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From Linked and Knotted Plasma, Electromagnetic, and Gravitational Radiation to Shockwaves on Rotating Black Holes

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From Linked and Knotted Plasma, Electromagnetic, and Gravitational Radiation to Shockwaves on Rotating Black Holes

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Physics

by

Joseph M. Swearngin

Committee in Charge:

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September 2022

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Professor Dirk Bouwmeester, Committee Chairperson

August 2022

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by

Joseph M. Swearngin

To bundles.

Acknowledgements

Our research thus far has been a profound and deeply personal journey. A few pages are far too small a space to sufficiently acknowledge all of the support I have received along the way. Collaborators and colleagues, this has been a life-altering journey; I am honored to have worked with everyone. Not all of our work can be in this thesis, but I am grateful for everything. The introduction can include only a tiny and biased summary of our story for narrative purposes and does not accurately represent the real-time collaborations and flows of ideas; of course, respect and all due credit are to be given to my coauthors and collaborators for each chapter. Here I will say a few brief words to express my appreciation for your impact on my life and academic career.

First and foremost, I want to thank my Ph.D. advisor and committee chair, Prof. Dirk Bouwmeester, for giving me the unique opportunity to work on this research project as it grew from early stages and to see the ideas bear fruit many years later. Some of my fondest memories are from my travels to Leiden University in the Netherlands, especially while earning my master's degree, which Dirk also supervised. It was during those visits that Dirk arranged for us to take a tour of Ehrenfest's house, attend the Marcel Grossmann Meeting in Stockholm, and meet with Roger Penrose, whose work was a large part of my inspiration for studying physics. I hope this thesis is a testament to the resources and mentor-ship you have provided me over the years. Without your intuition, guidance, and support, this work would not have been possible.

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Curriculum Vitæ

Joseph M. Swearngin

Education

2019	M.A. in Physics, UC Santa Barbara
2012	M.Sc. in Physics, Leiden University, The Netherlands
2011	B.S. in Physics, College of Creative Studies, UC Santa Barbara
2007	transferred to UCSB, Grossmont Community College

Teaching Experience

2017 – 2019	Adjunct Faculty, Physics & Astronomy, Santa Barbara City College
2017	Teaching Associate (Lecturer), Physics, UC Santa Barbara
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2013 – 2016	Teaching Assistant, Physics, UC Los Angeles
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2016 – 2022	Graduate Research Assistant, UC Santa Barbara
2011 – 2013	Graduate Research Assistant, Leiden University
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2010	Summer Research Fellow, SURF Program, UC Santa Barbara

Leadership

2018 – 2019	Graduate Mentor, UCSB Veterans Resource Center (VRC)
2017 – 2019	Founding Faculty Advisor, SBCC Student Veterans Organization (SVO)
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2007 – 2011	Founding Member, UCSB Student Veterans Organization (SVO)

Selected Honors

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2018	UCSB Graduate Opportunity Fellowship
2018	UCSB Physics Dept. Chair's Outstanding Service Award
2017	UCSB Physics Dept. Outstanding Teaching Assistant Award
2012	Leiden Excellence Scholarship
2011	UCSB Physics Dept. Academic Honors Award
2011	UCSB University Award of Distinction

Publications

August 2022	Eve Bodnia, Vlad Isenbaev, Kellan Colburn, Joe Swearngin, and Dirk Bouwmeester: "Conformal Cyclic Cosmology Signatures and Anomalies of the CMB Sky," In *arXiv/astro-ph.CO*.
February 2019	Yoni BenTov and Joe Swearngin: "Gravitational shockwaves on rotating black holes," In *General Relativity and Gravitation*.
August 2015	Chris Smiet, Simon Candelaresi, Amy Thompson, Joe Swearngin, J.W. Dalhuisen, and Dirk Bouwmeester: "Self-organizing knotted magnetic structures in plasma," In *Physical Review Letters*.
April 2015	Amy Thompson, Alex Wickes, Joe Swearngin, and Dirk Bouwmeester: "Classification of Electromagnetic and Gravitational Hopfions by Algebraic Type," In *Journal of Physics A: Mathematical and Theoretical*.
August 2014	Amy Thompson*, Joe Swearngin*, and Dirk Bouwmeester (*equal contributors): "Linked and Knotted Gravitational Radiation," In *Journal of Physics A: Mathematical and Theoretical*.
April 2014	Amy Thompson, Joe Swearngin, Alex Wickes, and Dirk Bouwmeester: "Constructing a class of topological solitons in magnetohydrodynamics," In *Physical Review E*.
February 2014	Joe Swearngin, Amy Thompson, Alex Wickes, J.W. Dalhuisen, and Dirk Bouwmeester: "Gravitational Hopfions," In *arXiv/gr-qc*.

Abstract

From Linked and Knotted Plasma, Electromagnetic, and Gravitational Radiation to Shockwaves on Rotating Black Holes

Joseph M. Swearngin

Building off senior and master thesis work, we present a class of topological plasma configurations characterized by their toroidal and poloidal winding numbers, n_t and n_p, respectively. The case of $n_t = 1$ and $n_p = 1$ corresponds to the Kamchatnov-Hopf soliton, a magnetic field configuration everywhere tangent to the fibers of a Hopf fibration so that the field lines are circular, linked precisely once, and form the surfaces of nested tori. We show that for $n_t \in \mathbb{Z}^+$ and $n_p = 1$, these configurations represent stable, localized solutions to the magnetohydrodynamic equations for an ideal incompressible fluid with infinite conductivity. Furthermore, we extend our stability analysis by considering a plasma with finite conductivity and estimate the soliton lifetime in such a medium as a function of the toroidal winding number.

Torus knot topology is also inherent in electromagnetic and gravitational radiation. We show this by constructing spin-N fields based on the elementary states of twistor theory. The twistor functions corresponding to the elementary states admit a parameterization in terms of the torus knots' poloidal and toroidal winding

numbers, allowing one to choose the degree of linking or knotting of the associated field configuration. We describe the topology of the gravitational fields and their physical interpretation in terms of their tidal and frame drag phenomenology. Using the gravito-electromagnetic formalism, we show that the torus knot structure is exhibited in the tendex and vortex lines for the analogous linearized gravitational solutions.

We extend the definition of hopfions to include a larger class of spin-h fields and use this to classify the electromagnetic and gravitational hopfions of different algebraic types. The fields are constructed through the Penrose contour integral transform; thus, the singularities of the generating functions are directly related to the geometry of the resulting physical fields. We discuss this relationship and how the topological structure of the fields is related to the Robinson congruence. Since the topology appears in the lines of force for both electromagnetism and gravity, the gravito-electromagnetic formalism is used to analyze the gravitational hopfions and describe the time evolution of their tendex and vortex lines. We thus obtain similar configurations based on the same topological structure but varying spin-h algebraic types. The null and type N fields propagate at the speed of light, while the non-null and type D fields radiate energy outward from the center. Finally, we discuss the type III gravitational hopfion, which has no direct electromagnetic analog, but find that it still exhibits some characteristic features common to the other hopfion fields.

In the complete non-linear theory we present an exact solution of Einstein's equation that describes the gravitational shockwave of a massless particle on the horizon of a Kerr-Newman black hole—generalizing the Dray-'t Hooft solution to the case of a rotating background. The back-reacted metric is of the generalized Kerr-Schild form and is Type II in the Petrov classification. We show that if the background tetrad is aligned with shear-free null geodesics, and the background Ricci tensor satisfies a simple condition, all nonlinearities in the perturbation will drop out of the curvature scalars. We reformulate Einstein-Hilbert gravity in the first-order formalism of Elie Cartan and find that the Riemann curvature tensor is the Yang-Mills gauge curvature of a local Lorentz group gauge connection. We derive and make heavy use of the method of spin coefficients (the Newman-Penrose formalism) in its compacted form (the Geroch-Held-Penrose formalism).

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Chapter 1

Introduction

Isaac Newton introduced the concept of an abstract space-filling gravitational force field to describe celestial motions in terms of the laws which govern the motion of bodies on Earth. Electromagnetism, introduced to describe the interactions and motion of electric charges and their fluid-like currents, was the first empirically accessible relativistic field theory to be developed. The intuitive physical descriptions of electromagnetic lines of force evolved alongside abstract mathematical constructs like vector calculus. As electromagnetism grew to accommodate more phenomena such as light and its interaction with matter, we had to develop even more abstract forms of algebra and calculus to write predictive models.

Electromagnetism is our most well-understood relativistic field theory because we can experience it in our everyday lives and describe it in common sense. We are all taught that electric and magnetic fields are to be initially defined operationally through their effects on charge and current distributions. Only much later, after relativity and Ricci's tensor calculus, did we begin to study the algebraic structure of the Faraday tensor $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. Additional layers of abstraction appear when considering the algebraic structures associated with the spinor representations of the Lorentz group $SO(1,3)$. This thesis is heavy in these spinor abstractions, yet we can keep in contact with the phenomenological world by turning our variables back into electric and magnetic fields. Since the spinor representation of $SO(1,3)$ is $SL(2,\mathbb{C})$ we will be dealing heavily in symmetric spinors with two $SL(2,\mathbb{C})$ indices. Since $SL(2,\mathbb{C}) \cong SO(3,\mathbb{C})$ in the sense of group homeomorphism we will frequently encounter expressions with complex linear combinations of the electric and magnetic fields in both vector $E_a + \imath B_a$ and tensor form $F_{\mu\nu} + \imath * F_{\mu\nu}$. The reward we reap for this level of abstraction is that we can solve field equations much more straightforwardly and construct solutions to higher spin equations from lower spin fields.

In contrast, General Relativity (GR) is far too weakly coupled to our daily lives for it to be intuitive beyond the Newtonian approximation. Humans don't experience the effects of curvature in a way that lends itself to common sense operational definitions[1]. Historically, this implies that building intuition in GR is hampered by the abstract data structures of differential geometry, which encode physical degrees of freedom in non-intuitive ways. In the case of gravity, we start

[1]This is true of Yang-Mills fields in the Standard Model.

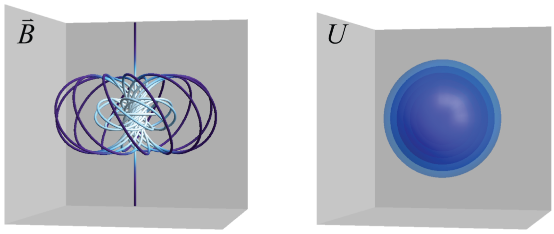

Figure 1.1: The (**a**) magnetic field and (**b**) energy density of Kamchatnov's Hopf Soliton.

with complicated arrays[2], and we deduce to find the operational degrees of freedom in them.

In this work, we study the relationship between electromagnetism and gravity. The presentation will be a bit ahistorical, as we first introduce and study the physics of knotted fields in plasma[3,4]. In the first work, we describe an infinite class of topological solitons. These solutions are sufficiently elementary as to be easily written down yet non-trivial enough that they possess a robust set of properties that are preserved when generalizing first to radiative EM fields and finally to gravitation by way of twistor theory[5]. We used these fields as a sort of mathe-

[2]Some, but not all arrays, are bonafide tensors. The Levi-Civita connection famously is not a tensor.

[3]I wrote two theses [1, 2] and one paper [3] on these topics is not contained in this thesis. The reader is encouraged to read that paper as a pre-introduction to this thesis. My two other theses are worth reading as well. They will be made available to the interested reader.

[4]We are proceeding in a pseudo-ahistorical fashion since Kamchatnov constructed his Hopf solitons in the early 1980s as part of a research program to generalize non-abelian magnetic monopoles [4]. We came across Kamchatnov's work, Fig. 1.1, after showing that Rañada's knots of light, Fig. 1.3, were spin-1 twistor elementary states which were introduced at least a decade before everything else [3, 5].

[5]We obtain a whole family of relativistic fields, each of which possesses the same characteristic structure as the plasma solitons. All fields will be finite energy and topologically non-trivial.

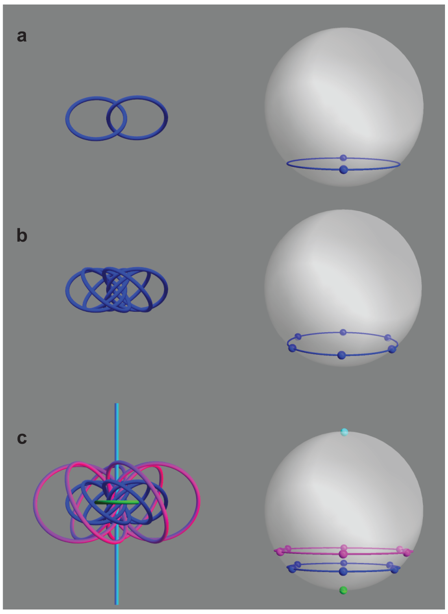

Figure 1.2: The Hopf fibration as a filling of $\mathbb{R}^3$ parameterized by points on a sphere. (**a**) Each point on the sphere is a circle and any two circles are linked. (**b**) All the points on a parallel of latitude fill out a torus. (**c**) The north and south poles are degenerate tori, each a circle, with the north pole mapping to a circle through infinity.

matical dye that we inject into an otherwise abstruse mathematical construction, allowing us to see what properties and phenomena survive the generalization process. The plasma articles [6, 7] primarily serve to establish the phenomenology of field line topology through the concept of magnetic helicity [8, 9] thought of as a sort of average linking density of magnetic field lines. A relativistic discussion of helicity, first introduced in [10], accounts for electric and magnetic helicities and their dynamic exchange.

Chapters 3 and 4 in this thesis are extensions of our previous work relating knots of light to elementary states in twistor theory [3]. We started by noting that

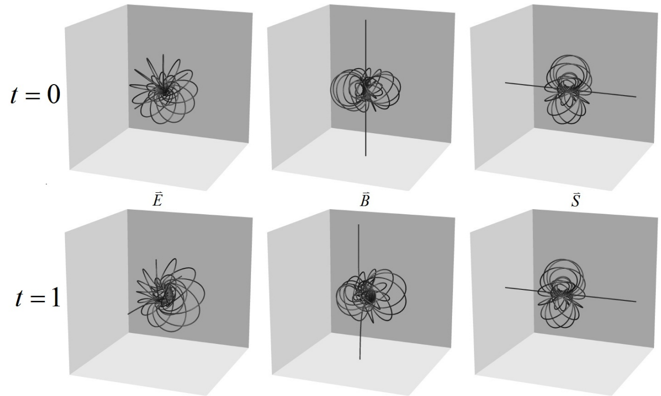

Figure 1.3: Rañada's knot of light at $t = 0$ (1^{st} row) and $t = 1$ (2^{nd} row).

the Hopf fibration, Fig. 1.2, played a fundamental role in two seemingly unrelated things - Rañada's knots of light[6] [11] and the Robinson congruence of twistor[7] theory [13]. We were studying them in parallel, hoping that we would eventually understand them well enough to see the connection we started collaborating with Jan Willem Dalhuisen. He was approaching the problem by considering Robinson's theorem, which relates solutions of the massless spin-N field equations[8] to null shear-free geodesic congruences.

Dalhuisen showed the correspondence held, and the knots of light could be obtained from Robinson's theorem as applied to the Robinson congruence [14].

[6]The electric, magnetic, and Poynting vector fields are each tangent to the fibers of three orthogonal Hopf fibrations, see Fig. 1.3, thus at each point $\{\vec{E}, \vec{B}, \vec{S}\}$ form an orthogonal basis for the lab frame.

[7]The Robinson congruence's spacial components are tangent to the twisted fibers of the Hopf fibration, and their central role in the formalism motivated the name twistor [12, 13].

[8]The massless spin-h field equations are linear differential equations for symmetric spinor fields given by $\nabla^{AA'_1}\varphi_{A_1\cdots A_{2h}}(x) = 0$ and its complex conjugate. Table 1.1 gives the physical interpretations of each field.

Spin	Field Eq.	Spinor Eq.	Physical Interpretation
0	$\Box\varphi = 0$	$\Box\varphi = 0$	Klein-Gordon Field
$-\frac{1}{2}$		$\nabla^{AA'}\varphi_A = 0$	Weyl Field (LH)
$+\frac{1}{2}$		$\nabla^{AA'}\overline{\varphi}_{A'} = 0$	Weyl Field (RH)
-1	$dF = 0$	$\nabla^{AA'}\varphi_{AB} = 0$	Electromagnetism (LH,ASD)
$+1$	$d * F = *J$	$\nabla^{AA'}\overline{\varphi}_{A'B'} = 0$	Electromagnetism (RH,SD)
$-\frac{3}{2}$		$\nabla^{AA'}\varphi_{ABC} = 0$	Rarita–Schwinger field (LH)
$+\frac{3}{2}$		$\nabla^{AA'}\overline{\varphi}_{A'B'C'} = 0$	Rarita–Schwinger field (RH)
-2	$\mathscr{D}\mathbf{C}^{ab} = 0$	$\nabla^{AA'}\varphi_{ABCD} = 0$	Gravity (LH,ASD)
$+2$	$\nabla_{[a}C_{bc]de} = 0$	$\nabla^{AA'}\overline{\varphi}_{A'B'C'D'} = 0$	Gravity (RH,SD)

Table 1.1: The massless spin-h field equations and their physical interpretation.

Roger Penrose was able to provide some important notes to us while he was visiting Leiden University as the Lorentz Chair. He suggested that the knot of light must be related to the elementary states and suggested a calculation for us to try. His suggestion bore fruit, and we found that we could construct the knots of light from the simplest elementary state employing a complex contour integral known as the Penrose transform. This result was my undergraduate honors thesis [1]. Having a foothold in the formalism, we rapidly generalized the results from spin-1 EM to spin-2 linearized gravitational fields called the type-N gravitational hopfions. We obtained an entire relativistic family of spin-N fields [2, 3] with the type-N gravitational hopfion being the spin-2 field. The twistor generalization of the knots of light and the resulting class of fields constituted my master's thesis at the University of Leiden.

Visualizations of the plasma and radiative EM fields were critical because the qualitative topology of the fields is far easier to process visually than mathematically grasp. The distinctive pattern of Villarceau circles[9] was the mathematical dye that was first present in the plasma fields in Fig 1.1. Once we had generalized the fields to gravitational radiation, we wanted to see where this dye emerged.

To visualize the curvature of these fields, we turned to the gravitoelectromagnetic (GEM) formalism which Kip Thorne's group at Caltech had been using to visualize space-time curvature [15–19]. In this version of GEM, one projects the curvature tensor onto a given lab frame in such a way that *all* the curvature degrees of freedom are encoded in two trace-less symmetric spatial tensors[10] called the gravitoelectric tidal tensor E_{ij}, and the gravitomagnetic frame drag tensor B_{ij}. This projection gives the 1+3 dimensional decomposition of the curvature tensor in the lab frame. The GE field's eigenvectors describe the two axes along which the gravitational field tidally stretches or compresses test objects[11]. The GM tensor's eigenvectors describe the two axes around which frame dragging causes clockwise or counterclockwise precession of test gyroscopes. The last eigenvector

[9]The fibers of the Hopf map project on to $\mathbb{R}^3$ are a collection of space-filling circles, each twisting around the surface of a torus, and each linked with every other exactly once, as in Fig. 1.2.

[10]Exactly these conditions define the spin-2 subset of the tensor algebra $u^j \bigotimes v^j$. The quadrupole tensor in a multipole expansion is exactly of the same form $Q_{ij} = \int \rho(\mathbf{r}) \left(\frac{3}{2} r_i r_j - \frac{1}{2} \|\vec{r}\|^2 \delta_{ij} \right) d^3\mathbf{r}$.

[11]A gravitational test object is an object which does not back react on the geometry by contributing stress-energy.

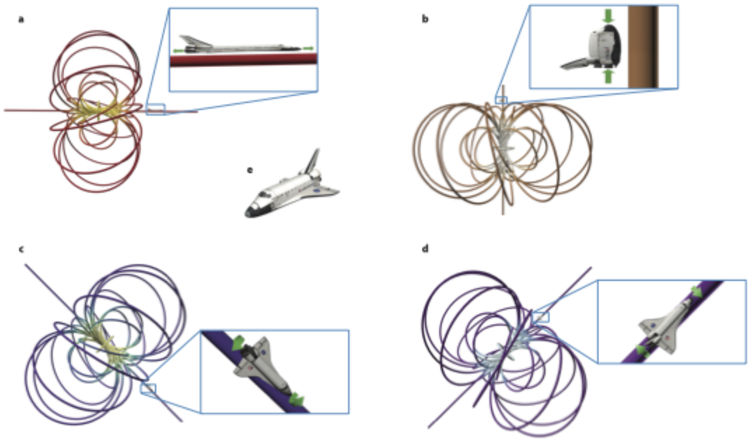

Figure 1.4: The kinetic effects of the GEM fields in the comoving frame of a spacecraft: (**a**) Tidal stretching (**b**) Tidal compression (**c**) Clockwise differential precession (**d**) Counterclockwise differential precession

for each GE and GM tensor describes the axis along which these kinetic effects vanish.

The type-N hopfion possesses a purely conformal curvature, so we decomposed the Weyl curvature tensor following the visualization method of Nichols', *et al.* [15–17] and we plotted the integral curves of the GE and GM eigenvectors to visualize the lines of force for gravitational fields. We were astonished by the plots when they came out. Every time we plotted the integral curves of the eigenvector fields, we obtained a collection of Villarceau circles[12]. The field lines for the Type-N gravitational hopfion were precisely two copies of a knot of light. The dye passed

[12]The conformal curvature spinor can also be thought of as the spin-2 $SO(3,\mathbb{C})$ quadrupole tensor $E_{ij}+\imath B_{ij}$ similarly to the Maxwell spinor being thought of as the $SO(3,\mathbb{C})$ vector $E_i+\imath B_i$ see Stephani, *et el.* [20].

through the abstruse artifice that is the twistor formalism and came out relatively unscathed, albeit with some doubling. Something topologically non-trivial was going on, but it was not the same as in EM. The time evolved field lines for the GE and GM fields were not the same as the time evolved fields for the knot of light[13]. For the knot of light, we could prove that the topology of the initial data is conserved under time evolution, yet we had no such way of knowing if this were true for the GEM fields. However, surveying the time-evolved plots convinced us that, more than likely, there was conservation occurring. This result was the topic of the second thesis [2] I wrote while obtaining my master's degree at the University of Leiden. This result is also presented in the theses of Dalhuisen [21] and Amy Thompson [22].

Chapter 3 begins right after we completed the analysis for the type-N gravitational hopfion. The Penrose transform takes as its argument a homogeneous holomorphic twistor function which acts as twistor space representation of the massless fields [13, 23, 24] and the homogeneity of the function determines the spin of the field on space-time. From our previous work, it was clear that the generating function for the knot of light could easily be changed to obtain a gravitational field by changing the homogeneity of the function but leaving the other

[13]The time evolution was closely related though. The time evolved fields were all related by local duality transformations. This relationship to local duality transformations will be meaningful later.

parameters fixed. The generating function was of the form

$$f(Z) = \frac{(\bar{C}_\gamma Z^\gamma)^c (\bar{D}_\delta Z^\delta)^d}{(\bar{A}_\alpha Z^\alpha)^a (\bar{B}_\beta Z^\beta)^b}.$$

The abstract properties of the elementary states have been known since the early years of twistor theory, yet we still know very little about the physical role of these fields. We conjectured that the exponents $\{a, b, c, d\}$ of the twistor generating functions were related to the integers which parameterized the solitons we introduced in Chapter 2. Tedious computations revealed that this was the case, and we deduced the relationship between the powers in the generating function and the toroidal and poloidal winding numbers of the solitons. In this way, we see the abstract pole structure of the generating function become the more physically intuitive winding numbers of the knot! We obtained torus knot EM and gravitational fields by computing the fields from the Penrose transforms. The EM fields were the same as the linked and knotted fields obtained by Kedia, *et al.* [25] confirming the conjecture.

