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Global Schrödinger maps

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GLOBAL SCHRÖDINGER MAPS IN DIMENSIONS $d \geq 2$: SMALL DATA IN THE CRITICAL SOBOLEV SPACES

I. BEJENARU, A. D. IONESCU, C. E. KENIG, AND D. TATARU

Abstract. We consider the Schrödinger map initial-value problem

$$\begin{cases} \partial_t \phi = \phi \times \Delta \phi \text{ on } \mathbb{R}^d \times \mathbb{R}; \\ \phi(0) = \phi_0, \end{cases}$$

where $\phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is a smooth function. In all dimensions $d \geq 2$, we prove that the Schrödinger map initial-value problem admits a unique global smooth solution $\phi \in C(\mathbb{R} : H^\infty_Q)$, $Q \in \mathbb{S}^2$, provided that the data $\phi_0 \in H^\infty_Q$ is smooth and satisfies the smallness condition $\|\phi_0 - Q\|_{\dot{H}^{d/2}} \ll 1$. We prove also that the solution operator extends continuously to the space of data in $\dot{H}^{d/2} \cap \dot{H}^{d/2-1}_Q$ with small $\dot{H}^{d/2}$ norm.

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1. Introduction

In this paper we consider the Schrödinger map initial-value problem

$$\begin{cases} \partial_t \phi = \phi \times \Delta \phi \text{ on } \mathbb{R}^d \times \mathbb{R}; \\ \phi(0) = \phi_0, \end{cases} \tag{1.1}$$

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where $d \geq 2$ and $\phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. The Schrödinger map equation has a rich geometric structure and arises in several different ways. For instance, it arises in ferromagnetism as the Heisenberg model for the ferromagnetic spin system whose classical spin ϕ, which belongs to $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, is given by (1.1) in dimensions $d = 1, 2, 3$; we refer the reader to [5], [35], [38], and [31] for more details. In this paper we are concerned with the issue of global well-posedness of the initial-value problem (1.1), in the case of data ϕ_0 which is small in the critical Sobolev spaces $\dot{H}^{d/2}$, $d \geq 2$ (see [5] for results in dimension $d = 1$). Our main result is the direct analogue in the setting of Schrödinger maps of the theorem of Tao [44] on global regularity of wave maps with small critical Sobolev norms. We also prove continuous dependence of solutions on the initial data in certain norms, as in [51].

We start with some notation. Let $\mathbb{Z}_+ = \{0, 1, \ldots\}$. For $\sigma \in [0, \infty)$ let $H^\sigma = H^\sigma(\mathbb{R}^d)$ denote the usual Sobolev spaces of complex valued functions on $\mathbb{R}^d$. For $Q \in \mathbb{S}^2$ we define the metric space

$$H^\sigma_Q = \{f : \mathbb{R}^d : \mathbb{R}^3) : |f(x)| \equiv 1 \text{ a. e. and } f - Q \in H^\sigma\}, \tag{1.2}$$

with the induced distance $d^\sigma_Q(f, g) = \|f - g\|_{H^\sigma}$. For simplicity of notation, let $\|f\|_{H^\sigma_Q} = d^\sigma_Q(f, Q)$ for $f \in H^\sigma_Q$. We also define the metric spaces

$$H^\infty = \bigcap_{\sigma \in \mathbb{Z}_+} H^\sigma \quad \text{and} \quad H^\infty_Q = \bigcap_{\sigma \in \mathbb{Z}_+} H^\sigma_Q,$$

with the induced distances.

Similarly, for $T \in (0, \infty)$ and $\sigma, \rho \in \mathbb{Z}_+$ let $H^{\sigma,\rho}(T)$ denote the Sobolev spaces of complex valued functions in $\mathbb{R}^d \times [-T.T]$ with norm

$$\|f\|_{H^{\sigma,\rho}(T)} = \sup_{t \in (-T,T)} \sum_{\rho'=0}^{\rho} \|\partial_t^{\rho'} f(., t)\|_{H^\sigma}$$

For $Q \in \mathbb{S}^2$ we also define the metric space

$$H^{\sigma,\rho}_Q(T) = \{f : \mathbb{R}^d \times (-T, T) : \mathbb{R}^3); |f(x, t)| \equiv 1 \text{ a. e. and } f - Q \in H^{\sigma,\rho}(T)\},$$

with the distance induced by the $H^{\sigma,\rho}(T)$ norm. Finally, we define the metric spaces

$$H^{\infty,\infty}(T) = \bigcap_{\sigma,\rho \in \mathbb{Z}_+} H^{\sigma,\rho}(T) \quad \text{and} \quad H^{\infty,\infty}_Q(T) = \bigcap_{\sigma,\rho \in \mathbb{Z}_+} H^{\sigma,\rho}_Q(T),$$

with the induced distances.

For $f \in H^\infty$ we define the homogeneous Sobolev norms

$$\|f\|_{\dot{H}^\sigma} = \|\mathcal{F}(f)(\xi) \cdot |\xi|^\sigma\|^2_{L^2}, \qquad \sigma \geq 0.$$

Our first main theorem concerns global existence and uniqueness of solutions of the initial-value problem (1.1) for data $\phi_0 \in H^\infty_Q$, with $\|\phi_0 - Q\|_{\dot{H}^{d/2}} \ll 1$.

Theorem 1.1. *(Global regularity) Assume $d \geq 2$ and $Q \in \mathbb{S}^2$. Then there is $\varepsilon_0(d) > 0$ such that for any $\phi_0 \in H_Q^\infty$ with $\|\phi_0 - Q\|_{\dot{H}^{d/2}} \leq \varepsilon_0(d)$ there is a unique solution $\phi = S_Q(\phi_0) \in C(\mathbb{R} : H_Q^\infty)$ of the initial-value problem* (1.1). *Moreover*

$$\sup_{t\in\mathbb{R}} \|\phi(t) - Q\|_{\dot{H}^{d/2}} \leq C\|\phi_0 - Q\|_{\dot{H}^{d/2}}, \tag{1.3}$$

and, for any $T \in [0,\infty)$ and $\sigma \in \mathbb{Z}_+$,

$$\sup_{t\in[-T,T]} \|\phi(t)\|_{H_Q^\sigma} \leq C(\sigma, T, \|\phi\|_{H_Q^\sigma}). \tag{1.4}$$

Theorem 1.1 was proved in dimensions $d \geq 4$ by the first three authors in [4]. In addition to this global regularity result, we also prove a uniform global bound on certain smooth norms and a well-posedness result, see Theorem 1.2 below. If $\sigma < d/2$ then the completion of H^∞ with respect to the $\dot{H}^\sigma$ norm is a space of distributions which we denote by $\dot{H}^\sigma$. As above, we set

$$\dot{H}_Q^\sigma = \{f : \mathbb{R}^d \to \mathbb{R}^3; f - Q \in \dot{H}^\sigma, \quad |f(x)| \equiv 1 \text{ a.e. in } \mathbb{R}^d\},$$

In the interesting case $\sigma = d/2$ this is no longer the case. Instead, the completion of H^∞ with respect to the $\dot{H}^{d/2}$ norm can be identified with a subspace of the quotient space of distributions modulo constants. For this and other technical reasons, in this article we do not consider the most general problem with initial data in $\dot{H}^{d/2}$ and instead we restrict ourselves to the smaller initial data space $\dot{H}^{d/2} \cap \dot{H}_Q^{d/2-1}$ where the above difficulty does not arise. More precisely, for $\sigma \geq d/2$ and $\varepsilon > 0$ we define

$$\mathcal{B}_\varepsilon^\sigma = \{\phi \in \dot{H}_Q^{d/2-1} \cap \dot{H}^\sigma : \|\phi - Q\|_{\dot{H}^{d/2}} \leq \varepsilon\}$$

with the distance induced by the space $\dot{H}_Q^{d/2-1} \cap \dot{H}^\sigma$. Our second main theorem concerns global wellposedness of the initial-value problem (1.1) for initial data in $\mathcal{B}_\varepsilon^\sigma$, $\sigma \geq d/2$, $\varepsilon \ll 1$.

Theorem 1.2. *(Uniform bounds and well-posedness) Assume $d \geq 2$, $Q \in \mathbb{S}^2$ and $\sigma_1 \geq d/2$. Then there is $\varepsilon_0(d, \sigma_1) \in (0, \varepsilon_0(d)]$ such that for any $\phi_0 \in H_Q^\infty$ with $\|\phi - Q\|_{\dot{H}^{d/2}} \leq \varepsilon_0(d, \sigma_1)$ the global solution $\phi = S_Q(\phi_0) \in C(\mathbb{R} : H_Q^\infty)$ constructed in Theorem 1.1 satisfies the uniform bound*

$$\sup_{t\in\mathbb{R}} \|\phi(t) - Q\|_{H^\sigma} \leq C_\sigma \|\phi_0 - Q\|_{H^\sigma}, \qquad d/2 \leq \sigma \leq \sigma_1. \tag{1.5}$$

In addition, for any $\sigma \in [d/2, \sigma_1]$ the operator S_Q admits a continuous extension

$$S_Q : \mathcal{B}_{\varepsilon_0(d,\sigma_1)}^\sigma \to C(\mathbb{R}, \dot{H}^\sigma \cap \dot{H}_Q^{d/2-1}).$$

Our analysis gives more information about the global solution ϕ; we can prove for instance that $\nabla\phi$ satisfies all the Strichartz estimates globally in time. The rough solutions obtained in Theorem 1.2 as uniform limits of smooth solutions can be also shown to satisfy the equation (1.1) in a suitable distributional sense. The global bound (1.5) is sometimes interpreted as the absence of "weak turbulence".

We record also two conservation laws for solutions of the Schrödinger map equation (1.1): if $\phi \in C((T_1, T_2) : H_Q^\infty)$ solves the equation $\partial_t \phi = \phi \times \Delta_x \phi$ on some time interval (T_1, T_2) then the quantities

$$E_0(t) = \int_{\mathbb{R}^d} |\phi(t) - Q|^2 \, dx \quad \text{and} \quad E_1(t) = \int_{\mathbb{R}^d} \sum_{m=1}^{d} |\partial_m \phi(t)|^2 \, dx \tag{1.6}$$

are conserved. In particular, with $S_Q(\phi_0)$ as in Theorem 1.2,

$$\|S_Q(\phi_0)(t)\|_{H_Q^0} = \|\phi_0\|_{H_Q^0}, \qquad \|S_Q(\phi_0)(t)\|_{H_Q^1} = \|\phi_0\|_{H_Q^1}, \quad t \in \mathbb{R}$$

As mentioned earlier, the direct analogue of Theorem 1.1 in the setting of wave maps is the theorem of Tao [44]. However, our proof of Theorem 1.1 is closer to that of [39], [34], [29], and [30], in the sense that we prove a priori bounds on the derivatives of the Schrödinger map ϕ, in a suitable gauge, rather than the Schrödinger map itself. See also [24], [26], [49], [50], [43], [25], [39], [34], [29], [30], and [51] for other local and global regularity (or well-posedness) theorems for wave maps. A complete account of the main ideas in the work on wave maps can be found in the book [48, Chapter 6].

We remark that, while from the geometric and algebraic points of view there are many similarities between wave maps and Schrödinger maps, there is a fundamental difference from the analytic point of view. This is mainly due to the fact that it is much more difficult to handle perturbatively derivatives in the nonlinearity for Schrödinger equations than for wave equations. This reflects the fact that wave equations have two time derivatives, while Schrödinger equations have only one, with corresponding effect on the Cauchy data (see [47, p. 268] for a related discussion). Thus, for wave equations, at least in high dimensions, there are large classes of Strichartz estimates which can be used to control derivative nonlinearities in a perturbative way. This is not the case for Schrödinger equations. To deal with this problem for Schrödinger equations, Kenig, Ponce, and Vega [19] introduced for the first time a method to obtain local well-posedness for general derivative nonlinearity Schrödinger equations. This method combines "local smoothing estimates", "inhomogeneous local smoothing estimates", which give the crucial gain of one derivative, and "maximal function estimates". Further results are in [20] and [21].

The initial-value problem (1.1) has been studied extensively (also in the case in which the sphere $\mathbb{S}^2$ is replaced by more general targets). It is known that sufficiently smooth solutions exist locally in time, even for large data (see, for example, [41], [5], [8], [33] and the references therein). Such theorems for (local in time) smooth solutions are proved using delicate geometric variants of the energy method. For low-regularity data, the initial-value problem (1.1) has been studied indirectly using the "modified Schrödinger map equations" and certain enhanced energy methods (see, for example, [5], [35], [36], [17], [15], and [16]), and directly, in the case of small data, using fixed point arguments in suitable spaces (see [13], [1]).

The first global well-posedness result for (1.1) in critical spaces (precisely, global well-posedness for small data in the critical Besov spaces in dimensions $d \geq 3$) was proved by two of the authors in [14], and independently by the first author [2]. This was later improved to global regularity for small data in the critical Sobolev spaces in dimensions $d \geq 4$ in [4]. In dimension $d = 2$, in the case of equivariant data with energy close to the energy of the equivariant harmonic map, the existence of global equivariant solutions (and asymptotic stability) was proved in [9]. In the case of radial or equivariant data of small energy, global well-posedness was proved in [5].

The global results in [14], [2], and [4] use in a fundamental way the strong "local smoothing", "inhomogeneous local smoothing", and "maximal function" spaces

$$L^{\infty,2}_{\mathbf{e}} \qquad L^{1,2}_{\mathbf{e}} \qquad L^{2,\infty}_{\mathbf{e}}. \tag{1.7}$$

See (3.3) for definitions and a longer discussion. These spaces were introduced earlier in the study of Schrödinger maps by two of the authors in [13], and replace the corresponding spaces in [19]–[21] (where everything was localized to finite cubes). It is essential to work with the strong spaces in (1.7) instead of their localized versions in order to be able to prove global in time results. The spaces in (1.7) were first used by Linares and Ponce [32] to study the local well-posedness of the Davey–Stewartson system. Other uses of such spaces (implicit or explicit) to prove local-wellposedness are in [22], [11], and [7]. In the case of global well-posedness, the strong spaces (1.7) were used for the first time by two of the authors in [12] in the study of the Benjamin–Ono equation in L^2.

As mentioned earlier, Theorem 1.1 was proved in dimensions $d \geq 4$ by the first three authors in [4]. It is likely that the proof in [4] can be extended to dimension $d = 3$, provided one uses some type of "dynamical separation" to bound High $\times$ High $\rightarrow$ Low frequency interactions that appear in the connection coefficients A_m of the Coulomb gauge (as in [29] and [30] in the case of wave maps). There are, however, two significant difficulties in dimension $d = 2$. The first main difficulty is related to the maximal function estimate

$$\|e^{it\Delta}\phi\|_{L^{2,\infty}_{\mathbf{e}}} \lesssim \|\phi\|_{L^2_x} \tag{1.8}$$

for functions $\phi \in L^2(\mathbb{R}^d)$ with $\mathcal{F}(\phi)$ supported in $\{\xi \in \mathbb{R}^d : |\xi| \in [1/2, 2]\}$. This estimate holds in dimensions $d \geq 3$ (see [13]) and plays a key role in the global results of [14], [4], and [2], but fails "logarithmically" in dimension $d = 2$. Because of this logarithmic failure, in dimension $d = 2$ we replace the space $L^{2,\infty}_{\mathbf{e}}$ in the left-hand side of (1.8) with a sum of Galilean transforms of it (see the precise definitions in Section 3). The idea of using such sums of spaces as substitutes for missing estimates in low dimensions is due to Tataru [50], in the setting of wave maps, where the Lorentz invariance and Strichartz spaces are used instead of the Galilean invariance and the maximal function space. These substitutes have played a key role in all the subsequent work on global wave maps in dimensions 2 and 3.

The second main difficulty is related to the choice of a suitable system of coordinates (or gauge) for perturbative analysis. The use of gauges for the equation (1.1) was pioneered in [5], where orthonormal frames were first used in the context of Schrödinger maps. These constructions have been presented by J. Shatah, his students and collaborators on several occasions (see [33], [37] and the references therein). Unlike in dimensions $d \geq 3$, in dimension $d = 2$ it appears that one cannot use the standard Coulomb gauge, even if the analysis of the Schrödinger equation is combined with elliptic techniques such as the dynamical separation mentioned earlier. We substitute the Coulomb gauge with Tao's caloric gauge introduced in [45], see section 2 and [48, Chapter 6] for a longer discussion on the various gauges used in the study of wave maps. As explained in [45], this caloric gauge leads to better estimates on the connection coefficients A_m than the Coulomb gauge, which allow us to close the perturbative part of the argument.

It is important to notice that the main components of the spaces we use for our perturbative analysis are the strong local smoothing, inhomogeneous local smoothing, and maximal function spaces in (1.7), as well as Galilean transformations of these spaces in dimension $d = 2$ (see Definition (3.3)). In particular, we do not use $X^{s,b}$-type structures that have been frequently used in the subject. All of our norms are defined in the physical space, without the use of the Fourier transform (except for dyadic localizations), and are very simple, see Definitions 3.6 and 3.7, at least when compared to the corresponding spaces used in the study of wave maps in dimensions 2 and 3. This reflects the cubic nature of the main part of the nonlinearity; see also [23] for another instance of this phenomenon. The simplicity of these spaces is due, in part, to the geometric nature and the efficiency of the caloric gauge, compared to other gauges used in the study of wave maps and Schrödinger maps.

Most of our construction is geometric and can be written in covariant form. There is one exception, however, namely the definition of the space H^0_Q, which depends on the Euclidean distance $|\phi(x) - Q|$. The supercritical quantity $\|\phi(t)\|_{H^0_Q} = E_0(t)$ is conserved through the Schrödinger map flow, see (1.6). It is useful to have control of such a supercritical quantity in the construction of the caloric gauge in Proposition 4.2, particularly in dimension $d = 2$, in order to be able to prove that the orthonormal frame v, w does indeed trivialize as the heat time s tends to infinity.

We prefer, however, to adopt the extrinsic point of view throughout the paper: we think of smooth maps $g : D \to \mathbb{S}^2$, where D is some domain, as maps $g : D \to \mathbb{R}^3$ (thus (3×1) matrices) with $|g| \equiv 1$. With this point of view, an orthonormal frame of $g^*T\mathbb{S}^2$ on D is simply a pair of smooth maps $v, w : D \to \mathbb{S}^2$ such that ${}^tv \cdot g = {}^tw \cdot g = {}^tv \cdot w = 0$ on D. See [48, Chapter 6] for a discussion on the relation between the intrinsic and the extrinsic points of view, in the setting of wave maps. The extrinsic formalism we use in this paper was explained to us by T. Tao [46].

For the sake of completeness we write the proof of the main theorems in all dimensions $d \geq 2$. We emphasize however that many of the difficulties are only present in dimension $d = 2$. In dimensions $d \geq 3$ the main normed spaces $F_k(T)$, $G_k(T)$, and $N_k(T)$ are simpler. Also the analysis related to High $\times$ High $\to$ Low frequency interactions, which motivates the use of the caloric gauge, is easier.

We thank T. Tao for a discussion on the benefits of the caloric gauge and for explaining us the elementary extrinsic formalism we use in this paper.

2. The differentiated equations and the caloric gauge

In this section we start with a smooth solution ϕ to the Schrödinger map equation and a smooth orthonormal frame (v, w) in $T_\phi\mathbb{S}^2$. Then we construct the fields ψ_m and the connection coefficients A_m, and derive the differentiated (modified) Schrödinger map equations satisfied by these functions (see [5], [35] where the modified Schrödinger map equations were introduced, and [37] for a detailed discussion on the connection between the modified Schrödinger maps and the original equation). Next we introduce the caloric gauge. This is done by solving first a covariant heat equation which leads to an extension of smooth Schrödinger maps to parabolic time $s \in [0, \infty)$. We then construct the orthonormal frame (v, w) by solving an ordinary differential equation with data prescribed at infinity in order to construct the orthonormal frame (v, w). This construction is due to Tao [45].

The Schrödinger map equation leads to the system (2.11) of d scalar Schrödinger equations satisfied by the fields ψ_m, $m = 1, \dots, d$, at heat time $s = 0$. The caloric gauge condition allows us to express the connection coefficients A_m in terms of the parabolic extensions of the differentiated fields ψ_m, see (2.20). Finally, we derive the linearized Schrödinger map equation and we express it in the frame form (2.25).

We begin with a smooth function $\phi : \mathbb{R}^d \times (-T, T) \to \mathbb{S}^2$. Instead of working directly on the equation (1.1) for the function ϕ it is convenient to study the equations satisfied by its derivatives $\partial_m\phi(x, t)$ for $m = 1, d + 1$, where $\partial_{d+1} = \partial_t$. These are tangent vectors to the sphere at $\phi(x, t)$. Suppose we have a smooth orthonormal frame $(v(t, x), w(t, x))$ in $T_{\phi(x,t)}\mathbb{S}^2$. Then we can introduce the differentiated variables,

$$\psi_m = {}^t v \cdot \partial_m\phi + i\ {}^t w \cdot \partial_m\phi. \tag{2.1}$$

Thus we can express $\partial_m\phi$ in the (v, w) frame as

$$\partial_m\phi = v\Re(\psi_m) + w\Im(\psi_m). \tag{2.2}$$

In order to write the equations for ψ_m we need to know how v and w vary as functions of (x, t). For this we introduce the real coefficients

$$A_m = {}^t w \cdot \partial_m v. \tag{2.3}$$

In particular this allows us to complement (2.2) with

$$\begin{cases} \partial_m v = -\phi \Re(\psi_m) + w A_m; \\ \partial_m w = -\phi \Im(\psi_m) - v A_m. \end{cases} \tag{2.4}$$

The variables ψ_m are not independent, instead they satisfy the curl type relations

$$(\partial_l + iA_l)\psi_m = (\partial_m + iA_m)\psi_l. \tag{2.5}$$

Thus with the notation $\mathbf{D}_m = \partial_m + iA_m$ we can rewrite this as

$$\mathbf{D}_l \psi_m = \mathbf{D}_m \psi_l. \tag{2.6}$$

A direct computation using the definition of A_m shows that

$$\partial_l A_m - \partial_m A_l = \Im(\psi_l \overline{\psi_m}) = q_{lm}. \tag{2.7}$$

Thus the curvature of the connection is given by

$$\mathbf{D}_l \mathbf{D}_m - \mathbf{D}_m \mathbf{D}_l = i q_{lm}. \tag{2.8}$$

Assume now that the smooth function ϕ satisfies the Schrödinger map equation $\partial_t \phi = \phi \times \Delta_x \phi$. Then we derive the Schrödinger equations for the functions ψ_m. A direct computation, using (2.5), (2.7), $\phi \times v = w$, and $\phi \times w = -v$, shows that

$$\psi_{d+1} = i \sum_{l=1}^{d} \mathbf{D}_l \psi_l. \tag{2.9}$$

Using (2.6) and (2.8), it follows that for $m = 1, \ldots, d$

$$\mathbf{D}_{d+1} \psi_m = i \sum_{l=1}^{d} \mathbf{D}_l \mathbf{D}_l \psi_m + \sum_{l=1}^{d} q_{lm} \psi_l, \tag{2.10}$$

which is equivalent to

$$(i\partial_t + \Delta_x)\psi_m = -2i \sum_{l=1}^{d} A_l \partial_l \psi_m + \big(A_{d+1} + \sum_{l=1}^{d} (A_l^2 - i\partial_l A_l)\big)\psi_m - i \sum_{l=1}^{d} \psi_l \Im(\overline{\psi_l} \psi_m). \tag{2.11}$$

Consider the system of equations which consists of (2.5), (2.7) and (2.11). The solution $\{\psi_m\}$ for the above system cannot be uniquely determined as it depends on the choice of the orthonormal frame (v, w). Precisely, it is invariant with respect to the gauge transformation

$$\psi_m \to e^{i\theta} \psi_m, \qquad A_m \to A_m + \partial_m \theta.$$

In order to obtain a well-posed system one needs to make a choice which uniquely determines the gauge. Ideally one may hope that this choice uniquely determines the A_m's in terms of the ψ_m's in a way that makes the nonlinearity in (2.11) perturbative.

A natural choice for this is the Coulomb gauge, where one adds the equation

$$\sum_{m=1}^{d} \partial_m A_m = 0$$

which in view of (2.7) leads to

$$A_m = \Delta^{-1} \sum_{l=1}^{d} \partial_l \Im(\bar{\psi}_l \psi_m), \qquad m = 1, \dots, d. \tag{2.12}$$

This choice works well in high dimension $d \geq 4$, see [4]; however, in low dimension there are difficulties caused by the contributions of two high frequencies in ψ to the low frequencies in A.

This is what causes us to look for a different choice of gauge, namely the caloric gauge. This was proposed in [45] in the context of the wave map equation, and then as a possible gauge for Schrödinger maps [46].

