Applying Math onto Mechanism: Investigating the Relationship Between Mechanistic and Mathematical Understanding

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Abstract

Manipulatives are commonly used to provide concrete bases for abstract mathematical concepts. However, it is unclear when concrete experiences benefit abstract thinking. We investigated whether understanding a manipulative's mechanism would affect mathematical use and understanding. Participants completed a robotics task that could be solved with either mathematical or non-mathematical strategies. Participants with higher mechanistic understanding were more likely to utilize complex mathematical strategies during the task, and understood the mathematical relationships within the robot better than participants with lower mechanistic understanding. The study provides evidence for a relationship between mechanistic and mathematical understanding, suggesting that mechanistic manipulatives, upon which mathematics can be applied, may foster mathematical understanding.

Keywords: mathematical understanding; mechanistic understanding; concreteness; manipulatives

Introduction

Mathematical proficiency is increasingly important for success in today's world. A popular strategy for promoting mathematical understanding is the use of manipulatives (e.g., blocks or fraction bars), which are intended to ground abstract mathematical concepts onto concrete experiences (Bruner, 1966). It is an interesting, if not counterintuitive, notion that concrete experiences should support abstract concepts; however, it remains unclear as to when concrete manipulatives improve abstract mathematical understanding. Many studies have found positive learning benefits from using manipulatives when compared to other types of instruction (e.g., Cass, Cates, Smith, & Jackson, 2003; Martin & Schwartz, 2005), but many have found no benefit, or even adverse effects (e.g., McNeil, Uttal, Jarvin, & Sternberg, 2009). In her review of 60 studies comparing manipulative instruction to other instructional types, Sowell (1989) found a large range of negative and positive effect sizes, though she concluded that mathematics achievement could be improved through concrete manipulative use.

Concrete manipulatives may be most beneficial for abstract mathematical learning when mathematics can be directly applied onto the manipulative. A manipulative's mechanism may be one way in which mathematics can be applied onto a manipulative. Mechanisms, as defined by Machamer, Darden, and Craver (2000), are "entities and activities organized such that they are productive of regular changes from start or set-up to finish or termination conditions," describing regular relationships between entities and the effects of activities on these entities. Similarly, mathematics involves understanding the relationships between quantities and the effects of operations on these quantities. We propose that the use of a concrete manipulative makes it more likely for students to discover and understand the mechanisms that exist within that manipulative. Mathematics can then be mapped onto the mechanism (e.g., via structure mapping; Gentner, 1983), which provides a perceptual basis for abstract mathematical concepts while emphasizing the regularity of mathematics (Figure 1).

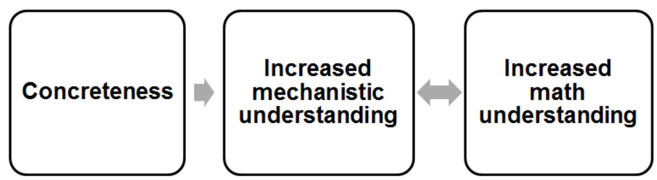

Figure 1. Proposed relationship between concreteness and math understanding.

In the current study, we tested whether students who understood the mechanism of a concrete manipulative would show greater mathematical understanding during a task using that manipulative. We used a robotics task, as robotics has been used to successfully improve mathematics performance (e.g., Petre & Price, 2004; Silk, 2011), and is a rich domain for integrating mathematics with other STEM domains. Our manipulative was the widely-used LEGO NXT robot, whose mechanism consists of three primary parts: the brick, which holds the robot's programs; the motors, which rotate depending on the program's input; and the wheels, which rotate when the motors rotate. Importantly, the motor and wheels are also proportionally related: one motor rotation will always cause the same number of wheel rotations, which will always cause the robot to travel the same distance. Initially, we manipulated participants' mechanistic understanding by providing a mechanistic or non-mechanistic explanation of the robot's functions. However, some participants who received the mechanistic explanation did not recognize the robot's causal mechanism, and vice versa. Because a stronger manipulation would likely affect more than mechanistic understanding alone, we elected to use a correlational study to investigate the relationship between mechanistic and mathematical understanding.