In Chapter 4 we continued to explore the gravitational fields through the Penrose transform. The Penrose transform is at its most basic level, incredibly clever use of Cauchy's residue theorem, which relates the value of a complex contour integral to the singularities of the integrand. The complex variable of integration is a complex homogeneous coordinate on S^2 of light null-directions at a point [2].

By controlling the residues, we were able to generalize the torus knot EM fields into any algebraic type and any spin, thus obtaining an entire relativistic family of torus knotted spin-N fields. The spin-1 EM fields and spin-2 gravitational fields both possessed strikingly similar field line configurations. When one considers the type-$(1, 1)$ EM fields, it is possible to observe the transfer of topology between the electric and magnetic fields precisely as one would have expected from the formalism of Arrayás and Trueba [10]. One can (see Fig. 4.2) arrange the initial field data such that the electric field lines are all Villarceau circles with no magnetic field and evolve it towards a state where the magnetic field is all Villarceau circles and the electric field vanishes, thus achieving complete conversion of all electric helicity to magnetic helicity in the limit as $t \to \infty$. The gravitational generalization of these fields exhibits precisely the same sort of phenomenon. However, there did not seem to exist any literature on what we believed should be called gravitational helicity.

Some clues seem to point towards a generalization of helicity to gravity. Electromagnetism is a gauge theory, and the geometry and topology of gauge theories are well studied. Our work thus far strongly suggested that the Riemann curvature tensor was more similar to the Faraday tensor than a traditional education would lead one to believe. The excess of indices in the Riemann tensor is not a problem since a Yang-Mills field strength tensor[14] has four indices if you expand the collectivized gauge group indices. Moreover, Arrayás and Trueba showed the topological currents associated with the electric and magnetic helicities to be Chern-Simons-like tensors for the vector potential A_μ and its field strength tensor $F_{\mu\nu}$. If we could find a way to identify the gauge connection which leads through a Yang-Mills equation to the Riemann curvature, we could construct the Yang-Mills Chern-Simons-like term for that connection and develop a theoretical construct for gravitational helicity.

Thus we find ourselves at the beginning of the Chapter 5. We needed to find a formulation of differential geometry similar to gauge theory. We found precisely this in Elie Cartan's version of differential geometry, which he named the method of moving frames. We derived the Newman-Penrose and the GHP equations from the Cartan structure equation, assuming the metric in the tangent space is of Newman-Penrose form. The details are given at the beginning of Chapter 5. Cartan's differential geometry confirmed that the spin connection was the local Lorentz gauge field hidden inside gravity and that its gauge curvature was indeed the Riemann curvature 2-form. Gauge fixing this local Lorentz group down to a local GHP subgroup gave us the GHP formalism associated with two new gauge covariant derivatives (þ and ð)[15].

[14] A Yang-Mills field is the generalization of an EM field to a non-abelian gauge group.

[15] A repeating theme in our work is that gravity tends to display properties which mimic the degrees of freedom of a pair of gauge fields.

Figure 1.5: From left to right Jan Willem Dalhusien, Alexander Burinskii, Joseph Swearngin, Amy Thompson, Dirk Bouwmeester.

During my first master's degree, I had the pleasure of studying with Alexander Burinskii[16]. Burinskii has studied the Kerr solution and its generalizations for his entire career. He wanted to know the relationship between the Kerr congruence and twistor theory. Specifically, he wondered if there existed a twistor generating function for the Kerr congruence and if it could be used, through the Penrose transform, to imbue a class of massless fields with the structure of the Kerr congruence in the same way that we had used that same transform to generate a class of massless fields with the Hopf structure of the Robinson congruence. As it turns out, Bramson had this idea much earlier, which he implemented via the rank-2 symmetric kinematic twistor [26]. While teaching us about the Kerr congruence,

[16]He was in attendance when I defended my first master's (see Fig. 1.5) who introduced me to Roy Kerr over breakfast at the Marcel Grossmann meeting that occurred the same year.

he also introduced two very important things that factor heavily into my next project. Kerr's theorem and its relation to the Debny-Kerr-Schild form of the general type-D metrics, $g_{\mu\nu} = \eta_{\mu\nu} + h(x) l_\mu l_\nu$ where l_μ is null with respect to both the full metric $g_{\mu\nu}$ *and* the flat metric $\eta_{\mu\nu}$.

The metric for a plane-fronted parallel-propagated wave, pp-wave space-time, is given in Kerr-Schild form[17] is

$$\begin{aligned} ds^2 &= -2dudv + 2P^{-2}d\xi d\bar{\xi} - 2H(u,\xi,\bar{\xi})dudu \\ &= -2du(dv - H(u,\xi,\bar{\xi})du) + 2P^{-2}d\xi d\bar{\xi} \\ &= -2\tilde{e}^1\tilde{e}^2 + 2\tilde{e}^3\tilde{e}^4 \end{aligned}$$

where the null frame $\tilde{e}^a$ is

$$\begin{aligned} \tilde{e}^1 &= du \qquad & \tilde{e}^2 &= dv - H(u,\xi,\bar{\xi})du \\ \tilde{e}^3 &= P^{-1}d\xi \qquad & \tilde{e}^4 &= P^{-1}d\bar{\xi}. \end{aligned}$$

[17]In flat space, the metric with respect to a null frame is

$$\begin{aligned} ds^2 &= -2dudv + 2P^{-2}d\xi d\bar{\xi} \\ &= -e^1e^2 + e^3e^4, \end{aligned}$$

where $e^1 = du$, $e^2 = dv$, $e^3 = P^1 d\xi$, and $e^4 = P^{-1}d\bar{\xi}$. P is a real function of the coordinates $x^\mu = \{u, v, \xi, \bar{\xi}\}$. A null Cartesian coordinate basis has $P = 1$ and a null spherical polar basis has $P = (1 + \xi\bar{\xi})$.

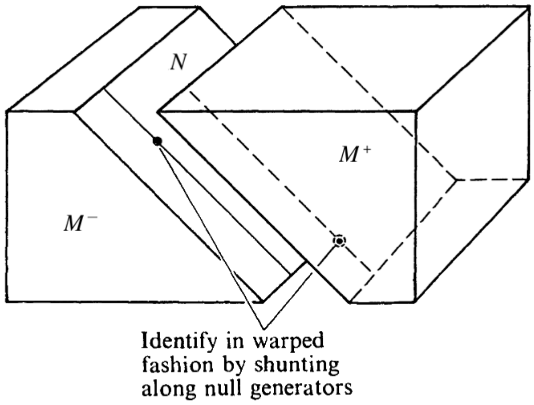

FIG. 2. A plane-fronted impulsive wave can be constructed by joining two Minkowski half-spaces together with a warp along a null hyperplane.

Figure 1.6: From Penrose's paper [27] depicting a plane fronted impulsive wave.

An impulsive wave or shockwave has $H(u, \xi, \bar{\xi}) = \delta(u)h(\xi, \bar{\xi})$ and the Einstein equation[18] reduces to the 2D Poisson equation $2\partial_\xi \partial_{\bar{\xi}} h = 8\pi T_{u\,u}$. For the Aichelberg-Sexl (AS) shockwave, $h(\xi, \bar{\xi}) = ln(\xi\bar{\xi})$, and the stress-energy tensor is given by $T_{\alpha\,\beta} = \delta(u)\delta(\xi)\partial_\alpha u\, \partial_\beta u$ which represents a massless particle moving along the outgoing null direction u located at $\xi = 0$. The AS shockwave was originally derived in [28] as the ultra-boost limit of a Schwarzschild black hole. An AS shockwave is an uncharged, non-rotating black hole moving at the speed of light.

Penrose describes these space-times [27] as two copies of Minkowski space glued together in a "warped and shunted fashion" along the null hyperplane $u = 0$. The

[18] The Weyl and Ricci curvature have only one component which are given respectively by $\Psi_4 = \delta(u)\partial_\xi \partial_{\bar{\xi}} h$ and $\Phi_{22} = \delta(u)\, \partial_\xi \partial_{\bar{\xi}} h$.

"warped and shunted fashion" by which the two flat halves are glued together in the cut and paste method shown in Fig. 1.6 is made manifest by identifying the frame $\tilde{e}^a$ as a shifted version of the flat frame e^a. Thus,

$$\tilde{e}^1 = e^1 \qquad \tilde{e}^2 = e^2 - H(u, \xi, \bar{\xi}) e^1 \tag{1.1}$$

$$\tilde{e}^3 = e^3 \qquad \tilde{e}^4 = e^4. \tag{1.2}$$

Gravitational shockwaves are a special class of metrics possessing a discontinuity that represent the solution to Einstein's equation for a propagating massless particle in a background space-time. Yoni Bentov, the teaching assistant for my first quantum field theory courses, was now a post-doc and had been recently studying gravitational shockwaves in the context of the black hole information problem. The Dray and t'Hooft solution [29] Fig. 1.7 represents an exact solution to Einstein's equations for a massless particle on the future horizon of a Schwarzschild black hole. The shockwave does not represent a test particle as it backreacts on the geometry and its presence perturbs the event horizon. This backreaction plays a fundamental role in the study of black hole statistical mechanics [30–33]. All these results utilized the Dray and t'Hooft solution, not the more complicated but astrophysically more relevant Kerr black hole.

Yoni says that when setting out to learn something, one must find a sufficiently complex problem to which one can apply it. The problem we chose was

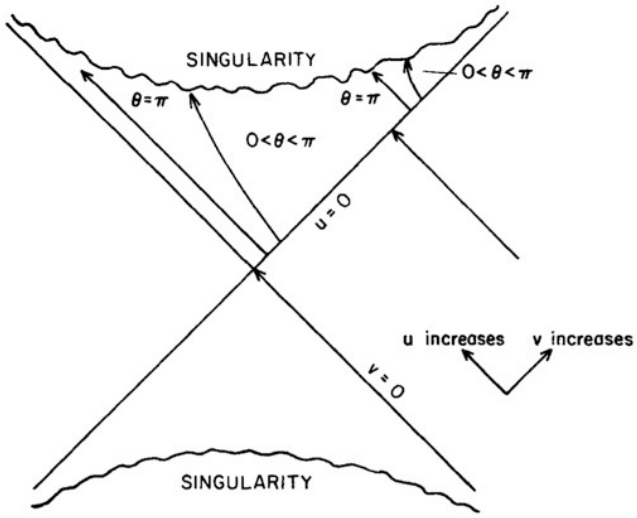

Fig. 1. The Dray-'t Hooft geometry consists of two Schwarzschild solutions joined at the (future) horizon. The horizon contains one singular null line, at coordinate $\theta = 0$. Infalling radial null geodesics cross the horizon to emerge in the inner Schwarzschild solution with an offset in coordinate v, and except for $\theta = \pi$, a deflection in θ. The incoming null rays with $\theta \neq 0$ connect up, for instance, to different rays on the inside of the horizon, depending on θ. Because rays with $\theta \neq \pi$ have some final θ velocity, these rays do not lie at 45° when projected into the $u - v$ plane for plotting here.

The geometry of the rays is the same at any crossing value of v. However, because the Schwarzschild solution is not independent of v, the strength of the singularity is best measured by the induced value of $\mathrm{d}\theta/\mathrm{d}r$ as the horizon is crossed. We find that for constant θ, $\mathrm{d}\theta/\mathrm{d}r \to 0$ in the limit of large v. Hence the strength of the singular source decays to zero for large v.

Figure 1.7: Behaviour of ray optics in the Dray-t'Hooft geometry *Reproduced from Ref. [34].* Here Matzner illustrates the cut and paste method of Penrose as applied to the Dray-t'Hooft solution.

to generalize Dray-t'Hooft from a Schwarzschild background to the Kerr-Newman background. Both the Dray-t'Hooft solution and its generalization by Sfetsos [35] to charged d-dimensional black holes could both be written in the generalized Kerr-Schild form $g_{\mu\nu} = \hat{g}_{\mu\nu} + S(x) l_\mu l_\nu$ where $\hat{g}_{\mu\nu}$ is the Schwarzschild metric for Dray-t'Hooft or the Reissner–Nordström metric for Sfetsos. In either case l_μ was the tangent vector to the Schwarzschild congruence. We reconstructed both metrics by writing the hatted metrics in terms of a principal frame and then shifting the principal frame precisely along the Schwarzschild congruence. Our method was a uniquely frame-centered approach that highlighted the role of a special null shear-free geodesic congruence associated with the hatted metric. The Kerr metric also possesses its own null shear-free geodesic congruence from which one may complete a principal frame for which the hatted line element takes the form $ds^2 = -e^1 e^2 + e^3 e^4$. Shifting the frame e^a, we construct a new frame $\tilde{e}^a$, thereby obtaining a new solution to Einstein's equation representing a massless particle on the future horizon of a rotating and charged black hole.

Chapter 2

Knotted Fields in Plasma

2.1 Topological Solitons in Condensed Matter Systems

Hopfions have been shown to represent localized topological solitons in many areas of physics - as a model for particles in classical field theory [36], fermionic solitons in superconductors [37], particle-like solitons in superfluid-He [38], knot-like solitons in spinor bose-einstein (BE) condensates [39] and ferromagnetic materials [40], and topological solitons in magnetohydrodynamics (MHD) [4]. The Hopf fibration can also be used in the construction of finite-energy radiative solutions to Maxwell's equations and linearized Einstein's equations [3]. Some examples are Rañada's null EM hopfion [11,41] and its generalization to torus knots [42,25,43].

Topological solitons are metastable states. They are not in an equilibrium, or lowest energy, state, but are shielded from decay by a conserved topological quantity. The energy E is a function of a scale factor, typically the size R of the soliton, so that the field could decrease its energy by changing this parameter. However, the topological invariant fixes the length scale and thus the energy. In condensed states (superconductors, superfluids, BE condensates, and ferromagnets) the topological structure is physically manifested in the order parameter, which is associated to a topological invariant. For example, the hopfion solutions in ferromagnets are such that the Hopf fibers correspond to the integral curves of the magnetization vector $\vec{m}$. The associated Hopf invariant is equal to the linking number of the integral curves of $\vec{m}$.

For many systems the solution can still decay by a continuous deformation while conserving the topological invariant. Another physical stabilization mechanism is needed to inhibit collapse [44]. For example, this can be achieved for superconductors with localized modes of a fermionic field [45], for superfluids by linear momentum conservation [38], for BE condensates with a phase separation from a second condensate [46], and for ferromagnets with conservation of the spin projection S_z [47].

2.2 Topological Solitons in MHD

In MHD, the topological structure is present in the magnetic field. The topological soliton of Kamchatnov has a magnetic field everywhere tangent to a Hopf fibration, so that the integral curves of the magnetic field lie on nested tori and form closed circles that are linked exactly once. The Hopf invariant is equal to the linking number of the integral curves of the magnetic field, which is proportional to the magnetic helicity. In addition to the topological invariant, another conserved quantity is required. MHD solitons can be stabilized if the magnetic field has a specific angular momentum configuration which will be discussed below.

Because of the importance of topology in plasma dynamics, there has previously been interest in generalizing the Kamchatnov-Hopf soliton [48]. The topology of field lines has been shown to be related to stability of flux tube configurations, with the helicity placing constraints on the relaxation of magnetic fields in plasma [49, 50]. Magnetic helicity gives a measure of the structure of a magnetic field, including properties such as twisting, kinking, knotting, and linking [8, 9]. Simulations have shown that magnetic flux tubes with linking possess a longer decay time than similar configurations with zero linking number [51–53]. Recently, higher order topological invariants have been shown to place additional constraints on the evolution of the system [49, 54, 55]. The work presented in this chapter distinguishes itself from these topological studies of discrete flux tubes in the sense that we are considering the topology and stability of continuous, space-filling magnetic field distributions. Furthermore, our results are analytic, rather than based on numerical simulations.

There are many applications where magnetic field topology has a significant effect on the stability and dynamics of plasma systems. For example, toroidal magnetic fields increase confinement in fusion reactors [56, 57], and solving for the behavior of some magnetic confinement systems is only tractable in a coordinate system based on a known parameterization of the nested magnetic surface topology [58, 59, 57]. In astrophysics, the ratio of the toroidal and poloidal winding of the internal magnetic fields impacts many properties of stars, including the shape [60, 61] and momentum of inertia [62], as well as the gravity wave signatures [63] and disk accretion [64] of neutron stars. The new class of stable, analytic MHD solutions presented in this chapter may be of use in the study of fusion reactions, stellar magnetic fields, and plasma dynamics in general.

The MHD topological soliton is intimately related to the radiative EM hopfion solution. The EM hopfion constructed by Rañada is a null EM solution with the property that the electric, magnetic, and Poynting vector fields are tangent to three orthogonal Hopf fibrations at $t = 0$. The electric and magnetic fields deform under time evolution, but their field lines remain closed and linked with linking number one. The Hopf structure of the Poynting vector propagates at the speed of light *without deformation.* The EM hopfion has been generalized to a set of null radiative fields based on torus knots with an identical Poynting vector structure [25]. The electric and magnetic fields of these toroidal solutions have integral curves that are not single rings, but rather each field line fills out the surface of a torus.

The time-independent magnetic field of the topological soliton is the magnetic field of the radiative EM hopfion at $t = 0$

$$\mathbf{B}_{\text{soliton}}(\mathbf{x}) = \mathbf{B}_{\text{hopfion}}(t = 0, \mathbf{x}). \tag{2.1}$$

The soliton field is then sourced by a stationary current

$$\mathbf{j}(\mathbf{x}) = \frac{1}{\mu_0}\nabla \times \mathbf{B}_{\text{soliton}}(\mathbf{x}). \tag{2.2}$$

We will use this relationship, along with the generalization of the EM hopfion to toroidal fields of higher linking number, in order to generalize the Kamchatnov-Hopf topological soliton to a class of stable topological solitons in MHD. We will also discuss how the helicity and angular momentum relate to the stability of these topological solitons.

2.3 Generalized Kamchatnov-Hopf Solitons

We construct the generalized topological soliton fields using Eqs. (2.1) and (2.2) applied to the null radiative torus knots. The time-independent magnetic field of the soliton is identical to the magnetic field of the radiative torus knots at $t = 0$. The magnetic field is sourced by a current, resulting in a stationary solution.

The torus knots are constructed from the Euler potentials[1]:

$$\alpha = \frac{(r^2 - t^2 - 1) + 2\imath z}{r^2 - (t - \imath)^2} \tag{2.3}$$

$$\beta = \frac{2(x - \imath y)}{r^2 - (t - \imath)^2}. \tag{2.4}$$

where $r^2 = x^2+y^2+z^2$. As Ref. [25] points out, at $t = 0$ these are the stereographic projection coordinates on $\mathbb{S}^3$. The magnetic field of the torus knots is obtained from the Euler potentials for the Riemann-Silberstein vector $\mathbf{F} = \mathbf{E} + \imath\mathbf{B}$.[2]

The solitons are found by taking the magnetic field of the torus knots at $t = 0$

$$\mathbf{B} = Im[\nabla\alpha^{n_t} \times \nabla\beta^{n_p}] \mid_{t=0} . \tag{2.5}$$

Each (n_t, n_p) with $n_t, n_p = 1, 2, 3...$ represents a solution to Maxwell's equations. A single magnetic field line fills the entire surface of a torus. These tori are nested and each degenerates down to a closed core field line that winds n_t times around the toroidal direction and n_p times around the poloidal direction,

[1] These Euler potentials are maps from $\mathbb{R}^3 \longrightarrow \mathbb{C}$. At each point in $\mathbb{R}^3$ the ordered pair $(\alpha, \beta) \in \mathbb{C}^2$. When $t = 0$ the ratio $\xi(x, y, z) = \beta(0, x, y, z)/\alpha(0, x, y, z)$ is a Hopf map composed with stereographic projection from $S^3 \longrightarrow \mathbb{R}^3$.

[2] Note that the Riemann-Silberstein construction is a non-standard use of Euler potentials. We are following the method in Ref. [25].

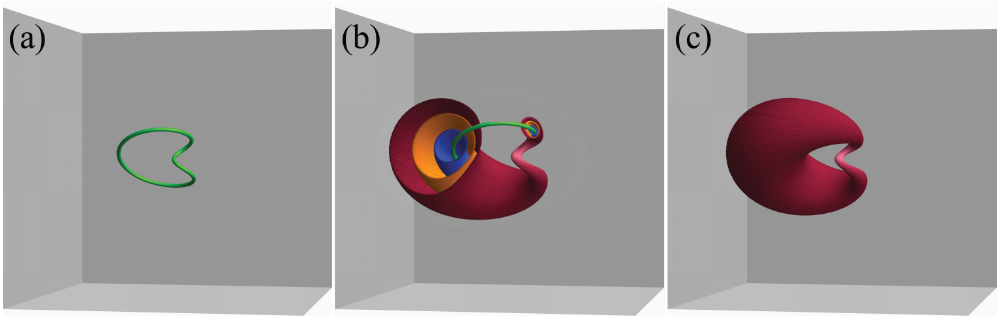

Figure 2.1: One lobe of the field configuration for $n_p = 1$ and $n_t = 2$. (a) A single, closed core magnetic field line. (b) The core field line is surrounded by nested toroidal surfaces, shown in cross section. (c) A complete magnetic surface filled entirely by one field line.

as illustrated in Figure 2.1. A complete solution for a given (n_t, n_p) is composed of pairs of these nested surfaces that are linked and fill all of space as shown in Figure 2.2. For $n_g = gcd(n_t, n_p)$, the solution is a magnetic field with $2n_g$ linked core field lines (knotted if $n_t > 1$ and $n_p > 1$). If $n_t = 1$ and $n_p = 1$, the solution is the Kamchatnov-Hopf soliton. We will analyze these fields and how the linking of field lines affects the stability of magentic fields in plasma. In particular, for $n_p = 1$ and $n_t \in \mathbb{Z}^+$, we will show that these fields can be used to construct a new class of stable topological solitons in ideal MHD. The solutions with $n_p \neq 1$ are not solitons in plasma, and their instability will be discussed in Section 2.4.1.

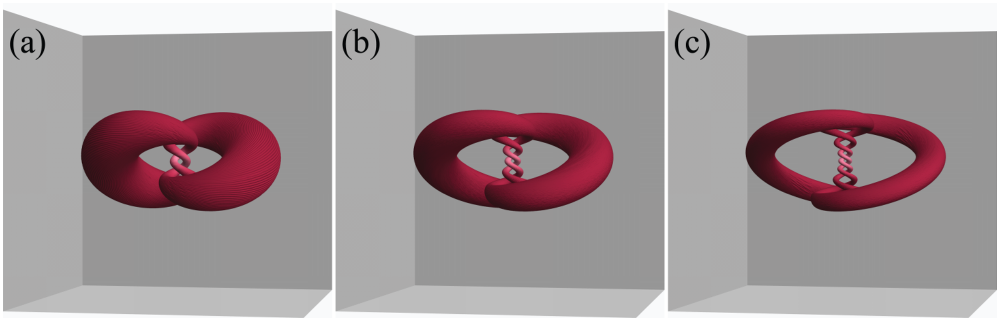

Figure 2.2: Topological solitons in MHD with $n_p = 1$ and (a)$n_t = 2$, (b)$n_t = 3$, and (c)$n_t = 4$. A single magnetic field line fills out each of the linked, toroidal surfaces.

