Title
Ping-pong in Hadamard manifolds

Journal
MUENSTER JOURNAL OF MATHEMATICS, 12(2)

ISSN
1867-5778

Authors
Dey, Subhadip
Kapovich, Michael
Liu, Beibei

Publication Date
2019

DOI
10.17879/53149722954

Peer reviewed

PING-PONG IN HADAMARD MANIFOLDS

SUBHADIP DEY, MICHAEL KAPOVICH, AND BEIBEI LIU

ABSTRACT. In this paper, we prove a quantitative version of the Tits alternative for negatively pinched manifolds X. Precisely, we prove that a nonelementary discrete isometry subgroup of $\mathrm{Isom}(X)$ generated by two non-elliptic isometries g, f contains a free subgroup of rank 2 generated by isometries f^N, h of uniformly bounded word length. Furthermore, we show that this free subgroup is convex-cocompact when f is hyperbolic.

1. INTRODUCTION

Let X be an n-dimensional negatively curved Hadamard manifold, with sectional curvature ranging between $-\kappa^2$ and 1, for some $\kappa \geq 1$. The main result of this note is the following quantitative version of the Tits alternative for X, which answers a question asked by Filippo Cerocchi during the Oberwolfach Workshop "Differentialgeometrie im Grossen", 2017, see also [**?**]:

Theorem 1.1. *There exists a function $\mathcal{L} = \mathcal{L}(n, \kappa)$ such that the following holds: Let f, g be non-elliptic isometries of X generating a nonelementary discrete subgroup Γ of $\mathrm{Isom}(X)$. Then there exists an element $h \in \Gamma$ whose word length (with respect to the generators f, g) is $\leq \mathcal{L}$ and a number $N \leq \mathcal{L}$ such that the subgroup of Γ generated by f^N, h is free of rank two.*

One can regard this theorem as a quantitative version of the Tits alternative for discrete subgroups of $\mathrm{Isom}(X)$. For other forms of the quantitative Tits alternative we refer to [**?**, **?**, **?**, **?**].

After replacing g with the element $g' := gfg^{-1}$, and noticing that the subgroup generated by f, g' is still discrete and nonelementary, we reduce the problem to the case when the isometries f and g are conjugate in $\mathrm{Isom}(X)$ which we will assume from now on.

The proof of Theorem **??** breaks into two cases which are handled by different arguments:

Case 1. f (and, hence, g) has translation length bounded below by some positive number λ. We discuss this case in Section **??**.

Case 2. f has translation length bounded above by some positive number λ. We discuss this case in Section **??**.

Date: February 28, 2019.

Remark 1.2. 1. For the constant λ we will take $\varepsilon(n,\kappa)/10$, where $\varepsilon(n,\kappa)$ is a positive lower bound for the Margulis constant of X.

2. We need to use a power of f only in Case 1, while in Case 2 we can take $N = 1$.

We also note that if f is hyperbolic, the free group $\langle f^N, h\rangle$ constructed in our proof is convex-cocompact. See Proposition **??** and Corollary **??**. One can also show that this subgroup is geometrically finite if f is parabolic but we will not prove it.

Acknowledgements. The second author was partly supported by the NSF grant DMS-16-04241 as well as by KIAS (the Korea Institute for Advanced Study) through the KIAS scholar program and by the Simons Foundation Fellowship, grant number 391602. The third author was partly supported by the NSF grant DMS-17-00814.

2. Definitions and notation

In a metric space (Y,d), we will be using the notation $B(a,R)$ to denote the open R-ball centered at $a \in Y$, and the notation $\bar{N}_R(A)$ to denote the closed R-neighborhood of a subset $A \subset X$. By

$$d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}$$

we denote the minimal distance between subsets $A, B \subset Y$.

If (Y,d) is a geodesic δ-hyperbolic metric space or a CAT(0) space, then $\partial_\infty Y$ will denote the visual boundary equipped with the visual topology, and we write $\bar{Y} := Y \cup \partial_\infty Y$. If Y is proper then $\bar{Y}$ is a compactification of Y. Given a pair of points x, y in (Y,d), we will use the notation xy to denote a geodesic segment in Y connecting x to y. For general δ-hyperbolic spaces this segment is *not* unique, but, since any two such segments are within distance δ from each other, this abuse of notation is harmless. We let $|xy| = d(x,y)$ denote the length of xy. Given points $A, B, C \in Y$, we let $\triangle ABC$ denote a geodesic triangle in Y with the vertices A, B, C. Similarly, if $y \in Y, \xi \in \partial_\infty Y$, then $y\xi$ will denote a geodesic ray emanating from y and asymptotic to ξ.

A subset A of Y is called λ-*quasiconvex* if every geodesic segment xy with the end-points in A is contained in $\bar{N}_\lambda(A)$.

A subset A in a metric space Y is called *starlike* with respect to a point $a \in A$ if for every $y \in A$ every geodesic segment ya is contained in A. More generally, if Y is δ-hyperbolic or a CAT(0) space then $A \subset Y$ is called *starlike* with respect to a point $\xi \in \partial_\infty Y$ if for every $y \in A$ every geodesic ray $y\xi$ is contained in A.

Throughout the paper, X will denote an n-dimensional Hadamard manifold with sectional curvature ranging between $-\kappa^2$ and -1, unless otherwise stated. Let d denote the Riemannian distance function on X. We use $\partial_\infty X$

to denote the visual boundary of X, and $\bar{X} := X \cup \partial_\infty X$ the visual compactification of X. Let $\mathrm{Isom}(X)$ denote the isometry group of X. We use $\varepsilon(n,\kappa)$ to denote a fixed positive lower bound on the Margulis constant for X; this number is known to depend only on n and κ, see e.g. [**?**].

Given a pair of points p, q in X we let $H(p,q)$ denote the closed half-space in X given by

$$H(p,q) = \{x \in X : d(x,p) \le d(x,q)\}.$$

Then $\mathrm{Bis}(p,q) = \mathrm{Bis}(q,p) := H(p,q) \cap H(q,p)$, is the equidistant set of p, q.

We use the notation $\mathrm{Hull}(A)$ for the closed convex hull of a subset $A \subset X$ which is the intersection of all closed convex subsets of X containing A.

For each isometry g of X we define its *translation length* $\tau(g)$ as

$$\tau(g) = \inf_{x \in X} d(x, g(x)). \tag{2.1}$$

Isometries of X are classified in terms of their translation lengths, see Section **??**.

A discrete subgroup $\Gamma < \mathrm{Isom}(X)$ is called *elementary* if either it fixes a point in $\bar{X}$ or preserves a geodesic in X.

3. Preliminary material

3.1. **Some** CAT(-1) **computations.** Let X be a CAT(-1) space. Recall that the hyperbolicity constant of X is $\le \delta = \cosh^{-1}(\sqrt{2})$.

Lemma 3.1. *Let $\triangle A_1A_2C$ be a triangle in X such that $\angle A_1CA_2 \ge \pi/2$. Then*

$$|A_1A_2| \ge |A_1C| + |A_2C| - 2\delta.$$

Proof. Let $D \in A_1A_2$ be the point closest to C. Then at least one of the angles $\angle A_iCD, i = 1,2$ is $\ge \pi/4$. The CAT(-1) property and the dual cosine law for the hyperbolic plane imply that

$$\cosh(|CD|)\sin(\frac{\pi}{4}) \le 1,$$

i.e.

$$|CD| \le \cosh^{-1}(\sqrt{2}) = \delta.$$

The rest follows from the triangle inequalities. □

Corollary 3.2. *Suppose that $x, x_+, \hat{x}_+, x'_+$ are points in X which lie on a common geodesic and appear on this geodesic in the given order. Assume that*

$$d(\hat{x}_+, x'_+) \ge d(x, x_+) + 2\cosh^{-1}(\sqrt{2}).$$

Then $H(x_+, \hat{x}_+) \subset H(x, x'_+)$.

