UC San Diego UC San Diego Electronic Theses and Dissertations

Title Informed Tuning Practice: A Practical Approach for Contrabass

Permalink https://escholarship.org/uc/item/49x9f3wc

Author Goodwin, Adam Nicholas

Publication Date 2012

Supplemental Material https://escholarship.org/uc/item/49x9f3wc#supplemental

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Informed Tuning Practice: A Practical Approach for Contrabassists

A thesis submitted in partial satisfaction of the requirements for the degree Master of Arts

in

Music

by

Adam Nicholas Goodwin

Committee in charge:

Professor Mark Dresser, Chair

Professor John Fonville

Professor Anthony Davis

The Thesis of Adam Nicholas Goodwin is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego 2012

TABLE OF CONTENTS

Signature Page	iii
Table of Contents	iv
List of Figures	v
List of Musical Examples	vi
List of Supplemental Materials	vii
Abstract	viii
Informed Tuning Practice: A Practical Approach for Contrabass	1
Creative Project #1 – Molten Lava Eyeball Fiend: "Junkyard Murder"	24
Creative Project #2 – Low Art Recital / "Nausea"	26
References	30

Recordings on File at Mandeville Special Collections Library

LIST OF FIGURES

Figure 1: Math of Intervals	4
Figure 2: Overtone Series of Open G String	6
Figure 3: List of Ratios and Intervals	6
Figure 4: Math of Transposed Major Third	7
Figure 5: Beating Tones	10
Figure 6: Sympathetic Vibrations	12
Figure 7: Comparison of Different Sounding "G's"	13

LIST OF MUSICAL EXAMPLES

Example 1: Interval Training	16
Example 2: Pitch Matching Exercise	17
Example 3: Interval Exercise Based on Harmonic Series of the D String	. 18
Example 4: Simple Intervals – "Beating and Stillness"	19
Example 5: Tuning Exercise	20
Example 6: Tuning Exercise for Singing Bassist	22

LIST OF SUPPLEMENTAL MATERIALS

Track 01: <01_ANIMALHELL.aif>

Track 02: <02_MERCURYDUST.aif>

Track 03: <03_BOVINESPONGIFORMPROTEIN.aif>

Track 04: <04_JUNKYARDMURDER.aif>

Track 05: <05_mortonsteamer.aif>

ABSTRACT OF THE THESIS

Informed Tuning Practice: A Practical Approach for Contrabassists

by

Adam Nicholas Goodwin

Master of Arts in Music

University of California, San Diego, 2012

Professor Mark Dresser, Chair

This paper seeks to provide a practical approach to understanding and executing intonation for the developing or advanced double bassist. Many of the ideas illustrated here are applicable to any player of a non-tempered instrument, but specific exercises will be presented within the context of the contrabass.

"Intonation" is a frequently mentioned concept within standard classical music training, yet many musicians have never learned the basic principles behind the tuning systems on which they rely daily. Instead, "perfect intonation" is a vague ideal, a goal which is constantly sought but never quite reached, namely because the musicians themselves are not truly aware of what they are pursuing.

In preparation for this paper, I managed to interview a small number of musicians about the way in which they learned to play in tune. Out of the interviews I conducted, I reached the conclusion that this topic is not thoroughly addressed in many educational contexts.

The following paper represents an effort to elucidate this complex topic by exploring and comparing the tuning systems of "just intonation" and "equal temperament," focusing on a tangible approach to playing just intervals on the contrabass through awareness of the overtone series and the natural harmonics of the open string.

ix

Informed Tuning Practice: A Practical Approach for Contrabass

by Adam Goodwin

This paper seeks to provide a practical approach to understanding and executing intonation for the developing contrabassist. Many of the ideas illustrated here are applicable to any string player, but specific exercises will be presented within the context of the contrabass.

"Intonation" is a frequently mentioned concept within standard classical music training, defined as "...the skillful ability of playing in tune" (Kopiez 383-84). Instrumentalists constantly struggle to achieve perfect intonation, yet it is a topic that is frequently neglected in terms of understanding and education. As such, this aspect of music making appears cryptic and mysterious, even to many highly trained professional musicians. Perfect intonation is viewed as a vague ideal, a goal which is constantly sought but never quite reached, namely because the musicians themselves are unaware of what they are pursuing. The following is an effort to elucidate this complex topic by exploring and comparing the tuning systems of "just intonation" and "equal temperament," focusing on a tangible approach to playing just intervals on the contrabass through awareness of the overtone series and the natural harmonics of the open strings.