2.4 Stability Analysis

In this section we assume the plasma is an ideal, perfectly conducting, incompressible fluid. In a fluid with finite conductivity, the magnetic field energy diffuses. Under this condition, one can estimate the lifetime of the soliton as will be shown in Section 2.5.

First we consider the case where the poloidal winding number $n_p = 1$ and the toroidal winding number n_t is any positive integer. These will be shown to represent stable topological solitons in ideal MHD. In the next section, we will consider the solutions with $n_p \neq 1$. Using the method in this chapter, these do not represent stable solitons, and we will discuss how this instability relates to the angular momentum.

To analyze the stability of these solutions, following the stability analysis in Ref. [4][3], we study the two scaled quantities of the system - the length scale R which corresponds to the size of the soliton and B_0 which is the magnetic field strength at the origin. (The length scale R is also the radius of the sphere $\mathbb{S}^3$ before stereographic projection.) First we change to dimensionful coordinates by taking

$$\{x, y, z\} \rightarrow \{\tfrac{x}{R}, \tfrac{y}{R}, \tfrac{z}{R}\} \tag{2.6}$$

$$|\mathbf{B}\,(0,0,0)\,| = B_0. \tag{2.7}$$

The stability depends on three quantities - energy, magnetic helicity, and angular momentum - which are functions of R and B_0. For a perfectly conducting plasma, the magnetic helicity h_m is an integral of motion and is thus conserved. The magnetic helicity is also a topological invariant proportional to the linking number of the magnetic field lines. If the field can evolve into a lower energy state by a continuous deformation (therefore preserving the topological invariant) then it will be unstable. However, we will show that such a deformation does not exist because the angular momentum M is also conserved and serves to inhibit the spreading of the soliton.

[3]Note that Ref. [4] uses CGS units and we use SI units in our analysis. The reference also has a typo - Eq. (45) should have a factor of R^2 instead of R.

The magnetic helicity is defined as

$$h_m = \int \mathbf{A} \cdot \mathbf{B} d^3x \tag{2.8}$$

where $\mathbf{A} = Im[\alpha^{n_t} \nabla \beta^{n_p}]$ is the vector potential. From Eqs. (2.3)-(2.5), it follows that

$$h_m = \frac{2n_t}{(n_t + 1)} \pi^2 B_0^2 R^4. \tag{2.9}$$

The MHD equations for stationary flow are satisfied for a fluid with velocity

$$\mathbf{v} = \pm \frac{\mathbf{B}}{(\mu_0 \rho)^{\frac{1}{2}}}. \tag{2.10}$$

The energy of the soliton is given by

$$\begin{aligned} E &= \int \left(\frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} \right) d^3x \\ &= \int \frac{B^2}{\mu_0} d^3x \end{aligned} \tag{2.11}$$

so that

$$\begin{aligned} E &= \frac{2n_t \pi^2}{\mu_0} B_0^2 R^3 \\ &\propto \frac{h_m}{R}. \end{aligned} \tag{2.12}$$

The angular momentum is

$$\begin{aligned}\mathbf{M} =& \rho \int [\mathbf{x} \times \mathbf{v}] d^3x \\ =& \left(\frac{\rho}{\mu_0}\right)^{1/2} 4 n_t \pi^2 B_0 R^4 \hat{y}\end{aligned} \tag{2.13}$$

where we took the positive velocity solution. We find that the conserved quantities h_m and $\mathbf{M}$ fix the values of R and B_0,

$$\begin{aligned}R =& \left(\frac{1}{8\pi^2 n_t(n_t+1)} \left(\frac{\mu_0}{\rho}\right) \frac{|M|^2}{h_m}\right)^{\frac{1}{4}}, \\ B_0 =& 2n_t(n_t+1) \left(\frac{\rho}{\mu_0}\right)^{1/2} \frac{h_m}{|M|},\end{aligned} \tag{2.14}$$

thus inhibiting energy dissipation. This shows that the solution given in Eqs. (2.3)-(2.5) (and shown in Figure 2.2) represents a class of topological solitons characterized by the parameter $n_t \in \mathbb{Z}^+$ for $n_p = 1$.

2.4.1 Angular Momentum and Instability for $n_p \neq 1$

For $n_p \neq 1$, the angular momentum for all n_t is zero. Some examples of fields with $n_t = 1$ and different n_p values are shown in Figure 2.3. The field lines fill two sets of linked surfaces. For a given pair of linked surfaces, the field in each lobe wraps around the surface in opposite directions. In Figure 2.3 the red and blue surfaces wind in opposite directions. This means that the contribution to the angular momentum of the two field lines cancels. In this case the length scale is not fixed by the conserved quantities. The energy can therefore decrease by increasing the radius and the fields are not solitons.

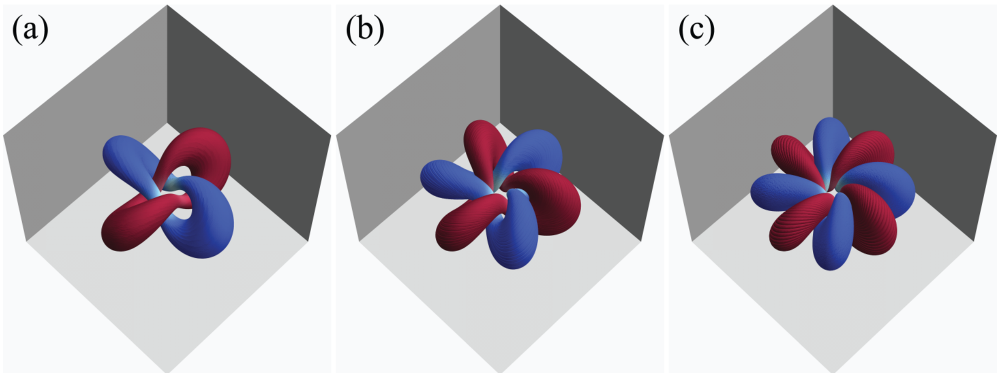

Figure 2.3: The magnetic surfaces for $n_t = 1$ and (a)$n_p = 2$, (b)$n_p = 3$, and (c)$n_p = 4$. Solutions with $n_p \neq 1$ have zero angular momentum and are therefore not stable solitons. The magnetic field lines in each lobe wind in opposite directions, represented by the red and blue surfaces.

2.5 Finite Conductivity and Soliton Lifetime

To include losses due to diffusion, we need to consider a plasma with finite conductivity. We can estimate the soliton lifetime by dividing the energy by dE/dt, calculated before any energy dissipation [4]. Since this is the maximum rate of energy dissipation, we can obtain a lower bound on the time it takes for the total energy to dissipate. Thus,

$$\frac{dE}{dt} = \frac{1}{\sigma}\int j^2 d^3x \tag{2.15}$$

$$= (3n_t + 7n_t^2 + 5n_t^3)\frac{\pi^2 B_0^2 R}{\mu_0^2 \sigma}. \tag{2.16}$$

The resulting lifetime is

$$t_{n_t} \geq \frac{3n_t}{3n_t + 7n_t^2 + 5n_t^3}\mu_0 \sigma R^2. \tag{2.17}$$

For higher n_t, the lifetime decreases although the helicity in Eq. (2.9) increases. This result is interesting as we would have expected from the results regarding flux tubes mentioned previously that the lifetime would increase with increasing helicity.

2.6 Conclusion

We have shown how to construct a new class of topological solitons in plasma. The solitons consist of two linked core field lines surrounded by nested tori that fill all of space. The solutions are characterized by the toroidal winding number of the core field lines and have poloidal winding number one in order to have non-zero angular momentum. We have shown that the conservation of linking number and angular momentum give stability to the solitons in the ideal case. We are currently studying the stability of these solutions in a resistive plasma using numerical simulations. Finally, we note that there may be related generalizations of the hopfion fields in other physical systems, such as superfluids, Bose-Einstein condensates, and ferromagnetic materials.

Chapter 3

Linked and Knotted Gravitational Radiation

3.1 Background

Knots and links are quite remarkable given that they are as old and ubiquitous as ropes and thread and yet have only relatively recently seen a rigorous formulation within mathematics. The study of knots and links has enjoyed a close relationship with physics since its inception by Gauss [65]. Today the application of these topological structures in theoretical physics is more widespread than it has ever been, from fault resistant quantum computing [66], hadron models [36, 67], topological MHD and fluid mechanics [4,6], classical field theories [41,42,3], quantum field theory [68,69], DNA topology [70], to nematic liquid crystals [71] just to name a few. In this chapter, we shall focus on the application of an important class of knots, torus knots, to classical electromagnetic and gravitational radiation.

A *hopfion* is a field configuration based on a topology derived from the Hopf fibration. The electromagnetic hopfion (EM hopfion) is a null solution to the source free Maxwell equations such that any two field lines associated to either the electric, magnetic, or Poynting vector fields (EBS fields) are closed and linked exactly once [41]. When an EM hopfion is decomposed onto hyperplanes of constant time there always exists a hyperplane wherein the EBS fields are tangent to the fibers of three orthogonal Hopf fibrations. If one extends the Poynting vector to be a future pointing light-like 4-vector its integral curves comprise a space filling shear-free null geodesic congruence dubbed the Robinson congruence by Roger Penrose.

Rañada [11] rediscovered the EM hopfion solution and noted that its topology was invariant under time evolution. The search for generalizations of the hopfion solution led to the introduction of a set of non-null EM solutions based on torus knots [72], but the topology was not preserved during time evolution. Around the same time, the Kerr-Robinson theorem was used to derive the hopfion from the Robinson congruence itself using 2-spinor methods [14]. Inspired by the role of the Robinson congruence in the evolution of the hopfion, the conservation of field line topology was tied to the shear-free property of the Robinson congruence [73].

3.2 Complex Analytic and Twistor Methods

The fundamental role of the Robinson congruence and its connection to the topology of physical systems leads one to consider its relationship to twistor theory. It has long been known that complex contour integral transforms can be used to find the solution to real PDEs. In 1903, Whittaker used this technique to construct the general solution to Laplace's equation [74]. In 1915, Bateman extended this method to give solutions to the vacuum Maxwell equations [75]. Twistor theory was developed by Roger Penrose in the late 1960's as an extension of the $sl(2, \mathbb{C})$ spinor algebra. From this perspective, the complex analytic structure of Bateman can be related to the geometry of spinor fields on space-time [76], which encode linear and angular momentum and are represented by two $SL(2, \mathbb{C})$ spinors $\pi_{A'}$ and ω^A. The linear momentum is the flagpole of $\pi_{A'}$ so that $p^a = \pi^{A'}\bar{\pi}^A$ and the angular momentum bivector is related to ω^A by

$$M^{ab} = i\omega^{(A}\bar{\pi}^{B)}\epsilon^{A'B'} + c.c.$$

These spinors are combined into a single object $Z^\alpha = \left(\omega^A, \pi_{A'}\right)$ called a twistor. In this formalism, massless linear relativistic fields are expressed in the form of symmetric spinor fields. In 1969, the general solution to the massless spin-N

field equations was given as a complex contour integral transform now called the Penrose transform [77].

Within the twistor framework the solutions of the massless spin-N equations are represented by the Penrose transform of homogeneous twistor functions (see the Appendix in Section 3.7). The elementary states are a canonical example of such functions whose singularities define Robinson congruences on Minkowski space $\mathbb{M}$. The space-time fields corresponding to the elementary states are finite-energy, and in the null case are everywhere non-singular [78]. For integer spin fields, the expansion of a solution over the elementary states in twistor space $\mathbb{T}$ is related to the expansion over spherical harmonics in $\mathbb{M}$ through the Penrose transform [79]. These properties have made the elementary states the topic of many studies [5,80,81], and for many problems it is assumed that considering the elementary states is sufficient to describe any solution [82].

While investigating these twistor functions and their connection to field topology in $\mathbb{M}$, we have previously shown that the EM hopfion and the analogous gravitational hopfion are elementary states of twistor theory [3]. Using the earlier construction for EM fields by Bateman [75], Kedia, *et al.* have shown that the EM hopfion is the simplest case in a set of null EM fields based on torus knots [25]. Here we show that field configurations based on all the torus knots are contained within the elementary states of twistor theory. The Hopf fibration appears as the degenerate case whereby the linked and knotted toroidal structure degenerates down to the linked hopfion configuration. This generalization leads to a construction for spin-N fields based on torus knots. We will focus our analysis on the spin-1 and spin-2 fields, where the topology is physically manifest in the field lines.[1]

The concept of tendex and vortex lines, gravitational lines of force for a particular observer, was developed by Nichols, *et al.* [15], who were motivated by the desire to understand the non-linear dynamics of curved space-time in a more intuitive, directly physical way than previous approaches. The physical understanding of the electromagnetic field is based upon the decomposition of the Faraday field strength tensor onto hyperplanes of constant time yielding two spacial vector fields interpreted as the electric and magnetic fields. Analogously, the Weyl curvature tensor admits a decomposition onto constant time hyperplanes yielding two spacial tensors called the gravito-electric (GE) and gravito-magnetic (GM) tensors. The integral curves of the eigenvector fields of these tensors are called tendex and vortex lines respectively and represent the gravitational analog of electromagnetic field lines. This method was elucidated through a series of papers where it was applied to (I) weak field solutions [15], (II) stationary black holes [16], and (III) weak perturbations of stationary black holes [17]. This method of GEM decomposition is well-suited to studying linked and knotted fields because, as we have

[1]For the Weyl fields, the linked and knotted topology appears in the current.

shown previously [3], the field topology is manifest in the lines of force for both the electromagnetic and analogous gravitational solutions.

3.3 Parameterization of the Elementary States

We will now relate the torus knot topology to the twistor elementary states to obtain solutions to the EM and gravitational spinor field equations.

Torus knots are closed curves on the surface of a torus which wind an integer number of times about the toroidal direction n_t and poloidal direction n_p as in Fig. 3.1, where n_t and n_p are coprime and both greater than one. If n_t and n_p are not coprime, then there are $n_g = gcd(n_t, n_p)$ linked curves, each corresponding to a (n_t, n_p) mod n_g torus knot. If either n_t or n_p is equal to one, then the knot is trivial with n_g linked curves.

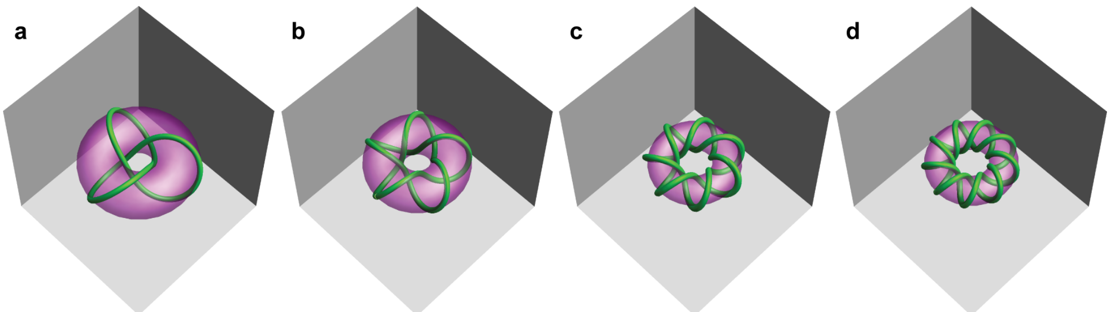

Figure 3.1: Torus knots (green) wind (n_t, n_p) times around a torus (purple) in the toroidal and poloidal directions, respectively. Shown here are the cases of (**a**) trefoil (2,3) knot, (**b**) cinquefoil (2,5) knot, (**c**) septafoil (2,7) knot, and (**d**) nonafoil (2,9) knot.

Following the twistor program, we represent solutions to the massless spin-N equations in $\mathbb{M}$ as the Penrose transform of functions $f(Z)$ in twistor space [13]. The details of the Penrose transform calculation for these fields are given in Section 3.7.

Consider the twistor functions corresponding to the elementary states [5]

$$f(Z) = \frac{(\bar{C}_\gamma Z^\gamma)^c (\bar{D}_\delta Z^\delta)^d}{(\bar{A}_\alpha Z^\alpha)^a (\bar{B}_\beta Z^\beta)^b} \tag{3.1}$$

where $(\bar{A}_\alpha Z^\alpha)$ is the $SU(2,2)$ twistor inner product. Choosing $a = 1$ yields null/Type N solutions and we must have $b = 2h + 1 + c + d$ to give the correct homogeneity $\hbar = -2h - 2$ for a solution with helicity h. We will show that the class of generating functions of the form[2]

$$f(Z) = \frac{(\bar{C}_\gamma Z^\gamma)^{h(n_p-1)} (\bar{D}_\delta Z^\delta)^{h(n_t-1)}}{(\bar{A}_\alpha Z^\alpha)(\bar{B}_\beta Z^\beta)^{h(n_p+n_t)+1}}, \tag{3.2}$$

lead to field configurations with a torus knot topology where n_p and n_t correspond to the poloidal and toroidal winding numbers.

[2]We use the conventions given in Eq. (3.18) of Section 3.7.

We choose the dual twistors

$$\bar{A}_\alpha = \imath(0, \sqrt{2}, 0, 1)$$
$$\bar{B}_\beta = \imath(-\sqrt{2}, 0, -1, 0)$$
$$\bar{C}_\gamma = (0, -\sqrt{2}, 0, 1)$$
$$\bar{D}_\delta = \imath(-\sqrt{2}, 0, 1, 0). \tag{3.3}$$

$\bar{A}_\alpha$ and $\bar{C}_\gamma$ correspond to Robinson congruences with opposite twist, both with central axes aligned along the $+\hat{z}$-direction. $\bar{B}_\beta$ and $\bar{D}_\delta$ correspond to Robinson congruences with opposite twist, but in the $-\hat{z}$-direction. This choice leads to spin-N fields which propagate $+\hat{z}$-direction with field line configurations that are based on a torus knot structure.

3.4 Electromagnetic Torus Knots

After applying the spin-1 Penrose transform to Eq. (3.2), the resulting spinor field is

$$\phi_{A'B'}(x) = \frac{(\mathcal{A}_{C'}\mathcal{C}^{C'})^{n_p-1}(\mathcal{A}_{D'}\mathcal{D}^{D'})^{n_t-1}}{(\mathcal{A}_{E'}\mathcal{B}^{E'})^{n_p+n_t+1}}\mathcal{A}_{A'}\mathcal{A}_{B'} \tag{3.4}$$

Note that the Latin script spinor variables are the spinors associated to the Latin twistor variable. Ergo, $\mathcal{A}_{A'}$ is defined implicitly by $\bar{A}_\alpha Z^\alpha = \mathcal{A}_{A'}\pi^{A'}$ (see Sec. 3.7). The solution in Eq. (3.4) satisfies the source-free spinor field equation by construction and yields the field strength spinor

$$\nabla^{AA'}\varphi_{A'B'} = 0,$$

$$F_{A'B'AB} = \varphi_{A'B'}\epsilon_{AB} + c.c.$$

The spin-1 fields are null torus knots with a Poynting vector that is everywhere tangent to a Hopf fibration and propagates in the $\hat{z}$-direction without deformation. The solutions have the same topology[3] as the electromagnetic fields in Ref. [25]. The electric and magnetic vector fields each have the following topological structure as shown in Fig. 3.2. There are $2n_g$ core field lines, where $n_g = gcd(n_t, n_p)$, which are linked (and knotted if $n_t, n_p > 1$). Each core line has the same configuration as the corresponding torus knot with (n_t, n_p) shown in Fig. 3.1. With the choice for C and D given in Eq. (3.3), the poloidal and toroidal winding numbers for the EM case are related to the exponents in Eq. (3.1) by $c = n_p - 1$ and $d = n_t - 1$. A single core field line is surrounded by nested, toroidal surfaces, each filled by one field line. A second core field line, also surrounded by nested surfaces, is linked with the first so that there are $2n_g$ sets of linked nested surfaces which fill all of space. A complete solution to Maxwell's equations consists of an

[3]There is an overall constant factor of $4n_tn_p$ in Ref. [25] that does not appear in our construction, but it does not affect the topology.

electric and a magnetic field orthogonal to each other, both with this field line structure. The (1,1) case corresponds to the electromagnetic hopfion.

At $t = 0$ the electric and magnetic fields are tangent to orthogonal torus knots, as shown in Fig. 3.3 (first row). The fields will deform under time evolution, but the topology will be conserved since $\vec{E} \cdot \vec{B} = 0$ [73, 10].

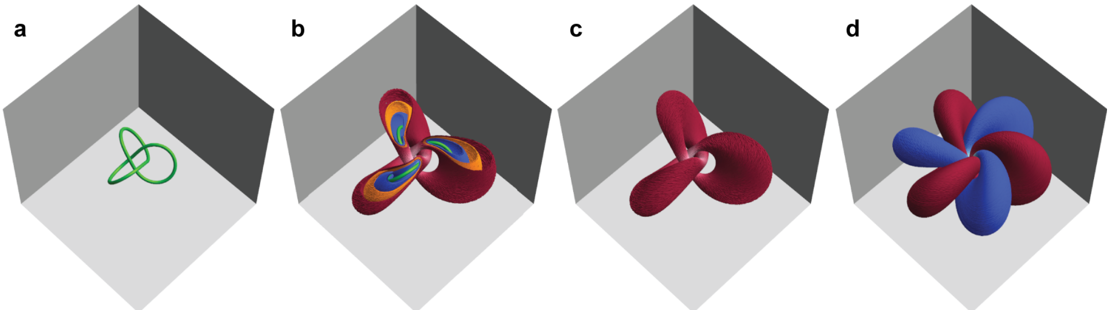

Figure 3.2: The field line structure based on (2,3) trefoil knot. (**a**) The core field line is a torus knot (green). (**b**) Each field line except the core lies on the surface of a nested, deformed torus. (**c**) One field line fills a complete surface (red). (**d**) Another field line fills a second surface (blue) linked with the first. The two linked core field lines and the nested surfaces around them fill all of space.

3.5 Gravito-electromagnetic Torus Knots

The spin-2 solutions will be analyzed in terms of the gravito-electromagnetic tidal tensors. The Weyl tensor C_{abcd} can be decomposed into an even-parity "electric" part E_{ij} corresponding to the tidal field and an odd-parity "magnetic" part B_{ij} for the frame-drag field, in direct analogy with the decomposition of the electromagnetic field strength tensor into an electric field and a magnetic field.

