $ker(e) = im(M) = \bigoplus_{n \in \mathbb{Z}} n\mathbb{Z}(1 \otimes x^n)$, we have that $H_1^{S^1}(LS^1) \cong \mathbb{H}_0(LS^1)/ker(e) \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/n\mathbb{Z}(1 \otimes x^n) \oplus \mathbb{Z}(1 \otimes 1)$. From the beginning of the Gysin sequence, we have $im(c) = ker(M) = \mathbb{Z}(a \otimes 1)$, and since c is injective, $H_2^{S^1}(LS^1) \cong \mathbb{Z}(a \otimes 1)$. Summarizing, we get the following remark. Remark 3.1.1.

$$\begin{aligned} H_0^{S^1}(LS^1) &\cong \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}(e(a \otimes x^n)) \\ H_1^{S^1}(LS^1) &\cong H_{2i+1}^{S^1}(LS^1) \cong \bigoplus_{n \in \mathbb{Z} - \{0\}} \mathbb{Z}/n\mathbb{Z}(1 \otimes x^n) \oplus \mathbb{Z}(1 \otimes 1), \quad i \ge 0 \\ H_2^{S^1}(LS^1) &\cong H_{2i}^{S^1}(LS^1) \cong \mathbb{Z}(a \otimes 1), \quad i \ge 1 \end{aligned}$$

The string bracket, $[-, -]: H_i^{S^1}(LS^1) \otimes H_j^{S^1}(LS^1) \to H_{i+j+1}^{S^1}(LS^1)$ is a degree +1 map, and it is only nontrivial on generators of degree zero since the marking map is trivial for generators of degree greater than zero. For $a \otimes x^n$, $a \otimes x^m$ in $H_0^{S^1}(LS^1)$,

$$[e(a \otimes x^n), e(a \otimes x^m)] = (-1)^{-1} e(M(e(a \otimes x^n)) \bullet M(e(a \otimes x^m)))$$
$$= -e(n(1 \otimes x^n) \bullet m(1 \otimes x^m))$$
$$= -nm(e(1 \otimes x^{n+m}))$$
$$= -nm(1 \otimes x^{n+m})$$

So $[a \otimes x^n, a \otimes x^m] = 0$ if $n + m \neq 0$ and n + m divides nm. If n + m = 0then $[a \otimes x^n, a \otimes x^m] = nm(1 \otimes 1)$. We can conclude that the bracket is only nontrivial for the torsion elements.

3.2 String Homology and String Bracket of S^3

We compute the equivariant homology of S^3 using the Gysin sequence for the circle bundle

$$S^1 \to LS^3 \times ES^1 \to LS^3 \times_{S^1} ES^1 \tag{3.1}$$

and the Serre homology spectral sequence for

$$LS^3 \to LS^3 \times_{S^1} ES^1 \to \mathbb{C}P^\infty$$

First we compute the equivariant cohomology of LS^3 and then translate it to equivariant homology. We also compute the erasing (e) and marking (M) maps, as in Chas and Sullivan's paper, to compute the String Bracket.

3.2.1 First Few Equivariant Homology Groups of LS^3

By equivariant homology, we mean the homology of the Borel construction from the natural action of S^1 on LS^3 by rotation, denoted by $H_*^{S^1}(LS^3) = H_*(LS^3 \times_{S^1} ES^1)$. We calculate the first few equivariant homology groups of LS^3 to aid in our computation of the equivariant cohomology of LS^3 . Recall that the non-equivariant homology of LS^3 is given as follows:

$$\mathbb{H}_*(LS^3) = \Lambda_{\mathbb{Z}}[\alpha] \otimes \mathbb{Z}[y], \ |\alpha| = -3, |y| = 2$$

where $\mathbb{H}_*(LS^3) = H_{*+3}(LS^3)$ and α corresponds to the dual of $[S^3]$ under the usual grading [Se], [CJY].

To compute the equivariant homology of LS^3 , we consider the Gysin sequence for the following fibration:

$$S^1 \to LS^3 \times ES^1 \to LS^3 \times_{S^1} ES^1.$$

and the BV-operator ($\Delta = M \circ e$), which acts on generators of $\mathbb{H}_*(LS^3)$ as follows [T], [Me]:

$$\Delta(\alpha \otimes y^i) = i(1 \otimes y^{i-1})$$
$$\Delta(1 \otimes y^i) = 0.$$
The Gysin exact sequence:

$$H_4^{S^1}(LS^3) \longrightarrow H_2^{S^1}(LS^3) \xrightarrow{M} \mathbb{H}_0(LS^3) \cong \mathbb{Z}(1 \otimes 1)$$

$$\stackrel{e}{\longrightarrow} H_3^{S^1}(LS^3) \longrightarrow H_1^{S^1}(LS^3) \xrightarrow{M} \mathbb{H}_{-1}(LS^3) \cong \mathbb{Z}(\alpha \otimes y)$$

$$\stackrel{e}{\longrightarrow} H_2^{S^1}(LS^3) \longrightarrow H_0^{S^1}(LS^3) \xrightarrow{M} \mathbb{H}_{-2}(LS^3) \cong 0$$

$$\stackrel{e}{\longrightarrow} H_1^{S^1}(LS^3) \longrightarrow 0 \xrightarrow{M} \mathbb{H}_{-3}(LS^3) \cong \mathbb{Z}(\alpha \otimes 1)$$

$$\stackrel{e}{\longrightarrow} H_0^{S^1}(LS^3) \longrightarrow 0 \xrightarrow{M} 0$$

The short exact sequence in the last two rows shows that $H_0^{S^1}(LS^3) \cong \mathbb{Z}(\alpha \otimes 1)$. The short exact sequence in the third and fourth row,

$$0 \to H_1^{S^1}(LS^3) \to 0$$

shows that $H_1^{S^1}(LS^3) \cong 0$. Thus, we obtain a short exact sequence from the second and third row,

$$0 \to \mathbb{Z}(\alpha \otimes y) \to H_2^{S^1}(LS^3) \to \mathbb{Z}(\alpha \otimes 1) \to 0.$$

Since the last non-zero term in the sequence is free, the sequence splits, giving $H_2^{S^1}(LS^3) \cong \mathbb{Z}(\alpha \otimes y) \oplus \mathbb{Z}(\alpha \otimes 1)$. To calculate $H_3^{S^1}(LS^3)$, we use the BV operator. The injective map e in the exact sequence (1.2) means that $e(\alpha \otimes y) = \alpha \otimes y$. Since $\Delta(\alpha \otimes y) = M \circ e(\alpha \otimes y) = 1 \otimes 1$, the map M in the first row of the Gysin sequence above is surjective, so the connecting map e from the first to the second row of the Gysin sequence has kernel $\mathbb{Z}(1 \otimes 1)$. Thus we have a short exact sequence,

$$0 \to H_3^{S^1}(LS^3) \to 0$$

So $H_3^{S^1}(LS^3) \cong 0$.

We may be able to continue computing the rest of the equivariant homology groups of LS^3 in this way, but we eventually reach extension issues. In summary, we have the following remark:

Remark 3.2.1.

$$H_0^{S^1}(LS^3) = \mathbb{Z}(\alpha \otimes 1)$$

$$H_1^{S^1}(LS^3) = 0$$

$$H_2^{S^1}(LS^3) = \mathbb{Z}(\alpha \otimes y) \oplus \mathbb{Z}(\alpha \otimes 1)$$

$$H_3^{S^1}(LS^3) = 0.$$

3.2.2 Equivariant Cohomology of LS^3

Consider the fibration

$$LS^3 \to LS^3 \times_{S^1} ES^1 \to \mathbb{C}P^\infty$$

and the cohomology Leray-Serre spectral sequence associated with it.

We use the fact that we know the ordinary cohomology of LS^3 and $\mathbb{C}P^{\infty}$,

since the E_{∞} page converges to $H_{S^1}^*(LS^3)$, the equivariant cohomology of S^3 . Remark 3.2.2. [CJY]

$$H^*(LS^3) \cong H^*(\Omega S^3) \otimes H^*(S^3) \cong \Gamma[y] \otimes \Lambda[a] \quad |a| = 3, |y| = 2, y_i = \frac{y^i}{i!}$$

Remark 3.2.3.

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x], \ |x| = 2$$

Below is the E_2 page of the spectral sequence. All of the nonzero entries are \mathbb{Z} generated by the entry. The arrows are the d_2 maps.

We can figure out the first few equivariant cohomology groups easily. It can immediately be seen that $H_{S^1}^0(LS^3) \cong \mathbb{Z}(1 \otimes 1)$ and $H_{S^1}^1(LS^3) \cong 0$. For $H_{S^1}^2(LS^3)$, the differential maps d_2 going to and from the generators along the diagonal, y_1 and x, are zero, so these generators survive to the E_{∞} page. In the filtration of $H_{S^1}^2(LS^3)$ corresponding to this spectral sequence, we obtain $0 \subset \mathbb{Z}y_1 \subset H_{S^1}^2(LS^3)$ where $H_{S^1}^2(LS^3)/\mathbb{Z}x \cong \mathbb{Z}y_1$. Thus $H_{S^1}^2(LS^3) \cong$ $\mathbb{Z}y_1 \oplus \mathbb{Z}x$.