We expected that participants who understood the causal mechanism between the robot's motors and wheels would be more likely to understand the robot's quantitative relationships and utilize mathematics based on these relationships. To test these hypotheses, participants were asked to navigate their robot through a maze, and to create a generalizable strategy that could navigate differently-sized robots through the same maze. Though non-mathematical methods (e.g., guessing and checking) could be used to navigate the maze, participants needed to use proportional reasoning to fulfill the strategy portion of the task. Thus, we hypothesized that: 1) Participants with high mechanistic understanding would use more frequent and more complex mathematics in their maze navigation strategies than participants with low mechanistic understanding; 2) Participants with high mechanistic understanding would show greater understanding of the quantitative relationships that exist within the robot than participants with low mechanistic understanding; and 3) Participants with greater understanding of the robot's quantitative relationships would be the same participants who use more frequent and more complex strategies during the maze navigation task. In addition, it is possible that participants with higher mechanistic understanding attend to proportionally relevant parts of the robot (e.g., the motors and wheels) more than those with lower mechanistic understanding, or that participants need to visualize the robot's mechanism before they can mathematically utilize their mechanistic understanding. Thus, we also explored participants' attention toward the robot and their spatial visualization ability as potential factors in the relationship between mechanistic and mathematical understanding.

Methods

Participants

Participants consisted of 50 undergraduate students recruited through our Psychology department's subject pool. Students majoring in robotics-related or math-heavy majors were not eligible to participate in the study.

Materials

Mechanistic Understanding Assessment Participants were asked two open-ended questions about how the robot functioned: "Please explain the process that the robot goes through to move, starting from its motor rotating" and "Please draw a diagram of the process". Participants were placed into the High Mechanistic group if they indicated, in at least one of the two questions, that the robot's motor rotations caused the robot's wheels to rotate (i.e., they recognized the fundamental causal mechanism that powers the robot), and the Low Mechanistic group if they could not.

Maze Task Participants learned to program a robot using a C-based language called ROBOTC. Commands told the robot which direction to move and the number of times to rotate its motor during each movement (e.g., forward(100), backward(150), turnRight(50), turnLeft(30)). Participants were asked to navigate their robot through a maze (see Figure 2), and to create a strategy that could navigate other robots with differently sized wheels through the same maze, without relying on guess-and-check methods. They were also given a tape measure to measure the maze or robot, though there was no requirement to use the tool. Participants had 30 minutes to complete the task. After creating their initial strategy (on average, after 26 minutes), participants were granted an additional 30 minutes to revise their initial strategy, with a recommendation to include a mathematical formula. They were also given access to a set of smaller robot wheels and were allowed to switch the smaller and larger wheels at will. Two raters coded participants' initial and final strategies based on the type of mathematization used (codes shown in Table 1), with Kappa = 1.0. The first two strategy types do not explicitly use mathematics, though prior research suggests that the Plausible Guesstimation strategy is a foundation upon which more sophisticated mathematical strategies can be built (Nouyvanisvong, 1999). The two later mathematical strategies are both relevant to the task, but only the last one can fully solve the task.

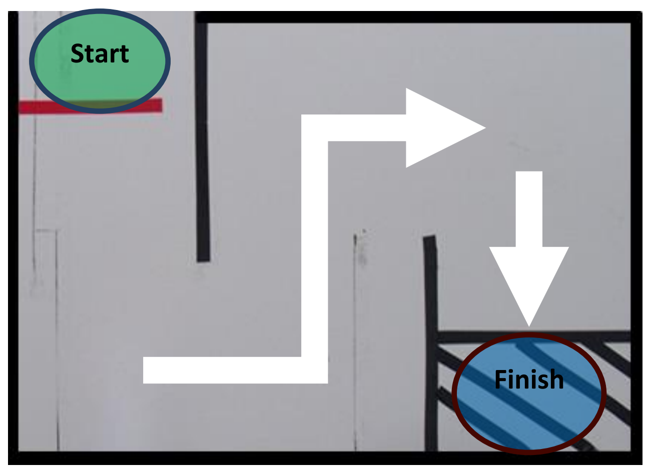

Figure 2. The maze used in the navigation task.