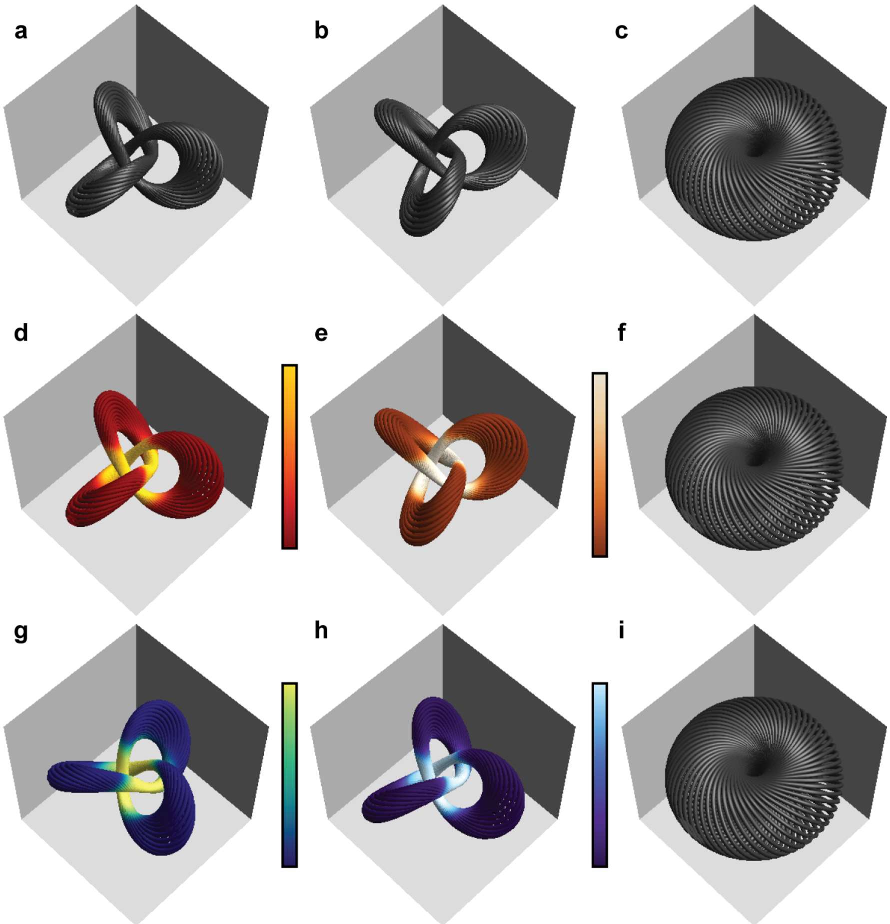

Figure 3.3: A comparison of the spin-1 (EM) and spin-2 (gravity) trefoil knots at $t = 0$. The first row is the EM trefoil knot: (**a**) the electric field, (**b**) the magnetic field, and (**c**) the Poynting vector field. The second row is the gravito-electric trefoil knot: (**d**) the negative eigenvalue field $\vec{e}_-$, (**e**) the positive eigenvalue field $\vec{e}_+$, and (**f**) the zero eigenvalue field $\vec{e}_0$.The third row is the gravito-magnetic trefoil knot: (**g**) the negative eigenvalue field $\vec{b}_-$, (**h**) the positive eigenvalue field $\vec{b}_+$, and (**i**) the zero eigenvalue field $\vec{b}_0$. The color scale indicates magnitude of the eigenvalue, with lighter colors indicating a higher magnitude.

For an observer at rest, this gives

$$E_{ij} = C_{i0j0} \tag{3.5}$$

$$B_{ij} = - * C_{i0j0}. \tag{3.6}$$

These tensors are symmetric and traceless, and are thus characterized entirely by their eigenvalues and eigenvectors. One may then study the eigensystem associated with these matrix-valued fields instead of the Weyl tensor itself, allowing for a more intuitive approach to understanding the gravitational field. The integral curves of the eigenvectors of the tidal tensor are called tendex lines and their eigenvalues define the tidal acceleration along these lines. The integral curves of the eigenvectors of the frame-drag tensor are called vortex lines and their eigenvalues define the gyroscope precession about the vortex lines. Together, the tendex and vortex lines are the analog of electromagnetic field lines. [15]

After applying the spin-2 Penrose transform to Eq. (3.2), the resulting spinor field is

$$\phi_{A'B'C'D'}(x) = \frac{(\mathcal{A}_{F'}\mathcal{C}^{F'})^{2(n_p-1)}(\mathcal{A}_{G'}\mathcal{D}^{G'})^{2(n_t-1)}}{(\mathcal{A}_{E'}\mathcal{B}^{E'})^{2(n_p+n_t)+1}}\mathcal{A}_{A'}\mathcal{A}_{B'}\mathcal{A}_{C'}\mathcal{A}_{D'}. \tag{3.7}$$

The source-free field equation and Weyl field strength spinor are

$$\nabla^{AA'}\varphi_{A'B'C'D'} = 0,$$
$$C_{A'B'C'D'ABCD} = \varphi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + c.c.$$

The Weyl tensor can then be decomposed into the GEM components. For Type N, the eigenvalues for both the GE and GM tensors take the form $\{-\Lambda, 0, +\Lambda\}$, with $-\Lambda(x) \leq 0 \leq +\Lambda(x)$ for all points x in space-time The magnitude of the eigenvalues is

$$\Lambda = \frac{2^{2n_p-3}(1+r^2+t^2-2tz)^2(r^2-z^2)^{n_p-1}(r^4-2r^2(1+t^2)+(1+t^2)^2+4z^2)^{n_t-1}}{(r^4-2r^2(-1+t^2)+(1+t^2)^2)^{\frac{5}{2}+n_t+n_p}} \tag{3.8}$$

We label the eigenvectors $\{\vec{e}_-, \vec{e}_0, \vec{e}_+\}$ and $\{\vec{b}_-, \vec{b}_0, \vec{b}_+\}$ corresponding to the eigenvalues for the tidal and frame-drag fields respectively. For the zero eigenvalue, the eigenvectors $\vec{e}_0$ and $\vec{b}_0$ are both aligned with the Poynting vector of the null EM torus knots. For the remaining eigenvectors, we can construct Riemann-Silberstein (RS) vectors $\vec{f}_e = \vec{e}_- + i\vec{e}_+$ and $\vec{f}_b = \vec{b}_- + i\vec{b}_+$ which are related to each other by

$$\vec{f}_e = e^{i\pi/4}\vec{f}_b. \tag{3.9}$$

At $t = 0$, the eigenvectors of the GE fields have precisely the same structure as the EM fields, and the GM eigenvector fields have the same structure but rotated by 45°. For the spin-2 case, the poloidal and toroidal winding numbers are related to the exponents in Eq. (3.1) by $c = 2(n_p - 1)$ and $d = 2(n_t - 1)$. The surfaces of the $\vec{e}_-$ eigenvector, color-scaled according to the magnitude of the eigenvalue, for different values of (n_t, n_p) are shown in Fig. 3.4. The other GEM fields can be constructed by rotating $\vec{e}_-$ according to Eq. (3.9): $\vec{e}_+$ is found by rotating $\vec{e}_-$ by 90° about the Poynting vector. $\vec{b}_-$ and $\vec{b}_+$ are found by rotating $\vec{e}_-$ and $\vec{e}_+$ by

45°, respectively. The eigenvalues of the GEM fields for a given (n_t, n_p) have the same magnitude (color-scaling) given by $|\Lambda(x)|$ in Eq. (3.8).

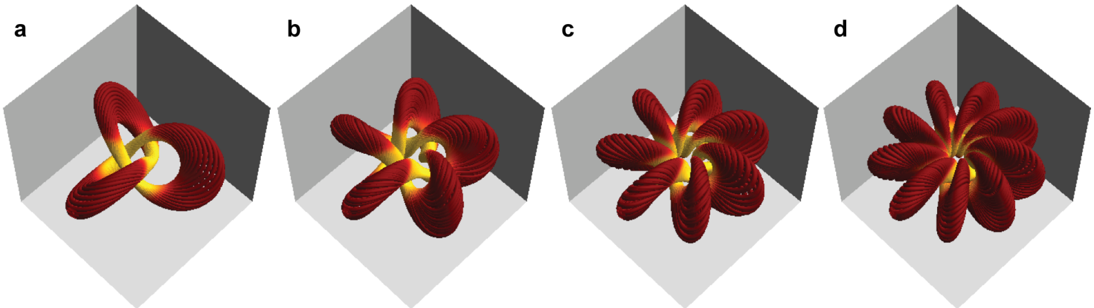

Figure 3.4: The eigenvector field $\vec{e}_-$ for the gravitational field based on the (**a**) trefoil (2,3) knot, (**b**) cinquefoil (2,5) knot, (**c**) septafoil (2,7) knot, and (**d**) nonafoil (2,9) knot. The color scaling is the same as in Fig. 3.3.

3.6 Conclusion

Here we have shown that the null EM torus knot solutions correspond to a class of elementary states characterized the poloidal and toroidal winding numbers of the associated field configuration. Using the relationship between fields of different spin in the twistor formalism, we constructed the analogous gravitational radiation configuration that possesses tendex and vortex lines based on a torus knot structure. Since the topology is manifest in the tendex and vortex lines, the gravito-electromagnetic tidal tensor decomposition is a straightforward method for characterizing these field configurations.

The elementary states were known as early as the 1970's [5], however the explicit forms of their associated spinor and tensor representations on $\mathbb{M}$ were never published.[4] The modern rediscovery of these solutions has raised interest in obtaining a more complete physical understanding of the topological properties of these fields. The parameterization of the twistor functions corresponding to the elementary states in terms of the poloidal and toroidal winding indicates that the torus knot structure is indeed inherent in the elementary states. For both electromagnetism and gravity, the topology appears in the configuration of the lines of force.

3.7 Appendix: The Penrose Transform for Spin-N Torus Knots

To obtain a spinor field $\varphi_{A'_1 \cdots A'_{2h}}(x)$ with helicity h which satisfies the spin-N massless field equation

$$\nabla^{AA'_1}\varphi_{A'_1 \cdots A'_{2h}}(x) = 0$$

we will calculate the Penrose transform

$$\varphi_{A'_1 \cdots A'_{2h}}(x) = \frac{1}{2\pi i}\oint_{\Gamma} \pi_{A'_1} \cdots \pi_{A'_{2h}} f(Z) \pi_{B'} d\pi^{B'} \tag{3.10}$$

[4] from private discussions with Roger Penrose

where Γ is a contour on the Celestial sphere of x that separates the poles of $f(Z)$. Consider the twistor function given by Eq. (3.2)

$$f(Z) = \frac{(\bar{C}_\gamma Z^\gamma)^{h(n_p-1)}(\bar{D}_\delta Z^\delta)^{h(n_t-1)}}{(\bar{A}_\alpha Z^\alpha)(\bar{B}_\beta Z^\beta)^{h(n_p+n_t)+1}}.$$

(For further review of the background material on twistors and the Penrose transform see Ref. [3]). Let $\overline{A}_\alpha = (\mu_A, \lambda^{A'})$ be a dual twistor such that

$$\begin{aligned} \overline{A}_\alpha Z^\alpha &= i\mu_A x^{AA'}\pi_{A'} + \lambda^{A'}\pi_{A'} \\ &\equiv \mathcal{A}^{A'}\pi_{A'} \end{aligned} \tag{3.11}$$

where $Z^\alpha = (ix^{AA'}\pi_{A'}, \pi_{A'})$. Similar relations hold for the other dual twistors $\overline{B}_\beta Z^\beta \equiv \mathcal{B}^{B'}\pi_{B'}$, $\overline{C}_\gamma Z^\gamma \equiv \mathcal{C}^{C'}\pi_{C'}$, and $\overline{D}_\delta Z^\delta \equiv \mathcal{D}^{D'}\pi_{D'}$. We want to write the Penrose transform as an integral over the $\mathbb{CP}^1$ coordinate $\zeta = \pi_{1'}/\pi_{0'}$, so we have

$$\begin{aligned} \pi_{C'}d\pi^{C'} &= \pi_{C'}d\pi_{D'}\epsilon^{D'C'} \\ &= \pi_{0'}d\pi_{1'} - \pi_{1'}d\pi_{0'} \\ &= (\pi_{0'})^2 d(\frac{\pi_{1'}}{\pi_{0'}}). \end{aligned} \tag{3.12}$$

Adopting the canonical spin bases $\{o_{A'}, \iota_{A'}\}$ we have that

$$\begin{aligned} \pi_{A'} &= \pi_{0'} o_{A'} + \pi_{1'} \iota_{A'} \\ &= \pi_{0'}(o_{A'} + (\frac{\pi_{1'}}{\pi_{0'}})\iota_{A'}). \end{aligned} \tag{3.13}$$

Observing that

$$\frac{1}{\pi_{0'}}\mathcal{A}^{A'}\pi_{A'} = \mathcal{A}^{0'} + \mathcal{A}^{1'}(\frac{\pi_{1'}}{\pi_{0'}}) \tag{3.14}$$

and similarly for $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, we see that the Penrose transform becomes an integral manifestly over $\mathbb{CP}^1$. Thus

$$\begin{aligned} \varphi_{A'_1 \cdots A'_{2h}}(x) &= \frac{1}{2\pi i}\oint_\Gamma f(Z)\pi_{A'_1}\cdots\pi_{A'_{2h}}\pi_{B'}d\pi^{B'} \\ &= \frac{1}{2\pi i}\oint_\Gamma \frac{(\mathcal{C}^{0'} + \mathcal{C}^{1'}(\frac{\pi_{1'}}{\pi_{0'}}))^{h(n_p-1)}(\mathcal{D}^{0'} + \mathcal{D}^{1'}(\frac{\pi_{1'}}{\pi_{0'}}))^{h(n_t-1)}}{(\mathcal{A}^{0'} + \mathcal{A}^{1'}(\frac{\pi_{1'}}{\pi_{0'}}))(\mathcal{B}^{0'} + \mathcal{B}^{1'}(\frac{\pi_{1'}}{\pi_{0'}}))^{h(n_p+n_t)+1}} \\ &\qquad \times (o_{A_1} + (\frac{\pi_{1'}}{\pi_{0'}})\iota_{A'_1})\cdots(o_{A_{2h}} + (\frac{\pi_{1'}}{\pi_{0'}})\iota_{A'_{2h}})d(\frac{\pi_{1'}}{\pi_{0'}}) \\ &= \frac{(\mathcal{C}^{1'})^{h(n_p-1)}(\mathcal{D}^{1'})^{h(n_t-1)}}{2\pi i\mathcal{A}^{1'}(\mathcal{B}^{1'})^{h(n_p+n_t)+1}}\oint_\Gamma \frac{(\rho+\zeta)^{h(n_p-1)}(\tau+\zeta)^{h(n_t-1)}}{(\mu+\zeta)(\nu+\zeta)^{h(n_p+n_t)+1}} \\ &\qquad \times (o_{A_1} + \zeta\iota_{A'_1})\cdots(o_{A_{2h}} + \zeta\iota_{A'_{2h}})d\zeta \end{aligned} \tag{3.15}$$

where $\mu = \mathcal{A}^{0'}/\mathcal{A}^{1'}$, $\nu = \mathcal{B}^{0'}/\mathcal{B}^{1'}$, $\rho = \mathcal{C}^{0'}/\mathcal{C}^{1'}$, and $\tau = \mathcal{D}^{0'}/\mathcal{D}^{1'}$.

The above change of variables leaves the contour integral in a form that is straightforward to calculate. The contour Γ is taken to enclose the pole $-\mu$ giving the result

$$
\begin{aligned}
\varphi_{A'_1\cdots A'_{2h}} &= \frac{(\mathcal{C}^{1'})^{h(n_p-1)}(\mathcal{D}^{1'})^{h(n_t-1)}}{\mathcal{A}^{1'}(\mathcal{B}^{1'})^{h(n_p+n_t)+1}} \operatorname*{Res}_{\zeta=-\mu} \frac{(\rho+\zeta)^{h(n_p-1)}(\tau+\zeta)^{h(n_t-1)}}{(\mu+\zeta)(\nu+\zeta)^{h(n_p+n_t)+1}} \\
&\qquad \times (o_{A_1}+\zeta\iota_{A'_1})\cdots(o_{A_{2h}}+\zeta\iota_{A'_{2h}}) \\
&= \frac{(\mathcal{C}^{1'})^{h(n_p-1)}(\mathcal{D}^{1'})^{h(n_t-1)}}{\mathcal{A}^{1'}(\mathcal{B}^{1'})^{h(n_p+n_t)+1}} \frac{(\rho-\mu)^{h(n_p-1)}(\tau-\mu)^{h(n_t-1)}}{(\nu-\mu)^{h(n_p+n_t)+1}} \\
&\qquad \times (o_{A'_1}-\mu\iota_{A'_1})\cdots(o_{A'_{2h}}-\mu\iota_{A'_{2h}}) \\
&= \frac{(\mathcal{A}^{1'}\mathcal{C}^{0'}-\mathcal{A}^{0'}\mathcal{C}^{1'})^{h(n_p-1)}(\mathcal{A}^{1'}\mathcal{D}^{0'}-\mathcal{A}^{0'}\mathcal{D}^{1'})^{h(n_t-1)}}{(\mathcal{A}^{1'}\mathcal{B}^{0'}-\mathcal{A}^{0'}\mathcal{B}^{1'})^{h(n_p+n_t)+1}} \\
&\qquad \times (\mathcal{A}^{1'}o_{A'_1}-\mathcal{A}^{0'}\iota_{A'_1})\cdots(\mathcal{A}^{1'}o_{A'_{2h}}-\mathcal{A}^{0'}\iota_{A'_{2h}}) \\
&= \frac{(\epsilon_{C'D'}\mathcal{A}^{C'}\mathcal{C}^{D'})^{h(n_p-1)}(\epsilon_{E'F'}\mathcal{A}^{E'}\mathcal{D}^{F'})^{h(n_t-1)}}{(\epsilon_{A'B'}\mathcal{A}^{A'}\mathcal{B}^{B'})^{h(n_p+n_t)+1}}\mathcal{A}_{A'_1}\cdots\mathcal{A}_{A'_{2h}} \\
&= \frac{(\mathcal{A}_{C'}\mathcal{C}^{C'})^{h(n_p-1)}(\mathcal{A}_{D'}\mathcal{D}^{D'})^{h(n_t-1)}}{(\mathcal{A}_{B'}\mathcal{B}^{B'})^{h(n_p+n_t)+1}}\mathcal{A}_{A'_1}\cdots\mathcal{A}_{A'_{2h}}.
\end{aligned} \tag{3.16}
$$

In the case of spin-1 and spin-2, the classical field strength spinors are given by

$$
\begin{aligned}
F_{A'_1A'_2\ A_1A_2} &= \varphi_{A'_1A'_2}\epsilon_{A_1A_2}+\overline{\varphi}_{A_1A_2}\epsilon_{A'_1A'_2} \\
C_{A'_1\cdots A'_4\ A_1\cdots A_4} &= \varphi_{A'_1\cdots A'_4}\ \epsilon_{A_1A_2}\ \epsilon_{A_3A_4}+\overline{\varphi}_{A_1\cdots A_4}\ \epsilon_{A'_1A'_2}\ \epsilon_{A'_3A'_4}.
\end{aligned} \tag{3.17}
$$

Choosing the standard basis to be the extended Pauli matrices we define the following symbols, referred to as the Infeld-van der Waerden symbols

$$\sigma_0^{AA'} \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1^{AA'} \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2^{AA'} \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_3^{AA'} \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.18}$$

The spinor fields are then related to the world tensor description by

$$\begin{aligned} F_{ab} &= F_{A_1'A_2' \ A_1A_2}\sigma_a^{A_1A_1'}\sigma_b^{A_2A_2'} \\ C_{abcd} &= C_{A_1' \cdots A_4' \ A_1 \cdots A_4}\sigma_a^{A_1A_1'} \cdots \sigma_d^{A_4A_4'}. \end{aligned} \tag{3.19}$$

Chapter 4

Generalization of Electromagnetic and Gravitational Hopfions by Algebraic Type

4.1 Background

Since the earliest days of electromagnetism, when Gauss introduced his integral for calculating the linking number of two curves in 1833, the topology of physical fields has been of interest [83]. Around the same time, Faraday developed the concept of lines of force to give a direct, physical description of electromagnetic phenomena [84]. By the turn of the 20th century "tangles" or knots in solar currents were already being observed and could be explained by magnetic effects [85,86]. Since then, the topological properties of lines of force, such as their linking, knotting, twisting, and kinking, have been studied in many physical systems [8,9].

Faraday had also hoped to find a similar intuitive description for gravitational lines of force. Although he and other scientists such as Maxwell gave much thought to finding this analogy connecting electromagnetism and gravity, they were never able to give an adequate explanation or mathematical formulation for this relationship [87]. In modern times, the connection between massless linear relativistic fields of different spin has been understood in terms of the $SL(2,\mathbb{C})$ spinor field equations. In this language, the spin-N equations - the Dirac, Maxwell, Rarita-Schwinger, and linear Einstein equations - take on a similar form [88]. Their solutions can be expressed in terms of complex contour integrals allowing one to construct analogous fields based on topologically non-trivial configurations for any spin.

The concept of field lines can be applied to gravity, as was once hoped by Faraday and Maxwell, by decomposing the curvature tensor into the forces experienced by a particular time-like observer allowing one to gain an intuitive understanding of the dynamics of curved space-time [15–17]. This gravito-electromagnetic analogy provides a direct way of analyzing linked and knotted fields, since the topology physically manifests in the lines of force for both electromagnetism and gravity. The topology of lines of force is also related to the dynamics of electromagnetic fields [73, 10, 6]. The spin-N correspondence suggests that topology may influence the dynamics of gravitational fields and the gravito-electromagnetic analogy may provide an avenue for investigating such effects.

In this work, we will study fields with a topological structure based on the Hopf map, which is a surjective map sending great circles on $\mathbb{S}^3$ to points on $\mathbb{S}^2$, denoted by

$$\mathbb{S}^3 \xrightarrow{\mathbb{S}^1} \mathbb{S}^2.$$

The great circles on $\mathbb{S}^3$ are called the fibers of the map, and when stereographically projected onto $\mathbb{R}^3$ correspond to the integral curves of physical fields. These circles lie on nested toroidal surfaces and each is linked with every other circle exactly once, creating the characteristic Hopf structure. The homotopy invariant of the map between two spheres is called the Hopf invariant, or Hopf index. This also corresponds to the topological invariant on $\mathbb{R}^3$, the linking number of the fibers.

Traditionally, the term *h*opfion has referred to soliton field configurations with Hopf index unity [89]. These kinds of hopfions have enjoyed a wide variety of application in many areas of physics [90]. In particle physics, hopfions manifest as stable knot-like structures in field theoretic descriptions of hadrons [36, 67]. Hopfions have also been shown to represent localized topological solitons in several physical systems, including superfluid He phases [38], spinor Bose-Einstein condensates [39], and MHD descriptions of plasma [4]. It has also been shown that there exists a radiative solution, referred to as the EM hopfion, which is a solution to the vacuum Maxwell equations with the property that the field lines belonging to either the electric, magnetic, and Poynting vector fields have linking number one and lie on the surfaces of nested tori [41].

An alternative construction for hopfion fields is based on the Robinson congruence, a null shear-free geodesic congruence that lies tangent to a Hopf fibration on all constant-time spatial foliations of Minkowski space. Thus a hopfion can be defined as a field configuration whose principal null directions (PNDs) all lie tangent to Robinson congruences. The twistor formalism provides a convenient method for obtaining these hopfions. Given a dual twistor[1] $A_\alpha = (\omega_A, \pi^{A'})$, there is an associated spinor field $\mathcal{A}_A = \omega_A - ix_{AA'}\pi^{A'}$ called the Robinson field whose flagpole is the Robinson congruence associated with the twistor [91]. This suggests that a hopfion may be represented entirely using twistors.