Proof. We observe that the CAT(-1) condition implies that for each z equidistant from $x_+, \hat{x}_+$ we have

$$\angle z x_+ \hat{x}_+ \le \pi/2, \quad \angle z \hat{x}_+ x_+ \le \pi/2.$$

Hence,

$$\angle x x_+ z \geq \pi/2, \quad \angle x'_+ \hat{x}_+ z \geq \pi/2.$$

Then the lemma and the triangle inequality implies that

$$d(z, x) \leq d(z, x'_+).$$

and, thus,

$$\mathrm{Bis}(x_+, \hat{x}_+) \subset H(x, x'_+).$$

Since every geodesic connecting $w \in H(x_+, \hat{x}_+)$ to x'_+ passes through some point $z \in \mathrm{Bis}(x_+, \hat{x}_+)$, it follows that

$$d(x, w) \leq d(w, x'_+). \qquad \square$$

3.2. **Quasiconvex and starlike subsets.**

Lemma 3.3. *Starlike subsets in a δ-hyperbolic space Y are δ-quasiconvex.*

Proof. We prove this for subsets $A \subset Y$ starlike with respect to $a \in A$; the proof in the case of starlike subsets with respect to $\xi \in \partial_\infty Y$ is similar and is left to the reader. Take $z_1, z_2 \in A$. Then, by the δ-hyperbolicity,

$$z_1 z_2 \subset \bar{N}_\delta \left(a z_1 \cup a z_2 \right) \subset \bar{N}_\delta \left(A \right). \qquad \square$$

Suppose now that X is a Hadamard manifold of negatively pinched curvature as above. Then, according to [**?**, Proposition 2.5.4], there exists $\mathfrak{q} = \mathfrak{q}(\kappa, \lambda)$ such that for every λ-quasiconvex subset $A \subset X$ we have

$$\mathrm{Hull}(A) \subset \bar{N}_{\mathfrak{q}}(A). \tag{3.1}$$

In particular:

Proposition 3.4. *For every starlike subset A in a Hadamard manifold X of negatively pinched curvature, the closed convex hull* Hull(A) *is contained in the $\mathfrak{q} = \mathfrak{q}(\kappa, \delta)$-neighborhood of A.*

In what follows, we will suppress the dependence of $\mathfrak{q}$ on κ and δ since these are fixed for our space X.

3.3. **Classification of isometries.**

Let X be a negatively curved Hadamard manifold. Isometries of X are classified into three types according to their translation lengths τ, see [**?**, **?**].

1. An isometry g of X is *hyperbolic* if $\tau(g) > 0$. Equivalently, the infimum in (**??**) is attained and is positive. In this case, the infimum is attained on a g-invariant geodesic, called the *axis* of g, and denoted by A_g.

2. An isometry g of X is *elliptic* of $\tau(g) = 0$ and the infimum in (**??**) is attained; the set where the infimum is attained is a totally geodesic submanifold of X fixed pointwise by g.

3. An isometry g of X is *parabolic* if the infimum in (**??**) is not attained. In this case, the infimum is necessarily equal to zero.

Thus, only parabolic and elliptic isometries have zero translation lengths. For any $g \in \mathrm{Isom}(X)$ and $m \in \mathbb{Z}$ we have

$$\tau(g^m) = |m|\tau(g). \tag{3.2}$$

The following theorem provides an alternative characterization of types of isometries of X, see [?].

Theorem 3.5. *Suppose that g is an isometry of X. Then:*

(1) *g is hyperbolic if and only if for some (equivalently, every) $x \in X$ the orbit map $N \to g^N x$ is a quasiisometric embedding $\mathbb{Z} \to X$.*
(2) *g is elliptic if and only if for some (equivalently, every) $x \in X$ the orbit map $N \to g^N x, N \in \mathbb{Z}$ has bounded image.*
(3) *g is parabolic if and only if for some (equivalently, every) $x \in X$ the orbit map $N \to g^N x, N \in \mathbb{Z}$ is proper and*

$$\lim_{N\to\infty} \frac{d(x, g^N(x))}{N} = 0.$$

If f, g are hyperbolic isometries of X generating a discrete subgroup of $\mathrm{Isom}(X)$, then either the ideal boundaries of the axes A_f, A_g are disjoint or $A_f = A_g$, (see [?], the argument for negatively curved Hadamard manifolds is similar).

3.4. **Margulis cusps and tubes.** Take $g \in \mathrm{Isom}(X)$. For each $\varepsilon \geq \tau(g)$ we define the following nonempty closed convex subset of X:

$$T_\varepsilon(g) = \{x \in X \mid d(x, g(x)) \leq \varepsilon\}.$$

Of primary importance are subsets $T_\varepsilon(g)$ for $\varepsilon < \varepsilon(n, \kappa)$. For any two isometries g, h of X we have

$$T_\varepsilon(hgh^{-1}) = h(T_\varepsilon(g)). \tag{3.3}$$

In particular, if g, h commute, then h preserves $T_\varepsilon(g)$.

For parabolic isometries g of X define the *Margulis cusp*

$$\mathcal{T}_\varepsilon(g) := \bigcup_{i\in\mathbb{Z}-\{0\}} T_\varepsilon(g^i).$$

(The same definition works for elliptic isometries of X, except the above region is not called a *cusp*.) This subset is $\langle g\rangle$-invariant.

Suppose that g is a hyperbolic isometry of X. Define m_g to be the (unique) positive integer such that

$$\tau(g^{m_g}) \leq \varepsilon/10, \quad \tau(g^{m_g+1}) > \varepsilon/10, \tag{3.4}$$

and set

$$\mathcal{T}_\varepsilon(g) := \bigcup_{1\leq i\leq m_g} T_\varepsilon(g^i) \subset X.$$

If $\tau(g) > \varepsilon/10$, then $\mathcal{T}_\varepsilon(g) = \emptyset$.

Since the subgroup $\langle g\rangle$ is abelian, in view of (**??**), we obtain:

Lemma 3.6. *The subgroup $\langle g \rangle$ preserves $\mathcal{T}_\varepsilon(g)$ and, hence, also preserves* $\mathrm{Hull}(\mathcal{T}_\varepsilon(g))$.

By convexity of the distance function, for any isometry $g \in \mathrm{Isom}(X)$, $T_\varepsilon(g)$ is convex. In particular, $\mathcal{T}_\varepsilon(g)$ is a starlike region with respect to any fixed point $p \in \bar{X}$ of g for general g, and with respect to and any point on the axis of g if g is hyperbolic. As a corollary to Lemma **??**, one obtains,

Corollary 3.7. *For every isometry $g \in \mathrm{Isom}(X)$, the set $\mathcal{T}_\varepsilon(g)$ is δ-quasiconvex.*

Proposition **??** then implies

Corollary 3.8. *For every isometry $g \in \mathrm{Isom}(X)$,*

$$\mathrm{Hull}(\mathcal{T}_\varepsilon(g)) \subset \bar{N}_{\mathfrak{q}}\left(\mathcal{T}_\varepsilon(g)\right),$$

where $\mathfrak{q}$ is as in Proposition **??**.