The derivation property of the differentials in the Serre spectral sequence makes the computation of the d_2 differentials easier. We only need to know the image of x, y_1 , and a through d_2 to know the image of the other generators in the E^2 grid. We see immediately that $d_2(x) = 0$ and $d_2(y_1) = 0$. From the multiplicative property of the sequence, we can conclude that $d_2(x^i) = 0$ and $d_2(y_i) = 0$ for all $i \ge 1$. We computed $H_3^{S^1}(LS^3) \cong 0$ above, and using the Universal Coefficient Theorem, we find that $H_{S^1}^3(LS^3) \cong 0$ also. This

10	Î	y_5	0	•••								
9		ay_3	$\overset{0}{\times}4$	÷	÷	÷	÷	÷	:	:	:	
8		y_4	8	$\mathbf{y}_4 x$	0	$y_4 x^2$	0	$y_4 x^3$	0	$y_4 x^4$	0	$y_4 x^5$
7		ay_2	$0 \\ \times 3$	ay_2x	0	ay_2x^2	0	ay_2x^3	0	ay_2x^4	0	ay_2x^5
6		y_3	0	• y ₃ x	0	$y_3 x^2$	0	$y_3 x^3$	0	$y_3 x^4$	0	$y_{3}x^{5}$
5		ay_1	$\overset{0}{\times 2}$	ay_1x	$\overset{0}{\times 2}$	ay_1x^2	$\overset{0}{\times 2}$	ay_1x^3	0	ay_1x^4	0	ay_1x^5
4		y_2	$\begin{array}{c} 0 \\ 0 \end{array}$	y ₂ x	00	$y_2 x^2$	0	$y_2 x^3$	0	$y_2 x^4$	0	$y_2 x^5$
3		a	Q	ax	<u>D</u>	ax^2	_Q	ax^3	0	ax^4	0	ax^5
2		y_1	00	y ₁ x	0	$y_1 x^2$	0	$y_1 x^3$	0	$y_1 x^4$	0	$y_1 x^5$
1		0	0	0	0	0	0	0	0	0	0	0
0		1	0	x	0	x^2	0	x^3	0	x^4	0	x^5
_						>						
E_2		0	1	2	3	4	5	6	7	8	9	10

Figure 3.1: The E_2 page of the spectral sequence.

means that on the E_{∞} page of the spectral sequence, there should only be zeros along the third diagonal. This gives that $d_2 : \mathbb{Z}a \to \mathbb{Z}y_1x$ should be an isomorphism. Since a and y_1x are the generators of these isomorphic groups $d_2(a) = \pm y_1x$. Let us assume $d_2(a) = y_1x$. Also,

$$d_2(ay_i) = d(a)y_i = y_1xy_i = y_1x\frac{y_1^i}{i!} = (i+1)y_{i+1}x$$

To summarize:

$$d_{2}(x^{i}) = 0$$

$$d_{0}(y_{i}) = 0$$

$$d_{2}(a) = y_{1}x$$

$$d_{2}(ay_{i}) = (i+1)y_{i+1}x.$$

These calculations correspond to the red arrows on the E_2 page above. The spectral sequence collapses at the E_3 page since there can never be nonzero differentials after the E_2 page because there is nothing for these differentials to hit, so $E_3 = E_{\infty}$.

Let's take a look at the E_{∞} page:

9	Î	:	:	•	:		÷	•	÷	•	÷
8		y_4	0	$\mathbb{Z}y_4x/4\mathbb{Z}$	0	$\mathbb{Z}y_4x^2/4\mathbb{Z}$	0	$\mathbb{Z}y_4x^3/4\mathbb{Z}$	0	$\mathbb{Z}y_4x^4/4\mathbb{Z}$	
7		0	0	0	0	0	0	0	0	0	
6		y_3	0	$\mathbb{Z}y_3x/3\mathbb{Z}$	0	$\mathbb{Z}y_3x^2/3\mathbb{Z}$	0	$\mathbb{Z}y_3x^3$ $3\mathbb{Z}$	0	$\mathbb{Z}y_3x^4/3\mathbb{Z}$	
5		0	0	0	0	0	0	0	0	0	
4		y_2	0	$\mathbb{Z}y_2x/2\mathbb{Z}$	0	$\mathbb{Z}y_2x^2/2\mathbb{Z}$	0	$\mathbb{Z}y_2x^3/2\mathbb{Z}$	0	$\mathbb{Z}y_2x^4/2\mathbb{Z}$	
3		0	0	0	0	0	0	0	0	0	
2		y_1	0	0	0	0	0	0	0	0	
1		0	0	0	0	0	0	0	0	0	
0		1	0	x	0	x^2	0	x^3	0	x^4	
E_{∞}		0	1	2	3	4	5	6	7	8	

Figure 3.2: The E_∞ page of the spectral sequence.

After some work, and with the assumption that $H^i_{S^1}(LS^3)$ is just the direct sum of the diagonal on the E_{∞} page shown above, we can make the following remark.

Remark 3.2.4.

$$H_{S^{1}}^{2i+1}(LS^{3}) = 0$$

$$H_{S^{1}}^{2i}(LS^{3}) = \mathbb{Z}y_{i} \oplus \mathbb{Z}x^{i} \oplus \sum_{j=2}^{i-1} \mathbb{Z}y_{j}x^{i-j}/j\mathbb{Z}, \quad i > 0$$

$$H_{S^{1}}^{0}(LS^{3}) = \mathbb{Z}1.$$

3.2.3 Equivariant Homology of LS^3

Using the results above and the Universal Coefficient Theorem, we get the following:

Theorem 3.2.5.

$$H_{2i}^{S^1}(LS^3) = \mathbb{Z}(\alpha \otimes y^i) \oplus \mathbb{Z}x_i, \quad i \ge 0$$
$$H_{2i+1}^{S^1}(LS^3) = \sum_{j=2}^i \mathbb{Z}(\alpha \otimes y^i)x/j\mathbb{Z}, \quad i \ge 2$$
$$H_1^{S^1}(LS^3) = H_3^{S^1}(LS^3) = 0.$$

Note that $\alpha \otimes y^i$ is dual to y_i and $1 \otimes y^i$ is dual to ay_i . This matches the findings of Basu [Ba] and Westerland [We] using the Universal Coefficient Theorem.



Figure 3.4: The E_{∞} page.

3.2.4 The Spectral Sequence Associated with the Gysin Sequence

To determine the erasing and marking maps, we will translate the Gysin sequence into a spectral sequence and see how they arise in the computation of the spectral sequence. We are using the fact that $H^*(S^1) = \Lambda \omega$ where $|\omega| = 1$. Note that on the E_{∞} page, $a = \omega y_1$. First, we will compute the differential maps for the fibration (1.1) in the spectral sequence. Note that the $E_{\infty} = \cdots = E_4 = E_3$ because all of the differential maps d_i for $i \geq 3$ are 0.

Since $H^1(LS^3) = 0$, the E_{∞} page has zeros along the 1 diagonal. This means for the map $d_2 : \mathbb{Z}\omega \to \mathbb{Z}y_1 \oplus \mathbb{Z}x$, $ker(d_2) = 0$ so the map is injective. Since the entry $E_{\infty}^{2,0} = \mathbb{Z}y_1$, the image of d_2 must be $\mathbb{Z}x$, so we can say that $d_2(\omega) = x$ (up to a sign). We have that $d_2(y_1) = 0$ and $d_2(x) = 0$ also. Using the multiplicative structure of the spectral sequence, we obtain the following remark.

Remark 3.2.6.

$$d_2(\omega y_i) = xy_i$$
$$d_2(\omega x^i) = x^{i+1}$$
$$d_2(\omega y_i x^j) = y_i x^{j+1}$$

3.2.5 The Erasing Map e

In the Gysin sequence above for the circle fibration 3.1, the erasing map e: can be viewed as the map induced by ϵ , the projection map in the fibration 3.1, so $e = \epsilon_*$. Since we are interested in seeing how the erasing map acts on specific generators, we will instead look at the dual map ϵ^* .

The map ϵ^* : $H^i_{S^1}(LS^3) \to H^i(LS^3)$ can be derived from the spectral

sequence of the above fibration 3.1. The map ϵ^* is the composition of the surjection map $E_2^{i,0} = H_{S^1}^i(LS^3) \to E_{\infty}^{i,0} = E_2^{i,0}/im(d_2)$ and the inclusion map $E_{\infty}^{i,0} \to H^i(LS^3)$ [Mc]. This means that an image of a generator in the bottom row of the E^2 page of the spectral sequence by ϵ^* is nonzero if it survives to the E^{∞} page, and a generator's image is zero if it does not survive. The following remark is immediate.

Remark 3.2.7.

$$\epsilon^*(x^i) = 0$$

$$\epsilon^*(y_i) = y_i$$

$$\epsilon^*(y_j x^i) = 0$$

(3.2)

To dualize ϵ^* to obtain e, we need the Kronecker pairing as in the computation of M.