Previously, we have used this formalism to construct the null EM hopfion solutions and find an analogous topologically non-trivial type N gravitational wave solution [3]. Under time evolution, these radiative fields deform, but maintain their linked structure while propagating at the speed of light. In this work, we investigate how twistor methods can be used to find the spin-h hopfion solutions of different spinor classifications. In particular, we will construct the non-null electromagnetic, type D and type III gravitational hopfion solutions, and then characterize the topology of their lines of force. The non-null and type D fields

[1]For our index conventions, we use lower case Latin letters for Lorentz indices, with i and j reserved for spatial indices, upper case Latin letters unprimed and primed for spinor and conjugate spinor indices respectively, and lower case Greek letters for twistor indices.

do not propagate, but radiate energy from the center of the configuration in all directions. The type III gravitational hopfion has no analogous EM counterpart, but can be written as a combination of the null and non-null hopfions. It also shares some of the distinctive traits of the other hopfions, for example they are localized, finite-energy fields with linked field line configurations.

4.2 Twistor Integral Methods

We consider the relationship between the singularities of the twistor generating functions and the geometry of the associated space-time fields. Since the fields are constructed through a contour integral transform, the poles of the twistor functions determine the spinor structure of the solutions in Minkowski space [92, 93]. By modifying the twistor function that generates the null EM hopfion, we find the generating function for a non-null EM solution with linked field lines. In a similar manner, we use the type N hopfion to construct gravitational hopfions of different Petrov classifications.

For our analysis, we will consider massless linear relativistic fields of helicity h, so their spin and helicity values are equivalent. Given a real-valued spin-h field $\varphi_{A'_1 \dots A'_{2h}}$ written in the self-dual $SL(2,\mathbb{C})$ spinor representation, the field can be decomposed into $2h$ single-index spinors called its principal spinors [12]. This is described by the relation

$$\varphi_{A'_1 \ldots A'_{2h}} \propto \mathcal{A}_{(A'_1} \cdots \mathcal{D}_{A'_{2h})}, \tag{4.1}$$

with the one-index spinors representing the field's PNDs via their flagpole directions. Thus, for a hopfion arising from the definition given above, the PNDs of the spinors $\mathcal{A}, \ldots, \mathcal{D}$ lie tangent to Robinson congruences.

For example, a null spin-1 field can be written as

$$\varphi_{A'B'}(x) = f(x)\mathcal{A}_{A'}\mathcal{A}_{B'} \tag{4.2}$$

where $\mathcal{A}_{A'}$ is an $SL(2, \mathbb{C})$ spinor which defines the doubly degenerate principal null direction of F_{ab} and $f(x)$ is a scalar function that is determined by the electromagnetic field equation.

This suggests for any helicity h there is an analogous spinor field

$$\varphi_{A'_1 \cdots A'_{2h}}(x) = f_h(x)\mathcal{A}_{A'_1} \cdots \mathcal{A}_{A'_{2h}} \tag{4.3}$$

with $2h$-fold degenerate PNDs, but we will need to find the scaling function $f_h(x)$ that satisfies the appropriate spin-h field equation.

The twistor formalism allows for the generalization of spin-1 fields to fields of any spin. The Penrose transform expresses solutions to the massless field equations as contour integrals over homogeneous twistor functions $F(Z)$

$$\varphi_{A'_1 \cdots A'_{2h}}(x) = \frac{1}{2\pi i} \oint_{\Gamma} \pi_{A'_1} \cdots \pi_{A'_{2h}} F_h(Z) \pi_{B'} d\pi^{B'} \tag{4.4}$$

where Γ is a contour on the Celestial sphere of x and

$$F_h(\lambda Z) = \lambda^{-2h-2} F_h(Z) \tag{4.5}$$

so that the degree of homogeneity is related to the helicity by $n_{hom} = -2h - 2$ [13]. In Section 4.6, we give the explicit calculation of the non-null hopfion fields as an example to demonstrate how the pole structure of $F(Z)$ determines the corresponding field geometry through the contour integral in Eq. (4.4).

The twistor function which gives way to the EM hopfion through the Penrose transform is given by

$$F_1(Z) = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)^3}. \tag{4.6}$$

where the contour is taken around the pole defined by A_α. To understand how the generating function relates to a particular space-time field configuration, consider the pole structure of Eq. (4.6). The term $(A \cdot Z)$ has a simple pole and the power of the $(B \cdot Z)$ term is chosen to give us homogeneity -4, and thus a spin-1 field.

This approach leads directly to the spin-h analogue. We keep the single pole for the $(A \cdot Z)$ term and the power of the $(B \cdot Z)$ term is determined by the relation between helicity and homogeneity $n_{hom} = -2h-2$ from Eq. (4.5). The generating function for the spin-h hopfion is then

$$F_h(Z) = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)^{2h+1}}. \tag{4.7}$$

We find the associated spinor field from the Penrose transform of this twistor function is

$$\varphi_{A'_1 \cdots A'_{2h}}(x) = \left(\frac{2}{\Omega |x-y|^2}\right)^{2h+1} \mathcal{A}_{A'_1} \cdots \mathcal{A}_{A'_{2h}} \tag{4.8}$$

where Ω is a constant scalar, y is a constant 4-vector determined by the specific values of A_α and B_β, and $\mathcal{A}_{A'}$ is the spinor field associated to the twistor A_α (see Sec. 4.6 for definitions). As expected, we find the solution has a $2h$-fold degeneracy in its PNDs and the Penrose transform has given us the correct form of the scalar function that will satisfy the field equations.

4.2.1 Petrov Variants

The twistor function in Eq. (4.7) had a single pole which resulted in a spinor field with all its PNDs degenerate (null EM or type N gravity fields). Changing the pole structure yields solutions of different Petrov classes. Because the Penrose transform is a contour integral, when transforming functions with a pole of order greater than one Cauchy's integral formula involves the derivative of $F(Z)$. This derivative brings the other spinor field, in this case $\mathcal{B}_{B'}$, into the numerator thus breaking the degeneracy of the PNDs. For example, the twistor function

$$F_h(Z) = \frac{1}{(A_\alpha Z^\alpha)^{h+1}(B_\beta Z^\beta)^{h+1}}. \tag{4.9}$$

results in non-null EM $\{11\}$ or type D gravity $\{22\}$ fields for $h = 1, 2$ given by

$$\varphi_{A'_1A'_2}(x) = f_1(x)\mathcal{A}_{(A'_1}\mathcal{B}_{A'_2)} \tag{4.10}$$

$$\varphi_{A'_1A'_2A'_3A'_4}(x) = f_2(x)\mathcal{A}_{(A'_1}\mathcal{A}_{A'_2}\mathcal{B}_{A'_3}\mathcal{B}_{A'_4)} \tag{4.11}$$

where $\mathcal{A}_{A'}$ and $\mathcal{B}_{A'}$ define the h-fold degenerate principal null directions. The details of the Penrose transform for the non-null spin-1 fields are given in Sec. 4.6, and the other field types are found by a similar calculation.

This gives all the classifications of the EM field strength tensor, which has two PNDs. For gravity we can also extend this to other classifications for which there is no EM analog, for example hopfion fields of Petrov type III are generated by

$$F_2(Z) = \frac{1}{(A_\alpha Z^\alpha)^2(B_\beta Z^\beta)^4}. \tag{4.12}$$

resulting in the field

$$\varphi_{A'_1 \cdots A'_4}(x) = \left(\frac{2}{\Omega |x-y|^2} \right)^{2h+1} \mathcal{A}_{(A'_1} \mathcal{A}_{A'_2} \mathcal{A}_{A'_3} \mathcal{B}_{A'_4)}. \tag{4.13}$$

In summary, the simplest hopfions correspond to homogeneous twistor functions of the form[2]

$$F(Z) = \frac{1}{(A \cdot Z)^{1+a} (B \cdot Z)^{1+b}}, \tag{4.14}$$

and the general solution is

$$\varphi_{A'_1 \ldots A'_{2h}} = \left(\frac{2}{\Omega |x-y|^2} \right)^{a+b+1} \mathcal{A}_{(A'_1} \cdots \mathcal{A}_{A'_b} \mathcal{B}_{A'_{b+1}} \cdots \mathcal{B}_{A'_{2h})}. \tag{4.15}$$

The spinor $\mathcal{A}_{A'}$ is the Robinson field of the twistor A_α, and similarly $\mathcal{B}_{A'}$ is the corresponding Robinson field for the twistor B_β. Thus the hopfions of different algebraic type are characterized by two quantities a and b, with $2h = a + b$.

We now see that the null EM and null (type N) gravitational hopfions which we studied in [3] are elementary hopfions with $a = 0$. In fact all null hopfions take this form, as seen from the expression in Eq. (4.15) where, in the case $a = 0$, the PNDs are all proportional to the flagpole of $\mathcal{A}_{A'}$ and thus completely degenerate.

[2]These functions, first introduced by Penrose [5], are a subset of the elementary states of twistor theory and play a fundamental role in solving problems in twistor space. For more discussion on the elementary states, see Refs. [78, 81, 82].

4.3 Electromagnetic Hopfions

Before we consider the gravitational hopfions, we give a brief overview of the electromagnetic hopfions. The EM solutions are characterized by $h = 1$, and thus have two distinct classifications: fields with two degenerate PNDs (null) and fields with two distinct PNDs (non-null). The null solution, originally due to Rañada [41], can be derived using the pullback of the area element on $\mathbb{S}^2$ via the Hopf map to generate a field configuration with linked field lines. The alternative derivation using twistor generating functions presented here can be used to generalize the null solution to fields of different algebraic type based on the same topological structure. We use this method to construct the non-null EM hopfion and explore its properties.

4.3.1 Null EM Hopfion

The null EM hopfion $\varphi_{A'B'} \sim \mathcal{A}_{A'}\mathcal{A}_{B'}$ is the simplest example of a hopfion, and exhibits a topologically non-trivial field line structure which is preserved as time evolves. It is a solution to the vacuum Maxwell equations such that any two field lines are closed and linked exactly once [94, 42]. When an EM hopfion is decomposed onto hyperplanes of constant time there always exists a hyperplane wherein the electric, magnetic, or Poynting vector fields are tangent to the fibers of three orthogonal Hopf fibrations [95]. The integral curves of the transport

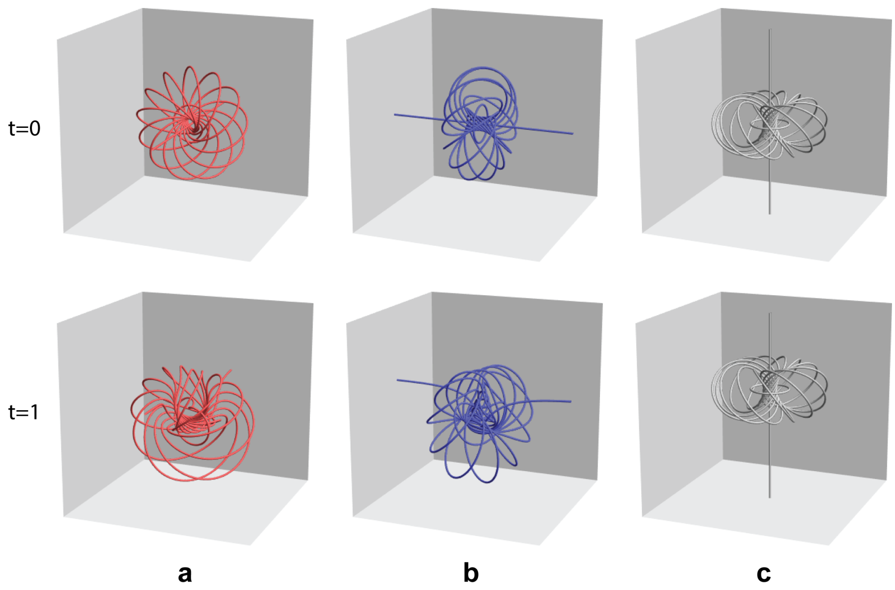

Figure 4.1: Field line structure of the null EM hopfion: (**a**) the electric field, (**b**) the magnetic field, and (**c**) the Poynting vector. Row 1 shows the fields at $t = 0$ are tangent to three orthogonal Hopf fibrations. The second row shows the $t = 1$ configuration, where the electric and magnetic fields have deformed, but the Poynting vector is still everywhere tangent to a Hopf fibration and is propagating at the speed of light.

velocity, defined as the ratio of the Poynting vector to the electromagnetic energy density, do not deform but translate at the speed of light, thus preserving the topological structure of the field. As time evolves, the electric and magnetic fields deform, but the topology is conserved, so that the field lines still lie on the surfaces of nested tori and have linking number one. The field line structure is illustrated in Figure 4.1.

To construct these fields from Eq. (4.8), we must first make a choice of the dual twistors A and B. The dual twistors

$$\begin{aligned} A_\alpha &= (0, \sqrt{2}, 0, 1) \\ B_\alpha &= (\sqrt{2}, 0, 1, 0) \end{aligned}$$

correspond to Robinson congruences oriented in the $+z$ and $-z$ directions, respectively. The same choice of A_α and B_α is used in conjunction with Eq. (4.15) to construct the hopfion fields of different algebraic type.

The expressions for the electric and magnetic fields are complicated, but the solution takes a simple form when written as a Riemann-Silberstein (RS) vector

$$\mathbf{F}_{\text{null}} = \mathbf{E} + i\mathbf{B} \tag{4.16}$$

$$= \frac{4}{\pi(-(t-i)^2 + r^2)^3} \begin{pmatrix} (x-iz)^2 - (t-i+y)^2 \\ 2(x-iz)(t-i+y) \\ i(x-iz)^2 + i(t-i+y)^2 \end{pmatrix}. \tag{4.17}$$

with $r^2 = x^2 + y^2 + z^2$. We will analyze the gravitational solutions in terms of the properties of the EM hopfions, such as the energy density u and Poynting vector

S. For the null case, these are given by

$$u_{\text{null}} = \frac{1}{(d(x))^3}(1 + x^2 + y^2 + (t-z)^2)^2, \tag{4.18}$$

$$\mathbf{S}_{\text{null}} = \frac{1}{(d(x))^3}(1 + x^2 + y^2 + (t-z)^2)\begin{pmatrix} 2(x(t-z)+y) \\ 2(y(t-z)-x) \\ x^2+y^2-(t-z)^2-1 \end{pmatrix}. \tag{4.19}$$

The term $d(x) = (1 + 2(t^2 + r^2) + (t^2 - r^2)^2)$ is a function related to the energy distribution of the field that appears in the expressions for all the hopfions. The overall scalar factor describes one of the quintessential features of these topological structures, namely that the energy density is concentrated at the center and falls off rapidly in the outward radial direction.

4.3.2 Non-null EM Hopfion

The non-null EM hopfion $\varphi_{A'B'} \sim \mathcal{A}_{(A'}\mathcal{B}_{B')}$ can also be neatly expressed with a RS vector,

$$\mathbf{F}_{\text{non-null}} = \frac{2}{(-(t-i)^2 + r^2)^3}\begin{pmatrix} -2(xz - iy(t-i)) \\ -2(yz + ix(t-i)) \\ (t-i)^2 + x^2 + y^2 - z^2 \end{pmatrix}. \tag{4.20}$$

At $t = 0$ the $\mathbf{E}$ field for this solution is everywhere tangent to a Hopf fibration, while $\mathbf{B}$ and hence $\mathbf{S}$ are identically zero since the RS vector is purely real. For $t \neq 0$, $\mathbf{B} \neq 0$ and the field line topologies for $\mathbf{E}$ and $\mathbf{B}$ are not preserved since $\mathbf{E} \cdot \mathbf{B} \neq 0$, however the fields are still finite energy. The field line structure of $\mathbf{S}$ shows that the solution does not propagate, but rather energy flows outward from the center of the configuration. These results are collected in Figure 4.2.

There are several known non-null EM configurations based on the Hopf map that are related to the fields in Eq. (4.20). A solution found using the pullback method of Rañada is given in Ref. [10]. It is dual to fields described here, so that $\mathbf{F}' \to -i\mathbf{F}_{\text{non-null}}$. Another configuration was constructed by Kiehn [96], which is related to Eq. (4.20) by a transformation involving a complex shift in time and a global phase of $\pi/4$, given by

$$\mathbf{F}' \to \tfrac{1}{\sqrt{2}} e^{i\pi/4} \mathbf{F}_{\text{non-null}} \Big|_{t \to t+i} . \tag{4.21}$$

The non-null EM hopfion has the following expressions for the energy density and Poynting vector

$$u_{\text{non-null}} = \frac{2}{(d(x))^3}(t^4 + 2t^2(1 + 3r^2 - 4z^2) + (1 + r^2)^2), \tag{4.22}$$

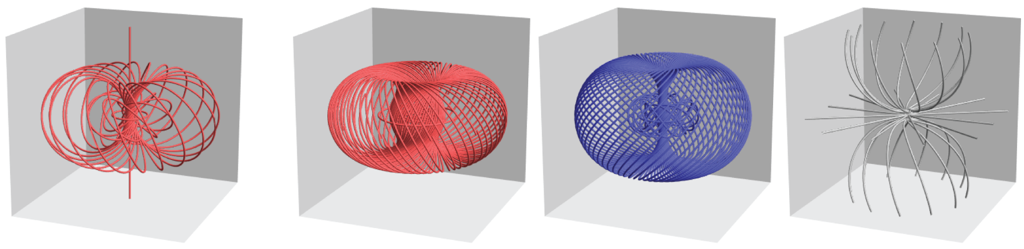

Figure 4.2: Field line structure of the non-null EM hopfion. (**a**) The electric field lines at $t = 0$ have the structure of the Hopf fibration. Not shown are $\mathbf{B}$ and $\mathbf{S}$, which are both identically zero at $t = 0$. For $t = 1$, (**b**) the electric field (**c**) the magnetic field, and (**d**) the Poynting vector.

$$\mathbf{S}_{\text{non-null}} = \frac{4t}{(d(x))^3} \begin{pmatrix} x^3 + 2yz + x(1 + t^2 + y^2 - z^2) \\ y^3 - 2xz + y(1 + t^2 + x^2 - z^2) \\ 2z(x^2 + y^2) \end{pmatrix} . \tag{4.23}$$

We will see in the next section that these are related to the analogous gravitational hopfion fields.

4.4 Gravitational Hopfions

The gravitational hopfions are characterized by $h = 2$, so there are a total of five distinct non-trivial gravitational hopfions, given by the Petrov types N, D, III, II, and I which classify the degeneracies of the PNDs. Here we review the type N, then present the type III and type D hopfions and analyze their structure using the GEM formalism.

4.4.1 Review of GEM Tidal Tensors

When analyzing the gravitational hopfions, it is useful to employ the gravito-electromagnetic (GEM) formalism, especially in cases where one can use this analogy to extract useful information about a solution from its electromagnetic counterpart.

For spin-1, the Penrose transform will generate a solution to the source-free field equation

$$\nabla^{AA'}\varphi_{A'B'} = 0 \tag{4.24}$$

from which we construct the field strength tensor[3] by

$$\begin{aligned} F_{A'B'AB} &= \varphi_{A'B'}\epsilon_{AB} + c.c. \\ F_{ab} &= F_{A'B'AB}\sigma_a^{AA'}\sigma_b^{BB'}. \end{aligned} \tag{4.25}$$

For an observer at rest, we can decompose this into the standard electric and magnetic fields

$$E_b = F_{b0}, \tag{4.26}$$

$$B_b = - * F_{b0} \tag{4.27}$$

[3]We use the conventions of Penrose as in Ref. [12], so that $x^a \leftrightarrow x^{AA'} \equiv x^a\sigma_a^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix}$.

The integral curves of E_b and B_b are the electromagnetic field lines.

For spin-2, the source-free field equation and Weyl curvature tensor are

$$\nabla^{AA'}\varphi_{A'B'C'D'} = 0,$$

$$C_{A'B'C'D'ABCD} = \varphi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + c.c.$$

$$C_{abcd} = C_{A'B'C'D'ABCD}\sigma_a^{AA'}\cdots\sigma_d^{DD'}. \tag{4.28}$$

In direct analogy with the decomposition of the electromagnetic field, the Weyl tensor C_{abcd} can be decomposed into an even-parity “electric" tensor E_{ij} called the tidal field and an odd-parity “magnetic" tensor B_{ij} called the frame-drag field [15]. For an observer at rest, these are

$$E_{ij} = C_{i0j0} \tag{4.29}$$

$$B_{ij} = - * C_{i0j0}. \tag{4.30}$$

These tensors are symmetric and traceless, and are thus characterized entirely by their eigensystem configuration. The GE tensor has eigenvectors whose integral curves define the tendex lines and the eigenvalues E_ℓ are the magnitude of the tidal acceleration along these lines, where the relative acceleration over a small spatial separation ℓ is given by

$$\Delta a = -E_\ell \ell.$$

Positive (negative) eigenvalues correspond to a compressive (stretching) force. The GM tensor has eigenvectors whose integral curves define the vortex lines and their eigenvalues B_ℓ are the magnitude of the gyroscope precession about the vortex lines

$$\Delta\Omega = B_\ell \ell.$$

Positive (negative) eigenvalues correspond to a clockwise (counter-clockwise) precession. The tendex and vortex lines are the gravitational lines of force for an observer at rest, which represent the analog of EM field lines.

In the GEM formalism, there are two local duality invariants analogous to the energy density and the Poynting vector in electromagnetism [19]. These are the super-energy density U and the super-Poynting vector $\mathbf{P}$, given by

$$U = \frac{1}{2}(E_{ij}E^{ij} + B_{ij}B^{ij}) \tag{4.31}$$

$$P_i = \varepsilon_{ijk} E^j{}_l B^{kl}. \tag{4.32}$$

4.4.2 Type N GEM Hopfion

The type N gravitational hopfion, previously studied in [3], has the form $\varphi_{A'B'C'D'} \sim \mathcal{A}_{A'}\mathcal{A}_{B'}\mathcal{A}_{C'}\mathcal{A}_{D'}$ and provides a good starting point for discussing the use of GEM in studying hopfions. For the type N hopfions, the Weyl decomposition has eigenvalues for both the GE and GM fields that take the form $\{\lambda_-, \lambda_0, \lambda_+\}$, with $\lambda_-(x) \leq \lambda_0(x) \leq \lambda_+(x)$ for all points x in space-time. We will label the eigenvectors $\{\mathbf{e}_-, \mathbf{e}_0, \mathbf{e}_+\}$ and $\{\mathbf{b}_-, \mathbf{b}_0, \mathbf{b}_+\}$ corresponding to the eigenvalues for the tidal and frame-drag fields respectively. The type N fields are purely radiative, so the eigenvalues take the simple form $\{-\Lambda, 0, \Lambda\}$ where $\Lambda(x)$ is a function on space-time.