Precisely, at each time t we solve a covariant heat equation with $\phi(t)$ as the initial data,

$$\begin{cases} \partial_s \widetilde{\phi} = \Delta_x \widetilde{\phi} + \widetilde{\phi} \cdot \sum_{m=1}^{d} |\partial_m \widetilde{\phi}|^2 \quad \text{ on } [0, \infty) \times \mathbb{R}^d; \\ \widetilde{\phi}(0, t, x) = \phi(t, x). \end{cases} \tag{2.13}$$

We heuristically remark that as the heat time s approaches infinity, the solution $\phi(s)$ approaches the equilibrium state Q. This is related to our assumption that the "mass" E_0 of ϕ_0 is finite, and would not necessarily be true otherwise. This would allow us to arbitrarily pick (v_∞, w_∞) at $s = \infty$ as an arbitrary orthonormal base in $T_Q\mathbb{S}^2$, independently of t and x. To define the orthonormal frame (v, w) for all $s \geq 0$ we pull back (v_∞, w_∞) along the backward heat flow using parallel transport. This translates into the relation

$$^t w \cdot \partial_s v = 0 \tag{2.14}$$

The existence of a global smooth solution for the caloric equation (4.7) and of the corresponding frame (v, w) is proved in Proposition 4.2. In particular for each $F \in \{\widetilde{\phi} - Q, v - v_\infty, w - w_\infty\}$ the following decay properties are valid:

$$|\partial_x^\alpha F(s)| \leq c_\alpha \langle s \rangle^{-(|\alpha|+1)/2}, \qquad s \geq 0 \tag{2.15}$$

Setting $\partial_0 = \partial_s$ we can define the functions ψ_m and A_m for all $s \in [0, \infty)$ and $m = 0, \cdots, d+1$ by

$$\begin{cases} \psi_m = {}^t v \cdot \partial_m \widetilde{\phi} + i\, {}^t w \cdot \partial_m \widetilde{\phi}; \\ A_m = {}^t w \cdot \partial_m v. \end{cases} \tag{2.16}$$

Then the relations (2.5)-(2.8) hold for all $l, m = 0, d+1$. In addition, the parallel transport relation ${}^t w \cdot \partial_s v = 0$ yields the main gauge condition

$$A_0 = 0. \tag{2.17}$$

As in the case of the Schrödinger equation, a direct computation using the heat equation (4.7) and (2.5), (2.7) shows that

$$\psi_0 = \sum_{l=1}^{d} \mathbf{D}_l \psi_l. \tag{2.18}$$

Thus, using again (2.7), for any $m = 1, \dots, d+1$

$$\begin{aligned}\partial_0 \psi_m = \mathbf{D}_m \psi_0 = \sum_{l=1}^{d} \mathbf{D}_m \mathbf{D}_l \psi_l &= \sum_{l=1}^{d} \mathbf{D}_l \mathbf{D}_m \psi_l + i \sum_{l=1}^{d} q_{ml} \psi_l \\ &= \sum_{l=1}^{d} \mathbf{D}_l \mathbf{D}_l \psi_m + i \sum_{l=1}^{d} \Im(\psi_m \overline{\psi_l}) \psi_l,\end{aligned}$$

which is equivalent to

$$(\partial_s - \Delta_x)\psi_m = 2i \sum_{l=1}^{d} A_l \partial_l \psi_m - \sum_{l=1}^{d} (A_l^2 - i\partial_l A_l)\psi_m + i \sum_{l=1}^{d} \Im(\psi_m \overline{\psi_l})\psi_l. \tag{2.19}$$

On the other hand from (2.7) we obtain

$$\partial_s A_m = \Im(\psi_0 \overline{\psi_m}).$$

Due to (4.12) and (4.13) we can integrate back from $s = \infty$ to obtain

$$A_m(s) = -\int_s^\infty \Im(\psi_0 \overline{\psi_m})(r)\, dr = -\sum_{l=1}^{d} \int_s^\infty \Im\big(\overline{\psi_m}(\partial_l \psi_l + iA_l \psi_l)\big)(r)\, dr, \tag{2.20}$$

for any $m = 1, \dots, d+1$ and $s \in [0, \infty)$. Then $A_{m|s=0}$ represents our choice of the gauge for the Schrödinger map equation. The reason we prefer the caloric gauge to the Coulomb gauge is the way the high-high frequency interactions are handled. Indeed, while (2.12) can be conceptually written in the form

$$A \approx \sum_{j<k} 2^{-k} P_j \psi P_k \psi + \sum_{j \le k} 2^{-j} P_j (P_k \psi P_k \psi),$$

substituting the first approximation $\psi(s) \approx e^{s\Delta}\psi(0)$ in (2.20) yields the relation

$$A \approx \sum_{j<k} 2^{-k} P_j \psi P_k \psi + \sum_{j \le k} 2^{-k} P_j (P_k \psi P_k \psi). \tag{2.21}$$

This has a better frequency factor in the high $\times$ high $\to$ low frequency interactions.

We consider now linearized Schrödinger map equations. This is necessary in order to establish the continuous dependence of the solutions on the initial data. The linearized equation along a Schrödinger map ϕ has the form

$$\partial_t \phi_{lin} = \phi_{lin} \times \Delta\phi + \phi \times \Delta\phi_{lin} \tag{2.22}$$

where

$$^t\phi_{lin} \cdot \phi = 0.$$

Then we can express ϕ_{lin} in the (v, w) frame by setting

$$\psi_{lin} = {}^t v \cdot \phi_{lin} + i\, {}^t w \cdot \phi_{lin}, \tag{2.23}$$

or equivalently

$$\phi_{lin} = v\Re(\psi_{lin}) + w\Im(\psi_{lin}). \tag{2.24}$$

The field ψ_{lin} satisfies the linearized equation

$$\begin{aligned}(i\partial_t + \Delta_x)\psi_{lin}\\ = -2i\sum_{l=1}^{d} A_l\partial_l\psi_{lin} + \big(A_{d+1} + \sum_{l=1}^{d}(A_l^2 - i\partial_l A_l)\big)\psi_{lin} - i\sum_{l=1}^{d}\psi_l\Im(\overline{\psi_l}\psi_{lin}).\end{aligned} \tag{2.25}$$

This can be derived by direct computations as before. Heuristically, one can also think of a one parameter family of solutions $\phi(h)$ for the Schrödinger map equation so that $\phi(0) = \phi$ and $\partial_h\phi_{|h=0} = \phi_{lin}$, and extend the frame (v, w) as h varies. Then we have $\psi_{lin} = \psi_h$ and we can use (2.10) with m replaced by h.

3. Function spaces

In this section we define our main function spaces and derive some of their properties. We define first several cutoff functions and (smooth) projection operators.

Definition 3.1. *We fix $\eta_0 : \mathbb{R} \to [0, 1]$ a smooth even function supported in the set $\{\mu \in \mathbb{R} : |\mu| \leq 8/5\}$ and equal to 1 in the set $\{\mu \in \mathbb{R} : |\mu| \leq 5/4\}$. We define*

$$\chi_j(\mu) = \eta_0(\mu/2^j) - \eta_0(\mu/2^{j-1}), \quad \chi_{\leq j} = \eta_0(\mu/2^j), \qquad j \in \mathbb{Z}.$$

Let P_k denote the operator on $L^\infty(\mathbb{R}^d)$ defined by the Fourier multiplier $\xi \to \chi_k(|\xi|)$. For any interval $I \subseteq \mathbb{R}$, let $\chi_I = \sum_{j\in I}\chi_j$ and let P_I denote the operator on $L^\infty(\mathbb{R}^d)$ defined by the Fourier multiplier $\xi \to \sum_{k\in I}\chi_k(|\xi|)$. For simplicity of notation, we define $P_{\leq k} = P_{(-\infty,k]}$. For any $\mathbf{e} \in \mathbb{S}^{d-1}$ and $k \in \mathbb{Z}$ we define the operators $P_{k,\mathbf{e}}$ by the Fourier multipliers $\xi \to \chi_k(\xi \cdot \mathbf{e})$.

To motivate our choice of spaces, recall the Schrödinger nonlinearities, see (2.11)

$$L_m = -2i\sum_{l=1}^{d} A_l\partial_l\psi_m + \big(A_{d+1} + \sum_{l=1}^{d}(A_l^2 - i\partial_l A_l)\big)\psi_m - i\sum_{l=1}^{d}\psi_l\Im(\overline{\psi}_l\psi_m). \tag{3.1}$$

We would like to analyze these nonlinearities perturbatively in suitable spaces. The main difficulty is caused by the magnetic terms $-2i\sum_{l=1}^{d} A_l\partial_l\psi_m$. Using (2.21) (for simplicity consider only the terms corresponding to $k = j$) they can be written schematically in the form

$$\sum_{k,k'\in\mathbb{Z}} 2^{-k}P_k\psi P_k\psi \cdot 2^{k'}P_{k'}\psi. \tag{3.2}$$

The difficulty is to estimate the sum over $k \leq k' - 100$ in (3.2). The main ingredient needed to estimate such magnetic terms in Schrödinger problems is the local smoothing phenomenon: we place the highest frequency factor of the nonlinearity in the local smoothing space $L^{\infty,2}_{\mathbf{e}}$ (see definition below) and attempt to estimate the nonlinearity in the inhomogeneous local smoothing space $L^{1,2}_{\mathbf{e}}$. This allows us to barely recover the full derivative loss of the magnetic nonlinearity. This approach, with certain local smoothing and inhomogeneous local smoothing spaces localized to cubes, was first used in [19] to study local well-posedness of Schrödinger equations with general derivative nonlinearities. To prove global results, it is essential to work with stronger local smoothing/inhomogeneous local smoothing spaces, which exploit better the geometry of Euclidean spaces. A low-regularity global result using such stronger local smoothing spaces was proved by two of the authors in [12].

This scheme was first used in the setting of Schrödinger maps by two of the authors in [13] and played a key role in all the global results in [14], [4], and [2]. In order for this scheme to work in the Schrödinger map problem we need to be able to control the $L^{2,\infty}_{\mathbf{e}}$ norm (in a scale invariant way) of the low frequency terms.

For a unit vector $\mathbf{e} \in \mathbb{S}^{d-1}$ we denote by $H_{\mathbf{e}}$ its orthogonal complement in $\mathbb{R}^d$ with the induced measure. We define the lateral spaces $L^{p,q}_{\mathbf{e}}$ with norms

$$\|h\|_{L^{p,q}_{\mathbf{e}}} = \Big[\int_{\mathbb{R}} \Big[\int_{H_{\mathbf{e}} \times \mathbb{R}} |h(x_1\mathbf{e} + x', t)|^q \, dx'dt \Big]^{p/q} dx_1 \Big]^{1/p} \tag{3.3}$$

with the usual modifications when $p = \infty$ or $q = \infty$. The key spaces in this family are the local smoothing space $L^{\infty,2}_{\mathbf{e}}$ and the inhomogeneous local smoothing space $L^{1,2}_{\mathbf{e}}$. These are the only spaces in this family that can be used to analyze perturbatively magnetic nonlinearities, such as L_m. This is in sharp contrast with the case of wave equations, where large classes of Strichartz estimates can be used to control magnetic nonlinearities, at least in high dimensions. To make the transition between the local smoothing and the inhomogeneous local smoothing spaces we use the maximal function spaces $L^{2,\infty}_{\mathbf{e}}$.

The following local smoothing/maximal function estimates were proved by two of the authors in [13] and [14]:

Lemma 3.2. *If $f \in L^2(\mathbb{R}^d)$, $k \in \mathbb{Z}$, and $\mathbf{e} \in \mathbb{S}^{d-1}$ then*

$$\|e^{it\Delta} P_{k,\mathbf{e}} f\|_{L^{\infty,2}_{\mathbf{e}}} \lesssim 2^{-k/2} \|f\|_{L^2}. \tag{3.4}$$

In addition, if $d \geq 3$ then

$$\|e^{it\Delta} P_k f\|_{L^{2,\infty}_{\mathbf{e}}} \lesssim 2^{k(d-1)/2} \|f\|_{L^2}. \tag{3.5}$$

It is easy to see that the two bounds (3.4) and (3.5) cooperate in the right way to allow us to estimate the expression in (3.2) in the inhomogeneous local smoothing space $L^{1,2}_{\mathbf{e}}$. The bounds (3.4) and (3.5) are not hard to prove. They depend, however, on delicate global properties of the Euclidean geometry.

Unfortunately, the maximal function bound (3.5) fails in dimension $d = 2$, which causes considerable difficulties. To handle this case we need to use the Galilean invariance: if f solves $(i\partial_t + \Delta_x)f = 0$ in $\mathbb{R}^d \times \mathbb{R}$, then $T_w(f)$ solves the same equation, where T_w, $w \in \mathbb{R}^d$, is the Galilean operator defined below.

Definition 3.3. *Assume $d = 2$, $p, q \in [1, \infty]$, $\mathbf{e} \in \mathbb{S}^1$, $\lambda \in \mathbb{R}$ and $W \subseteq \mathbb{R}$ finite. We define the spaces $L^{p,q}_{e,\lambda}$ using the norms*

$$\begin{aligned} &\|h\|_{L^{p,q}_{\mathbf{e},\lambda}} = \|T_{\lambda\mathbf{e}}(h)\|_{L^{p,q}_{\mathbf{e}}} = \Big[\int_{\mathbb{R}} \Big[\int_{H_{\mathbf{e}}\times\mathbb{R}} |h((x_1 + \lambda t)\mathbf{e} + x', t)|^q \, dx' dt\Big]^{p/q} dx_1\Big]^{1/p};\\ &T_w(h)(x, t) = e^{-ix\cdot w/2} e^{-it|w|^2/4} h(x + tw, t). \end{aligned} \tag{3.6}$$

with the usual modifications when $p = \infty$ or $q = \infty$. Then we define the spaces $L^{p,q}_{\mathbf{e},W}$

$$L^{p,q}_{\mathbf{e},W} = \sum_{\lambda\in W} L^{p,q}_{\mathbf{e},\lambda}, \qquad \|h\|_{L^{p,q}_{\mathbf{e},W}} = \inf_{h=\sum_{\lambda\in W} h_\lambda} \sum_{\lambda\in W} \|h_\lambda\|_{L^{p,q}_{\mathbf{e},\lambda}}.$$

In what follows we fix some large integer $\mathcal{K}$ and define, for $k \in \mathbb{Z}$,

$$W_k = W_k(\mathcal{K}) = \{\lambda \in [-2^k, 2^k] : 2^{k+2\mathcal{K}}\lambda \in \mathbb{Z}\}.$$

Lemma 3.4. *Let $d = 2$. For any $f \in L^2$, $k \in \mathbb{Z}$, and $\mathbf{e} \in \mathbb{S}^1$ we have*

$$\|e^{it\Delta} P_{k,\mathbf{e}} f\|_{L^{\infty,2}_{\mathbf{e},\lambda}} \lesssim 2^{-k/2} \|f\|_{L^2}, \qquad |\lambda| \le 2^{k-40}. \tag{3.7}$$

In addition, if $T \in (0, 2^{2\mathcal{K}}]$ then

$$\|1_{[-T,T]}(t) e^{it\Delta} P_k f\|_{L^{2,\infty}_{\mathbf{e},W_{k+40}}} \lesssim 2^{k/2} \|f\|_{L^2}. \tag{3.8}$$

The bound (3.7) is a straightforward consequence of (3.4) via a Galilean transformation. The main novelty is the estimate (3.8), which provides a usable replacement to (3.5) in dimension $d = 2$. Indeed, it is easy to see that the bounds (3.7) and (3.8) can still be used to estimate the expression in (3.2) in the space $L^{1,2}_{\mathbf{e},W_{k-40}}$, which is an acceptable inhomogeneous local smoothing space in dimension $d = 2$. The idea of using sums of spaces such as $L^{2,\infty}_{\mathbf{e},W_{k+40}}$ as substitutes for missing estimates in the setting of wave maps is due to Tataru [50].

Limiting the time T to the interval $(0, 2^{2\mathcal{K}}]$ is what allows us to use the discretization which is given by the W_k sets. One could also allow T to be arbitrarily large, at the expense of replacing the discrete sums in the definition of the $L^{2,\infty}_{\mathbf{e},W_{k+40}}$ with a continuous counterpart. We do not pursue this here in order to avoid distracting technicalities.

Once the main terms of the Schrödinger nonlinearities (3.1) are under control, we can use various Strichartz-type estimates to estimate the remaining terms. We state below the Strichartz-type estimates we need in this paper; at this stage many variations are possible. Let $p_d = (2d + 4)/d$ denote the Strichartz exponent.

Lemma 3.5. *If $f \in L^2(\mathbb{R}^d)$ then we have the Strichartz estimates*

$$\|e^{it\Delta} f\|_{L^{p_d}_{x,t}} \lesssim \|f\|_{L^2},$$

and the maximal function bounds

$$\|e^{it\Delta} P_k f\|_{L^{p_d}_x L^\infty_t} \lesssim 2^{kd/(d+2)} \|f\|_{L^2}, \qquad k \in \mathbb{Z}$$

In addition, if $2/p + d/q = d/2$ and $2 \leq q < 2d(d-2)$ then

$$\sup_{\mathbf{e} \in \mathbb{S}^1} \|e^{it\Delta} P_k f\|_{L^{p,q}_{\mathbf{e}}} \lesssim_p 2^{k(\frac{d+2}{dp} - \frac{1}{2})} \|f\|_{L^2}, \qquad p \leq q,$$

respectively

$$\sup_{\mathbf{e} \in \mathbb{S}^1} \|e^{it\Delta} P_{k,\mathbf{e}} f\|_{L^{p,q}_{\mathbf{e}}} \lesssim 2^{k(\frac{d+2}{dp} - \frac{1}{2})} \|f\|_{L^2}, \qquad p \geq q.$$

The first bound in Lemma 3.5 is the original Strichartz estimate [40]. The second bound follows by scaling. The last two bounds, which we call lateral Strichartz estimates, follow informally by interpolation between the L^{p_d} Strichartz estimate and the local smoothing/maximal function estimates of Lemma 3.4. The results stated in Lemma 3.2, Lemma 3.4, and Lemma 3.5 are summarized and proved later in Lemma 7.1 (we prove in fact a slightly stronger version of (3.8), which is needed to prove full inhomogeneous estimates).

We are now ready to define our main dyadic function spaces $F_k(T)$, $G_k(T)$ and $N_k(T)$. Assume that $T \in \mathbb{R}$. For $k \in \mathbb{Z}$ let $I_k = \{\xi \in \mathbb{R}^d : |\xi| \in [2^{k-1}, 2^{k+1}]\}$ and

$$L^2_k(T) = \{f \in L^2(\mathbb{R}^d \times [-T, T]) : \mathcal{F}(f) \text{ is supported in } I_k \times \mathbb{R}\},$$

Definition 3.6. *Assume $d \geq 3$, $T \in \mathbb{R}$, and $k \in \mathbb{Z}$. Then $F_k(T)$, $G_k(T)$ and $N_k(T)$ are the Banach spaces of functions in $L^2_k(T)$ for which the corresponding norms are finite:*

$$\|\phi\|_{F_k(T)} = \|\phi\|_{L^\infty_t L^2_x} + \|\phi\|_{L^{p_d}} + 2^{-kd/(d+2)} \|\phi\|_{L^{p_d}_x L^\infty_t} + 2^{-k(d-1)/2} \sup_{\mathbf{e} \in \mathbb{S}^{d-1}} \|\phi\|_{L^{2,\infty}_{\mathbf{e}}}, \tag{3.9}$$

$$\|\phi\|_{G_k(T)} = \|\phi\|_{F_k} + 2^{k/2} \sup_{|j-k| \leq 20} \sup_{\mathbf{e} \in \mathbb{S}^{d-1}} \|P_{j,\mathbf{e}} \phi\|_{L^{\infty,2}_{\mathbf{e}}}, \tag{3.10}$$

respectively

$$\|f\|_{N_k(T)} = \inf_{f = f_1 + f_2} \|f_1\|_{L^{p_d'}} + 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^{d-1}} \|f_2\|_{L^{1,2}_{\mathbf{e}}}. \tag{3.11}$$

Definition 3.7. *Assume that $d = 2$, $k \in \mathbb{Z}$, $\mathcal{K} \in \mathbb{Z}_+$, and $T \in (0, 2^{2\mathcal{K}}]$. Recall the definition $W_k = \{\lambda \in [-2^k, 2^k] : 2^{k+2\mathcal{K}} \lambda \in \mathbb{Z}\}$. For $\phi \in L^2(\mathbb{R}^d \times [-T, T])$ let*

$$\|\phi\|_{F^0_k(T)} = \|\phi\|_{L^\infty_t L^2_x} + \|\phi\|_{L^4} + 2^{-k/2} \|\phi\|_{L^4_x L^\infty_t} + 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^1} \|\phi\|_{L^{2,\infty}_{\mathbf{e}, W_{k+40}}}. \tag{3.12}$$

We define $F_k(T)$, $G_k(T)$ and $N_k(T)$ as the normed spaces of functions in $L^2_k(T)$ for which the corresponding norms are finite:

$$\|\phi\|_{F_k(T)} = \inf_{J,m_1,\dots,m_J\in\mathbb{Z}_+} \inf_{f=f_{m_1}+\dots+f_{m_J}} \sum_{j=1}^{J} 2^{m_j}\|f_{m_j}\|_{F^0_{k+m_j}}, \tag{3.13}$$

$$\begin{aligned}\|\phi\|_{G_k(T)} = \|\phi\|_{F^0_k} &+ 2^{-k/6}\sup_{\mathbf{e}\in\mathbb{S}^1}\|\phi\|_{L^{3,6}_{\mathbf{e}}} + 2^{k/6}\sup_{|j-k|\le 20}\sup_{\mathbf{e}\in\mathbb{S}^1}\|P_{j,\mathbf{e}}\phi\|_{L^{6,3}_{\mathbf{e}}}\\ &+ 2^{k/2}\sup_{|j-k|\le 20}\sup_{\mathbf{e}\in\mathbb{S}^1}\sup_{|\lambda|<2^{k-40}}\|P_{j,\mathbf{e}}\phi\|_{L^{\infty,2}_{\mathbf{e},\lambda}},\end{aligned} \tag{3.14}$$

respectively

$$\|f\|_{N_k(T)} = \inf_{f=f_1+f_2+f_3+f_4}\|f_1\|_{L^{\frac{4}{3}}} + 2^{\frac{k}{6}}\|f_2\|_{L^{\frac{3}{2},\frac{6}{5}}_{\mathbf{e}_1}} + 2^{\frac{k}{6}}\|f_3\|_{L^{\frac{3}{2},\frac{6}{5}}_{\mathbf{e}_2}} + 2^{-\frac{k}{2}}\sup_{\mathbf{e}\in\mathbb{S}^1}\|f_4\|_{L^{1,2}_{\mathbf{e},W_{k-40}}}, \tag{3.15}$$

where $(\mathbf{e}_1,\mathbf{e}_2)$ is the canonical basis in $\mathbb{R}^2$.

In all dimensions $d\ge 2$ the spaces $N_k(T)$ and $G_k(T)$ and related by the following linear estimate, which is proved in Section 7.

Proposition 3.8. *(Main linear estimate) Assume $\mathcal{K}\in\mathbb{Z}_+$, $T\in(0,2^{2\mathcal{K}}]$ and $k\in\mathbb{Z}$. Then for each $u_0\in L^2$ which is frequency localized in I_k and any $h\in N_k(T)$ the solution u to*

$$(i\partial_t+\Delta_x)u = h, \qquad u(0)=u_0$$

satisfies

$$\|u\|_{G_k(T)} \lesssim \|u(0)\|_{L^2_x} + \|h\|_{N_k(T)}$$

We describe now the structure of the normed spaces $G_k(T)$, $N_k(T)$, and $F_k(T)$. As Proposition 3.8 suggests, we use the spaces $G_k(T)$ to measure solutions of Schrödinger equations. The main components of the spaces $G_k(T)$ are given by the local smoothing/maximal function spaces $L^{2,\infty}_{\mathbf{e},W_{k+40}}$ and $L^{\infty,2}_{\mathbf{e},\lambda}$ in dimension $d=2$, respectively $L^{2,\infty}_{\mathbf{e}}$ and $L^{\infty,2}_{\mathbf{e}}$ in dimensions $d\ge 3$ (compare with Lemma 3.2 and Lemma 3.4). The other components of the spaces $G_k(T)$ are Strichartz-type spaces, compare with Lemma 3.5. These components are much more flexible.

As Proposition 3.8 suggests, we use the spaces $N_k(T)$ to measure nonlinearities of Schrödinger equations. The key components of the spaces $N_k(T)$ are the inhomogeneous local smoothing spaces $L^{1,2}_{\mathbf{e},W_{k-40}}$ in dimension $d=2$, and $L^{1,2}_{\mathbf{e}}$ in dimension $d\ge 3$, which are the only spaces that can be used to bound the difficult magnetic parts of the Schrödinger nonlinearities. The other components of the spaces $N_k(T)$ are Strichartz-type spaces, and are chosen in a way that matches the Strichartz spaces of $G_k(T)$.

We discuss now the spaces $F_k(T)$. It is clear from the definition that $G_k(T)\hookrightarrow F_k(T)$. The larger spaces $F_k(T)$ have an important advantage over the spaces $G_k(T)$: for any $k\in\mathbb{Z}$ and $f\in F_k(T)\cap F_{k+1}(T)$ we have $\|f\|_{F_k(T)}\approx\|f\|_{F_{k+1}(T)}$. This is easy to see by examining the definitions and noticing that $\|\phi\|_{F^0_{k+1}(T)} \lesssim \|\phi\|_{F^0_k(T)}$ for any $\phi \in L^2(\mathbb{R}^d \times [-T,T])$ if $d = 2$. Moreover, if $k, k' \in \mathbb{Z}$, $|k - k'| \leq 20$, $u \in F_{k'}(T)$, and $v \in L^\infty(\mathbb{R}^d \times (-T,T))$ then

$$\|P_k(uv)\|_{F_k(T)} \lesssim \|u\|_{F_{k'}(T)}\|v\|_{L^\infty_{x,t}}. \tag{3.16}$$

The spaces $G_k(T)$ do not have this important property, proved in Lemma 5.1, mostly because of the local smoothing norms which require certain frequency localizations.

We use the spaces $G_k(T)$ to measure the fields ψ_m, $m = 1, \ldots, d$, at parabolic time $s = 0$. This is consistent with the Schrödinger equations (2.11) satisfied by these fields. We use, however, the weaker spaces $F_k(T)$ to measure the fields $\psi_m(s)$ for $s > 0$, as well as the connection coefficients $A_m(s)$. The fields $\psi_m(s)$ satisfy the covariant heat equations (2.19), and we are able to propagate control of these fields along the heat flow, with suitable parabolic decay, only in the larger spaces $F_k(T)$. Fortunately, in the perturbative analysis of Schrödinger equations, it is not necessary to control the connection coefficients $A_m(0)$ in the missing local smoothing norms.

To bound products of functions in $F_k(T)$ we often use a more relaxed criterion. Precisely, since for $\mathbf{e} \in \mathbb{S}^1$ we have

$$\|f\|_{L^{2,\infty}_{\mathbf{e},W_{k+m_j}}} \leq \|f\|_{L^{2,\infty}_{\mathbf{e}}} \lesssim 2^{k(d-1)/2}\|f\|_{L^2_x L^\infty_t}$$

it follows that, in all dimensions $d \geq 2$,

$$\|f\|_{F_k(T)} \lesssim \|f\|_{L^2_x L^\infty_t} + \|f\|_{L^{p_d}}. \tag{3.17}$$

This criterion is often used to estimate bilinear expressions, by exploiting the $L^{p_d}_x L^\infty_t$ norms in the spaces $F_k(T)$.

We also need to evolve $F_k(T)$ functions along the heat flow. Since the $F_k(T)$ norm is translation invariant it immediately follows that if $h \in F_k(T)$ then

$$\|e^{s\Delta_x}h\|_{F_k(T)} \lesssim (1 + s2^{2k})^{-20}\|h\|_{F_k(T)}, \qquad s \geq 0. \tag{3.18}$$

To prove useful bounds on the connection coefficients A_m, $m = 1, \ldots, d$, for $k \in \mathbb{Z}$ and $\omega \in [0, 1/2]$ we define the normed spaces $S^\omega_k(T)$ of functions in $L^2_k(T)$ for which

$$\|f\|_{S^\omega_k(T)} = 2^{k\omega}(\|f\|_{L^\infty_t L^{2_\omega}_x} + \|f\|_{L^{p_d}_t L^{p_{d,\omega}}_x} + 2^{-kd/(d+2)}\|f\|_{L^{p_{d,\omega}}_x L^\infty_t}) < \infty, \tag{3.19}$$

where the exponents 2_ω and $p_{d,\omega}$ are such that

$$\frac{1}{2_\omega} - \frac{1}{2} = \frac{1}{p_{d,\omega}} - \frac{1}{p_d} = \frac{\omega}{d}.$$

The spaces $S^\omega_k(T)$ are at the same scale as the spaces $F_k(T)$ and $F_k(T) \hookrightarrow S^0_k(T)$. By Sobolev embeddings we have

$$\|f\|_{S^{\omega'}_k(T)} \lesssim \|f\|_{S^\omega_k(T)} \quad \text{ if } \omega' \leq \omega. \tag{3.20}$$

Thus the spaces $S^\omega_k(T)$ can be interpreted as refinements of the Strichartz part of the spaces $F_k(T)$ (which corresponds to $S^0_k(T)$). It is important to be able to prove bounds on the coefficients A_m, $m = 1, \ldots, d$, in both spaces $F_k(T)$ and $S^{1/2}_k(T)$. These bounds quantify an essential gain of smoothness of the coefficients A_m compared to the fields ψ_m. This is proved in Lemma 5.2 and used in many estimates in sections 5 and 6.