Lemma 3.2.8.

$$e = \epsilon_* : H_*(LS^3) \to H_*^{S^1}(LS^3)$$
$$\alpha \otimes y^i \mapsto \alpha \otimes y^i$$
$$1 \otimes y^i \mapsto (\alpha \otimes y^{i+1})x$$
$$1 \otimes 1 \mapsto 0.$$

Proof. Since $|\alpha \otimes y^i| = 2i$, $e(\alpha \otimes y^i) = kx_i + l(\alpha \otimes y^i)$ for $k, l \in \mathbb{Z}$. We have,

$$<\epsilon^{*}(y_{i}), \alpha \otimes y^{i} > = < y_{i}, \alpha \otimes y^{i} > = 1$$
$$= < y_{i}, \epsilon_{*}(\alpha \otimes y^{i}) >$$
$$= < y_{i}, kx_{i} + l(\alpha \otimes y^{i}) >$$
$$= k < y_{i}, x_{i} > + l < y_{i}, \alpha \otimes y^{i} > = l$$

and

$$<\epsilon^{*}(x^{i}), \alpha \otimes y^{i} > =< 0, \alpha \otimes y^{i} >= 0$$
$$=< x^{i}, \epsilon_{*}(\alpha \otimes y^{i}) >$$
$$=< y^{i}, kx_{i} + \alpha \otimes y^{i} >$$
$$= k < x^{i}, x_{i} > +l < x^{i}, \alpha \otimes y^{i} >= k.$$

Therefore $e(\alpha \otimes y^i) = \epsilon_*(\alpha \otimes y^i) = \alpha \otimes y^i$.

Since $H_3^{S^1}(LS^3) = 0$ and $|1 \otimes 1| = 3$, we must have $e(1 \otimes 1) = 0$. Since $|1 \otimes y^i| = 3 + 2i$, these generators are of odd degree so they cannot be paired with generators in cohomology since $H_{S^1}^{2i+1}(LS^3) = 0$ for $i \ge 0$, so we need to use another technique to find the image of $1 \otimes y^i$. For this we will go back to the Gysin sequence for the fibration (1.1).

For i = 1 we look at the following piece of the Gysin sequence.

Since e is surjective and $ker(e) = im(M) \cong 2\mathbb{Z}(1 \otimes y)$ by Lemma 6.2, we have that $im(e) \cong \mathbb{Z}/2\mathbb{Z}(1 \otimes y)$. Thus $e(1 \otimes y) = \pm (\alpha \otimes y^2)x$.

In general, we have the following.

We have that $ker(e) = im(M) = (i+1)\mathbb{Z}(1 \otimes y^i)$, so $im(e) \cong (\mathbb{Z}_{i+1}\mathbb{Z})(1 \otimes y^i)$. Since the map c is given by the cap product with $x \in H^2_{S^1}(LS^3)$, $c((\alpha \otimes y^j)x_{i-j+2}) = (\alpha \otimes y^j)x_{i-j+1}$. So $ker(c) = im(e) \cong \mathbb{Z}_j(\alpha \otimes y^{i+1})x$. We can conclude that $e(1 \otimes y^i) = (\alpha \otimes y^{i+1})x$.

3.2.6 The Marking Map M

We consider the dual of the marking map, $M^* : H^i(LS^3) \to H^{i-1}_{S^1}(LS^3)$. This can be derived from the spectral sequence of the circle fibration 3.1. M_* is the composition of the surjective map $H^i(LS^3) \longrightarrow E^{i-1,1}_{\infty} \cong H^i(LS^3)/E^{i,0}_{\infty}$ and the injective map $E^{i-1,1}_{\infty} \cong ker(d_2) \longrightarrow E^{i-1,1}_2$ [Mc]. Remark 3.2.9.

$$M^*(y_i) = 0$$
$$M^*(ay_i) = (i+1)y_{i+1}$$

Proof. Since the kernel of the differential $d_2 : (E_2^{-1,1} \cong 0) \longrightarrow (E_2^{1,0} \cong 0)$ is 0, for $1 \in H^0(LS^3)$, $M^*(1) = 0$. To find the image of y_i , we consider the composition $M^* : (H^{2i}(LS^3) \cong \mathbb{Z}y_i) \longrightarrow (E_{\infty}^{2i-1,1} \cong H^{2i}(LS^3)/E_{\infty}^{2i,0} \cong$ $\mathbb{Z}y_i/\mathbb{Z}y_i \cong 0$), so $M^*(y_i) = 0$. The image of ay_i can be determined by identifying ay_i with $(i+1)\omega y_{i+1}$. Then,

$$M^*: H^{2i+1} \to E_{\infty}^{2i+2,1} \cong ker(d_2) \longrightarrow E_2^{2i+2,1}$$
$$ay_i \mapsto (i+1)\omega y_{i+1} \longmapsto (i+1)\omega y_{i+1} = (i+1)y_{i+1}$$

so $M^*(ay_i) = (i+1)y_{i+1}$.

To dualize M_* to obtain M, we need the Kronecker pairing [Br].

Definition 3.2.10. The Kronecker pairing is a map

$$< -, ->: H^i(X) \otimes H_i(X) \to \mathbb{Z}$$

such that for $\alpha = [f] \in H^i(X)$ and $\gamma = [c] \in H_i(X)$ then

$$< \alpha, \gamma >= f(c).$$

Alternatively, for $\beta : H^i(X) \to Hom(H_i(X))$, the map from the universal coefficient theorem,

$$< \alpha, \gamma >= f(c) \in \mathbb{Z}.$$

The Kronecker pairing satisfies the following property, which will be used to dualize the map M^* :

$$\langle f^*(\alpha), \gamma \rangle = \langle \alpha, f_*(\gamma) \rangle$$

Lemma 3.2.11.

$$M = M_* : H_*^{S^1}(LS^3) \longrightarrow H_{*+1}(LS^3)$$
$$\alpha \otimes 1 \longmapsto 0$$
$$\alpha \otimes y^i \longmapsto i(1 \otimes y^{i-1})$$
$$x_i \longmapsto 0$$
$$(\alpha \otimes y^j) x_i \longmapsto 0$$

Proof. Since $H_1(LS^3) = 0$, it is immediate that $M(\alpha \otimes 1) = 0$. To find the image of $\alpha \otimes y^i \in H_{2i}^{S^1}(LS^3)$, since M is a map of degree +1, the only possible generator of $H_*(LS^3)$ of degree 2i+1 is $1 \otimes y^{i-1}$. Let $M_*(\alpha \otimes y^i) = k(1 \otimes y^{i-1})$

for $k \in \mathbb{Z}$. Then

$$< M^*(ay_{i-1}), y^i > = < ay_{i-1}, M_*(\alpha \otimes y^i) >$$
$$= < ay_{i-1}, k(1 \otimes y^{i-1}) > = k$$
$$= < iy_i, \alpha \otimes y^i > = i.$$

This implies k = i, so $M(\alpha \otimes y^i) = i(1 \otimes y^{i-1})$.

We must have $M(x_i) = 0$ since $|x_i| = 2i$ and the only generator on cohomology that it can be paired with is ay^{i-1} , which is not dual to x_i . Similarly, $(\alpha \otimes y^j)x_i$ gets sent to zero by M since it is torsion, mapping into a free group.

3.2.7 The String Bracket [-, -]

Recall the string bracket from Definition 2.2.2

$$[a,b] = (-1)^{(|a|-3)} e(M(a) \bullet M(b))$$

of degree -1 for LS^3 .

The only possible non-zero bracket is from the pair $\alpha \otimes y^i, \alpha \otimes y^j$, as M maps all other generators of $H^{S^1}_*(LS^3)$ to 0, thus the string bracket of these generators are also 0. We see that when $i \ge 1$ and $j \ge 1$,

$$\begin{split} [\alpha \otimes y^i, \alpha \otimes y^j] &= (-1)^{2i-3} e(M(\alpha \otimes y^i) \bullet M(\alpha \otimes y^j)) \\ &= -e(i(1 \otimes y^{i-1}) \cdot j(1 \otimes y^{j-1})) \\ &= -e((ij)(1 \otimes y^{i+j-2})) \\ &= -ij(\alpha \otimes y^{i+j-1})x. \end{split}$$

So the bracket is equal to zero if both i = 1 and j = 1 or if (i + j - 1) | ijand non-zero in all other cases.

As it turns out, the only non-zero brackets are torsion elements, which corresponds to the findings of [Ba], which are that the brackets are all trivial when considering rational string homology of S^3 .

3.3 String Homology and the String Bracket of Odd Spheres

3.3.1 String Homology for Odd Spheres

We try to compute the string homology for odd spheres using only the Gysin sequence for the following fibration:

$$S^1 \to LS^n \times ES^1 \to LS^n \times_{S^1} ES^1 \tag{3.3}$$

for n odd.