The eigenvectors $\mathbf{e}_0$ and $\mathbf{b}_0$ are both equivalent to the Poynting vector in Eq. (4.19) for the null EM hopfion, up to an overall scalar. For the remaining eigenvector fields, we can construct RS vectors $\mathbf{f}_e = \mathbf{e}_- + i\mathbf{e}_+$ and $\mathbf{f}_b = \mathbf{b}_- + i\mathbf{b}_+$ which are related to the RS vector of the null EM hopfion from Eq. (4.16) by

$$\mathbf{f}_e = e^{i\pi/4}\mathbf{f}_b \tag{4.33}$$

$$= e^{i\,\mathrm{Arg}\,\theta}\mathbf{F}_{\mathrm{null}} \tag{4.34}$$

where

$$\theta(x) = \sqrt{-(t-i)^2 + r^2}. \tag{4.35}$$

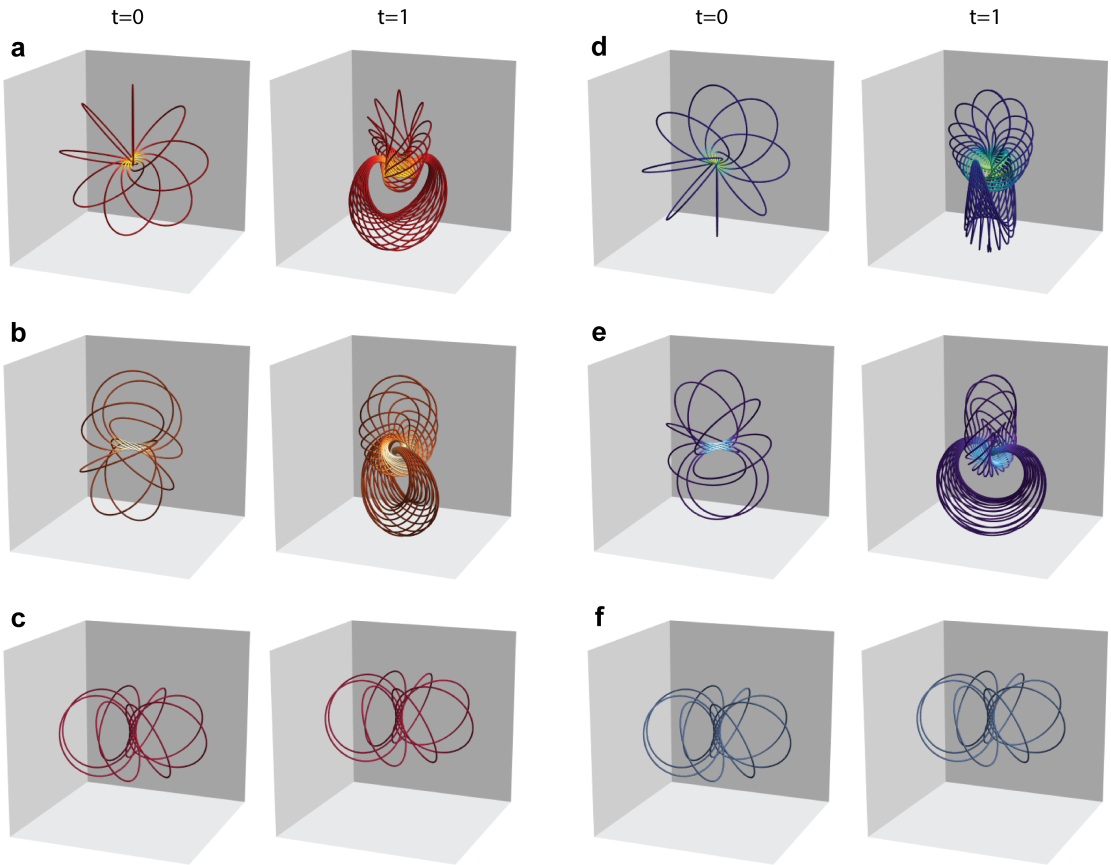

Figure 4.3: The type N gravitational hopfion at $t = 0$ and $t = 1$: the tidal fields (**a**) $\mathbf{e}_-$, (**b**) $\mathbf{e}_+$, and (**c**) $\mathbf{e}_0$; and the frame-drag fields (**d**) $\mathbf{b}_-$, (**e**) $\mathbf{b}_+$, and (**f**) $\mathbf{b}_0$. The field lines are colored by the relative magnitude of their eigenvalues, with lighter colors indicating greater magnitude.

The tendex and vortex lines have been plotted numerically in Figure 4.3.

The super-energy and super-Poynting vector for this field are related to the duality invariants of the null EM hopfion by

$$U_{\mathrm{N}} = \frac{1}{2} d(x) u_{\mathrm{null}}^2, \tag{4.36}$$

$$\mathbf{P}_{\mathrm{N}} = \frac{1}{2} d(x) u_{\mathrm{null}} \mathbf{S}_{\mathrm{null}}. \tag{4.37}$$

Thus, we find the same Hopf structure that propagates at the speed of light. The surfaces of constant energy are concentric spheres, as in the spin-1 case, but the magnitude drops off more quickly in the radial direction.

4.4.3 Type D GEM Hopfion

The type D gravitational hopfion $\varphi_{A'B'C'D'} \sim \mathcal{A}_{(A'}\mathcal{A}_{B'}\mathcal{B}_{C'}\mathcal{B}_{D')}$ is the gravitational analog of the non-null EM hopfion, in that its PNDs are split evenly into two sets. For spin-2, the two sets consist of pairs of doubly degenerate PNDs. This hopfion has eigenvalue structure $\{2\Lambda, -\Lambda + \lambda, -\Lambda - \lambda\}$, where $\lambda = 0$ at $t = 0$ simplifying the eigenvalue structure to $\{2\Lambda, -\Lambda, -\Lambda\}$. Note that $\Lambda(x)$ used here represents a different function than the $\Lambda(x)$ used before to describe the type N hopfion; we use the symbol only to describe the overall structure of the eigenvalues. This eigenvalue configuration is interesting because at $t = 0$ the eigenvalues $-\Lambda \pm \lambda$ coincide, so their eigenvectors collapse into a doubly degenerate eigenspace. Furthermore, at $t = 0$, the GE field $\mathbf{e}_{2\Lambda}$ is exactly tangent to a Hopf fibration and the frame-drag field vanishes, hence the GM eigenvalues and eigenvectors vanish as well. The values of Λ and λ are rather complicated, so we will not present them here. The tendex and vortex lines have been plotted numerically in Figure 4.4.

The expressions for the super-energy and super-Poynting vector of the type D hopfion are quite long, but they take a simpler form when written in terms of the

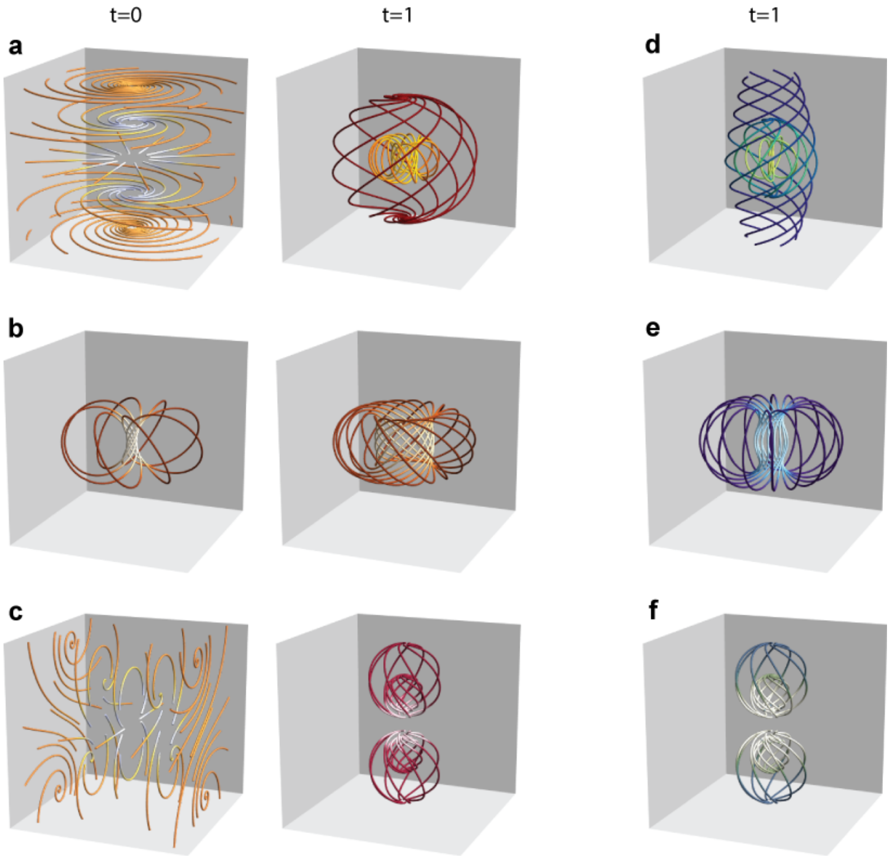

Figure 4.4: The type D gravitational hopfion at $t = 0$ and $t = 1$: the tidal fields (**a**) $e_{-\Lambda+\lambda}$, (**b**) $e_{2\Lambda}$, and (**c**) $e_{-\Lambda-\lambda}$; and the frame-drag fields (**d**) $b_{-\Lambda+\lambda}$, (**e**) $b_{2\Lambda}$, and (**f**) $b_{-\Lambda-\lambda}$. The frame-drag fields at $t = 0$ are omitted because they are all vanishing then. As before the field lines are colored by the relative magnitude of their eigenvalues, with lighter colors indicating greater magnitude. The fields at $t = 0$ in (**a**) and (**c**) are presented with the same color scheme to convey the fact that they really represent a degenerate eigenspace together.

duality invariants of the non-null EM hopfion

$$U_{\text{D}} = \frac{1}{48} d(x) g_{\text{D}}(x) u_{\text{non-null}}^2 \tag{4.38}$$

where

$$g_{\mathrm{D}}(x) = \frac{(t^4 + 2t^2(1 + 5r^2 - 6z^2) + (1 + r^2)^2)^2 - 48t^4(x^2 + y^2)^2}{(t^4 + 2t^2(1 + 3r^2 - 4z^2) + (1 + r^2)^2)^2} \tag{4.39}$$

and

$$\mathbf{P}_{\mathrm{D}} = \frac{1}{32} d(x) u_{\text{non-null}} \mathbf{S}_{\text{non-null}}. \tag{4.40}$$

Similarly to the non-null EM case, the super-Poynting vector indicates that the field configuration radiates energy outward from the center in all directions, but the overall structure does not propagate.

4.4.4 Type III GEM Hopfion

The type III gravitational hopfion $\varphi_{A'B'C'D'} \sim \mathcal{A}_{(A'}\mathcal{A}_{B'}\mathcal{A}_{C'}\mathcal{B}_{D')}$ has one set of triply degenerate PNDs and one unique PND. This hopfion has eigenvalue structure $\{\lambda_-, \lambda_0, \lambda_+\} = \{-\Lambda, \lambda, \Lambda - \lambda\}$, where $\lambda = 0$ at $t = 0$. We again note that the functions $\Lambda(x)$ and $\lambda(x)$ used here represent different functions than those used to describe the type N and type D hopfions. At $t = 0$, both the GE and GM fields are tangent to three orthogonal Hopf fibrations, but with different orientations than the type N configuration. For type III, the eigenvectors $\mathbf{e}_0$ and $\mathbf{b}_0$ are not aligned with the super-Poynting vector, but rather are orthogonal to

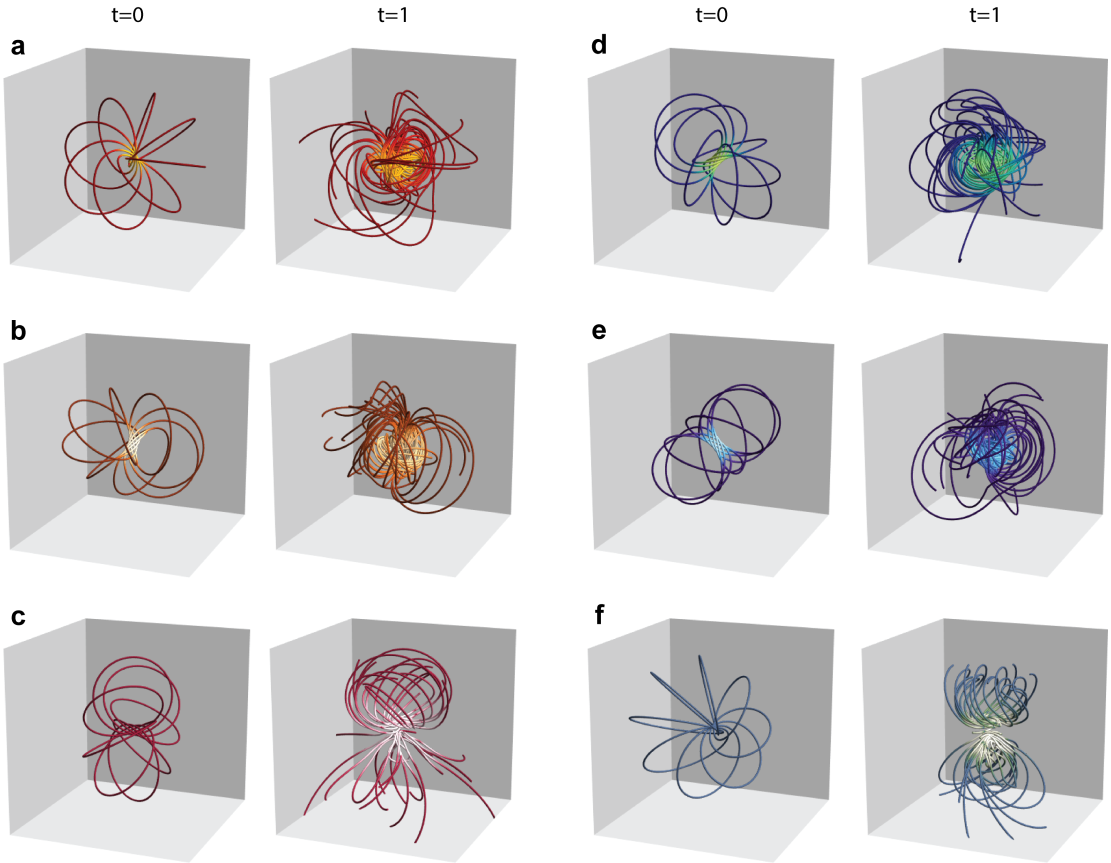

Figure 4.5: The type III gravitational hopfion at $t = 0$ and $t = 1$: the tidal fields (**a**) $\mathbf{e}_-$, (**b**) $\mathbf{e}_+$, and (**c**) $\mathbf{e}_0$; and the frame-drag fields (**d**) $\mathbf{b}_-$, (**e**) $\mathbf{b}_+$, and (**f**) $\mathbf{b}_0$. The field lines are colored by the relative magnitude of their eigenvalues, with lighter colors indicating greater magnitude.

it (and each other). The tendex and vortex lines have been plotted numerically in Figure 4.5.

Comparing the type III hopfion to the EM hopfions, we see that the two sets of local duality invariants are similar. The super-energy is related by

$$U_{\text{III}} = \frac{1}{16} d(x) g_{\text{III}}(x) u_{\text{null}} u_{\text{non-null}} \tag{4.41}$$

where

$$g_{\text{III}}(x) = \frac{(t^4 + 2t^2(1 + 7r^2 - 8z^2) + (1 + r^2)^2)}{(t^4 + 2t^2(1 + 3r^2 - 4z^2) + (1 + r^2)^2)}. \tag{4.42}$$

The super-Poynting vector can be written in terms of two vector terms that are exactly the same as the vector terms in the Poynting vectors of the EM fields, from Eqs. (4.19) and (4.23), up to the overall scalar factors

$$\begin{aligned}\mathbf{P}_{\text{III}} = \frac{1}{32} d(x) g_{\text{III}}(x) u_{\text{non-null}} \mathbf{S}_{\text{null}} \\ + \frac{1}{16} d(x) u_{\text{null}} \mathbf{S}_{\text{non-null}}.\end{aligned} \tag{4.43}$$

Thus we see the field is comprised of two distinct structures. The first term in Eq. (4.43) corresponds to a component of the field that propagates at the speed of light, which is proportional to the Poynting vector of the null EM hopfion from Figure 4.1(c). The second term is a component that radiates energy outward from the center, which is proportional to the Poynting vector of the non-null EM hopfion from Figure 4.2(d). The two terms combined create the configuration in 4.5, so that as time evolves the linked structure propagates in the $+\hat{z}$-direction, leaving open field lines radiating energy in the $-\hat{z}$-direction. This can be seen by comparing the visualizations of the EM Poynting vectors to the type III gravity fields in Figures 4.5(c) and 4.5(f).

4.4.5 Type II and Type I Fields

Finally, we briefly mention the type II and type I fields. It is not possible to generate algebraically special fields of these types from a twistor function of the form in Eq. (4.14). Type II fields contain three distinct PNDs, therefore one must introduce a $(C \cdot Z)$ term into Eq. (4.14), where the twistor C_γ has the associated spinor field $\mathcal{C}_{A'}$ which becomes one PND. However, when you apply Cauchy's integral theorem to the contour integral the derivative requires you use the product rule, thus the result includes multiple terms. The solution is then a linear combination of different type II fields. A similar situation arises for type I.

4.5 Conclusion

The beauty of the Penrose transform lies in its complex contour integral nature, which allows for the application of Cauchy's theorem to bring out the spinor structure of solutions in $\mathbb{M}$. We used this method to modify the generating functions corresponding to the null EM and type N GEM hopfions and construct a class of spin-h fields, including the non-null electromagnetic, type D and type III gravitational hopfions. The gravito-electromagnetic formalism was used to characterize the tendex and vortex structure of the gravitational hopfions and show that the linked configuration of the Hopf fibration appears in the lines of force.

The fields based on the Hopf fibration studied here represent some of the most basic topological structures found in continuous, space-filling configurations. The methods we have presented could potentially be extended to the construction of classical electromagnetic and gravitational fields based on more intricate topological structures. For example, the radiative hopfions - the null electromagnetic and type N gravitational hopfions - have been shown to be the simplest case in a class of solutions with field line structures based on torus knots [25,97]. After the torus knots, the twist knots are considered to be the next simplest class of knots [98], and have already been observed in other areas of physics such as polymer materials [99,100], DNA organization [101,102], and quantum field theory [103,104]. Identifying new field configurations and studying their properties could open new physical applications, and deepening our understanding of field line topology gives us insight into the structure and dynamics of physical systems.

4.6 Appendix: Penrose Transform for Non-null Hopfions

When the Penrose transform is written as an integral over a $\mathbb{CP}^1$ coordinate, the application of Cauchy's theorem to generating functions with poles of order greater than one results in derivative terms which break the degeneracy of the

PNDs. Here we show the Penrose transform for the non-null spin-1 case, but the type D and type III spin-2 calculations follow in a similar manner. The calculation for the null and type N fields is given in Ref. [3].

We will calculate the Penrose transform

$$\phi(X)_{A'B'} = \frac{1}{2\pi i}\oint_{\Gamma} \pi_{A'}\pi_{B'}F(Z)\pi_{B'}d\pi^{B'}$$

with $f(Z)$ given by Eq. (4.9) with $h = 1$

$$F(Z) = \frac{1}{(A_\alpha Z^\alpha)^2(B_\beta Z^\beta)^2}.$$

Let A_α and B_α be dual twistors associated to the spinor fields $\mathcal{A}^{A'}$ and $\mathcal{B}^{B'}$ according to

$$\begin{aligned} A_\alpha Z^\alpha &= iA_A x^{AA'}\pi_{A'} + A^{A'}\pi_{A'} \\ &\equiv \mathcal{A}^{A'}\pi_{A'} \\ B_\beta Z^\beta &= iB_B x^{BB'}\pi_{B'} + B^{B'}\pi_{B'} \\ &\equiv \mathcal{B}^{A'}\pi_{B'}, \end{aligned}$$

The measure can be written in the form

$$\pi_{C'}d\pi^{C'} = (\pi_{0'})^2 d\zeta.$$

We also have the relations

$$\frac{1}{\pi_{0'}}\mathcal{A}^{A'}\pi_{A'} = \mathcal{A}^{0'} + \mathcal{A}^{1'}\zeta,$$
$$\frac{1}{\pi_{0'}}\mathcal{B}^{A'}\pi_{A'} = \mathcal{B}^{0'} + \mathcal{B}^{1'}\zeta.$$

Introducing the canonical spin bases $\{o_{A'}, \iota_{A'}\}$ into the primed spin space S' we have that

$$\begin{aligned}\pi_{A'} &= \pi_{0'}o_{A'} + \pi_{1'}\iota_{A'} \\ &= \pi_{0'}(o_{A'} + (\frac{\pi_{1'}}{\pi_{0'}})\iota_{A'}) \\ &= \pi_{0'}(o_{A'} + \zeta\iota_{A'}). \end{aligned} \tag{4.44}$$

Thus

$$\begin{aligned}\varphi_{A'B'}(x) &= \frac{1}{2\pi i}\oint_\Gamma \frac{(o_{A'} + (\frac{\pi_{1'}}{\pi_{0'}})\iota_{A'})(o_{B'} + (\frac{\pi_{1'}}{\pi_{0'}})\iota_{B'})}{(\mathcal{A}^{0'} + \mathcal{A}^{1'}(\frac{\pi_{1'}}{\pi_{0'}}))^2(\mathcal{B}^{0'} + \mathcal{B}^{1'}(\frac{\pi_{1'}}{\pi_{0'}}))^2} d(\frac{\pi_{1'}}{\pi_{0'}}) \\ &= \frac{1}{2\pi i(\mathcal{A}^{1'})^2(\mathcal{B}^{1'})^2}\oint_\Gamma \frac{(o_{A'} + \zeta\iota_{A'})(o_{B'} + \zeta\iota_{B'})}{(\mu+\zeta)^2(\nu+\zeta)^2} d\zeta \end{aligned} \tag{4.45}$$

where $\zeta = \pi_{1'}/\pi_{0'}$, $\mu = \mathcal{A}^{0'}/\mathcal{A}^{1'}$, and $\nu = \mathcal{B}^{0'}/\mathcal{B}^{1'}$ represent the projective coordinate and poles respectively.