For a more detailed discussion of the regions $\mathcal{T}_\varepsilon(g)$, see [**?**, **?**].

3.5. **Displacement estimates.** In this subsection, we let X be a CAT(-1) geodesic metric space. For each pair of points $A, B \in \mathbb{H}^2$ and each circle $S \subset \mathbb{H}^2$ passing through these points, we let $\widehat{AB}^S$ denote the (hyperbolic) length of the shorter arc into which A, B divide the circle S.

Lemma 3.9. *If $d(A,B) \le D$ then for every circle S as above, then the length ℓ of $\widehat{AB}^S$ satisfies the inequality:*

$$d(A,B) \le \ell \le \frac{2\pi \tanh(D/4)}{1 - \tanh^2(D/4)}.$$

Proof. The first inequality is clear, so we verify the second. We want to maximize the length of $\widehat{AB}^S$ among all circles S passing through A, B. We claim that the maximum is achieved on the circle S_o whose center o is the midpoint of AB. This follows from the fact that given any other circle S, we have the radial projection from $\widehat{AB}^{S_o}$ to $\widehat{AB}^S$ (with the center of the projection at o). Since this radial projection is distance-decreasing (by convexity), the claim follows. The rest of the proof amounts to a computation of the length of the hyperbolic half-circle with the given diameter. ☐

Lemma 3.10. *There exists a function $c(D)$ so that the following holds. Consider an isosceles triangle ABC in X with $d(A,C) = d(B,C)$, $d(A,B) \le D$, and an isosceles subtriangle $A'B'C$ with $A' \in AC, B' \in BC$, $d(A,A') = d(B,B') = \tau$. Then*

$$d(A',B') \le c(D)e^{-\tau}.$$

Proof. In view of the CAT(-1) assumption, it suffices to consider the case when $X = \mathbb{H}^2$. We will work with the unit disk model of the hyperbolic plane where C is the center of the disk as in Figure **??**. Let α denote the angle $\angle(ACB)$. Set $T := d(C,A) = d(C,B)$. For points $A_t \in CA, B_t \in CB$ such that $d(C,A_t) = d(C,B_t) = t$ we let l_t denote the hyperbolic length of the (shorter) circular arc $\widehat{A_tB_t} = \widehat{A_tB_t}^{S_t}$ of the angular measure α, centered

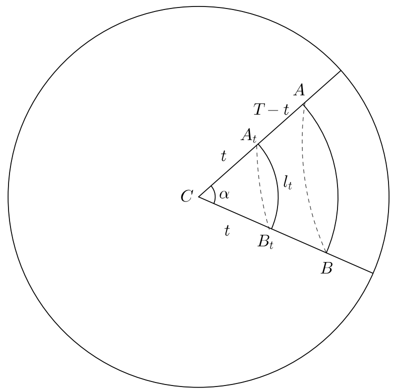

FIGURE 1

at C and connecting A_t to B_t. (Here S_t is the circle centered at C and of the hyperbolic radius t.) Let R_t denote the Euclidean distance between C and A_t (same for B_t). Then

$$l_t = \frac{2\alpha R_t}{1-R_t^2},$$

$$R_t = \tanh(t/2).$$

Thus, for $\tau = T - t$,

$$\begin{aligned}\frac{l_t}{l_T} &= \frac{R_t}{R_T}\frac{1-R_T^2}{1-R_t^2} \leq \frac{1-R_T^2}{1-R_t^2} \leq 2\frac{1-R_T}{1-R_t} \\ &= 2\frac{1-\tanh(T/2)}{1-\tanh(t/2)} = 2\frac{e^t+1}{e^T+1} = 2\frac{e^{-T}+e^{-\tau}}{e^{-T}+1} \leq 4e^{-\tau}.\end{aligned}$$

In other words,

$$d(A_t, B_t) \leq l_t \leq 4e^{-\tau} l_T.$$

Combining this inequality with Lemma **??**, we obtain

$$l_t \leq 4e^{-\tau}\frac{2\pi\tanh(d(A,B)/4)}{1-\tanh^2(d(A,B)/4)} \leq 4e^{-\tau}\frac{2\pi\tanh(D/4)}{1-\tanh^2(D/4)}.$$

Lastly, setting $A' = A_t, B' = B_t, A = A_T, B = B_T$, we get:

$$d(A', B') \leq 4\frac{2\pi\tanh(D/4)}{1-\tanh^2(D/4)}e^{-\tau} = c(D)e^{-\tau}.$$ □

Corollary 3.11. *There exists a function $\mathfrak{r}(\varepsilon)$ such that for any hyperbolic isometry $h \in \mathrm{Isom}(X)$ with translation length*

$$\tau(h) = l \leq \varepsilon/10,$$

if $A \in X$ satisfies $d(A, h(A)) = \varepsilon$, then there exists $B \in X$ such that $d(B, h(B)) = \varepsilon/3$, $d(A, B) \leq \mathfrak{r} = \mathfrak{r}(\varepsilon)$ and B lies of the shortest geodesic segment connecting A to the axis A_h of h.

Proof. Let $C \in A_h$ be the closest point to A in A_h. By the convexity of the distance function, there exists a point $B \in AC$ such that $d(B, h(B)) = \varepsilon/3$. Suppose that $d(A, B) = d(h(A), h(B)) = t$ and $d(A, C) = d(h(A), h(C))) = T$ as shown in Figure **??**. Then $d(C, h(A)) \leq T + l \leq T + \varepsilon/10$. There exist points D, E in the segment $h(A)C$ such that $d(C, D) = d(C, B) = T - t$, $d(h(A), E) = t$ and $d(A', C) = d(A, C) = T$.

Then $d(A, A') \leq \varepsilon + l \leq 11\varepsilon/10$. By Lemma **??**, $c(11\varepsilon/10)$ (defined in that lemma) satisfies

$$d(B, D) \leq c(d(A, A'))e^{-t} \leq c(11\varepsilon/10)e^{-t}.$$

Similarly, by taking the point $A'' \in h(A)C$ satisfying $d(A'', h(A)) = T$, $d(h(C), A'') \leq 2l$, considering the isosceles triangle $\triangle h(C)A''h(A)$ and its subtriangle $\triangle h(B)Eh(A)$, we obtain:

$$d(h(B), E) \leq c(2l)e^{t-T}.$$

Since $l \leq \varepsilon/10$ and $d(B, h(B)) = \varepsilon/3$, convexity of the distance function implies that $T - t > t$. Thus,

$$\begin{aligned}\varepsilon/3 = d(B, h(B)) &\leq d(B, D) + d(D, E) + d(E, h(B)) \\ &\leq c(11\varepsilon/10)e^{-t} + l + c(2l)e^{t-T} \leq c(11\varepsilon/10)e^{-t} + \frac{\varepsilon}{10} + c(\varepsilon/5)e^{-t},\end{aligned}$$

which simplifies to

$$\frac{7}{30}\varepsilon \leq \left(c(11\varepsilon/10) + c(\varepsilon/5)\right) e^{-t},$$

and consequently

$$d(A, B) = t \leq \mathfrak{r}(\varepsilon) := \log\left([c(11\varepsilon/10) + c(\varepsilon/5)]\frac{30}{7}\varepsilon^{-1}\right). \qquad \square$$