Recall that the loop homology is given as follows by [CJY]:

$$\mathbb{H}_*(LS^n) = \Lambda[a] \otimes \mathbb{Z}[u]$$

where a corresponds to the dual of $[S^n]$, so |a| = -n and |u| = n - 1 after re-grading.

The BV-operator acts on the generators as follows, [Me]:

$$\Delta(a \otimes u^i) = i(1 \otimes u^{i-1})$$
$$\Delta(1 \otimes u^i) = 0.$$

Throughout this section, we consider n to be odd. Consider the bottom of the Gysin sequence. Let $H_i^{S^1}$ denote $H_i^{S^1}(LS^n)$ and \mathbb{H}_i denote $\mathbb{H}_i(LS^n)$



The maps $H_i^{S^1} \longrightarrow H_{i-2}^{S^1}$ are given by the cap product with the class generator $x \in H^2(\mathbb{C}P^\infty)$. Since $H_2^{S^1}(LS^n) \cong \mathbb{Z}$, we denote the generator by γ , which is dual to x. We use the notation $\gamma_i = \frac{\gamma^i}{i!}$, dual to x^i . Since the maps given by

the cap product are isomorphisms between where the loop homology is zero, we have that

$$H_{2i+1}^{S^1}(LS^n) = 0, \quad 0 \le i \le \frac{n-3}{2}$$
$$H_{2i}^{S^1}(LS^n) = \mathbb{Z}(\gamma_i), \quad 1 \le i \le \frac{n-3}{2}$$

Note that for even degrees, the generator γ_i increases subscript as isomorphisms in the sequence are given by cap product with x, dual to the cup product with x.

To determine $H_{n-1}^{S^1}(LS^n)$, note that we have a short exact sequence,

$$0 \longrightarrow \mathbb{Z}(a \otimes u) \longrightarrow H_{n-1}^{S^1} \longrightarrow \mathbb{Z}(\gamma_{\frac{n-3}{2}}) \longrightarrow 0$$

that splits since the last term is free. Thus $H_{n-1}^{S^1}(LS^n) \cong \mathbb{Z}e(a \otimes u) \oplus (\gamma_{\frac{n-1}{2}})$. We use the notation of e(-) to denote that the generator comes from the erasing map. Using the BV-operator to determing the marking map M: $H_{n-1}^{S^1} \to \mathbb{H}_0$, we have that $M(a \otimes u) = 1 \otimes 1$, so the erasing map $e : \mathbb{H}_0 \to H_n^{S^1}$ is zero, thus $H_n^{S^1}(LS^n) \cong 0$.

Lemma 3.3.1. $M(\gamma_{\frac{n-1}{2}}) = 0$, or more generally, the marking map sends generators coming from $H_*(\mathbb{C}P^\infty)$ to zero.

Proof. In the circle bundle (3.3) the marking map is an umkehr map coming from the projection map. Notice that $\mathbb{C}P^{\infty} = BS^1 = \{pt\} \times ES^1 \subset LS^n \times_{S^1}$ ES^1 . Since $\pi^{-1}(\{pt\} \times_{S^1} ES^1) = \{pt\} \times ES^1$, which is contractible, then M maps generators from $\mathbb{C}P^{\infty}$ into a contractible space, thus $M(\gamma_i) = 0$ for any i, where γ_i denotes a generator coming from the homology of $\mathbb{C}P^{\infty}$. \Box

With the knowledge that $M(\gamma_{\frac{n-1}{2}}) = 0$, the cap product map $H_{n+1}^{S^1} \to H_{n-1}^{S^1}$ is injective with image isomorphic to $\mathbb{Z}(\gamma_{\frac{n-1}{2}})$ so $H_{n+1}^{S^1} \cong \mathbb{Z}(\gamma_{\frac{n+1}{2}})$.

Now consider the next piece of the Gysin sequence where loop homology is non-zero.



In the third and fourth row above, we have a short exact sequence with $H_{2n-2}^{S^1}$ in the center, which splits, so $H_{2n-2}^{S^1} \cong \mathbb{Z}(e(a \otimes u^2)) \oplus \mathbb{Z}(\gamma_{n-1})$. Mapping $H_{2n-2}^{S^1}$ through M, we have $M(e(a \otimes u^2)) = 2(1 \otimes u)$ given by the BV-operator.

Thus $H_{2n-1}^{S^1} \cong \mathbb{Z}/2\mathbb{Z}(e(1 \otimes u))$. Since the cap product map $H_{2n}^{S^1} \to H_{2n-2}^{S^1}$ is injective with image $\mathbb{Z}(\gamma_{n-1}), H_{2n}^{S^1} \cong \mathbb{Z}(\gamma_n)$. Summarizing, we have

$$H_{2i+1}^{S^{1}}(LS^{n}) \cong 0, \qquad \qquad \frac{n-1}{2} \le i \le n-2$$

$$H_{2i}^{S^{1}}(LS^{n}) \cong \mathbb{Z}(\gamma_{i}), \qquad \qquad \frac{n+1}{2} \le i \le n-2$$

$$H_{n-2}^{S^{1}}(LS^{n}) \cong \mathbb{Z}e(a \otimes u^{2}) \oplus \mathbb{Z}(\gamma_{n-1})$$

$$H_{n-1}^{S^{1}}(LS^{n}) \cong \mathbb{Z}/2\mathbb{Z}(e(1 \otimes u))$$

$$H_{2n}^{S^{1}}(LS^{n}) \cong \mathbb{Z}(\gamma_{n})$$

$$H_{2i+1}^{S^{1}} \cong \mathbb{Z}_{2}(1 \otimes u)\gamma_{i-n-1} \qquad n-1 \le i \le \frac{3n-5}{2}$$

$$H_{2i}^{S^{1}} \cong \mathbb{Z}\gamma_{i} \qquad \qquad \frac{n}{2} \le i \le \frac{3n-5}{2}$$

Now assume the following holds for all $k \in \mathbb{N}$:

$$H_{2i}^{S^1} \cong \begin{cases} \mathbb{Z}\gamma_i & \text{if } (n-1) \nmid 2i, \\ \mathbb{Z}\gamma_i \oplus \mathbb{Z}(e(a \otimes u^i)) & \text{if } (n-1)|2i. \end{cases}$$
(3.4)

for $1 \le i \le \frac{(k+1)(n-1)-2}{2}$ and

$$H_{2i+1}^{S^1} \cong t_k$$

for $\frac{k(n-1)}{2} \leq i \leq \frac{(k+1)(n-1)-2}{2}$, where t_k is a torsion group of order k!. (We would like to be able to say that $H_{2i+1}^{S^1} \cong \mathbb{Z}_k(e(1 \otimes u^{k-1}))\gamma_{i-\frac{k}{2}(n-1)} \oplus \cdots \oplus \mathbb{Z}_3 e(1 \otimes u^2)\gamma_{i-\frac{3}{2}(n-1)} \oplus \mathbb{Z}_2 e(1 \otimes u)\gamma_{i-n+1}$, but there are extension issues that are difficult to resolve, so we cannot say which torsion group $H_{2i+1}^{S^1}$ should be.)



Thus, we can extract a short exact sequence from the last two lines of the Gysin sequence above, giving us $H^{S^1}_{(k+1)(n-1)} \cong \mathbb{Z}(e(a \otimes u^{k+1})) \oplus \mathbb{Z}(\gamma_{\frac{(k+1)(n-1)}{2}}).$

It can be seen that $H_{(k+1)(n-1)+2}^{S^1} \cong \mathbb{Z}(\gamma_{(k+1)(n-1)+2})$ and that $torsion(H_{(k+1)(n-1)+1}^{S^1}) \cong t_{k+1}$. Since $H_{(k+1)(n-1)+1}^{S^1}$ is all torsion since it is sandwiched between a short exact sequence of torsion groups, we have $H_{(k+1)(n-1)+1}^{S^1} \cong t_{k+1}$. Since loop homology $\mathbb{H}_i(LS^n)$ is zero for $(k+1)(n-1)+2-n \leq i \leq (k+2)(n-1)-1-n$, we obtain the analogous statements of (3.4) for k+1. In summary, we get the following theorem.

Theorem 3.3.2. For n odd,

$$H_{2i}^{S^1}(LS^n) \cong \begin{cases} \mathbb{Z}\gamma_i & \text{if } (n-1) \nmid 2i \\ \mathbb{Z}\gamma_i \oplus \mathbb{Z}(e(a \otimes u^i)) & \text{if } (n-1)|2i. \end{cases}$$

for $1 \le i \le \frac{(k+1)(n-1)-2}{2}$ and

$$H_{2i+1}^{S^1}(LS^n) \cong t_k$$

for $\frac{k(n-1)}{2} \leq i \leq \frac{(k+1)(n-1)-2}{2}$, where t_k is a torsion group of order k!. All other j that does not fall into the above categories, we have that $H_i^{S^1}(LS^n) \cong H_{i-2}^{S^1}(LS^n)$

3.3.2 The String Bracket for Odd Spheres

The string bracket is a degree 2 - n map. The only possible non-zero bracket is of the generators $e(a \otimes u^i)$, since the marking map M sends all other generators to zero.

