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A decomposition theorem for noncommutative  $L_p$ -spaces and a new symmetric monoidal bicategory of von Neumann algebras

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A decomposition theorem for noncommutative  $L_p$ -spaces  
and a new symmetric monoidal bicategory of von Neumann algebras  
by  
Dmitri Pavlov

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## Abstract

A decomposition theorem for noncommutative  $L_p$ -spaces  
and a new symmetric monoidal bicategory of von Neumann algebras  
by Dmitri Pavlov

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The classical  $L_p$ -spaces were introduced by Riesz in 1910. Even earlier Rogers in 1888 and Hölder in 1889 proved the fundamental inequality for  $L_p$ -spaces: If  $f \in L_p$  and  $g \in L_q$ , then  $fg \in L_{p+q}$  and  $\|fg\| \leq \|f\| \cdot \|g\|$ . (Here we denote  $L_p := L^{1/p}$ , in particular  $L_0 = L^\infty$ ,  $L_{1/2} = L^2$ , and  $L_1 = L^1$ . The necessity of such a change is clear once we start exploring the algebraic structure of  $L_p$ -spaces. In particular, the above result essentially states that  $L_p$ -spaces form a graded algebra, which would be completely wrong in the traditional notation.) Even though the original definition of  $L_p$ -spaces depends on the choice of a measure, we can easily get rid of this choice by identifying  $L_p$ -spaces for different measures using the Radon-Nikodym theorem.

A variant of the Gelfand-Neumark theorem says that the category of measurable spaces is contravariantly equivalent to the category of commutative von Neumann algebras via the functor that sends a measurable space to its algebra of bounded measurable functions. Measure theory can be reformulated exclusively in terms of commutative von Neumann algebras.

We can now drop the commutativity condition and ask what theorems of measure theory can be extended to the noncommutative case. This was first done by von Neumann in 1929. Even though some aspects of measure theory such as the theory of  $L_1$ -space (also known as the predual) were worked out fairly quickly, and  $L_p$ -spaces were defined for some special cases like bounded operators on Hilbert spaces (Schatten-von Neumann classes), it took 50 years before in 1979 Haagerup defined noncommutative  $L_p$ -spaces for arbitrary von Neumann algebras. Just as in the commutative case, noncommutative  $L_p$ -spaces form a unital  $*$ -algebra graded by complex numbers with a nonnegative real part. In particular, every noncommutative  $L_p$ -space of a von Neumann algebra  $M$  is an  $M$ - $M$ -bimodule because  $L_0(M) = M$ . The operations of the unital  $*$ -algebra mentioned above together with the appropriate form of functional calculus can be used to define a norm (or a quasi-norm if  $\Re p > 1$ ) on  $L_p$ -spaces, which coincides with the usual norm in the commutative case. In 1984 Kosaki extended the classical inequality by Rogers and Hölder to Haagerup's noncommutative  $L_p$ -spaces equipped with the (quasi-)norm mentioned above.

Even though Kosaki's theorem closed the question of extending Hölder's inequality to the noncommutative case, many algebraic questions remained. For example, consider the multiplication map  $L_p(M) \otimes_M L_q(M) \rightarrow L_{p+q}(M)$ . What is its kernel and cokernel? What kind of tensor product should we use and should we complete it? Amazingly enough it turns out that this map is an isomorphism if we use the usual algebraic tensor product without any kind of completion. In particular, the algebraic tensor product  $L_p(M) \otimes_M L_q(M)$  is automatically complete. If we equip it with the projective (quasi-)norm, then the multiplication map becomes an isometry. A similar result is true for another form of multiplication

map: The map  $L_p(M) \rightarrow \text{Hom}_M(L_q(M), L_{p+q}(M))$  ( $x \mapsto (y \mapsto xy)$ ) is an isometric isomorphism, where  $\text{Hom}_M$  denotes the space of  $M$ -linear algebraic homomorphisms without any kind of continuity restrictions. In particular, every element of  $\text{Hom}_M(L_q(M), L_{p+q}(M))$  is automatically bounded. The above two theorems form the first main result of the dissertation. (The result for  $\text{Hom}$  for the case of *bounded* homomorphisms was proved by Junge and Sherman in 2005, but the automatic continuity part is new.) We summarize the above results as follows:

**Theorem.** *For any von Neumann algebra  $M$  and for any complex numbers  $a$  and  $b$  with a nonnegative real part the multiplication map  $L_a(M) \otimes_M L_b(M) \rightarrow L_{a+b}(M)$  and the left multiplication map  $L_a(M) \rightarrow \text{Hom}_M(L_b(M), L_{a+b}(M))$  are isometric isomorphisms of (quasi-)Banach  $M$ - $M$ -bimodules. Here  $\otimes_M$  denotes the algebraic tensor product (without any kind of completion) and  $\text{Hom}_M$  denotes the algebraic inner hom (without any kind of continuity restriction).*

The second main result of this dissertation is concerned with extension of the above results to  $L_p$ -modules, which were defined by Junge and Sherman in 2005. An  $L_p(M)$ -module over a von Neumann algebra  $M$  is an algebraic  $M$ -module equipped with an inner product with values in  $L_{2p}(M)$ . A typical example is given by the space  $L_p(M)$  itself with the inner product  $(x, y) = x^*y$ . The phenomena of automatic completeness and continuity extend to  $L_p$ -modules. In particular, if  $X$  is an  $L_p(M)$ -module, then  $X \otimes_M L_q(M)$  is an  $L_q(M)$ -module. Similarly, if  $Y$  is an  $L_{p+q}(M)$ -module, then  $\text{Hom}_M(L_q(M), Y)$  is an  $L_p(M)$ -module.

Of particular importance are the cases  $p = 0$  and  $p = 1/2$ .  $L_0(M)$ -modules are also known as Hilbert  $W^*$ -modules. The category of  $L_{1/2}(M)$ -modules is equivalent to the category of representations of  $M$  on Hilbert spaces and the equivalence preserves the underlying algebraic  $M$ -module. We can summarize these results as follows:

**Theorem.** *For any von Neumann algebra  $M$  and for any nonnegative real numbers  $d$  and  $e$  the category of right  $L_d(M)$ -modules is equivalent to the category of right  $L_{d+e}(M)$ -modules. The equivalences are implemented by the algebraic tensor product and the algebraic inner hom with  $L_e(M)$ . In particular, all categories of right  $L_d(M)$ -modules are equivalent to each other and to the category of representations of  $M$  on Hilbert spaces.*

This theorem can also be extended to bimodules. An  $M$ - $L_d(N)$ -bimodule is a right  $L_d(N)$ -module  $X$  equipped with a morphism of von Neumann algebras  $M \rightarrow \text{End}(X)$ . Here  $\text{End}(X)$  denotes the space of all continuous  $N$ -linear endomorphisms of  $X$ . An  $L_d(M)$ - $N$ -bimodule is defined similarly.

In particular, the category of  $M$ - $L_{1/2}(N)$ -bimodules is equivalent to the category of commuting representations (birepresentations) of  $M$  and  $N$  on Hilbert spaces. The latter category is also equivalent to the category of  $L_{1/2}(M)$ - $N$ -bimodules. These equivalences preserve the underlying algebraic  $M$ - $N$ -bimodules, in particular, every  $M$ - $L_{1/2}(N)$ -bimodule is also an  $L_{1/2}(M)$ - $N$ -bimodule.

**Theorem.** *The categories of  $L_d(M)$ - $N$ -bimodules,  $M$ - $L_d(N)$ -bimodules, and commuting representations of  $M$  and  $N$  on Hilbert spaces are all equivalent to each other. The equivalences for different values of  $d$  are implemented as usual by the algebraic tensor*

product and the algebraic inner hom with the relevant space  $L_e(M)$ . The equivalence between  $L_d(M)$ - $N$ -bimodules and  $M$ - $L_d(N)$ -bimodules is implemented by passing from an  $L_d(M)$ - $N$ -bimodule to an  $L_{1/2}(M)$ - $N$ -bimodule, then reinterpreting the latter module as an  $M$ - $N$ -birepresentation, then passing to an  $M$ - $L_{1/2}(N)$ -bimodule, and finally passing to an  $M$ - $L_d(N)$ -bimodule.

(A weaker form of this result relating  $L_0$ -bimodules and birepresentations was proved earlier by Baillet, Denizeau, and Havet, who used the completed tensor product and the continuous inner hom.) Note that passing from  $L_d(M)$ - $N$ -bimodule to  $M$ - $L_d(N)$ -bimodule can completely change the underlying algebraic bimodule structure. For example, take  $d = 0$ ,  $M = \mathbf{C}$  (the field of complex numbers) and  $N = B(H)$  for some Hilbert space  $H$ . Then  $B(H)$  is a  $\mathbf{C}$ - $L_0(B(H))$ -bimodule. The corresponding  $L_0(\mathbf{C})$ - $B(H)$ -bimodule is  $L_{1/2}(B(H))$  (the space of Hilbert-Schmidt operators on  $H$ ), which is completely different from  $B(H)$ .

The above equivalences allow us to pass freely between different categories, choosing whatever category is the most convenient for the current problem. For example, Connes fusion can be most easily defined for  $M$ - $L_0(N)$ -bimodules, where it is simply the completed tensor product. In fact, the easiest way to define the “classical” Connes fusion (Connes fusion of birepresentations) is to pass from birepresentations of  $M$  and  $N$  to  $M$ - $L_{1/2}(N)$ -modules, then to  $M$ - $L_0(N)$ -modules, then compute the completed tensor product, and then pass back to birepresentations.

One of the reasons for studying  $L_d$ -bimodules is that they form a target category for 2|1-dimensional Euclidean field theories, which conjecturally describe the cohomology theory known as TMF (topological modular forms). More precisely, a Euclidean field theory is a 2-functor from a certain 2-category of 2|1-dimensional Euclidean bordisms to some algebraic 2-category, which in this case should consist of algebras, bimodules, and intertwiners of some sort. Thus we are naturally forced to organize von Neumann algebras, right  $L_d$ -bimodules, and their morphisms into some sort of a 2-category (more precisely, a framed double category):

**Theorem.** *There is a framed double category whose category of objects is the category of von Neumann algebras and their isomorphisms and the category of morphisms is the category of right  $L_d$ -bimodules (for all values of  $d$ ) and their morphisms. The composition of morphisms is given by the Connes fusion (i.e., the completed tensor product) of bimodules.*

However, one important aspect of Euclidean field theories is still missing from our description. Namely, Euclidean field theories are *symmetric monoidal* functors, where the symmetric monoidal structure on bordisms is given by the disjoint union and on the target category it should come from some kind of tensor product.

Thus we are naturally led into the question of constructing a suitable symmetric monoidal structure on the double category of von Neumann algebras and bimodules. This involves constructing a tensor product of von Neumann algebras and an *external* tensor product of bimodules (which should not be confused with the *internal* tensor product of bimodules, i.e., the Connes fusion). In terms of pure algebra, the external tensor product takes an  $M$ - $N$ -bimodule  $X$  and a  $P$ - $Q$ -bimodule  $Y$  and spits out an  $M \otimes P$ - $N \otimes Q$ -

bimodule  $X \odot Y$ . (The internal tensor product takes an  $L$ - $M$ -bimodule  $X$  and an  $M$ - $N$ -bimodule  $Y$  and spits out an  $L$ - $N$ -bimodule  $X \otimes_M Y$ .)

Naïvely, one might expect that the usual spatial tensor product of von Neumann algebras combined with the spatial external tensor product of bimodules should suffice. Unfortunately, the resulting symmetric monoidal structure is not flexible enough. In particular, we often need to move actions around, i.e., we want to be able to pass from an  $L \otimes M$ - $N$ -bimodule to an  $L$ - $M^{\text{op}} \otimes N$ -bimodule and vice versa. (Here for simplicity we suppress  $L_d$  from our notation.) This is not possible with the spatial monoidal structure. For example, the algebra  $M$  itself is an  $M$ - $M$ -bimodule, but it almost never is a  $\mathbf{C}$ - $M^{\text{op}} \otimes M$ -bimodule. Thus we are forced to look for a different monoidal structure.

It turns out that the relevant tensor product on the level of algebras was defined by Guichardet in 1966. We call it the *categorical tensor product*, because it has good categorical universal properties. We construct a new external tensor product of bimodules, whose properties can be summarized as follows:

**Theorem.** *The symmetric monoidal category of von Neumann algebras and their isomorphisms equipped with the categorical tensor product together with the symmetric monoidal category of  $L_0$ -bimodules and their morphisms equipped with the categorical external tensor product form a symmetric monoidal framed double category.*

This symmetric monoidal structure has good properties, in particular, we can move actions around. In fact, every von Neumann algebra is *dualizable* in this monoidal structure:

**Theorem.** *In the above symmetric monoidal framed double category every von Neumann algebra  $M$  is dualizable, with the dual von Neumann algebra being  $M^{\text{op}}$ , the unit morphism being  $L_{1/2}(M)$  as a  $\mathbf{C}$ - $M^{\text{op}} \otimes M$ -birepresentation (more precisely, we take the corresponding  $L_0$ -bimodule) and the counit morphism being  $L_{1/2}(M)$  as an  $M \otimes M^{\text{op}}$ - $\mathbf{C}$ -birepresentation.*

We can compute categorified traces (shadows) of arbitrary endomorphisms of any dualizable object in a symmetric monoidal double category (or a bicategory) in the same way we compute the trace of a dualizable object in a symmetric monoidal category. In our case we can compute the shadow of any  $A$ - $A$ -bimodule, which turns out to be a  $\mathbf{C}$ - $\mathbf{C}$ -bimodule, i.e., a complex vector space. Of particular importance are the shadows of identity bimodules:

**Theorem.** *For any von Neumann algebra  $M$  the shadow of  $M$  as an  $M$ - $M$ -bimodule is isomorphic to  $L_{1/2}(Z(M))$ , where  $Z(M)$  denotes the center of  $M$ .*

The general theory developed by Ponto and Shulman allows us to take traces of arbitrary endomorphisms of dualizable 1-morphisms in any bicategory equipped with a shadow. It is a well-known fact that dualizable 1-morphisms in the bicategory of von Neumann algebras,  $L_0$ -bimodules, and intertwiners are precisely *finite index bimodules*. If  $f$  is an endomorphism of an  $M$ - $N$ -bimodule, then the left trace of  $f$  is a morphism  $L_{1/2}(Z(M)) \rightarrow L_{1/2}(Z(N))$  and the right trace of  $f$  is a morphism in the opposite direction. Since the right trace is the adjoint of the left trace, we concentrate exclusively on the left trace.

If  $M$  and  $N$  are factors, then  $L_{1/2}(Z(M)) = L_{1/2}(Z(N)) = \mathbf{C}$ , thus the left trace is a number and we recover the classical Jones index as the trace of the identity endomorphism:

**Theorem.** *If  $M$  and  $N$  are factors and  $X$  is a dualizable  $M$ - $N$ -bimodule, then the trace of the identity endomorphism of  $X$  is equal to the Jones index of  $X$ .*

In the general case, the trace of the identity endomorphism is a refinement of the Jones index. One should think of  $M$  and  $N$  as direct integrals of factors (von Neumann algebras with trivial centers) over the measurable spaces corresponding to  $Z(M)$  and  $Z(N)$  respectively. Then an  $M$ - $N$ -bimodule  $X$  can be decomposed as a direct integral of bimodules over the corresponding factors over the product  $W$  of measurable spaces corresponding to  $Z(M)$  and  $Z(N)$ . Now for every point of  $W$  compute the index of the bimodule over this point, obtaining thus a function on  $W$ . We should think of this function as the Schwartz kernel of the left trace, which is an operator  $L_{1/2}(Z(M)) \rightarrow L_{1/2}(Z(N))$ . The above theorem provides a rigorous foundation for this intuitive picture.

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## Algebraic tensor products and inner homs of $L_p$ -spaces and $L_p$ -modules.

### Abstract.

We prove that the canonical multiplication map  $L_a(M) \otimes_M L_b(M) \rightarrow L_{a+b}(M)$  is an isometric isomorphism of (quasi-)Banach  $M$ - $M$ -bimodules. Here  $L_a(M) = L^{1/a}(M)$  is the noncommutative  $L_p$ -space for an arbitrary von Neumann algebra  $M$  and  $\otimes_M$  denotes the algebraic tensor product over  $M$  equipped with a generalized version of the projective tensor norm, but without any kind of completion. We also prove the corresponding statement for algebraic homomorphisms: The canonical left multiplication map  $L_a(M) \rightarrow \text{Hom}_M(L_b(M), L_{a+b}(M))$  is an isometric isomorphism of (quasi-)Banach  $M$ - $M$ -bimodules. We then extend these results to  $L_p$ -modules and prove that the categories of  $M$ - $L_p(N)$ -bimodules,  $L_p(M)$ - $N$ -bimodules, and Hilbert spaces equipped with a left action of  $M$  and a right action of  $N$  are canonically equivalent to each other.

### Notation.

Throughout this paper we denote by  $L_a$  what is usually denoted by  $L^{1/a}$ . Thus we denote by  $L_0$  and  $L_{1/2}$  what is usually denoted by  $L^\infty$  and  $L^2$ . Elements of  $L_a$  are called *a-densities*. To avoid any confusion that might arise from this non-standard notation we never use the letter  $p$  as an index for  $L$  except for the title and the abstract of this paper.

By a von Neumann algebra  $M$  we mean a complex  $W^*$ -algebra, i.e., a complex  $C^*$ -algebra that admits a predual. We do not assume that  $M$  is represented on a Hilbert space. The symbol  $\otimes_M$  denotes the algebraic tensor product of a right  $M$ -module and a left  $M$ -module over  $M$  and  $\otimes$  denotes  $\otimes_{\mathbf{C}}$ .  $\text{Hom}_M$  denotes the algebraic inner hom of right  $M$ -modules and  ${}_M\text{Hom}$  denotes the algebraic inner hom of left  $M$ -modules.  $\mathbf{R}_{\geq 0}$  denotes the set of nonnegative real numbers and  $\mathbf{C}_{\Re \geq 0}$  denotes the set of complex numbers with a nonnegative real part. By a morphism of von Neumann algebras we mean a normal ( $\sigma$ -weakly continuous) unital  $*$ -homomorphism. All weights, states, traces, and operator valued weights are normal and semifinite by definition. We denote by  $\hat{L}_1^+(M)$  the set of all weights on a von Neumann algebra  $M$  and by  $\hat{L}_1^\sharp(M)$  the set of all faithful weights on  $M$  (recall that a weight  $\mu$  is faithful if  $\mu(x) = 0$  implies  $x = 0$  for all  $x \geq 0$  in the domain of  $\mu$ ).

Denote by  $\mathbf{I}$  the set of imaginary complex numbers  $\{z \in \mathbf{C} \mid \Re z = 0\}$ . Denote by  $\Re: \mathbf{C} \rightarrow \mathbf{R}$  and  $\Im: \mathbf{C} \rightarrow \mathbf{I}$  the projections of  $\mathbf{C} \cong \mathbf{R} \oplus \mathbf{I}$  onto  $\mathbf{R}$  and  $\mathbf{I}$  respectively. We also assume that modular automorphism groups and Connes' Radon-Nikodym cocycle derivatives are parametrized by elements of  $\mathbf{I}$ , not  $\mathbf{R}$ . Finally, denote by  $\mathbf{U}$  the set of complex numbers with absolute value 1.