1

In most of my educational upbringing as a classical musician, intonation has been addressed as an element of music making that is either performed correctly or incorrectly. The topic is usually taught by providing aural examples to be learned and repeated, frequently without any further explanation. Throughout the process of exploring intonation, however, I have realized that tuning can be approached from many different angles and represents a vast continuum of possibilities. In an effort to determine whether my educational experience was unique or representative of a larger issue, I interviewed a handful of musicians with diverse educational backgrounds and amounts of musical training about their experiences in the realm of intonation education. Out of the musicians I spoke with, everyone echoed the same sentiment - that "being in tune" is rarely taught with more focus than the simple dichotomy of right and wrong (Balliet, Ensey, Gilbert, Ro). As such, I have realized that this area of study is often sorely neglected from an educational standpoint and hope to clarify the topic for potential teachers and students alike.

INTONATION SYSTEMS

The system of intonation most widely used and taught today is known as equal temperament. Within this system, each octave is divided into twelve equal parts, enabling the performer to play in any key, or out of a key, without

2

any noticeable difference of intervallic relationships. In other words, every chromatic half step is the same distance apart. This system was considered a novel invention "...more than 250 years ago [when] Johann Sebastian Bach composed 'Das wohl temperirte Clavier' (the well tempered piano) to celebrate an achievement combining music, mathematics, and science. Finally, a method of tuning musical instruments had been devised which allowed playing pieces in all 12 major and all 12 minor scales on the same instrument without retuning" (Neuwirth 79). This is the intonation system used to tune most instruments today, albeit with slight fluctuations, such as the piano, vibraphone, guitar, and essentially every other modern instrument with immutable pitches. To learn intervals in equal temperament, performers must train their ears with a piano or chromatic tuner, teaching themselves the sounds of the different intervals. In Kopiez' study analyzing instrumentalists' tuning abilities, he mentions that "the surprisingly successful adaptation to the 'unnatural' ET system shows that deliberate practice is required to obtain task adaptation" (Kopiez 408). Performers can learn the sound of these intervals successfully without any need to understand the mathematical derivation of the pitches, but the math behind this tuning system is quite complex. Regardless, this system is guite useful for playing atonal music or modulating guickly between different keys, and sounds relatively agreeable due to its prominence in most forms of current popular music.

Another system of intonation that has been explored in a variety of historical and contemporary contexts is known as just intonation. While equal temperament derives its pitches by equally dividing the octave, in "...just-intonation (pure) tuning, pitches are given as fractions, which are actually ratios between the named pitch and a constant fundamental" (Gann). These numbers correspond to the harmonic series of a string, and "the smaller the number in an interval's ratio, the more consonant (sweet-sounding) it is, and the more useful it is for purposes of musical intelligibility...The most consonant interval besides the unison (1/1) is the octave (2/1), next the perfect fifth (3/2)" (Gann). So, because the relationship between two notes an octave apart is 2:1, the higher octave is double the frequency of the first. Similarly, a perfect fifth is represented by the ratio 3:2, because one and a half times the frequency of any given pitch produces a fifth.



Figure 1 – Math of Intervals

Intervals played in just intonation have a very different sonic effect than equal tempered intervals, resulting in a "greatly heightened purity and clarity of sound for two reasons: it eliminates acoustic beats to the maximum possible, and second, it exploits resonance by utilizing harmonically simple combinations of pitches" (Fonville 107). The concepts of acoustic beats and simple, pure intervals will be discussed in more detail in the following paragraphs.

HARMONIC SERIES AND RATIOS

An overtone is defined by the Grove Dictionary of Music as "one of the frequency components of a sound other than that of lowest frequency. Usually overtones are numbered consecutively in ascending order of frequency" (Bain 2). On a string the overtones exists in consecutive whole numbers, creating a sequence of pitches known as the "harmonic series." These numbers correspond with the physical properties of a string, which "vibrates at whole number multiples of a fundamental frequency. One note on the bass is not solely one pitch but rather a constellation of harmonic partials vibrating at different amplitudes" (Dresser). These nodes are where natural harmonics exist on a string, and "the partial number also translates into an interval relationship with the fundamental. They are all multiple frequencies of the fundamental" (Dresser). The ratio of the partial numbers represents the fraction of the string that is vibrating, while the inverse ratio of the partials represents the mathematical relationship of the frequencies. Due to the fact

that the harmonic series of a string is comprised of whole number ratios, the intervals represented by the harmonics are in just intonation. The distance between the fourth and fifth partial of the G string (a sounding G and B, respectively) is a just major third, or 5:4, because the frequency of the B is 5/4's as much as the G. The following shows the harmonic series of the open G string, followed by a list of the most basic just intervals and the ratios associated with them.