Theorem 3.3.3.

$$[e(a \otimes u^i), e(a \otimes u^j)] = ije(1 \otimes u^{i+j-2})$$

where $e(1 \otimes u^{i+j-2})$ is a generator of \mathbb{Z}_{i+j-1} , so the bracket is only zero when i+j-1 divides ij

Proof.

$$\begin{split} [e(a \otimes u^i), e(a \otimes u^j)] = & e(M(e(a \otimes u^i)) \bullet M(e(a \otimes u^j))) \\ = & e(i(1 \otimes u^{i-1}) \bullet j(1 \otimes u^{j-1})) \\ = & e(ij(1 \otimes u^{i+j-2})) = . \end{split}$$

3.4 String Homology and the String Bracket of Even Spheres

3.4.1 Computations for S^4

Here we only use information from the Gysin sequence. As before, we know that

$$\mathbb{H}_*(LS^4, \mathbb{Z}) = \frac{\Lambda(b) \otimes \mathbb{Z}[a, v]}{(a^2, ab, 2av)}$$

where |a| = -4, |b| = -1, and |v| = 6, so all of the generators look like av^k , bv^k , v^k , where $|av^k| = -4 + 6k$, $|bv^k| = -1 + 6k$, $|v^k| = 6^k$ [CJY].

We also know how the BV-operator acts,

$$\Delta(v^k) = 0$$
$$\Delta(av^k) = 0$$
$$\Delta(bv^k) = (2k+1)v^k$$

from [Me]. Let's consider the bottom of the Gysin sequence: $H_{7}^{S^{1}}(LS^{4}) \longrightarrow H_{5}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{2}(LS^{4}) \cong \mathbb{Z}_{2}(av)$ $\stackrel{e}{\longrightarrow} H_{6}^{S^{1}}(LS^{4}) \longrightarrow H_{4}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{1}(LS^{4}) \cong 0$ $\stackrel{e}{\longrightarrow} H_{5}^{S^{1}}(LS^{4}) \longrightarrow H_{3}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{-1}(LS^{4}) \cong \mathbb{Z}(1)$ $\stackrel{e}{\longrightarrow} H_{3}^{S^{1}}(LS^{4}) \longrightarrow H_{1}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{-2}(LS^{4}) \cong 0$ $\stackrel{e}{\longrightarrow} H_{2}^{S^{1}}(LS^{4}) \longrightarrow H_{0}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{-3}(LS^{4}) \cong 0$ $\stackrel{e}{\longrightarrow} H_{1}^{S^{1}}(LS^{4}) \longrightarrow 0 \stackrel{M}{\longrightarrow} \mathbb{H}_{-4}(LS^{4}) \cong \mathbb{Z}(a)$ The Gysin sequence, along with the BV-operator, allow us to determine that

$$H_0^{S^1}(LS^4) \cong \mathbb{Z}(e(a))$$

$$H_1^{S^1}(LS^4) \cong 0$$

$$H_2^{S^1}(LS^4) \cong \mathbb{Z}(\gamma)$$

$$H_3^{S^1}(LS^4) \cong \mathbb{Z}(e(b))$$

$$H_4^{S^1}(LS^4) \cong \mathbb{Z}(\gamma_2)$$

$$H_5^{S^1}(LS^4) \cong 0$$

$$H_6^{S^1}(LS^4) \cong \mathbb{Z}(\gamma_3) \oplus \mathbb{Z}_2(e(av))$$

$$H_7^{S^1}(LS^4) \cong 0.$$

Here, γ is the generator from $H_2(\mathbb{C}P^{\infty})$. Continuing up the Gysin sequence inductively, the *k*th piece of the sequence is as follows.

$$H_{6k+7}^{S^{1}}(LS^{4}) \to H_{6k+5}^{S^{1}}(LS^{4}) \stackrel{M}{\to} \mathbb{H}_{6k+2}(LS^{4}) \cong \mathbb{Z}_{2}(av^{k+1})$$

$$\stackrel{e}{\to} H_{6k+6}^{S^{1}}(LS^{4}) \to H_{6k+4}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{6k+1}(LS^{4}) \cong 0$$

$$\stackrel{e}{\to} H_{6k+5}^{S^{1}}(LS^{4}) \to H_{6k+3}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{6k}(LS^{4}) \cong \mathbb{Z}(v^{k})$$

$$\stackrel{e}{\to} H_{6k+4}^{S^{1}}(LS^{4}) \to H_{6k+2}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{6k-1}(LS^{4}) \cong \mathbb{Z}(bv^{k})$$

$$\stackrel{e}{\to} H_{6k+3}^{S^{1}}(LS^{4}) \to H_{6k+1}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{6k-2}(LS^{4}) \cong 0$$

$$\stackrel{e}{\to} H_{6k+2}^{S^{1}}(LS^{4}) \stackrel{\cong}{\to} H_{6k}^{S^{1}}(LS^{4}) \stackrel{M}{\longrightarrow} \mathbb{H}_{6k-3}(LS^{4}) \cong 0$$

Using the Poincaré polynomial for $H^{S^1}_*(LS^4, \mathbb{Z}_2)$ from [We],

$$\left(\frac{1}{1-t^6}\right)\left(t^3 + \frac{1+t^7}{1-t^2}\right)$$

which we can rewrite in a more useful way as follows

$$\sum_{k=0}^{\infty} (k+1)(t^{6k} + t^{6k+2} + t^{6k+3} + t^{6k+4}) + \sum_{k=1}^{\infty} k(t^{6k+1} + t^{6k+5}).$$

From this, we see that $H^{S^1}_{6k+2}(LS^4,\mathbb{Z}_2) \cong \bigoplus_{i=1}^k \mathbb{Z}_2$ Using the Universal Coefficient Theorem,

$$0 \to H^{S^1}_{6k+2}(LS^4, \mathbb{Z}) \otimes \mathbb{Z}_2 \to \bigoplus_{i=1}^k \mathbb{Z}_2 \to Tor(H^{S^1}_{6k+1}(LS^4, \mathbb{Z}), \mathbb{Z}_2) \to 0$$

Since $H_{6k+1}^{S^1}(LS^4), \mathbb{Z}) = 0$, We have that $H_{6k+2}^{S^1}(LS^4, \mathbb{Z}) \otimes \mathbb{Z}_2 \cong \bigoplus_{i=1}^k \mathbb{Z}_2$. We knot that $H_{6k+2}^{S^1}(LS^4, \mathbb{Z})$ has a summand $\bigoplus_i \mathbb{Z}_{2^l}$ where $\sum l = k$, then we have

$$H_{6k+2}^{S^1}(LS^4,\mathbb{Z})\otimes\mathbb{Z}_2\cong\bigoplus_i\mathbb{Z}_{2^l}\cong\bigoplus_{i=1}^k\mathbb{Z}_2.$$

So we must have that each l = 1. Therefore, we have the following proposition.

Proposition 3.4.1.

$$H_{6k+7}^{S^{1}} \cong 0$$

$$H_{6k+6}^{S^{1}} \cong \mathbb{Z}(\gamma_{3k+3}) \oplus \mathbb{Z}_{2}(e(av^{k+1})) \bigoplus_{i=1}^{k+1} [\mathbb{Z}_{2}(e(av^{i})\gamma)] \oplus C_{k}$$

$$H_{6k+5}^{S^{1}} \cong 0$$

$$H_{6k+4}^{S^{1}} \cong \mathbb{Z}(\gamma_{3k+2}) \bigoplus_{i=1}^{k+1} [\mathbb{Z}_{2}(e(av^{i})\gamma)] \oplus C_{k}$$

$$H_{6k+3}^{S^{1}} \cong \mathbb{Z}e(bv^{k})$$

$$H_{6k+2}^{S^{1}} \cong H_{6k}^{S^{1}} \cong \mathbb{Z}(\gamma_{3k}) \oplus C_{k-1}.$$

 C_k is some torsion group of order $\prod_{i=1}^k (2i+1)$. The torsion comes from the fact that $M(e(bv^k)) = \Delta(bv^k) = (2k+1)v^k$, which is where all of the odd torsion groups come from.

3.4.2 The String Bracket for S^4

The string bracket for S^4 is of degree -2. We know that the marking map M maps generators from $H_*(\mathbb{C}P^{\infty})$ to zero (Lemma 3.3.1), and it takes all the torsion to zero since for those cases, M maps into zero or into a free group. Thus, the only possible case for the bracket to be nonzero is for the generators $e(bv^k)$. We have that

$$[e(bv^{k}), e(bv^{l})] = (-1)e(\Delta(bv^{k}) \bullet \Delta(bv^{l}))$$
$$= -e((2k+1)v^{k} \bullet (2l+1)v^{l})$$
$$= -(2k+1)(2l+1)e(v^{k+l})$$
$$= -4kle(v^{k+l}).$$

We know that $e(v^{k+l})$ has order 2(k+l) + 1, so the bracket is zero when $k \neq 0$, $l \neq 0$, and (2k+2l+1)|(4kl+2k+2l+1), or when 4kl|(2k+2l+1), but the latter number is odd, so this can never happen. Thus, the bracket is always nontrivial in this case.