Math Understanding Questionnaire The questionnaire consisted of eight open-ended questions about how the robot's motor rotations, wheel rotations, and distances are quantitatively related (e.g., "Are the number of wheel rotations related to the distance that the robot moves forward?"). Each response was scored for the number of accurate mathematical relationships included in the answer, such that a higher score signified greater quantitative understanding (with a maximum score of 16 points). The Cronbach's alpha (α) for this questionnaire was 0.64, suggesting that an understanding of each of the robot's mathematical relationships was not strongly dependent on one another. However, the overall alpha is sufficiently high to justify analysis as one overall construct.

Robot Drawing Task To determine the features of the robot to which participants attended during the task, participants were asked to draw the robot they had programmed from memory, including the most important parts of the robot in their drawing. A second drawing task, to control for drawing ability differences across participants, was also given with the same instructions, except that participants could look at the robot as a reference while they drew. Both memory and control drawings were coded for the number of accurately drawn wheels and the number of motors included in the drawings (proportionally-relevant features), and whether or not the drawing included a detailed depiction of the robot's screen (a proportionally-irrelevant feature).

Paper Folding Test This test (Ekstrom, French, Harman, & Dermen, 1976) measures spatial visualization ability. A series of pictures depicts one to three folds made in a piece of paper, and the final picture shows a hole punched into the paper. Participants selected which of five options illustrated the reopened piece of paper. The test consisted of two parts with 10 questions each ($\alpha = 0.84$).

Motivation Questionnaire As a control variable, participants answered nine questions about their level of motivation during the maze task, building upon theories and measures of engagement (the Intrinsic Motivation Inventory, e.g., Ryan, 1982) and achievement goals (Elliot & Church, 1997). Three questions involved the participants' level of engagement ($\alpha = 0.91$), three questions involved the participants' level of performance-approach goals ($\alpha = 0.86$), and three questions involved participants' level of mastery-approach goals ($\alpha = 0.80$). Participants were asked to rate their agreement with each statement on a scale of 1 (Strongly disagree) to 7 (Strongly agree).

Procedure

Participants first completed the Paper Folding test. Next, an experimenter gave a brief, verbal introduction to the LEGO NXT robot. Participants then began the maze task, which was introduced as a programming task and included basic programming instructions. Afterwards, participants were given time to revise their initial strategy while working with a robot with smaller-sized wheels. After the maze task, participants were given the Robot Drawing and Control Drawing tasks. They then filled out the Motivation, Mechanism Understanding, and Math Understanding questionnaires.

Results

Based on the Mechanistic Understanding Questionnaire, 31 participants were placed in the High Mechanistic group, and 19 participants were placed in the Low Mechanistic group.

Maze Strategies

Analyses were computed separately for initial strategies (completed before an experimenter recommended the use of a math formula in their strategy) and final strategies. Table 2 shows the percentage of participants in each group who created each type of initial and final strategy (i.e., Guessing, Plausible Guesstimation, Specific Proportional, and General Proportional).

We examined whether the frequency of mathematical strategies (i.e., the proportion of strategies coded as either Specific Proportional or General Proportional, as opposed to Guessing or Plausible Guesstimation) differed between groups. For initial strategies, individuals with higher mechanistic understanding were somewhat more likely to use mathematical strategies prior to any prompting to use mathematics: 49% of High Mechanistic participants used a mathematical strategy, as compared to 21% of Low Mechanistic participants [X^2 (1, N = 50) = 3.74, p = .053]. After an experimenter recommended the use of a mathematical strategy, individuals with higher mechanistic understanding generated more mathematical strategies than individuals with lower mechanistic understanding: 77% of High Mechanistic participants used a mathematical strategy, as compared to 48% of Low Mechanistic participants [X^2 (1, N = 50) = 4.74, p = .029].