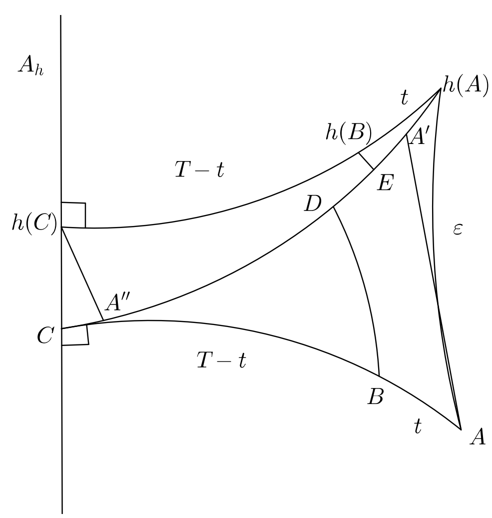

FIGURE 2

3.6. **Local-to-global principle for quasigeodesics in X.** For a piecewise-geodesic path consisting of alternating 'long' arcs and 'short' segments such that adjacent geodesic segments meet at the angles $\geq \pi/2$, we construct a quasigeodesic in X by making the long segments sufficiently long, given a lower bound on the lengths of the short arcs. More precisely, according to [**?**, Proposition 7.3]:

Proposition 3.12. *There are functions $\lambda = \lambda(\varepsilon) \geq 1, \alpha = \alpha(\varepsilon) \geq 0$ and $L = L(\varepsilon) > \varepsilon > 0$ such that the following holds. Suppose that $\gamma = \cdots \gamma_{-1} * \gamma_0 * \gamma_1 * \cdots * \gamma_n \ldots \subseteq X$ is a piecewise geodesic path such that:*

1. *Each geodesic arc γ_j has length either at least ε or at least L.*
2. *If γ_j has length $< L$, then the adjacent geodesic arcs γ_{j-1} and γ_{j+1} have lengths at least L.*
3. *All adjacent geodesic segments meet at the angles $\geq \pi/2$.*

Then γ is a (λ, α)-quasigeodesic in X.

3.7. **Ping-pong.**

Proposition 3.13. *Suppose that $g, h \in \mathrm{Isom}(X)$ are parabolic/hyperbolic elements with equal translation lengths $\leq \varepsilon/10$, and*

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(g)), \mathrm{Hull}(\mathcal{T}_\varepsilon(f))) \geq L, \tag{3.5}$$

where $L = L(\varepsilon/10)$ is as in Proposition **??**. *Then $\Phi := \langle g, h\rangle < \mathrm{Isom}(X)$ is a free subgroup of rank 2.*

Proof. To simplify notation, for a non-elliptic element $f \in \mathrm{Isom}(X)$, we denote $\mathrm{Hull}(\mathcal{T}_\varepsilon(f))$ by $\widehat{\mathcal{T}}_\varepsilon(f)$.

Using Lemma **??**, (**??**), and the definition of $\mathcal{T}_\varepsilon$, we obtain

$$d(\widehat{\mathcal{T}}_\varepsilon(g), g^k \widehat{\mathcal{T}}_\varepsilon(h)) = d(\widehat{\mathcal{T}}_\varepsilon(g), \widehat{\mathcal{T}}_\varepsilon(h)) \geq L, k \in \mathbb{Z}.$$

Our goal is to show that every nonempty word $w(g, h)$ represents a non-trivial element of $\mathrm{Isom}(X)$. It suffices to consider cyclically reduced words w which are not powers of g, h.

We will consider a cyclically reduced word

$$w = w(g, h) = g^{m_k} h^{m_{k-1}} g^{m_{k-2}} h^{m_{k-3}} \ldots g^{m_2} h^{m_1}, \tag{3.6}$$

words with the last letter g are treated by relabeling. Since w is cyclically reduced and is not a power of g, h, the number k is ≥ 2 and all of the m_i's in this equation are nonzero.

For each $N \geq 1$ we define the *r-suffix* of w^N as the following subword of w^N:

$$w_r = \begin{cases} g^{m_r} h^{m_{r-1}} g^{m_{r-2}} h^{m_{r-3}} \ldots g^{m_2} h^{m_1}, & r \text{ even} \\ h^{m_r} g^{m_{r-1}} h^{m_{r-2}} \ldots g^{m_2} h^{m_1}, & r \text{ odd} \end{cases} \tag{3.7}$$

where, of course, $m_i \equiv m_j$ modulo N. Since w is reduced, each w_r is reduced as well.

We will prove that the map

$$\mathbb{Z} \to X, \quad N \mapsto w^N x$$

is a quasiisometric embedding. This will imply that $w(g,h)$ is nontrivial. In fact, this will also show that $w(g,h)$ is hyperbolic, see Theorem **??**.

Let $l = yz$ be the unique shortest geodesic segment connecting points in $\widehat{\mathcal{T}}_\varepsilon(g)$ and $\widehat{\mathcal{T}}_\varepsilon(h)$, where $y \in \widehat{\mathcal{T}}_\varepsilon(g)$ and $z \in \widehat{\mathcal{T}}_\varepsilon(h)$. For $r \geq 0$, we denote $w_r l$, $w_r y$ and $w_r z$ by l_r, y_r and z_r, respectively. In particular, $y_0 = y, z_0 = z$ and $l_0 = l$.

Since l is the shortest segment between $\widehat{\mathcal{T}}_\varepsilon(g), \widehat{\mathcal{T}}_\varepsilon(h)$ and these are convex subsets of X, for every $y' \in \widehat{\mathcal{T}}_\varepsilon(g)$ (resp. $z' \in \widehat{\mathcal{T}}_\varepsilon(h)$),

$$\angle y'yz \geq \pi/2, \quad \text{(resp. } \angle yzz' \geq \pi/2). \tag{3.8}$$

Since g and h have equal translation lengths, h is parabolic (resp. hyperbolic) if and only if g is parabolic (resp. hyperbolic). When both of them are hyperbolic, since y and z are not in the interior of $\mathcal{T}_\varepsilon(g)$ and $\mathcal{T}_\varepsilon(h)$, respectively, $d(y, g^i y), d(z, h^j z) \geq \varepsilon$, for all $1 \leq i \leq m_g$, $1 \leq j \leq m_h$. Also, when $i > m_g$, $j > m_h$, it follows from (**??**) and (**??**) that

$$\min\left(d(y, g^i y), d(z, h^j z)\right) \geq \frac{\varepsilon}{10}.$$

Moreover, when both g and h are parabolic, $d(y, g^i y), d(z, h^j z) \geq \varepsilon$, for all $1 \leq i$, $1 \leq j$. Therefore, in the general case,

$$\min\left(d(y, g^i y), d(z, h^j z)\right) \geq \frac{\varepsilon}{10}, \quad \forall i \geq 1, \forall j \geq 1. \tag{3.9}$$

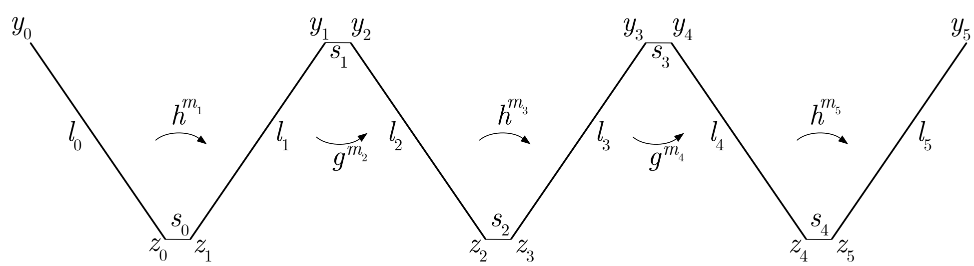

FIGURE 3

Let s_r be the segment

$$s_r = \begin{cases} y_r y_{r+1}, & \text{when } r \text{ is odd} \\ z_r z_{r+1}, & \text{when } r \text{ is even} \end{cases}.$$

See the arrangement of the points and segments in Figure **??**.