Figure 2 - Overtone Series of Open G String (up to 13th partial)

1/1 - Unison2/1 - Octave $3/2 - \text{Perfect 5}^{\text{th}}$ $4/3 - \text{Perfect 4}^{\text{th}}$ $5/4 - \text{Major 3}^{\text{rd}}$ $6/5 - \text{Minor 3}^{\text{rd}}$

Figure 3 – List of Ratios and Intervals

The ratios represented by the harmonics on the open strings hold true regardless of the given fundamental. A 5:4 will always represent the same interval, whether the fundamental pitch is an open string or not.



Figure 4 – Math of Transposed Major Third

Instead of simply memorizing and repeating intervals, in just intonation the acoustical model of the string can be used to hear and confirm pure intervals.

DIFFERENCES AND PROBLEMS

One noticeable difference between these two tuning systems occurs in the physical manner that sound waves of different frequencies interact with each other. Just intervals are understood as consonant, due to "the well defined periodicity of the resultant wave, a clear consequence of the condition of consonance" (Goto 11). In other words, the sound waves of consonant intervals eventually align, because the frequencies are related by a whole number fraction. This whole number relationship is something that we can hear and feel, and is thus a useful tool for tuning pure intervals. The intervals in equal temperament, however, excluding the octave, are founded on irrational ratios derived from the twelfth root of 2. This means that the waves never line up with each other, and "dissonance is characterized by the absence of periodicity" (Goto 11). The phenomenon that accompanies these irrational relationships is known as beating, and does not occur when playing in just intonation. Beating tones are experienced as a buzz or pulsing sensation, and learning to avoid them can be an invaluable tool in tuning perfect intervals, as will be addressed in more detail later.

While certain ensembles may play exclusively in one or the other of these systems, a fair amount of conflict exists regarding how to tune certain intervals, especially within large music ensembles. A study analyzing the tendencies of a capella quartets "made use of an equal or a non-equal tempered tuning system, indicated that the latter was the preference, and that singers tended to a form of just intonation for their tuning" (Howard). In my personal observations in orchestras and chamber ensembles, the natural tendency of most string players when playing tonal music seems to be to tune thirds, fourths, fifths, and sixths justly, while more or less maintaining equal tempered minor sevenths, tri-tones, and seconds. As such, problems can arise in the case of a piano quintet, for example, where the string players must conform to equal tempered intervals in order to match the sonority of the piano. In the context of a traditional string quartet, however, the players would most likely play any piece of tonal music using just thirds, fourths, fifths, and sixths. Similarly, in the orchestra it is common for string players to tune these same just intervals, but not just minor sevenths or seconds. Thus, if a given musician does not know his or her role in the context of a specific chord or piece, or how to tune the interval within that situation, the result might be quite cacophonous against the rest of the section or ensemble. I would argue that many professional orchestras play in a mix of equal temperament and just intonation, unconsciously adjusting intervals depending on the context of the pitch they are playing. With a greater understanding of how and when to use these different systems, the unity of ensemble playing could be greatly enhanced.

Knowing how to tune and hear just intervals can be a great benefit for every musician, as it facilitates deep listening and a greater understanding of natural acoustics. Simple just intervals can be used in a variety of contexts and are much more historically accurate in many tonal situations than equal temperament (Balliet). Similarly, the sound of a string section playing just chords blends together beautifully due to the absence of beating and the connection to the natural acoustics of the strings. There is no correct way to tune, but I believe that intonation should be consciously considered depending on historical and harmonic context, rather than fixing each pitch as an immobile landmark.