## Introduction.

The classical inequality discovered by Rogers in 1888 and Hölder in 1889 can be stated in the modern language as follows: If  $x \in L_a(M)$  and  $y \in L_b(M)$  then  $xy \in L_{a+b}(M)$  and  $\|xy\| \leq \|x\| \cdot \|y\|$ , where  $M$  is an arbitrary von Neumann algebra and  $a$  and  $b$  belong to  $\mathbf{C}_{\Re \geq 0}$ . Of course, Rogers and Hölder stated their results in terms of functions on the real line. Only in 1984 Kosaki [35] extended their inequality to arbitrary von Neumann algebras using Haagerup's definition [15] of  $L_a(M)$  for arbitrary von Neumann algebra  $M$ .

This inequality can be expressed in terms of two maps  $L_a(M) \otimes L_b(M) \rightarrow L_{a+b}(M)$  ( $x \otimes y \mapsto xy$ ) and  $L_a(M) \rightarrow \text{Hom}(L_b(M), L_{a+b}(M))$  ( $x \mapsto (y \mapsto xy)$ ). The inequality then says that these maps are contractive (i.e., preserve or decrease the norm). The multiplication is associative, hence the above maps factor through the maps  $L_a(M) \otimes L_b(M) \rightarrow L_a(M) \otimes_M L_b(M)$  and  $\text{Hom}_M(L_b(M), L_{a+b}(M)) \rightarrow \text{Hom}(L_a(M), L_b(M))$ . We equip  $L_a(M) \otimes_M L_b(M)$  with the factor-(quasi-)norm and  $\text{Hom}_M(L_b(M), L_{a+b}(M))$  with the restriction (quasi-)norm.

The tensor product and the inner hom above turn out to be automatically complete and the maps  $L_a(M) \otimes_M L_b(M) \rightarrow L_{a+b}(M)$  and  $L_a(M) \rightarrow \text{Hom}_M(L_b(M), L_{a+b}(M))$  are isometric isomorphisms of  $M$ - $M$ -bimodules. This is the first main result of this paper:

**Theorem.** For any von Neumann algebra  $M$  and for any  $a$  and  $b$  in  $\mathbf{C}_{\Re \geq 0}$  the multiplication map  $L_a(M) \otimes_M L_b(M) \rightarrow L_{a+b}(M)$  and the left multiplication map  $L_a(M) \rightarrow \text{Hom}_M(L_b(M), L_{a+b}(M))$  are isometric isomorphisms of (quasi-)Banach  $M$ - $M$ -bimodules. Here  $\otimes_M$  denotes the algebraic tensor product (without any kind of completion) and  $\text{Hom}_M$  denotes the algebraic inner hom (without any kind of continuity restriction).

The underlying intuition behind these claims is that the spaces  $L_a(M)$  for any  $a \in \mathbf{C}_{\Re \geq 0}$  and any von Neumann algebra  $M$  should be thought of as (algebraically) cyclic right  $M$ -modules. Strictly speaking this is true only for  $\Re a = 0$ . For  $\Re a > 0$  only weaker versions of this statement are true (e.g.,  $L_a(M)$  is *topologically* cyclic (i.e., admits a dense cyclic submodule) whenever  $M$  is  $\sigma$ -finite and every finitely generated algebraic submodule of  $L_a(M)$  is cyclic).

Another source of intuition is the smooth counterpart of the above theory. Suppose  $V$  is a finite-dimensional complex vector space. Then  $\text{Dens}_a(V)$  is the one-dimensional complex vector space consisting of set-theoretical functions  $x: \det(V) \setminus \{0\} \rightarrow \mathbf{C}$  such that  $x(pg) = |p|^a x(g)$  for all  $p \in \mathbf{C}^*$  and  $g \in \det(V) \setminus \{0\}$ , where  $\det(V)$  denotes the top exterior power of  $V$ . The vector spaces  $\text{Dens}_a(V)$  can be organized into a  $\mathbf{C}$ -graded unital  $*$ -algebra using pointwise multiplication and conjugation. We can already see the above isomorphisms in this case: the maps  $\text{Dens}_a(V) \otimes \text{Dens}_b(V) \rightarrow \text{Dens}_{a+b}(V)$  and  $\text{Dens}_a(V) \rightarrow \text{Hom}(\text{Dens}_b(V), \text{Dens}_{a+b}(V))$  are isomorphisms. Here  $a$  and  $b$  can be arbitrary complex numbers. Later we will require that  $\Re a \geq 0$  and  $\Re b \geq 0$ .

The vector space  $\text{Dens}_0(V)$  is canonically isomorphic to  $\mathbf{C}$  and  $\text{Dens}_1(V)$  is canonically isomorphic to  $\det(V)^* \otimes \text{or}(V)$ , where  $\text{or}(V)$  denotes the orientation line of  $V$ , i.e., the vector space of all set-theoretical functions  $x: \det(V) \rightarrow \mathbf{C}$  such that  $x(pg) = u(p)^* x(g)$  for all  $p \in \mathbf{C}^*$  and  $g \in \det(V) \setminus \{0\}$ , where  $u(p)$  denotes the unitary part of  $p$ .

If  $\Im a = 0$  then it makes sense to talk about the positive part  $\text{Dens}_a^+(V)$  of  $\text{Dens}_a(V)$ , which consists of all functions in  $\text{Dens}_a(V)$  with values in  $\mathbf{R}_{\geq 0}$ . In particular, for every

$b \in \mathbf{C}$  we have a power map  $\text{Dens}_a^+(V) \rightarrow \text{Dens}_{ab}(V)$ , which lands in the positive part of  $\text{Dens}_{ab}(V)$  whenever  $\Im b = 0$ .

The above constructions can be done in smooth families. In particular, if we apply them to the tangent bundle of a smooth manifold  $X$  we obtain the line bundles of densities  $\text{Dens}_a(X)$  for every  $a \in \mathbf{C}$ .

For any smooth manifold  $X$  we have an integration map  $f: C_{\text{cs}}^\infty(\text{Dens}_1(X)) \rightarrow \mathbf{C}$ , which can be defined as the composition of the canonical factor map  $C_{\text{cs}}^\infty(\text{Dens}_1(X)) \rightarrow \text{H}_{\text{cs}}^{\text{top}}(X, \text{or}(X))$ , the Poincaré duality map  $\text{H}_{\text{cs}}^{\text{top}}(X, \text{or}(X)) \rightarrow \text{H}_0(X)$ , and the pushforward map in homology  $\text{H}_0(X) \rightarrow \text{H}_0(\text{pt}) = \mathbf{C}$ . In particular, we have a canonical pairing  $C_{\text{cs}}^\infty(\text{Dens}_a(X)) \times C^\infty(\text{Dens}_{1-a}(X)) \rightarrow \mathbf{C}$  given by  $x \times y \mapsto f(xy)$ , where  $\Re a \in [0, 1]$ . The integration map preserves positivity:  $f(\text{Dens}_1^+(X)) \subset \mathbf{R}_{\geq 0}$ .

If  $\Re a = 0$ , then  $L_a(X)$  is canonically isomorphic to the completion of  $C_{\text{cs}}^\infty(\text{Dens}_a(X))$  with respect to the weak topology induced by  $C^\infty(\text{Dens}_{1-a}(X))$ . For  $\Re a > 0$  the space  $L_a(X)$  is the completion of  $C_{\text{cs}}^\infty(\text{Dens}_a(X))$  in the norm  $x \mapsto f(xx^*)^{1/2\Re a}$ . We refer to the above topologies on  $C_{\text{cs}}^\infty(\text{Dens}_a(X))$  as the *measurable topologies*. There is no reasonable topology on  $C_{\text{cs}}^\infty(\text{Dens}_a(X))$  for  $\Re a < 0$ , which explains why the spaces  $L_a$  are defined only for  $\Re a \geq 0$ .

The isomorphisms  $C_{\text{cs}}^\infty(\text{Dens}_a(X)) \otimes_{C^\infty(X)} C_{\text{cs}}^\infty(\text{Dens}_b(X)) \rightarrow C_{\text{cs}}^\infty(\text{Dens}_{a+b}(X))$  and  $C_{\text{cs}}^\infty(\text{Dens}_a(X)) \rightarrow \text{Hom}_{C^\infty(X)}(C_{\text{cs}}^\infty(\text{Dens}_b(X)), C_{\text{cs}}^\infty(\text{Dens}_{a+b}(X)))$  can now be extended to the corresponding completions in the measurable topology, yielding the above isomorphisms for all commutative von Neumann algebras, because every measurable space is the underlying measurable space of some smooth manifold.

The second main result of this paper extends these two isomorphisms to the case of modules. Again the smooth case provides a good source of intuition. Consider a smooth bundle  $E$  of Hilbert spaces over a smooth manifold  $X$ . The inner product on  $E$  can be expressed as a morphism of bundles  $\bar{E} \otimes E \rightarrow \text{Dens}_0(X)$ , where  $\bar{E}$  denotes the conjugate bundle of  $E$ . We now consider vector bundles equipped with a more general type of inner product with values in  $\text{Dens}_{2d}(X)$  for some  $d \in \mathbf{R}_{\geq 0}$  (here it is essential that  $d$  is real). Such an inner product equips every fiber of  $E$  with an inner product with values in some one-dimensional vector space  $W$  and we require that all fibers are complete with respect to this inner product. In particular, for  $d = 0$  the space  $W$  is canonically isomorphic to  $\mathbf{C}$  and we get the usual smooth bundles of Hilbert spaces. We refer to such bundles as *d-bundles*. Examples of *d-bundles* abound in differential geometry. For example, the Dirac operator on a conformal spin  $n$ -manifold is a differential operator from a  $(1/2 - 1/2n)$ -bundle to a  $(1/2 + 1/2n)$ -bundle as explained by Stolz and Teichner in Definition 2.3.9 of [1].

The easiest example of a *d-bundle* is supplied by  $\text{Dens}_d(X)$  itself with the inner product  $(x, y) \in C_{\text{cs}}^\infty(\text{Dens}_d(X)) \times C_{\text{cs}}^\infty(\text{Dens}_d(X)) \mapsto x^*y \in C_{\text{cs}}^\infty(\text{Dens}_{2d}(X))$ . (This example explains why we consider *d-bundles* with real values of  $d$ : If  $d$  is complex, then  $\text{Dens}_d(X)$  is still a  $\Re d$ -bundle via the above formula.) Other examples can be obtained by tensoring a bundle of Hilbert spaces with  $\text{Dens}_d(X)$ . In fact, these examples exhaust all possible *d-bundles*, for if  $E$  is a *d-bundle*, then  $\bar{E} \otimes \text{Dens}_{-d}(X) = \text{Hom}(\text{Dens}_d(X), E)$  is a 0-bundle.

More generally, if  $E$  is a *d-bundle*, then  $E \otimes \text{Dens}_e$  is a  $(d+e)$ -bundle for any  $e \in \mathbf{R}_{\geq 0}$ . This correspondence extends to a functor, which establishes an equivalence between the categories of *d-bundles* and  $(d+e)$ -bundles, in particular, all categories of *d-bundles* for

various values of  $d$  are canonically equivalent to each other.

Just as compactly supported smooth sections of bundles of  $d$ -densities of  $X$  can be completed to the space  $L_d(X)$ , compactly supported smooth sections of an arbitrary  $d$ -bundle on  $X$  can be completed to an  $L_d(X)$ -module.

More precisely, if  $M$  is a von Neumann algebra, then a *right*  $L_d(M)$ -module is an algebraic right  $M$ -module  $E$  equipped with an inner product with values in  $L_{2d}(M)$  satisfying the usual algebraic properties (bilinearity, positivity, non-degeneracy) that is complete in the *measurable topology*, which is the weakest topology on  $E$  such that all maps  $y \in E \mapsto (x, y) \in L_{2d}(M)$  are continuous for any  $x \in E$  if  $L_{2d}(M)$  is equipped with the measurable topology. A *morphism* of  $L_d(M)$ -modules is a continuous morphism of algebraic  $M$ -modules. The category of  $L_d(M)$ -modules can be equipped with a structure of a  $W^*$ -category as defined by Ghez, Lima, and Roberts in [2].

Suppose  $X$  is a smooth manifold. Then  $L_0(X)$  is a von Neumann algebra and by an  $L_d(X)$ -module we mean an  $L_d(L_0(X))$ -module. The space of compactly supported smooth sections of a  $d$ -bundle can be equipped with a (smooth version of) measurable topology in a similar way. Completing this space gives us an  $L_d(X)$ -module with the measurable topology.

Combining together equivalences of categories of  $d$ -bundles and facts about algebraic tensor products and inner homs of spaces  $L_a(M)$  we arrive at the following statement, which is the second main result of this paper: (1) If  $E$  is a right  $L_d(M)$ -module, then  $E \otimes_M L_e(M)$  can be equipped in a natural way with a structure of a right  $L_{d+e}(M)$ -module, in particular it is automatically complete; (2) If  $E$  is a right  $L_{d+e}(M)$ -module, then  $\text{Hom}_M(L_e(M), E)$  is naturally an  $L_d(M)$ -module, in particular it is automatically complete; (3) The above constructions can be extended to an adjoint unitary  $W^*$ -equivalence of  $W^*$ -categories. Here  $d$  and  $e$  are arbitrary elements of  $\mathbf{R}_{\geq 0}$ . The above results can be summarized as follows:

**Theorem.** For any von Neumann algebra  $M$  and for any  $d$  and  $e$  in  $\mathbf{R}_{\geq 0}$  the category of right  $L_d(M)$ -modules is equivalent to the category of right  $L_{d+e}(M)$ -modules. The equivalences are implemented by the algebraic tensor product and the algebraic inner hom with  $L_e(M)$ . In particular, all categories of right  $L_d(M)$ -modules are equivalent to each other and to the category of representations of  $M$  on Hilbert spaces.

This theorem can also be extended to bimodules. An  $M$ - $L_d(N)$ -bimodule is a right  $L_d(N)$ -module  $X$  equipped with a morphism of von Neumann algebras  $M \rightarrow \text{End}(X)$ . Here  $\text{End}(X)$  denotes the space of all continuous  $N$ -linear endomorphisms of  $X$ . An  $L_d(M)$ - $N$ -bimodule is defined similarly.

**Theorem.** The categories of  $L_d(M)$ - $N$ -bimodules,  $M$ - $L_d(N)$ -bimodules, and commuting representations of  $M$  and  $N$  on Hilbert spaces are all equivalent to each other. The equivalences for different values of  $d$  are implemented by the algebraic tensor product and the algebraic inner hom with the relevant space  $L_e(M)$ . The equivalence between  $L_d(M)$ - $N$ -bimodules and  $M$ - $L_d(N)$ -bimodules is implemented by passing from an  $L_d(M)$ - $N$ -bimodule to an  $L_{1/2}(M)$ - $N$ -bimodule, then reinterpreting the latter module as an  $M$ - $N$ -birepresentation, then passing to an  $M$ - $L_{1/2}(N)$ -bimodule, and finally passing to an  $M$ - $L_d(N)$ -bimodule.

## History.

In 1910 F. Riesz defined  $L_a$  for the case of ordinary (commutative) measurable spaces. In 1929 von Neumann introduced noncommutative measurable spaces (von Neumann algebras), which raised the problem of extending Riesz' definition to the case of noncommutative measurable spaces. In 1953 I. Segal [7] defined noncommutative  $L_1$  and  $L_{1/2}$  spaces. Ogasawara and Yoshinaga [8, 9] explored some of their properties. Later Kunze [10] defined  $L_a$  for  $a \in [0, 1]$  and Stinespring [11] developed the theory further. Nelson [12] wrote a nice exposition of the theory using the measure topology introduced by Stinespring. Dixmier [13] published his own definition of  $L_a$  for  $a \in (0, 1]$  with an accompanying theory simultaneously with Segal. These authors assumed that the von Neumann algebra under consideration is semifinite. Haagerup [15] was the first one to define  $L_a$  for  $a \in [0, 1]$  for an arbitrary von Neumann algebra, using crossed products. Terp [16] wrote up an elaborate exposition of his construction and the related theory. Meanwhile Connes [20] gave his own definition using spatial products. Hilsum [21] wrote up the corresponding theory. Araki and Masuda [23] gave a definition for the  $\sigma$ -finite case based on the relative modular theory and the canonical representation of a von Neumann algebra, which was soon extended by Masuda [24] to the general case.

All these definitions depend on a choice of a faithful weight  $\mu$  on  $M$ . However, the spaces  $L_a(M, \mu)$  for different choices of a faithful weight  $\mu$  are canonically isomorphic to each other. Thus we can define  $L_a(M)$  as the limit (or the colimit, because all maps are isomorphisms) of  $L_a(M, \mu)$  for all faithful weights  $\mu$  on  $M$ . See the next section for the details of this construction in the commutative case. Kosaki in his thesis [22] gave a weight-independent definition using a different approach. Earlier Haagerup in his thesis [14] gave a weight-independent definition of  $L_{1/2}$  for an arbitrary von Neumann algebra.

Another series of approaches uses Calderón's complex interpolation method. Kosaki [25] gave the first definition, which was restricted to the case of  $\sigma$ -finite von Neumann algebras. Terp [26] extended Kosaki's construction to all von Neumann algebras. Izumi [27, 28, 29] defined for every  $a \in (0, 1)$  a one-parameter family of spaces functorially isomorphic to each other in such a way that the constructions of Kosaki and Terp correspond to two particular values of the parameter. Leinert [30] developed another interpolation-based approach by defining a noncommutative analog of the upper integral in the semifinite case and later [31] extended it to the general case. Pisier and Xu [32] wrote an extensive survey of the theory summarizing the results mentioned above.

Yamagami [17] reformulated the original results of Haagerup [15] in a more convenient algebraic setting of modular algebras and defined  $L_a(M)$  for arbitrary  $a \in \mathbf{C}_{\Re \geq 0}$ . Sherman [19] along with Falcone and Takesaki [18] give detailed expositions. The principal idea of the modular approach is to construct a unital  $*$ -algebra that contains the spaces  $L_a$  for all  $a \in \mathbf{C}_{\Re \geq 0}$  and then extract  $L_a$  from this algebra by algebraic means. We use the language of modular algebras systematically throughout this paper.

## The commutative case.

To help the reader develop a better intuition for the noncommutative case, in this section we briefly review the relevant constructions in the commutative case.

A *measurable space* is a triple  $(X, M, N)$ , where  $X$  is a set,  $M$  is a  $\sigma$ -algebra of *measurable subsets* of  $X$ , and  $N \subset M$  is a  $\sigma$ -ideal of *null sets*. For the sake of simplicity we assume that our measurable spaces are *complete*, i.e., every subset of an element of  $N$  is again an element of  $N$ . The inclusion functor from the category of complete measurable spaces to the category of measurable spaces is an equivalence, thus we do not lose anything by restricting ourselves to complete measurable spaces.

If  $(X, M, N)$  and  $(Y, P, Q)$  are measurable spaces, then a map  $f: X \rightarrow Y$  is *measurable* if the preimage of every element of  $P$  is an element of  $M$ . Measurable maps are closed under composition. A measurable map is *non-singular* if the preimage of every element of  $Q$  is an element of  $N$ . Non-singular measurable maps are also closed under composition. Two measurable maps  $f$  and  $g$  are *equivalent* if  $\{x \in X \mid f(x) \neq g(x)\} \in N$ . Composition of non-singular measurable maps preserves this equivalence relation. We define a *morphism of measurable spaces* as an equivalence class of non-singular measurable maps.

A *measure* on a measurable space  $(X, M, N)$  is a  $\sigma$ -additive map  $m: M \rightarrow [0, \infty]$  such that  $m(A) = 0$  for all  $A \in N$  and the union of all  $A \in M$  such that  $m(A) \neq \infty$  equals  $X$ . A measure  $m$  is *faithful* if  $m(A) = 0$  implies  $A \in N$  for all  $A \in M$ . Abusing the language we say that a (complex valued) *finite measure* on a measurable space  $(X, M, N)$  is a  $\sigma$ -additive map  $m: M \rightarrow \mathbf{C}$  such that  $m(A) = 0$  for all  $A \in N$ . Denote the complex vector space of all finite measures on a measurable space  $Z$  by  $L_1(Z)$ , which is a Banach space via the norm defined in the next paragraph.