Let $\tilde{l}_N$ be the concatenation of the segments l_r's and s_r's as shown in Figure **??**, $0 \leq r \leq kN$. According to (**??**), the length of each segment s_r is at least $\varepsilon/10$, while by the assumption, the length of each l_r is $\geq L = L(\varepsilon/10)$. Moreover, according to (**??**), the angle between any two consecutive segments in $\tilde{l}_N$ is at least $\pi/2$. Using Proposition **??**, we conclude that $\tilde{l}_N$ is a (λ, α)-quasigeodesic.

Consequently,

$$d(w^N x, x) \geq \frac{1}{\lambda}\left(\sum_{i=0}^{kN-1} |s_i| + NkL\right) - \alpha \geq \frac{kL}{\lambda} N - \alpha. \tag{3.10}$$

From this inequality it follows that the map $\mathbb{Z} \to X, N \mapsto w^N x$ is a quasi-isometric embedding. □

Remark 3.14. In fact, this proof also shows that every nontrivial element of the subgroup $\Phi < \mathrm{Isom}(X)$ is either conjugate to one of the generators or is hyperbolic.

For the next proposition and the subsequent remark, one needs the notions of *convex-cocompact* and *geometrically finite* subgroups of $\mathrm{Isom}(X)$. We refer to [**?**] for several equivalent definitions, see also [**?**, section 1]. For now, it suffices to say that a subgroup Γ in $\mathrm{Isom}(X)$ is *convex-cocompact* if it is finitely generated and for some (equivalently, every) $x \in X$, the orbit map $\Gamma \to \Gamma x \subset X$ is a quasiisometric embedding, where Γ is equipped with a word metric.

Proposition 3.15. *Let $g, h \in \mathrm{Isom}(X)$ be hyperbolic isometries satisfying the hypothesis of Proposition* **??***. Then the subgroup $\Phi = \langle g, h\rangle < \mathrm{Isom}(X)$ is convex-cocompact.*

Proof. We equip the free group $\mathbb{F}_2$ on two generators (denoted g, h) with the word metric corresponding to this free generating set. Since g, h are hyperbolic, by (**??**) the lengths of the segments s_r's in the proof of Proposition **??** are $\geq \tau|m_{r+1}|$, where

$$\tau = \tau(g) = \tau(h).$$

Then, for $N = 1$, $r = k$, and a reduced but not necessarily cyclically reduced word w, the inequality (**??**) becomes

$$d(wy, y) \geq \frac{1}{\lambda}\left(\sum_{i=0}^{k-1} |s_i|\right) - \alpha \geq \frac{\tau}{\lambda}|w| - \alpha, \tag{3.11}$$

where $|w| \geq |m_1| + |m_2| + \cdots + |m_k|$ is the (word) length of w. Therefore, the orbit map $\mathbb{F}_2 \to \Phi y \subset X$ is a quasiisometric embedding. □

Remark 3.16. One can also show that if g, h are parabolic then the subgroup Φ is geometrically finite. We will not prove it in this paper since a proof requires further geometric background material on geometrically finite groups.

4. Case 1: Displacement bounded below

In this section we consider discrete nonelementary subgroups of $\mathrm{Isom}(X)$ generated by two hyperbolic elements g, h whose translation lengths are equal to $\tau \geq \lambda$. Our goal is to show that in this case the subgroup $\langle g^N, h^N\rangle$ is free of rank 2 provided that N is greater than some constant depending only on the Margulis constant of X and on λ. The strategy is to bound from above how 'long' the axes A_g, A_h of g and h can stay 'close to each other' in terms of the constant λ. Once we get such an estimate, we find a uniform upper bound on N such that Dirichlet domains for $\langle g^N\rangle, \langle h^N\rangle$ (based at some points on A_g, A_h) have disjoint complements. This implies that g^N, h^N generate a free subgroup of rank two by a classical ping-pong argument.

Let α, β be complete geodesics in the Hadamard manifold X. These geodesics eventually will be the axes of g and h, hence, we assume that these geodesics do not share ideal end-points. Let x_-x_+ denote the (nearest point) projection of β to α and let y_-y_+ denote the projection of x_-x_+ to β. Let x, y denote the mid-points of x_-x_+ and y_-y_+ respectively.

Then

$$L_\beta := d(y_-, y_+) \le L_\alpha := d(x_-, x_+).$$

Fix some $T \ge 0$, and let $\hat{x}_-\hat{x}_+$ and $\hat{y}_-\hat{y}_+$ denote the subsegments of α and β containing x_-x_+ and y_-y_+ respectively, such that

$$d(x_\pm, \hat{x}_\pm) = T, \quad d(y_\pm, \hat{y}_\pm) = T. \tag{4.1}$$

We let $U_\pm$ and $V_\pm$ denote the 'half-spaces' in X equal to $H(\hat{x}_\pm, x_\pm)$ and $H(\hat{y}_\pm, y_\pm)$ respectively. See Figure **??**.

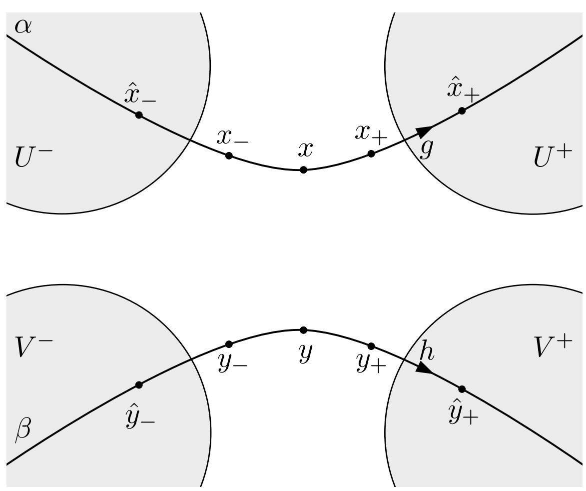

FIGURE 4

The following is proven in [**?**, Appendix]:

Lemma 4.1. *If $T \ge 5$ then the sets $U_\pm$, $V_\pm$ are pairwise disjoint.*

Suppose now that g, h are hyperbolic isometries of X with the axes α, β respectively, and equal translation length $\tau(g) = \tau(h) = \tau \ge \lambda > 0$. We let $\Gamma = \langle g, h\rangle < \mathrm{Isom}(X)$ denote the, necessarily nonelementary (but not necessarily discrete), subgroup of isometries of X generated by g and h.