BEATING TONES

One way of knowing that an interval is perfectly in tune is the absence of beating tones. A beating tone, as mentioned previously, is an acoustic phenomenon that occurs when two pitches are irrationally related to each other (Goto 11). The resulting beating tone is the difference of frequency between the two pitches, and is experienced as a physical pulsing sensation, which speeds up or slows down depending on the interval. This is especially noticeable when two pitches are extremely close to one another, as the difference of the two frequencies is below the threshold of human hearing, creating the stated physical experience. When tuning a unison interval, one can use beating tones to truly feel the difference between "in" and "out" of tune. To experience beating tones, simply bow a double stop with an open G and a unison G on the D string, and then very slowly glissando down to an F on the D string.



Figure 5 – Beating Tones

You should notice a strong pulsing sensation upon moving away from the unison interval. The pulsing will speed up as you move further away from G, until it is so fast that you cannot discern the individual pulses. To better feel this, you can press your torso against the body of the instrument or place your head against the back of the neck while playing the double stops. The vibrations can also be observed by watching the strings. When the G is perfectly in tune, the pulsing stops, and one can physically feel the sound waves lock in place.

The main point here is to use the vibrations coming through the strings, the bow, the instrument, the floor, the room, the air, and your body, as a tool for tuning perfect intervals and fully experiencing your sound.

SYMPATHETIC VIBRATIONS

One more occurrence that can be used for tuning purposes is that of "sympathetic vibrations." The idea is that any given fundamental will cause other pitches within its overtone series to vibrate. This can be experienced by playing a low G that is perfectly in tune with the open G string. When the low G is perfectly in tune the G string will start vibrating.

This also works the other way around. By bowing a higher partial of an open string on an adjacent string, the fundamental and corresponding overtone may be excited in the open string. For instance, if an F# is bowed on the G string, the fifth partial of the D string will ring, and one can both see the D string vibrate and hear the resulting overtone.



Figure 6 – Sympathetic Vibrations

HARMONICS

Due to the extremely long string length of a double bass string, bass players are for once at a great advantage regarding the possibility of playing harmonics, and thus experiencing just intonation in a very precise manner. It has long been in fashion for composers to write bass pieces using much higher partials than they would for a violin, for instance, and many "standard" bass concertos were originally played using mainly open strings and harmonics (ie. the Dittersrdorf Concerto). Similarly, it is not so uncommon to see simple melodies that are written to be played as all natural harmonics in beginning bass books.

While harmonics offer vast musical opportunities on their own, they also offer great potential for learning to tune perfect intervals. The harmonics represent the pitches of each open string's harmonic series, and therefore exhibit perfect just intervals in relation to the string. This information is quite useful, as "knowing the multiple locations of the harmonics...is an entryway to harmony, timbre potentials, intonation, and an approach to microtonality" (Dresser 1). The pitch of each harmonic exists in direct relationship to the open string, meaning that two notes that are notated as the same pitch may actually vary significantly, depending on the partial number of the harmonic and its fundamental. For example, the seventh partial of the A string, which is a sounding G, will create a very different pitch than a sounding G harmonic on the G string.



Figure 7 – Comparison of Different Sounding "G's"

This can prove to be problematic when playing in an orchestral setting and trying to tune harmonics with stopped pitches, but it also provides a wealth of potential for string players in regards to understanding and improving intonation. While an instrument such as the piano is tuned more or less in equal temperament, "musical instruments whose pitch control is not fixed to specific notes, such as non-fretted stringed instruments and the human voice, have the freedom to vary the intonation of each note individually" (Howard). At first this approach to intonation can be quite daunting, but it actually gives string players and especially bassists an incredible advantage in learning to tune, and "...in the long run it gives us a comprehensive understanding of where to place our fingers and bow as well as an expectation of the resultant pitches and timbres" (Dresser 2). These pitches can then be matched to stopped pitches in the context of double stops or ensemble playing and serve as flawless guides to realizing just intervals. The intervals can be transposed and played based on fundamental, using the intervallic relationships of the overtones as a guide. This approach can help bassists develop informed tuning abilities with a built in reference in the strings.

By combining and informed, intellectual approach to tuning with an awareness of the natural acoustics of strings, string players can develop an approach to intonation which exhibits incredible accuracy and eliminates the arbitrary placement of pitches found in so many musical contexts today. Specific decisions regarding intonation must be made by the individual musician, but hopefully this paper, along with the following musical exercises, will help to clarify the complex world of string intonation.