When k = 0 or l = 0, the bracket is zero.

3.4.3 Computations for S^2

We just state results for the string homology computations for S^2 , as the computations for S^4 are more illustrative. From [CJY], we have

$$\mathbb{H}_*(LS^2) = \frac{\Lambda(b) \otimes \mathbb{Z}[a, v]}{a^2, ab, 2av}$$

where |a| = -2, |b| = -1, and |v| = 2. The BV-operator acts as follows,

$$\Delta(v^k) = 0$$

$$\Delta(av^k) = 0$$

$$\Delta(bv^k) = (2k+1)v^k + av^{k+1}$$

Proposition 3.4.2.

$$H_0^{S^1}(LS^2) \cong \mathbb{Z}e(a)$$
$$H_2^{S^1}(LS^2) \cong \mathbb{Z}e(a)\gamma \oplus \mathbb{Z}_2e(av)$$
$$H_4^{S^1}(LS^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$$
$$H_{2i+1}^{S^1}(LS^2) \cong \mathbb{Z}e(bv^i)$$
$$H_{2j}^{S^1}(LS^2) \cong \mathbb{Z} \oplus C_k$$

where $i \ge 0$, $j \ge 3$, and C_k is a torsion group of order $\prod_{k=1}^{j-1} (4j - 2 - 4k)$. The string bracket is only non-zero in odd degrees. Proposition 3.4.3.

$$[e(bv^i), e(bv^j)] = -4ije(v^{i+j})$$

which is not always zero since $e(v^{i+j})$ is torsion, and all other brackets are zero.

Proof.

$$\begin{split} [e(bv^{i}), e(bv^{j})] &= (-1)^{2i+1-2} e(\Delta(bv^{i}) \bullet \Delta(bv^{j})) \\ &= -e(((2i+1)v^{i} + av^{i+1}) \bullet ((2j+1)v^{j} + av^{j+1})) \\ &= -e((2i+1)(2j+1)v^{i+j} + (2i+2j+2)av^{i+j+1}) \\ &= -(2i+1)(2j+1)e(v^{i+j}) \\ &= -4ije(v^{i+j}). \end{split}$$

3.4.4 String Homology for Even Spheres

(CLEAN UP THIS SECTION) We have from [CJY] that, for n even.

$$\mathbb{H}_*(LS^n, \mathbb{Z}) \cong \frac{\Lambda(b) \otimes \mathbb{Z}[a, v]}{(a^2, ab, 2av)}$$

where |a| = -n, |b| = -1 and |v| = 2n - 2. By [Me] we have

$$\Delta(v^k) = 0$$

$$\Delta(av^k) = 0$$

$$\Delta(bv^k) = (2k+1)v^k.$$

To keep track of things, $|av^k| = k(2n - 2) - n$, $|bv^k| = k(2n - 2) - 1$, $|v^k| = k(2n - 2)$. Let us consider the bottom of the Gysin sequence:



from this sequence and knowledge of the BV-operator, we get

$$H_0^{S^1}(LS^n) \cong \mathbb{Z}a$$

$$H_1^{S^1}(LS^n) \cong 0$$

$$H_2^{S^1}(LS^n) \cong \mathbb{Z}\gamma$$

$$H_3^{S^1}(LS^n) \cong 0$$

$$H_4^{S^1}(LS^n) \cong \mathbb{Z}\gamma_2$$

$$\vdots$$

$$H_{n-1}^{S^1}(LS^n) \cong \mathbb{Z}e(b)$$

$$H_n^{S^1}(LS^n) \cong \mathbb{Z}e(v) \oplus \mathbb{Z}\gamma_{\frac{n}{2}}$$

$$\vdots$$

$$H_{2n-2}^{S^1}(LS^n) \cong \mathbb{Z}_2e(av) \oplus \mathbb{Z} \oplus \mathbb{Z}\gamma_{\frac{2n-2}{2}}$$

$$H_{2n-1}^{S^1}(LS^n) \cong 0$$

$$H_{2n}^{S^1}(LS^n) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$\vdots$$

$$H_{3n-3}^{S^1}(LS^n) \cong \mathbb{Z}e(bv)$$

$$H_{3n-2}^{S^1}(LS^n) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3e(v^3) \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$H_{4n-4}^{S^{1}}(LS^{n}) \cong \mathbb{Z}_{2} \oplus \mathbb{Z}_{2} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z} \oplus \mathbb{Z}$$
$$H_{4n-3}^{S^{1}}(LS^{n}) \cong 0$$
$$\vdots$$
$$H_{5n-5}^{S^{1}}(LS^{n}) \cong \mathbb{Z}e(bv^{2})$$
$$H_{5n-4}^{S^{1}}(LS^{n}) \cong \mathbb{Z}_{2} \oplus \mathbb{Z}_{2} \oplus \mathbb{Z}_{2} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z} \oplus \mathbb{Z}$$
$$\vdots$$

where all of the odd degree homology are isomorphic, and all even degree homology are isomorphic, or $H_i^{S^1} \cong H_{i-2}^{S^1}$ m in the gaps denoted by the vertical dots. The *k*-th piece of the sequence is as follows:

$$H_{k(2n-2)+n+1}^{S^{1}} \xrightarrow{\cong} H_{k(2n-2)+n-1}^{S^{1}} \xrightarrow{M} \mathbb{H}_{k(2n-2)} \cong \mathbb{Z}(v^{k})$$

$$\xrightarrow{e}$$

$$H_{k(2n-2)+n}^{S^{1}} \xrightarrow{\cong} H_{k(2n-2)-2+n}^{S^{1}} \xrightarrow{M} \mathbb{H}_{k(2n-2)-1} \cong \mathbb{Z}(bv^{k})$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$H_{k(2n-2)+1}^{S^{1}} \longrightarrow H_{k(2n-2)-1}^{S^{1}} \cong 0 \xrightarrow{M} \mathbb{H}_{k(2n-2)-n} \cong \mathbb{Z}_{2}(av^{k})$$

inductively, we have that

$$H_{k(2n-2)-2}^{S^1} \cong \mathbb{Z}_2^{k-1} \oplus C_k \oplus \mathbb{Z} \oplus \mathbb{Z}$$

where C_k is a torsion group of order $\prod_{i=1}^{k-1} (2i+1)$. The bottom of the above Gysin sequence gives the short exact sequence

$$0 \to \mathbb{Z}_2 a v^k \to H^{S^1}_{k(2n-2)} \to H^{S^1}_{k(2n-2)-2} \to 0$$

which gives

$$H_{k(2n-2)}^{S^1} \cong \mathbb{Z}_2 e(av^k) \oplus \mathbb{Z}_2^{k-1} \oplus \left(\text{torsion group of order } \sum_{i=1}^{k-1} (2i+1) \right) \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Note that the even torsion can be resolved using the results by Westerland in [We] (as in the above example for S^4). From the top of the above Gysin sequence, we get the following.

$$H_{k(2n-2)+1}^{S^1}(LS^n) \cong 0$$

$$\vdots$$

$$H_{k(2n-2)+n-1}^{S^1}(LS^n) \cong \mathbb{Z}e(bv^k)$$

$$H_{k(2n-2)+n}^{S^1}(LS^n) \cong \mathbb{Z}_2^k \oplus C_k \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Summarizing, we get the following theorem.
Theorem 3.4.4. Suppose n is even.

$$H_{k(2n-2)-2}^{S^1}(LS^n) \cong \mathbb{Z}_2^{k-1} \oplus C_k \oplus \mathbb{Z} \oplus \mathbb{Z}$$
$$H_{k(2n-2)+1}^{S^1}(LS^n) \cong 0$$
$$\vdots$$
$$H_{k(2n-2)+n-1}^{S^1}(LS^n) \cong \mathbb{Z}e(bv^k)$$
$$H_{k(2n-2)+n}^{S^1}(LS^n) \cong \mathbb{Z}_2^k \oplus C_k \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

where all other unstated homology, we have $H_i^{S^1}(LS^n)\cong H_{i-1}^{S^1}(LS^n).$.

3.4.5 The String Bracket for Even Spheres

Theorem 3.4.5. The string bracket is always zero except on the generators $e(bv^j)$,

$$[e(bv^k), e(bv^l)] = -(4kl + 2k + 2l + 1)e(v^{k+l})$$

but $e(v^{k+l})$ has order 2(k+l) + 1 so it is not always zero.

Proof.

$$[e(bv^k), e(bv^l)] = (-1)^{k(2n-2)-1-n} e(M(e(bv^k)) \bullet M(e(bv^l)))$$
$$= -(4kl + 2k + 2l + 1)e(v^{k+l})$$

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Chapter 4

String Homology and String Bracket Computations for Surfaces

In this chapter we compute the string homology and string bracket for surfaces.