4. Outline of the proof

This section contains an outline of the proofs of the main theorems. We observe first that it suffices to construct the solution ϕ on the time interval $(-2^{2\mathcal{K}}, 2^{2\mathcal{K}})$, for any given $\mathcal{K} \in \mathbb{Z}_+$, and prove the bounds (1.3) and (1.5) uniformly in $\mathcal{K}$. We therefore fix[1] once and for all $\mathcal{K} \gg 1 \in \mathbb{Z}_+$ and assume $T \in (0, 2^{2\mathcal{K}}]$.

We start with a solution $\phi \in C((-T, T) : H^\infty_Q)$ of (1.1) on some time interval $(-T, T)$, where $T \in (0, 2^{2\mathcal{K}}]$. Our main goal is to prove a priori estimates on

$$\sup_{t \in (-T,T)} \|\phi(t)\|_{\dot{H}^{d/2}} \quad \text{and} \quad \sup_{t \in (-T,T)} \|\phi(t)\|_{H^\sigma_Q},$$

for σ in a fixed interval $\sigma \in [d/2, \sigma_1]$. We use the notion of frequency envelopes. We fix a small parameter δ (for instance $\delta = 1/(20d)$ suffices).

Definition 4.1. *A positive sequence $\{b_k\}_{k \in \mathbb{Z}}$ is a frequency envelope if it is l^2 bounded*

$$\sum_{k \in \mathbb{Z}} b_k^2 < \infty \tag{4.1}$$

and slowly varying,

$$b_k \le b_j 2^{\delta|k-j|}, \qquad k, j \in \mathbb{Z}. \tag{4.2}$$

An ϵ-frequency envelope $\{b_k\}_{k \in \mathbb{Z}}$ satisfies the additional relation

$$\sum_{k \in \mathbb{Z}} b_k^2 < \epsilon^2. \tag{4.3}$$

Given an l^2 bounded nonnegative sequence $\{\alpha_k\}_{k \in \mathbb{Z}}$ we often define the frequency envelope

$$\alpha'_k = \sup_{k' \in \mathbb{Z}} \alpha_{k'} 2^{-\delta|k-k'|}.$$

Clearly, we have $\alpha_k \le \alpha'_k$ and $\alpha'_k = \alpha_k$ if $\{\alpha_k\}_{k \in \mathbb{Z}}$ is already a frequency envelope. In addition, $\sum_{k \in \mathbb{Z}} [\alpha'_k]^2 \lesssim \sum_{k \in \mathbb{Z}} \alpha_k^2$.

Given $\sigma_1 > d/2$ as in Theorem 1.2, $T > 0$ and $\phi \in H^{\infty,\infty}_Q(T)$ we define the frequency envelopes

$$\gamma_k(\sigma) = \sup_{k' \in \mathbb{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'} \|P_{k'}\phi\|_{L^\infty_t L^2_x}, \qquad \sigma \in [0, \sigma_1], \qquad \delta = 1/(20d). \tag{4.4}$$

[1]The value of $\mathcal{K}$ does not appear in any of the effective bounds; it is useful, however, to have $\mathcal{K} < \infty$ in some of the continuity arguments and to be able to define the sets $W_k(\mathcal{K})$ in Definition 3.7 as finite sets. Weak bounds, such as (4.10), may depend implicitly on the value of $\mathcal{K}$.

We also set $\gamma_k = \gamma_k(d/2)$. Then we have $\gamma_{k'}(\sigma) \leq 2^{\delta|k-k'|}\gamma_k(\sigma)$ for any $k, k' \in \mathbb{Z}$, and

$$\|P_k\phi\|_{L^\infty_t L^2_x} \leq 2^{-\sigma k}\gamma_k(\sigma), \qquad \sigma \in [0, \sigma_1]. \tag{4.5}$$

Proposition 4.2. *(Construction of the caloric gauge) Assume that $T \in (0,\infty)$ and $Q \in \mathbb{S}^2$. Let $\phi \in H_Q^{\infty,\infty}(T)$ which satisfies the smallness condition*

$$\sum_{k\in\mathbb{Z}} 2^{kd}\|P_k\phi\|^2_{L^\infty_t L^2_x} = \gamma^2 \ll 1. \tag{4.6}$$

Then there is a unique smooth solution $\widetilde{\phi} \in C([0,\infty) : H_Q^{\infty,\infty}(T))$ of the covariant heat equation

$$\begin{cases} \partial_s\widetilde{\phi} = \Delta_x\widetilde{\phi} + \widetilde{\phi}\cdot\sum_{m=1}^d |\partial_m\widetilde{\phi}|^2 & \text{on } [0,\infty)\times\mathbb{R}^d\times(-T,T); \\ \widetilde{\phi}(0,x,t) = \phi(x,t). \end{cases} \tag{4.7}$$

In addition, there are smooth functions $v, w : [0,\infty)\times\mathbb{R}^d\times(-T,T) \to \mathbb{S}^2$ with the properties

$${}^t v\cdot\widetilde{\phi} = {}^t w\cdot\widetilde{\phi} = {}^t v\cdot w = {}^t w\cdot\partial_s v = 0 \text{ on } [0,\infty)\times\mathbb{R}^d\times(-T,T). \tag{4.8}$$

For any $F \in \{\widetilde{\phi}, v, w\}$ we have the bounds

$$\|P_k F(s)\|_{L^\infty_t L^2_x} \lesssim \gamma_k(\sigma)(1+s2^{2k})^{-20}2^{-\sigma k}, \qquad \sigma \in [d/2, \sigma_1] \tag{4.9}$$

with $\gamma_k(\sigma)$ defined by (4.4), and, for any $\sigma, \rho \in \mathbb{Z}_+$,

$$\sup_{k\in\mathbb{Z}}\sup_{s\in[0,\infty)} (s+1)^{\sigma/2}2^{\sigma k}\|P_k\partial_t^\rho F(s)\|_{L^\infty_t L^2_x} < \infty. \tag{4.10}$$

The key caloric gauge condition is the last identity in (4.8), namely ${}^t w\cdot\partial_s v \equiv 0$, which leads to the identity $A_0 \equiv 0$. It is also important that the functions $\widetilde{\phi}, v, w$ become trivial as $s \to \infty$, in the sense of (4.10). Proposition 4.2 is due to Tao [45]; we give a complete proof of Proposition 4.2 in section 8.

Most of our analysis is done at the level of the fields ψ_m and the connection coefficients A_m. From (4.9) and (8.4) we obtain

$$\|P_k\psi_m(s)\|_{L^\infty_t L^2_x} + \|P_k A_m(s)\|_{L^\infty_t L^2_x} \lesssim \gamma_k(\sigma)(1+s2^{2k})^{-20}2^{-(\sigma-1)k}, \tag{4.11}$$

for $m = 1,\ldots,d$, $\sigma \in [d/2,\sigma_1]$, $s \in [0,\infty)$, $k \in \mathbb{Z}$. By (4.10) we also have

$$\sup_{k\in\mathbb{Z}}\sup_{s\in[0,\infty)} (s+1)^{\sigma/2}2^{k\sigma}2^{-k}\big[\|P_k(\partial_t^\rho\psi_m(s))\|_{L^\infty_t L^2_x} + \|P_k(\partial_t^\rho A_m(s))\|_{L^\infty_t L^2_x}\big] < \infty \tag{4.12}$$

for $m = 1,\ldots,d$, and

$$\sup_{k\in\mathbb{Z}}\sup_{s\in[0,\infty)} (s+1)^{\sigma/2}2^{k\sigma}\big[\|P_k(\partial_t^\rho\psi_{d+1}(s))\|_{L^\infty_t L^2_x} + \|P_k(\partial_t^\rho A_{d+1}(s))\|_{L^\infty_t L^2_x}\big] < \infty. \tag{4.13}$$

Given an initial data $\phi_0 \in H_Q^\infty$ for the Schrödinger map equation, for $\sigma \in [(d-2)/2, \sigma_1 - 1]$ we introduce the frequency envelopes

$$c_k(\sigma) = \sup_{k' \in \mathbb{Z}} \|P_{k'} \nabla \phi_0\|_{L^2_x} 2^{\sigma k'} 2^{-\delta|k-k'|}. \tag{4.14}$$

Then we have the relations, for any $\sigma \in [(d-2)/2, \sigma_1 - 1]$ and $k \in \mathbb{Z}$,

$$\|\nabla \phi_0\|^2_{\dot{H}^\sigma} \approx \sum_{k \in \mathbb{Z}} c_k^2(\sigma) \qquad \text{and} \qquad \|P_k \nabla \phi_0\|_{L^2} \leq c_k(\sigma) 2^{-\sigma k}. \tag{4.15}$$

Let $c_k = c_k(d/2-1)$. If $\|\phi_0\|_{\dot{H}^{d/2}} \leq \varepsilon_0$ then $\sum_{k \in \mathbb{Z}} c_k^2 \lesssim \varepsilon_0^2$.

The bounds in Proposition 4.2 in the energy space $L^\infty_t L^2_x$ are far from sufficient for the study of the Schrödinger equation. We need suitable bounds in the $F_k(T)$ spaces. In the next proposition we fix some $\sigma_0 \in [d/2-1, \sigma_1 - 1]$ and use two frequency envelopes b_k and $b_k(\sigma_0)$. The envelope $b_k = b_k((d-2)/2)$ is used to measure critical regularity and carries a smallness condition. The envelope $b_k(\sigma_0)$ is always used to measure noncritical regularity. To these two envelopes we associate the sequences

$$b_{>k} = \big(\sum_{j \geq k} b_j^2\big)^{1/2}, \qquad b_{>k}(\sigma_0) = \big(\sum_{j \geq k} b_j b_j(\sigma_0) 2^{(k-j)(\sigma_0 - (d-2)/2)}\big)^{1/2}.$$

Proposition 4.3. *(Heat flow bootstrap estimates) Assume that $T \in (0, \infty)$ and $Q \in \mathbb{S}^2$. Given $\phi \in H_Q^{\infty,\infty}(T)$ satisfying (4.6) we consider $\widetilde{\phi}, v, w$ as in Proposition 4.2, and ψ_m and A_m the associated fields and connection coefficients. Let $b_k = b_k(d/2-1)$ be an ε-frequency envelope with small ε, and $b_k(\sigma_0)$ be another frequency envelope.*

(a) Suppose that the functions $\{\psi_m\}_{m=1,d}$ satisfy

$$\|P_k \psi_m(0)\|_{F_k(T)} \leq b_k(\sigma) 2^{-\sigma k}, \qquad \sigma \in \{(d-2)/2, \sigma_0\} \tag{4.16}$$

as well as the bootstrap condition

$$\|P_k \psi_m(s)\|_{F_k(T)} \leq \varepsilon^{-1/2} b_k 2^{-k(d-2)/2} (1 + s2^{2k})^{-4}. \tag{4.17}$$

Then for $\sigma \in \{(d-2)/2, \sigma_0\}$ we have

$$\|P_k \psi_m(s)\|_{F_k(T)} \lesssim b_k(\sigma) 2^{-\sigma k} (1 + s2^{2k})^{-4}, \qquad \sigma \in \{(d-2)/2, \sigma_0\}. \tag{4.18}$$

Also, for $l, m = 1, \cdots, d$ and $\sigma \in \{(d-2)/2, \sigma_0\}$, we have the $F_k(T)$ bounds

$$\|P_k(A_m(s)\psi_l(s))\|_{F_k(T)} \lesssim b_k(\sigma) 2^{-(\sigma-1)k} (2^{2k}s)^{-\frac{3}{8}} (1 + s2^{2k})^{-2}, \tag{4.19}$$

as well as the L^{p_d} estimate at $s = 0$

$$\|P_k A_m(0)\|_{L^{p_d}} \lesssim b_k(\sigma) 2^{-\sigma k}. \tag{4.20}$$

(b) Assume in addition that

$$\|P_k \psi_{d+1}(0)\|_{L^{p_d}} \lesssim b_k(\sigma) 2^{-(\sigma-1)k}, \qquad \sigma \in \{(d-2)/2, \sigma_0\}. \tag{4.21}$$

Then for $\sigma \in \{(d-2)/2, \sigma_0\}$ we have

$$\|P_k \psi_{d+1}(s)\|_{L^{p_d}} \lesssim b_k(\sigma) 2^{-(\sigma-1)k} (1 + 2^{2k}s)^{-2}, \tag{4.22}$$

and the connection coefficient A_{d+1} satisfies the L^2 estimate at $s=0$

$$\|P_k A_{d+1}(0)\|_{L^2} \lesssim \varepsilon b_k(\sigma) 2^{-\sigma k}, \qquad d \geq 3 \tag{4.23}$$

respectively

$$\|A_{d+1}(0)\|_{L^2} \lesssim \varepsilon^2, \qquad \|P_k A_{d+1}(0)\|_{L^2} \lesssim b^2_{>k}(\sigma) 2^{-\sigma k}, \qquad d = 2. \tag{4.24}$$

Proposition 4.3 is proved in Section 5. Here we note the following improvement:

Corollary 4.4. *The result in Proposition 4.3 remains valid as well if the bootstrap assumption* (4.17) *is dropped.*

Proof of Corollary 4.4. Define the function

$$\Psi(T') = \sup_{k\in\mathbb{Z}} \sup_{s\geq 0} b_k^{-1} 2^{k(d-2)/2} (1+s2^{2k})^4 \|P_k \psi_m(s)\|_{F_k(T')}, \qquad 0 < T' \leq T.$$

It follows easily from (4.12) with $\rho = 1$ that Ψ is an increasing, continuous function of T'. We will prove that

$$\lim_{T'\to 0} \Psi(T') \lesssim 1. \tag{4.25}$$

Since $\Psi : (0,T] \to [0,\infty)$ is a continuous increasing function and $\Psi(T') \leq \varepsilon^{-1/2}$ implies $\Psi(T') \lesssim 1$ (see (4.18)), the corollary follows easily from (4.25). To prove (4.25), let $\psi_{m,0}(s,x) = \psi_m(s,x,0)$. Using (3.17) and (4.12), it suffices to prove that

$$\sup_{k\in\mathbb{Z}} \sup_{s\geq 0} b_k^{-1} 2^{k(d-2)/2} (1+s2^{2k})^4 \|P_k \psi_{m,0}(s)\|_{L^2_x} \lesssim 1. \tag{4.26}$$

It follows from (4.16) that

$$2^{k(d-2)/2} \|P_k \psi_{m,0}(0)\|_{L^2_x} \leq b_k. \tag{4.27}$$

We need to extend this bound to $s > 0$ with suitable parabolic decay.

Recall the coefficients $c_k = c_k(d/2-1)$ defined in (4.14). We apply Proposition 4.2 on sufficiently short time intervals $(-T,T)$; using (4.9)

$$2^{dk/2}(1+s2^{2k})^{20} \left[\|P_k(v_0(s))\|_{L^2_x} + \|P_k(w_0(s))\|_{L^2_x} \right] \lesssim c_k, \tag{4.28}$$

where $v_0(s,x) = v(s,x,0)$ and $w_0(s,x) = w(s,x,0)$. We use this bound at $s = 0$, the identity (2.2), and the bounds (4.27) and (8.4); it follows that

$$2^{k(d-2)/2} \|P_k \nabla \phi_0\|_{L^2_x} \lesssim b_k.$$

Since $\{b_k\}_{k\in\mathbb{Z}}$ is a frequency envelope, it follows that $c_k \lesssim b_k$, see definition (4.14). It follows from (4.11) that

$$\|P_k \psi_{m,0}(s)\|_{L^2_x} \lesssim c_k (1+s2^{2k})^{-20} 2^{-k(d-2)/2}.$$

The bound (4.26) follows since $c_k \lesssim b_k$. □

Next we turn our attention to the Schrödinger equations (2.11). Our main Schrödinger bootstrap result is the following.

Proposition 4.5. *(Schrödinger bootstrap estimates) Assume that* $T \in (0, 2^{2\mathcal{K}}]$ *and* $Q \in \mathbb{S}^2$*. Let* $\{c_k\}_{k\in\mathbb{Z}}$ *be an* ε_0*-frequency envelope with* $\varepsilon_0 \in (0, \varepsilon_0(d)]$*, and* $\{c_k(\sigma_0)\}_{k\in\mathbb{Z}}$ *another frequency envelope. Let* $\phi \in H_Q^{\infty,\infty}(T)$ *be a solution of the Schrödinger map equation* (1.1) *whose initial data satisfies*

$$\|P_k \nabla \phi_0\|_{L^2_x} \leq c_k(\sigma) 2^{\sigma k}, \qquad \sigma \in \{(d-2)/2, \sigma_0\}. \tag{4.29}$$

Assume that ϕ *satisfies the bootstrap condition*

$$\|P_k \nabla \phi\|_{L^\infty_t L^2_x} \leq \varepsilon_0^{-1/2} 2^{-(d-2)k/2} c_k \tag{4.30}$$

and let (ϕ, v, w) *be the caloric extension of* ϕ *given by Proposition 4.2, with the corresponding fields* ψ_m*,* A_m*. Suppose also that at the initial parabolic time* $s = 0$ *the functions* $\{\psi_m\}_{m=1,d}$ *satisfy the additional bootstrap condition*

$$\|P_k \psi_m(0)\|_{G_k(T)} \leq \varepsilon_0^{-1/2} 2^{-(d-2)k/2} c_k. \tag{4.31}$$

Then we have

$$\|P_k \psi_m(0)\|_{G_k(T)} \lesssim c_k(\sigma) 2^{-\sigma k}, \qquad \sigma \in \{(d-2)/2, \sigma_0\}. \tag{4.32}$$

The above proposition is proved in Section 6 by applying the linear result in Proposition 3.8 to the equation (2.11). The right hand side in (2.11) is estimated in the $N_k(T)$ spaces using the bounds in Proposition 4.3 for the differentiated fields ψ_m and the connection coefficients A_m. In what follows we show that Proposition 4.5 implies Theorem 1.1 and the bound (1.5).

Proof of Theorem 1.1 and the bound (1.5)**.** Consider an initial data $\phi_0 \in H_Q^\infty$ for the Schrödinger map equation (1.1) which satisfies $\|\phi_0\|_{\dot{H}^{d/2}} \ll 1$. Our starting point is the local existence and uniqueness of smooth solutions of the Schrödinger map equation (see, for example, [33]): if $\phi_0 \in H_Q^\infty$ then there is $T = T(\|\phi_0\|_{H_Q^{2d+20}}) > 0$ and a unique solution $\phi \in C((-T,T) : H_Q^\infty)$ of the initial value problem (1.1).

In order to prove Theorem 1.1 we take $\sigma_1 = 2d+20$ in what follows. For Theorem 1.2, on the other hand, we allow σ_1 to be arbitrarily large. It suffices to prove that the solution $\phi \in C((-T,T) : H_Q^\infty)$ of the initial value problem (1.1) satisfies the bounds (1.3) and (1.5) with constants which are independent of T. In preparation for the proof of the well-posedness part of Theorem 1.2, we prove stronger bounds for the ϕ in terms of the frequency envelopes of ϕ_0.

Define the frequency envelopes $c_k(\sigma)$ as in (4.14). Our goal for the rest of this proof is to use Proposition 4.5 in order to prove that

$$\sup_{t\in(-T,T)} \|P_k \nabla \phi(0,.,t)\|_{L^2_x} \lesssim c_k(\sigma) 2^{-\sigma k}, \qquad \sigma \in [(d-2)/2, \sigma_1 - 1], \tag{4.33}$$

with implicit constants which depend only on d and σ_1. In view of (4.15), and (1.6), this suffices to establish (1.3) and (1.5). Then (1.4) follows from (1.5) for σ up to $2d+20$ and from the result in [33] for larger σ.

For $T' \in (0,T]$ let

$$\Psi(T') = \sup_{k\in\mathbb{Z}} c_k^{-1} 2^{(d-2)k/2} (\|P_k \nabla\phi\|_{L^\infty_{T'} L^2_x} + \|P_k \psi_m(0)\|_{G_k(T')}), \quad 0 < T' \leq T.$$

The function $\Psi : (0,T] \to [0,\infty)$ is well-defined, increasing and continuous, using (4.12), the fact that $\phi \in C((-T,T) : H^\infty_Q)$, and the fact that c_k is a frequency envelope. We show now that

$$\begin{cases} \text{ if } \Psi(T') \leq \varepsilon_0^{-1/2} \text{ then } \Psi(T') \lesssim 1; \\ \lim_{T'\to 0} \Psi(T') \lesssim 1. \end{cases} \tag{4.34}$$

The limit in the second line of (4.34) follows from the definition of the coefficients c_k, see (4.15), and (4.11) (we apply Proposition 4.2 on sufficiently short intervals $(-T',T')$). Also, using Proposition 4.5, if $\Psi(T') \leq \varepsilon_0^{-1/2}$ then

$$\sup_{k\in\mathbb{Z}} c_k^{-1} 2^{(d-2)k/2} \|P_k \psi_m(0)\|_{G_k(T')} \lesssim 1.$$

Assuming $\Psi(T') \leq \varepsilon_0^{-1/2}$, we apply Proposition 4.2 to conclude that

$$\|P_k \nabla v(0)\|_{L^\infty_{T'} L^2_x} + \|P_k \nabla w(0)\|_{L^\infty_{T'} L^2_x} \lesssim \varepsilon_0^{-1/2} 2^{-(d-2)k/2} c_k. \tag{4.35}$$

On the other hand from (4.32)

$$\|P_k \psi_m(0)\|_{L^\infty_{T'} L^2_x} \lesssim 2^{-\sigma k} c_k(\sigma), \qquad \sigma \in [(d-2)/2, \sigma_1 - 1]. \tag{4.36}$$

We use the relation, see (2.2), $\partial_m \phi = v\Re(\psi_m) + w\Im(\psi_m)$. From (4.36) with $\sigma = (d-2)/2$ and (4.35), using (8.4), we obtain

$$\|P_k \nabla\phi\|_{L^\infty_{T'} L^2} \lesssim 2^{-(d-2)k/2} c_k.$$

The implication in the first line of (4.34) follows.

It follows from (4.34) and the continuity of Ψ that $\Psi(T) \lesssim 1$. Thus

$$\|P_k \nabla\phi\|_{L^\infty_T L^2_x} + \|P_k \psi_m(0)\|_{G_k(T)} \lesssim c_k 2^{-k(d-2)/2}. \tag{4.37}$$

This suffices to prove the bound (4.33) for $\sigma = (d-2)/2$. To establish (4.33) for a different σ we denote by $B > 0$ the best constant so that

$$\|P_k \nabla\phi\|_{L^\infty_T L^2_x} \leq B c_k(\sigma) 2^{-\sigma k}. \tag{4.38}$$

Such a constant exists because of the smoothness of ϕ and the fact that $c_k(\sigma)$ is a frequency envelope. Using (4.37) and Proposition 4.2 we have

$$\|P_k v(0)\|_{L^\infty_T L^2_x} + \|P_k w(0)\|_{L^\infty_T L^2_x} \lesssim B c_k(\sigma) 2^{-(\sigma+1)k}. \tag{4.39}$$

From (4.36) and (4.39), by (8.4), we obtain

$$\|P_k \nabla\phi\|_{L^\infty L^2} \lesssim (1+\varepsilon B) 2^{-\sigma k} c_k(\sigma). \tag{4.40}$$

By the optimality of B in (4.38) we conclude that $B \lesssim 1 + \varepsilon B$, which yields $B \lesssim 1$. Thus (4.33) is proved. □

To define rough solutions and study the dependence of solutions on the initial data we consider the linearized equation (2.25) and prove that it is well-posed in $\dot{H}^{(d-2)/2}$.

Proposition 4.6. *Let $\phi_0 \in H_Q^\infty$ be an initial data for the Schrödinger map equation which satisfies the smallness condition $\|\phi_0\|_{\dot{H}^{d/2}} \ll 1$. Let ϕ be the corresponding global solution to* (1.1)*, (ϕ, u, v) its caloric extension and ψ_m, A_m as before. Then for each initial data $\psi_{lin}(0) \in H^\infty$ there exists an unique solution $\psi_{lin} \in C(\mathbb{R}, H^\infty)$ for* (2.25)*, which satisfies the bounds*

$$\sum_k 2^{(d-2)k} \|P_k \psi_{lin}\|^2_{G_k(T)} \lesssim \|\psi_{lin}(0)\|^2_{\dot{H}^{\frac{d-2}{2}}} \tag{4.41}$$

The proof of this result is identical to the proof of Proposition 4.3. As a consequence of this we obtain the Lipschitz dependence of solutions to (1.1) in terms of the initial data in a weaker topology:

Proposition 4.7. *Consider two initial data ϕ_0^0 and ϕ_0^1 in H_Q^∞ which satisfy the smallness condition $\|\phi_0^h\|_{\dot{H}^{\frac{d}{2}}} \ll 1$, $h = 0, 1$, and let ϕ^0, ϕ^1 be the corresponding global solutions for* (2.25)*. Then*

$$\sum_k 2^{(d-2)k} \|P_k(\phi^0 - \phi^1)\|^2_{L^\infty L^2} \lesssim \|\phi_0^0 - \phi_0^1\|^2_{\dot{H}^{\frac{d-2}{2}}} \tag{4.42}$$

Proof. By (4.41) we have the global in time bound

$$\sum_k 2^{(d-2)k} \|P_k \psi_{lin}\|^2_{L^\infty L^2} \lesssim \|\psi_{lin}(0)\|^2_{\dot{H}^{\frac{d-2}{2}}}$$

As in the proof of Theorem 1.1, this bound easily transfers to the functions ϕ_{lin}, and we obtain

$$\sum_k 2^{(d-2)k} \|P_k \phi_{lin}\|^2_{L^\infty L^2} \lesssim \|\phi_{lin}(0)\|^2_{\dot{H}^{\frac{d-2}{2}}} \tag{4.43}$$

for all solutions $\phi_{lin} \in C(\mathbb{R}, H^\infty)$ to (2.22).