A (complex valued) *function* on a measurable space  $Z$  is an equivalence class of (possibly singular) measurable maps from  $Z$  to the set of complex numbers equipped with the  $\sigma$ -algebra of Lebesgue measurable sets and the standard  $\sigma$ -ideal of null sets. A function is *bounded* if at least one representative of its equivalence class is bounded. Denote the set of all bounded functions on a measurable space  $Z$  by  $L_0(Z)$ , which is a  $C^*$ -algebra in a natural way. Every element  $m$  of  $L_1(Z)$  yields a unique norm-continuous linear functional on  $L_0(Z)$ , whose value on the characteristic function of a set  $A$  is  $m(A)$ . Now we equip  $L_1(Z)$  with the norm  $\|m\| := \sup_{f \in L_0(Z)} m(f)$ , where  $\|f\| \leq 1$ .

I. Segal proved in [38] that for a measurable space  $Z = (X, M, N)$  the following properties are equivalent:

- The Boolean algebra  $M/N$  of equivalence classes of measurable sets is complete.
- The lattice of all real functions on  $Z$  is Dedekind complete. An ordered set  $S$  is *Dedekind complete* if every nonempty subset of  $S$  bounded from above has a supremum and every nonempty subset bounded from below has an infimum.
- The lattice of all bounded real functions on  $Z$  is Dedekind complete.
- $Z$  has the Radon-Nikodym property: For any two faithful measures  $\mu$  and  $\nu$  on  $Z$  we have  $\mu = f\nu$  for a (unique unbounded strictly positive) function  $f$  on  $Z$ .
- $Z$  has the Riesz representability property: The functorial evaluation map from  $L_0(Z)$  to  $L_1(Z)^*$  is an isomorphism.
- $L_0(Z)$  is a von Neumann algebra.
- $Z$  is isomorphic to a coproduct (disjoint union) of points and real lines.

If  $Z$  satisfies any of these properties, then we say that  $Z$  is *localizable*. In particular, every  $\sigma$ -finite measurable space (i.e., a measurable space that admits a faithful measure) is localizable, but not vice versa. Thus Segal's theorem shows that it is best to work with localizable measurable spaces. Moreover, the category of localizable measurable spaces is contravariantly equivalent to the category of commutative von Neumann algebras. The equivalence functor sends a measurable space  $Z$  to  $L_0(Z)$  and a morphism of measurable spaces to the corresponding pullback map for functions. Moreover,  $L_1(Z)$  is canonically isomorphic to the predual of  $L_0(Z)$  and the dual of  $L_0(Z)$  in the weak topology induced by  $L_1(Z)$  (the  $\sigma$ -weak topology) is again canonically isomorphic to  $L_1(Z)$ . Measures on  $Z$  are canonically identified with weights on  $L_0(Z)$  ( $\sigma$ -weakly lower semi-continuous  $[0, \infty]$ -valued functionals on  $L_0^+(Z)$ ). Henceforth we include the property of localizability in the definition of a measurable space.

If  $\mu$  is a faithful measure on a measurable space  $Z$ , define  $L_1(Z, \mu)$  as the space of all functions on  $Z$  such that  $\mu(|f|)$  is finite. Then  $L_1(Z, \mu)$  and the space of finite measures  $L_1(Z)$  are functorially isometrically isomorphic via the multiplication map  $u \in L_1(Z, \mu) \mapsto u\mu \in L_1(Z)$ .

For an arbitrary  $a \in \mathbf{C}_{\Re \geq 0}$  we define  $L_a(Z, \mu)$  as the set of all functions  $f$  on  $Z$  such that  $\mu(|f|^{1/\Re a})$  is finite if  $\Re a > 0$  or  $f$  is bounded if  $\Re a = 0$ . If  $\mu$  and  $\nu$  are two faithful measures on  $Z$ , then Radon-Nikodym theorem gives a unique (strictly positive) function  $u$  on  $Z$  such that  $\mu = u\nu$ . We have a canonical isometric isomorphism  $f \in L_a(Z, \mu) \mapsto fu^a \in L_a(Z, \nu)$ . The space of  $a$ -densities  $L_a(Z)$  is the limit (or the colimit, because all maps are isomorphisms) of  $L_a(Z, \mu)$  for all  $\mu$ . The individual spaces  $L_a(Z, \mu)$  do not depend on the imaginary part of  $a$ , but the isomorphisms between them do, hence  $L_a(Z)$  is non-canonically isomorphic to  $L_{\Re a}(Z)$ , and choosing such an isomorphism is equivalent to choosing a measure on  $Z$ . The spaces  $L_a(Z)$  and  $L_{\Re a}(Z)$  are no longer isomorphic as  $L_0(Z)$ - $L_0(Z)$ -bimodules in the noncommutative case.

### The noncommutative case.

In this section we define a functor  $M \mapsto \hat{M}$  from the category of von Neumann algebras and faithful operator valued weights to the category of  $\mathbf{I}$ -graded von Neumann algebras (see below for the definition of gradings). We define  $L_a(M)$  for  $a \in \mathbf{I}$  as the  $a$ -graded component of  $\hat{M}$ . Furthermore,  $\hat{M}$  has a canonical trace  $\tau$ , which we use to complete  $\hat{M}$  in the  $\tau$ -measurable topology and obtain a  $\mathbf{C}$ -graded unital  $*$ -algebra (not a von Neumann algebra)  $\bar{M}$ , whose  $a$ -graded component is the space  $L_a(M)$  for  $a \in \mathbf{C}_{\Re \geq 0}$  and zero for all other  $a$ . These spaces keep the usual properties of their commutative versions, in particular, they have a (quasi-)norm that turns them into (quasi-)Banach spaces and their elements have left and right polar decompositions and supports.

Since continuous gradings are essential in the discussion below, we discuss them first.

**Definition.** If  $G$  is an abelian locally compact topological group, then a  $G$ -grading on a von Neumann algebra  $M$  is a morphism of groups  $\theta: \hat{G} \rightarrow \text{Aut}(M)$  such that the map  $\hat{g} \in \hat{G} \mapsto \theta_{\hat{g}}(p) \in M$  is  $\sigma$ -weakly continuous for all  $p \in M$ . Here  $\hat{G} := \text{Hom}(G, \mathbf{U})$  is the dual group of  $G$  equipped with the compact-open topology (recall that we denote by  $\mathbf{U}$  the group of unitary complex numbers). A *morphism* of  $G$ -graded von Neumann algebras is a morphism of von Neumann algebras that commutes with  $\theta$ . For  $g \in G$  we define the

$g$ -graded component of  $M$  as the set of all elements  $p \in M$  such that  $\theta_{\hat{g}}(p) = \hat{g}(g) \cdot p$  for all  $\hat{g} \in \hat{G}$ . This construction extends to a functor: A morphism of  $G$ -graded von Neumann algebras  $M \rightarrow N$  induces a morphism of their  $g$ -graded components because it commutes with the grading.

In our case  $G = \mathbf{I}$  and  $\hat{G} = \text{Hom}(\mathbf{I}, \mathbf{U})$  is identified with  $\mathbf{R}$  via the following map:  $s \in \mathbf{R} \mapsto (t \in \mathbf{I} \mapsto \exp(-st) \in \mathbf{U}) \in \text{Hom}(\mathbf{I}, \mathbf{U})$ . The sign is present for purely historical reasons. The  $\mathbf{I}$ -grading on the algebra  $\hat{M}$  constructed below is also known as the *scaling automorphism group* or the *noncommutative flow of weights*.

We can now informally describe the core of a von Neumann algebra  $M$  as follows: The core of  $M$  is the  $\mathbf{I}$ -graded von Neumann algebra  $\hat{M}$  generated by  $M$  in grading 0 and symbols  $\mu^t$  in grading  $t$ , where  $t \in \mathbf{I}$  and  $\mu \in \hat{L}_1^+(M)$ , subject to the following relations:

- For all  $\mu \in \hat{L}_1^+(M)$  the map  $t \in \mathbf{I} \mapsto \mu^t \in \mathbf{U}(p\hat{M}p)$  is a continuous morphism of groups. Recall that  $\hat{L}_1^+(M)$  denotes the set of weights on  $M$ , i.e., normal additive positive homogeneous maps  $M^+ \rightarrow [0, \infty]$ , which are the noncommutative analogs of measures. Here  $p$  is the support of  $\mu$  (i.e., the maximum projection  $q$  such that  $q\mu q = \mu$ ) and  $\mathbf{U}(N)$  denotes the group of unitary elements of a von Neumann algebra  $N$  equipped with the  $\sigma$ -weak topology.
- For all  $\mu \in \hat{L}_1^+(M)$  and  $t \in \mathbf{I}$  we have  $\mu^t x \mu^{-t} = \sigma_t^\mu(x)$ , where  $\sigma_t^\mu$  denotes the modular automorphism group (see Definition VIII.1.3 in Takesaki [41]) of  $\mu$ .
- For all  $\mu \in \hat{L}_1^+(M)$ ,  $\nu \in \hat{L}_1^\sharp(M)$ , and  $t \in \mathbf{I}$  we have  $\mu^t \nu^{-t} = (D\mu : D\nu)_t$ , where  $(D\mu : D\nu)$  denotes Connes' Radon-Nikodym cocycle derivative (see Definition VIII.3.20 in Takesaki [41]) of  $\mu$  with respect to  $\nu$ . Recall that  $\hat{L}_1^\sharp(M)$  denotes the set of faithful weights on  $M$ , i.e., weights with support 1.

The theory of representable functors assigns a precise meaning to the notion of an  $\mathbf{I}$ -graded von Neumann algebra generated by a family of generators and relations. First we define for every  $\mathbf{I}$ -graded von Neumann algebra  $N$  the set of morphisms from  $\hat{M}$  to  $N$  and for every morphism  $f: N \rightarrow O$  of  $\mathbf{I}$ -graded von Neumann algebras the map of sets  $\text{Mor}(\hat{M}, N) \rightarrow \text{Mor}(\hat{M}, O)$  given by the composition with  $f$ , thus obtaining a functor  $F$  from the category of  $\mathbf{I}$ -graded von Neumann algebras to the category of sets. Then we prove that the functor  $F$  is representable, i.e., isomorphic to a functor of the form  $N \mapsto \text{Mor}(X, N)$  for some  $\mathbf{I}$ -graded von Neumann algebra  $X$ . Then we recover  $\hat{M}$  from this functor using Yoneda lemma (i.e.,  $\hat{M}$  is canonically isomorphic to  $X$ ).

For a given von Neumann algebra  $M$  we define a functor  $F$  from the category of  $\mathbf{I}$ -graded von Neumann algebras to the category of sets by sending an  $\mathbf{I}$ -graded von Neumann algebra  $N$  to the set of all pairs  $(f, g)$  such that  $f$  is a morphism from  $M$  to the 0-graded component of  $N$ ,  $g$  is a map that sends every  $\mu \in \hat{L}_1^+(M)$  to a continuous morphism of groups  $v: \mathbf{I} \rightarrow \mathbf{U}(f(p)Nf(p))$  ( $p$  is the support of  $\mu$ ) such that  $v(t)$  has grading  $t$  for all  $t \in \mathbf{I}$ , and finally the pair  $(f, g)$  satisfies the two relations above concerning the modular automorphism group and the cocycle derivative. Likewise we send a morphism  $h: N \rightarrow O$  to the map given by the composition of  $f$  and  $g$  with  $h$ .

**Theorem.** For any von Neumann algebra  $M$  the functor  $F$  defined above is representable.

*Proof.* The representable functor theorem (see Theorem 2.9.1 in Pareigis [5]) states that a functor  $F$  from a complete category  $C$  to the category of sets is representable if and only



if  $F$  is continuous (preserves small limits) and satisfies the solution set condition: There is a set  $A$  of objects in  $C$  such that for every object  $X$  in  $C$  and for every  $x \in F(X)$  there are  $W \in A$ ,  $w \in F(W)$ , and  $h: W \rightarrow X$  such that  $x = F(h)(w)$ . We apply this theorem to the case when  $C$  is the category of  $\mathbf{I}$ -graded von Neumann algebras and  $F$  is the functor constructed above.

The category of  $\mathbf{I}$ -graded von Neumann algebras is complete (Guichardet [6] proves the result for ordinary von Neumann algebras, which extends word for word to the  $\mathbf{I}$ -graded case). To prove that the functor  $F$  preserves small limits it is sufficient to prove that  $F$  preserves small products and equalizers. The functor  $F$  preserves small products because a weight on a product decomposes into a family of weights on the factors, and the same is true for one-parameter groups of unitary elements. It preserves equalizers because the equalizer of two von Neumann algebras is their set-theoretical equalizer equipped with the restriction of relevant structures. Finally,  $F$  satisfies the solution set condition for a set  $A$  that contains one isomorphism class of every  $\mathbf{I}$ -graded von Neumann algebra of cardinality at most the cardinality of  $M$  because for every  $\mathbf{I}$ -graded von Neumann algebra  $X$  and every element  $x \in F(X)$  the  $\mathbf{I}$ -graded von Neumann subalgebra of  $X$  generated by the image of  $M$  and elements in the one-parameter families corresponding to all weights on  $M$  is bounded in cardinality uniformly with respect to  $X$ , and all  $\mathbf{I}$ -graded von Neumann algebras with cardinality at most some cardinal form a set. Thus the functor  $F$  is representable and we define the core of  $M$  as the representing object:

**Definition.** If  $M$  is a von Neumann algebra, then the *core* of  $M$  is the von Neumann algebra  $\hat{M}$  that represents the functor  $F$  defined above.

There are alternative ways to prove representability. For example, the universal property of the crossed product (see Theorem 2 in Landstad [40]) allows us to prove that the crossed product of  $M$  by the modular automorphism group of an arbitrary faithful weight represents  $F$ . In particular, all of these crossed products for different weights are functorially isomorphic to each other and to the core. Hence the limit (or the colimit) of all such crossed products also represents the functor  $F$  and gives us a functorial construction of the core that does not depend on a choice of a faithful weight. This construction is essentially the same as the first construction, except that here we allow weights to be distinct at a later stage.

Another way to prove representability is to construct an algebraically  $\mathbf{I}$ -graded unital  $*$ -algebra  $\check{M}$  generated by the above generators and relations and take its completion in the weakest topology that makes all of its representations continuous. Here a representation is a morphism of  $\mathbf{I}$ -graded unital  $*$ -algebras from  $\check{M}$  to an arbitrary  $\mathbf{I}$ -graded von Neumann algebra  $N$  corresponding to some element of  $F(N)$ . This construction is essentially an expansion of a proof of the representable functor theorem.

The algebra  $\hat{M}$  has a canonical faithful operator valued weight and a faithful trace, which we describe briefly. Denote by  $\epsilon$  the faithful operator valued weight from  $\hat{M}$  to  $M$  corresponding to the embedding of  $M$  into  $\hat{M}$  and defined by the equality  $\epsilon(x) = \int_{\mathbf{R}} (s \in \mathbf{R} \mapsto \theta_s(x) \in \hat{M}^+)$  for all  $x \in \hat{M}^+$  and by  $\tau$  the faithful trace on  $\hat{M}$  defined by the equality  $(D(\mu \circ \epsilon) : D\tau)_t = \mu^t$  for all  $t \in \mathbf{I}$  and  $\mu \in \hat{L}_1^+(M)$ . We have  $\tau \circ \theta_s = \exp(-s)\tau$ . See Yamagami [17], Falcone and Takesaki [18], as well as §XII.6 of Takesaki [41] for the

relevant proofs.

We now extend the construction of the core to a functor. First we need to define an appropriate notion of a morphism between von Neumann algebras. It turns out that in this situation a morphism from  $M$  to  $N$  is a pair  $(f, T)$ , where  $f$  is a usual morphism of von Neumann algebras from  $M$  to  $N$  and  $T$  is a faithful operator valued weight from  $N$  to  $M$  associated to the morphism  $f$ . Composition is componentwise. Recall that the composition of faithful operator valued weights is again faithful. See Takesaki [41] for the relevant facts about operator valued weights. Given a morphism  $(f, T)$  from  $M$  to  $N$ , we define a map  $\hat{f}$  from  $\hat{M}$  to  $\hat{N}$  by the following formulas:  $\hat{f}(x) = f(x)$  for all  $x \in M$  and  $\hat{f}(\mu^t) = (\mu \circ T)^t$  for all  $\mu \in \hat{L}_1^+(M)$  and  $t \in \mathbf{I}$ . Note that  $\mu \circ T \in \hat{L}_1^+(N)$ , because  $\mu$  is an operator valued weight from  $M$  to  $\mathbf{C}$  associated to the morphism from  $\mathbf{C}$  to  $M$ . The theory of operator valued weights immediately implies that  $\hat{f}$  preserves all relations between generators (here we use the fact that  $T$  is faithful) and hence defines a morphism from  $\hat{M}$  to  $\hat{N}$  by the universal property of  $\hat{M}$ .

Another way to define a morphism of the above category is to consider a pair of morphisms  $(f: M \rightarrow N, \hat{f}: \hat{M} \rightarrow \hat{N})$  that makes the square diagram consisting of  $f$  and  $\hat{f}$  and embeddings  $M \rightarrow \hat{M}$  and  $N \rightarrow \hat{N}$  commute and such that the morphism  $\hat{f}$  preserves the grading. From the theory of operator valued weights it follows that there is a bijective correspondence between pairs  $(f, \hat{f})$  and pairs  $(f, T)$  defined in the previous paragraph.

Thus we obtain a functor from the category of von Neumann algebras and their morphisms equipped with faithful operator valued weights to the category of  $\mathbf{I}$ -graded von Neumann algebras, which turn out to be semifinite and are equipped with algebraic gadgets like  $\tau$  and  $\epsilon$  interacting in a certain way. The core functor is fully faithful, and hence it is an equivalence of the domain category and its essential image. Thus the study of the category of arbitrary von Neumann algebras and faithful operator valued weights reduces to the study of the category of  $\mathbf{I}$ -graded semifinite von Neumann algebras.

We define the space  $L_a(M)$  to be the  $a$ -graded component of  $\hat{M}$  for all  $a \in \mathbf{I}$ . We now explain how to define  $L_a(M)$  for  $\Re a > 0$ . For this we need to introduce unbounded elements. See Nelson [12] or Terp [16] for the relevant definitions and proofs. Denote by  $\bar{M}$  the completion of  $\hat{M}$  in the  $\tau$ -measurable topology. It turns out that all algebraic operations (including grading) on  $\hat{M}$  are continuous in this topology, hence  $\bar{M}$  is a unital  $*$ -algebra. We define positive elements of  $\bar{M}$  in the same way as for von Neumann algebras. The set of all positive elements of  $\bar{M}$  is the closure of  $M^+$  in the  $\tau$ -measurable topology. The  $\mathbf{I}$ -grading on  $\bar{M}$  extends analytically to a  $\mathbf{C}$ -grading. For every  $a \in \mathbf{C}_{\Re \geq 0}$  we define  $L_a(M)$  as the  $a$ -graded component of  $\bar{M}$ . We require  $a \in \mathbf{C}_{\Re \geq 0}$  because all other graded components are zero. See Terp [16] and Yamagami [17] for details.