As an application of the above lemma, as in [**?**, Appendix], we obtain:

Lemma 4.2. *If $N\tau \geq L_\alpha + 5 + 2\delta$ then the half-spaces $H(g^{\pm N}x, x)$, $H(h^{\pm N}y, y)$ are pairwise disjoint.*

Proof. The inequality

$$N\tau \geq L_\alpha + 5 + 2\delta \geq L_\beta + 5 + 2\delta.$$

implies that the quadruples

$$(x, x_+, \hat{x}_+, g^N(x)), (x, x_-, \hat{x}_-, g^{-N}(x)), (y, y_+, \hat{y}_+, h^N(y)), (y, y_-, \hat{y}_-, h^{-N}(y))$$

satisfy the assumptions of Corollary **??** where $\hat{x}_\pm$ and $\hat{y}_\pm$ are given by taking $T = 5$ in (**??**). Therefore, according to this corollary, we have

$$H(g^{\pm N}(x), x) \subset U^\pm, \quad H(h^{\pm N}(y), y) \subset V^\pm.$$

Now, the assertion of the lemma follows from Lemma **??**. □

Corollary 4.3. *If*

$$N\tau \geq L_\alpha + 5 + 2\delta \tag{4.2}$$

then the subgroup $\Gamma_N < \Gamma$ generated by g^N, h^N is free with the basis g^N, h^N.

Proof. We have

$$g^{\pm N}\left(H(h^{-N}(y), y) \ \cup \ H(h^{+N}(y), y)\right) \subset H(g^{\pm N}x, x)$$

and

$$h^{\pm N}\left(H(g^{-N}(x), x) \ \cup \ H(g^{+N}(x), x)\right) \subset H(h^{\pm N}y, y).$$

Thus, the conditions of the standard ping-pong lemma (see e.g. [**?**, **?**]) are satisfied and, hence, Γ_N is free with the basis g^N, h^N. □

Let $\eta = d(\alpha, \beta)$ denote the minimal distance between α, β and pick some $\eta_0 > 0$ (we will eventually take $\eta_0 = 0.01\varepsilon(n, \kappa)$). Let $\beta_0 = z_-^0 z_+^0 \subset \beta$ be the (possibly empty!) maximal closed subinterval such that the distance from the end-points of β_0 to α is $\leq \eta_0$. Thus, $\beta_0 \subset \bar{N}_{\eta_0}(\alpha)$.

Remark 4.4. $\beta_0 = \emptyset$ if and only if $\eta_0 < \eta$.

Let $\alpha_0 = x_-^0 x_+^0$ denote the projection of β_0 to α, let $2L_0$ denote the length of α_0. Hence, the intervals α_0, β_0 are within Hausdorff distance η_0 from each other.

Furthermore, $\angle\beta(-\infty)z_-^0 x_-^0 \geq \pi/2$ and $\angle\beta(-\infty)z_+^0 x_+^0 \geq \pi/2$; see Figure **??**. Hence, according to [**?**, Corollary 3.7], for

$$L_1 = \sinh^{-1}\left(\frac{1}{\sinh(\eta_0)}\right),$$

we have

$$d(x_-, x_-^0) \leq L_1, \quad d(x_+, x_+^0) \leq L_1.$$

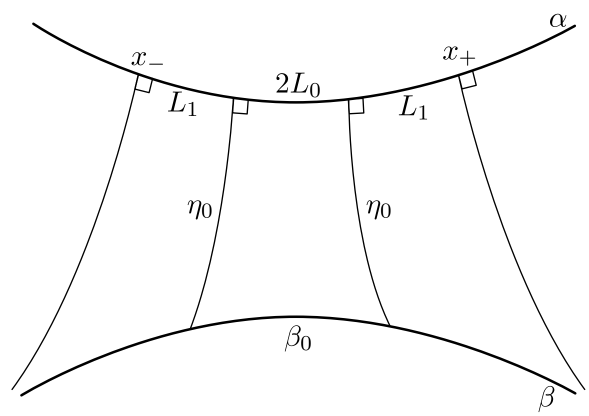

FIGURE 5

Thus, the interval x_-x_+ breaks into the union of two subintervals of length $\leq L_1 = L_1(\eta_0)$ and the interval α_0 of the length $2L_0$. In other words, $L_\alpha = 2(L_0 + L_1)$.

Most of our discussion below deals with the case when the interval β_0 is nonempty.

Our goal is to bound from above L_α in terms of λ, η_0 and the Margulis constant $\varepsilon(n,\kappa)$ of X, provided that $\eta_0 = 0.01\varepsilon(n,\kappa)$ and Γ is discrete.

Lemma 4.5. *Let $S \subset \Gamma$ be the subset consisting of elements of word-length ≤ 4 with respect to the generating set g, h. Let $P_-P_+ \subset \alpha_0$ be the middle subinterval of α_0 whose length is $\frac{2}{9}L_0$. Assume that $\tau \leq d(P_-, P_+)$. Then for each $\gamma \in S$ the interval $\gamma(P_-P_+)$ is contained in the $3\eta_0$-neighborhood of α_0.*

Proof. The proof is a straightforward application of the triangle inequalities taking into account the fact that the Hausdorff distance between α_0 and β_0 is $\leq \eta_0$. □

Then, arguing as in the proof of [**?**, Theorem 10.24][1], we obtain that each of the commutators

$$[g^{\pm 1}, h^{\pm 1}], \quad [h^{\pm 1}, g^{\pm 1}]$$

moves each point of P_-P_+ by at most

$$28 \times 3\eta_0 \leq 100\eta_0.$$

Therefore, by applying the Margulis Lemma as in the proof of [**?**, Theorem 10.24], we obtain:

[1] In fact, the argument there is a variation on a proof due to Culler-Shalen-Morgan and Bestvina, Paulin

Corollary 4.6. *If Γ is discrete and $\eta_0 = 0.01\varepsilon(n,\kappa)$, then*

$$\tau \geq \frac{2}{9}L_0 = \frac{1}{9}(L_\alpha - 2L_1).$$

Corollary 4.7. *If Γ is discrete and $\tau \geq \lambda$, then the subgroup $\langle g^N, h^N\rangle = \Gamma_N < \Gamma$ is free of rank 2 whenever one of the following holds:*

i. Either $L_\alpha \leq 3L_1$ and

$$N \geq \frac{5+2\delta+3L_1}{\lambda}.$$

ii. Or $L_\alpha \geq 3L_1$ and

$$N \geq 27 + \frac{9(5+2\delta)}{L_1}.$$

Proof. In view of Corollary **??**, it suffices to ensure that the inequality (**??**) holds.

(i) Suppose first that $L_\alpha \leq 3L_1$, hence, $L_\beta \leq 3L_1$. Then, in view of the inequality $\tau \geq \lambda > 0$, the inequality (**??**) will follow from

$$N \geq \frac{5+2\delta+3L_1}{\lambda}.$$

(ii) Suppose now that $L_\alpha \geq 3L_1$. The function

$$\frac{9(t+5+2\delta)}{t-2L_1}$$

attains its maximum on the interval $[3L_1, \infty)$ at $t = 3L_1$. Therefore,

$$\frac{9(L_\alpha+5+2\delta)}{L_\alpha - 2L_1} \leq 27 + \frac{9(5+2\delta)}{L_1}.$$

Thus, the inequality

$$\tau \geq \frac{L_\alpha - 2L_1}{9}$$

implies that for any

$$N \geq 27 + \frac{9(5+2\delta)}{L_1},$$

we have $N\tau \geq L_\alpha + 5 + 2\delta$. □

Consider now the remaining case when for $\eta_0 := \frac{1}{100}\varepsilon(n,\kappa)$, the subinterval β_0 is empty, i.e. $\eta > \eta_0 = \frac{1}{100}\varepsilon(n,\kappa)$. Then, as above, the length L_α of the segment x_-x_+ is at most $2L_1$. Therefore, similarly to the case (i) of Corollary **??**, in order for N to satisfy the inequality (**??**), it suffices to get

$$N \geq \frac{5+2\delta+3L_1}{\lambda}.$$

To conclude:

Theorem 4.8. *Suppose that g, h are hyperbolic isometries of X generating a discrete nonelementary subgroup, whose translation lengths are equal to some $\tau \geq \lambda > 0$. Let L_1 be such that*

$$\sinh(L_1) \sinh\left(\frac{1}{100}\varepsilon\right) = 1,$$

where $\varepsilon = \varepsilon(n, \kappa)$. Then for every

$$N \geq \max\left(\frac{5 + 2\delta + 3L_1}{\lambda}, 27 + \frac{9(5 + 2\delta)}{L_1}\right) \tag{4.3}$$

the group generated by g^N, h^N is free of rank 2.