MUSICAL EXAMPLES

The following will present exercises designed to encourage physical awareness while playing, as well as approaches to tuning simple just intervals that can be used in many traditional musical settings, such as orchestras, bands, choirs, and chamber ensembles. Compositions, etudes, and visual examples are presented as tools for contrabassists to explore just intonation through observation of natural harmonics and acoustical properties of the vibrating string. This exercise is designed to demonstrate just thirds, fourths, and fifths. The interval is first demonstrated with harmonics, which are then matched to stopped pitches, resulting in perfect just intervals.





The following exercise explores different partials of the A and E strings. Stopped pitches should be tuned to the preceding harmonic or open string.

Example 2 – Pitch Matching Exercise



This etude explores the harmonic series of the D string. It should be played extremely slowly, always taking time to listen to the resonance of the strings.

Example 3 – Interval Exercise Based on Harmonic Series of the D String

Slow and Free (at your own tempo) sul pont. norm sul pont. norm I I MPI I I 7 T noim. sul pont. Pπ Pπ I π 1 I I L I norm. sul pont. sul pont. ΠЦ P≖ mp I π 1 9 Ξ I П I τ μ I 11 π π , Ŧ I ... mp I - --- - - - m- $\mathbf{\hat{}}$ Ş I

The only intervals that should beat in this exercise are the occasional tri-tone or major 7th.

Example 4 – Simple Intervals – "Beating and Stillness"



The top line in this exercise is always left hand pizzicato and the bottom line is always arco. Every interval should be played in just intonation, meaning that no intervals will beat.

Example 5 – Tuning Exercise



Tuning exercise, continued



The top line of this piece is sung (by the bassist), and should be tuned to the preceding pitch played by the bass. In turn, the second pitch played by the bass in each measure should be tuned to the sung pitch. Some chords may beat, but each successive interval should be tuned justly.





Tuning Exercise for Singing Bassist, continued



CREATIVE PROJECT #1 - MOLTEN LAVA EYEBALL FIEND - "JUNKYARD MURDER"

(audio recording)

Adam Nicholas Goodwin

MOLTEN LAVA EYEBALL FIEND is a collaboration between myself and sound artist Joe Mariglio. We have been performing regularly around San Diego for the last two years, and have played a number of shows in California, Texas, and New Mexico. MOLTEN LAVA EYEBALL FIEND generally performs long-form improvisations realized on a vast assortment of acoustic and electronic instruments, most of which are either found or homemade. This ranges from analog synthesizers to suspended, amplified metal, as well as sound sculptures with imbedded speakers, pickups, and amplifiers. I play amplified double bass in the group, but also function as a percussionist, vocalist, and instrument builder.

"JUNKYARD MURDER" was recorded this past year at our apartment, which has a full recording and mixing setup. The tracks were edited out of two much longer recordings, mixed, and mastered at the apartment. The album features six tracks, all performed on our collection of metal and homemade instruments along with double bass. The second track, "Mercury Dust," heavily features the bass, although the nature of this group's sound tends to obfuscate

24

the aural distinction of individual instruments. The fifth track, "Mortonsteamer," focuses heavily on feedback loops created through the large metal plates attached to our homemade sound sculpture.

This recording represents an aesthetic palette that has been developed through continuous collaboration, as well as an approach to sound exploration that exists outside of the necessity for expensive equipment and resources. Instead, we seek to create high quality music as cheaply as possible, always maintaining a positive attitude towards the sharing of ideas, recordings and merchandise with the general population and fellow artists.

CREATIVE PROJECT #2 - LOW ART RECITAL / "NAUSEA"

Adam Nicholas Goodwin

I presented my Low Art Recital in March of this year, featuring the world premiere of my composition, "Nausea," performed by myself and five colleagues from the UCSD music department. The piece features acoustic and electronic sections and alternates between through-composed material and improvisation. The concert took place about a month before my "High Art Recital," which featured well-known compositions from the contemporary solo bass literature. The following will briefly discuss the musical concepts behind "Nausea" and the Low Art Recital.

As an improviser "trained" in the Western classical tradition, I am constantly searching for ways to explore the boundary between composition and improvisation in both my writing and performance practice. I am of the opinion that every live performance involves an element of improvisation, but the balance between control and spontaneity must be constantly shifted depending on the context of the piece and performance.

In "Nausea," I experiment with various approaches to improvisation, including text instructions, traditional notation, and minimal graphics. The

26

overall form of the piece is dictated by a score, but sections alternate between boxes with improvisational guidelines and traditionally notated, composed material. Transitions between sections depend heavily on aural and sometimes physical cues, necessitating deep listening and communication between the performers. Furthermore, the role of both the conductor and the composer are questioned, creating a greater potential for spontaneous communication and exchange of ideas during performance.