The construction  $M \rightarrow \bar{M}$  is functorial if we restrict ourselves to the subcategory of von Neumann algebras whose morphisms are *bounded* faithful operator valued weights. An operator valued weight is *bounded* if it sends bounded elements to bounded elements. An element of the extended positive cone of  $M$  is called *bounded* if it belongs to  $M$ . Alternatively, a bounded operator valued weight associated to a morphism  $f: M \rightarrow N$  of von Neumann algebras is simply a continuous positive morphism of bimodules  ${}_M N_M \rightarrow {}_M M_M$ , where the  $M$ -actions on  $N$  come from  $f$ . Just as operator valued weights are generalizations of weights, bounded operator valued weights are generalizations of positive

elements of the predual. The reason for this boundedness condition is that the relevant map from  $L_1^+(M)$  to  $L_1^+(N)$  is given by the formula  $\bar{f}(\mu) = \mu \circ T$  for  $\mu \in L_1^+(M)$ . This formula requires that  $\mu \circ T \in L_1^+(N)$ , and if  $\mu \circ T \in L_1^+(N)$  for all  $\mu \in L_1^+(M)$  then  $T$  is bounded. Morphisms of the restricted category also admit a conceptual definition similar to the one discussed above if we replace  $\hat{M}$  by  $\bar{M}$ . Thus we obtain a fully faithful functor from the category of von Neumann algebras and faithful bounded operator valued weights to the category of  $\mathbf{C}_{\Re \geq 0}$ -graded topological unital  $*$ -algebras.

For all  $a \in \mathbf{I}$  the space  $L_a(M)$  defined above is a functor from the category of von Neumann algebras and their morphisms equipped with a faithful operator valued weight to the category of Banach spaces, which is the composition of the core functor and the functor that extracts the relevant grading from an  $\mathbf{I}$ -graded von Neumann algebra. Likewise, the space  $L_a(M)$  for  $a \in \mathbf{C}_{\Re \geq 0}$  is a functor from the category of von Neumann algebras and faithful bounded operator valued weights to the category of (quasi-)Banach spaces.

As expected,  $L_0(M)$  and  $L_1(M)$  are naturally isomorphic to  $M$  and  $M_*$ . The definition of  $L_a(M)$  implies that the multiplication on  $\bar{M}$  induces a bilinear map  $L_a(M) \times L_b(M) \rightarrow L_{a+b}(M)$  for all  $a$  and  $b$  in  $\mathbf{C}_{\Re \geq 0}$ . The spaces  $L_a(M)$  together with these bilinear maps form an algebraically  $\mathbf{C}_{\Re \geq 0}$ -graded ring. In particular, every complex vector space  $L_a(M)$  is an  $M$ - $M$ -bimodule. Moreover, the involution on  $\bar{M}$  restricts to an anti-isomorphism of  $M$ - $M$ -bimodules  $L_a(M)$  and  $L_{\bar{a}}(M)$ , hence the graded ring introduced above is a unital algebraically  $\mathbf{C}_{\Re \geq 0}$ -graded  $*$ -algebra. Since the multiplication in  $\bar{M}$  is associative, we have a functorial map  $m: L_a(M) \otimes_M L_b(M) \rightarrow L_{a+b}(M)$ . The first main theorem of this paper states that  $m$  is an isomorphism of algebraic  $M$ - $M$ -bimodules. Even though  $L_a(M)$  is not finitely generated as a right  $M$ -module, there is no need to complete the tensor product  $L_a(M) \otimes_M L_b(M)$ .

The second main theorem states that this isomorphism is an isometry. To make sense of this statement we introduce (quasi-)norms on  $L_a(M)$  and on the tensor product. Suppose for a moment that  $\Re a \leq 1$ . Then there is a natural norm on  $L_a(M)$ . If  $\Re a = 0$ , this norm is the restriction of the norm on  $\hat{M}$ . (In this case  $L_a(M)$  is a subset of  $\hat{M}$ .) If  $\Re a > 0$ , then the norm is given by the map  $x \in L_a(M) \mapsto ((x^*x)^{1/2\Re a}(1))^{\Re a}$ . Here  $x \in L_a(M)$  and  $x^* \in L_{\bar{a}}(M)$ , hence  $x^*x \in L_{2\Re a}^+(M)$ . By  $L_d^+(M)$  we denote the intersection of  $\bar{M}^+$  with  $L_d(M)$  for arbitrary  $d \in \mathbf{R}_{\geq 0}$ . (If  $d$  is not real, then the intersection is zero.) The Borel functional calculus extended to  $\bar{M}$  and applied to the function  $t \in \mathbf{R}_{\geq 0} \mapsto t^e \in \mathbf{R}_{\geq 0}$  yields a map  $L_d^+(M) \rightarrow L_{de}^+(M)$  for all real  $d \geq 0$  and  $e \geq 0$ , which is a bijection for  $e > 0$ . Thus  $y = (x^*x)^{1/2\Re a} \in L_1^+(M)$ . Since  $L_1^+(M) = M_*^+$  (here  $M_*^+$  is the set of all positive functionals in  $M_*$ ), we have  $y(1) \geq 0$  and  $\|x\| = (y(1))^{\Re a} \geq 0$ . This norm turns  $L_a(M)$  into a Banach space. If  $\Re(a+b) \leq 1$ , then we equip the tensor product  $L_a(M) \otimes_M L_b(M)$  with the projective tensor norm and the second main theorem states that the map  $m$  is an isometry of Banach spaces.

If  $\Re a > 1$ , then the same formula as above gives a quasi-norm on  $L_a(M)$ . Quasi-norms satisfy a relaxed triangle inequality  $\|x + y\| \leq c(\|x\| + \|y\|)$  for some  $c \geq 1$ . In our case we can take  $c = 2^{\Re a - 1}$ . The space  $L_a(M)$  is complete with respect to this quasi-norm and therefore is a quasi-Banach space. If  $\Re(a+b) > 1$ , then we equip the tensor product  $L_a(M) \otimes_M L_b(M)$  with the generalization of projective tensor norm by Turpin [34] and the second main theorem states that the map  $m$  is an isometry of quasi-Banach spaces.

The left multiplication map  $L_a(M) \rightarrow \text{Hom}_M(L_b(M), L_{a+b}(M))$  that sends an element  $x \in L_a(M)$  to the map  $y \in L_b(M) \mapsto xy \in L_{a+b}(M)$  is also an isomorphism of algebraic  $M$ - $M$ -bimodules. Similarly to the tensor product, all elements in the algebraic inner hom above are automatically continuous in the norm topology, in particular they are bounded and can be equipped with the usual norm, which turns the above isomorphism into an isometry. This result was known before for the case of continuous inner hom, only the automatic continuity part is new.

We now recall some properties of the left and the right support and of the left and the right polar decomposition of an element  $x \in M$ . The right support of  $x$  is the unique projection that generates the left annihilator of the right annihilator of  $x$ . It can also be defined as the infimum of all projections  $p$  such that  $x = xp$ . The left support of  $x$  is the right support of  $x$  in the opposite algebra of  $M$ . The involution exchanges the left and the right support. The left and the right support of  $x$  are equal if and only if  $x$  is normal. In this case we refer to both of them as the support of  $x$ . The first definition of the right support of  $x$  implies that  $xy = 0$  if and only if  $py = 0$ , where  $p$  is the right support of  $x$ . Likewise for the left support.

The right polar decomposition of  $x$  is the unique pair  $(y, z) \in M \times M$  such that  $x = yz$ ,  $y$  is a partial isometry,  $z \geq 0$ , and the right support of  $y$  equals the right support of  $x$ . It follows that  $y^*x = z = (x^*x)^{1/2}$ , the left support of  $y$  equals the left support of  $x$ , and the right support of  $y$  equals the support of  $z$ . Recall that for any partial isometry  $y$  its left support is equal to  $yy^*$  and its right support is equal to  $y^*y$ . The left polar decomposition of  $x$  is the right polar decomposition of  $x$  in the opposite algebra of  $M$ . If  $x = yz$  is the right polar decomposition of  $x$ , then  $x = (yz y^*)y$  is the left polar decomposition of  $x$ . Thus the partial isometry parts of both polar decompositions coincide and we refer to both of them as the partial isometry part of  $x$ .

The notions of supports and polar decompositions extend to elements of  $\tilde{M}$ , where  $\tilde{M}$  is the completion of a semifinite von Neumann algebra  $M$  with respect to the  $\tau$ -measurable topology for some faithful trace  $\tau$ . The definitions of all notions are the same. Recall that all bounded elements of  $\tilde{M}$  (an element  $z \in \tilde{M}$  is bounded if  $z^*z \leq d \cdot 1$  for some  $d \in \mathbf{R}_{\geq 0}$ ) belong to  $M$ , in particular all projections and partial isometries in  $\tilde{M}$  belong to  $M$ . Hence the left and the right support of any element belong to  $M$ . See below for the construction of polar decompositions.

For the case of the core of  $M$  with its canonical trace we can say more about homogeneous elements, i.e., elements of  $L_a(M)$  for some  $a \in \mathbf{C}_{\Re \geq 0}$ . It turns out that if  $w \in L_a(M)$  for some  $a \in \mathbf{C}_{\Re \geq 0}$ , then both supports of  $w$  are in  $M$  (and not only in  $\hat{M}$ ), the partial isometry part of  $w$  is in  $L_{\Im a}(M)$  (recall that the imaginary part  $\Im a$  of  $a$  belongs to  $\mathbf{I}$ , the space of imaginary complex numbers), and the positive parts are in  $L_{\Re a}^+(M)$ . See below for a proof of all these statements.

## Algebraic tensor product and algebraic inner hom of $L_a$ and $L_b$ .

First of all we need the following version of Douglas lemma (see Douglas [39] for the original Douglas lemma). Various bits of this result are scattered throughout the literature, see the formula (2.1) and Lemma 4.2 in Junge and Sherman [33], Proposition 6b in Appendix B of Chapter 5 in Connes [20], Lemma 2.13 and Lemma 3.5 in Yamagami [17], Sections 1.2, 1.4, and 1.5 and Lemmas 2.2 and 2.4 in Schmitt [36], Lemma VII.1.6 (i) in Takesaki [41].

**Douglas lemma and polar decomposition for measurable operators.** For any von Neumann algebra  $N$  with a faithful trace  $\tau$  and any elements  $x$  and  $y$  in  $\tilde{N}$ , where  $\tilde{N}$  is the completion of  $N$  with respect to the  $\tau$ -measurable topology, there exists an element  $p \in N$  such that  $px = y$  if and only if there exists a  $c \in \mathbf{R}_{\geq 0}$  such that  $c^2 x^* x \geq y^* y$ . Moreover, if the right support of  $p$  is at most the left support of  $x$ , then such  $p$  is unique. In this case the norm of  $p$  equals the smallest possible value of  $c$ , and the left support of  $p$  equals the left support of  $y$ .

If  $x$  and  $y$  satisfy the stronger condition  $x^* x = y^* y$ , then  $p$  is a partial isometry, the right support of  $p$  equals the left support of  $x$ , and  $p^* y = x$ . In particular all elements of  $\tilde{N}$  have a unique left polar decomposition with the standard properties if we set  $x = (y^* y)^{1/2}$ .

*Proof.* To prove uniqueness, suppose that  $px = qx$ , where the right supports of  $p \in N$  and  $q \in N$  are at most the left support of  $x$ . We have  $(p-q)x = 0$ , therefore  $(p-q)z = 0$ , where  $z$  is the left support of  $x$ . But  $pz = p$  and  $qz = q$ , because the right supports of  $p$  and  $q$  are at most  $z$ , and therefore  $p = q$ . Uniqueness implies that the left support of  $p$  equals the left support of  $y$ , because we can multiply  $p$  by the left support of  $y$  from the left.

Suppose that  $px = y$  for some  $p \in N$ , hence  $y^* y = x^* p^* p x$ . Also  $p^* p \leq \|p^* p\| = \|p\|^2 1$  and  $x^* p^* p x \leq \|p\|^2 x^* x$ , because conjugation preserves inequalities. Therefore for  $c = \|p\|$  we have  $c^2 x^* x \geq y^* y$ , thus the smallest possible value of  $c$  is at most  $\|p\|$ .

Once we have an element  $p \in N$  such that  $px = y$ , then the element  $pq \in N$ , where  $q$  is the left support of  $x$ , satisfies the condition  $pqx = y$  and its right support is at most the left support of  $x$ . Hence it is enough to construct  $p$  without the additional condition on its right support.

To construct  $p$ , assume first that  $x \geq 0$ . Set  $z_\epsilon = f_\epsilon(x)$ , where  $f_\epsilon(t) = t^{-1}$  for all  $t \geq \epsilon$  and  $f_\epsilon(t) = 0$  for all other  $t$ . Note that  $z_\epsilon \in N$  for all  $\epsilon > 0$ . Now  $z_\epsilon^* y^* y z_\epsilon \leq z_\epsilon^* c^2 x^* x z_\epsilon \leq c^2$ , hence  $yz_\epsilon \in N$  and its norm is at most  $c$ . Set  $p = \lim_{\epsilon \rightarrow 0} yz_\epsilon$ . Here the limit is taken over all  $\epsilon > 0$  in the  $\sigma$ -weak topology. Recall that the unit ball is compact in the  $\sigma$ -weak topology, therefore the limit exists,  $px = y$ , and  $\|p\| \leq c$ .

Suppose now that  $x^* x = y^* y$  and  $px = y$  for some  $p \in N$  such that the right support of  $p$  is at most  $q$ , where  $q$  is the left support of  $x$ . We have  $x^*(p^* p - 1)x = (px)^* px - x^* x = y^* y - y^* y = 0$ , hence  $q^*(p^* p - 1)q = 0$ , which implies that  $p^* p = (pq)^* pq = q^* q = q$ , therefore  $p^* p$  is a projection and  $p$  is a partial isometry. We also have  $p^* p = q$ , hence the right support of  $p$  equals the left support of  $x$  and  $p^* y = x$ .

To prove the right polar decomposition theorem simply apply the previous argument to  $x = (y^* y)^{1/2}$ , where  $y$  is an arbitrary element of  $\tilde{N}$ .

For the case of general  $x$  denote by  $(r, u)$  the right polar decomposition of  $x = ru$  with  $r$  being a partial isometry in  $N$  and  $u$  being an element of  $\tilde{N}^+$ . We use the construction for

positive  $u$  to find  $q \in N$  such that  $qu = y$ , which is possible because  $c^2 u^* u = c^2 x^* x \geq y^* y$ . For  $p = qr^*$  we have  $p \in N$  and  $px = qr^* x = qu = y$ .

Finally, if  $x^* x = y^* y$  for arbitrary  $x$  and  $y$  in  $\hat{N}$ , then the construction in the previous paragraph gives us a  $p \in N$  such that  $px = y$  and the right support of  $p$  is at most the left support of  $x$ , hence  $p$  is a partial isometry by the argument above and the right support of  $p$  equals the left support of  $x$  and  $p^* y = x$ .

**Corollary.** If  $x \in L_a(M)$  and  $y \in L_b(M)$  for some  $a$  and  $b$  in  $\mathbf{C}_{\mathfrak{R} \geq 0}$  satisfy the conditions of the previous theorem for the core of  $M$  with its canonical trace, then  $\mathfrak{R}a = \mathfrak{R}b$  (unless  $y = 0$ ) and  $p \in L_{b-a}(M)$ . In particular, all elements of  $L_a(M)$  have a unique left polar decomposition as a product of a partial isometry in  $L_{\mathfrak{S}a}(M)$  and a positive element in  $L_{\mathfrak{R}a}(M)$ . Hence the left and the right support of an arbitrary element of  $L_a(M)$  belong to  $M$ , because for an arbitrary partial isometry  $u \in L_{\mathfrak{S}a}(M)$  the elements  $u^* u$  and  $uu^*$  belong to  $M$ .

*Proof.* We have  $c^2 x^* x \in L_{2\mathfrak{R}a}(M)$ ,  $y^* y \in L_{2\mathfrak{R}b}(M)$ , and  $c^2 x^* x \geq y^* y$ . The inequality is preserved under the action of  $\theta_s$  for all  $s \in \mathbf{R}$ , hence  $\mathfrak{R}a = \mathfrak{R}b$  unless  $y = 0$ . Suppose  $px = y$  for some  $p \in \hat{M}$ . For all real  $s$  we have  $\theta_s(px) = \theta_s(p)\theta_s(x) = \theta_s(p)\exp(-sa)x = \theta_s(y) = \exp(-sb)y$ , hence  $\exp(sb - sa)\theta_s(p)x = y$ . Thus  $(p - \exp(sb - sa)\theta_s(p))x = 0$  and therefore  $(p - \exp(sb - sa)\theta_s(p))e = 0$  where  $e$  is the left support of  $x$ . Hence  $pe = \exp(sb - sa)\theta_s(p)e$  and  $\exp(sa - sb)p = \theta_s(p)e$ . Since  $e \in M$  and  $p = pe$  we have  $\theta_s(p) = \theta_s(p)e = \exp(-(b - a)s)p$ , thus  $p \in L_{b-a}(M)$ .

**Lemma.** If  $M$  is a von Neumann algebra and  $a \in \mathbf{C}_{\mathfrak{R} \geq 0}$ , then any finitely generated left  $M$ -submodule  $U$  of  $L_a(M)$  is generated by one element. Likewise for right  $M$ -submodules.

*Proof.* Suppose that  $U$  is generated by a family  $u: I \rightarrow L_a(M)$  for some finite set  $I$ . Set  $x = (\sum_{i \in I} u_i^* u_i)^{1/2}$ . Note that  $u_i^* u_i \in L_{2\mathfrak{R}a}^+(M)$  for all  $i \in I$ , therefore the sum is also in  $L_{2\mathfrak{R}a}^+(M)$ , hence  $x \in L_{\mathfrak{R}a}^+(M)$ . We have  $x^* x = \sum_{i \in I} u_i^* u_i \geq u_i^* u_i$  for all  $i \in I$ . By the corollary above  $u_i = q_i x$  for some  $q_i \in L_{\mathfrak{S}a}(M)$  for all  $i \in I$ . Choose an arbitrary faithful weight  $\mu \in \hat{L}_1^+(M)$ . Now  $\mu^{-\mathfrak{S}a} \mu^{\mathfrak{S}a} = 1$  and therefore  $u_i = q_i x = (q_i \mu^{-\mathfrak{S}a})(\mu^{\mathfrak{S}a} x)$ . Thus all  $u_i$  are left  $M$ -multiples of  $y = \mu^{\mathfrak{S}a} x$ .

The proof will be complete when we show that  $y \in U$ . If  $I = \emptyset$ , then  $y = 0 \in U$ . Otherwise fix an element  $k \in I$  and set  $N = M \otimes \text{End}(\mathbf{C}^I)$ . Consider two elements  $Y$  and  $Z$  in  $L_a(N)$  such that  $Y_{k,k} = y$ ,  $Z_{i,k} = u_i$  for all  $i \in I$  and all other entries of  $Y$  and  $Z$  are 0. Since  $Y^* Y = Z^* Z$ , there is a partial isometry  $P \in N$  such that  $Y = PZ$ . In particular,  $y = \sum_{i \in I} p_{k,i} u_i$ , hence  $y \in U$ .

**Rank 1 theorem.** For any von Neumann algebra  $M$ , any right  $M$ -module  $X$ , any  $a \in \mathbf{C}_{\mathfrak{R} \geq 0}$ , and any element  $z \in X \otimes_M L_a(M)$  there exist  $x \in X$  and  $y \in L_a(M)$  such that  $z = x \otimes_M y$ .

*Proof.* Represent  $z$  as  $\sum_{i \in I} u_i \otimes_M v_i$  for some finite set  $I$  and some finite families  $u: I \rightarrow X$  and  $v: I \rightarrow L_a(M)$ . By the lemma above there exists an element  $y \in L_a(M)$  and a finite family  $q: I \rightarrow M$  such that  $v_i = q_i y$  for all  $i \in I$ . Now  $z = \sum_{i \in I} u_i \otimes_M v_i = \sum_{i \in I} u_i \otimes_M q_i y = \sum_{i \in I} u_i q_i \otimes_M y = (\sum_{i \in I} u_i q_i) \otimes_M y$ . Hence  $x = \sum_{i \in I} u_i q_i$  and  $y$  satisfy the requirements of the theorem.