4.1 The Torus

We can compute the loop homology of the torus T quite easily since we already know the loop homology of S^1 . We compute the loop homology and BV-operator using the following,

$$\mathbb{H}_*(LT) \cong \mathbb{H}_*(\mathbb{LS}^{\mathbb{H}}) \otimes \mathbb{H}_*(\mathbb{LS}^{\mathbb{H}})$$

and that we can compute the BV-operator as follows:

$$\Delta_T(a \otimes b) = \Delta(a) \otimes b + (-1)^{|a|+1} a \otimes \Delta(b).$$

We obtain the following:

$$\mathbb{H}_{-2}(LT) \cong \bigoplus_{(n,m)\in\mathbb{Z}^2} \mathbb{Z}(1_{nm})$$
$$\mathbb{H}_{-1}(LT) \cong \bigoplus_{(n,m)\in\mathbb{Z}^2} \mathbb{Z}x_{nm} \bigoplus_{(n,m)\in\mathbb{Z}^2} \mathbb{Z}y_{nm}$$
$$\mathbb{H}_0(LT) \cong \bigoplus_{(n,m)\in\mathbb{Z}^2} \mathbb{Z}z_{nm}$$

the loop product:

$$x_{nm} \bullet y_{kl} = 1_{n+k,m+l}$$
$$x_{nm} \bullet z_{kl} = x_{n+k,m+l}$$
$$y_{nm} \bullet x_{kl} = y_{n+k,m+l}$$
$$z_{nm} \bullet z_{kl} = z_{n+k,m+l}$$

and the BV-operator:

$$\Delta(1_{nm}) = nx_{nm} + my_{nm}$$
$$\Delta(x_{nm}) = nz_{nm}$$
$$\Delta(y_{nm}) = -mz_{nm}$$
$$\Delta(z_{nm}) = 0.$$

To calculate string homology of the torus, we use the Gysin sequence:

$$\begin{array}{c} 0 \\ & H_{3}^{S^{1}}(LT) \stackrel{c}{\star} H_{1}^{S^{1}}(LT) \stackrel{M}{\longrightarrow} \mathbb{H}_{0}(LT) \cong \bigoplus_{(n,m)\in\mathbb{Z}^{2}} \mathbb{Z}z_{nm} \\ & e \\ & H_{2}^{S^{1}}(LT) \star H_{0}^{S^{1}}(LT) \stackrel{M}{\star} \mathbb{H}_{-1}(LT) \cong \bigoplus_{(n,m)\in\mathbb{Z}^{2}} \mathbb{Z}x_{nm} \bigoplus_{(n,m)\in\mathbb{Z}^{2}} \mathbb{Z}y_{nm} \\ & e \\ & H_{1}^{S^{1}}(LT) \longrightarrow 0 \stackrel{M}{\longrightarrow} \mathbb{H}_{-2}(LT) \cong \bigoplus_{(n,m)\in\mathbb{Z}^{1}} \mathbb{Z}(1_{nm}) \\ & e \\ & H_{0}^{S^{1}}(LT) \longrightarrow 0 \end{array}$$

we get the following:

$$\begin{aligned} H_0^{S^1}(LT) &\cong \bigoplus_{(n,m)\in\mathbb{Z}^2} \mathbb{Z}e(1_{nm}) \\ H_1^{S^1}(LT) &\cong \bigoplus_{\substack{(n,m)\in\mathbb{Z}^2\\gcd(n,m)=d\\d\neq n,m}} \mathbb{Z}_d(q_{n_1}e(x_{nm}) + q_{m_1}e(y_{nm})) \bigoplus_{\substack{n|m\\nd=m}} \mathbb{Z}_n(e(x_nm) + de(y_{nm})) \\ &\bigoplus_{\substack{m|n\\md=n}} \mathbb{Z}_m(de(x_{nm}) + e(y_{nm})) \bigoplus_{(n,m)\in\Delta} \mathbb{Z}_n(e(x_{nn}) + e(y_nn)) \\ &\bigoplus_{\substack{gcd(n,m)=1}} \mathbb{Z}(q_ne(x_{nm}) + q_me(y_{nm})) \bigoplus_{\substack{gcd(n,m)=d}} \mathbb{Z}(q_ne(x_{nm}) + q_me(y_{nm})) \\ &\bigoplus_{n|m} \mathbb{Z}_e(y_{nm}) \bigoplus_{m|n} e(x_{nm}) \bigoplus_{(n,n)\in\Delta} \mathbb{Z}e(y_{nn}) \oplus \mathbb{Z}e(x_{00}) \oplus e(y_{00}) \end{aligned}$$

where for $H_1^{S^1}(LT)$, $(n,m) \in \mathbb{Z} - (0,0)$ for all n,m, and Δ is the diagonal in $\mathbb{Z} \times \mathbb{Z}$, and each q_i are polynomials in the quotients that show up in the division algorithm for n, m. Note that the generator in the third to the last summand could also have been chosen to be $\mathbb{Z}e(x_nn)$. To calculate $H_2^{S^1}(LT)$, we look at the second and third line in the Gysin sequence above. Using that $\mathbb{H}_0(LT)$ is free, and Δ maps torsion to zero, the marking map is nonzero only on the free elements of $H_1^{S^1}(LT)$, we can extract the following short exact sequence:

$$0 \to \frac{\mathbb{H}_0(LT)}{im(M) = im(\Delta)} \to H_2^{S^1}(LT) \to \mathbb{Z}1_{00} \to 0$$

where the last part of the sequence comes from the kernel of M being generated by 1_{00} . We get that

$$\frac{\mathbb{H}_0(LT)}{im(M)} = \bigoplus_{(n,n)\in\Delta-(0,0)} \mathbb{Z}_n z_{nn} \bigoplus_{m|n} \mathbb{Z}_n z_{nm} \bigoplus_{n|m} \mathbb{Z}_m z_{nm} \bigoplus_{\substack{gcd(m,n)=d\\d\neq n,m}} \mathbb{Z}_{q_nn-mq_m} z_{nm}.$$

Thus, we have

$$H_2^{S^1}(LT) \cong \frac{\mathbb{H}_0(LT)}{im(M)} \oplus \mathbb{Z}1_{00}\gamma$$

From the very top of the Gysin sequence pictured above, we get that $H_3^{S^1}(LT) \cong ker(M)$, thus we have

$$H_3^{S^1}(LT) \cong tor(H_1^{S^1}(LT)) \oplus \mathbb{Z}x_{00}\gamma \oplus \mathbb{Z}y_{00}\gamma.$$

Since loop homology higher than two is zero, we obtain the following isomorphisms:

$$H_{2k}^{S^1}(LT) \cong H_2^{S^1}(LT)$$

 $H_{2k+1}^{S^1}(LT) \cong H_3^{S^1}(LT)$

for all $k \geq 2$.

Torus Bracket Computations

For $[-,-]: H_0^{S^1}(LT) \otimes H_0^{S^1}(LT) \to H_0^{S^1}(LT),$

$$[e(1_{nm}), e(1_{kl})] = (nl - mk)e(1_{(n+k),(m+l)}).$$

which corresponds to the Goldman bracket for the torus in Proposition 1.1.3. For $[-,-]: H_0^{S^1}(LT) \otimes H_1^{S^1}(LT) \to H_1^{S^1}(LT)$,

$$[e(1_nm), e(q_1x_{nm} + q_2y_{kl})] = (q_1k - q_2l)e(nx_{n+k,m+l} + my_{n+k,m+l})$$

For $[-,-]: H_1^{S^1}(LT) \otimes H_1^{S^1}(LT) \to H_2^{S^1}(LT),$

$$[e(q_1x_{nm} + q_2y_{nm}), e(q_3x_{kl} + q_4y_{kl})] = (q_1n - q_2m)(q_3k - q_4l)e(z_{n+k,m+l})$$

which is torsion, and not always zero. All other brackets are zero.

String Lie Algebra Structure on the Torus

The center of the String Lie algebra can be directly computed from the above bracket results. Let $\mathfrak{g}(T)$ denote the String Lie algebra for the closed

torus, the center of the Lie algebra is as follows:

$$Z(\mathfrak{g}(T)) \cong H^{(S^{1})}_{*}(L_{0}(T) \bigoplus_{k \in \mathbb{Z} - \{0\}} \mathbb{Z}e(y_{k0}) \bigoplus_{l \in \mathbb{Z} - \{0\}} \mathbb{Z}e(x_{0l})$$
$$\bigoplus_{(m,n) \in \mathbb{Z}^{2} - \{(0,0)\}} \mathbb{Z}e(z_{mn}) \oplus \operatorname{tor}(H^{S^{1}}_{1}(LT)) \oplus H^{S^{1}}_{2}(LT)$$

where $L_0(T)$ denotes the connected component of the loop space LT containing the contractible loops.