Any two initial data ϕ_0^0 and ϕ_0^1 in H_Q^∞ which satisfy the smallness condition $\|\phi_0^h\|_{\dot{H}^{\frac{d}{2}}} \ll 1$, $h = 0, 1$ can be joined with a one parameter family of initial data as follows:

Lemma 4.8 (Proposition 3.13, [50])**.** *Consider two functions $\phi_0^0, \phi_0^1 \in H_Q^\infty$ so that $\|\phi_0^h\|_{\dot{H}^{\frac{d}{2}}} \ll 1$, $h = 0, 1$. Then there exists a smooth one parameter family of initial data $\{\phi_0^h\}_{h \in [0,1]} \in C^\infty([0,1]; H^\infty)$, taking values in H_Q^∞, which joins them, so that the smallness condition $\|\phi_0^h\|_{\dot{H}^{\frac{d}{2}}} \ll 1$, $h \in [0,1]$ is satisfied uniformly and*

$$\int_0^1 \|\partial_h \phi_0^h\|_{\dot{H}^{\frac{d-2}{2}}} \approx \|\phi_0^0 - \phi_0^1\|_{\dot{H}^{\frac{d-2}{2}}} \tag{4.44}$$

The corresponding family of global solutions ϕ^h is smooth with respect to h, and the functions $\partial_h \phi^h$ solve the linearized equation (2.25) with ϕ replaced by ϕ_h. Applying (4.43) to $\partial_h \phi^h$ we obtain

$$\sum_k 2^{(d-2)k} \|P_k \partial_h \phi^h\|^2_{L^\infty L^2} \lesssim \|\partial_h \phi^h(0)\|^2_{\dot H^{\frac{d-2}{2}}}$$

The estimate (4.42) is obtained by integrating with respect to h due to (4.44). □

The above proposition allows us to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. The bound (1.5) was proved earlier. To show that the map S_Q admits an unique continuous extension

$$S_Q : \dot H^{\frac{d}{2}} \cap \dot H_Q^{\frac{d-2}{2}} \to C(\mathbb{R}; \dot H^{\frac{d}{2}} \cap \dot H_Q^{\frac{d-2}{2}})$$

it suffices to consider a sequence of smooth initial data $\phi_0^n \in H_Q^\infty$ which satisfy uniformly the smallness condition $\|\phi_0^n\|_{\dot H^{\frac{d}{2}}} \ll 1$ and so that $\phi_0^n \to \phi_0$ in $\dot H^{\frac{d}{2}} \cap \dot H_Q^{\frac{d-2}{2}}$, and show that the corresponding sequence of global solutions is Cauchy in the space in $C(\mathbb{R}; \dot H^{\frac{d}{2}} \cap \dot H_Q^{\frac{d-2}{2}})$. By Proposition 4.7 it follows the ϕ^n is Cauchy in $C(\mathbb{R}; \dot H_Q^{\frac{d-2}{2}})$,

$$\lim_{n,m\to\infty} \|\phi^n - \phi^m\|_{C(\mathbb{R}; \dot H^{\frac{d-2}{2}})} = 0 \tag{4.45}$$

Consider frequency envelopes $\{c_k^n\}$ associated as in (4.14) to ϕ_0^n. Since ϕ_0^n is convergent in $\dot H^{\frac{d}{2}}$ we can choose the corresponding envelopes $\{c_k^n\}$ to converge in l^2. Then we have the uniform summability property

$$\lim_{k_0\to\infty} \sup_n \sum_{k>k_0} (c_k^n)^2 = 0 \tag{4.46}$$

Now we use (4.33) to estimate

$$\begin{aligned}\|\phi^n - \phi^m\|_{C(\mathbb{R};\dot H^{\frac{d}{2}})} \leq & \|P_{\leq k_0}(\phi^n - \phi^m)\|_{C(\mathbb{R};\dot H^{\frac{d}{2}})} + \|P_{>k_0}\phi^n\|_{C(\mathbb{R};\dot H^{\frac{d}{2}})} + \|P_{>k_0}\phi^m\|_{C(\mathbb{R};\dot H^{\frac{d}{2}})} \\ \lesssim & \, 2^{k_0} \|P_{\leq k_0}(\phi^n - \phi^m)\|_{C(\mathbb{R};\dot H^{\frac{d-2}{2}})} + \sum_{k>k_0} (c_k^n)^2 + (c_k^n)^2\end{aligned}$$

Hence using (4.45) we have

$$\limsup_{n,m\to\infty} \|\phi^n - \phi^m\|_{C(\mathbb{R};\dot H^{\frac{d}{2}})} \lesssim \sup_n \sum_{k>k_0} (c_k^n)^2$$

Letting $k_0 \to \infty$, by (4.46) we obtain

$$\limsup_{n,m\to\infty} \|\phi^n - \phi^m\|_{C(\mathbb{R};\dot H^{\frac{d}{2}})} = 0$$

and the argument is concluded.

The continuity of the solution operator S_Q in higher Sobolev spaces

$$S_Q : \dot{H}^\sigma \cap \dot{H}_Q^{\frac{d-2}{2}} \to C(\mathbb{R}; \dot{H}^\sigma \cap \dot{H}_Q^{\frac{d-2}{2}}), \qquad \frac{d}{2} < \sigma \leq \sigma_1$$

is obtained in the same manner. The proof of Theorem 1.2 is concluded. $\square$

5. The heat flow: proof of Proposition 4.3.

For $k \in \mathbb{Z}$ we denote

$$a(k) = \sup_{s\in[0,\infty)} (1 + s2^{2k})^4 \sum_{m=1}^{d} \|P_k\psi_m(s)\|_{F_k(T)}. \tag{5.1}$$

For $\sigma \in \{(d-2)/2, \sigma_0\}$ we introduce the frequency envelopes

$$a_k(\sigma) = \sup_{j\in\mathbb{Z}} 2^{\sigma j} 2^{-\delta|k-j|} a(j). \tag{5.2}$$

These are finite and belong to l^2 due to (4.12) and (3.17). For (4.18) we need to show that $a_k(\sigma) \lesssim b_k(\sigma)$. On the other hand from the bootstrap assumption (4.17) we know that $a_k = a_k(d/2-1) \leq \varepsilon^{-1/2} b_k$. In particular this implies that

$$\sum_{k\in\mathbb{Z}} a_k^2 \leq \varepsilon. \tag{5.3}$$

We prove first a bilinear estimate.

Lemma 5.1. *Assume that $T \in (0, 2^{2\mathcal{K}}]$, $f, g \in H^{\infty,\infty}(T)$, $P_k f \in F_k(T) \cap S_k^\omega(T)$, $P_k g \in F_k(T)$ for some $\omega \in [0, 1/2]$ and any $k \in \mathbb{Z}$, and*

$$\alpha_k = \sum_{|j-k|\leq 20} \|P_j f\|_{F_j(T)\cap S_j^\omega(T)}, \quad \beta_k = \sum_{|j-k|\leq 20} \|P_j g\|_{F_j(T)}.$$

Then, for any $k \in \mathbb{Z}$

$$\|P_k(fg)\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim \sum_{j\leq k} 2^{\frac{jd}{2}}(\beta_k\alpha_j + \alpha_k\beta_j) + 2^{\frac{kd}{2}} \sum_{j\geq k} 2^{(j-k)(\frac{2d}{d+2}-\omega)}\alpha_j\beta_j. \tag{5.4}$$

Proof of Lemma 5.1. We observe first that if $k, j \in \mathbb{Z}$ with $|k-j| \leq 20$ and $\omega' \in [0, 1/2]$ then

$$\|P_k(uv)\|_{F_k(T)} \lesssim \|u\|_{F_j(T)} \|v\|_{L^\infty_{x,t}} \tag{5.5}$$

and

$$\|P_k(uv)\|_{S_k^{\omega'}(T)} \lesssim \|u\|_{F_j(T)} 2^{k\omega'} \|v\|_{L_x^{d/\omega'} L_t^\infty}. \tag{5.6}$$

both of which follow directly the definitions. For the second factor on the right in both (5.5) and (5.6) we observe that for v which is localized at frequency 2^k we have by Sobolev embeddings

$$\|v\|_{L^\infty_{x,t}} + 2^{k\omega'} \|v\|_{L_x^{d/\omega'} L_t^\infty} \leq C 2^{dk/2} \|v\|_{F_k(T)} \tag{5.7}$$

We show now that if $|k_1 - k_2| \leq 8$, and f_{k_1}, g_{k_2} are localized at frequency 2^{k_1}, respectively 2^{k_2} then

$$\|P_k(f_{k_1}g_{k_2})\|_{F_k(T)\cap S_k^{1/2}(T)} \leq C2^{kd/2}2^{(k_2-k)(2d/(d+2)-\omega)}\|f_{k_1}\|_{S^{\omega}_{k_1}(T)}\|g_{k_2}\|_{S^0_{k_2}(T)}. \tag{5.8}$$

To prove this we use (3.17) and Sobolev embeddings:

$$\begin{aligned}\|P_k(f_{k_1}g_{k_2})\|_{F_k(T)\cap S_k^{1/2}(T)} &\lesssim \|P_k(f_{k_1}g_{k_2})\|_{L^2_xL^\infty_t} + 2^{k/2}\|P_k(f_{k_1}g_{k_2})\|_{L^\infty_t L_x^{2^{1/2}} \cap L_t^{p_d} L_x^{p_{d,1/2}}} \\ &\lesssim 2^{kd(1/p_d+1/p_{d,\omega}-1/2)}\|f_{k_1}\|_{L_x^{p_{d,\omega}}L_t^\infty}\|g_{k_2}\|_{L_x^{p_d}L_t^\infty} + 2^{kd/2}\|f_{k_1}\|_{L^\infty_t L^2_x}\|g_{k_2}\|_{L^\infty_t L^2_x\cap L^{p_d}} \\ &\lesssim 2^{kd/2}(2^{|k_2-k|(2d/(d+2)-\omega)}\|f_{k_1}\|_{S^\omega_{k_1}(T)}\|g_{k_2}\|_{S^0_{k_2}(T)} + \|f_{k_1}\|_{S^0_{k_1}(T)}\|g_{k_2}\|_{S^0_{k_2}(T)}),\end{aligned}$$

To prove the estimate (5.4) we use a bilinear Littlewood-Paley decomposition

$$P_k(fg) = \sum_{k_1\leq k-4}^{|k_2-k|\leq 4} P_k(P_{k_1}fP_{k_2}g) + \sum_{k_2\leq k-4}^{|k_1-k|\leq 4} P_k(P_{k_1}fP_{k_2}g) + \sum_{k_1,k_2\geq k-4}^{|k_1-k_2|\leq 8} P_k(P_{k_1}fP_{k_2}g) \tag{5.9}$$

and bound each of the terms on the right in $F_k(T)\cap S_k^{1/2}(T)$. For the first two we use (5.5), (5.6) and (5.7). For the third we use (5.8) instead. The bound (5.4) follows. □

We prove now our main estimates on the connection coefficients A_m, $m = 1, \ldots, d$.

Lemma 5.2. *(Bounds on $A_m(s)$) For any $k \in \mathbb{Z}$, $s \in [0, \infty)$, and $m = 1, \ldots, d$*

$$\|P_k(A_m(s))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim 2^{-\sigma k}(1 + s2^{2k})^{-4}b_{k,s}(\sigma), \tag{5.10}$$

where, if $s \in [2^{2k_0-1}, 2^{2k_0+1})$, $k_0 \in \mathbb{Z}$, then

$$b_{k,s}(\sigma) = \begin{cases} 2^{k+k_0}a_{-k_0}a_k(\sigma) \text{ if } k + k_0 \geq 0; \\ \sum_{j=k}^{-k_0} a_ja_j(\sigma) \text{ if } k + k_0 \leq 0. \end{cases} \tag{5.11}$$

Proof of Lemma 5.2. We use the identity (2.20)

$$A_m(s) = -\sum_{l=1}^{d}\int_s^\infty \Im\big(\overline{\psi_m}(\partial_l\psi_l + iA_l\psi_l)\big)(r)\,dr. \tag{5.12}$$

To prove (5.10), let B_1 denote the smallest number in $[1, \infty)$ with the property that for any $s \in [0, \infty)$, $k \in \mathbb{Z}$, $m = 1, \ldots, d$, and $\sigma \in \{(d-2)/2, \sigma_1 - 1\}$

$$\|P_k(A_m(s))\|_{F_k(T)} \leq B_1 2^{-\sigma k}(1 + s2^{2k})^{-4}b_{k,s}(\sigma). \tag{5.13}$$

We observe first that for any $f, g \in \{\psi_m, \overline{\psi_m} : m = 1, \ldots d\}$, $r \in [2^{2j-2}, 2^{2j+2}]$, $j \in \mathbb{Z}$, $l = 1, \ldots, d$, and $\sigma \in \{(d-2)/2, \sigma_1 - 1\}$ we have the bounds

$$\|P_k(f(r)g(r))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim 2^{-\sigma k}(1 + 2^{2k+2j})^{-4}2^{-j}a_{-j}a_{\max(k,-j)}(\sigma) \tag{5.14}$$

$$\|P_k(f(r)\partial_l g(r))\|_{F_k(T)\cap S_k^{1/2}(T)} \lesssim 2^{-\sigma k}(1+2^{2k+2j})^{-4}2^{-j}a_{-j}(2^ka_k(\sigma)+2^{-j}a_{-j}(\sigma)) \tag{5.15}$$

To prove this we consider the cases $k+j\leq 0$ and $k+j\geq 0$, use the bounds (4.17) and (5.4) with $\omega=0$, and simplify the resulting expressions using the fact that a_k is slowly varying.

We apply P_k to (5.12) to conclude that for any $s\in[0,\infty)$, $k\in\mathbb{Z}$, $m=1,\dots,d$

$$\begin{aligned}\|P_k(A_m(s))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim&\sum_{\alpha,\beta,\gamma=1}^{d}\int_s^\infty\|P_k(\overline{\psi_\alpha}(r)\partial_\beta\psi_\gamma(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\,dr\\&+\sum_{\alpha,\beta,\gamma=1}^{d}\int_s^\infty\|P_k(\overline{\psi_\alpha}(r)\psi_\beta(r)A_\gamma(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\,dr\end{aligned}\tag{5.16}$$

With k_0 as before, using (5.15), the first term in the right-hand side of (5.16) is dominated by

$$\begin{aligned}&\sum_{\alpha,\beta,\gamma=1}^{d}\sum_{j\geq k_0}\int_{2^{2j-1}}^{2^{2j+1}}\|P_k(\overline{\psi_\alpha}(r)\partial_\beta\psi_\gamma(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\,dr\\&\lesssim\sum_{j\geq k_0}2^{-\sigma k}(1+2^{2k+2j})^{-4}2^j a_{-j}(2^k a_k(\sigma)+2^{-j}a_{-j}(\sigma))\\&\lesssim 2^{-\sigma k}\sum_{j\geq k_0}(1+2^{2k+2j})^{-4}(2^{j+k}a_{-j}a_k(\sigma)+a_{-j}a_{-j}(\sigma))\\&\lesssim 2^{-\sigma k}(1+s2^{2k})^{-4}b_{k,s}(\sigma),\end{aligned}\tag{5.17}$$

where the last inequality follows easily from (5.11) by checking the two cases $k+k_0\geq 0$ and $k+k_0\leq 0$.

We estimate now $\|P_k(\overline{\psi_\alpha}(r)\psi_\beta(r)\cdot A_\gamma(r))\|_{F_k(T)}$ for $r\in[2^{2j-1},2^{2j+1}]$. It follows from (5.14) that for any $k'\in\mathbb{Z}$

$$\|P_{k'}(\overline{\psi_\alpha}(r)\psi_\beta(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim 2^{-\sigma k'}(1+2^{2k'+2j})^{-4}2^{-j}a_{-j}a_{\max(k',-j)}(\sigma).\tag{5.18}$$

It follows from (5.4), (5.13) and (5.3) that

$$\|P_k(\overline{\psi_\alpha}(r)\psi_\beta(r)\cdot A_\gamma(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim B_1 2^{-\sigma k}\varepsilon 2^{-2j}a_{-j}a_{-j}(\sigma)$$

if $k+j\leq 0$, and

$$\|P_k(\overline{\psi_\alpha}(r)\psi_\beta(r)\cdot A_\gamma(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim B_1 2^{-\sigma k}\varepsilon(1+2^{2k+2j})^{-4}2^{-2j}b_{k,r}(\sigma)$$

if $k+j\geq 0$. Thus, with k_0 as before, the second term in the right-hand side of (5.16) is dominated by

$$\begin{aligned}&\sum_{\alpha,\beta,\gamma=1}^{d}\sum_{j\geq k_0}\int_{2^{2j-1}}^{2^{2j+1}}\|P_k(\overline{\psi_\alpha}(r)\psi_\beta(r)\cdot A_\gamma(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\,dr\\&\lesssim B_1 2^{-\sigma k}\varepsilon\sum_{j\geq k_0}(1+2^{2k+2j})^{-4}(\mathbf{1}_-(k+j)a_{-j}a_{-j}(\sigma)+\mathbf{1}_+(k+j)b_{k,2^{2j}}(\sigma))\\&\lesssim B_1 2^{-\sigma k}\varepsilon(1+2^{2k+2k_0})^{-4}b_{k,2^{2k_0}}(\sigma).\end{aligned}$$

Thus, by (5.16) and (5.17), for any $s\in[0,\infty)$ and $\sigma\in\{(d-2)/2,\sigma_0\}$ we obtain

$$\|P_k(A_m(s))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim 2^{-\sigma k}(1+s2^{2k})^{-4}b_{k,s}(\sigma)(1+B_1\varepsilon),$$

which shows that $B_1\lesssim 1+B_1\varepsilon$ and further $B_1\lesssim 1$. This completes the proof of (5.10). □

We prove now bounds on nonlinearity of the heat equation (2.19)

$$K_m=2i\sum_{l=1}^{d}\partial_l(A_l\psi_m)-\sum_{l=1}^{d}(A_l^2+i\partial_lA_l)\psi_m+i\sum_{l=1}^{d}\Im(\psi_m\overline{\psi_l})\psi_l. \tag{5.19}$$

Lemma 5.3. *(Control of the heat nonlinearities) For any $s\in[0,\infty)$, $k\in\mathbb{Z}$, $m=1,\dots,d$ and $\sigma\in\{(d-2)/2,\sigma_0\}$*

$$\Big\|\int_0^s e^{(s-r)\Delta_x}P_k(K_m(r))\,dr\Big\|_{F_k(T)}\lesssim\varepsilon(1+s2^{2k})^{-4}2^{-\sigma k}a_k(\sigma). \tag{5.20}$$

Proof of Lemma 5.3. Assume $r\in[2^{2j-2},2^{2j+2}]$ for some $j\in\mathbb{Z}$ and assume that

$$F\in\big\{A_l^2,\partial_lA_l,fg:l=1,\dots,d,\ f,g\in\{\psi_n,\overline{\psi}_n:n=1,\dots,d\}\big\}. \tag{5.21}$$

We show first that for F as in (5.21) we have

$$\|P_k(F(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim 2^{-\sigma k}(1+2^{2k+2j})^{-4}c_{k,j}(\sigma) \tag{5.22}$$

where

$$c_{k,j}(\sigma)=\begin{cases}2^{-j}a_{-j}a_{-j}(\sigma) & \text{if } k+j\leq 0;\\ 2^{2k+j}a_{-j}a_k(\sigma) & \text{if } k+j\geq 0.\end{cases} \tag{5.23}$$

If F is of the form ∂_lA_l or fg then the bound (5.22) follows from (5.10) and (5.11), respectively (5.14) (recall that $a_k(\sigma)$ is slowly varying). To prove this bound for $F=A_l^2$ we use (5.10) and Lemma 5.1 with $\omega=0$: if $k+j\leq 0$ then

$$\|P_k(A_l^2(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim\varepsilon 2^{-\sigma k}2^{-j}a_{-j}a_{-j}(\sigma),$$

and if $k+j\geq 0$ then

$$\|P_k(A_l^2(r))\|_{F_k(T)\cap S_k^{1/2}(T)}\lesssim\varepsilon 2^{-\sigma k}2^{-j}b_{k,2^{2j}}(\sigma).$$

These bounds suffice to prove (5.22).

We prove now that, with $r \in [2^{2j-2}, 2^{2j+2}]$ as before,

$$\|P_k(K_m(r))\|_{F_k(T)} \lesssim \varepsilon 2^{-\sigma k} 2^{2k} (1 + 2^{2k+2j})^{-4} [a_k(\sigma) + 2^{-3(k+j)/2} a_{-j}(\sigma)]. \tag{5.24}$$

In view of the formula (5.19), it suffices to prove that

$$\begin{aligned} &\|P_k(F(r)f(r))\|_{F_k(T)} + 2^k \|P_k(A_l(r)f(r))\|_{F_k(T)} \\ &\quad \lesssim \varepsilon 2^{-\sigma k} 2^{2k} (1 + 2^{2k+2j})^{-4} [a_k(\sigma) + 2^{-3(k+j)/2} a_{-j}(\sigma)], \end{aligned} \tag{5.25}$$

for any F as in (5.21) and $f \in \{\psi_n, \overline{\psi}_n : n = 1, \ldots, d\}$. In the proof of (5.25) we need to use Lemma 5.1 with $\omega = 1/2$. From (4.17) we have

$$\|P_k(f(r))\|_{F_k(T)} \leq 2^{-\sigma k} a_k(\sigma) (1 + 2^{2k+2j})^{-4}. \tag{5.26}$$

We combine this with (5.22), using Lemma 5.1 with $\omega = 1/2$, to obtain

$$\|P_k(F(r)f(r))\|_{F_k(T)} \lesssim 2^{-\sigma k} 2^{k-j} 2^{-(k+j)/2} a_{-j}^2 a_{-j}(\sigma), \qquad k + j \leq 0$$

$$\|P_k(F(r)f(r))\|_{F_k(T)} \lesssim 2^{-\sigma k} (1 + 2^{2k+2j})^{-4} 2^{2k} a_{-j}^2 a_k(\sigma) \qquad k + j > 0$$

which imply (5.25) for the first term. By (5.10), (5.26), and Lemma 5.1 with $\omega = 1/2$

$$2^k \|P_k(A_l(r)f(r))\|_{F_k(T)} \lesssim 2^{2k} 2^{-\sigma k} 2^{-(k+j)/2} a_{-j}^2 a_{-j}(\sigma) \qquad k + j \leq 0$$

$$2^k \|P_k(A_l(r)f(r))\|_{F_k(T)} \leq C 2^{-\sigma k} (1 + 2^{2k+2j})^{-4} 2^{2k} a_{-j}^2 a_k(\sigma) \qquad k + j > 0$$

These bounds imply (5.25) for the second term.

We use now (5.24) to prove (5.20). Assume $s \in [2^{2k_0-1}, 2^{2k_0+1})$ for some $k_0 \in \mathbb{Z}$. We use (3.18). If $k + k_0 \leq 0$ then

$$\begin{aligned} \Big\| \int_0^s e^{(s-r)\Delta_x} P_k(K_m(r))\, dr \Big\|_{F_k(T)} &\lesssim \sum_{j \leq k_0} \int_{2^{2j-1}}^{2^{2j+1}} \|P_k(K_m(r))\|_{F_k(T)}\, dr \\ &\lesssim \sum_{j \leq k_0} 2^{2j} \varepsilon 2^{-\sigma k} 2^{2k} 2^{-3(k+j)/2} a_{-j}(\sigma) \lesssim \varepsilon 2^{-\sigma k} 2^{(k+k_0)/2} a_{-k_0}(\sigma), \end{aligned}$$

which suffices. If $k+k_0 \geq 0$ then, with $d_{k,j}$ as in the right-hand side of (5.24),

$$\begin{aligned}
&\Big\| \int_0^s e^{(s-r)\Delta_x} P_k(K_m(r))\,dr \Big\|_{F_k(T)} \\
&\leq \int_0^{s/2} \|e^{(s-r)\Delta_x} P_k(K_m(r))\|_{F_k(T)}\,dr + \int_{s/2}^s \|e^{(s-r)\Delta_x} P_k(K_m(r))\|_{F_k(T)}\,dr \\
&\lesssim \sum_{j \leq k_0} 2^{-20(k+k_0)} 2^{2j} d_{k,j} + 2^{-2k} d_{k,k_0} \\
&\lesssim 2^{-20(k+k_0)} \sum_{j \leq k_0} \varepsilon 2^{-\sigma k} 2^{2k} (1+2^{2k+2j})^{-4} [2^{2j} a_k(\sigma) + 2^{j/2} 2^{-3k/2} a_{-j}(\sigma)] \\
&\quad + \varepsilon 2^{-\sigma k} (1+2^{2k+2k_0})^{-4} a_k(\sigma) \\
&\lesssim \varepsilon 2^{-\sigma k} (1+2^{2k+2k_0})^{-4} a_k(\sigma)
\end{aligned}$$

which suffices. This completes the proof of the lemma. □

We are now able to prove the $F_k(T)$ bounds (4.18) in Proposition 4.3. In view of (2.19) we have

$$P_k(\psi_m(s)) = e^{s\Delta_x} P_k(\psi_m(0)) + \int_0^s e^{(s-r)\Delta_x} P_k(K_m(r))\,dr.$$

Thus, from Lemma 5.3 and (4.16),(3.18) we obtain

$$\|P_k \psi_m(s)\|_{F_k(T)} \lesssim 2^{-\sigma k} (1+s2^{2k})^{-4} (b_k(\sigma) + \varepsilon a_k(\sigma)), \qquad \sigma \in \{(d-2)/2, \sigma_0\}$$

Due to the definition of $a_k(\sigma)$ in (5.2) this implies that $a_k(\sigma) \lesssim b_k(\sigma) + \varepsilon a_k(\sigma)$, and further $a_k(\sigma) \lesssim b_k(\sigma)$. Then (4.18) follows.

Next we consider the F_k bound (4.19) for the functions $P_k(A_m(s)\psi_l(s))$. It follows from Lemma 5.1 (with $\omega = 1/2$), Lemma 5.2 and (4.17) that for any $l, m = 1, \ldots, d$, and $r \in [2^{2j-2}, 2^{2j+2}]$

$$\|P_k(A_l(r)\psi_m(r))\|_{F_k(T)} \lesssim 2^{-\sigma k} 2^k 2^{-(k+j)/2} a_{-j}^2 a_{-j}(\sigma), \qquad k \leq -j$$

respectively

$$\|P_k(A_l(r)\psi_m(r))\|_{F_k(T)} \lesssim 2^{-\sigma k} 2^k (1+2^{2j+2k})^{-4} a_{-j}^2 a_k(\sigma), \qquad k \geq -j.$$

Then (4.19) follows since a_k, $a_k(\sigma)$ are slowly varying.

Next we turn our attention to the L^{p_d} bounds in Proposition 4.3. We start with a general lemma, similar to Lemma 5.1.