We are ready to prove the first main result of this paper.

**Algebraic tensor product isomorphism theorem.** For any von Neumann algebra  $M$  and any  $a$  and  $b$  in  $\mathbf{C}_{\Re \geq 0}$ , the multiplication map  $m: L_a(M) \otimes_M L_b(M) \rightarrow L_{a+b}(M)$  is an isomorphism of algebraic  $M$ - $M$ -bimodules. The inverse map is denoted by  $n$  and is called the *comultiplication map*. Recall that  $\otimes_M$  denotes the algebraic tensor product over  $M$ .

*Proof.* Since  $m$  is a morphism of algebraic  $M$ - $M$ -bimodules, it is enough to prove that  $m$  is injective and surjective. To prove injectivity, suppose that we have an element  $z \in L_a(M) \otimes_M L_b(M)$  such that  $m(z) = 0$ . By the rank 1 theorem there exist  $x \in L_a(M)$  and  $y \in L_b(M)$  such that  $z = x \otimes_M y$ . We want to prove that  $x \otimes_M y = 0$ . Note that  $xy = m(x \otimes_M y) = m(z) = 0$ . Since  $xy = 0$ , we have  $py = 0$ , where  $p \in M$  is the right support of  $x$ . We have  $z = x \otimes_M y = xp \otimes_M y = x \otimes_M py = x \otimes_M 0 = 0$ .

To prove surjectivity, suppose that we have an element  $z \in L_{a+b}(M)$ . Using the right polar decomposition, choose  $t \in L_{\Im(a+b)}(M)$  and  $h \in L_1^+(M)$  such that  $z = th^{\Re(a+b)}$  is the right polar decomposition of  $z$ . Note that  $th^{\Re a - \Im b} \in L_a(M)$ ,  $h^b \in L_b(M)$ , and  $m(th^{\Re a - \Im b} \otimes_M h^b) = z$ . Hence, the map  $m$  is surjective and therefore bijective. This also proves that the map  $n: L_{a+b}(M) \rightarrow L_a(M) \otimes_M L_b(M)$  defined by  $n(z) = th^{\Re a - \Im b} \otimes_M h^b$  with  $z$ ,  $t$ , and  $h$  as above is the inverse of the map  $m$ , in particular it is linear.

To prove the second main result of this paper we need to recall some basic definitions of the theory of quasi-Banach spaces and their tensor products. See Kalton [42] for a nice survey of this area. If  $p \geq 1$  is a real number, then a  $p$ -seminorm on a complex vector space  $V$  is a function  $x \in V \mapsto \|x\| \in \mathbf{R}_{\geq 0}$  such that  $\|ax\| = |a| \cdot \|x\|$  for any  $a \in \mathbf{C}$  and any  $x \in V$ , and  $\|x + y\|^{1/p} \leq \|x\|^{1/p} + \|y\|^{1/p}$  for all  $x$  and  $y$  in  $V$ . A  $p$ -norm is a  $p$ -seminorm satisfying the usual nondegeneracy condition: For all  $x \in V$  the relation  $\|x\| = 0$  implies  $x = 0$ . In particular,  $p = 1$  gives the usual definition of norm. If  $a \in \mathbf{C}_{\Re \geq 0}$  and  $\Re a \geq 1$ , then  $L_a(M)$  is a complete  $\Re a$ -normed complex vector space. Recall that for  $\Re a \geq 1$  the space  $L_a(M)$  is a complete normed complex vector space, i.e., a complex Banach space.

A  $p$ -norm is an example of a quasi-norm, which we define in the same way as a  $p$ -norm, but replace the last condition by  $\|x + y\| \leq c(\|x\| + \|y\|)$ , where  $c \geq 1$  is some constant. In the case of a  $p$ -norm we can take  $c = 2^{p-1}$ . A theorem by Aoki and Rolewicz states that every quasi-norm is equivalent to a  $p$ -norm for  $p = 1 + \log_2 c$ .

If  $X$  is a  $p$ -normed space for some  $p \geq 1$ ,  $Y$  is a  $q$ -normed space for some  $q \geq 1$ , and  $r \geq 1$  is a real number, then we can introduce an  $r$ -seminorm on  $X \otimes Y$  as follows:  $u \in X \otimes Y \mapsto \|u\|_r = \sup_B \|B(u)\| \in \mathbf{R}_{\geq 0}$ , where  $B$  ranges over all linear maps from  $X \otimes Y$  to some  $r$ -normed space  $Z$  such that for all  $x \in X$  and  $y \in Y$  we have  $\|B(x \otimes y)\| \leq \|x\| \cdot \|y\|$ . A theorem by Turpin [34] states that for  $r \geq p + q - 1$  this  $r$ -seminorm is an  $r$ -norm. In particular, for  $p = q = r = 1$  we get the usual projective tensor norm.

**Algebraic tensor product isometry theorem.** In the notation of the previous theorem, equip the space  $L_a(M) \otimes_M L_b(M)$  with the factor-norm of the Turpin tensor  $r$ -norm on  $L_a(M) \otimes L_b(M)$ , where  $r = \max(1, \Re(a + b))$ . Then both  $m$  and  $n$  are isometries.

*Proof.* It is enough to prove that both  $m$  and  $n$  are norm-decreasing. To prove that  $n$  is norm-decreasing, suppose that  $z \in L_{a+b}(M)$ . In the notation of the previous proof we have  $\|n(z)\| = \|th^{\Re a - \Im b} \otimes_M h^b\| \leq \|th^{\Re a - \Im b}\| \cdot \|h^b\| = \|h^{\Re a}\| \cdot \|h^{\Re b}\| = h(1)^{\Re a} \cdot h(1)^{\Re b} = h(1)^{\Re(a+b)} = \|h^{\Re(a+b)}\| = \|th^{a+b}\| = \|z\|$ . Note that  $\|th^{\Re a - \Im b}\| = \|th^{\Re a}\| = \|h^{\Re a}\|$  because  $t$  is a partial isometry in  $M$  such that  $t^*t$  equals the support of  $h$ .

The map  $m$  satisfies the condition on  $B$  in the definition of Turpin norm because of the Hölder-Kosaki inequality (Kosaki [35]):  $\|xy\| \leq \|x\| \cdot \|y\|$  for all  $x \in L_a(M)$  and  $y \in L_b(M)$ . Hence  $m$  is norm-decreasing.

**Remark.** The (quasi-)norm on  $L_a(M)$  determines  $\mathfrak{R}a$  unless  $M = \mathbf{C}$  or  $M = 0$ . (For  $\mathfrak{R}a > 1$  this follows from Aoki-Rolewicz theorem and for  $\mathfrak{R}a \leq 1$  from Clarkson's inequalities.) Thus in the generic case the number  $r$  in the statement of the above theorem is determined by the Banach structures on  $L_a(M)$  and  $L_b(M)$ . In the two special cases the norm on  $L_{a+b}(M)$  coincides with the norms on  $L_a(M)$  and  $L_b(M)$ . Thus it is always possible to define the quasi-norm in the above theorem using just the Banach structures without any reference to  $a$  and  $b$ .

We now prove the corresponding result for algebraic homomorphisms. The only novelty of the theorem below is that we remove the boundedness condition on homomorphisms. Everything else has been proved before, see Theorem 2.5 in Junge and Sherman [33], Proposition 2.10 in Yamagami [17], and Proposition II.35 in Terp [16]. We start by proving an automatic continuity lemma, which then implies the desired result.

**Automatic continuity lemma.** Suppose  $M$  is a von Neumann algebra,  $a \in \mathbf{C}_{\mathfrak{R} \geq 0}$ , and  $X$  is a topological vector space equipped with a structure of a right  $M$ -module such that for any  $x \in X$  the map  $p \in M \mapsto xp \in X$  is continuous in the norm topology on  $M$ . Then every morphism of algebraic right  $M$ -modules from  $L_a(M)$  to  $X$  is continuous if  $L_a(M)$  is equipped with the norm topology.

*Proof.* Suppose  $T \in \text{Hom}_M(L_a(M), X)$ . Consider a family  $u: I \rightarrow L_a(M)$  such that the sum  $\sum_{i \in I} \|u_i\|^2$  exists. Set  $v = (\sum_{i \in I} u_i u_i^*)^{1/2} r$ , where  $r$  is an fixed arbitrary unitary element of  $L_{\mathfrak{S}a}(M)$ . The sum converges in the norm topology due to the properties of  $u$ . Since  $vv^* \geq u_i u_i^*$  for all  $i \in I$ , Douglas lemma gives us a family  $p: I \rightarrow M$  such that for all  $i \in I$  we have  $u_i = vp_i$  and  $\|p_i\| \leq 1$ . Now  $T(u_i) = T(vp_i) = T(v)p_i$ . The map  $q \in M \mapsto T(v)q \in X$  is continuous, therefore the preimage of any neighborhood  $V$  of 0 in  $X$  must contain some ball of radius  $\lambda > 0$  centered at the origin, in particular  $T(au_i) \in V$  for all  $a \in [0, \lambda]$  and all  $i \in I$  because  $\|ap_i\| \leq \lambda$ .

If  $T$  is not continuous, then there is at least one neighborhood  $V$  of zero such that its preimage under  $T$  does not contain any ball of a positive radius, in particular for any  $\kappa > 0$  we can find  $w \in L_a(M)$  such that  $\|w\| \leq \kappa$  and  $T(w) \notin V$ . We now fix such a neighborhood  $V$  and let  $\kappa$  run through a sequence of rapidly decreasing values (e.g.,  $\kappa = n^{-2}$  for consecutive positive integer  $n$ ), which allows us to construct a sequence  $u$  such that  $\sum_{i \in I} \|u_i\|^2$  exists and  $T(\|u_i\| \cdot u_i) \notin V$  (e.g., we can take  $u_i := \|w_i\|^{-1/2} w_i$ , where  $w_i$  is constructed from  $\kappa$ ), which contradicts the statement proved above.

**Algebraic inner hom isomorphism theorem.** For any von Neumann algebra  $M$  and any  $a$  and  $b$  in  $\mathbf{C}_{\mathfrak{R} \geq 0}$  the left multiplication map  $L_a(M) \rightarrow \text{Hom}_M(L_b(M), L_{a+b}(M))$  is an isomorphism of algebraic  $M$ - $M$ -bimodules. The inverse map is called the *left comultiplication map*. Recall that  $\text{Hom}_M$  denotes the algebraic inner hom of right  $M$ -modules, the left  $M$ -action on the inner hom is induced by the left  $M$ -action on  $L_{a+b}(M)$ , and the right  $M$ -action is induced by the left  $M$ -action on  $L_b(M)$ .



*Proof.* If for some  $x \in L_a(M)$  we have  $xy = 0$  for all  $y \in L_b(M)$ , then  $xp = 0$  for all projections  $p \in M$ . If  $p$  equals the right support of  $x$ , then  $x = xp = 0$ . Hence the map is injective.

The space  $L_{a+b}(M)$  satisfies the conditions of the automatic continuity lemma because  $\|mx\|^2 = \|x^*m^*mx\| \leq \|m\|^2\|x^*x\| = \|m\|^2\|x\|^2$  for all  $m \in M$  and  $x \in L_{a+b}(M)$ . Hence all elements of  $\text{Hom}_M(L_b(M), L_{a+b}(M))$  are continuous in the norm topology (i.e., bounded). The proof is finished by the Theorem 2.5 in Junge and Sherman [33], which states that every bounded homomorphism of right  $M$ -modules from  $L_b(M)$  to  $L_{a+b}(M)$  is the left multiplication by an element of  $L_a(M)$ .

**Algebraic inner hom isometry theorem.** In the notation of the previous theorem, equip the space  $\text{Hom}_M(L_b(M), L_{a+b}(M))$  with the quasi-norm  $\|f\| = \sup_{\|y\| \leq 1} \|f(y)\|$ , where  $f: L_b(M) \rightarrow L_{a+b}(M)$  is an element of the above inner hom and  $y \in L_b(M)$ . Then both  $m$  and  $n$  are isometries.

*Proof.* For  $x \in L_a(M)$  we have  $\|m(x)\| = \sup_{\|y\| \leq 1} \|xy\| \leq \sup_{\|y\| \leq 1} \|x\| \cdot \|y\| \leq \|x\|$ , thus  $\|m(x)\| \leq \|x\|$ . It remains to prove that  $\|x\| \leq \|m(x)\|$ . We can assume that  $x \neq 0$ .

If  $\Re a \neq 0$ , then we construct  $y \in L_b(M)$  such that  $\|y\| \neq 0$  and  $\|xy\| = \|x\| \cdot \|y\|$ . Set  $z = (x^*x)^{1/2\Re a}$  and  $y = z^b$ . Note that  $x = uz^a$  for some partial isometry  $u \in M$ . Now  $\|xy\| = \|uz^a y\| = \|z^a y\| = \|z^a z^b\| = \|z^{a+b}\| = \|z\|^{\Re a + \Re b} = \|z\|^{\Re a} \|z\|^{\Re b} = \|z^a\| \cdot \|z^b\| = \|uz^a\| \cdot \|y\| = \|x\| \cdot \|y\|$ .

In the case  $\Re a = 0$  the above strategy does not work because such an element  $y$  might not exist. Instead for every  $c \in [0, \|x\|)$  we construct a nonzero  $y \in L_b(M)$  such that  $\|xy\| \geq c\|y\|$ . Set  $z = (xx^*)^{1/2}$ . Note that  $x = zu$  for some partial isometry  $u \in L_a(M)$ . Denote by  $f$  the function  $\mathbf{R} \rightarrow \mathbf{R}$  that equals 1 on  $[c, \infty)$  and is 0 elsewhere. Note that  $p = f(z)$  is a nonzero projection and  $zp \geq cp$ . Choose a nonzero  $w \in L_1^+(M)$  with the support at most  $p$  and set  $y = u^*w^{a+b}$ . Now  $\|xy\| = \|z u u^* w^{a+b}\| = \|z w^{a+b}\| = \|z p w^{a+b}\| = \|(w^{a+b})^* p z z p w^{a+b}\|^{1/2} \geq \|(w^{a+b})^* c^2 w^{a+b}\|^{1/2} = c \|w^{a+b}\| = c \|u^* w^{a+b}\| = c \|y\|$ .

**Remark.** For any von Neumann algebra  $M$  the spaces  $L_a(M)$  for all  $a \in \mathbf{C}_{\Re \geq 0}$  can be organized into a smooth bundle  $L$  of quasi-Banach  $M$ - $M$ -bimodules over  $\mathbf{C}_{\Re \geq 0}$  by postulating that a section of this bundle is smooth if it is locally of the form  $a \in \mathbf{C}_{\Re \geq 0} \mapsto f(a)\mu^a$  for some smooth function  $f: \mathbf{C}_{\Re \geq 0} \rightarrow M$  and some  $\mu \in L_1^+(M)$ . The tensor product isomorphisms can be combined into a smooth bundle isomorphism  $i^*L \otimes_M j^*L \rightarrow k^*L$ , where  $i, j, k: \mathbf{C}_{\Re \geq 0} \times \mathbf{C}_{\Re \geq 0} \rightarrow \mathbf{C}_{\Re \geq 0}$  are respectively projections on the first and the second component and the addition map. Likewise, the inner hom isomorphisms can be combined into a smooth isomorphism  $i^*L \rightarrow \text{Hom}_M(j^*L, k^*L)$ . In particular, if we restrict the tensor product isomorphism from  $\mathbf{C}_{\Re \geq 0}$  to  $\mathbf{I}$  and allow  $\mu \in \hat{L}_1^+(M)$  in the definition of smooth sections, we obtain a convolution product on the space of distributional sections of  $L$  restricted to  $\mathbf{I}$  with bounded Fourier transform, which turns it into an  $\mathbf{I}$ -graded von Neumann algebra (the grading is the composition of the isomorphism  $\mathbf{R} \rightarrow \text{Hom}(\mathbf{I}, \mathbf{U})$  and the left multiplication action), which is canonically isomorphic to  $\hat{M}$ . This is essentially the approach used by Falcone and Takesaki [18] to construct  $\hat{M}$ . A similar result is true for  $\bar{M}$  if we do not restrict to  $\mathbf{I}$ .

### $L_a$ -modules.

We now extend the results of the previous section to the case of modules. The case of densities 0 and 1/2 (without automatic completeness or continuity) is covered by Theorem 2.2 in Baillel, Denizeau, and Havet [4]. Many ideas were already present in Rieffel's paper [3].

**Remark.** If  $M$  is a complex  $*$ -algebra and  $X$  is an algebraic right  $M$ -module, then denote by  $X^\sharp$  the algebraic left  $M$ -module whose underlying abelian group is that of  $X$  and the left multiplication map is the composition  $M \otimes X \rightarrow M \otimes X \rightarrow X \otimes M \rightarrow X$ , where the first map is the tensor product of the involution on  $M$  and the identity map on  $X$ , the second map is the braiding map, and the third map is the right multiplication map on  $M$ . There is a canonical complex antilinear isomorphism of vector spaces  $X \rightarrow X^\sharp$ , which equals the identity on the underlying abelian groups. Henceforth we denote this isomorphism by  $x \mapsto x^\sharp$ . The functor  $X \mapsto X^\sharp$  is an equivalence of the categories of right and left algebraic  $M$ -modules. Whenever we equip  $M$ -modules with additional structures or properties we extend the functor  $X \mapsto X^\sharp$  to the new category of modules without explicitly mentioning it.

**Definition.** (Junge and Sherman, [33].) Suppose  $d \in \mathbf{R}_{\geq 0}$  and  $M$  is an arbitrary von Neumann algebra. A *right pre- $L_d(M)$ -module* is an algebraic right  $M$ -module  $X$  together with an  $M$ - $M$ -bilinear inner product  $\mu: X^\sharp \otimes X \rightarrow L_{2d}(M)$  such that for all  $u$  and  $v$  in  $X$  we have  $(u, v)^* = (v, u)$  and for all  $w \in X$  we have  $(w, w) \geq 0$  and  $(w, w) = 0$  implies  $w = 0$ . Here  $(x, y) := \mu(x^\sharp \otimes y)$ . *Left pre- $L_d(M)$ -modules* are defined similarly.

**Remark.** A canonical example of a right  $L_d(M)$ -module is given by the space  $L_a(M)$ , where  $\Re a = d$  and the inner product is given by  $(x, y) := x^*y$ . This explains why we require  $d$  to be real.

**Definition.** Suppose  $d \in \mathbf{R}_{\geq 0}$ ,  $M$  is a von Neumann algebra, and  $X$  and  $Y$  are right pre- $L_d(M)$ -modules. A morphism  $f$  from  $X$  to  $Y$  is a morphism of the underlying algebraic right  $M$ -modules of  $X$  and  $Y$  that is continuous in the topologies given by the quasi-norm  $x \in X \mapsto \|(x, x)^{1/2}\| \in \mathbf{R}_{\geq 0}$  on  $X$  and likewise for  $Y$ . See Proposition 3.2 in Junge and Sherman [33] for a proof that this is a quasi-norm.

**Definition.** Suppose  $M$  is a von Neumann algebra and  $d \in \mathbf{R}_{\geq 0}$ . A *right  $L_d(M)$ -module* is a right pre- $L_d(M)$ -module such that every bounded (in the corresponding quasi-norms) morphism from  $X$  to  $L_{2d}(M)$  has the form  $y \in X \mapsto (x, y) \in L_{2d}(M)$  for some  $x \in X$ . The category of right  $L_d(M)$ -modules is the full subcategory of the category of right pre- $L_d(M)$ -modules consisting of  $L_d(M)$ -modules. We also refer to right  $L_0(M)$ -modules as  *$W^*$ -modules over  $M$* .