We note that proving that (some powers of) g and h generate a free subsemigroup, is easier, see [**?**] and [**?**, section 11].

Corollary 4.9. *Given g, h as in Theorem* **??**, *and any N satisfying (**??**), the free group $\Gamma_N = \langle g^N, h^N \rangle$ is convex-cocompact.*

Proof. Let $\mathcal{U}^{\pm} = H(g^{\pm N}x, x)$ and $\mathcal{V}^{\pm} = H(h^{\pm N}y, y)$. Observe that

$$g^{\pm N}(X \setminus \mathcal{U}^{\mp}) \subset \mathcal{U}^{\pm}$$

and

$$h^{\pm N}(X \setminus \mathcal{V}^{\mp}) \subset \mathcal{V}^{\pm}.$$

We let $\mathfrak{D}_{g^N}, \mathfrak{D}_{h^N}$ denote the closures in $\bar{X}$ of the domains

$$X \setminus (\mathcal{U}^- \cup \mathcal{U}^+), \quad X \setminus (\mathcal{V}^- \cup \mathcal{V}^+)$$

respectively and set

$$\mathfrak{D} = \mathfrak{D}_{g^N} \cap \mathfrak{D}_{h^N}.$$

It is easy to see (cf. [**?**]) that this intersection is a fundamental domain for the action of Γ_N on the complement $\bar{X} \setminus \Lambda$ to its limit set Λ. Therefore, $(\bar{X} \setminus \Lambda)/\Gamma_N$ is compact. Hence, Γ_N is convex-cocompact (see [**?**]). $\square$

Remark 4.10. It is also not hard to see directly that the orbit maps $\Gamma_N \to \Gamma_N x \subset X$ are quasiisometric embeddings by following the proofs in [**?**, section 7] and counting the number of bisectors crossed by geodesics connecting points in Γx.

5. Case 2: Displacement bounded above

The strategy in this case is to find an element g' conjugate to g (by some uniformly bounded power of f) such that the Margulis regions of g, g' are sufficiently far apart, i.e. are at distance $\geq L$, where L is given by the local-to-global principle for piecewise-geodesic paths in X, see Proposition **??**.

Proposition 5.1. *There exists a function*

$$\mathfrak{k} : [0, \infty) \times (0, \varepsilon] \to \mathbb{N}$$

for $0 < \varepsilon \le \varepsilon(n, \kappa)$ *with the following property: Let* $g_1, \cdots, g_k$ *be nonelliptic isometries of the same type (hyperbolic or parabolic) with translation lengths* $\le \varepsilon/10$ *and*

$$k \ge \mathfrak{k}(L, \varepsilon).$$

Suppose that $\langle g_i, g_j \rangle$ *are nonelementary discrete subgroup for all* $i \ne j$*. Then, there exists a pair of indices* $i, j \in \{1, \ldots, k\}$*,* $i \ne j$ *such that*

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)), \mathrm{Hull}(\mathcal{T}_\varepsilon(g_j))) > L.$$

Proof. If all the isometries g_i are parabolic, then the proposition is established in [**?**, Proposition 8.3]. Therefore, we only consider the case when all these isometries are hyperbolic. Our proof follows closely the proof of [**?**, Proposition 8.3].

Since for all $i \ne j$ the subgroup $\langle g_i, g_j \rangle$ is a discrete and nonelementary, and $\varepsilon \le \varepsilon(n, \kappa)$, we have

$$\mathcal{T}_\varepsilon(g_i) \cap \mathcal{T}_\varepsilon(g_j) = \emptyset.$$

Given $L > 0$, suppose that

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)), \mathrm{Hull}(\mathcal{T}_\varepsilon(g_j))) \le L, \quad \forall i, j \in \{1, \ldots, k\}$$

Our goal is to get a uniform upper bound of k.

Consider the $L/2$-neighborhoods $\bar{N}_{L/2}(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)))$. They are convex in X, and have nonempty pairwise intersections. Thus, by [**?**, Proposition 8.2], there exists a point $x \in X$ such that

$$d(x, \mathcal{T}_\varepsilon(g_i)) \le R_1 := n\delta + L/2 + \mathfrak{q}, \quad i = 1, \ldots, k,$$

where δ is the hyperbolicity constant of X and $\mathfrak{q}$ is as in Proposition **??**. Then

$$\mathcal{T}_\varepsilon(g_i) \cap B(x, R_1) \ne \emptyset, \quad i = 1, \ldots, k.$$

For each $i = 1, \ldots, k$ take a point $x_i \in \mathcal{T}_\varepsilon(g_i) \cap B(x, R_1)$ satisfying $d(x_i, g_i^{p_i}(x_i)) = \varepsilon$ for some $0 < p_i \le m_{g_i}$. Since the translation lengths of the elements $g_i^{p_i}$ are $\le \varepsilon/10$, by Corollary **??** there exist points $y_i \in X$ such that

$$d(y_i, g_i^{p_i}(y_i)) = \varepsilon/3, \quad d(x_i, y_i) \le \mathfrak{r}(\varepsilon).$$

Consider the $\varepsilon/3$-balls $B(y_i, \varepsilon/3)$. Then $B(y_i, \varepsilon/3) \subset \mathcal{T}_\varepsilon(g_i)$ since

$$d(z, g_i^{p_i}(z)) \le d(z, y_i) + d(y_i, g_i^{p_i}(y_i)) + d(g_i^{p_i}(y_i), g_i^{p_i}(z)) \le \varepsilon$$

for any point $z \in B(y_i, \varepsilon/3)$. Thus, the balls $B(y_i, \varepsilon/3)$ are pairwise disjoint. Observe that $B(y_i, \varepsilon/3) \subset B(x, R_2)$ where $R_2 = R_1 + \mathfrak{r}(\varepsilon) + \varepsilon/3$.

Let $V(r, n)$ denote the volume of the r-ball in $\mathbb{H}^n$. Then for each i, $\mathrm{Vol}(B(y_i, \varepsilon/3))$ is at least $V(\varepsilon/3, n)$, see [**?**, Proposition 1.1.12]. Moreover, the volume of $B(x, R_2)$ is at most $V(\kappa R_2, n)/\kappa^n$, see [?, Proposition 1.2.4]. Let

$$\mathfrak{k}(L, \varepsilon) := \frac{V(\kappa R_2, n)/\kappa^n}{V(\varepsilon/3, n)} + 1.$$

Then $k < \mathfrak{k}(L, \varepsilon)$, because otherwise we would obtain

$$\mathrm{Vol}\left(\bigcup_{i=1}^{k} B(y_i, \varepsilon/3)\right) > \mathrm{Vol}(B(x, R_2)),$$

where the union of the balls on the left side of this inequality is contained in $B(x, R_2)$, which is a contradiction.