To examine the two groups' strategy complexity, we more finely compared the mathematizations used in participants' initial and final strategies (with Guessing being the least complex strategy possible, and General Proportional being the most complex strategy possible). For initial strategies, the High Mechanistic group (mean rank = 29.4) was more likely to create complex strategies than the Low Mechanistic group (mean rank = 19.1) [Mann-Whitney U: U = 173.5, p = .012, r = .36]. For final strategies, the High Mechanistic group (mean rank = 30.7) was more likely to create complex strategies than the Low Mechanistic group (mean rank = 17.0) [U = 133.0, p = .001, r = .48]. Overall, participants with higher mechanistic understanding created strategies that were more mathematically complex (and consequently more accurate), while participants with lower mechanistic understanding relied on simpler guessing or estimation strategies.

Math Understanding Questionnaire

The High Mechanistic group (M = 5.52, SD = 2.42) had significantly higher scores on the Math Understanding Questionnaire than the Low Mechanistic group (M = 3.32, SD = 2.36) [$t(48)$ = -3.15, p = .003, d = .92], showing greater understanding of the quantitative relationships that exist within the robot. Participants' scores on the questionnaire also positively correlated with the complexity of their final maze strategy [$r(50)$ = .375, p = .007], indicating that participants who created more mathematically complex strategies in the maze task were those who possessed greater understanding of the quantitative relationships within the robot.

Robot Drawing Tasks

There were no differences between groups for the number of wheels included (controlling for the number of wheels in control drawings) [$F(1, 47)$ = .055, p = .82, η_p^2 = .001] or the number of motors included (controlling for the number of motors included in control drawings) [$F(1, 47) = .30$, $p = .59$, $\eta_p^2 = .006$]. Similarly, no differences were found in the likelihood of including screen details in the drawing [X^2 (1, $N = 50$) = .42, $p = .52$]. In addition, no correlations existed between Math Relationship Understanding Questionnaire score and the number of wheels [$r(50) = .075$, $p = .61$], the number of motors [$r(50) = .15$, $p = .29$], or whether the screen was included [$r(50) = .17$, $p = .24$] in drawings. Thus, both level of mechanistic understanding and level of mathematical understanding appeared unrelated to level of attention toward basic features of the robot.

Controlling for Individual Differences

Spatial Ability There were no significant correlations between participants' scores on the Paper Folding Test and their initial strategy complexity [$r(50) = .23$, $p = .10$] or their final strategy complexity [$r(50) = .23$, $p = .11$]. There was no correlation between Paper Folding Test score and Math Relationship Understanding Questionnaire score [$r(50) = .21$, $p = .14$]. Thus, spatial visualization ability did not appear to factor into participants' ability to effectively mathematize their understanding of the robot's mechanism.

Motivation The High Mechanistic group reported significantly higher engagement during the robotics task (M = 5.69, SD = 1.04) than the Low Mechanistic group (M = 4.58, SD = 1.70) [$t(26) = -2.56$, $p = .016$, $d = .79$]. There were no differences in reported levels of mastery goals [$t(48) = -.97$, $p = .34$, $d = .28$] or performance goals [$t(47) = -.30$, $p = .76$, $d = .08$] (Figure 3). However, level of engagement did not explain the relationship found between mechanistic understanding and strategy complexity or mathematical understanding: after controlling for engagement, the main effect of mechanistic understanding level was still significant for initial maze strategy [$F(1, 47) = 5.41$, $p = .024$, $\eta_p^2 = .10$], final maze strategy [$F(1, 47) = 11.6$, $p = .001$, $\eta_p^2 = .20$], and Math Relationship Understanding Questionnaire score [$F(1, 47) = 6.41$, $p = .015$, $\eta_p^2 = .12$]. To determine whether the creation of more successful maze strategies or better understanding of the robot's mathematical relationships explained increased engagement during the robots tasks, we conducted three ANCOVAs on engagement, using initial maze strategy complexity, final maze strategy complexity, and Math Relationship Understanding Questionnaire scores as covariates, respectively. In all three tests, the main effect of level of mechanistic understanding remained significant [initial: $F(1, 47) = 6.0$, $p = .018$, $\eta_p^2 = .11$; final: $F(1, 47) = 4.70$, $p = .035$, $\eta_p^2 = .09$; math understanding: $F(1, 47) = 4.84$, $p = .033$, $\eta_p^2 = .09$], while main effects of initial strategy, final strategy, and math understanding were not. Therefore, level of mechanistic understanding related to differences in engagement, but engagement itself was not directly related to task performance or greater mathematical understanding of the robot.