After the variable substitutions, the integral is straightforward. Taking the contour Γ to enclose the pole $-\mu$ and integrating each of the above terms in turn we arrive at

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_\Gamma o_{A'}o_{B'}\frac{d\zeta}{(\mu+\zeta)^2(\nu+\zeta)^2} &= o_{A'}o_{B'}\frac{2}{(\mu-\nu)^3}\\
\frac{1}{2\pi i}\oint_\Gamma o_{A'}\imath_{B'}\frac{\zeta d\zeta}{(\mu+\zeta)^2(\nu+\zeta)^2} &= o_{A'}\imath_{B'}\frac{-(\mu+\nu)}{(\mu-\nu)^3}\\
\frac{1}{2\pi i}\oint_\Gamma o_{B'}\imath_{A'}\frac{\zeta d\zeta}{(\mu+\zeta)^2(\nu+\zeta)^2} &= o_{B'}\imath_{A'}\frac{-(\mu+\nu)}{(\mu-\nu)^3}\\
\frac{1}{2\pi i}\oint_\Gamma \imath_{B'}\imath_{A'}\frac{\zeta^2 d\zeta}{(\mu+\zeta)^2(\nu+\zeta)^2} &= \imath_{B'}\imath_{A'}\frac{2\mu\nu}{(\mu-\nu)^3},
\end{aligned}
$$

which we reassemble to give the spinor field $\phi_{A'B'}$ as

$$
\begin{aligned}
\phi_{A'B'} = &\frac{1}{\left(\mathcal{A}^{1'}\right)^2\left(\mathcal{B}^{1'}\right)^2}\frac{1}{(\mu-\nu)^3}(2o_{A'}o_{B'}-(\mu+\nu)(o_{A'}\imath_{B'}+o_{B'}\imath_{A'})+2\mu\nu\imath_{B'}\imath_{A'})\\
=&\frac{(\mathcal{A}^{1'})(\mathcal{B}^{1'})}{(\varepsilon_{A'B'}\mathcal{A}^{A'}\mathcal{B}^{B'})^3}(2o_{A'}o_{B'}-(\mu+\nu)(o_{A'}\imath_{B'}+o_{B'}\imath_{A'})+2\mu\nu\imath_{B'}\imath_{A'})\\
=&\frac{(\mathcal{A}^{1'})(\mathcal{B}^{1'})}{(A_AB^A(x^a-y^a)(x_a-y_a))^3}(2o_{A'}o_{B'}-(\mu+\nu)(o_{A'}\imath_{B'}+o_{B'}\imath_{A'})+2\mu\nu\imath_{B'}\imath_{A'})\\
=&\frac{2}{(A_AB^A(|x-y|^2)^3}(\mathcal{A}^{1'}\mathcal{B}^{1'}o_{A'}o_{B'}-\tfrac{1}{2}(\mathcal{A}^{0'}\mathcal{B}^{1'}+\mathcal{A}^{1'}\mathcal{B}^{0'})(o_{A'}\imath_{B'}+o_{B'}\imath_{A'})+\mathcal{A}^{0'}\mathcal{B}^{0'}\imath_{B'}\imath_{A'})\\
=&\frac{2}{(A_AB^A(|x-y|^2)^3}(\mathcal{A}_{0'}\mathcal{B}_{0'}o_{A'}o_{B'}+\tfrac{1}{2}(\mathcal{A}_{1'}\mathcal{B}_{0'}+\mathcal{A}_{0'}\mathcal{B}_{1'})(o_{A'}\imath_{B'}+o_{B'}\imath_{A'})+\mathcal{A}_{1'}\mathcal{B}_{1'}\imath_{B'}\imath_{A'})\\
=&\frac{2}{(\Omega|x-y|^2)^3}\mathcal{A}_{(A'}\mathcal{B}_{B')}
\end{aligned}
$$

where $\Omega = A_AB^A$ is a constant scalar and the point y is given by

$$
y^{AA'} = i\frac{A^{A'}B^A - B^{A'}A^A}{A_BB^B}. \tag{4.46}
$$

Chapter 5

Gravitational Shockwaves on Rotating Black Holes

5.1 Motivation

Black holes are thermodynamic systems whose microscopic description we still do not understand. After the original work on black hole thermodynamics by Christodoulou [105], Penrose and Floyd [106], Carter [107], Bekenstein [108], and Bardeen, Carter, and Hawking [109], Hawking justified the analogy between the surface gravity α and a temperature T by predicting that an isolated black hole will radiate as a black body at the expected temperature $T = \frac{\alpha}{2\pi}$ [110,111]. About 20 years later, Strominger and Vafa vindicated the analogy between the horizon area A and an entropy S by enumerating microstates in string theory to derive the expected result $S = \frac{1}{4}A$ for extremal black holes in $4+1$ dimensions [112].

We will not recount the subsequent history of microstate counting. Suffice it to say that the calculations from string theory, while eminently laudable, are restricted to black holes near extremality and may not provide enough insight into the statistical mechanics behind the conventional black holes of general relativity for generic values of their parameters. It would be helpful to establish a complementary strategy for black hole statistical mechanics tailored to an expansion around the Schwarzschild solution.

One such alternative is the S-matrix approach of 't Hooft [30, 31]. Motivated by this and by Shenker and Stanford's investigation of the butterfly effect [32, 33], Kitaev recently proposed a quantum field theory in $0+1$ dimensions [113] whose low-energy effective action is that of dilaton gravity[1] in $1+1$ dimensions [115, 116]. Details of this model were explored further by Maldacena and Stanford [117]. Since the equations of motion derived from the effective action admit the AdS_2 black hole as a solution [118], we now have an explicit statistical mechanical model of black hole thermodynamics.

Kitaev's calculation demonstrates, for the first time, that the thermodynamic limit of a quantum mechanical model[2] can produce a bona fide black hole horizon,

[1]For a reference on black holes in dilaton gravity in $1+1$ dimensions, see the work by Callan, Giddings, Harvey, and Strominger [114].

[2]As remarked by Witten, "the average of a quantum system over quenched disorder is not really a quantum system" [119]. Strictly speaking it is only a quantum mechanical model if the average captures the physics of a single realization with fixed couplings $J_{jk\ell m}$. We thank Yonah Lemonik for a discussion about this important point.

albeit in lower-dimensional scalar-tensor gravity, not in $(3 + 1)$-dimensional Einstein gravity. Between this calculation and Maldacena's conjecture that Type-IIB string theory is defined nonperturbatively by the partition function of $\mathscr{N} = 4$ supersymmetric Yang-Mills theory [120], the evidence is strong that Hawking's famous prediction of information loss [110, 111] points to a deficiency in general relativity rather than in quantum mechanics.[3]

The core of 't Hooft's reasoning is based not on philosophical prognostication about theories of everything but on an exact solution of Einstein's equation that describes the gravitational backreaction of a massless particle on the future horizon of a Schwarzschild black hole [29]. We will call this solution the Dray-'t Hooft gravitational shockwave.[4]

[3]Raised on a wholesome diet of statistics, quantum mechanics, and field theory, we espouse the view that when a classical theory conflicts with a quantum theory we simply jettison the classical theory. There is no information "paradox" in the sense of a contradiction, there is only our inability to coarse grain correctly. We agree with Mathur that this proclamation does not solve anything, since "[y]ou are welcome to do your analysis in either the CFT or the gravity theory, but at the end you must show me what happens when a black hole forms and evaporates in the gravity description" [121]. We further agree that "being able to compute the Page curve in some particular theory is what it means to have solved the black hole information problem," a characterization attributed (without citation) to Strominger in the review by Harlow [122]. We only mean to endorse the search for models in which quantum mechanics and special relativity coexist unadulterated and whose low-energy effective action is that of general relativity coupled to gauge fields and matter. From our point of view, the salient logical import of Maldacena's conjecture is that $\mathscr{N} = 4$ supersymmetric Yang-Mills theory in $3 + 1$ dimensions with gauge group $U(N)$ is such a model.

[4]This solution can be viewed on the one hand as the generalization of the Aichelburg-Sexl shockwave [28] to curved space-time, or on the other hand as an application of Penrose's "scissors-and-paste" method for gluing together known solutions of Einstein's equation to form new solutions [27].

The solution was generalized to the Reissner-Nordström (RN) black hole by Alonso and Zamorano [123] and by Sfetsos [35], who also adapted the shockwave to other static backgrounds. Kiem, Verlinde, and Verlinde [124] deployed a perturbative variant of the Dray-'t Hooft result to investigate the effect of gravitational interactions on black hole evaporation. Polchinski [125] revisited the solution to refine 't Hooft's "relation between a given black hole S-matrix element and another with an additional ingoing particle" and in the process reformulated arguments for the firewall [126, 127].[5]

In his exposition of the S-matrix framework, 't Hooft did not concern himself with more general black hole backgrounds, opining that "[c]onceptually, generalization of everything we say to these cases should be straightforward" [31]. Perhaps, but in this paper our principal ambition is to galvanize the search for a statistical mechanics[6] underlying astrophysical black holes [135], whose equilibrium field configurations are described by the Kerr geometry.[7] So if we intend to adapt

[5]As far as we can tell, the analysis of Kiem, Verlinde, and Verlinde [124] seems to corroborate, rather than refute, some of the arguments by Marolf and Polchinski [127]. But the former authors advocated *f*or black hole complementarity, while the latter authors spearheaded the consortium that famously proclaimed "complementarity is not enough" [126].

[6]It has long been suggested, on the grounds of matching absorption cross sections with two-point functions in the appropriate thermodynamic ensemble, that the microscopic model should be a conformal field theory. See the seminal works by Maldacena and Strominger [128] and Cvetič and Larsen [129], and the more recent papers by Castro, Maloney, and Strominger [130], Bredberg, Keeler, Lysov, and Strominger [131], and Hartman, Song, and Strominger [132]. We would characterize this proposal as a conjectured equivalence at the level of eigenstate thermalization [133, 134].

[7]This is not quite the most realistic situation, since a black hole found in nature is expected to be formed from the collapse of a star, while the Kerr solution describes a black hole that always was and always will be. We will acquiesce to this criticism and leave gravitational collapse for another day.

't Hooft's blueprint and Kitaev's recent insights to the microscopics of rotating black holes, then our very first preliminary step must be to generalize 't Hooft's formula for the transition amplitude.

That is what we do here: We generalize the Dray-'t Hooft gravitational shockwave to the Kerr-Newman background, which is the most general asymptotically flat black hole in four space-time dimensions. Readers who know about the recent work on gravitational shockwaves and are intimately familiar with the method of spin coefficients could skip to our metric ansatz described by Eqs. (5.119) and (5.125), and then to our main result: the Ricci tensor in Eq. (5.217), the Ricci scalar Φ_{22} in Eq. (5.225), and the differential operator in Eq. (5.226). We acknowledge that this provides only the most tentative intimation toward a microscopic theory of the Kerr-Newman space-time, but it is a new exact solution of Einstein's equation and therefore deserves to be studied in its own right.

We should say that at the late stages of our calculation[8] we learned that Balasin generalized the Ricci tensor for the Dray-'t Hooft solution with the express aim of including rotation in the formalism [136]. But he did not perform the explicit calculation for the Kerr-Newman family of backgrounds, stating only that "it would be interesting to apply it to a rotating, i.e. Kerr black hole" and that "[w]ork in this direction is currently in progress." Similar comments were made by

[8]We found Balasin's paper after we had already computed the Ricci tensor but before we managed to express it in the relatively compact and geometrical form described by Eqs. (5.217), (5.225), and (5.226).

Alonso and Zamorano [123] and by Taub [137]. We have not found later articles by any of these authors that contain our results.

5.2 The Kerr-Newman black hole

To establish our conventions and provide the necessary context for our result, we will first review the Kerr-Newman black hole using the method of spin coefficients. This method was introduced by Newman and Penrose (NP) [138] and later refined into a "compacted" version by Geroch, Held, and Penrose (GHP) [139].[9] Since we would like our work to be accessible to field theorists who do not necessarily live and breathe general relativity, we will present explicit formulas and concrete calculations instead of intuitive explanations that are often designed more to impress than to educate.

[9] Perhaps other theorists could have performed our calculation using ordinary tensor calculus, but for us the formalism was indispensable. We presume that it is not taught in the standard graduate curricula only because of its formidable notation and its wanting presentation in the textbooks that bother to use it. Even among seasoned relativists, it seems that the GHP incarnation has largely fallen by the wayside. While the original NP paper may be difficult to parse by modern pedagogical standards, the original GHP paper is, despite some lingering notational deformities, clear and enlightening. We highly recommend it to any student interested in the method of spin coefficients.

5.2.1 Null tetrad

Our description of the space-time will begin with a set of "frame fields"[10]

$$\{e_\mu^a(x)\}_{a=1}^4 \,, \tag{5.1}$$

collectively called a "frame," "tetrad," or "vierbein." The frame is designed to bring the metric tensor $g_{\mu\nu}$ to its locally flat form η_{ab} at each point in space-time:

$$g_{\mu\nu}(x) \equiv \eta_{ab}\, e_\mu^a(x)\, e_\nu^b(x) \;. \tag{5.2}$$

The essential insight of Newman and Penrose [138] was to recognize that in general relativity we often care about the local lightcone structure of the space-time, so it may be geometrically advantageous to work with a *n*ull tetrad instead of the usual orthonormal one. Motivated by this and by spinorial considerations of no concern to us here, they introduced a metric in the tangent space that is strictly

[10] Those who refer to 1-forms as "co-vectors" might prefer to reserve the terminology "frame fields" for the quantities e_a^μ that define the inverse metric, in which case the quantities e_μ^a that define the metric could be called "co-frame fields." We prefer to use "frame fields" as a catch-all term that encompasses both objects, just as we prefer to use the symbol "e" for both e_μ^a and e_a^μ.

off-diagonal:[11]

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{5.3}$$

The notation we will use for the frame fields is[12]

$$e_\mu^{\ a} \equiv (-l'_\mu,\, -l_\mu,\, m'_\mu,\, m_\mu) \;, \tag{5.4}$$

where l_μ and l'_μ are independent and real, while m_μ is complex with $m'_\mu \equiv (m_\mu)^*$.

[11]We display this elementary definition up front for two reasons. First, one of us, raised as a particle physicist, was so used to "η_{ab}" being diagonal that it took time to readjust his raising and lowering reflexes. (Those accustomed to the lightcone are already halfway fluent but must make peace with the off-diagonal spatial block.) Second, we change metric signature with respect to the classical references (see Sec. 5.A). A desire to caution, not to be old-fashioned, is what motivated us to retain the traditional range 1-4 instead of 0-3 for indices in the tangent space [recall Eq. (5.1)]. In this section we wish to preempt potential pitfalls and will therefore provide some prophylaxis for the uninitiated. Brevity may be the soul of wit, but it is a scourge to pedagogy.

[12]This awkward convention is defined so that $e_a^{\ \mu} \equiv g^{\mu\nu}\eta_{ab}\, e_\nu^{\ b} = (l^\mu, l'^\mu, m^\mu, m'^\mu)$. This is because the traditional presentation of the spin coefficient formalism focuses on the vectors tangent to null trajectories, and it treats the 1-forms as derived by matrix inversion. For us it is more convenient to privilege the 1-forms and derive the vectors by matrix inversion. To keep the notation consistent, we are stuck with $e_\mu^{\ a} = (-l'_\mu, -l_\mu, m'_\mu, m_\mu)$. Note that lowering the tangent space index on the collection of 1-forms gives $e_{a\mu} = (l_\mu, l'_\mu, m_\mu, m'_\mu)$. Despite its geometrical obscurity, this is an extremely convenient object to work with.

Given a coordinate basis of differential 1-forms[13] dx^μ, we define a collection of frame field 1-forms[14]

$$e^a \equiv e^a_\mu \, dx^\mu \equiv (-l', \, -l, \, m', \, m) \; . \tag{5.5}$$

In terms of these, the metric is[15]

$$ds^2 = -2ll' + 2mm' \; . \tag{5.6}$$

A tactical advantage of deploying the tetrad formulation is to never have to look at a metric tensor, so we will *n*ot show ds^2 explicitly. Instead, we will only show the functional form of the tetrad. We will first use the standard "Schwarzschild-like" coordinates (t, r, θ, φ) of Boyer and Lindquist [141], which are appropriate for describing the experience of an observer far from the black hole.[16]

The three physical parameters of the black hole are its mass M, its charge[17] Q, and its angular momentum J. It is standard practice to trade J for the ratio

---

[13]It is the dx^μ, not the x^μ, that transform linearly under general coordinate transformations. See pp. 312-313 of Zee [140].

[14]The traditional notation for the form l' is n, but we prefer a notation that is manifestly covariant under the exchange of primed and unprimed quantities [we will get to this in Eq. (5.27)].

[15]Our convention is such that ordinary multiplication of differential forms denotes the symmetric tensor product: for example, $ll' \equiv \frac{1}{2}(l \otimes l' + l' \otimes l)$. When we intend the antisymmetric tensor product, we will write the wedge symbol explicitly: $l \wedge l' \equiv \frac{1}{2}(l \otimes l' - l' \otimes l)$.

[16]These coordinates are technically applicable anywhere outside the horizon, but their relation to our ordinary notions of space and time becomes obscure in the ergoregion.

[17]Some numerical factors are absorbed into Q, whose precise relation to the electric charge of a test particle depends on a choice of units. We also set $c = G = 1$.

$a \equiv J/M$. It is also both customary and convenient[18] to define the "horizon function" [142]

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2 \equiv (r - r_+)(r - r_-) \ . \tag{5.7}$$

The inner horizon $r_- \equiv M - \sqrt{M^2 - a^2 - Q^2}$ and the outer horizon $r_+ \equiv M + \sqrt{M^2 - a^2 - Q^2}$ are defined as the solutions to $\Delta = 0$. In this chapter, we will be concerned exclusively with the region $r \geq r_+$, so when we refer to "the" horizon, we will always mean the outer one.

We will also define a complex function $R(r, \theta)$ that generalizes the radial coordinate, along with the value of that function at the north pole, as follows:

$$R \equiv r + ia\cos\theta \ , \quad R_0 \equiv r + ia \ . \tag{5.8}$$

[18]It is beyond dispute that defining the horizon function is useful computationally, but its interpretation should be treated with caution. Consider the situation $a = 0$ where we recover the Reissner-Nordström geometry. In Schwarzschild-like coordinates, the metric is analytic near $Q = 0$. The series $g^{\rm RN}_{\mu\nu}(Q) = g^{\rm RN}_{\mu\nu}(0) + Q\ \partial_Q g^{\rm RN}_{\mu\nu}\Big|_{Q=0} + O(Q^2)$ is well-defined, and in these coordinates we have $g^{\rm RN}_{\mu\nu}(0) = g^{\rm Schwarzschild}_{\mu\nu}$. But the singularity in the maximally extended Reissner-Nordström solution is timelike, while the singularity in the maximally extended Schwarzschild solution is spacelike. The point is that the singularity at $r = 0$ is never null. So it is not correct to look at the horizon function $\Delta = (r - r_+)(r - r_-)$ and conclude from it that in the limit $Q \to 0$, the inner horizon r_- merges with the singularity at $r = 0$: the limit is discontinuous. While the two metrics in Schwarzschild-like coordinates are smoothly connected to each other, the maximally extended geometries are not.

In the above notation, the following null 1-forms describe the Kerr-Newman black hole:[19]

$$l = -dt + \frac{|R|^2}{\Delta} dr + a \sin^2\theta \, d\varphi \ , \quad l' = \frac{\Delta}{2|R|^2} \left(-dt - \frac{|R|^2}{\Delta} dr + a \sin^2\theta \, d\varphi \right) \ ,$$

$$m = \frac{1}{R\sqrt{2}} \left(|R|^2 \, d\theta + i|R_0|^2 \sin\theta \, d\varphi - ia \sin\theta \, dt \right) \ , \quad m' = m^* \ . \tag{5.9}$$

Given these 1-forms, we solve the matrix inversion problem[20]

$$e_\mu^{\ a} e_a^{\ \nu} \equiv \delta_\mu^{\ \nu} \ , \quad e_a^{\ \mu} e_\mu^{\ b} \equiv \delta_a^{\ b} \tag{5.11}$$

for the vectors[21] $e_a^{\ \mu} = (l^\mu, l'^\mu, m^\mu, m'^\mu)$. With these, we can define the Newman-Penrose directional covariant derivatives:[22]

$$D \equiv l^\mu \nabla_\mu \ , \quad D' \equiv l'^\mu \nabla_\mu \ , \quad \delta \equiv m^\mu \nabla_\mu \ , \quad \delta' \equiv m'^\mu \nabla_\mu \ . \tag{5.12}$$

[19]In the limit $M \to 0$ and $J \to 0$ with $a \equiv J/M$ held fixed, these 1-forms describe the "Kerr congruence" in flat space-time [143]. See, however, the recent work by Gibbons and Volkov [144].

[20]This is equivalent to solving the "Newman-Penrose normalization conditions"

$$\begin{aligned} & l^\mu l_\mu = l'^\mu l'_\mu = m^\mu m_\mu = m'^\mu m'_\mu = 0 \qquad \text{(nullity)} \\ & l^\mu l'_\mu = -1 \ , \quad m^\mu m'_\mu = +1 \qquad \text{(normalization)} \\ & l^\mu m_\mu = l'^\mu m_\mu = l^\mu m'_\mu = l'^\mu m'_\mu = 0 \qquad \text{(orthogonality)} \ . \end{aligned} \tag{5.10}$$

[21]We will often refer to the components of vectors as vectors themselves. Since the explicit index structure will clarify whether we mean the components $e_a^{\ \mu}$ or the basis-independent object $e_a^{\ \mu}\partial_\mu$, there should be no risk of confusion.

[22]Unfortunately, Newman and Penrose reserved the overloaded symbol "δ" for the directional derivative along m^μ. Accordingly, we will not use that symbol to denote any sort of variation. We will, however, continue to express the Dirac delta function in its standard notation and trust that the reader will not get confused.

Once the basic equations of differential geometry are cast in spin coefficient form, all of the dynamical variables will be invariant under coordinate transformations on the base space. So the tensors of general relativity will be replaced by scalar fields.

Without loss of generality, we can then replace the covariant derivatives by partial derivatives and treat the operators D, D', δ, δ' as ordinary vector fields. For the coordinates in which we expressed the 1-forms in Eq. (5.9), we have:

$$D = l^\mu \partial_\mu = \frac{|R_0|^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\varphi \ , \quad D' = l'^\mu \partial_\mu = \frac{\Delta}{2|R|^2}\left(\frac{|R_0|^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\varphi\right) \ ,$$
$$\delta = m^\mu \partial_\mu = \frac{1}{R\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\,\partial_\varphi + ia\,\sin\theta\,\partial_t\right) \ , \quad \delta' = \delta^* . \tag{5.13}$$

For lack of a better name, we will say that the forms in Eq. (5.9) and the vectors in Eq. (5.13) compose the "asymptotic" tetrad. The vector l^μ is aligned with a special class of *o*utgoing null geodesics in the Kerr-Newman geometry, and the vector l'^μ is aligned with the corresponding class of *i*ngoing geodesics. The real and imaginary parts of m^μ span the orthogonal spacelike plane.

5.2.2 Gauge theory of gravity

There are two ways to express the classical field theory of gravitation, distinguished by whether a local invariance under $SO(3,1)$ is $\imath$mposed or $\imath$nferred.[23] Drastically oversimplifying a complicated history, we will say that the former is Cartan's approach, while the latter is Einstein's.[24]

In Cartan's gravity, we introduce a frame field e^a_μ, without any reference to a metric tensor, and $\imath$mpose invariance under

$$SO(3,1): \qquad e^a(x) \to O^a{}_b(x)\, e^b(x) \ . \tag{5.14}$$

The exterior derivative d does not transform covariantly, so we introduce an $SO(3,1)$ algebra-valued 1-form gauge field[25]

$$\omega^a{}_b \equiv (\omega_\mu)^a{}_b\, dx^\mu \tag{5.15}$$

[23] Kinnersley felt compelled to explain this dichotomy to his doctoral committee almost 50 years ago [145], and we feel obligated to reiterate it today.

[24] For example, Penrose and Rindler refer to what we call "Cartan's approach" as the "Einstein-Cartan-Sciama-Kibble theory" (see Sec. 4.7 in *Spinors and Space-time* [12,13]).