Therefore, whenever $k \geq \mathfrak{k}(L, \varepsilon)$, there exist a pair of indices i, j such that

$$d\left(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)), \mathrm{Hull}(\mathcal{T}_\varepsilon(g_j))\right) > L. \qquad \square$$

Remark 5.2. Proposition **??** also holds for isometries of mixed types (i.e. some g_i's are parabolic and some are hyperbolic). The proof is similar to the one given above.

Theorem 5.3. *For every nonelementary discrete subgroup* $\Gamma = \langle g, h\rangle <$ $\mathrm{Isom}(X)$ *with* g, h *nonelliptic isometries satisfying*

$$\tau(g) \leq \varepsilon/10 \leq \varepsilon(n, \kappa)/10,$$

there exists i*,* $1 \leq i \leq \mathfrak{k}(L(\varepsilon/10), \varepsilon)$*, such that* $\langle g, h^i g h^{-i}\rangle$ *is a free subgroup of rank 2, where* $\mathfrak{k}$ *is the function given by Proposition* **??** *and* $L(\varepsilon/10)$ *is the constant in Proposition* **??**.

Proof. Consider isometries $g_i := h^i g h^{-i}$, $i \geq 1$. We first claim that no pair g_i, g_j, $i \neq j$, generates an elementary subgroup of $\mathrm{Isom}(X)$. There are two cases to consider:

(i) Suppose that g is parabolic with the fixed point $p \in \partial_\infty X$. We claim that for all $i \neq j$, $h^i(p) \neq h^j(p)$. Otherwise, $h^{j-i}(p) = p$, and p would be a fixed point of h. But this would imply that Γ is elementary, contradicting our hypothesis.

(ii) The proof in the case when g is hyperbolic is similar. The axis of g_i equals $h^i(A_g)$. If hyperbolic isometries g_i, g_j, $i \neq j$, generate a discrete elementary subgroup of Γ, then they have to share the axis, and we would obtain $h^i(A_g) = h^j(A_g)$. Then $h^{j-i}(A_g) = A_g$. Since h^{j-i} is nonelliptic, it cannot swap the fixed points of g, hence, it fixes both of these points. Therefore, g, h have common axis, contradicting the hypothesis that Γ is nonelementary.

All the isometries g_i have equal translation lengths $\leq \varepsilon/10$. Therefore, by Proposition **??**, there exists a pair of natural numbers $i, j \leq \mathfrak{k}(L(\varepsilon/10), \varepsilon)$ such that

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(h^i g h^{-i})), \mathrm{Hull}(\mathcal{T}_\varepsilon(h^j g h^{-j}))) > L(\varepsilon/10)$$

where $\mathfrak{k}(L(\varepsilon/10),\varepsilon)$ is the function as in Proposition **??**. It follows that

$$d(\mathrm{Hull}(\mathcal{T}_\varepsilon(h^{j-i}gh^{i-j})), \mathrm{Hull}(\mathcal{T}_\varepsilon(g))) > L(\varepsilon/10).$$

Setting $f := h^{j-i}gh^{i-j}$, and applying Proposition **??** to the isometries f, g, we conclude that the subgroup $\langle f, g\rangle < \Gamma$ is free of rank 2. The word length of f is at most

$$2|j-i| + 1 \leq 2\mathfrak{k}(L(\varepsilon/10),\varepsilon) + 1. \qquad \square$$

6. Conclusion

Now we are in a position to complete the proof of Theorem **??**.

Proof of Theorem **??**. We set $\lambda := \varepsilon/10$, where $\varepsilon = \varepsilon(n,\kappa)$ is the Margulis constant. Let g, h be non-elliptic isometries of X generating a discrete nonelementary subgroup of $\mathrm{Isom}(X)$, such that $\tau(g) = \tau(h) = \tau$.

If $\tau \geq \lambda$, then by Theorem **??**, the subgroup $\Gamma_N < \Gamma$ generated by g^N, h^N is free of rank 2, where

$$N := \left\lceil \max\left(\frac{5 + 2\delta + 3L_1}{\lambda}, 27 + \frac{9(5+2\delta)}{L_1}\right)\right\rceil.$$

Here $\delta = \cosh^{-1}(\sqrt{2})$, and

$$L_1 = \sinh^{-1}\left(\frac{1}{\sinh(\varepsilon/100)}\right).$$

If $\tau \leq \lambda$, then by Theorem **??** there exists $i \in [1, \mathfrak{k}(L(\lambda),\varepsilon)]$ such that $\langle g, h^i g h^{-i}\rangle$ is free of rank 2 where $\mathfrak{k}(L(\lambda),\varepsilon)$ is a constant as in Theorem **??**. $\square$

References

[1] W. Ballmann, M. Gromov, V. Schroeder, "Manifolds of nonpositive curvature." Progr. Math. 61, Birkhäuser, Boston, 1985.

[2] G. Besson, G. Courtois, S. Gallot, *Uniform growth of groups acting on Cartan-Hadamard spaces*, J. Eur. Math. Soc. 13 (2011), no. 5, 1343–1371.

[3] B. H. Bowditch, *Geometrical finiteness for hyperbolic groups.* J. Funct. Anal. 113 (1993), 245-317.

[4] B. H. Bowditch, *Geometrical finiteness with variable negative curvature.* Duke Math. J. 77 (1995), no. 1, 229–274.

[5] E. Breuillard, *A strong Tits alternative*, arXiv:0804.1395

[6] E. Breuillard, K. Fujiwara, *On the joint spectral radius for isometries of non-positively curved spaces and uniform growth*, arXiv:1804.00748.

[7] M. Coornaert, T. Delzant, A. Papadopoulos, "Géométrie et Théorie des Groupes: Les Groupes Hyperboliques de Gromov." Lecture Notes in Math 1441 (Springer, Berlin, 1980).

[8] E. Breuillard, T. Gelander, *Uniform independence in linear groups*, Invent. math. 173 (2008) 225–263.

[9] P. de la Harpe, "Topics in geometric group theory." University of Chicago Press, 2000.

[10] F. Cerocchi, A. Sambusetti, *Entropy and finiteness of groups with acylindrical splittings*, arXiv:1711.06210.

[11] C. Drutu, M. Kapovich, "Geometric group theory." With an appendix by Bogdan Nica. American Mathematical Society Colloquium Publications, 63. American Mathematical Society, Providence, RI, 2018.

[12] M. Kapovich, "Hyperbolic manifolds and discrete groups. Reprint of the 2001 edition." Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009.

[13] M. Kapovich, B. Leeb, *Discrete isometry groups of symmetric spaces*, Spring 2015 MSRI Lecture Notes. To appear in Volume IV of Handbook of Group Actions. The ALM series, International Press, Eds. L.Ji, A.Papadopoulos, S-T.Yau.

[14] M. Kapovich and B. Liu, *Geometric finiteness in negatively pinched Hadamard manifolds*. Preprint, arXiv:1801.08239.

[15] B. Maskit, "Kleinian groups." Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 287. Springer-Verlag, Berlin, 1988.

Department of Mathematics, UC Davis, One Shields Avenue, Davis, CA 95616, USA

Email address: `sdey@math.ucdavis.edu`

Department of Mathematics, UC Davis, One Shields Avenue, Davis, CA 95616, USA

Email address: `kapovich@math.ucdavis.edu`

Department of Mathematics, UC Davis, One Shields Avenue, Davis, CA 95616, USA

Email address: `bxliu@math.ucdavis.edu`