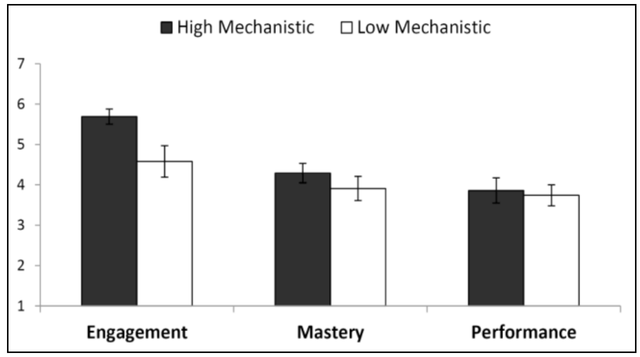

Figure 3. Mean scores on the Motivation Questionnaire.

Discussion

The current study investigated whether understanding the mechanism of a robot would be associated with higher frequency and complexity of mathematics used in a robotics task, and increased understanding of the mathematical relationships within the robot. We found that participants who understood the robot's mechanism showed greater understanding of the robot's quantitative relationships. Furthermore, these participants were more likely to use math when navigating the robot through a maze, and were able to use more complex mathematizations for the task. We also found that higher mechanistic understanding was associated with greater engagement in the robotics task, which was not explained by higher mathematical understanding or better performance on the task, suggesting that mechanistic understanding *per se* may play a motivational role as well. The shared regularities between mechanism and mathematics may have allowed mathematics to be readily applied to the robot, encouraging mathematical use and understanding.

We found no association between participants' mechanistic understanding and their attention to details: both groups were equally likely to include proportionally relevant and irrelevant parts of the robot in drawings, suggesting that they recognized and encoded the parts of the mechanism. Thus, attention to mechanistic details is insufficient for discovering mechanistic and mathematical relationships. It is possible that the High Mechanistic group primarily differed from the Low Mechanistic group in the importance placed on the robot's functional mechanism (as opposed to the individual static parts of the mechanism), which may consequently have influenced whether they mentioned the motor-wheel relationship when asked about the mechanism (see Schwank, 1993).

Spatial visualization ability also did not correlate with mechanistic understanding, mathematical use, or mathematical understanding. This result appears to contradict previous findings that spatial ability correlates with accuracy on mechanistic reasoning problems (Hegarty & Sims, 1994). However, research by Schwartz and Black (1996) suggests that people initially use mental simulations and mechanistic reasoning until a suitable rule is discovered, at which point people shift toward rule-based reasoning instead. In the current study's robotics task, participants' mechanistic understanding may have initially helped them to discover the proportional relationship between the robot's motor rotations and distance movements. Once that relationship was found, participants may have stopped relying on mechanistic reasoning and shifted to other non-mechanistic strategies, such as rule-based reasoning, allowing them to avoid simulations of the motor-wheel relationship. Because participants would not have to rely as heavily on visualization of the mechanism, spatial ability may have played less of a role, explaining the lack of correlation between spatial ability and our mechanism and mathematical understanding measures.