Lemma 5.4. *Assume that $T \in (0, 2^{2\mathcal{K}}]$, $f, g \in H^{\infty,\infty}(T)$, $P_k f \in S_k^\omega(T)$, $P_k g \in L_{t,x}^{p_d}$ for some $\omega \in [0, 1/2]$ and any $k \in \mathbb{Z}$, and*

$$\mu_k = \sum_{|k'-k| \leq 20} \|P_{k'} f\|_{S_{k'}^\omega(T)}, \quad \nu_k = \sum_{|k'-k| \leq 20} \|P_{k'} g\|_{L_{t,x}^{p_d}}.$$

Then, for any $k \in \mathbb{Z}$

$$\|P_k(fg)\|_{L^{p_d}_{t,x}} \lesssim \sum_{k' \le k} 2^{\frac{k'd}{2}}(\mu_{k'}\nu_k + 2^{\frac{d}{d+2}(k-k')}\mu_k\nu_{k'}) + 2^{\frac{kd}{2}}\sum_{k' \ge k} 2^{-\omega(k'-k)}\mu_{k'}\nu_{k'}. \quad (5.27)$$

Proof of Lemma 5.4. We use the same bilinear Littlewood-Paley decomposition (5.9) as in the proof of Lemma 5.1, and estimate each term. If $|k_2-k| \le 4$ and $k_1 \le k-4$ then by the Sobolev embedding $\|P_k f\|_{L^\infty_{t,x}} \lesssim 2^{dk/2}\mu_k$ we have

$$\|P_k(P_{k_1}f \cdot P_{k_2}g)\|_{L^{p_d}_{t,x}} \lesssim \|P_{k_1}f\|_{L^\infty_{t,x}}\|P_{k_2}g\|_{L^{p_d}_{t,x}} \lesssim 2^{k_1 d/2}\mu_{k_1}\nu_k.$$

If $|k_1-k| \le 4$, $k_2 \le k-4$ then we use the Sobolev embeddings $\|P_k f\|_{L^\infty_t L^{p_d}_x} \lesssim 2^{\frac{d}{d+2}k}\mu_k$ and $\|P_k g\|_{L^{p_d}_t L^\infty_x} \lesssim 2^{(\frac{d}{2}-\frac{d}{d+2})k}\nu_k$ to obtain

$$\|P_k(P_{k_1}f \cdot P_{k_2}g)\|_{L^{p_d}_{t,x}} \lesssim \|P_{k_1}f\|_{L^\infty_t L^{p_d}_x}\|P_{k_2}g\|_{L^{p_d}_t L^\infty_x} \lesssim 2^{dk_2/2}2^{\frac{d}{d+2}(k-k')}\mu_k\nu_{k_2}.$$

Finally if $k_1, k_2 \ge k-4$ and $|k_1-k_2| \le 8$ then we similarly have

$$\|P_k(P_{k_1}f \cdot P_{k_2}g)\|_{L^{p_d}_{t,x}} \lesssim 2^{k(d/2+\omega)}\|P_{k_1}f\|_{L^\infty_t L^{2\omega}_x}\|P_{k_2}g\|_{L^{p_d}_{t,x}} \lesssim 2^{k(d/2+\omega)}2^{-\omega k_1}\mu_{k_1}\nu_{k_1}.$$

The bound (5.27) follows by summing up the three cases above. □

We prove now $L^{p_d}_{t,x}$ bounds on the connection coefficients $A_m(0)$, $m = 1, \dots, d$.

Lemma 5.5. *For any $k \in \mathbb{Z}$, $m = 1, \dots, d$ and $\sigma \in \{(d-2)/2, \sigma_0\}$*

$$\|P_k(A_m(0))\|_{L^{p_d}_{t,x}} \lesssim 2^{-\sigma k}b_k(\sigma). \quad (5.28)$$

Proof of Lemma 5.5. We start from the identity (2.20)

$$A_m(0) = -\sum_{l=1}^{d}\int_0^\infty \Im\big(\overline{\psi_m}\mathbf{D}_l\psi_l\big)(s)\,ds. \quad (5.29)$$

where, as before, $\mathbf{D}_l\psi_l = \partial_l\psi_l + iA_l\psi_l$. From (4.18) we have

$$\|P_k\psi_m(s)\|_{S^0_k} \lesssim 2^{-\sigma k}(1+s2^{2k})^{-4}b_k(\sigma)$$

while from (4.18), (4.19) we obtain

$$\|P_k(\mathbf{D}_l\psi_l(s))\|_{L^{p_d}_{t,x}} \lesssim 2^k 2^{-\sigma k}(s2^{2k})^{-3/8}(1+s2^{2k})^{-3}b_k(\sigma).$$

We use now (5.27) with $\omega = 0$ to estimate

$$\begin{aligned}
\|P_k(A_m(0))\|_{L^{p_d}_{t,x}} &\lesssim \sum_{l=1}^d \int_0^\infty \|\overline{\psi_m(s)}\mathbf{D}_l\psi_l(s)\|_{L^{p_d}_{t,x}} ds \\
&\lesssim 2^{-\sigma k} \sum_{k' \leq k} b_k(\sigma) b_{k'} 2^{k'+k} \int_0^\infty (s2^{2k})^{-3/8}(1+s2^{2k})^{-3} ds \\
&\quad + 2^{-\sigma k} \sum_{k' \leq k} b_k(\sigma) b_{k'} 2^{2k'} 2^{\frac{d}{d+2}(k-k')} \int_0^\infty (s2^{2k'})^{-3/8}(1+s2^{2k})^{-4} ds \\
&\quad + \sum_{k' \geq k} 2^{-\sigma k'} b_{k'}(\sigma) b_{k'} 2^{(k-k')d/2} 2^{2k'} \int_0^\infty (s2^{2k'})^{-3/8}(1+s2^{2k'})^{-7} ds \\
&\lesssim 2^{-\sigma k} b_k(\sigma) \sum_{k' \leq k} b_{k'} 2^{(k'-k)/4} + \sum_{k' \geq k} b_{k'}(\sigma) b_{k'} 2^{-\sigma k'} 2^{(k-k')d/2} \\
&\lesssim 2^{-\sigma k} b_k b_k(\sigma).
\end{aligned}$$

Thus (5.28) follows. □

This concludes the proof of part (a) of Proposition 4.3. We next turn our attention to part (b). We first prove L^{p_d} bounds on the field $\psi_{d+1}(s)$.

Lemma 5.6. *For any $k \in \mathbb{Z}$ and $\sigma \in \{(d-2)/2, \sigma_1 - 1\}$*

$$\|P_k(\psi_{d+1}(s))\|_{L^{p_d}_{t,x}} \leq C 2^k 2^{-\sigma k} b_k(\sigma)(1+s2^{2k})^{-2} \tag{5.30}$$

Proof of Lemma 5.6. We use the heat equation (2.19) for ψ_{d+1},

$$\begin{aligned}
&(\partial_s - \Delta_x)\psi_{d+1} = K(\psi_{d+1}); \\
&K(\psi) = 2i\sum_{l=1}^d \partial_l(A_l\psi) - \sum_{l=1}^d (A_l^2 + i\partial_l A_l)\psi + i\sum_{l=1}^d \Im(\psi\overline{\psi_l})\psi_l.
\end{aligned} \tag{5.31}$$

We rewrite this equation in the form

$$\psi_{d+1}(s) = e^{s\Delta}\psi_{d+1}(0) + \int_0^s e^{(s-r)\Delta} K(\psi_{d+1})(r) dr. \tag{5.32}$$

Assuming that

$$\|P_k\psi(s)\|_{L^{p_d}_{t,x}} \lesssim 2^{-(\sigma-1)k} b_k(\sigma)(1+s2^{2k})^{-2} \tag{5.33}$$

we claim the following

$$\left\| \int_0^s e^{(s-r)\Delta_x} P_k K(\psi)(r)\, dr \right\|_{L^{p_d}_{t,x}} \lesssim \varepsilon^2 2^{-(\sigma-1)k} b_k(\sigma)(1+s2^{2k})^{-2}. \tag{5.34}$$

By (4.21) the function $e^{s\Delta_x}\psi_{d+1}(0)$ satisfies (5.33). Then a standard iteration argument shows that the solution ψ_{d+1} to (5.32) also satisfies (5.33). We note that by standard L^∞ bounds for the heat equation, (5.31) admits an unique bounded solution on each interval $[0, S]$, with $S > 0$. Therefore the solution obtained iteratively must coincide with ψ_{d+1}.

It remains to prove our claim. As in the proof of Lemma 5.3, assume that

$$F \in \big\{A_l^2, \partial_l A_l, fg : l = 1, \dots, d,\ f, g \in \{\psi_n, \overline{\psi}_n : n = 1, \dots, d\}\big\},$$

Due to (5.22) and (5.23) we have

$$\|P_k F(r)\|_{S_k^{1/2}(T)} \lesssim 2^{-(\sigma-1)k}(1 + s2^{2k})^{-2}(s2^{2k})^{-\frac{5}{8}} b_k b_k(\sigma). \tag{5.35}$$

Also, by Proposition 5.2,

$$\|P_k A_l(r)\|_{S_k^{1/2}(T)} \lesssim 2^{-\sigma k}(1 + s2^{2k})^{-3}(s2^{2k})^{-\frac{1}{8}} b_k b_k(\sigma). \tag{5.36}$$

Using (5.27) (with $\omega = 1/2$), (5.35), (5.36) and (5.33) it follows that

$$\|P_k(F(r)\psi(r))\|_{L^{p_d}_{t,x}} + 2^k\|P_k(A_l(r)\psi(r))\|_{L^{p_d}_{t,x}} \lesssim 2^{-(\sigma+1)k}(1 + s2^{2k})^{-2}(s2^{2k})^{-\frac{7}{8}} b_k^2 b_k(\sigma)$$

for any $k \in \mathbb{Z}$, $l = 1, \dots, d$, and $\sigma \in \{(d-2)/2, \sigma_0\}$. Since $b_k^2 \le \varepsilon^2$ we get

$$\|P_k K(\phi)\|_{L^{p_d}_{t,x}} \lesssim \varepsilon^2 2^{-(\sigma-3)k}(1 + s2^{2k})^{-2}(s2^{2k})^{-\frac{7}{8}} b_k(\sigma).$$

This implies (5.34) after integration with respect to s since

$$\int_0^s (1 + (s-r)2^{2k})^{-N}(1 + r2^{2k})^{-2}(r2^{2k})^{-\frac{7}{8}} dr \lesssim 2^{-2k}(1 + s2^{2k})^{-2}.$$

□

We conclude the proof of Proposition 4.3 with the L^2 bounds on $P_k A_{d+1}(0)$.

Lemma 5.7. *The connection coefficient A_{d+1} satisfies*

$$\|P_k(A_{d+1}(0))\|_{L^2_{t,x}} \lesssim \varepsilon 2^{-\sigma k} b_k(\sigma), \qquad d \ge 3 \tag{5.37}$$

respectively

$$\|A_{d+1}(0)\|_{L^2_{t,x}} \lesssim \varepsilon^2, \qquad \|P_k(A_{d+1}(0))\|_{L^2_{t,x}} \lesssim 2^{-\sigma k} b_{>k}^2(\sigma), \qquad d = 2 \tag{5.38}$$

Proof of Lemma 5.38. To bound A_{d+1} we start from the identity (2.20)

$$A_{d+1}(0) = -\sum_{l=1}^d \int_0^\infty \Im\big(\overline{\psi_{d+1}} \mathbf{D}_l \psi_l\big)(s)\, ds. \tag{5.39}$$

For ψ_{d+1} we use the bound (5.30). For $\mathbf{D}_l\psi_l$, by (4.18) and (4.19),

$$\|\mathbf{D}_l\psi_l(s)\|_{L_t^\infty L_x^2 \cap L^{p_d}_{t,x}} \lesssim 2^k 2^{-\sigma k} a_k(\sigma)(s2^{2k})^{-3/8}(1 + s2^{2k})^{-2}. \tag{5.40}$$

To multiply $L_t^\infty L_x^2 \cap L^{p_d}$ and L^{p_d} functions we will use the following bound: if

$$\mu_k = \sum_{|k'-k|\le 20} \|P_{k'} f\|_{L^{p_d}_{t,x} \cap L_t^\infty L_x^2}, \quad \nu_k = \sum_{|k'-k|\le 20} \|P_{k'} g\|_{L^{p_d}_{t,x}}$$

then, for any $k \in \mathbb{Z}$,

$$\|P_k(fg)\|_{L^2_{t,x}} \lesssim \sum_{j\leq k} 2^{j(d-2)/2}(\mu_j\nu_k + \mu_k\nu_j) + \sum_{j\geq k} 2^{k(d-2)/2}\mu_j\nu_j. \tag{5.41}$$

This is easy to prove as in Lemma 5.4 using a bilinear Littlewood-Paley decomposition and Sobolev embeddings.

In dimension $d \geq 3$ we estimate using (5.41), (5.30) and (5.40):

$$\begin{aligned}
\|P_k A_{d+1}(0)\|_{L^2_{t,x}} &\lesssim \sum_{l=1}^{d} \int_0^\infty \|P_k\big(\overline{\psi_{d+1}}\mathbf{D}_l\psi_l\big)(s)\|_{L^2_{t,x}} ds \\
&\lesssim 2^{-\sigma k} \sum_{j\leq k} 2^{j+k} b_k(\sigma) b_j \int_0^\infty (s2^{2j})^{-3/8}(1+s2^{2k})^{-2} ds \\
&\quad + \sum_{j\geq k} 2^{-\sigma j}2^{2j}2^{(k-j)(d-2)/2} b_j(\sigma) b_j \int_0^\infty (s2^{2j})^{-3/8}(1+s2^{2j})^{-4} ds \\
&\lesssim 2^{-\sigma k} b_k(\sigma) \sum_{j\leq k} b_j 2^{(j-k)/4} + \sum_{j\geq k} b_j(\sigma) b_j 2^{-\sigma j} 2^{(k-j)(d-2)/2} \\
&\lesssim 2^{-\sigma k} b_k b_k(\sigma).
\end{aligned}$$

In dimension $d = 2$ the same computation applies, with the only difference that the last sum can only be bounded by $b^2_{>k}(\sigma)2^{-\sigma k}$. This gives the second part of (5.37). For the first part we replace (5.41) by

$$\|fg\|_{L^2_{t,x}} \lesssim \sum_k \mu_k \sum_{j\leq k} \nu_j + \sum_k \nu_k \sum_{j\leq k} \mu_j. \tag{5.42}$$

Then repeating the above computation we obtain

$$\|A_{d+1}(0)\|_{L^2_{t,x}} \lesssim \sum_k b_k \sum_{j\leq k} b_j 2^{(j-k)/4} \lesssim \sum_k b_k^2.$$

□

6. Perturbative analysis of the Schrödinger equation

In this section we prove Proposition 4.5. For $k \in \mathbb{Z}$ we denote

$$b(k) = \sum_{m=1}^{d} \|P_k\psi_m(0)\|_{G_k(T)}. \tag{6.1}$$

For $\sigma \in \{(d-2)/2, \sigma_0\}$ we introduce the frequency envelopes

$$b_k(\sigma) = \sup_{j\in\mathbb{Z}} 2^{\sigma j}2^{-\delta|k-j|}b(j). \tag{6.2}$$

These are finite and belong to l^2 due to (4.12) and Sobolev embeddings. We also have

$$\|P_k\psi_m(0)\|_{G_k(T)} \lesssim 2^{-\sigma k} b_k(\sigma). \tag{6.3}$$

For (4.32) we need to show that $b_k(\sigma) \lesssim c_k(\sigma)$. On the other hand from the bootstrap assumption (4.31) we know that $b_k \leq \varepsilon_0^{-\frac{1}{2}} c_k$. In particular

$$\sum_{k\in\mathbb{Z}} b_k^2 \leq \varepsilon_0. \tag{6.4}$$

For the connection coefficients A_m we use Proposition 4.3 with $\varepsilon = \varepsilon_0^{\frac{1}{2}}$. The assumption (4.16) follows from the inclusion $G_k \subset F_k$. We also need to verify that the assumption (4.21) in Proposition 4.3 follows from (4.16) if ϕ solves the Schrödinger map equation:

Lemma 6.1. *If $b_k(\sigma)$ are as above then the field $\psi_{d+1}(0)$ satisfies the bounds*

$$\|P_k\psi_{d+1}(0)\|_{L^{p_d}_{t,x}} \lesssim b_k(\sigma) 2^{-(\sigma-1)k}. \tag{6.5}$$

Proof of Lemma 6.1. We use the identity (2.9)

$$\psi_{d+1}(0) = i\sum_{l=1}^{d}(\partial_l\psi_l(0) + iA_l(0)\psi_l(0)).$$

From (4.18), (4.20) and (4.11) we have

$$\|P_k\psi_l(0)\|_{L^\infty_t L^2_x\cap L^{p_d}_{t,x}} + \|P_kA_l(0)\|_{L^\infty_t L^2_x\cap L^{p_d}_{t,x}} \lesssim 2^{-\sigma k} b_k(\sigma). \tag{6.6}$$

The $L^{p_d}_{t,x}$ bound for the first term $\partial_l\psi_l(0)$ immediately follows. The second term $A_l(0)\psi_l(0)$ can also be estimated by (6.6) using (5.41), except in dimension $d = 2$. If $d = 2$ then from (5.41) and (6.6) we still obtain

$$\|P_k(P_{\leq k+4}A_l(0)\psi_l(0))\|_{L^2_{t,x}} \lesssim 2^{-\sigma k} b_k(\sigma).$$

However, in order to handle the high-high frequency interactions we need a stronger bound on A_l which follows from Lemma 5.2, namely

$$2^{\frac{k}{2}}\|P_kA_l(0)\|_{L^4_t L^2_x} \lesssim \|P_k(A_l(0))\|_{S^{1/2}_k(T)} \lesssim \varepsilon.$$

Combining this with (6.6) for A_l we easily obtain the remaining bound

$$\|P_k(P_{>k+4}A_l(0)\psi_l(0))\|_{L^2_{t,x}} \lesssim 2^{-\sigma k} b_k(\sigma).$$

□

Thus we can apply Proposition 4.3 and Corollary 4.4. For convenience we summarize the two main ingredients which are to be used in the sequel. On one hand, for $l = 1, \cdots, m$ we have the bounds

$$\begin{cases} \|\psi_l(s)\|_{F_k(T)} \lesssim 2^{-\sigma k} b_k(\sigma)(1 + s2^{2k})^{-4} \\ \\ \|P_k \mathbf{D}_l \psi_l(s)\|_{F_k(T)} \lesssim 2^k 2^{-\sigma k} b_k(\sigma)(s2^{2k})^{-\frac{3}{8}}(1 + s2^{2k})^{-2} \end{cases} \tag{6.7}$$

which follow from (4.18) and (4.19). On the other hand, for each

$$F \in \{\psi_m(0)\overline{\psi_l}(0), A_l^2(0), A_{d+1}(0)\}$$

we have the bounds

$$\begin{cases} \|P_k F\|_{L^2_{t,x}} \lesssim \varepsilon_0^{\frac{1}{2}} 2^{-\sigma k} b_k(\sigma), & d \geq 3 \\ \\ \|F\|_{L^2_{t,x}} \lesssim \varepsilon_0, \qquad \|P_k F\|_{L^2_{t,x}} \lesssim 2^{-\sigma k} b^2_{>k}(\sigma), & d = 2 \end{cases} \tag{6.8}$$

Also by Sobolev embeddings and Littlewood-Paley theory,

$$\|F\|_{L^2_t L^d_x} \lesssim [\sum_{k \in \mathbb{Z}} \|P_k F\|^2_{L^2_t L^d_x}]^{1/2} \lesssim [\sum_{k \in \mathbb{Z}} \varepsilon_0 b_k^2]^{1/2} \lesssim \varepsilon_0. \tag{6.9}$$

Here the A_{d+1} bound is from (4.23) and (4.24), while the $\psi_m(0)\overline{\psi_l}(0)$ and the $A_l^2(0)$ bounds follow from (6.6) due to (5.41).

For $m = 1, \ldots, d$ we denote the nonlinearity of the Schrödinger equation (2.11)

$$L_m = -2i \sum_{l=1}^{d} A_l \partial_l \psi_m + \big(A_{d+1} + \sum_{l=1}^{d} (A_l^2 - i\partial_l A_l)\big)\psi_m - i \sum_{l=1}^{d} \psi_l \Im(\overline{\psi_l}\psi_m). \tag{6.10}$$

For simplicity of notation, in this section we use sometimes ψ_m for $\psi_m(0)$ and A_m for $A_m(0)$.

Proposition 6.2. *(Control of the Schrödinger nonlinearities) For any $m = 1, \ldots, d$ and $\sigma \in \{(d-2)/2, \sigma_0\}$ we have*

$$\|P_k(L_m)\|_{N_k(T)} \lesssim \varepsilon_0 2^{-\sigma k} b_k(\sigma). \tag{6.11}$$

Before proving the above proposition we show how to use it to conclude the proof of Proposition 4.5. Applying Proposition 3.8 for the equations (2.11), by (6.11) and (4.29) we obtain

$$\|P_k \psi_m(0)\|_{G_k(T)} \lesssim 2^{\sigma k}(c_k(\sigma) + \varepsilon_0 b_k(\sigma)). \tag{6.12}$$

By the definition of $b_k(\sigma)$ this implies that

$$b_k(\sigma) \lesssim c_k(\sigma) + \varepsilon_0 b_k(\sigma).$$

Hence

$$b_k(\sigma) \lesssim c_k(\sigma)$$

which combined with (6.12) gives (4.32), concluding the proof of Proposition 4.5.

The rest of this section is concerned with the proof of Proposition 6.2, which follows from Lemma 6.4 and Lemma 6.7. We begin our analysis with some bilinear estimates:

Lemma 6.3. *(a) If* $|k_1 - k| \leq 80$ *and* $f \in F_{k_1}(T)$ *then*

$$\|P_k(Ff)\|_{N_k(T)} \lesssim \|F\|_{L^2_t L^d_x} \|f\|_{F_{k_1}(T)}. \tag{6.13}$$

(b) If $k_1 \leq k - 80$ *and* $f \in F_{k_1}(T)$ *then*

$$\|P_k(Ff)\|_{N_k(T)} \leq 2^{k_1(d-2)/2} 2^{(k_1-k)/2} \|F\|_{L^2_{t,x}} \|f\|_{F_{k_1}(T)}. \tag{6.14}$$

(c) If $k \leq k_1 - 80$ *and* $g \in G_{k_1}(T)$ *then*

$$\|P_k(Fg)\|_{N_k(T)} \leq 2^{k_1(d-2)/2} 2^{(k-k_1)/6} \|F\|_{L^2_{t,x}} \|g\|_{G_{k_1}(T)}. \tag{6.15}$$

Proof of Lemma 6.3. Part (a) follows from the definition of $F_k(T)$, $N_k(T)$, and

$$\|Ff\|_{L^{p_d'}_{t,x}} \lesssim \|F\|_{L^2_t L^d_x} \|f\|_{L^\infty_t L^2_x \cap L^{p_d}_{t,x}}.$$

Part (b) also follows directly from the definitions, since

$$\|P_k(Ff)\|_{N_k(T)} \lesssim 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^{d-1}} \|Ff\|_{L^{1,2}_{\mathbf{e},W_{k-40}}} \lesssim 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^{d-1}} \|f\|_{L^{2,\infty}_{\mathbf{e},W_{k_1+40}}} \|F\|_{L^2_{x,t}}.$$

Finally for part (c) we use Sobolev embeddings if $d \geq 3$,

$$\|P_k(Fg)\|_{N_k(T)} \lesssim \|P_k(Fg)\|_{L^{p_d'}_{t,x}} \lesssim 2^{k(d-2)/2} \|F\|_{L^2_{t,x}} \|g\|_{L^\infty_t L^2_x \cap L^{p_d}_{t,x}}.$$

If $d = 2$ we need to use the lateral Strichartz estimates. Using an angular partition of unity in frequency we can write

$$g = g_1 + g_2, \qquad \|g_1\|_{L^{6,3}_{\mathbf{e}_1}} + \|g_2\|_{L^{6,3}_{\mathbf{e}_2}} \lesssim 2^{-k_1/6} \|g\|_{G_k(T)}.$$

Then we have

$$\begin{aligned}
\|P_k(Fg)\|_{N_k(T)} &\lesssim 2^{k/6} \big(\|Fg_1\|_{L^{\frac{3}{2},\frac{6}{5}}_{\mathbf{e}_1}} + \|Fg_2\|_{L^{\frac{3}{2},\frac{6}{5}}_{\mathbf{e}_2}} \big) \\
&\lesssim 2^{k/6} \|F\|_{L^2} \big(\|g_1\|_{L^{6,3}_{\mathbf{e}_1}} + \|g_2\|_{L^{6,3}_{\mathbf{e}_2}} \big) \\
&\lesssim 2^{(k-k_1)/6} \|F\|_{L^2} \|g\|_{G_{k_1}(T)}.
\end{aligned}$$

□

The above lemma suffices in order to estimate the easier component of L_m:

$$L_{m,1} = \big(A_{d+1} + \sum_{l=1}^d A_l^2 \big) \psi_m - i \sum_{l=1}^d \psi_l \Im(\overline{\psi_l} \psi_m). \tag{6.16}$$

Lemma 6.4. *Let* F *satisfy* (6.8) *and* $\psi \in \{\psi_m,\ m = 1, \cdots, d\}$. *Then*

$$\|P_k(\psi F)\|_{N_k(T)} \lesssim \varepsilon_0 2^{-\sigma k} b_k(\sigma). \tag{6.17}$$

Proof of Lemma 6.4. We use a bilinear Littlewood-Paley decomposition

$$P_k(F\psi) = P_k(P_{<k-80}FP_{[k-4,k+4]}\psi) + \sum_{|k_1-k|\le 4}^{k_2<k-80} P_k(P_{k_1}FP_{k_2}\psi) + \sum_{k_1,k_2\ge k-80}^{|k_1-k_2|\le 90} P_k(P_{k_1}FP_{k_2}\psi).$$

The first term is estimated using (6.13) and (6.9). For the second by (6.14),

$$\|P_k(P_{k_1}F\,P_{k_2}\psi)\|_{N_k(T)} \lesssim 2^{k_2(d-2)/2}2^{\frac{k_2-k}{2}}\|P_{k_1}F\|_{L^2_{t,x}}\|P_{k_2}\psi\|_{G_{k_2}(T)} \lesssim \varepsilon_0 2^{\frac{k_2-k}{2}}2^{-\sigma k}b_k(\sigma).$$

The summation with respect to $k_2 < k-80$ is straightforward.