**Corollary.** Suppose  $M$  is a von Neumann algebra and  $d \in \mathbf{R}_{\geq 0}$ . The full subcategory of right  $L_d(M)$ -modules in the category of right pre- $L_d(M)$ -modules is reflective. The reflector (i.e., the left adjoint to the inclusion functor) sends a right pre- $L_d(M)$ -module to its completion in the measurable topology (which can also be described algebraically as  $\text{CHom}(X, L_{2d}(M))^\sharp$ ) and a morphism of right pre- $L_d(M)$ -module to its unique extension to completed modules (which admits a similar algebraic description). The unit of the

adjunction embeds a right pre- $L_d(M)$ -module into its completion. This embedding of categories is exact (the reflector preserves finite products) and bireflective (the unit of the adjunction is both a monomorphism and an epimorphism).

**Proposition.** Suppose  $M$  is a von Neumann algebra and  $d \in \mathbf{R}_{\geq 0}$ . Consider the contravariant endofunctor  $*$  on the category of right  $L_d(M)$ -modules that sends every object to itself and every morphism  $f: X \rightarrow Y$  of right  $L_d(M)$ -modules  $X$  and  $Y$  to the morphism  $f^*$  given by the composition  $Y \rightarrow \text{CHom}_M(Y, L_{2d}(M))^\sharp \rightarrow \text{CHom}_M(X, L_{2d}(M))^\sharp \rightarrow X$ , where the middle morphism is given by the precomposition with  $f$  and the other two isomorphisms come from the definition of right  $L_d(M)$ -modules. Here  $\text{CHom}$  denotes morphisms that are continuous in the quasi-norm topology (i.e., bounded). This functor is an involutive contravariant endoequivalence of the category of right  $L_d(M)$ -modules, i.e., the category of right  $L_d(M)$ -modules is a  $*$ -category. Moreover, the morphism  $f^*$  is uniquely characterized by the equation  $(f(x), y) = (x, f^*(y))$  for all  $x \in X$  and  $y \in Y$ .

*Proof.* If there is another such map  $g$ , then we have  $(x, (f^* - g)(y)) = 0$  for all  $x \in X$  and  $y \in Y$ , in particular, for  $x = (f^* - g)(y)$  we have  $((f^* - g)(y), (f^* - g)(y)) = 0$ , hence  $(f^* - g)(y) = 0$  for all  $y \in Y$ , therefore  $f^* = g$ .

Denote the three morphisms in the above composition by  $v$ ,  $h$ , and  $u$ . Observe that  $(x, f^*(y)) = (x, uhv(y)) = (uhv(y), x)^* = h(v(y))(x^\sharp)^* = v(y)(f^\sharp(x^\sharp))^* = v(y)(f(x)^\sharp)^* = (y, f(x))^* = (f(x), y)$ , as desired.

The map  $f \mapsto f^*$  preserves identities and composition, hence it defines a contravariant endofunctor. Now  $(x, f^{**}(y)) = (f^*(x), y) = (y, f^*(x))^* = (f(y), x)^* = (x, f(y))$  for all  $x \in X$ , hence  $f^{**}(y) = f(y)$  for all  $y \in X$ , thus  $f^{**} = f$  and the functor  $*$  is an involutive equivalence. Since it is also  $\mathbf{C}$ -antilinear on morphisms, it turns the category of right  $L_d(M)$ -modules into a  $*$ -category.

**Proposition.** Suppose  $M$  is a von Neumann algebra,  $d \in \mathbf{R}_{\geq 0}$ , and  $e \in \mathbf{R}_{\geq 0}$ . If  $X$  is a right  $L_d(M)$ -module with the inner product  $\mu: X^\sharp \otimes X \rightarrow L_{2d}(M)$ , then  $X \otimes_M L_e(M)$  has an inner product  $\nu$  given by the composition  $(X \otimes_M L_e(M))^\sharp \otimes (X \otimes_M L_e(M)) \rightarrow L_e(M)^\sharp \otimes_M X^\sharp \otimes X \otimes_M L_e(M) \rightarrow L_e(M)^\sharp \otimes_M L_{2d}(M) \otimes_M L_e(M) \rightarrow L_{2(d+e)}(M)$ , where the first map comes from the functor  $\sharp$ , the second map is the inner product on  $X$  tensored with identity maps, and the last map is the multiplication map combined with the canonical isomorphism  $L_e(M)^\sharp \rightarrow L_e(M)$  given by the involution. This inner product turns  $X \otimes_M L_e(M)$  into a right  $L_{d+e}(M)$ -module. Furthermore, a morphism  $f: X \rightarrow Y$  of right  $L_d(M)$ -modules induces a morphism  $f \otimes_M \text{id}_{L_e(M)}: X \otimes_M L_e(M) \rightarrow Y \otimes_M L_e(M)$  of right  $L_{d+e}(M)$ -modules. The above constructions combine into a  $*$ -functor from the category of right  $L_d(M)$ -modules to the category of right  $L_{d+e}(M)$ -modules.

*Proof.* First we prove that the morphism  $\nu$  defined above is an inner product on  $X \otimes_M L_e(M)$ . By the rank 1 theorem every element of  $X \otimes_M L_e(M)$  can be represented in the form  $x \otimes_M u$  for some  $x \in X$  and  $u \in L_e(M)$ . We have  $(x \otimes_M u, y \otimes_M v)^* = (u^*(x, y)v)^* = v^*(x, y)^*u = v^*(y, x)u = (y \otimes_M v, x \otimes_M u)$  for all  $x$  and  $y$  in  $X$  and  $u$  and  $v$  in  $L_e(M)$ . Moreover,  $(x \otimes_M u, x \otimes_M u) = u^*(x, x)u \geq 0$  because  $(x, x) \geq 0$  and conjugation preserves positivity. Finally, if  $(x \otimes_M u, x \otimes_M u) = u^*(x, x)u = 0$ , then  $p^*(x, x)p = 0$ , where  $p$  is the left support of  $u$ , hence  $p^*(x, x)p = (x \otimes_M p, x \otimes_M p) = (xp, xp) = 0$ , therefore  $xp = 0$ , which implies that  $x \otimes_M u = x \otimes_M pu = xp \otimes_M u = 0 \otimes_M u = 0$ .

The module  $X \otimes_M L_e(M)$  is complete because every bounded map from it to  $L_{2d+2e}(M)$  is given by an inner product with some element.

Recall that a morphism  $f: X \rightarrow Y$  of algebraic right  $M$ -modules is continuous if and only if it has an adjoint. For  $f \otimes_M \text{id}_{L_e(M)}$  we have  $((f \otimes_M \text{id}_{L_e(M)})(x \otimes_M u), y \otimes_M v) = u^*(f(x), y)v = u^*(x, f^*(y))v(x \otimes_M u, f^*(y) \otimes_M v) = (x \otimes_M u, (f^* \otimes_M \text{id}_{L_e(M)})(y \otimes_M v))$  for all  $x \in X, y \in Y, u$  and  $v$  in  $L_e(M)$ , therefore the adjoint of  $f \otimes_M \text{id}_{L_e(M)}$  is  $f^* \otimes_M \text{id}_{L_e(M)}$ , hence  $f \otimes_M \text{id}_{L_e(M)}$  is continuous.

Finally, the map  $f \mapsto f \otimes_M \text{id}_{L_e(M)}$  preserves the involution and therefore is a  $*$ -functor:  $((f \otimes_M \text{id}_{L_e(M)})(x \otimes_M u), y \otimes_M v) = (f(x) \otimes_M u, y \otimes_M v) = u^*(f(x), y)v = u^*(x, f^*(y))v = (x \otimes_M u, (f^* \otimes_M \text{id}_{L_e(M)})(y \otimes_M v))$  for all  $x \in X, y \in Y$ , and  $u$  and  $v$  in  $L_e(M)$ , hence  $(f \otimes_M \text{id}_{L_e(M)})^* = (f^* \otimes_M \text{id}_{L_e(M)})$ .

**Proposition.** Suppose  $M$  is a von Neumann algebra,  $d \in \mathbf{R}_{\geq 0}$ , and  $e \in \mathbf{R}_{\geq 0}$ . If  $X$  is a right  $L_{d+e}(M)$ -module with the inner product  $\mu: X^\sharp \otimes X \rightarrow L_{2d+2e}(M)$ , then  $\text{Hom}_M(L_e(M), X)$  has an inner product  $\nu$  given by the composition  $\text{Hom}_M(L_e(M), X)^\sharp \otimes \text{Hom}_M(L_e(M), X) \rightarrow {}_M\text{Hom}(L_e(M)^\sharp, X^\sharp) \otimes \text{Hom}_M(L_e(M), X) \rightarrow {}_M\text{Hom}_M(L_e(M)^\sharp \otimes L_e(M), X^\sharp \otimes X) \rightarrow {}_M\text{Hom}_M(L_e(M)^\sharp \otimes L_e(M), L_{2d+2e}(M)) \rightarrow \text{Hom}_M(L_e(M), {}_M\text{Hom}(L_e(M)^\sharp, L_{2d+2e}(M))) \rightarrow \text{Hom}_M(L_e(M), L_{2d+e}(M)) \rightarrow L_{2d}(M)$ , where the first map comes from the functor  $\sharp$ , the second map is the tensor product of morphisms, the third map is the composition with the inner product on  $X$ , the fourth map is the usual tensor-hom adjunction map, the fifth map is given by the algebraic hom isomorphism theorem combined with the canonical isomorphism  $L_e(M)^\sharp \rightarrow L_e(M)$  given by the involution, and the last map is again given by the algebraic hom isomorphism theorem. Alternatively, if  $x$  and  $y$  are in  $\text{Hom}_M(L_e(M), X)$ , then their inner product is the unique element  $w \in L_{2d}(M)$  such that  $(x(u), y(v)) = u^*wv$  for all  $u$  and  $v$  in  $L_e(M)$ . This inner product turns  $\text{Hom}_M(L_e(M), X)$  into a right  $L_d(M)$ -module. Furthermore, a morphism  $f: X \rightarrow Y$  of right  $L_{d+e}(M)$ -modules induces via composition a morphism  $\text{Hom}_M(L_e(M), f): \text{Hom}_M(L_e(M), X) \rightarrow \text{Hom}_M(L_e(M), Y)$  of right  $L_{d+e}(M)$ -modules. The above constructions combine into a  $*$ -functor from the category of right  $L_{d+e}(M)$ -modules to the category of right  $L_d(M)$ -modules.

*Proof.* Suppose that  $x$  and  $y$  are in  $\text{Hom}_M(L_e(M), X)$ . We have  $u^*(x, y)v = (x(u), y(v))$  for all  $u$  and  $v$  in  $L_e(M)$ , therefore  $v^*(x, y)^*u = (y(v), x(u))$ , hence by the alternative definition of the inner product we have  $(x, y)^* = (y, x)$ . If  $x \in \text{Hom}_M(L_e(M), X)$ , then  $u^*(x, x)u = (x(u), x(u)) \geq 0$  for all  $u \in L_e(M)$ . An element of  $L_{2d}(M)$  whose conjugation by any element of  $L_e(M)$  is positive must itself be positive, hence  $(x, x) \geq 0$ . Finally, if  $(x, x) = 0$ , then  $(x(u), x(u)) = u^*(x, x)u = 0$ , hence  $x(u) = 0$  for all  $u \in L_e(M)$ , therefore  $x = 0$ .

The module  $\text{Hom}_M(L_e(M), X)$  is complete because every bounded map from it to the module  $L_d(M)$  comes from an inner product with some element.

The morphism  $\text{Hom}_M(L_e(M), f)$  is continuous for every morphism  $f$ , because  $u^*(\text{Hom}_M(L_e(M), f)(x), y)v = (f(x(u)), y(v)) = (x(u), f^*(y(v))) = u^*(x, \text{Hom}_M(L_e(M), f^*)(y))v$  for all  $x \in X, y \in Y, u$  and  $v$  in  $L_e(M)$ , therefore the adjoint of  $\text{Hom}_M(L_e(M), f)$  is  $\text{Hom}_M(L_e(M), f^*)$ , and maps that admit an adjoint are continuous.

Finally, the map  $f \mapsto \text{Hom}_M(L_e(M), f)$  preserves the involution:  $u^*(\text{Hom}_M(L_e(M), f)(x), y)v = (f(x(u)), y(v)) = (x(u), f^*(y(v))) = u^*(x, \text{Hom}_M(L_e(M), f^*)(y))v$  for all

$u$  and  $v$  in  $L_e(M)$ , hence we have  $(\text{Hom}_M(L_e(M), f)(x), y) = (x, \text{Hom}_M(L_e(M), f^*)(y))$  and  $\text{Hom}_M(L_e(M), f)^* = \text{Hom}_M(L_e(M), f^*)$  for all  $f: X \rightarrow Y$ ,  $x \in X$ , and  $y \in Y$ .

**Proposition.** Suppose  $M$  is a von Neumann algebra,  $d \in \mathbf{R}_{\geq 0}$ , and  $e \in \mathbf{R}_{\geq 0}$ . If  $X$  is a right  $L_{d+e}(M)$ -module, then the evaluation map  $\text{ev}: \text{Hom}_M(L_e(M), X) \otimes_M L_e(M) \rightarrow X$  is a unitary isomorphism. Moreover, these maps combine into a unitary natural isomorphism of  $*$ -functors.

*Proof.* The alternative definition of the inner product on  $\text{Hom}_M(L_e(M), X)$  immediately proves that the evaluation map preserves the inner product, in particular it is injective:  $(\text{ev}(x \otimes_M u), \text{ev}(y \otimes_M v)) = (x(u), y(v)) = u^*(x, y)v = (x \otimes_M u, y \otimes_M v)$  for all  $x$  and  $y$  in  $X$  and  $u$  and  $v$  in  $L_e(M)$ .

Consider an arbitrary element  $x \in X$  and denote by  $Z$  the closed submodule of  $X$  generated by  $x$ . The map  $q$  from the algebraic submodule of  $Z$  generated by  $x$  to the algebraic right submodule of  $L_{d+e}(M)$  generated by  $(x, x)^{1/2}$  given by sending an element of the form  $xp \in X$  for some  $p \in M$  to the element  $(x, x)^{1/2}p \in L_{d+e}(M)$  is well-defined with respect to the choice of  $p$  and preserves the inner product, hence it extends to an isomorphism from  $Z$  to  $sL_{d+e}(M)$ , where  $s \in M$  is the right support of  $x$ , i.e., the support of  $(x, x)^{1/2}$ . We now prove that the restriction of  $\text{ev}$  to the map of the form  $\text{Hom}_M(L_e(M), Z) \otimes_M L_e(M) \rightarrow Z$  is an isomorphism. We have  $sL_d(M) \otimes_M L_e(M) \rightarrow \text{Hom}_M(L_e(M), sL_{d+e}(M)) \otimes_M L_e(M) \rightarrow \text{Hom}_M(L_e(M), Z) \otimes_M L_e(M) \rightarrow Z \rightarrow sL_{d+e}(M)$  is given by the multiplication map, which is an isomorphism, hence the above restriction of  $\text{ev}$  is an isomorphism and therefore  $\text{ev}$  is a surjection.

**Proposition.** Suppose  $M$  is a von Neumann algebra,  $d \in \mathbf{R}_{\geq 0}$ , and  $e \in \mathbf{R}_{\geq 0}$ . If  $X$  is a right  $L_d(M)$ -module, then the left multiplication map  $\text{lm}: X \rightarrow \text{Hom}_M(L_e(M), X \otimes_M L_e(M))$  is a unitary isomorphism. Moreover, these maps combine into a unitary natural isomorphism of  $*$ -functors.

*Proof.* The left multiplication map preserves the inner product, in particular it is injective:  $u^*(\text{lm}(x), \text{lm}(y))v = (\text{lm}(x)(u), \text{lm}(y)(v)) = (x \otimes_M u, y \otimes_M v) = u^*(x, y)v$  for all  $x$  and  $y$  in  $X$  and  $u$  and  $v$  in  $L_e(M)$ . Since  $u$  and  $v$  are arbitrary, it follows that  $(\text{lm}(x), \text{lm}(y)) = (x, y)$ . Finally, the left multiplication map is surjective and hence it is an isomorphism.

**Theorem.** Suppose  $M$  is a von Neumann algebra,  $d \in \mathbf{R}_{\geq 0}$ , and  $e \in \mathbf{R}_{\geq 0}$ . The  $*$ -category of right  $L_d(M)$ -modules is a  $W^*$ -category. Moreover, the functors of tensor product and inner hom with  $L_e(M)$  between the categories of right  $L_d(M)$ -modules and  $L_{d+e}(M)$ -modules together with the unitary natural isomorphisms of evaluation and left multiplication form an adjoint unitary  $W^*$ -equivalence of the  $W^*$ -categories of right  $L_d(M)$ -modules and right  $L_{d+e}(M)$ -modules.

*Proof.* The above propositions establish that the functors and natural isomorphisms under consideration constitute a unitary  $*$ -equivalence of the corresponding  $*$ -categories. The  $*$ -category of right  $L_0(M)$ -modules is a  $W^*$ -category, hence the  $*$ -category of right  $L_d(M)$ -modules is also a  $W^*$ -category. A  $*$ -equivalence of  $W^*$ -categories is automatically normal, i.e., it is a  $W^*$ -equivalence, and a unitary natural transformation is automatically bounded, i.e., it is a unitary  $W^*$ -natural transformation. Thus we only have to prove the adjunction

property. This amounts to checking the unit-counit equations. For equivalences, one of the equations implies the other one, but here we check them both.

The first property states that the composition of the morphisms  $X \otimes_M L_e(M) \rightarrow \text{Hom}_M(L_e(M), X \otimes_M L_e(M)) \otimes_M L_e(M)$  (the tensor product of the left multiplication map of  $X$  and the identity morphism of  $L_e(M)$ ) and  $\text{Hom}_M(L_e(M), X \otimes_M L_e(M)) \otimes_M L_e(M) \rightarrow X \otimes_M L_e(M)$  (the evaluation map of  $X \otimes_M L_e(M)$ ) is the identity morphism of  $X \otimes_M L_e(M)$ . The first map sends an element  $x \otimes_M u \in X \otimes_M L_e(M)$  to the element  $(v \in L_e(M) \mapsto x \otimes_M v \in X \otimes_M L_e(M)) \otimes_M u$ , which is then evaluated to  $x \otimes_M u$ .

The second property states that the composition of the morphism  $\text{Hom}_M(L_e(M), X) \rightarrow \text{Hom}_M(L_e(M), \text{Hom}_M(L_e(M), X) \otimes_M L_e(M))$  (the left multiplication map of the module  $\text{Hom}_M(L_e(M), X)$ ) and the morphism  $\text{Hom}_M(L_e(M), \text{Hom}_M(L_e(M), X) \otimes_M L_e(M)) \rightarrow \text{Hom}_M(L_e(M), X)$  (the composition with the evaluation map of  $X$ ) is the identity morphism of  $\text{Hom}_M(L_e(M), X)$ . The first map sends an element  $f \in \text{Hom}_M(L_e(M), X)$  to  $u \in L_e(M) \mapsto f \otimes_M u \in \text{Hom}_M(L_e(M), X) \otimes_M L_e(M)$ , which is then mapped to  $(u \in L_e(M) \mapsto f(u) \in X) = f$ .