4.2 Genus g > 1

Using that $L(BG) = \prod_{[\gamma]} BC(\gamma)$, for a closed, orientable surface Σ_g of genus g > 1, we have

$$L\Sigma_g = LB(\pi_1(\Sigma_g, *)) = \prod_{[\gamma] \in \hat{\pi}} BC(\gamma)$$

where $[\gamma]$ is a conjugacy class in $\pi_1(\Sigma_g)$, or an element of $\hat{\pi}$. By Kupers [Ku], $C(\gamma) \cong \mathbb{Z}$ for $\gamma \neq e$, where e is the identity in $\pi_1(\Sigma_g)$. This gives us that $BC(\gamma) = S^1$. For the centralizer of e, $C(e) = \pi_1(\Sigma_g)$ since everything commutes with e, so $BC(e) = \Sigma_g$. Therefore, we have

$$L\Sigma_g = \coprod_{[\gamma] \neq e \in \hat{\pi}} S^1 \cup \Sigma_g.$$

We can follow [Ku], and also knowing the loop homology of S^1 and Σ_g , we get the following

Theorem 4.2.1. (Kupers, [Ku], Theorem 2.2)

$$\begin{split} \mathbb{H}_{-2}(L\Sigma_g) &= \bigoplus_{\gamma \in \hat{\pi}} \mathbb{Z}[\gamma] \\ \mathbb{H}_{-1}(L\Sigma_g) &= H_1(\Sigma_g) \bigoplus_{\gamma \in \hat{\pi} - \{e\}} \mathbb{Z}\beta_\gamma \\ \mathbb{H}_0(L\Sigma_g) &= \mathbb{Z}1 \\ \mathbb{H}_i(L\Sigma_g) &= 0 \quad for \ i > 0. \end{split}$$

Let $H_1(\Sigma_g) = \bigoplus_{i=1}^g \mathbb{Z}a_i \bigoplus_{j=1}^g \mathbb{Z}b_i$. Let κ_{γ} be the generator of $C[\gamma]$, then $\gamma = \kappa_{\gamma}^{l_{\gamma}}$,

where $l_{\gamma} \in \mathbb{Z}$. The loop product can be computed as follows [Ku],

$$1 \bullet 1 = 1 \tag{4.1}$$

$$\beta_{\gamma} \bullet 1 = \beta_{\gamma} \tag{4.2}$$

$$a_i \bullet 1 = a_i \tag{4.3}$$

$$b_j \bullet 1 = b_i \tag{4.4}$$

$$a_i \bullet a_j = 0 \tag{4.5}$$

$$b_i \bullet b_j = 0 \tag{4.6}$$

$$a_i \bullet b_j = \delta_{ij} \tag{4.7}$$

$$[\gamma] \bullet 1 = [\gamma] \tag{4.8}$$

$$\beta_{[\gamma_1]} \bullet \beta_{[\gamma_2]} = \frac{[\beta_{[\gamma_1]}, \beta_{[\gamma_2]}]}{l_{\gamma_1} \cdot l_{\gamma_2}}$$
(4.9)

where (4.3)-(4.6) are just the homology intersection product on $H_1(\Sigma_g)$ and the bracket in (4.9) is the Goldman bracket. We also can compute the BVoperator as in [Ku], where the only non-trivial result is $\Delta([\gamma]) = l_{\gamma}\beta_{\gamma}$.

As before, we can use the Gysin sequence to compute string homology. Consider the bottom of the Gysin sequence,

$$H_{3}^{S^{1}}(L\Sigma_{g}) \xrightarrow{c} H_{1}^{S^{1}}(L\Sigma_{g}) \xrightarrow{M} \mathbb{H}_{0}(L\Sigma_{g}) \cong \mathbb{Z}1$$

$$H_{2}^{S^{1}}(L\Sigma_{g}) \xrightarrow{c} H_{0}^{S^{1}}(L\Sigma_{g}) \xrightarrow{M} \mathbb{H}_{-1}(L\Sigma_{g}) \cong \bigoplus_{\gamma \in \hat{\pi} - \{e\}} \mathbb{Z}\beta_{\gamma} \oplus H_{1}(L\Sigma_{g})$$

$$\xrightarrow{e} H_{1}^{S^{1}}(L\Sigma_{g}) \longrightarrow 0 \xrightarrow{M} \mathbb{H}_{-2}(L\Sigma_{g}) \cong \bigoplus_{\gamma \in \hat{\pi}} \mathbb{Z}[\gamma]$$

$$H_{0}^{S^{1}}(L\Sigma_{g}) \longrightarrow 0$$

we get that $H_0^{S^1}(L\Sigma_g) \cong \bigoplus_{\gamma \in \hat{\pi}} \mathbb{Z}e([\gamma])$. Since $M \circ e([\gamma]) = \Delta([\gamma]) = l_{\gamma}\beta_{[\gamma]}$ for $[\gamma] \neq e, \ \Delta([e]) = 0$ and $\Delta(a_i) = \Delta(b_j) = 0$, we get that $H_1^{S^1}(L\Sigma_g) \cong$ $\mathbb{H}_{-1}(L\Sigma_g) / \bigoplus_{\gamma \in \hat{\pi} - \{e\}} (\mathbb{Z}l_{\gamma}\beta_{\gamma})$. Since $M : H_1^{S^1}(L\Sigma_g) \to \mathbb{Z}1$ is the zero map, $H_2^{S^1}(L\Sigma_g)$ sits in the following short exact sequence

$$0 \to \mathbb{Z}_1 \to H_2^{S^1}(L\Sigma_g) \to im(c) \to 0.$$

Since $im(c) = ker(M) = \mathbb{Z}e([e])$, then the above exact sequence splits and $H_2^{S^1}(L\Sigma_g) \cong \mathbb{Z}e(1) \oplus \mathbb{Z}s$ (EXPLAIN s). Therefore, we have the following:

Proposition 4.2.2.

$$H_0^{S^1}(L\Sigma_g) \cong \bigoplus_{\gamma \in \hat{\pi}} \mathbb{Z}e([\gamma])$$

$$H_1^{S^1}(L\Sigma_g) \cong \bigoplus_{i=1}^g \mathbb{Z}e(a_i) \bigoplus_{j=1}^g \mathbb{Z}e(b_i) \bigoplus_{\gamma \in \hat{\pi} - \{e\}} \mathbb{Z}_{l_{\gamma}}e(\beta_{\gamma})$$

$$H_2^{S^1}(L\Sigma_g) \cong \mathbb{Z}e(1) \oplus \mathbb{Z}s$$

$$H_{2i+1}^{S^1}(L\Sigma_g) \cong H_1^{S^1}(L\Sigma_g), \quad i \ge 1$$

$$H_{2i}^{S^1}(L\Sigma_g) \cong H_2^{S^1}(L\Sigma_g), \quad i \ge 1.$$

From this, we can compute the string bracket for Σ_g . The only non-trivial bracket is the Goldman bracket.

Proposition 4.2.3.

$$[e([\gamma_1]), e([\gamma_2])] = [\gamma_1, \gamma_2]$$
(4.10)

The second bracket in 4.10 is the Goldman bracket. So the only non-trivial string bracket of string homology of Σ_g is the Goldman bracket as in Chapter 1.

Bibliography

- [Ba] S. Basu, Transversal String Topology and Invariants of Manifolds, (2011)
- [Br] G.E. Bredon, Topology and Geometry, Graduate Texts in Mathematics; 139 (1993).
- [Ch] M. Chas, The Goldman bracket and the intersection of curves on surfaces, Contemporary Mathematics, Vol. 639, 2015
- [CS] M. Chas and D. Sullivan, String Topology, Ann. Math., (1999)
- [CJY] R. Cohen, J.D.S. Jones, J. Yan, The Loop Homology Algebra of Spheres and Projective Spaces, *Progress in Mathematics*, Vol. 215, 77-92 (2002)
- [CHV] R. Cohen, K. Hess, A. A. Voronov, String Topology and Cyclic Homology, Advanced courses in mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, (2006)
- [Et] P. Etingof, Casimirs of the Goldman Lie algebra of a closed surface, Int. Math Res. Notices, Vol. 2006, (2006)
- [Go] W.M. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, *Invent. math*, 85, 263-302 (1986)
- [He] R. Hepworth, String Topology for Lie Groups
- [Ka] A. Kabiraj, Center of the Goldman Algebra, arXiv:1412.2331 [math.GT], (2014)
- [KKT] N. Kawazumi, Y. Kuno, K. Toda, Generators of the Homological Goldman Lie Algebra, Osaka K. Math, 51, 665-671, (2014)

- [Ku] A.P.M. Kupers, An elementary proof of the string topology structure of compact oriented surfaces, arXiv:1110.1158 [math.AT], (2011)
- [Mc] J. McCleary, A User's Guide to Spectral Sequences, Cambridge University Press, (2001)
- [Me] L. Menichi, String topology for spheres, *Comment. Math. Helv.* 84, no 1, 135-157, (2009)
- [Se] N. Seeliger, Loop homology of spheres and complex projective spaces, arXiv:1104.5219 [math.AT], (2011)
- [T] H. Tamanoi, Batalin-Vilkovisky Lie algebra structure on the loop homology of complex Stiefel manifolds, *International Mathematics Research Notices*, Vol. 2006, Article ID 97193, 1-23 (2006)
- [We] C. Westerland, String homology of spheres and projective spaces, Algebraic and Geometric Topology, 7, 309-325, (2007)