Finally for the third term we use (6.15)

$$\|P_k(P_{k_1}F\cdot P_{k_2}\psi)\|_{N_k(T)} \lesssim 2^{\frac{k-k_2}{6}}2^{(d-2)k_2/2}\|P_{k_1}F\|_{L^2_{t,x}}\|P_{k_2}\psi\|_{G_{k_2}(T)}.$$

If $d \ge 3$, then using (6.8) and that $\sigma \ge \frac{1}{2}$, the third sum is easily estimated. If $d = 2$, one needs to distinguish between two cases. If $0 \le \sigma \le 1/12$ then we bound the right hand side by

$$2^{\frac{k-k_2}{6}}\varepsilon_0 2^{-\sigma k_2}b_{k_2}(\sigma) \lesssim \varepsilon_0 2^{-\sigma k}2^{\frac{k-k_2}{12}}b_k(\sigma)$$

and the summation with respect to $k_1, k_2 \ge k-80$ is straightforward. If $\sigma \ge 1/12$ then we bound the right hand side by

$$2^{\frac{k-k_2}{6}}2^{-\sigma k}b^2_{>k}(\sigma)b_{k_2} \lesssim 2^{\frac{k-k_2}{6}}2^{-\sigma k}b_k(\sigma)b_k b_{k_2}$$

and the summation with respect to $k_1, k_2 \ge k-80$ is again straightforward. □

It remains to estimate the second part of L_m, namely

$$L_{m,2} = -2i\sum_{l=1}^d A_l\partial_l\psi_m - i\sum_{l=1}^d \partial_l A_l\cdot\psi_m = -i\sum_{l=1}^d \partial_l(A_l\psi_m) - i\sum_{l=1}^d A_l\partial_l\psi_m. \quad (6.18)$$

For this we first complement Lemma 6.3 with two L^2 bilinear estimates:

Lemma 6.5. *(a) If $k_1 \le k_2$, $f_1 \in F_{k_1}(T)$, and $f_2 \in F_{k_2}(T)$ then*

$$\|f_1\cdot f_2\|_{L^2_{t,x}} \lesssim 2^{k_1(d-2)/2}\|f_1\|_{F_{k_1}(T)}\|f_2\|_{F_{k_2}(T)}. \quad (6.19)$$

(b) If $k_1 \le k_2$, $f \in F_{k_1}(T)$, and $g \in G_{k_2}(T)$ then

$$\|f\cdot g\|_{L^2_{t,x}} \lesssim 2^{k_1(d-2)/2}2^{(k_1-k_2)/2}\|f\|_{F_{k_1}(T)}\|g\|_{G_{k_2}(T)}. \quad (6.20)$$

Proof of Lemma 6.5. Part (a) follows by Sobolev embeddings from

$$\|f_1\cdot f_2\|_{L^2_{t,x}} \lesssim \|f_1\|_{L^4_tL^{2d}_x}\|f_2\|_{L^\infty_tL^2_x\cap L^{p_d}_{t,x}}.$$

For part (b), we first observe that, using a smooth partition of 1 in the frequency space, we may assume that $\mathcal{F}(g)$ is supported in the set

$$\{(\xi,\tau) : |\xi| \in [2^{k_2-1}, 2^{k_2+1}] \text{ and } \xi\cdot\mathbf{e}_0 \ge 2^{k_2-5}\}$$

for some vector $\mathbf{e}_0 \in \mathbb{S}^{d-1}$. Thus

$$\|g\|_{L^{\infty,2}_{\mathbf{e}_0,\lambda}} \lesssim 2^{-k_2/2}\|g\|_{G_{k_2}(T)} \quad \text{if } |\lambda| \leq 2^{k_2-40}. \tag{6.21}$$

Then in dimension $d \geq 3$ we have

$$\|f \cdot g\|_{L^2_{t,x}} \lesssim \|f\|_{L^{2,\infty}_{\mathbf{e}}}\|g\|_{L^{\infty,2}_{\mathbf{e}}} \lesssim 2^{k_1(d-2)/2}2^{(k_1-k_2)/2}\|f\|_{F_{k_1}(T)}\|g\|_{G_{k_2}(T)}.$$

The argument is more involved if $d = 2$. Given the definition (3.13) of the $F_k(T)$ space in terms of $F^0_k(T)$, it suffices to show that the following bounds hold for $F^0_k(T)$:

$$\|f \cdot g\|_{L^2} \lesssim \|f\|_{F^0_{k_1}(T)}\|g\|_{G_{k_2}(T)}, \qquad k_1 \geq k_2 - 100 \tag{6.22}$$

respectively

$$\|f \cdot g\|_{L^2} \lesssim 2^{(k_1-k_2)/2}\|f\|_{F^0_{k_1}(T)}\|g\|_{G_{k_2}(T)}, \qquad k_1 < k_2 - 100. \tag{6.23}$$

The bound (6.22) follows easily by estimating both factors in $L^4_{t,x}$. For (6.23), on the other hand, we use the local smoothing/maximal function spaces. Precisely, for g as in (6.21) we have

$$\|f \cdot g\|_{L^2_{t,x}} \lesssim \|f\|_{L^{2,\infty}_{\mathbf{e}_0,W_{k_1+40}}} \sup_{|\lambda|<2^{k_2-40}} \|P_{k_2}g\|_{L^{\infty,2}_{\mathbf{e}_0,\lambda}} \lesssim 2^{(k_1-k_2)/2}\|f\|_{F^0_{k_1}(T)}\|g\|_{G_{k_2}(T)},$$

as desired. □

The bilinear estimates in Lemmas 6.3,6.5 allow us to obtain corresponding trilinear estimates. We denote by $C(k, k_1, k_2, k_3)$ the best constant C in the estimate

$$\begin{aligned} &\|P_k(P_{k_1}f_1P_{k_2}f_2P_{k_3}g)\|_{N_k(T)} \\ &\lesssim C2^{\frac{d-2}{2}(k_1+k_2+k_3-k)}\|P_{k_1}f_1\|_{F_{k_1}(T)}\|P_{k_2}f_2\|_{F_{k_2}(T)}\|P_{k_3}g\|_{G_{k_3}(T)}. \end{aligned} \tag{6.24}$$

Using the $L^\infty_t L^2_x \cap L^{p_d}_{t,x}$ norm for each of the three factors and the $L^{p_d'}_{t,x}$ norm for the output, by Sobolev embeddings one can easily show that

$$C(k, k_1, k_2, k_3) \lesssim 1.$$

We seek to improve this with certain off-diagonal gains:

Lemma 6.6. *The best constant $C = C(k, k_1, k_2, k_3)$ in* (6.24) *satisfies the following bounds:*

$$C(k, k_1, k_2, k_3) \lesssim \begin{cases} 2^{(k_1+k_2)/2-k} & k_1, k_2 \leq k - 40 \\ 2^{-|k-k_3|/6} & k, k_3 \leq k_1 - 40 \\ 2^{-|\Delta k|/6} & \text{otherwise} \end{cases} \tag{6.25}$$

where $\Delta k = \max\{k, k_1, k_2, k_3\} - \min\{k, k_1, k_2, k_3\}$.

Proof. In the case $k_1, k_2 \leq k-40$ we must also have $|k_3-k| \leq 4$. Then we successively apply (6.20) and (6.14).

In the case $k, k_3 \leq k_1 - 40$ we apply first (6.19) and then conclude with (6.15) if $k \leq k_3$, respectively (6.14) if $k > k_3$.

In the remaining case we can assume without any restriction in generality that $k_1 \leq k_2$. Then there are two possibilities:

(i) $k_3 \leq k$ and $|k-k_2| \leq 40$. If $k_1 \leq k_3$ then we use (6.19) for $P_{k_2}f_2P_{k_3}g$ and then conclude with (6.14). If $k_3 < k_1$ then we use (6.19) for $P_{k_1}f_1P_{k_2}f_2$ and then conclude with (6.14).

(ii) $k_3 > k$ and $|k_3-k_2| \leq 40$. If $k_1 \leq k$ then we use (6.20) for $P_{k_1}f_1P_{k_3}g$ and then conclude using only Strichartz norms. If $k_{min} = k$ then we use (6.19) for $P_{k_1}f_1P_{k_2}f_2$ and then conclude with (6.15).

□

Lemma 6.7. *The following estimate holds:*

$$\|P_kL_{m,2}\|_{N_k(T)} \lesssim \varepsilon_0 2^{-\sigma k} b_k(\sigma). \tag{6.26}$$

Proof of Lemma 6.7. To bound $L_{m,2}$ we use the representation (2.20) for the connection coefficients A_l. Thus using the short notations A, $\mathbf{D}\psi$ and ψ for A_m, $\mathbf{D}_m\psi_m$, and ψ_m with $m = 1, d$ we can write

$$\begin{aligned}
\|P_k\partial_x(A\psi)\|_{N_k(T)} &\lesssim \int_0^\infty \|\partial_x P_k(\psi(s)\mathbf{D}\psi(s)\psi)\|_{N_k(T)} ds \\
&\lesssim \sum_{k_1,k_2,k_3} \int_0^\infty 2^k \|P_k(P_{k_1}\psi(s)P_{k_2}(\mathbf{D}\psi(s))P_{k_3}\psi)\|_{N_k(T)} ds \\
&\lesssim \sum_{k_1,k_2,k_3} 2^k C_0 \|P_{k_3}\psi\|_{G_{k_3}(T)} \int_0^\infty \|P_{k_1}\psi(s)\|_{F_{k_1}(T)} \|P_{k_2}(\mathbf{D}\psi(s))\|_{F_{k_3}(T)} ds
\end{aligned}$$

where $C_0 = C(k,k_1,k_2,k_3)2^{\frac{d-2}{2}(k_1+k_2+k_3-k)}$. For the term $A\partial_x\psi$ we obtain a similar bound but with 2^k replaced by 2^{k_3} since $\|\partial_x P_{k_3}\psi\|_{G_{k_3}(T)} \lesssim 2^{k_3}\|P_{k_3}\psi\|_{G_{k_3}(T)}$. Thus

$$\begin{aligned}
\|P_kL_{m,2}\|_{N_k(T)} \lesssim &\sum_{k_1,k_2,k_3} 2^{\max\{k,k_3\}} C_0 \|P_{k_3}\psi\|_{G_{k_3}(T)} \\
&\times \int_0^\infty \|P_{k_1}\psi(s)\|_{F_{k_1}(T)} \|P_{k_2}(\mathbf{D}\psi(s))\|_{F_{k_3}(T)} ds.
\end{aligned}$$

For the last two factors we use (6.7). Thus we need to evaluate the integrals

$$I_{k_1k_2} = \int_0^\infty (1+s2^{2k_1})^{-4} 2^{k_2} (s2^{2k_2})^{-3/8} (1+s2^{2k_2})^{-2} ds \lesssim 2^{-\max\{k_1,k_2\}}.$$

Taking this into account, from (6.7) and (6.3) we obtain

$$\begin{aligned}
\|P_kL_{m,2}\|_{N_k(T)} &\lesssim 2^{\sigma k} \sum_{k_1,k_2,k_3} C(k,k_1,k_2,k_3) 2^{\max\{k,k_3\}-\max\{k_1,k_2\}} b_{k_{min}} b_{k_{mid}} b_{k_{max}}(\sigma) \\
&= 2^{\sigma k}(S_1+S_2+S_3)
\end{aligned}$$

where the sums S_1, S_2 and S_3 correspond to the three cases in (6.25) and the indices $\{k_{min}, k_{mid}, k_{max}\}$ represent the increasing rearrangement of $\{k_1, k_2, k_3\}$. Then

$$S_1 \lesssim \sum_{k_1,k_2\le k-4} 2^{-|k_1-k_2|/2} b_{k_1} b_{k_2} b_k(\sigma) \lesssim \varepsilon_0 b_k(\sigma).$$

In the second case we have full off-diagonal decay

$$S_2 \lesssim \sum_{k_1\ge k}^{k_3\le k_1} 2^{\max\{k,k_3\}-k_1} 2^{-|k-k_3|/6} b_{k_3} b_{k_1} b_{k_1}(\sigma) \lesssim \varepsilon_0 b_k(\sigma).$$

In the third case $\max\{k, k_3\} = \max\{k_1, k_2\}$ therefore we obtain

$$S_3 \lesssim \sum_{k_1,k_2,k_3} 2^{-|\Delta k|/6} b_{k_{min}} b_{k_{mid}} b_{k_{max}}(\sigma) \lesssim \varepsilon_0 b_k(\sigma).$$

□

7. The main linear estimates

In this section we prove Proposition 3.8. We use the notation of section 3, see in particular the definitions 3.3, 3.6 and 3.7. We define two more classes of spaces, which are used only in this section. Given a finite subset $W \subseteq \mathbb{R}$ and $r \in [1, \infty]$ we define the spaces $\sum^r L^{p,q}_{\mathbf{e},W}$ and $\bigcap^r L^{p,q}_{\mathbf{e},W}$ using the norms

$$\|\phi\|^r_{\sum^r L^{p,q}_{\mathbf{e},W}} = |W|^{r-1} \inf_{\phi=\sum_{\lambda\in W}\phi_\lambda} \sum_{\lambda\in W} \|\phi_\lambda\|^r_{L^{p,q}_{\mathbf{e},\lambda}} \tag{7.1}$$

and

$$\|\phi\|^r_{\bigcap^r L^{p,q}_{\mathbf{e},W}} = |W|^{-1} \sum_{\lambda\in W} \|\phi\|^r_{L^{p,q}_{\mathbf{e},\lambda}}. \tag{7.2}$$

Clearly, $\sum^1 L^{p,q}_{\mathbf{e},W} = L^{p,q}_{\mathbf{e},W}$ (compare with definition 3.3) and

$$\|\phi\|_{\sum^r L^{p,q}_{\mathbf{e},W}} \le \|\phi\|_{\sum^{r'} L^{p,q}_{\mathbf{e},W}} \qquad \text{if } r \le r'. \tag{7.3}$$

We first consider the homogeneous equation

$$(i\partial_t + \Delta_x)u = 0, \qquad u(0) = f \in L^2(\mathbb{R}^d)$$

which has the solution $u(t) = e^{it\Delta} f$. For this we have the following:

Lemma 7.1. *Assume $f \in L^2(\mathbb{R}^d)$, $k \in \mathbb{Z}$, and $\mathbf{e} \in \mathbb{S}^{d-1}$. We have:*
(i) Local smoothing estimate

$$\sup_{|\lambda|\le 2^{k-5}} \|e^{it\Delta} P_{k,\mathbf{e}} f\|_{L^{\infty,2}_{\mathbf{e},\lambda}} \lesssim \|f\|_{L^2}. \tag{7.4}$$

(ii) Maximal function estimates:

$$\|e^{it\Delta} P_k f\|_{L^{2,\infty}_{\mathbf{e}}} \lesssim 2^{k(d-1)/2} \|f\|_{L^2}, \qquad d \ge 3, \tag{7.5}$$

and, for any $\mathcal{K} \in \mathbb{Z}_+$,

$$\|1_{[-2^{2\mathcal{K}},2^{2\mathcal{K}}]}(t)e^{it\Delta}P_k f\|_{\sum^2 L^{2,\infty}_{\mathbf{e},W_{k+5}}} \lesssim 2^{k/2}\|f\|_{L^2}, \qquad d=2. \tag{7.6}$$

(iii) Strichartz-type estimates:

$$\|e^{it\Delta}f\|_{L^{p_d}_{x,t}} \lesssim \|f\|_{L^2}, \tag{7.7}$$

and

$$\|e^{it\Delta}P_k f\|_{L^{p_d}_x L^\infty_t} \lesssim 2^{kd/(d+2)}\|f\|_{L^2}. \tag{7.8}$$

If $d=2$ and $(p,q) \in (2,\infty] \times [2,\infty]$, $1/p+1/q=1/2$, then

$$\|e^{it\Delta}P_{k,\mathbf{e}}f\|_{L^{p,q}_{\mathbf{e}}} \lesssim 2^{k(2/p-1/2)}\|f\|_{L^2}, \qquad p \ge q, \tag{7.9}$$

and

$$\|e^{it\Delta}P_k f\|_{L^{p,q}_{\mathbf{e}}} \lesssim_p 2^{k(2/p-1/2)}\|f\|_{L^2}, \qquad p \le q. \tag{7.10}$$

Proof of Lemma 7.1. The Strichartz estimate (7.7) is well-known, see [40]. As a consequence we obtain

$$\|e^{it\Delta}P_k f\|_{L^{p_d}_t L^{p_d}_x} + 2^{-2k}\|\partial_t e^{it\Delta}P_k f\|_{L^{p_d}_t L^{p_d}_x} \lesssim \|f\|_{L^2}.$$

The maximal Strichartz estimate (7.8) follows. The local smoothing estimate (7.4) is proved, for example, in [13] and [14] for $\lambda = 0$. The general case follows using the Galilean transformation $T_{\lambda\mathbf{e}}$, see the definition (3.6). The maximal function estimate (7.5) is known, see for example [13] or [14]. The lateral Strichartz estimate (7.9) follows by interpolation between (7.4) and (7.7). The lateral Strichartz estimate (7.10) follows using a standard TT^* argument and the Hardy-Littlewood-Sobolev inequality. We prove below the maximal function estimate (7.6), which represents our main new contribution to the linear theory.

Proof of (7.6)**.** We fix $\mathbf{e} \in \mathbb{S}^1$. By rescaling we can assume that $\mathcal{K} = 0$. We may also assume that $k \ge 1$, since for $k \le 0$ one has the stronger bound

$$\|\mathbf{1}_{[-1,1]}(t)e^{it\Delta}P_k f\|_{L^2_x L^\infty_t} \lesssim \|f\|_{L^2}.$$

We need to show that

$$\|1_{[-1,1]}(t)e^{it\Delta}P_k f\|_{\sum^2 L^{2,\infty}_{\mathbf{e},W_{k+5}}} \lesssim 2^{k/2}\|f\|_{L^2}. \tag{7.11}$$

We show first that if $\|g\|_{\bigcap^2 L^{2,1}_{\mathbf{e},W_{k+5}}} \le 1$ then

$$\Big|\int_{\mathbb{R}^2\times\mathbb{R}} \overline{g(x,t)}\mathbf{1}_{[-1,1]}(t)(e^{it\Delta}P_k f)(x,t)\,dxdt\Big| \lesssim 2^{k/2}\|f\|_{L^2}. \tag{7.12}$$

This can be rewritten as

$$\Big|\int_{\mathbb{R}^2\times\mathbb{R}} \overline{(e^{-it\Delta}P_k g(t))(x)}\mathbf{1}_{[-1,1]}(t)f(x)\,dtdx\Big| \lesssim 2^{k/2}\|f\|_{L^2}.$$

or equivalently

$$\left\| \int_{-1}^{1} e^{-it\Delta} P_k g(t) \right\|_{L^2} \lesssim 2^{k/2}$$

Hence it suffices to show that

$$\Big| \int_{\mathbb{R}^2\times\mathbb{R}} \int_{\mathbb{R}^2\times\mathbb{R}} g(x,t)\mathbf{1}_{[-1,1]}(t)\overline{g}(y,s)\mathbf{1}_{[-1,1]}(s)K_k(x-y,t-s)\, dxdtdyds \Big| \lesssim 2^k \quad (7.13)$$

where

$$K_k(x,t) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{-it|\xi|^2} \chi_k(|\xi|)^2\, d\xi. \quad (7.14)$$

By stationary phase

$$|K_k(t,x)| \lesssim \begin{cases} 2^{2k}(1+2^{2k}|t|)^{-1} & |x| \leq 2^{k+4}|t|; \\ 2^{2k}(1+2^k|x|)^{-N} & |x| \geq 2^{k+4}|t|. \end{cases}$$

The key idea is to foliate K_k in the $\mathbf{e}$ direction with respect to (thickened) rays with speed less than 2^{k+5}. We observe that for $t \in [-2,2]$

$$|K_k(t,x)| \lesssim \sum_{\lambda\in W_{k+5}} K_{k,\lambda}(t,x), \qquad K_{k,\lambda}(t,x) = (1+2^k|x\cdot\mathbf{e}-\lambda t|)^{-N}.$$

Hence the left hand side of (7.13) can be bounded by

$$\sum_{\lambda\in W_{k+5}} \int_{-1}^{1}\int_{-1}^{1} K_{k,\lambda}(t-s,x-y)|g(y,s)||g(x,t)|dxdydsdt$$

$$\lesssim \sum_{\lambda\in W_{k+5}} \|K_{k,\lambda}\|_{L^{1,\infty}_{\mathbf{e},\lambda}} \|g\|_{L^{2,1}_{\mathbf{e},\lambda}} \|g\|_{L^{2,1}_{\mathbf{e},\lambda}} \lesssim 2^{-k} \sum_{\lambda\in W_{k+5}} \|g\|^2_{L^{2,1}_{\mathbf{e},\lambda}} \lesssim 2^k \|g\|^2_{\bigcap^2 L^{2,1}_{\mathbf{e},W_{k+5}}},$$

where we used the fact that $|W_{k+5}| \approx 2^{2k}$. Thus (7.12) follows.

Formally the main bound (7.11) would follow from (7.12) and the duality relation

$$\left(\bigcap\nolimits^2 L^{2,1}_{\mathbf{e},W_{k+5}}\right)' = \sum\nolimits^2 L^{2,\infty}_{\mathbf{e},W_{k+5}}.$$

However this duality relation is not entirely straightforward so we provide a direct argument. Let $\chi(t)$ be a function which has the following properties:

a) $\mathcal{F}\chi$ is smooth and with compact support.

b) $\chi(t) \neq 0$ for $t \in [-1,1]$.

Then χ is rapidly decreasing at infinity, therefore from (7.12) it follows that

$$\Big| \int_{\mathbb{R}^2\times\mathbb{R}} \overline{g(x,t)}\chi(t)(e^{it\Delta}P_k f)(x,t)\, dxdt \Big| \lesssim 2^{k/2}\|f\|_{L^2} \quad (7.15)$$

whenever $\|g\|_{\bigcap^2 L^{2,1}_{\mathbf{e},W_{k+5}}} \leq 1$. We will use this to prove a stronger form of (7.11), namely

$$\|\chi(t)e^{it\Delta}P_k f\|_{\sum^2 L^{2,\infty}_{\mathbf{e},W_{k+5}}} \lesssim 2^{k/2}\|f\|_{L^2}. \quad (7.16)$$

The advantage is that the function $u = \chi(t)e^{it\Delta}P_k f$ has a compactly supported Fourier transform. To obtain (7.16) from (7.15) it suffices to show that

$$\|u\|_{\sum^2 L^{2,\infty}_{\mathbf{e},W_{k+5}}} \lesssim \sup \left\{ \int_{\mathbb{R}^2\times\mathbb{R}} u(x,t)\overline{v}(x,t)dxdt;\ \|g\|_{\bigcap^2 L^{2,1}_{\mathbf{e},W_{k+5}}} \le 1 \right\} \tag{7.17}$$

Indeed, suppose that the space-time Fourier transform $\mathcal{F}u$ is supported inside a ball B. We define the normed space

$$X_B = \{h \in L^2(\mathbb{R}^2\times\mathbb{R}) : \mathcal{F}(h) \text{ supported in } B \text{ and } \|h\|_A = \|h\|_{\Sigma^2 L^{2,\infty}_{\mathbf{e},W_{k+5}}} < \infty\}.$$

We also use a larger ball $2B$ and the corresponding space X_{2B}.

Since $u \in X_B$, by the Hahn-Banach theorem there is $\Lambda \in X_B^*$ so that $\|u\|_{X_B} = \Lambda(u)$ and $\|\Lambda\|_{X_B^*} = 1$. By the Hahn-Banach theorem we can extend Λ to X_{2B}. On the other hand, for any $h \in X_{2B}$

$$\|h\|_{X_{2B}} \lesssim_B \|h\|_{L^2}.$$

Thus, using the Hahn-Banach theorem again, there is a linear functional $\Lambda' : L^2 \to \mathbb{C}$ such that $\Lambda'(h) = \Lambda(h)$ for any $h \in A \cap L^2$ and $|\Lambda'(h)| \lesssim_B \|h\|_{L^2}$ for any $h \in L^2$. Therefore there is $g \in L^2(\mathbb{R}^d\times\mathbb{R})$ with the property that

$$\Lambda(h) = \int_{\mathbb{R}^2\times\mathbb{R}} \bar{g}\cdot h\, dxdt \text{ for any } h \in X_{2B}\cap L^2.$$

We consider a space-time multiplier $\chi_B(D)$ whose symbol is smooth, supported in $2B$ and equals 1 in B. By the choice of Λ we have

$$\|u\|_{\sum^2 L^{2,\infty}_{\mathbf{e},W_{k+5}}} = \Lambda(u) = \int_{\mathbb{R}^2\times\mathbb{R}} \bar{g}\cdot u\, dxdt = \int_{\mathbb{R}^2\times\mathbb{R}} \overline{\chi_B(D)g}\cdot u\, dxdt$$

Hence for (7.17) it suffices to prove that

$$\|\chi_B(D)g\|_{\bigcap^2 L^{2,1}_{\mathbf{e},W_{k+5}}} \lesssim 1. \tag{7.18}$$

Since $\chi_B(D)$ is a bounded operator on $\sum^2 L^{2,\infty}_{\mathbf{e},W_{k+5}}$ and Λ is bounded in X_{2B} we have

$$\left|\int_{\mathbb{R}^2\times\mathbb{R}} \overline{\chi_B(D)g} h dxdt\right| = \left|\int_{\mathbb{R}^2\times\mathbb{R}} \overline{g}\chi_B(D)h dxdt\right| \lesssim \|h\|_{\sum^2 L^{2,\infty}_{\mathbf{e},W_{k+5}}}$$

for all $h \in \sum^2 L^{2,\infty}_{\mathbf{e},W_{k+5}} \cap L^2$. In view of the duality relations $(L^{2,1}_{\mathbf{e},\lambda})' = L^{2,\infty}_{\mathbf{e},\lambda}$ we can optimize the choice of h above to obtain (7.18). In order to guarantee that $h \in L^2$ one can carry out this analysis first in a compact set, and then expand it to infinity. □

We return to the proof of Proposition 3.8, which we restate here for convenience.

Proposition 7.2. *(Main linear estimate) Assume $\mathcal{K} \in \mathbb{Z}_+$, $T \in (0, 2^{2\mathcal{K}}]$, and $k \in \mathbb{Z}$. Then for any $u_0 \in L^2$ localized at frequency 2^k and $h \in N_k(T)$, the solution u to*

$$(i\partial_t + \Delta_x)u = h, \qquad u(0) = u_0 \tag{7.19}$$

satisfies

$$\|u\|_{G_k(T)} \lesssim \|u_0\|_{L^2} + \|h\|_{N_k(T)}. \tag{7.20}$$

The proposition follows from Lemma 7.3 and Lemma 7.4 below. If $d = 2$ we define $\widetilde{G}_k(T)$ as the normed spaces of functions in $L^2_k(T)$ for which the norm

$$\begin{aligned}\|\phi\|_{\widetilde{G}_k(T)} = \|\phi\|_{G_k(T)} + 2^{-k/2} \sup_{\mathbf{e}\in S^1} \|\phi\|_{\Sigma^2 L^{2,\infty}_{\mathbf{e},W_{k+35}}} + 2^{k/6} \sup_{|j-k|\leq 25} \sup_{\mathbf{e}\in\mathbb{S}^1} \|P_{j,\mathbf{e}}\phi\|_{L^{6,3}_{\mathbf{e}}} \\ + 2^{k/2} \sup_{|j-k|\leq 25} \sup_{\mathbf{e}\in\mathbb{S}^1} \sup_{|\lambda|\leq 2^{k-35}} \|P_{j,\mathbf{e}}\phi\|_{L^{\infty,2}_{\mathbf{e},\lambda}}\end{aligned}$$

is finite. In other words, we replace the norm $L^{2,\infty}_{\mathbf{e},W_{k+40}} = \Sigma^1 L^{2,\infty}_{\mathbf{e},W_{k+40}}$ in (3.12) with the stronger norm $\Sigma^2 L^{2,\infty}_{\mathbf{e},W_{k+35}}$, compare with Definition 3.7 and (7.3), and readjust slightly the ranges of j and λ. Similarly, if $d \geq 3$ let

$$\|\phi\|_{\widetilde{G}_k(T)} = \|\phi\|_{G_k(T)} + 2^{k/2} \sup_{|j-k|\leq 25} \sup_{\mathbf{e}\in\mathbb{S}^{d-1}} \|P_{j,\mathbf{e}}\phi\|_{L^{\infty,2}_{\mathbf{e}}}.$$

It is easy to check from the definition that

$$\|v\|^2_{\widetilde{G}_k(T)} \leq \sum_{i=0}^{m-1} \|\mathbf{1}_{[t_i,t_{i+1})}(t)\cdot v\|^2_{\widetilde{G}_k(T)} \tag{7.21}$$

for any partition $\{-T = t_0 < \ldots < t_m = T\}$ of the interval $[-T, T]$ and any $v \in \widetilde{G}_k(T)$. This property does not hold for the spaces $G_k(T)$ if $d = 2$.