The overall arc of the piece is a progression from control and restraint to freedom and self-organization. The first portion is entirely acoustic and largely composed, but upon the entrance of electronics the music gradually shifts into free improvisation for an unspecified duration.

"Nausea" opens with a brief improvisation performed solely on metal objects, which are both bowed and struck. This moves fluidly into ten minutes of composed material, beginning with the same instrumentation and eventually moving into a duo between bowed metal and double bass. The bassist has a choice of four harmonics (one on each string) and two low stopped pitches, which are the only pitches used throughout the duo. Similarly, the percussionist chooses six metal objects with different pitches to be used for this portion.

The stillness of the opening ritual is broken by an aggressive violin solo, written as an irreverent homage to neo-complexity. This sort of violent

27

interruption recurs periodically throughout the piece, somewhat foreshadowing the ultimate release into freedom and spontaneity at the end of the performance, as well as representing my personal struggle to balance so many diverse interests and seemingly contradictory ideas.

The violin solo transitions into a composed trio with brief sections of improvisation for violin, voice, and double bass. The trio is an exploration in the world of intonation and color, utilizing just intervals along with more dissonant chords and textures. Every interval in the piece must be tuned by listening to the other players, but the resulting chords exhibit various levels of tension and perceived interference. All three performers in this section had prior experience with extended just intonation, which enabled a beautiful realization of the trio.

While the trio is happening, the sousaphone and bass clarinet enter from the back of the audience, instructed only to improvise with "non-pitched" sounds. At the end of the trio, all five performers hit one fortissimo chord together, which leads into the next section. From this point on, the piece becomes much more improvised and relies much less on the score than on listening and spontaneous communication.

The improvisatory sections are mainly designated with text descriptions, such as "sparse, quiet improvisation." Periodically, percussion will make a forceful entrance, either with an improvised solo or traditionally notated rhythmic gestures, which serve as cues to the rest of the improvising ensemble. At the end of the piece, no instructions are given to the performers other than "ad libitum."

I was quite satisfied with the performance of the piece, and feel that all of the involved musicians contributed unique and personal musical ideas that could never be repeated by any other group of performers. In addition, I noticed a developing sense of freedom and a loss of general inhibitions throughout the 70 minute performance amongst some of the musicians with less experience improvising. As such, this piece of music was created by all six musicians, rather than existing solely as a reflection of the composer.

REFERENCES

Bain, Reginald. "The Harmonic Series. A Path to Understanding Musical Intervals, Scales, Tuning and Timbre". 2003.

Balliet, Doug. Personal interview. 25 May 2012.

Dresser, Mark. "GUTS Demonstration transcription".

Dresser, Mark. Personal interview. 2012.

Ensey, Paul. Personal interview. 10 May 2012.

Fonville, John. "Ben Johnston's Extended Just Intonation: A Guide for Interpreters". Perspectives of New Music 1991: Vol. 29, No. 2.

Fonville, John. "Changing Scales: Seven Voice Pitch Collections Using Seventh, Eleventh and Thirteenth Partial Chroma".

Fonville, John. Personal interview. 28 April 2012.

Gann, Kyle. "Just Intonation Explained". 1997.

Gilbert, Liz. Personal interview. 07 May 2012.

Goto, Mario. "Physical Consonance Law of Sound Waves". arXiv 2004.

- Howard, David M. "Equal or non-equal temperament in a capella SATB singing". Logopedics Phoniatrics Vocology 2007: Vol. 32, No. 2.
- Kopiez, Reinhard. "Intonation of Harmonic Intervals: Adaptability of Expert Musicians to Equal Temperament and Just Intonation". Music Perception Summer 2003: Vol. 20, No. 4. 383-410.
- Neuwirth, Erich. "The Mathematics of Tuning Musical Instruments a Simple Toolkit for Experiments". Mathematics and Art: Mathematical Visualization in Art and Education. Ed. Paul Claude Bruter. Springer, 2002.
- Page, Michael F. "Perfect harmony: a mathematical analysis of four historical tunings". Journal of the Acoustical Society of America 2004. Volume: 116, Issue: 4 Pt 1. ASA.
- Ro, Sophia. Personal interview. 28 January 2012.