The Direction of the Mechanistic and Mathematical Understanding Relationship

Although the current study posited that increased mechanistic understanding would lead to greater mathematical understanding, it was not possible to conclusively test the direction of this relationship due to the correlational nature of the study. Mathematical studies often focus on the direction of concrete experiences to abstract mathematics, but the alternative exists that mathematical understanding may lead to increased mechanistic understanding. Indeed, previous research has suggested that mathematics can be used to make sense of concrete experiences (e.g., Martin & Schwartz, 2005; Schwartz, Martin, & Pfaffman, 2005; Sherin, 1996). In our study, participants may have used their mechanistic understanding to generate mathematical strategies and inform their mathematical understanding of the robot (i.e., mechanism to math); or, they may have first discovered the mathematical patterns between their inputted motor rotations and the robot's traveled distance and used that to conceptualize the robot's mechanism (i.e., math to mechanism); or, there may have been a constant conversation between mechanism understanding and mathematical understanding, where discoveries about mechanistic and/or mathematical patterns were used to inform and revise their understanding of the other (i.e., a reciprocal mechanism and math relationship). This directionality question could be answered with future studies investigating the steps through which students proceed as they generate their mathematical strategies. Such data would also provide additional information about whether there are differences between students who begin with mechanistic or mathematical understanding in creating their strategies.

The Mechanism Underlying the Mechanistic and Mathematical Understanding Relationship

A second open question involves how mechanistic and mathematical understanding affect one another. One possibility is that mechanistic understanding allows students to integrate mechanistic details into their mental representation of a situation. These representations can be used in mental simulations during mathematical problem solving; however, this is unlikely, as we found no correlation between spatial visualization ability and mathematical understanding. Mechanistic understanding may also lead to the priming of a specific math schema, such that a student will always know which mathematical procedure to use in a given mechanistic situation. As a slight alternative, mechanistic understanding may instead prime multiple plausible math schemas. Students may be able to use the results predicted by their mechanistic schema to verify which mathematics schema is correct (i.e., which mathematical schema also leads to the same result as the mechanistic schema). Future work can help to distinguish the cognitive mechanism underlying the connection between mechanistic and mathematical understanding.

In sum, the current study shows that mechanistic understanding is associated with greater mathematical understanding and use. Teaching mathematics with mechanistic manipulatives may provide several mathematical benefits, including increased use and complexity of mathematical strategies. Grounding mathematical concepts in concrete mechanisms and taking advantage of the regularities in both mechanical and mathematical systems allows students to see the applicability of mathematics in concrete situations, ultimately leading to a better understanding of both mechanism and mathematics, and the connections between the two.

Table 1: Codes used for the maze navigation task.

Code	Description	Example
Non-Math: Guessing	Participant created a guess-and-check strategy with no clear basis for guessed numbers	"Go straight direction, forward(100). turnLeft(28), 28 is still too large to turn, 100 is too long. Go straight like first step, but the length is a little shorter, forward(100)."
Non-Math: Plausible guesstimation	Participant created a guess-and-check strategy; guessed numbers were estimated using some situational basis	"Guess + test was my main strategy. After I learned that it took the robot 150 (approx.) motor rotations to go one straight stretch of the maze + 30 (approx.) motor rotations to make a turn in the maze, I just entered in the numbers in the computer until finally the robot got through the maze."
Math: Specific proportional	Participant created a strategy utilizing proportional reasoning; values were specific to their robot	"It is 0.1 inch per motor-rotation. […] Measure the distance for each straight trait which is divided by 0.1 to get the number of motor-rotations for each straight trait."

Math: General proportional	Participant created a strategy utilizing proportional reasoning that could be generalized to other robots	"Start off with a given value for motor rotations (R1) and measure the distance the robot travelled for that number of rotations (D1). Measure the distance you would like the robot to travel to reach its intended destination (D2). Calculate the number of rotations it will take the robot to travel this distance using the formula R1/R2 = D1/D2."

Table 2: Percentage of participants using each strategy type.

Strategy	Mechanistic Level	Guessing	Plausible Guesstimation	Specific Proportional	General Proportional
Initial	High Mechanistic	16%	35%	26%	23%
	Low Mechanistic	42%	37%	21%	0%
Final	High Mechanistic	0%	23%	29%	48%
	Low Mechanistic	26%	26%	42%	6%

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