The solution u for (7.19) can be represented as

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}h(s)ds.$$

As a consequence of the Lemma 7.1, we immediately obtain (7.20) for the first term. More precisely, for any $f \in L^2(\mathbb{R}^d)$ localized at frequency 2^k we have

$$\|e^{it\Delta}f\|_{\widetilde{G}_k(T)} \lesssim \|f\|_{L^2}. \tag{7.22}$$

It remains to make the transition to the full inhomogeneous problem. We divide the $N_k(T)$ space into two components, $N_k(T) = N^0_k(T) + N^1_k(T)$ with norms

$$\begin{aligned}\|f\|_{N^0_k(T)} &= \inf_{f=f_1+f_2+f_3} \|f_1\|_{L^{\frac{4}{3}}} + 2^{\frac{k}{6}}\|f_2\|_{L^{\frac{3}{2},\frac{6}{5}}_{\mathbf{e}_1}} + 2^{\frac{k}{6}}\|f_3\|_{L^{\frac{3}{2},\frac{6}{5}}_{\mathbf{e}_2}} \\ \|f\|_{N^1_k(T)} &= 2^{-k/2} \sup_{\mathbf{e}\in\mathbb{S}^1} \|f\|_{L^{1,2}_{\mathbf{e},W_{k-40}}}\end{aligned} \tag{7.23}$$

in dimension $d = 2$, and

$$\|f\|_{N^0_k(T)} = \|f\|_{L^{p'_d}}, \qquad \|f\|_{N^1_k(T)} = 2^{-k/2} \sup_{\mathbf{e}\in\mathbb{S}^{d-1}} \|f\|_{L^{1,2}_{\mathbf{e}}} \tag{7.24}$$

in dimensions $d \geq 3$. The spaces $N_k^0(T)$ have the property that if $f \in N_k^0(T)$ and $-T = t_0 < t_1 < \ldots < t_m = T$ is a partition of the interval $[-T, T]$ then

$$\sum_{i=0}^{m-1} \|f \cdot 1_{[t_i,t_{i+1})}(t)\|_{N_k^0(T)}^{p_1} \leq \|f\|_{N_k^0(T)}^{p_1} \text{ for some } p_1 < 2. \tag{7.25}$$

This property is easy to verify for $p_1 = p_d'$ if $d \geq 3$ and $p_1 = 3/2$ if $d = 2$. The spaces $N_k^1(T)$ do not have this property. From Lemma 7.1, by duality, we obtain the energy bound

$$\left\| \int_0^t e^{i(t-s)\Delta} h(s) ds \right\|_{L_t^\infty L_x^2} \lesssim \|h\|_{N_k^0(T)} \quad \text{ for any } h \in N_k^0(T). \tag{7.26}$$

Lemma 7.3. *Assume that* $u \in L_k^2(T)$ *satisfies*

$$(i\partial_t + \Delta_x)u = h \text{ on } \mathbb{R}^d \times [-T, T], \quad u(0) = 0.$$

Then

$$\|u\|_{\widetilde{G}_k(T)} \lesssim \|h\|_{N_k^0(T)}.$$

Proof of Lemma 7.3. This lemma is an abstract consequence of the homogeneous bound (7.22), the energy bound (7.26), and the bounds (7.21) and (7.25).

The simplest way to prove this lemma is by using the U_Δ^p and V_Δ^p spaces associated to the Schrödinger evolution. These spaces have been first introduced in unpublished work of the last author, as substitutes for Bourgain's $X^{s,b}$ spaces which are better suited for the study of dispersive evolutions in critical Sobolev spaces. They have been successfully used for instance in [27], [28], [3], [10].

For convenience we recall their definition. V_Δ^p is the space of right continuous L^2 valued functions with bounded p-variation along the Schrödinger flow,

$$\|u\|_{V_\Delta^p}^p = \|u\|_{L_t^\infty L_x^2}^p + \sup_{t_k \nearrow} \sum_{k \in \mathbb{Z}} \|u(t_{k+1} - e^{i(t_{k+1}-t_k)\Delta} u(t_k)\|_{L^2}^p$$

where the supremum is taken over all increasing sequences t_k. On the other hand U_Δ^p is the atomic space generated by a family $\mathcal{A}_p$ of atoms a which have the form

$$a(t) = e^{it\Delta} \sum_k 1_{[t_k,t_k+1)} u_k, \qquad \sum_k \|u_k\|_{L^2}^p \leq 1,$$

where the sequence $\{t_k\}$ is increasing. Precisely, we have

$$U_\Delta^p = \{u = \sum c_k a_k : \ \sum_k |c_k| < \infty, \quad a_k \in \mathcal{A}_p\}.$$

The above sum converges uniformly in L^2; it also converges in the stronger V_Δ^p topology. The U_Δ^p norm is defined by

$$\|u\|_{U_\Delta^p} = \inf\{\sum_k |c_k| : \ u = \sum c_k a_k, \ a_k \in \mathcal{A}_p\}.$$

These spaces are related as follows:

$$U^p_\Delta \subset V^p_\Delta \subset U^q_\Delta, \qquad 1 \leq p < q < \infty. \tag{7.27}$$

The first inclusion is straightforward, but the second is not. As a consequence of the bounds (7.22) and (7.21) we have

$$\|u\|_{\widetilde{G}_k(T)} \lesssim \|u\|_{U^2_\Delta} \tag{7.28}$$

for all $u \in U^2_\Delta$ localized at frequency 2^k. On the other hand, as a consequence of the bounds (7.26) and (7.25) we have

$$\|u\|_{V^{p_1}_\Delta} \lesssim \|h\|_{N^0_k}, \qquad u(t) = \int_0^t e^{i(t-s)\Delta} h(s) ds. \tag{7.29}$$

The lemma follows using (7.27) and $p_1 < 2$.

One could also provide a self-contained proof which does not use the spaces U^p_Δ and V^p_Δ, in the spirit of the Christ-Kiselev lemma [6], or of the proof of the second inclusion in (7.27) (see [27, Lemma 6.4]).

□

We estimate now the contribution of terms in $N^1_k(T)$.

Lemma 7.4. *Assume that* $v \in L^2_k(T)$ *satisfies*

$$(i\partial_t + \Delta_x)u = h \text{ on } \mathbb{R}^d \times [-T,T], \quad u(0) = 0.$$

Then

$$\|u\|_{G_k(T)} \lesssim \|h\|_{N^1_k(T)}.$$

Proof of Lemma 7.4. The spaces $N^1_k(T)$ do not satisfy (7.25), so the general argument given in Lemma 7.3 does not apply. We give a direct argument starting from the definition of the spaces $L^{1,2}_{\mathbf{e},\lambda}$. Using a smooth angular partition of unity in frequency we can assume without any restriction in generality that h is frequency localized to a region $\xi \cdot \mathbf{e} \in [2^{k-2}, 2^{k+2}]$ for some $\mathbf{e} \in \mathbb{S}^{d-1}$. After a Galilean transformation, we may assume that $h \in L^{1,2}_{\mathbf{e}}$ and it suffices to prove the stronger bound

$$\|P_k P_{j,\mathbf{e}} u\|_{\widetilde{G}_k(T)} \lesssim 2^{-k/2} \|h\|_{L^{1,2}_{\mathbf{e}}}, \qquad |j-k| \leq 4. \tag{7.30}$$

Suppose $\mathbf{e} = \mathbf{e}_1$. The solution u can be expressed via the fundamental solution K_0 for the Schrödinger equation as

$$\begin{aligned} u(t,x) &= \int_{s<t} \int_{\mathbb{R}^d} K_0(t-s, x-y) h(s,y) dy ds \\ &= \int_{s<t} \int_{\mathbb{R}^d} K_0(t-s, x_1-y_1, x'-y') h(s,y_1,y') dy_1 dy' ds \\ &= \int_{\mathbb{R}} u_{y_1}(t,x) dy_1 \end{aligned}$$

where

$$u_{y_1}(t,x) = \int_{s<t}\int_{\mathbb{R}^{d-1}} K_0(t-s, x_1-y_1, x'-y')h(s,y_1,y')dy'ds.$$

It suffices to show that

$$\|P_kP_{j,\mathbf{e}_1}u_{y_1}\|_{\widetilde{G}_k(T)} \lesssim 2^{-k/2}\|h(y_1)\|_{L^2}. \tag{7.31}$$

This is a consequence of the following:

Lemma 7.5. *Suppose that $|j-k|\leq 4$. Then the function $P_kP_{j,\mathbf{e}_1}u_{y_1}$ can be represented as*

$$\begin{aligned}&P_kP_{j,\mathbf{e}_1}u_{y_1}(t,x) = (P_{<k-40,\mathbf{e}_1}1_{\{x_1>y_1\}})\cdot P_kP_{j,\mathbf{e}_1}e^{it\Delta}v_0 + w(t,x);\\ &\|v_0\|_{L^2} + 2^{-k}(\|\Delta w\|_{L^2} + \|\partial_t w\|_{L^2}) \lesssim 2^{-k/2}\|h(y_1)\|_{L^2}.\end{aligned} \tag{7.32}$$

To see that this implies (7.31), notice that, since w is also localized at frequency 2^k, the bound for $\|w\|_{\widetilde{G}_k(T)}$ is obtained from Sobolev embeddings. We observe also that the function $P_{<k-40,\mathbf{e}_1}1_{\{x_1<y_1\}}$ is bounded while $v = P_kP_{j,\mathbf{e}_1}e^{it\Delta}v_0$ is an L^2 solution for the homogeneous equation which is localized at frequency 2^k. By Lemma 7.1 this allows us to directly estimate all components of its $\widetilde{G}_k(T)$ norm except for the $L^{\infty,2}_{\mathbf{e},\lambda}$ and the $L^{6,3}_{\mathbf{e}}$ bounds if $d=2$, respectively the $L^{\infty,2}_{\mathbf{e}}$ bound if $d\geq 3$. For these we need an additional step: if $|k_1-k|\leq 25$ and $\mathbf{e}\in\mathcal{S}^{d-1}$ we take advantage of the frequency localization of the above cutoff function in order to write

$$P_{k_1,\mathbf{e}}[(P_{<k-40,\mathbf{e}_1}1_{\{x_1>y_1\}})\cdot v] = \sum_{|k_2-k|\leq 30} P_{k_1,\mathbf{e}}[(P_{<k-40,\mathbf{e}_1}1_{\{x_1>y_1\}})\cdot P_{k_2,\mathbf{e}}v].$$

This suffices to prove (7.31), in view of Lemma 7.1. It remains to prove the last lemma.

Proof of Lemma 7.5. By translation invariance we can set $y_1 = 0$ and drop the parameter y_1 from the notations. The Fourier transform of $P_kP_{j,\mathbf{e}_1}u$ is

$$(\mathcal{F}(P_kP_{j,\mathbf{e}_1}u))(\tau,\xi) = \frac{\chi_k(|\xi|)\chi_j(\xi_1)}{-\tau-|\xi|^2+i0}\hat{h}(\tau,\xi').$$

On the other hand the Fourier transform of $v = P_kP_{j,\mathbf{e}_1}e^{it\Delta}v_0$ equals

$$(\mathcal{F}v)(\tau,\xi) = \chi_k(|\xi|)\chi_j(\xi_1)v_0(\xi)\delta_{\tau+|\xi|^2}.$$

Assuming that $\hat{v}$ is supported in the region $\{|\xi|\approx 2^k,\ \xi_1\approx 2^j\}$, after truncation we obtain

$$(\mathcal{F}((P_{<k-40,\mathbf{e}_1}1_{\{x_1>y_1\}})v))(\tau,\xi) = (\chi_k(|\xi|)\chi_j(\xi_1)v_0(\xi)\delta_{\tau+|\xi|^2}) * \frac{\chi_{<k-40}(\xi_1)}{\xi_1-i0}.$$

On the $\{\xi_1 > 0\}$ part of the paraboloid we can write

$$\delta_{\tau+|\xi|^2} = \frac{1}{2\sqrt{-\tau - |\xi'|^2}} \delta_{\xi_1 - \sqrt{-\tau-|\xi'|^2}}.$$

Hence, with the notation $\tilde{\xi}_1 = \sqrt{-\tau - |\xi'|^2}$, the above convolution gives

$$\begin{aligned}(\mathcal{F}((P_{<k-40,\mathbf{e}_1} 1_{\{x_1>y_1\}})v))(\tau,\xi) = &\ \chi_k(|(\tilde{\xi}_1,\xi')|)\chi_j(\xi_1)v(\tilde{\xi}_1,\xi')\frac{1}{2\tilde{\xi}_1}\frac{\chi_{<k-40}(\xi_1-\tilde{\xi}_1)}{\xi_1-\tilde{\xi}_1-i0} \\ = &\ -\chi_k(|(\tilde{\xi}_1,\xi')|)\chi_j(\xi_1)v(\tilde{\xi}_1,\xi')\frac{\xi_1+\tilde{\xi}_1}{2\tilde{\xi}_1}\frac{\chi_{<k-40}(\xi_1-\tilde{\xi}_1)}{-\tau-|\xi|^2+i0}.\end{aligned}$$

Matching the two expressions on the paraboloid $\tau + |\xi|^2 = 0$ we see that it is natural to choose v_0 of the form

$$v_0(\xi_1,\xi') = \begin{cases} \hat{h}(|\xi|^2,\xi') & \xi \in \operatorname{supp}\ \chi_k(|\xi|)\chi_j(\xi_1) \\ 0 & \text{otherwise.}\end{cases}$$

Changing variables we obtain the required bound for v, namely

$$\|v_0\|_{L^2} \lesssim 2^{-\frac{k}{2}}\|h\|_{L^2}.$$

It remains to consider w, whose Fourier transform is obtained by taking the difference, $\hat{w}(\tau,\xi) = \hat{h}(\tau,\xi')q(\tau,\xi)$ where

$$q(\tau,\xi) = \left(\chi_k(|\xi|)\chi_j(\xi_1) - \chi_k(\tilde{\xi}_1,\xi')\chi_j(\tilde{\xi}_1)\chi_{<k-40}(\xi_1-\tilde{\xi}_1)\frac{\xi_1+\tilde{\xi}_1}{2\tilde{\xi}_1}\right)\frac{1}{-\tau-|\xi|^2+i0}.$$

The expression in the bracket is supported in $\{\xi_1 \approx 2^k, |\xi'| \lesssim 2^k\}$ and vanishes on the paraboloid $\{\tau + |\xi|^2 = 0\}$, canceling the singularity. Thus we obtain

$$|q(\tau,\xi)| \lesssim (|\tau| + |\xi|^2)^{-1}.$$

The bounds for w follow. □

□

8. Proof of Proposition 4.2

In this section we prove Proposition 4.2. Recall that the frequency envelopes $\gamma_k(\sigma)$ and γ_k have the properties

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\gamma_k^2 \approx \gamma^2 \ll 1; \\ &\|P_k\phi\|_{L^\infty_t L^2_x} \leq 2^{-\sigma k}\gamma_k(\sigma) \qquad \text{for any } \sigma \in [0,\infty) \text{ and } k \in \mathbb{Z}.\end{aligned} \tag{8.1}$$

We start with two technical lemmas.

Lemma 8.1. *Assume* $f, g \in L^\infty(\mathbb{R}^d \times (-T,T) : \mathbb{C})$*,* $P_k f, P_k g \in L^\infty_t L^2_x$ *and define, for any* $k \in \mathbb{Z}$*,*

$$\alpha_k = \sum_{|k'-k|\leq 20} 2^{dk'/2} \|P_{k'} f\|_{L^\infty_t L^2_x}, \quad \beta_k = \sum_{|k'-k|\leq 20} 2^{dk'/2} \|P_{k'} g\|_{L^\infty_t L^2_x}. \tag{8.2}$$

Then, for any $k \in \mathbb{Z}$*,*

$$2^{dk/2} \|P_k(fg)\|_{L^\infty_t L^2_x} \lesssim \alpha_k \sum_{k'\leq k} \beta_{k'} + \beta_k \sum_{k'\leq k} \alpha_{k'} + \sum_{k'\geq k} 2^{-d|k'-k|} \alpha_{k'} \beta_{k'}. \tag{8.3}$$

In addition, if $\|g\|_{L^\infty_{x,t}} \leq 1$ *then, for any* $k \in \mathbb{Z}$

$$2^{dk/2} \|P_k(fg)\|_{L^\infty_t L^2_x} \lesssim \alpha_k + \beta_k \sum_{k'\leq k} \alpha_{k'} + \sum_{k'\geq k} 2^{-d|k'-k|} \alpha_{k'} \beta_{k'}. \tag{8.4}$$

Finally, if $\|f\|_{L^\infty_{x,t}} + \|g\|_{L^\infty_{x,t}} \leq 1$ *then, for any* $k \in \mathbb{Z}$

$$2^{dk/2} \|P_k(fg)\|_{L^\infty_t L^2_x} \lesssim \alpha_k + \beta_k + \sum_{k'\geq k} 2^{-d|k'-k|} \alpha_{k'} \beta_{k'}. \tag{8.5}$$

Proof of Lemma 8.1. We decompose the expressions to be estimated into Low $\times$ High $\to$ High frequency interactions and High$\times$High $\to$ Low frequency interactions. We estimate Low $\times$ High $\to$ High frequency interactions using the $L^\infty_{x,t}$ norm for the low frequency factor and the $L^\infty_t L^2_x$ norm for the high frequency factor. We estimate High $\times$ High $\to$ Low frequency interactions using the $L^\infty_t L^2_x$ norms for both factors, and Sobolev embedding.

For example, for (8.3) we estimate

$$\begin{aligned}
\|P_k(fg)\|_{L^\infty L^2} \lesssim & \sum_{|k_2-k|\leq 4} \|P_k(P_{\leq k-5} f \cdot P_{k_2} g)\|_{L^\infty_t L^2_x} + \sum_{|k_1-k|\leq 4} \|P_k(P_{k_1} f \cdot P_{\leq k-5} g)\|_{L^\infty_t L^2_x} \\
& + \sum_{k_1,k_2\geq k-4, |k_1-k_2|\leq 8} \|P_k(P_{k_1} f \cdot P_{k_2} g)\|_{L^\infty_t L^2_x} \\
\lesssim & \sum_{|k_2-k|\leq 4} \|P_{\leq k-5} f\|_{L^\infty_{x,t}} \|P_{k_2} g\|_{L^\infty_t L^2_x} + \sum_{|k_1-k|\leq 4} \|P_{k_1} f\|_{L^\infty_t L^2_x} \|P_{\leq k-5} g\|_{L^\infty_{x,t}} \\
& + \sum_{k_1,k_2\geq k-4, |k_1-k_2|\leq 8} 2^{dk/2} \|P_{k_1} f \cdot P_{k_2} g\|_{L^\infty_t L^1_x} \\
\lesssim & \sum_{k_1\leq k} \alpha_{k_1} 2^{-dk/2} \beta_k + \sum_{k_2\leq k} \beta_{k_2} 2^{-dk/2} \alpha_k + 2^{dk/2} \sum_{k'\geq k} 2^{-dk'} \alpha_{k'} \beta_{k'},
\end{aligned}$$

and the bound (8.3) follows. To prove (8.4) we use a similar argument, but estimate $\|P_{\leq k-5} g\|_{L^\infty_{x,t}} \lesssim 1$ instead of $\|P_{\leq k-5} g\|_{L^\infty_{x,t}} \lesssim \sum_{k_2\leq k} \beta_{k_2}$. For (8.5), we also estimate $\|P_{\leq k-5} f\|_{L^\infty_{x,t}} \lesssim 1$ instead of $\|P_{\leq k-5} f\|_{L^\infty_{x,t}} \lesssim \sum_{k_2\leq k} \alpha_{k_2}$. $\square$

Lemma 8.2. *Assume* $f, g, h \in C^\infty(\mathbb{R}^d \times (-T, T) : \mathbb{C})$ *and* $P_k f, P_k g, P_k h \in L^\infty_t L^2_x$ *for any* $k \in \mathbb{Z}$*. Let*

$$\mu_k = \sum_{|k'-k|\leq 20} 2^{dk'/2}(\|P_{k'}f\|_{L^\infty_t L^2_x} + \|P_{k'}g\|_{L^\infty_t L^2_x} + \|P_{k'}h\|_{L^\infty_t L^2_x}).$$

We assume that $\|f\|_{L^\infty_{x,t}} + \|g\|_{L^\infty_{x,t}} + \|h\|_{L^\infty_{x,t}} \leq 1$ *and* $\sup_{k\in\mathbb{Z}} \mu_k \leq 1$*. Then, for any* $k \in \mathbb{Z}$ *and* $l, m = 1, \ldots, d$

$$2^{dk/2}\|P_k(f\partial_l g\partial_m h)\|_{L^\infty_t L^2_x} \lesssim 2^{2k}\big[\mu_k \sum_{k'\leq k} 2^{-|k'-k|}\mu_{k'} + \sum_{k'\geq k} 2^{-(d-2)|k'-k|}\mu_{k'}^2\big]. \tag{8.6}$$

Proof of Lemma 8.2. By symmetry, it suffices to estimate

$$\sum_{k_2\in\mathbb{Z}}\sum_{k_1\leq k_2} 2^{dk/2}\|P_k(f \cdot P_{k_1}(\partial_l g) \cdot P_{k_2}(\partial_m h))\|_{L^\infty_t L^2_x},$$

which is dominated by

$$\begin{aligned}
&\sum_{k_2\geq k-4}\sum_{k_1\leq k_2} 2^{dk/2}\|P_k(P_{\leq k_2-5}(f) \cdot P_{k_1}(\partial_l g) \cdot P_{k_2}(\partial_m h))\|_{L^\infty_t L^2_x}\\
&+\sum_{k_2\in\mathbb{Z}}\sum_{k_1\leq k_2}\sum_{k_3\geq k_2+5} 2^{dk/2}\|P_k(P_{k_3}(f) \cdot P_{k_1}(\partial_l g) \cdot P_{k_2}(\partial_m h))\|_{L^\infty_t L^2_x}\\
&+\sum_{k_2\in\mathbb{Z}}\sum_{k_1\leq k_2}\sum_{|k_3-k_2|\leq 4} 2^{dk/2}\|P_k(P_{k_3}(f) \cdot P_{k_1}(\partial_l g) \cdot P_{k_2}(\partial_m h))\|_{L^\infty_t L^2_x} = I + II + III.
\end{aligned}$$

We estimate

$$\begin{aligned}
I \leq &\sum_{|k_2-k|\leq 4}\sum_{k_1\leq k_2} 2^{dk/2}\|P_k(P_{\leq k_2-5}(f) \cdot P_{k_1}(\partial_l g) \cdot P_{k_2}(\partial_m h))\|_{L^\infty_t L^2_x}\\
&+\sum_{k_2\geq k+5}\sum_{|k_1-k_2|\leq 4} 2^{dk/2}\|P_k(P_{\leq k_2-5}(f) \cdot P_{k_1}(\partial_l g) \cdot P_{k_2}(\partial_m h))\|_{L^\infty_t L^2_x}\\
&\lesssim \sum_{|k_2-k|\leq 4}\sum_{k_1\leq k_2} 2^{dk/2}\|P_{\leq k_2-5}(f)\|_{L^\infty_{x,t}}\|P_{k_1}(\partial_l g)\|_{L^\infty_{x,t}}\|P_{k_2}(\partial_m h)\|_{L^\infty_t L^2_x}\\
&+\sum_{k_2\geq k+5}\sum_{|k_1-k_2|\leq 4} 2^{dk/2}2^{dk/2}\|P_{\leq k_2-5}(f)\|_{L^\infty_{x,t}}\|P_{k_1}(\partial_l g)\|_{L^\infty_t L^2_x}\|P_{k_2}(\partial_m h)\|_{L^\infty_t L^2_x}\\
&\lesssim \sum_{k_1\leq k} 2^{k_1}\mu_{k_1}2^k\mu_k + C\sum_{k_2\geq k} 2^{-d|k_2-k|}2^{2k_2}\mu_{k_2}^2,
\end{aligned}$$

which is dominated by the right-hand side of (8.6). We estimate

$$\begin{aligned}II &\leq \sum_{|k_3-k|\leq 5}\sum_{k_1,k_2\leq k+5} 2^{dk/2}\|P_k(P_{k_3}(f)\cdot P_{k_1}(\partial_l g)\cdot P_{k_2}(\partial_m h))\|_{L^\infty_t L^2_x}\\ &\lesssim \sum_{|k_3-k|\leq 5}\sum_{k_1,k_2\leq k+5} 2^{dk/2}\|P_{k_3}(f)\|_{L^\infty_t L^2_x}\|P_{k_1}(\partial_l g)\|_{L^\infty_{x,t}}\|P_{k_2}(\partial_m h))\|_{L^\infty_{x,t}}\\ &\lesssim \mu_k\big(\sum_{k_1\leq k} 2^{k_1}\mu_{k_1}\big)^2,\end{aligned}$$

which is dominated by the right-hand side of (8.6) since $\sup_{k\in\mathbb{Z}}\mu_k\leq 1$. Finally, we estimate

$$\begin{aligned}III &\leq \sum_{k_2\geq k-8}\sum_{k_1\leq k_2}\sum_{|k_3-k_2|\leq 4} 2^{dk/2}\|P_k(P_{k_3}(f)\cdot P_{k_1}(\partial_l g)\cdot P_{k_2}(\partial_m h))\|_{L^\infty_t L^2_x}\\ &\lesssim \sum_{k_2\geq k-8}\sum_{k_1\leq k_2}\sum_{|k_3-k_2|\leq 4} 2^{dk}\|P_{k_3}(f)\|_{L^\infty_t L^2_x}\|P_{k_1}(\partial_l g)\|_{L^\infty_{x,t}}\|P_{k_2}(\partial_m h))\|_{L^\infty_t L^2_x}\\ &\lesssim \sum_{k_2\geq k} 2^{dk}2^{-dk_2}2^{k_2}\mu_{k_2}^2\sum_{k_1\leq k_2}2^{k_1}\mu_{k_1}\end{aligned}$$

which is dominated by the right-hand side of (8.6) since $\sup_{k\in\mathbb{Z}}\mu_k\leq 1$. This completes the proof of (8.6). □

We construct now the function $\widetilde{\phi}$.