**Definition.** Suppose  $M$  and  $N$  are von Neumann algebras and  $d \in \mathbf{R}_{\geq 0}$ . An  $M$ - $L_d(N)$ -bimodule is a right  $L_d(N)$ -module  $X$  equipped with a morphism of von Neumann algebras  $M \rightarrow \text{End}(X)$ . A morphism of  $M$ - $L_d(N)$ -bimodules is a morphism of the underlying right  $L_d(N)$ -modules that commutes with the left action of  $M$ . Similarly, an  $L_d(M)$ - $N$ -bimodule is a left  $L_d(M)$ -module  $X$  equipped with a morphism of von Neumann algebras  $N \rightarrow \text{End}(X)$ . We also refer to right  $M$ - $L_0(N)$ -bimodules as *right  $M$ - $N$ - $W^*$ -bimodules* and similarly for left bimodules.

**Theorem.** For any von Neumann algebras  $M$  and  $N$  and any  $d \in \mathbf{R}_{\geq 0}$  and  $e \in \mathbf{R}_{\geq 0}$  there are canonical adjoint unitary  $W^*$ -equivalences of the  $W^*$ -categories of  $M$ - $L_d(N)$ -bimodules and  $M$ - $L_e(N)$ -bimodules.

*Proof.* All functors and natural isomorphisms under consideration are  $W^*$ -functors and unitary  $W^*$ -natural isomorphisms and therefore they preserve the left action of  $M$ . and give an adjoint unitary equivalence of the corresponding  $W^*$ -categories.

We briefly review the equivalence of  $L_{1/2}(M)$ -modules and representations of  $M$  on a Hilbert space. See Example 3.4.(ii) in Junge and Sherman [33].

**Definition.** A *right representation* of a von Neumann algebra  $M$  on a Hilbert space  $H$  is a morphism of von Neumann algebras  $M \rightarrow B(H)^{\text{op}}$ , where multiplication of elements of  $B(H)$  corresponds to the composition of operators in the usual reverse order (i.e.,  $xy$  means apply  $y$  first, then apply  $x$ ). We denote the  $W^*$ -category of right representations of  $M$  and their bounded intertwiners by  $\text{Rep}_M$ . Likewise, a *left representation* is a morphism  $M \rightarrow B(H)$  and all left representations form a category  ${}_M\text{Rep}$ . Finally, a *birepresentation* of von Neumann algebras  $M$  and  $N$  on a Hilbert space  $H$  is a pair of morphisms  $M \rightarrow B(H)$  and  $N \rightarrow B(H)^{\text{op}}$  with commuting images.

**Definition.** For every von Neumann algebra  $M$  we define a  $W^*$ -functor  $F: \text{Mod}_{L_{1/2}(M)} \rightarrow \text{Rep}_M$  as follows: If  $X$  is a right  $L_{1/2}(M)$ -module, then we turn it into a Hilbert space with the inner product given by composing the  $L_1(M)$ -valued inner product on  $X$  with the Haagerup trace  $L_1(M) \rightarrow \mathbf{C}$ . We also define a  $W^*$ -functor going in the opposite direction

$\text{Rep}_M \rightarrow \text{Mod}_{L_{1/2}(M)}$  as follows: If  $X$  is a right representation of  $M$ , then for a pair of elements  $u$  and  $v$  in  $X$  the value of  $L_1(M)$ -valued inner product  $(u, v)$  evaluated at  $p \in M$  is  $\langle u, vp \rangle$ . Observe that we have unitary  $W^*$ -natural isomorphisms  $\text{id} \rightarrow GF$  and  $FG \rightarrow \text{id}$ .

**Theorem.** Two  $W^*$ -functors defined above together with the corresponding  $W^*$ -natural isomorphisms form an adjoint unitary  $W^*$ -equivalence of  $W^*$ -categories of right  $L_{1/2}(M)$ -modules and right representations of  $M$ . Likewise for left modules and representations. Moreover, the above adjoint unitary  $W^*$ -equivalences of  $W^*$ -categories yield adjoint unitary  $W^*$ -equivalences of the  $W^*$ -categories of  $M$ - $L_{1/2}(N)$ -bimodules, birepresentations of  $M$  and  $N$ , and  $L_{1/2}(M)$ - $N$ -bimodules.

**Corollary.** There are canonical adjoint unitary  $W^*$ -equivalences of the  $W^*$ -categories of  $M$ - $L_d(N)$ -bimodules,  $L_d(M)$ - $N$ -bimodules, and birepresentations of  $M$  and  $N$  for all von Neumann algebras  $M$  and  $N$  and for all  $d \in \mathbf{R}_{\geq 0}$ .

If we pass from an  $M$ - $L_d(N)$ -bimodule  $X$  to the corresponding  $L_d(M)$ - $N$ -bimodule  $Y$ , then  $X$  is generally not isomorphic to  $Y$  as an algebraic  $M$ - $N$ -bimodule unless  $d = 1/2$ . For example, take  $d = 0$ ,  $M = \mathbf{C}$ ,  $N = B(H)$ ,  $X = B(H)$  for some infinite dimensional Hilbert space  $H$ . Then  $Y = L_{1/2}(B(H))$  as a  $L_{1/2}(\mathbf{C})$ - $B(H)$ -bimodule and  $Y$  is not isomorphic to  $X$  as an algebraic right  $B(H)$ -module.

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## Symmetric monoidal framed double category of von Neumann algebras.

### Abstract.

We construct a new external tensor product of  $W^*$ -bimodules, which, combined with the categorical tensor product of von Neumann algebras constructed by Guichardet in 1966, yields a symmetric monoidal structure on the  $W^*$ -bicategory of von Neumann algebras,  $W^*$ -bimodules, and intertwiners. We prove that every von Neumann algebra is dualizable with respect to this monoidal structure, thus obtaining a shadow on the above bicategory in the sense of Shulman and Ponto, which allows us to compute traces of endomorphisms of finite index bimodules. The trace of the identity endomorphism gives a new invariant of finite index bimodules, which coincides with the Jones index in the case of factors.

### The category of von Neumann algebras.

Denote by  $\text{Ban}$  the category of complex Banach spaces and contractive linear maps (i.e., linear maps that do not increase the norm).

**Definition.** A *von Neumann algebra*  $A$  is a  $C^*$ -algebra that admits a predual, i.e., a Banach space  $Z$  such that  $Z^*$  is isomorphic to the underlying Banach space of  $A$  in the category  $\text{Ban}$ .

**Definition.** A *morphism*  $f: A \rightarrow B$  of von Neumann algebras is a morphism of the underlying  $C^*$ -algebras that admits a predual, i.e., a morphism of Banach spaces  $p: Z \rightarrow Y$  such that  $p^*: Y^* \rightarrow Z^*$  is isomorphic to the underlying morphism of Banach spaces of  $f$  in the category of morphisms of  $\text{Ban}$ .

**Notation.** Denote by  $W^*$  the category of von Neumann algebras. Denote by  $W^*C^*: W^* \rightarrow C^*$  the faithful forgetful functor from the category of von Neumann algebras to the category of  $C^*$ -algebras.

**Theorem.** For any von Neumann algebra  $A$  all preduals of  $A$  induce the same weak topology on  $A$ , which we call the *ultraweak topology*. In particular, the predual is unique up to unique isomorphism and is canonically isomorphic to the dual of  $A$  in the ultraweak topology.

*Proof.* See Corollary 1.13.3 in Sakai [43].

**Corollary.** The predual of a morphism  $f: A \rightarrow B$  of von Neumann algebras is also unique up to unique isomorphism and is canonically isomorphic to the dual of  $f$  in the ultraweak topology.

**Corollary.** The functor  $W^*C^*$  reflects isomorphisms.

**Remark.** As explained in the first part of this thesis, the predual of  $A$  is canonically isomorphic to  $L_1(A)$ .

**Notation.** Denote by  $L_1: W^{*\text{op}} \rightarrow \text{Ban}$  the functor that sends a von Neumann algebra  $A$  to its predual, i.e., the dual of  $A$  in the ultraweak topology, and likewise for morphisms.

**Remark.** The predual possesses additional algebraic structures that come from the respective algebraic structures on the original von Neumann algebra  $A$ . More precisely, the



predual of the unit  $\mathbf{k} \rightarrow A$  is the Haagerup trace  $\text{tr}: A_* \rightarrow \mathbf{k}$ , the predual of the involution  $*$ :  $A \rightarrow A$  is the modular conjugation  $*$ :  $A_* \rightarrow A_*$ , and the predual of the multiplication  $A \otimes_{\sigma\text{h}} A \rightarrow A$  is the comultiplication  $A_* \rightarrow A_* \otimes_{\text{eh}} A_*$ , where  $\otimes_{\sigma\text{h}}$  is the normal Haagerup tensor product and  $\otimes_{\text{eh}}$  is the extended Haagerup tensor product. See Section 2 in Effros and Ruan [45] for a discussion of these tensor products. If we refine the codomain of  $L_1$  to the category of involutive comonoids in the category of operator spaces with the involutive monoidal structure coming from the extended Haagerup tensor product, then the resulting functor is fully faithful, in particular it becomes an equivalence of categories if we restrict its codomain to its essential image.

**Theorem.** The category  $W^*$  is complete and the forgetful functor  $W^*C^*: W^* \rightarrow C^*$  preserves and reflects limits.

*Proof.* Suppose  $D: I \rightarrow W^*$  is a small diagram. Denote by  $A$  the limit of  $W^*C^*D$ . Denote by  $Z$  the colimit of  $L_1D$ . From the construction of limits in  $C^*$  it follows that  $A$  is a von Neumann algebra and  $Z$  is its predual. Now for any cone from  $B$  to  $D$  we have a canonical morphism  $A_* \rightarrow B_*$ . Dualizing this morphism we obtain a morphism of von Neumann algebras and their corresponding cones from  $B$  to  $A$ . This morphism is unique because  $A$  is the limit of  $W^*C^*D$ . Since  $W^*$  is complete and  $W^*C^*$  preserves limits and reflects isomorphisms,  $W^*C^*$  also reflects limits.

We now summarize the categorical properties of bimodules over von Neumann algebras.

**Theorem.** The category of von Neumann algebras and their isomorphisms together with the category of right  $L_0$ -bimodules and their morphisms forms a framed double category in the sense of Shulman [47].

*Proof.* Brouwer wrote up a full account of the bicategory of von Neumann algebras in [49]. By Theorem 4.1 in Shulman [47] the additional structure of a framed double category is given by assigning to every isomorphism of von Neumann algebras  $A \rightarrow B$  the  $A$ - $B$ -bimodule  $A$  and the  $B$ - $A$ -bimodule  $B$  together with the corresponding morphisms of bimodules. These are given by the identities and the isomorphism  $A \rightarrow B$ , which satisfy the relevant identities for trivial reasons.

## The categorical tensor product of von Neumann algebras.

The categorical tensor product was constructed in 1966 by Guichardet [6]. Unlike the spatial tensor product it has a nice universal property, which allows us to construct various kinds of bimodules below.

**Definition.** Suppose that  $f: A \rightarrow C$  and  $g: B \rightarrow C$  are morphisms of von Neumann algebras. We say that  $f$  and  $g$  *commute* if for all  $a \in A$  and  $b \in B$  we have  $f(a)g(b) = g(b)f(a)$ .

**Theorem.** For any von Neumann algebras  $A$  and  $B$  the following functor  $T: W^* \rightarrow \text{Set}$  is representable:  $T$  sends a von Neumann algebra  $E$  to the set of all pairs of commuting morphisms  $f: A \rightarrow E$  and  $g: B \rightarrow E$ . It sends a morphism  $h: E \rightarrow F$  of von Neumann algebras to the morphism of sets that sends a pair  $(f, g)$  to  $(hf, hg)$ . We denote the representing object by  $A \otimes B$  and call it the *categorical tensor product* of  $A$  and  $B$ .

*Proof.* First we prove that the functor  $T$  preserves limits. Consider a small diagram  $D: I \rightarrow W^*$  together with its limit  $E$  and the corresponding cone. To prove the universal property of limit for the image of this cone under  $T$  it is enough to consider one-element sets. A cone from a one-element set to  $TD$  is a compatible system of pairs of commuting morphisms, which can be interpreted as a pair of cones from  $A$  respectively  $B$  to  $D$ . By the universal property of limit we obtain a pair of morphisms from  $A$  respectively  $B$  to  $E$ . This pair commutes because composing this pair with all possible projections from the limit to individual objects of the diagram  $D$  gives a commuting pair of morphisms.

**Remark.** By the universal property of the categorical tensor product we have a canonical epimorphism  $A \otimes B \rightarrow A \bar{\otimes} B$  for any von Neumann algebras  $A$  and  $B$ , where  $\bar{\otimes}$  is the spatial tensor product of von Neumann algebras. (The universal map is an epimorphism because the spatial tensor product is generated by the images of  $A$  and  $B$ .) This epimorphism is an isomorphism if and only if  $A$  or  $B$  is type I with atomic center. See Lemme 8.2, Proposition 8.5, and Proposition 8.6 in Guichardet [6].

**Remark.** In the notation of the above theorem the element of  $T(A \otimes B)$  corresponding to the identity morphism of  $A \otimes B$  gives us a canonical pair of commuting morphisms  $i_{A,B}: A \rightarrow A \otimes B$  and  $j_{A,B}: B \rightarrow A \otimes B$ . Precomposing a morphism  $A \otimes B \rightarrow C$  with  $i_{A,B}$  or  $j_{A,B}$  allows one to extract the underlying pair of commuting morphisms  $A \rightarrow C$  and  $B \rightarrow C$ .

**Remark.** The universal property of the categorical tensor product implies that the categorical tensor product is generated by the images of  $i_{A,B}$  and  $j_{A,B}$ .

Now we embed the categorical tensor product into a symmetric monoidal structure on the category of von Neumann algebras. We identify the relevant data and properties in the following list:

- If  $f: A \rightarrow E$  and  $g: B \rightarrow F$  are morphisms of von Neumann algebras, then the commuting pair of morphisms  $(i_{E,F}f, j_{E,F}g)$  defines a morphism  $f \otimes g: A \otimes B \rightarrow E \otimes F$ .
- By the universal property of the categorical tensor product for any von Neumann algebras  $A$  and  $B$  we have  $\text{id}_A \otimes \text{id}_B = \text{id}_{A \otimes B}$ , because both morphisms come from the commuting pair of morphisms  $(i_{A,B}, j_{A,B})$ . Likewise, if  $e: A \rightarrow B$ ,  $f: B \rightarrow C$ ,

$g: E \rightarrow F$ , and  $h: F \rightarrow G$  are morphisms of von Neumann algebras, then  $fe \otimes hg = (f \otimes h)(e \otimes g)$ , because both morphisms come from the commuting pair of morphisms  $(i_{C,G}fe, j_{C,G}hg)$ .

- The field of scalars  $\mathbf{k}$  is the monoidal unit: For any von Neumann algebra  $A$  we have canonical isomorphisms  $\lambda_A: \mathbf{k} \otimes A \rightarrow A$  and  $\rho_A: A \otimes \mathbf{k} \rightarrow A$  (left and right unitor respectively). These isomorphisms are functorial:  $\lambda_B(\text{id}_{\mathbf{k}} \otimes f) = f\lambda_A$  and  $\rho_B(f \otimes \text{id}_{\mathbf{k}}) = f\rho_A$  for any morphism of von Neumann algebras  $f: A \rightarrow B$ .
- If  $A, B$ , and  $C$  are von Neumann algebras, then the commuting pair of morphisms  $(k, j_{A,B \otimes C}j_{B,C})$  defines a canonical isomorphism (the associator)  $\alpha_{A,B,C}: (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$ , whose inverse is constructed in a similar way. Here  $k: A \otimes B \rightarrow A \otimes (B \otimes C)$  is the morphism defined by the commuting pair  $(i_{A,B \otimes C}, j_{A,B \otimes C}i_{B,C})$ .
- The associator is functorial: For any morphisms of von Neumann algebras  $f: A \rightarrow E$ ,  $g: B \rightarrow F$ , and  $h: C \rightarrow G$  we have  $\alpha_{E,F,G}((f \otimes g) \otimes h) = (f \otimes (g \otimes h))\alpha_{A,B,C}$ , because the underlying triple of commuting morphisms is in both cases  $(i_{E,F \otimes G}f, j_{E,F \otimes G}i_{F,G}g, j_{E,F \otimes G}j_{F,G}h)$ .
- Unitors are compatible with the associator: For any von Neumann algebras  $A$  and  $B$  we have  $(\text{id}_A \otimes \lambda_B)\alpha_{A,\mathbf{k},B} = \rho_A \otimes \text{id}_B$ , because the underlying triple of commuting morphisms is in both cases  $(i_{A,B}, \mathbf{k} \rightarrow A \otimes B, j_{A,B})$ .
- The associator satisfies the pentagon identity: For any von Neumann algebras  $A, B, E$ , and  $F$  we have  $\alpha_{A,B,E \otimes F}\alpha_{A \otimes B,E,F} = (\text{id}_A \otimes \alpha_{B,E,F})\alpha_{A,B \otimes E,F}\alpha_{A,B,E \otimes F}$ , because the underlying 4-tuple of commuting morphisms is in both cases  $(i_{A,B \otimes (E \otimes F)}, j_{A,B \otimes (E \otimes F)}i_{B,E \otimes F}, j_{A,B \otimes (E \otimes F)}j_{B,E \otimes F}i_{E,F}, j_{A,B \otimes (E \otimes F)}j_{B,E \otimes F}j_{E,F})$ .
- If  $A$  and  $B$  are two von Neumann algebras, then the commuting pair of morphisms  $(j_{B,A}, i_{B,A})$  defines a canonical morphism (the braiding)  $\gamma_{A,B}: A \otimes B \rightarrow B \otimes A$ .
- The braiding is functorial: For any morphisms of von Neumann algebras  $f: A \rightarrow E$  and  $g: B \rightarrow F$  we have  $(g \otimes f)\gamma_{A,B} = \gamma_{E,F}(f \otimes g)$ , because the underlying pair of morphisms is in both cases  $(j_{F,E}f, i_{F,E}g)$ .
- The braiding is symmetric:  $\gamma_{B,A}\gamma_{A,B} = \text{id}_{A \otimes B}$ , because both morphisms are represented by the commuting pair  $(i_{A,B}, j_{A,B})$ .
- The braiding satisfies the hexagon identities:  $(\text{id}_B \otimes \gamma_{A,C})\alpha_{B,A,C}(\gamma_{A,B} \otimes \text{id}_C) = \alpha_{B,C,A}\gamma_{A,B \otimes C}\alpha_{A,B,C}$ , because both morphisms are represented by the commuting triple  $(j_{B,C \otimes A}j_{C,A}, i_{B,C \otimes A}, j_{B,C \otimes A}i_{C,A})$ . The other hexagon identity follows automatically because the braiding is symmetric.

The above constructions and proofs can be summarized as follows:

**Theorem.** The category of von Neumann algebras equipped with the additional structures described above is a symmetric monoidal category.

**The categorical external tensor product of  $W^*$ -modules and  $W^*$ -bimodules.**

Recall the following universal property of the  $W^*$ -category of  $W^*$ -modules over a von Neumann algebra (see Theorem 7.13 in Ghez, Lima, and Roberts [2]):

**Theorem.** The  $W^*$ -category of  $W^*$ -modules over a von Neumann algebra  $M$  is a free  $W^*$ -category with direct sums and sufficient subobjects on one object with endomorphism algebra  $M$ . More precisely, consider the  $W^*$ -category  $*_M$  with one object whose endomorphism algebra is  $M$ . Embed this category into the  $W^*$ -category  $\text{Mod}_M$  of  $W^*$ -modules over  $M$  by sending the only object to  $M_M$  and the endomorphisms of this object to the left action of  $M$  on  $M_M$ . Suppose  $C$  is a  $W^*$ -category with direct sums and sufficient subobjects. The restriction  $W^*$ -functor from the  $W^*$ -category of  $W^*$ -functors from  $\text{Mod}_M$  to  $C$  to the  $W^*$ -category of  $W^*$ -functors from  $*_M$  to  $C$  given by the embedding  $*_M \rightarrow \text{Mod}_M$  is a  $W^*$ -equivalence of  $W^*$ -categories.