Lemma 8.3. *Assume $\phi\in H^{\infty,\infty}_Q(T)$ and $\{\gamma_k(\sigma)\}_{k\in\mathbb{Z}}$ be as in (8.1). Then there is a unique global solution $\widetilde{\phi}\in C([0,\infty):H^{\infty,\infty}_Q(T))$ of the initial-value problem*

$$\begin{cases}\partial_s\widetilde{\phi}=\Delta_x\widetilde{\phi}+\widetilde{\phi}\cdot\sum_{m=1}^d|\partial_m\widetilde{\phi}|^2 & \text{on } [0,\infty)\times\mathbb{R}^d\times(-T,T);\\ \widetilde{\phi}(0)=\phi.\end{cases}\tag{8.7}$$

In addition, for any $k\in\mathbb{Z}$, $s\in[0,\infty)$, and $\sigma\in[0,\sigma_1]$

$$\|P_k(\widetilde{\phi}(s))\|_{L^\infty_t L^2_x}\lesssim(1+s2^{2k})^{-\sigma_1}2^{-\sigma k}\gamma_k(\sigma),\tag{8.8}$$

and, for any $\sigma,\rho\in\mathbb{Z}_+$,

$$\sup_{s\in[0,\infty)}(s+1)^{\sigma/2}\|\nabla^\sigma_x\partial^\rho_t(\widetilde{\phi}(s)-Q)\|_{L^\infty_t L^2_x}<\infty.\tag{8.9}$$

Proof of Proposition 8.3. Let $M=\sum_{k\in\mathbb{Z}}(2^{2\sigma_1 k}+1)\|P_k(\phi)\|^2_{L^\infty_t L^2_x}$. A simple fixed-point argument shows that there is $S=S(M)>0$ and a unique smooth solution $\widetilde{\phi}$ for (8.7), with $\widetilde{\phi}-Q\in C([0,S]:H^{\infty,\infty}(T))$. In addition, for any $\sigma,\rho\in\mathbb{Z}_+$

$$\sup_{s\in[0,S]}\|\widetilde{\phi}(s)-Q\|_{H^{\sigma,\rho}(T)}\leq C(M,s,\rho,\|\phi-Q\|_{H^{\sigma,\rho}(T)}),\tag{8.10}$$

and

$$\partial_s({}^t\widetilde{\phi}\cdot\phi-1)=2{}^t\widetilde{\phi}\cdot\Delta_x\widetilde{\phi}+2\sum_{m=1}^d|\partial_m\widetilde{\phi}|^2+2({}^t\widetilde{\phi}\cdot\phi-1)\sum_{m=1}^d|\partial_m\widetilde{\phi}|^2.$$

This shows that $|\widetilde{\phi}|\equiv 1$, thus $\widetilde{\phi}\in C([0,S]:H_Q^{\infty,\infty}(T))$. We prove now a priori estimates on the solution $\widetilde{\phi}$.

For any $S'\in[0,S]$ we define

$$B_1(S')=\sup_{s\in[0,S']}\sup_{k\in\mathbb{Z}}(1+s2^{2k})^{\sigma_1}2^{kd/2}\gamma_k^{-1}\|P_k(\widetilde{\phi}(s))\|_{L_t^\infty L_x^2}. \tag{8.11}$$

It is easy to see that $B_1:[0,S]\to(0,\infty)$ is a well-defined continuous nondecreasing function and $\lim_{S'\to 0}B_1(S')\leq 1$.

Using Duhamel's principle, for any $k\in\mathbb{Z}$ and $s\in[0,S]$

$$P_k(\widetilde{\phi}(s))=e^{s\Delta}(P_k\phi)+\int_0^s e^{(s-s')\Delta}\Big[P_k\big[\widetilde{\phi}(s')\cdot\sum_{m=1}^d|\partial_m\widetilde{\phi}(s')|^2\big]\Big]\,ds'. \tag{8.12}$$

Hence for any $k\in\mathbb{Z}$ and $s\in[0,S']$

$$\begin{aligned}2^{dk/2}\|P_k(\widetilde{\phi}(s))\|_{L_t^\infty L_x^2}&\leq e^{-s2^{2k-2}}2^{dk/2}\|P_k\phi\|_{L_t^\infty L_x^2}\\&+\int_0^s e^{-(s-s')2^{2k-2}}2^{dk/2}\Big\|P_k\big[\widetilde{\phi}(s')\cdot\sum_{m=1}^d|\partial_m\widetilde{\phi}(s')|^2\big]\Big\|_{L_t^\infty L_x^2}\,ds'.\end{aligned} \tag{8.13}$$

In view of the definition (8.11), for any $s'\in[0,S']$ and $k'\in\mathbb{Z}$

$$2^{dk'/2}\|P_{k'}(\widetilde{\phi}(s'))\|_{L_t^\infty L_x^2}\leq B_1(S')(1+s'2^{2k'})^{-\sigma_1}\gamma_{k'}.$$

It follows from (8.6) (since $d\geq 2$) that, for any $k\in\mathbb{Z}$ and $s'\in[0,S']$,

$$2^{dk/2}\Big\|P_k\big[\widetilde{\phi}(s')\cdot\sum_{m=1}^d|\partial_m\widetilde{\phi}(s')|^2\big]\Big\|_{L_t^\infty L_x^2}\lesssim 2^{2k}B_1(S')^2\sum_{k'\geq k}\gamma_{k'}^2(1+s'2^{2k'})^{-\sigma_1}. \tag{8.14}$$

We substitute this bound into (8.13) and integrate in s'. We notice that

$$\int_0^s e^{-(s-s')\lambda}(1+s'\lambda')^{-\sigma_1}\,ds'\lesssim s(1+\lambda s)^{-\sigma_1}(1+\lambda' s)^{-1} \tag{8.15}$$

for any $s\geq 0$ and $0\leq\lambda\leq\lambda'$. Using (8.1), for any $s\in[0,S']$ and $k\in\mathbb{Z}$ we get

$$\begin{aligned}&2^{dk/2}\|P_k(\widetilde{\phi}(s))\|_{L_t^\infty L_x^2}(1+s2^{2k})^{\sigma_1}\\&\lesssim\gamma_k+B_1(S')^22^{2k}(1+s2^{2k})^{\sigma_1}\int_0^s e^{-(s-s')2^{2k}/4}\sum_{k'\geq k}\gamma_{k'}^2(1+s'2^{2k'})^{-\sigma_1}\,ds'\\&\lesssim\gamma_k+B_1(S')^22^{2k}s\sum_{k'\geq k}\gamma_{k'}^2(1+2^{2k'}s)^{-1}\\&\lesssim\gamma_k+B_1(S')^2\gamma\gamma_k,\end{aligned} \tag{8.16}$$

which gives

$$B_1(S') \lesssim 1 + \gamma B_1(S')^2$$

Since B_1 is continuous and $B_1(0) \leq 1$, it follows that $B_1(S') \lesssim 1$ for any $S' \in [0, S]$ (provided that γ is sufficiently small). Thus for any $k \in \mathbb{Z}$ and $s \in [0, S]$

$$2^{dk/2} \| P_k(\widetilde{\phi}(s)) \|_{L_t^\infty L_x^2} \leq C(1 + s2^{2k})^{-\sigma_1} \gamma_k. \tag{8.17}$$

We control now $(1 + s2^{2k})^{\sigma_1} 2^{\sigma k} \| P_k(\widetilde{\phi}(s)) \|_{L_t^\infty L_x^2}$, $\sigma \in [0, \sigma_1]$. We define

$$B_2(S) = \sup_{\sigma \in \{0, \sigma_1\}} \sup_{s \in [0,S]} \sup_{k \in \mathbb{Z}} (1 + s2^{2k})^{\sigma_1} 2^{\sigma k} \gamma_k(\sigma)^{-1} \| P_k(\widetilde{\phi}(s)) \|_{L_t^\infty L_x^2}.$$

It is easy to see that $B_2(S) < \infty$. It follows from (8.12) that

$$\begin{aligned} &\| P_k(\widetilde{\phi}(s)) \|_{L_t^\infty L_x^2} 2^{\sigma k} (1 + s2^{2k})^{\sigma_1} \leq (1 + s2^{2k})^{\sigma_1} e^{-s2^{2k-2}} \gamma_k(\sigma) \\ &+ (1 + s2^{2k})^{\sigma_1} \int_0^s e^{-(s-s')2^{2k-2}} 2^{\sigma k} \Big\| P_k\big[\widetilde{\phi}(s') \cdot \sum_{m=1}^d |\partial_m \widetilde{\phi}(s')|^2\big] \Big\|_{L_t^\infty L_x^2} ds' \end{aligned} \tag{8.18}$$

for any $s \in [0, S]$. Using the definition of $B_2(S)$ we have

$$2^{dk'/2} \| P_{k'}(\widetilde{\phi}(s')) \|_{L_t^\infty L_x^2} \lesssim B_2(S)(1 + s'2^{2k'})^{-\sigma_1} 2^{dk'/2} 2^{-\sigma k'} \gamma_{k'}(\sigma). \tag{8.19}$$

We combine this with (8.17). It follows from (8.6) that

$$2^{\sigma k} \Big\| P_k\big[\widetilde{\phi}(s') \cdot \sum_{m=1}^d |\partial_m \widetilde{\phi}(s')|^2\big] \Big\|_{L_t^\infty L_x^2} \lesssim 2^{2k} B_2(S) \sum_{k' \geq k} 2^{|k'-k|} (1 + s'2^{2k'})^{-\sigma_1} \gamma_{k'} \gamma_{k'}(\sigma),$$

for any $k \in \mathbb{Z}$ and $s \in [0, S]$. Using (8.15)

$$\begin{aligned} &(1 + s2^{2k})^{\sigma_1} \int_0^s e^{-(s-s')2^{2k-2}} 2^{\sigma k} \Big\| P_k\big[\widetilde{\phi}(s') \cdot \sum_{m=1}^d |\partial_m \widetilde{\phi}(s')|^2\big] \Big\|_{L_t^\infty L_x^2} ds' \\ &\lesssim s B_2(S) \sum_{k' \geq k} 2^{k-k'} \gamma_{k'} \gamma_{k'}(\sigma) (1 + s2^{2k'})^{-1} \lesssim B_2(S) \gamma_k \gamma_k(\sigma). \end{aligned}$$

It follows from (8.18) that

$$B_2(s) \lesssim \sup_{k \in \mathbb{Z}} \sup_{s \in [0,S]} \gamma_k(\sigma)^{-1} \| P_k(\widetilde{\phi}(s)) \|_{L_t^\infty L_x^2} 2^{\sigma k} (1 + s2^{2k})^{\sigma_1} \lesssim 1 + \gamma B_2(S),$$

Since γ is small this gives $B_2(S) \lesssim 1$. Thus for any $k \in \mathbb{Z}$, $s \in [0, S]$, $\sigma \in [0, \sigma_1]$,

$$2^{\sigma k} \| P_k(\widetilde{\phi}(s)) \|_{L_t^\infty L^2} \lesssim \gamma_k(\sigma)(1 + s2^{2k})^{-\sigma_1}. \tag{8.20}$$

As a consequence, the solution $\widetilde{\phi}$ can be extended globally to a smooth solution $\widetilde{\phi} \in C([0, \infty) : H_Q^{\infty,\infty}(T))$ of (8.7). The bound (8.8) follows from (8.17) and (8.20).

It remains to prove the bound (8.9), which follows from (8.8) (applied for $\sigma = 0$ and $\sigma = \sigma_1$) for $\rho = 0$ and $\sigma \le \sigma_1$. It follows also from (8.8) that for any $s \ge S_0 = (M/\gamma)^4 \gg 1$, $\sigma \in [0, \sigma_1 + 10] \cap \mathbb{Z}$, and $m_1, \dots, m_\sigma \in \{1, \dots, d\}$

$$\begin{aligned}\|\partial_{m_1}\dots\partial_{m_\sigma}(\widetilde{\phi}(s) - Q)\|_{L^\infty_{x,t}} &\lesssim \sum_{k\in\mathbb{Z}} 2^{dk/2}2^{k\sigma}\|P_k(\widetilde{\phi}(s) - Q)\|_{L^\infty_t L^2_x}\\ &\lesssim \sum_{k\in\mathbb{Z}} 2^{dk/2}2^{k\sigma}(1 + s2^{2k})^{-\sigma_1}M \\ &\lesssim Ms^{-1/2}s^{-\sigma/2}\end{aligned} \tag{8.21}$$

and

$$\|\partial_{m_1}\dots\partial_{m_\sigma}(\widetilde{\phi}(s)-Q)\|_{L^\infty_t L^2_x} \lesssim \big[\sum_{k\in\mathbb{Z}} 2^{2k\sigma}\|P_k(\widetilde{\phi}(s)-Q)\|^2_{L^\infty_t L^2_x}\big]^{1/2} \lesssim Ms^{-\sigma/2}. \tag{8.22}$$

For $S \ge S_0$ let

$$M_{\rho,\sigma}(S) = M + 1 + \sum_{\sigma'\le\sigma}\sum_{\rho'\le\rho}\sup_{s\in[S_0,S]} s^{\sigma'/2}\|\nabla_x^{\sigma'}\partial_t^{\rho'}(\widetilde{\phi}(s) - Q)\|_{L^\infty_t L^2_x}.$$

As in (8.21),

$$s^{\sigma'/2}\|\nabla_x^{\sigma'}\partial_t^{\rho'}(\widetilde{\phi}(s) - Q)\|_{L^\infty_{x,t}} \lesssim s^{-1/2}M_{\rho,\sigma}(S) \tag{8.23}$$

for any $s \in [S_0, S]$, $\rho' \le \rho$, and $\sigma' < \sigma - d/2$.

We prove first (8.9) for $\rho = 0$ and $\sigma \ge \sigma_1 + 1$. We use induction over σ, (8.21), (8.23), and (8.7) to estimate, for $s \ge 2S_0$,

$$\begin{aligned}s^{\sigma/2}\|\nabla_x^\sigma(\widetilde{\phi}(s))\|_{L^\infty_t L^2_x} &\le s^{\sigma/2}\|e^{(s-S_0)\Delta}(\nabla_x^\sigma\widetilde{\phi}(S_0))\|_{L^\infty_t L^2_x}\\ &\quad + s^{\sigma/2}\int_{S_0}^s \big\|e^{(s-s')\Delta}[\nabla_x^\sigma(\widetilde{\phi}(s')\cdot\sum_{m=1}^d|\partial_m\widetilde{\phi}(s')|^2)]\big\|_{L^\infty_t L^2_x}\,ds'\\ &\lesssim \|\widetilde{\phi}(S_0)\|_{L^\infty_t L^2_x} + \int_{S_0}^{s/2}\big\|\widetilde{\phi}(s')\cdot\sum_{m=1}^d|\partial_m\widetilde{\phi}(s')|^2)]\big\|_{L^\infty_t L^2_x}\,ds'\\ &\quad + s^{\sigma/2}\int_{s/2}^s\frac{1}{(s-s')^{1/2}}\big\|\nabla_x^{\sigma-1}(\widetilde{\phi}(s')\cdot\sum_{m=1}^d|\partial_m\widetilde{\phi}(s')|^2)\big\|_{L^\infty_t L^2_x}\,ds'\\ &\lesssim (C_\sigma Ms^{-1/2})M_{0,\sigma}(s) + C_\sigma(M_{0,\sigma-1}(s))^3.\end{aligned} \tag{8.24}$$

It follows by induction that $\sup_{s\ge S_0} M_{0,\sigma}(s) < \infty$ for any $\sigma \in \mathbb{Z}_+$, and (8.9) follows for $\rho = 0$.

To prove (8.9) for a pair $(\rho, \sigma) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $\rho \ge 1$, we may assume by induction that $\sup_{s\ge S_0} M_{\rho-1,\sigma'}(s) < \infty$ for any $\sigma' \in \mathbb{Z}$ and $\sup_{s\ge S_0} M_{\rho,\sigma-1}(s) < \infty$. In view of

(8.7), the function $v_\rho = \partial_t^\rho \widetilde{\phi}$ solves the heat equation

$$\begin{aligned}(\partial_s - \Delta_x)v_\rho &= v_\rho \cdot \sum_{m=1}^d |\partial_m \widetilde{\phi}|^2 + 2\widetilde{\phi} \cdot \sum_{m=1}^d {}^t(\partial_m \widetilde{\phi}) \cdot \partial_m v_\rho + E_{\rho-1} \\ &= \sum_{m=1}^d \big[|\partial_m \widetilde{\phi}|^2 I_3 - 2\partial_m(\widetilde{\phi} \cdot {}^t(\partial_m \widetilde{\phi}))\big] \cdot v_\rho + \sum_{m=1}^d \partial_m[2\widetilde{\phi} \cdot {}^t(\partial_m \widetilde{\phi}) \cdot v_\rho] + E_{\rho-1} \\ &= P \cdot v_\rho + \sum_{m=1}^d \partial_m(Q_m \cdot v_\rho) + E_{\rho-1}.\end{aligned} \tag{8.25}$$

It follows from (8.21) that for any $s \geq S_0$

$$s^{3/2}\|P\|_{L^\infty_{x,t}} + s\sum_{m=1}^d \|Q_m\|_{L^\infty_{x,t}} \lesssim M. \tag{8.26}$$

In addition, it follows from (8.23) and the induction hypothesis that

$$\sup_{s \geq S_0} \big[s^{3/2}s^{\sigma'/2}\|\nabla_x^{\sigma'} P\|_{L^\infty_{x,t}} + s^{\sigma'/2+1}\|\nabla_x^{\sigma'} Q_m\|_{L^\infty_{x,t}} + s^{3/2}s^{\sigma'/2}\|\nabla_x^{\sigma'} E_{\rho-1}\|_{L^\infty_t L^2_x}\big] < \infty$$

for any $\sigma' \in \mathbb{Z}_+$ and $m = 1, \ldots d$. An estimate similar to (8.24) shows that $\sup_{s \geq S_0} M_{\rho,\sigma}(s) < \infty$, which completes the proof of the lemma. □

We construct now the function v.

Lemma 8.4. *There is a smooth function $v : [0,\infty) \times \mathbb{R}^d \times (-T,T) \to \mathbb{S}^2$ such that*

$$ {}^t v \cdot \widetilde{\phi} = 0 \quad \text{and} \quad \partial_s v = [\partial_s \widetilde{\phi} \cdot {}^t\widetilde{\phi} - \widetilde{\phi} \cdot {}^t(\partial_s \widetilde{\phi})] \cdot v \tag{8.27}$$

on $[0,\infty) \times \mathbb{R}^d \times (-T,T)$. In addition, for any $k \in \mathbb{Z}$, $s \in [0,\infty)$, $\sigma \in [d/2, \sigma_1]$,

$$2^{\sigma k}\|P_k(v(s))\|_{L^\infty_t L^2_x} \lesssim (1 + s2^{2k})^{-\sigma_1+1}\gamma_k(\sigma), \tag{8.28}$$

and, for any $\sigma, \rho \in \mathbb{Z}_+$,

$$\sup_{s \in [0,\infty)} \sup_{k \in \mathbb{Z}} (s+1)^{\sigma/2} 2^{k\sigma}\|P_k(\partial_t^\rho(v(s))\|_{L^\infty_t L^2_x} < \infty. \tag{8.29}$$

Proof of Lemma 8.4. Let R denote the 3×3 matrix

$$R = \partial_s\widetilde{\phi} \cdot {}^t\widetilde{\phi} - \widetilde{\phi} \cdot {}^t(\partial_s\widetilde{\phi}) = \Delta_x\widetilde{\phi} \cdot {}^t\widetilde{\phi} - \widetilde{\phi} \cdot {}^t(\Delta_x\widetilde{\phi}) = \sum_{m=1}^d \partial_m[\partial_m\widetilde{\phi} \cdot {}^t\widetilde{\phi} - \widetilde{\phi} \cdot {}^t(\partial_m\widetilde{\phi})], \tag{8.30}$$

where one of the identities follows from (8.7). It follows from (8.8) and (8.4) that

$$2^{\sigma k}\|P_k(R(s))\|_{L^\infty_t L^2_x} \lesssim 2^{2k}(1 + s2^{2k})^{-\sigma_1}\gamma_k(\sigma) \tag{8.31}$$

for any $k \in \mathbb{Z}$, $s \in [0,\infty)$, $\sigma \in [d/2, \sigma_1]$. It follows from (8.9) and (8.23) that

$$\sup_{s \in [0,\infty)} \big[(s+1)^{(\sigma+2)/2}\|\nabla_x^\sigma \partial_t^\rho(R(s))\|_{L^\infty_t L^2_x} + (s+1)^{(\sigma+3)/2}\|\nabla_x^\sigma \partial_t^\rho(R(s))\|_{L^\infty_{x,t}}\big] < \infty \tag{8.32}$$

for any $\sigma, \rho \in \mathbb{Z}_+$.

We prove first the existence of a smooth function $v : [0,\infty) \times \mathbb{R}^d \times (-T,T) \to \mathbb{S}^2$ satisfying (8.27). We fix $Q' \in \mathbb{S}^{d-1}$ with the property that ${}^tQ \cdot Q' = 0$. By (8.32) we have

$$\int_0^\infty \|R(s)\|_{L^\infty_{x,t}}\, ds < \infty.$$

This allows us to construct $v : [0,\infty) \times \mathbb{R}^d \times (-T,T) \to \mathbb{R}^3$ (by a simple fixed point argument) as the unique solution of the ODE

$$\partial_s v = R(s) \cdot v \text{ and } v(\infty) = Q' \tag{8.33}$$

for any $(x,t) \in \mathbb{R}^d \times (-T,T)$.

Since $\int_0^\infty \|\nabla_x^\sigma \partial_t^\rho (R(s))\|_{L^\infty_{x,t}}\, ds < \infty$ for any $\sigma, \rho \in \mathbb{Z}_+$, the function v constructed as a solution of (8.33) is smooth on $[0,\infty) \times \mathbb{R}^d \times (-T,T)$ and

$$\sup_{s \in [0,\infty)} (s+1)^{(\sigma+1)/2} \|\nabla_x^\sigma \partial_t^\rho (v(s) - Q')\|_{L^\infty_{x,t}} < \infty \tag{8.34}$$

for any $\sigma, \rho \in \mathbb{Z}_+$. Using (8.33) and ${}^t(\partial_s \widetilde{\phi}) \cdot \widetilde{\phi} = 0$, it is easy to see that

$$\partial_s({}^tv \cdot \widetilde{\phi}) = {}^tv \cdot [\widetilde{\phi} \cdot {}^t(\partial_s \widetilde{\phi}) - \partial_s \widetilde{\phi} \cdot {}^t\widetilde{\phi}] \cdot \widetilde{\phi} + {}^tv \cdot \partial_s \widetilde{\phi} = 0.$$

Since $\lim_{s\to\infty} {}^tv(s) \cdot \widetilde{\phi}(s) = 0$, it follows that ${}^tv \cdot \widetilde{\phi} \equiv 0$ on $[0,\infty) \times \mathbb{R}^d \times (-T,T)$. Thus, using (8.33) again,

$$\partial_s({}^tv \cdot v) = 2{}^tv \cdot [\partial_s \widetilde{\phi} \cdot {}^t\widetilde{\phi} - \widetilde{\phi} \cdot {}^t(\partial_s \widetilde{\phi})] \cdot v = 0.$$

Since $\lim_{s\to\infty} {}^tv(s) \cdot v(s) = 1$ it follows that ${}^tv \cdot v \equiv 1$ on $[0,\infty) \times \mathbb{R}^d \times (-T,T)$. To summarize, we constructed a smooth function $v : [0,\infty) \times \mathbb{R}^d \times (-T,T) \to \mathbb{S}^2$ that satisfies (8.27).

We prove now (8.29). In view of (8.33), we have

$$v(s) - Q' + \int_s^\infty R(s') \cdot Q'\, ds' = -\int_s^\infty R(s') \cdot (v(s') - Q')\, ds'.$$

Thus, using (8.32) and (8.34),

$$\sup_{s \in [0,\infty)} (s+1)^{(\sigma+1)/2} \Big\| \nabla_x^\sigma \partial_t^\rho (v(s) - Q') + \int_s^\infty \nabla_x^\sigma \partial_t^\rho (R(s')) \cdot Q'\, ds' \Big\|_{L^\infty_t L^2_x} < \infty$$

for any $\sigma, \rho \in \mathbb{Z}_+$. The bound (8.29) follows from (8.32).

Finally, we prove (8.28). It follows from (8.33) that

$$P_k(v(s)) = -\int_s^\infty P_k(R(s') \cdot v(s'))\, ds'. \tag{8.35}$$

For any $S \in [0,\infty)$ let

$$B_3(S) = 1 + \sup_{\sigma \in [d/2, \sigma_1]} \sup_{s' \in [S,\infty)} \sup_{k \in \mathbb{Z}} \gamma_k(\sigma)^{-1} (1 + s' 2^{2k})^{\sigma_1 - 1} 2^{\sigma k} \|P_k(v(s'))\|_{L^\infty_t L^2_x}.$$

We have $B_3(S)<\infty$ for any $S\in[0,\infty)$, using (8.29) and $\sup_{k\in\mathbb{Z}}\gamma_k(\sigma)^{-1}2^{-\delta_0|k|}<\infty$. Also,

$$2^{\sigma k'}\|P_{k'}(v(s'))\|_{L^\infty_t L^2_x}\leq B_3(S)\gamma_{k'}(\sigma)(1+s'2^{2k'})^{-\sigma_1+1}$$

for any $\sigma\in[d/2,\sigma_1]$, $s'\geq S$, and $k'\in\mathbb{Z}$. It follows from (8.4) and (8.31) that

$$2^{\sigma k}\|P_k(R(s')\cdot v(s'))\|_{L^\infty_t L^2_x}\lesssim 2^{2k}\big(\gamma_k(\sigma)+\gamma B_3(S)\sum_{2^{2k}\leq 2^{2k'}\leq 1/s'}\gamma_{k'}(\sigma)\big)$$

if $s'2^{2k}\leq 1$, and

$$2^{\sigma k}\|P_k(R(s')\cdot v(s'))\|_{L^\infty_t L^2_x}\lesssim 2^{2k}(s'2^{2k})^{-\sigma_1}\gamma_k(\sigma)(1+\gamma B_3(S))$$

if $s'2^{2k}\geq 1$. Thus, for $s\geq S$ and $k\in\mathbb{Z}$

$$\int_s^\infty 2^{\sigma k}\|P_k(R(s')\cdot v(s'))\|_{L^\infty_t L^2_x}\,ds'\lesssim\gamma_k(\sigma)(1+s2^{2k})^{-\sigma_1+1}(1+\gamma B_3(S)).$$

It follows from (8.35) that $B_3(S)\lesssim(1+\gamma B_3(S))$, which gives (8.28). □

We complete now the proof of Proposition 4.2. We define the smooth function $w=\widetilde{\phi}\times v:[0,\infty)\times\mathbb{R}^d\times(-T,T)\to\mathbb{S}^2$. It follows from (8.5), (8.8), and (8.28) that

$$2^{\sigma k}\|P_k(w(s))\|_{L^\infty_t L^2_x}\lesssim(1+s2^{2k})^{-\sigma_1+1}2^{-\sigma k}\gamma_k(\sigma)$$

for any $k\in\mathbb{Z}$, $s\in[0,\infty)$, $\sigma\in[d/2,\sigma_1]$. It follows easily from (8.9) and (8.29) that

$$\sup_{s\in[0,\infty)}\sup_{k\in\mathbb{Z}}(s+1)^{\sigma/2}2^{k\sigma}\|P_k(\partial_t^\rho(w(s))\|_{L^\infty_t L^2_x}<\infty$$

for any $\rho,\sigma\in\mathbb{Z}_+$. Finally, the identities (4.8) follow from (8.27) and $w=\widetilde{\phi}\times v$.

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