The categorical external tensor product of  $W^*$ -modules will be a  $W^*$ -functor  $\text{Mod}_M \otimes \text{Mod}_N \rightarrow \text{Mod}_{M \otimes N}$ . First we extend the categorical tensor product of von Neumann algebras (i.e.,  $W^*$ -categories with one object) to arbitrary  $W^*$ -categories.

**Definition.** Suppose  $C$  and  $D$  are  $W^*$ -categories. The categorical tensor product  $C \otimes D$  is defined as follows: Objects of  $C \otimes D$  are pairs of objects of  $C$  and  $D$ . We denote the object corresponding to the pair  $(c, d)$  as  $c \otimes d$ . If  $a \otimes b$  and  $c \otimes d$  are two objects in  $C \otimes D$ , then  $\text{Hom}(a \otimes b, c \otimes d) = \text{Hom}(a, c) \otimes \text{Hom}(b, d)$ . Here we use the categorical tensor product of corners of von Neumann algebras, which is defined as follows. Consider the von Neumann algebra  $M$  of  $2 \times 2$ -matrices with entries in  $\text{End}(a)$ ,  $\text{Hom}(a, c)$ ,  $\text{Hom}(c, a)$ , and  $\text{End}(c)$ . We have  $\text{Hom}(a, c) = pMq$  for some projections  $p$  and  $q$  in  $M$  such that  $p + q = 1$ . Do the same trick with  $\text{Hom}(b, d) = rNs$ . Now  $\text{Hom}(a, c) \otimes \text{Hom}(b, d) := (p \otimes r)(M \otimes N)(q \otimes s)$ . The resulting category is a  $W^*$ -category because the algebra of  $2 \times 2$ -matrices corresponding to every pair of objects is a  $W^*$ -algebra. More precisely, in the above notation it is the algebra  $(p \otimes r + q \otimes s)(M \otimes N)(p \otimes r + q \otimes s)$ .

**Proposition.** If  $C$  is a  $W^*$ -category with a generator  $A$  and  $D$  is a  $W^*$ -category with a generator  $B$ , then  $A \otimes B$  is a generator of  $C \otimes D$ .

*Proof.* By Proposition 7.3 in Ghez, Lima, and Roberts [2] an object  $E$  of a  $W^*$ -category is a generator if and only for any object  $F$  there is a family of partial isometries  $r: K \rightarrow \text{Hom}(F, E)$  such that  $\text{id}_F = \sum(k \in K \mapsto r_k^* r_k \in \text{End}(F))$ . Consider an arbitrary object  $X \otimes Y$  of  $C \otimes D$ . Choose families of partial isometries  $p: I \rightarrow \text{Hom}(X, A)$  and  $q: J \rightarrow \text{Hom}(Y, B)$  that satisfy the above property. Then the family  $p \otimes q: (i, j) \in I \times J \mapsto p_i \otimes q_j \in \text{Hom}(X \otimes Y, A \otimes B)$  satisfies  $\sum(i, j) \in I \times J \mapsto (p_i \otimes q_j)^*(p_i \otimes q_j) \in \text{End}(X \otimes Y) = \sum(i, j) \in I \times J \mapsto (p_i^* p_i \otimes q_j^* q_j) \in \text{End}(X \otimes Y) = (\sum i \in I \mapsto p_i^* p_i \in \text{End}(X)) \otimes (\sum j \in J \mapsto q_j^* q_j \in \text{End}(Y)) = \text{id}_X \otimes \text{id}_Y = \text{id}_{X \otimes Y}$ .

**Theorem.** For any two von Neumann algebras  $M$  and  $N$  there is a unique  $W^*$ -functor  $\odot: \text{Mod}_M \otimes \text{Mod}_N \rightarrow \text{Mod}_{M \otimes N}$  that sends  $M_M \otimes N_N$  to  $(M \otimes N)_{M \otimes N}$  and maps the endomorphism algebra of  $M_M \otimes N_N$  isomorphically to  $M \otimes N$ . We call it the *categorical external tensor product functor*.

*Proof.* The  $W^*$ -category  $\text{Mod}_M \otimes \text{Mod}_N$  has a generator  $M_M \otimes N_N$  and the  $W^*$ -category  $\text{Mod}_{M \otimes N}$  has direct sums and sufficient subobjects, hence the conditions of the above

theorem are satisfied and there is an equivalence of  $W^*$ -categories  $\text{Hom}(\text{Mod}_M \otimes \text{Mod}_N, \text{Mod}_{M \otimes N}) \rightarrow \text{Hom}(M_M \odot N_N, \text{Mod}_{M \otimes N})$  given by the restriction to the full subcategory of  $\text{Mod}_M \otimes \text{Mod}_N$  consisting of  $M_M \odot N_N$ .

**Theorem.** The above functor extends to bimodules as follows:

$$\odot: {}_K\text{Bimod}_M \otimes {}_L\text{Bimod}_N \rightarrow {}_{K \otimes L}\text{Bimod}_{M \otimes N}.$$

*Proof.* An object of  ${}_K\text{Bimod}_M$  is an object  $X$  of  $\text{Mod}_M$  equipped with a morphism  $K \rightarrow \text{End}(X)$ . For  ${}_L\text{Bimod}_N$  we have a similar pair  $(Y, L \rightarrow \text{End}(Y))$ . We define  $(X, K \rightarrow \text{End}(X)) \odot (Y, L \rightarrow \text{End}(Y)) = (X \odot Y, K \otimes L \rightarrow \text{End}(X) \otimes \text{End}(Y) = \text{End}(X \odot Y))$ . A morphism of bimodules is a morphism of the underlying right modules that commutes with the left action. The external tensor product of morphisms of the underlying right  $M$ -modules and  $N$ -modules commutes with the left action of  $K \otimes L$ , hence we get a functor  $\odot: {}_K\text{Bimod}_M \otimes {}_L\text{Bimod}_N \rightarrow {}_{K \otimes L}\text{Bimod}_{M \otimes N}$ .

**Remark.** The external tensor product of two invertible bimodules is again an invertible bimodule.

**Definition.** Denote by  $\text{Bimod}$  the category whose objects are triples  $(M, N, X)$ , where  $M$  and  $N$  are von Neumann algebras and  $X$  is an  $M$ - $N$ -bimodule. Consider two arbitrary objects  ${}_K X_M$  and  ${}_L Y_N$ . A morphism from  $X$  to  $Y$  is a triple  $(f, g, h)$ , where  $f: K \rightarrow L$  and  $g: M \rightarrow N$  are isomorphisms of von Neumann algebras, and  $h$  is a morphism of  $K$ - $M$ -bimodules from  $X$  to  ${}_f Y_g$ , where  ${}_f Y_g$  denotes  $Y$  with the left action composed with  $f$ , the right action composed with  $g$ , and the inner product composed with  $g^{-1}$ . We extend the categorical external tensor product to  $\text{Bimod}$ .

Now we embed the categorical external tensor product into a symmetric monoidal structure on the category  $\text{Bimod}$  of bimodules over von Neumann algebras. For the sake of brevity we often establish necessary properties for the case of modules, automatically extending them to bimodules. We also work with morphisms of bimodules over the same pair of algebras, automatically extending them to other morphisms. We identify the relevant data and properties in the following list:

- The identity bimodule  $\text{id}_{\mathbf{k}}$  over the field of scalars  $\mathbf{k}$  is the monoidal unit: For any bimodule  $X$  we have canonical isomorphisms  $\lambda_X: \text{id}_{\mathbf{k}} \odot X \rightarrow X$  and  $\rho_X: X \odot \mathbf{k} \rightarrow X$  (left and right unitor respectively). These isomorphisms are functorial:  $\lambda_Y(\text{id}_{\mathbf{k}} \odot f) = f \lambda_X$  and  $\rho_Y(f \odot \text{id}_{\mathbf{k}}) = f \rho_X$  for any morphism of bimodules  $f: X \rightarrow Y$ .
- The associator is the unique natural isomorphism that satisfies the identity  $(L_L \odot M_M) \odot N_N = L_L \odot (M_M \odot N_N)$ . The inverse associator is constructed in a similar way.
- The associator is functorial: For any morphisms of bimodules  $f: U \rightarrow X$ ,  $g: V \rightarrow Y$ , and  $h: W \rightarrow Z$  we have  $\alpha_{X,Y,Z}((f \odot g) \odot h) = (f \odot (g \odot h)) \alpha_{U,V,W}$ , because  $(f \otimes g) \otimes h = f \otimes (g \otimes h)$  as an element of  $\text{Hom}(U, X) \otimes \text{Hom}(V, Y) \otimes \text{Hom}(W, Z) = \text{Hom}(U \odot V \odot W, X \odot Y \odot Z)$ .
- Unitors are compatible with the associator: For any bimodules  $X$  and  $Y$  we have  $(\text{id}_X \odot \lambda_Y) \alpha_{X,\mathbf{k},Y} = \rho_X \odot \text{id}_Y$ .

- The associator satisfies the pentagon identity: For any bimodules  $W$ ,  $X$ ,  $Y$ , and  $Z$  we have  $\alpha_{W,X,Y} \odot_Z \alpha_{W \odot X, Y, Z} = (\text{id}_W \odot \alpha_{X,Y,Z}) \alpha_{W,X \odot Y, Z} \alpha_{W,X,Y} \odot_Z$ .
- The braiding  $\gamma$  is defined as the unique morphism of the corresponding functors that sends  $M_M \odot N_N$  to  $N_N \odot M_M$ .
- The braiding is functorial: For any morphisms of bimodules  $f: U \rightarrow X$  and  $g: V \rightarrow Y$  we have  $(g \odot f) \gamma_{U,V} = \gamma_{X,Y} (f \odot g)$ .
- The braiding is symmetric:  $\gamma_{Y,X} \gamma_{X,Y} = \text{id}_{X \odot Y}$ .
- The braiding satisfies the hexagon identities:  $(\text{id}_Y \odot \gamma_{X,Z}) \alpha_{Y,X,Z} (\gamma_{X,Y} \odot \text{id}_Z) = \alpha_{Y,Z,X} \gamma_{X,Y} \odot_Z \alpha_{X,Y,Z}$ . The other hexagon identity follows automatically because the braiding is symmetric.

The above constructions and proofs can be summarized as follows:

**Theorem.** The category  $\text{Bimod}$  has a symmetric monoidal structure given by the structures defined above.

### Symmetric monoidal framed double category of von Neumann algebras.

**Theorem.** The symmetric monoidal category  $\hat{W}^*$  of von Neumann algebras and their isomorphisms together with the symmetric monoidal category of  $W^*$ -bimodules and their morphisms forms a (strong) symmetric monoidal framed double category.

*Proof.* We only have to show that the source, target, identity, and composition functors are strong symmetric monoidal functors and both unitors and the associator are symmetric monoidal natural isomorphisms. The source and target functors are symmetric monoidal by definition. The identity functor is symmetric monoidal because  ${}_M M_M \odot_N N_N = {}_{M \otimes N} M \otimes N_{M \otimes N}$ . Finally, the composition functor is symmetric monoidal because the categorical tensor product of von Neumann algebras is a functor.

For the following theorems it will be more convenient to use the category of density  $1/2$  bimodules instead of  $W^*$ -bimodules (density 0 bimodules). Recall that these two categories are equivalent via the algebraic tensor product and the algebraic inner hom with  $L_{1/2}$  as explained in the first part of this thesis.

**Theorem.** All von Neumann algebras are dualizable as objects of  $\hat{W}^*$ . For a von Neumann algebra  $M$  the dual algebra is  $M^{\text{op}}$ , the unit/evaluation  $u$  is given by  ${}_C L_{1/2}(M)_{M^{\text{op}} \otimes M}$  and the counit/coevaluation  $v$  is given by  ${}_{M \otimes M^{\text{op}}} L_{1/2}(M)_C$ .

*Proof.* We have to construct the two isomorphisms given by the usual triangle diagrams. Due to the symmetry between the unit and the counit it is enough to construct just one of the triangle isomorphisms, for example,  $(\text{id}_M \odot u) \otimes (v \odot \text{id}_M) \rightarrow \text{id}_M$ .

Denote by  $p$  the support of the right action of  $M^{\text{op}} \otimes M$  on  $u$ , which is a central projection in  $M^{\text{op}} \otimes M$ . In the commutative case,  $p$  can be described geometrically as the projection corresponding to the diagonal of the product of measurable spaces corresponding to  $M^{\text{op}}$  and  $M$ . The support of the right action of  $M \otimes M^{\text{op}} \otimes M$  on  $\text{id}_M \odot u$  is  $1 \otimes p$ . Similarly, the support of the left action of  $M \otimes M^{\text{op}} \otimes M$  on  $v \odot \text{id}_M$  is  $p \otimes 1$ . When we compute the tensor product, we can reduce both actions by the product of these two

projections, which is  $q = (1 \otimes p)(p \otimes 1)$ . In the commutative case  $q$  is simply the projection on the diagonal in the product of measurable spaces corresponding to  $M$ ,  $M^{\text{op}}$ , and  $M$ .

The bimodule  $\text{id}_M \odot u$  with the right action reduced by  $q$  is invertible, which can be seen as follows: The bimodule  $\text{id}_M$  is invertible and  $u$  becomes invertible once we make  $Z(M)$  act on the left and reduce the right action by its kernel. The external tensor product of the above modified bimodules is thus also invertible and the bimodule  $\text{id}_M \odot u$  with the reduced right action can be obtained from this bimodule by replacing the left action of  $M \otimes Z(M)$  by the action of  $M$  alone and reducing the right action by  $q$ . Thus the commutant of the right action of  $\text{id}_M \odot u$  is  $M \otimes Z(M)$ . When we reduce the right action by  $q$  the algebras  $M$  and  $Z(M)$  now act identically, hence the commutant is contained in  $M$  and therefore equals  $M$ .

Thus  $\text{id}_M \odot u$  and  $v \odot \text{id}_M$  both become invertible when we reduce the actions in the middle by  $q$ . Due to the symmetry the bimodules above are contragredient to each other, hence their tensor product is canonically isomorphic to  $\text{id}_M$ .

The definition of dualizable objects in higher categories does not require us to prove that the triangle isomorphisms satisfy higher coherence conditions (even though they do), hence the proof is complete.

**Remark.** Consider the bicategory of von Neumann algebras, bimodules, and intertwiners equipped with the symmetric monoidal structure coming from the spatial tensor product of von Neumann algebras and the spatial external tensor product of bimodules. Dualizable objects in this bicategory are type I von Neumann algebras with atomic center. This explains the choice of categorical tensor products for the monoidal structure.

### Shadows and refinement of the Jones index.

Recall that 1-endomorphisms of dualizable objects in symmetric monoidal bicategories have *shadows*, which categorify the usual traces of endomorphisms of dualizable objects in symmetric monoidal category and are defined in exactly the same way. Shadows allow us to define traces of endomorphisms of dualizable 1-morphisms. See the paper by Ponto and Shulman [46] for details.

**Theorem.** In the categorical symmetric monoidal structure on the bicategory of von Neumann algebras the shadow of the identity bimodule over a von Neumann algebra  $M$  is canonically isomorphic to  $L_{1/2}(Z(M))$  as a  $\mathbf{C}$ - $\mathbf{C}$ -bimodule.

**Remark.** All definitions depend only on the underlying symmetric monoidal category of bimodules, hence the end result is independent of the choice of a particular model of bimodules. In particular, the space  $L_{1/2}(Z(M))$  pops up even in the case of density 0 bimodules.

*Proof.* By definition the shadow under consideration is  ${}_{\mathbf{C}}L_{1/2}(M)_{M \otimes M^{\text{op}}} \otimes_{M \otimes M^{\text{op}}} L_{1/2}(M)_{\mathbf{C}}$ . Observe that  ${}_{\mathbf{C}}L_{1/2}(M)_{M \otimes M^{\text{op}}} = {}_{\mathbf{C}}L_{1/2}(Z(M))_{Z(M)} \otimes_{Z(M)} L_{1/2}(M)_{M \otimes M^{\text{op}}}$  and similarly for  ${}_{M \otimes M^{\text{op}}}L_{1/2}(M)_{\mathbf{C}}$ . Thus  ${}_{\mathbf{C}}L_{1/2}(M)_{M \otimes M^{\text{op}}} \otimes_{M \otimes M^{\text{op}}} L_{1/2}(M)_{\mathbf{C}} = {}_{\mathbf{C}}L_{1/2}(Z(M))_{Z(M)} \otimes_{Z(M)} L_{1/2}(M)_{M \otimes M^{\text{op}}} \otimes_{M \otimes M^{\text{op}}} L_{1/2}(M)_{Z(M)} \otimes_{Z(M)} L_{1/2}(Z(M))_{\mathbf{C}}$ . The first and the last bimodule simply adjust the left and the right action from  $Z(M)$  to  $\mathbf{C}$ . The two bimodules in the middle are almost invertible, except that the actions of  $M \otimes M^{\text{op}}$  are not faithful.

However, the non-faithful parts of both actions are the same and we can simply throw them away, obtaining a pair of invertible bimodules, which are the inverses of each other. Therefore their tensor product is canonically isomorphic to  ${}_{Z(M)}L_{1/2}(Z(M))_{Z(M)}$ . Thus the entire tensor product is canonically isomorphic to the **C-C**-bimodule  $L_{1/2}(Z(M))$ .

**Remark.** The shadow of an arbitrary  $M$ - $M$ -bimodule  $X$  is canonically isomorphic to the **C-C**-bimodule  $X/[X, M]$  (here  $X$  must have density  $1/2$  and  $[X, M]$  denotes the closure of the linear span of all commutators of the form  $xm - mx$ ). Connes in [48] showed that every  $M$ - $M$ -bimodule  $X$  (of density  $1/2$ ) canonically decomposes in a direct sum  $Z(X) \oplus [X, M]$ , where  $Z(X)$  is the set of all elements in  $X$  such that for all  $m \in M$  we have  $xm = mx$ . Thus the shadow of  $X$  is canonically isomorphic to  $Z(X)$  as a **C-C**-bimodule, i.e., a Hilbert space.

Recall that dualizable bimodules in  $\hat{W}^*$  are precisely finite index bimodules.

**Theorem.** Suppose  $M$  is a type  $II_1$  factor and  $N$  is a finite index subfactor of  $M$ . Denote by  $X$  the associated  $M$ - $N$ -bimodule, which is  $L_{1/2}(M)$  equipped with the standard left action of  $M$  and the right action of  $N$  coming from the inclusion of  $N$  into  $M$ . The trace of the identity endomorphism of  $X$  is equal to the Jones index of  $X$ .

*Proof.* We use the language of density 0 bimodules. Identify (the density 0 counterpart of)  $X$  with  $M$  as an  $M$ - $N$ -bimodule, where the right inner product is given by the canonical conditional expectation associated to the morphism  $N \rightarrow M$ . Choose a Pimsner-Popa basis  $R$  for  $X$ . The trace of  $\text{id}_X$  is the composition of the shadow of the coevaluation map of  $X$ , the cyclic morphism, and the shadow of the evaluation map. We identify shadows of bimodules with their central elements. The coevaluation map sends  $1 \in \text{id}_N$  to  $\sum_{r \in R} r^* \otimes r$ . The cyclic morphism sends this element to  $\sum_{r \in R} r \otimes r^*$  and the evaluation maps sends it to  $\sum_{r \in R} rr^*$ , which is the Jones index of the inclusion  $N \rightarrow M$ . See Théorème 3.5 in Baillet, Denizeau, and Havet [4] for the relevant facts about index.

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