[25] We intentionally evoke Sec. IX.7 of Zee [140] to portray the spin coefficient formalism as a natural addendum to the usual Cartan formulation of gravity. Along the lines of the "3+1 decomposition" suitable for numerical relativity, we could call the Newman-Penrose notation a "2+2 decomposition" or a "lightcone 3+1 decomposition" (see Sec. 5.2.8).

to repair it. The field ${\omega^a}_b$ is called the "spin connection," and the renovated derivative will be denoted

$$\mathcal{D}^a{}_b \equiv \delta^a{}_b\, d + \omega^a{}_b \ . \tag{5.16}$$

Demanding covariance of Eq. (5.16) establishes the transformation law

$$SO(3,1): \qquad \omega^a{}_b \to O^a{}_c(x)(\delta^c{}_d\, d + \omega^c{}_d(x))(O^{-1})^d{}_b(x) \ . \tag{5.17}$$

We then insist that whatever action we choose[26] is to be invariant under Eqs. (5.14) and (5.17), with e^a_μ and $(\omega_\mu)^a{}_b$ being independent variables.

This logic parallels the contemporary presentation of Yang-Mills theory: Step 1 is to introduce a group G and an algebra-valued gauge field $A^a{}_b$ that behaves as $A^a{}_b \to O^a{}_c\, A^c{}_d\, (O^{-1})^d{}_b + O^a{}_c\, d(O^{-1})^c{}_b$ under local G transformations. Step 2 is to *i*mpose, by sheer decree, invariance of the action under this transformation.[27] But gravity is a theory of *t*wo classical fields, e^a and $\omega^a{}_b$, whereas Yang-Mills theory has only one classical field, $A^a{}_b$.[28] In both cases we are discussing only the "pure" gauge theory,[29] without introducing any matter fields.

[26] To write down the action we must first introduce curvature, which we defer until Sec. 5.2.11. The reader who is well acclimated to curvature may consult Eq. (5.86).

[27] This philosophy is also the state of the art in particle physics.

[28] It is possible to consider a field theory only of the spin connection $\omega^a{}_b$ without a frame e^a, but we decline to call that "gravitation." See, however, a paper by Chamseddine (and references therein) on how to embed gravity into a higher-dimensional Chern-Simons theory [146].

[29] Gravity and Yang-Mills theory are to physicists what frame bundles and principal bundles are to mathematicians.

In contrast to this conception of gravity, Einstein's approach is to formulate a classical field theory of the metric tensor $g_{\mu\nu}(x)$, which in physical terms is called the gravitational field.[30] If, as in Eq. (5.2), we decide to rewrite the metric in its locally diagonalized form, then we *i*nfer an inherent ambiguity in the description: one frame $e^a_\mu(x)$ and another frame $e'^a_\mu(x) \equiv O^a{}_b(x)\, e^b_\mu(x)$ define the very same gravitational field

$$g_{\mu\nu}(x) \equiv \eta_{ab}\, e^a_\mu(x)\, e^b_\nu(x) = \eta_{ab}\, e'^a_\mu(x)\, e'^b_\nu(x) \tag{5.18}$$

if and only if $\eta_{ab}\, O^a{}_c(x)\, O^b{}_d(x) = \eta_{cd}$. This is the definition of gauge invariance in classical field theory, in this case with gauge group $SO(3,1)$.

[30] There are three different quantities that could justifiably be called the "gravitational field." Consider the analogy to electromagnetism. In the vernacular, the objects $E^i = F^{0i}$ and $B^i = \frac{1}{2}\varepsilon^{0ijk}F_{jk}$ are the electric and magnetic fields. Accordingly, it is customary to refer to A_0 and A_i as the scalar and vector potentials. But members of high society know that matter fields couple directly to A_μ. So in field theory (whether classical or quantum), it is more enlightening to call A_μ the electromagnetic (or gauge) field instead of the electromagnetic (or gauge) potential. In gravity, the situation is even more confusing, since there are three classical fields to consider: the metric $g_{\mu\nu}$, the Christoffel symbols $\Gamma^\mu_{\nu\rho}$, and the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$. (Or, in the tetrad formalism, we have the frame field e^a_μ, the spin connection $(\omega_\mu)^a{}_b$, and the curvature $(\Omega_{\mu\nu})^a{}_b$.) Just as matter fields couple directly to A_μ, matter fields also couple directly to $g_{\mu\nu}$; or, to use the incisive lexicon of Zee [140,147], just as charge is defined as what A_μ listens to, energy-momentum is defined as what $g_{\mu\nu}$ listens to. So it is not only consonant with mathematical protocol but also with physical intuition to refer to $g_{\mu\nu}$ as the gravitational field. Electrodynamics aficionados who would cling to $\vec{E}$ and $\vec{B}$ until their dying breath could refer to $R^\mu{}_{\nu\rho\sigma}$ as the gravitational field, to $\Gamma^\mu_{\nu\rho}$ as the gravitational potential, and to $g_{\mu\nu}$ as the gravitational pre-potential. But they should not. Instead they should come to grips with reality and accept that $\vec{E}$ and $\vec{B}$ are parts of the electromagnetic curvature. Finally, since the Christoffel symbols (or the spin connection) arise from the definition of parallel transport, the $\Gamma^\mu_{\nu\rho}$ are most mathematically analogous to A_μ and hence could also be called the gravitational (or gauge) field.

For all considerations in this chapter, Cartan's and Einstein's perspectives will describe the same physics.[31] So we can feel free to gloss over whether the $SO(3,1)$ gauge invariance is imposed or inferred. Either way, however, we should emphasize that this local symmetry has nothing to do with coordinate transformations on the base space: infinitesimal transformations $x^\mu \to x^\mu + \xi^\mu$ transform the metric as $g_{\mu\nu} \to g_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$, while gauge transformations leave $g_{\mu\nu}$ unchanged.

5.2.3 Partial gauge fixing

After Newman and Penrose developed the original spin coefficient formalism, Geroch, Held, and Penrose recognized that specifying a tetrad $e_a^{\ \mu} = (l^\mu, l'^\mu, m^\mu, m'^\mu)$ that satisfies the normalization conditions in Eq. (5.11) only partially fixes the gauge.[32]

The remaining ambiguity comprises a boost along the outgoing congruence, the inverse boost along the ingoing congruence, and a rotation of the transverse

[31] This will happen when and only when our bestiary comprises matter fields that transform as scalars, vectors, or tensors under coordinate transformations on the base space. The instant we obtain an import license for the exotic specimen known as spinor, Einstein's theory is done. Supergravity, for example, must be formulated according to Cartan, and its study would compel us to abandon general relativity. In this work, we emphasize only the computational framework of classical field theory, which is why we bring up supergravity instead of the Standard Model. But the real world is quantum mechanical and made up of fermions. So with or without supersymmetry, coupling fundamental particles to gravity requires the Cartan formulation.

[32] Readers who object to this characterization should revisit the original paper [139]. Also, we should say that only toward the end of our project did we encounter Harnett's work [148], which privileges differential forms as we do.

plane:

$$\text{GHP}: \qquad l^\mu \to r(x)\, l^\mu \ , \ \ l'^\mu \to \frac{1}{r(x)}\, l'^\mu \ , \ \ m^\mu \to e^{i\vartheta(x)}\, m^\mu \ , \ \ m'^\mu \to e^{-i\vartheta(x)}\, m'^\mu \ . \tag{5.19}$$

Here $r(x)$ and $\vartheta(x)$ are arbitrary real functions, with the restriction that $r(x) \neq 0$. We will say that this transformation generates the "GHP group." Note that the subgroup of transformations for which we rephase m^μ while holding l^μ and l'^μ fixed is the little group for the outgoing and ingoing congruences.

It is convenient to define the complex function

$$\lambda \equiv r^{1/2} e^{i\vartheta/2} \tag{5.20}$$

and to rewrite Eq. (5.19) as

$$\text{GHP}: \qquad l^\mu \to \lambda\lambda^*\, l^\mu \ , \ \ l'^\mu \to \lambda^{-1}\lambda^{*\,-1}\, l'^\mu \ , \ \ m^\mu \to \lambda\lambda^{*\,-1}\, m^\mu \ , \ \ m'^\mu \to \lambda^{-1}\lambda^*\, m'^\mu \ . \tag{5.21}$$

We will say that a function $f_{h,\bar{h}}$ transforms as the representation[33] $(h, \bar{h})$ of the GHP group if its transformation law under Eq. (5.21) has the form:

$$f_{h,\bar{h}} \to \lambda^{2h} \lambda^{*\,2\bar{h}} f_{h,\bar{h}} \ . \tag{5.22}$$

[33] The bar is part of the name of the weight and does not denote any sort of conjugation. Readers who dislike this could use an alternative notation such as (h, k).

To express this we will use the physicist's standard notation in representation theory:

$$f_{h,\bar{h}} \sim (h, \bar{h}) \ . \tag{5.23}$$

The numbers $(h, \bar{h})$ are called the weights[34] of the function $f_{h,\bar{h}}$. Such a function is called "weighted." An object that cannot be assigned a transformation law of the form in Eq. (5.22) for any values of $(h, \bar{h})$ will be called "nonweighted."[35] In the language of Eq. (5.23), we summarize Eq. (5.21) as

$$l^\mu \sim (\tfrac{1}{2}, \tfrac{1}{2}) \ , \ \ l'^\mu \sim (-\tfrac{1}{2}, -\tfrac{1}{2}) \ , \ \ m^\mu \sim (\tfrac{1}{2}, -\tfrac{1}{2}) \ , \ \ m'^\mu \sim (-\tfrac{1}{2}, \tfrac{1}{2}) \ . \tag{5.24}$$

The compacted spin coefficient formalism is just a convenient repackaging of the original spin coefficient formalism into a presentation that is manifestly covariant under the GHP group: all quantities that appear explicitly in the equations transform according to Eq. (5.22) for some values of h and $\bar{h}$. In such a formulation, only objects with the same weights can be added, and the weights of a product of

[34]The standard notation in the relativity community is to define $p \equiv 2h$ and $q \equiv 2\bar{h}$. The factors of 2 are a matter of taste. Either way, the "boost weight" and the "spin weight" are defined as $\frac{1}{2}(p+q) = h + \bar{h}$ and $\frac{1}{2}(p-q) = h - \bar{h}$ respectively [139]. As a matter of grammatical construction, we will sometimes refer to $(h, \bar{h})$ as a singular noun ("weight") or as a plural noun ("weights") depending on whether we describe the representation as a whole or home in on the particular values of h and $\bar{h}$ separately.

[35]Something invariant under Eq. (5.22) is considered to be weighted with weight zero, not nonweighted.

objects are the sums of the weights of each object:

$$f_{h_1,\bar{h}_1} \sim (h_1, \bar{h}_1) \ , \ \ g_{h_2,\bar{h}_2} \sim (h_2, \bar{h}_2) \implies f_{h_1,\bar{h}_1} g_{h_2,\bar{h}_2} \sim (h_1 + h_2, \bar{h}_1 + \bar{h}_2) \ . \tag{5.25}$$

From Eq. (5.22) we deduce that complex conjugation exchanges the weights:

$$f_{h,\bar{h}} \sim (h, \bar{h}) \implies (f_{h,\bar{h}})^* \sim (\bar{h}, h) \ . \tag{5.26}$$

In addition to complex conjugation, there are two discrete transformations under which the compacted formalism is covariant. The first is the *p*riming transformation, which is defined to exchange primed and unprimed quantities:

$$' : \qquad l_\mu \leftrightarrow l'_\mu \ , \ \ m_\mu \leftrightarrow m'_\mu \ . \tag{5.27}$$

In this way we elevate the notation from Eq. (5.4) to an operation. From Eq. (5.21) we deduce that priming flips the signs of the weights:

$$f_{h,\bar{h}} \sim (h, \bar{h}) \implies (f_{h,\bar{h}})' \sim (-h, -\bar{h}) \ . \tag{5.28}$$

The second discrete transformation is the *S*achs operation,[36] which is an analog of Hodge duality:[37]

$$s: \qquad (l_\mu, l'_\mu, m_\mu, m'_\mu) \to (m_\mu, -m'_\mu, -l_\mu, l'_\mu) \ . \tag{5.29}$$

Unlike priming, the Sachs operation does not commute with complex conjugation. It is extremely convenient to streamline the spin coefficient formalism by using a notation that is manifestly covariant under priming. The Sachs operation will instead help us establish geometrical meaning.[38]

[36]As far as we know, the first mention of this transformation is in the original GHP paper [139], in which the authors cited "private communication" with Sachs. In Sec. 4.12 of *Spinors and Space-time* [12], Penrose and Rindler cite two papers by Sachs [149,150]. Unless our eyes deceive us, these papers do not explicitly mention the transformation defined in Eq. (5.29). Presumably some paper by Sachs does, but we have not conducted an exhaustive investigation.

[37]The discerning reader will be puzzled by this transformation, since taken literally it instructs us to exchange the real quantities l_μ and l'_μ with the complex ones m_μ and m'_μ. We will say that this operation is to be defined "algebraically," which is a formal way of saying just do it anyway. If it was good enough for Penrose it will be good enough for us.

[38]Let us make the following series of tangential observations, which we have not seen anywhere else. First, the composition of priming with complex conjugation exchanges $l_\mu \leftrightarrow l'_\mu$ without touching m_μ and m'_μ. Therefore, if we align l^μ and l'^μ with the trajectories of outgoing and ingoing null rays, then the combined operation $'*$ acts as a GHP version of time reversal. Second, along the same lines, complex conjugation exchanges $m_\mu \leftrightarrow m'_\mu$ without touching l_μ or l'_μ; if we define for the moment $X^\mu \equiv \mathrm{Re}(m^\mu)$ and $Y^\mu \equiv \mathrm{Im}(m^\mu)$, then complex conjugation acts as $(X^\mu, Y^\mu) \to (X^\mu, -Y^\mu)$. This is a GHP version of parity, defined as reflection in a mirror along one of the spatial directions. Third, we are naturally tempted to identify the Sachs operation as some kind of GHP notion of "charge conjugation," which in particle physics is defined to exchange representations of internal symmetry groups with their conjugate representations. We do not know if there is any deeper significance to this kind of "CPT" transformation in the compacted spin coefficient formalism.

5.2.4 Spin coefficients

The components of the spin connection contracted with the vectors of the null tetrad are called the *s*pin coefficients.[39]

Cartan's first equation of structure[40] states that the torsion of space-time is given by the covariant derivative of the frame fields. Mathematically this is expressed by defining the torsion 2-form

$$\Theta^a \equiv \mathcal{D}^a{}_b\, e^b = de^a + \omega^a{}_b\, \wedge e^b \, . \tag{5.30}$$

Following Chandrasekhar [142], we lower an index and define the tangent-space decomposition of the first term as:

$$de_a = \partial_\mu e_{a\nu}\, dx^\mu \wedge dx^\nu = \tfrac{1}{2} e^\mu_b e^\nu_c (\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu})\, e^b \wedge e^c \equiv -\tfrac{1}{2}\lambda_{bac}\, e^b \wedge e^c \, . \tag{5.31}$$

We then follow Newman and Penrose [138] and define the spin coefficients γ_{abc} as

$$\gamma_{abc} \equiv (\omega_\mu)_{ab}\, e^\mu_c \, . \tag{5.32}$$

[39] They are also called the "Ricci rotation coefficients." We prefer "spin coefficients" partially because it matches the term "spin connection," and partially because we find it confusing to assign Ricci's name to elements of the *c*onnection in addition to its usual association with parts of the curvature.

[40] We insist on this stilted ordering of words to highlight the difference between an "equation of structure" (a geometrical consequence of transporting the frame) and an "equation of motion" (an output of the principle of least action).

The crucial property of antisymmetry in the first two indices,

$$\gamma_{abc} = -\gamma_{bac} \ , \tag{5.33}$$

comes from the definition of $\omega^a{}_b$ in Eq. (5.15) as an $SO(3,1)$ gauge field, which is a linear combination of the antisymmetric (when both matrix indices are either up or down) 4-by-4 matrices that generate infinitesimal Lorentz transformations.

In terms of the γ_{abc} and the λ_{abc}, the torsion 2-form becomes:

$$\Theta_a = \tfrac{1}{2}(\gamma_{bac} - \gamma_{cab} - \lambda_{bac})\ e^b \wedge e^c \ . \tag{5.34}$$

The fundamental assumption of general relativity is[41]

$$\Theta_a = 0 \ . \tag{5.35}$$

We can solve this equation for the spin coefficients in terms of the frame fields:

$$\gamma_{abc} = \tfrac{1}{2}(\lambda_{abc} + \lambda_{cab} - \lambda_{bca}) = e_a^{\ \mu}\, e_c^{\ \nu}\, \nabla_\nu\, e_{b\mu} \ . \tag{5.36}$$

[41]Like many intuitively obvious aspects of gravitation, this property is neither obvious nor intuitive. It is best thought of as the vacuum equation of motion obtained from varying the action with respect to the spin connection. As per an earlier footnote about the action, we must introduce curvature before explaining this further.

This is the tetrad version of metric compatibility. The first form of the γ_{abc} in Eq. (5.36) involves only ordinary partial derivatives and provides a convenient way to code the spin coefficients in Mathematica.

The second form, involving space-time-covariant derivatives, provides a convenient way to calculate the transformation law for each spin coefficient under the GHP group [recall Eq. (5.21)]. By computing their behavior under GHP transformations, we learn that the 12 independent γ_{abc} fall naturally into three sets: weighted quantities associated with l_μ, weighted quantities associated with l'_μ, and nonweighted quantities that transform as gauge fields.

The weighted spin coefficients associated with l_μ, along with their weights, are:

$$\kappa \equiv \gamma_{311} \sim \left(\tfrac{3}{2},\, \tfrac{1}{2}\right) , \; \tau \equiv \gamma_{312} \sim \left(\tfrac{1}{2},\, -\tfrac{1}{2}\right) , \; \sigma \equiv \gamma_{313} \sim \left(\tfrac{3}{2},\, -\tfrac{1}{2}\right) , \; \rho \equiv \gamma_{314} \sim \left(\tfrac{1}{2},\, \tfrac{1}{2}\right) \; . \tag{5.37}$$

The weighted spin coefficients associated with l'_μ are defined by priming,[42] which flips the signs of the weights:[43]

$$\kappa' \equiv \gamma_{422} \sim \left(-\tfrac{3}{2},\, -\tfrac{1}{2}\right) , \; \tau' \equiv \gamma_{421} \sim \left(-\tfrac{1}{2},\, \tfrac{1}{2}\right) , \; \sigma' \equiv \gamma_{424} \sim \left(-\tfrac{3}{2},\, \tfrac{1}{2}\right) , \; \rho' \equiv \gamma_{423} \sim \left(-\tfrac{1}{2},\, -\tfrac{1}{2}\right) \; . \tag{5.38}$$

[42]Some authors prefer the notation $\nu \equiv -\kappa'$, $\pi \equiv -\tau'$, $\lambda \equiv -\sigma'$, $\mu \equiv -\rho'$, $\gamma \equiv -\varepsilon'$, and $\alpha \equiv -\beta'$, as originally presented by Newman and Penrose [138]. As Geroch, Held, and Penrose observed, the priming-covariant notation "not only halves the number of Greek letters needed, but also effectively halves the number of equations" [139].

[43]Priming acts on the tetrad labels by exchanging $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. Complex conjugation leaves 1 and 2 fixed while exchanging $3 \leftrightarrow 4$.

The gauge fields of the spin coefficient formalism are defined as

$$\varepsilon \equiv \tfrac{1}{2}(-\gamma_{121} + \gamma_{341}) \ , \quad \beta \equiv \tfrac{1}{2}(-\gamma_{123} + \gamma_{343}) \ , \quad \varepsilon' \equiv \tfrac{1}{2}(-\gamma_{212} + \gamma_{432}) \ , \quad \beta' \equiv \tfrac{1}{2}(-\gamma_{214} + \gamma_{434}) \ . \tag{5.39}$$

These are gauge fields in the sense that they combine with the NP derivatives of Eq. (5.13) to form weighted derivatives:[44]

$$
\begin{aligned}
&\text{þ} \equiv D + 2h\,\varepsilon + 2\bar{h}\,\varepsilon^* \,, \quad \eth \equiv \delta + 2h\,\beta - 2\bar{h}\,\beta'^* \,, \\
&\text{þ}' \equiv D' - 2h\,\varepsilon' - 2\bar{h}\,\varepsilon'^* \,, \quad \eth' \equiv \delta' - 2h\,\beta' + 2\bar{h}\,\beta^* \,.
\end{aligned} \tag{5.43}
$$

[44]Since our goal is to teach physics instead of cryptography, let us show explicitly that þ is a weighted derivative. From the second form of the γ_{abc} in Eq. (5.36) and the definition of ε in Eq. (5.39), we compute the GHP transformation as instructed by Eq. (5.21):

$$
\begin{aligned}
\varepsilon = \tfrac{1}{2}\left(l'^\mu D l_\mu + m^\mu D m'_\mu\right) &\to \tfrac{1}{2}\left[l'^\mu D(\lambda\lambda^* l_\mu) + \lambda^2 m^\mu D(\lambda^{-1}\lambda^* m'_\mu)\right] \\
&= \lambda\lambda^*\varepsilon + \tfrac{1}{2}\left[l'^\mu l_\mu D(\lambda\lambda^*) + \lambda^2 m^\mu m'_\mu D(\lambda^{-1}\lambda^*)\right] \\
&= \lambda\lambda^*\varepsilon + \tfrac{1}{2}\left[(-1)(\lambda^* D\lambda + \lambda D\lambda^*) + \lambda^2(+1)(\lambda^* D\lambda^{-1} + \lambda^{-1} D\lambda^*)\right] \\
&= \lambda\lambda^*\varepsilon - \lambda^* D\lambda = \lambda\lambda^*\left(\varepsilon - \lambda^{-1} D\lambda\right) \,.
\end{aligned} \tag{5.40}
$$

Meanwhile, for the NP derivative D acting on a weighted function $f_{h,\bar{h}} \sim (h,\bar{h})$, we have:

$$
\begin{aligned}
D f_{h,\bar{h}} \to \lambda\lambda^* D(\lambda^{2h}\lambda^{*2\bar{h}} f_{h,\bar{h}}) &= \lambda^{2h+1}\lambda^{*2\bar{h}+1} D f_{h,\bar{h}} + \lambda\lambda^* D(\lambda^{2h}\lambda^{*2\bar{h}}) f_{h,\bar{h}} \\
&= \lambda^{2h+1}\lambda^{*2\bar{h}+1} D f_{h,\bar{h}} + \lambda\lambda^*\left(2h\lambda^{2h-1}\lambda^{*2\bar{h}} D\lambda + 2\bar{h}\lambda^{2h}\lambda^{*2\bar{h}-1}\right) f_{h,\bar{h}} \\
&= \lambda^{2h+1}\lambda^{*2\bar{h}+1} D f_{h,\bar{h}} + \left(2h\,\lambda^{2h}\lambda^{*2\bar{h}+1} D\lambda + 2\bar{h}\,\lambda^{2h+1}\lambda^{*2\bar{h}} D\lambda^*\right) f_{h,\bar{h}} \\
&= \lambda^{2h+1}\lambda^{*2\bar{h}+1}\left(D + 2h\,\lambda^{-1} D\lambda + 2\bar{h}\,\lambda^{*-1} D\lambda^*\right) f_{h,\bar{h}} \,.
\end{aligned} \tag{5.41}
$$

So the terms with $D\lambda$ and $D\lambda^*$ cancel out of the transformation law for the combination $\text{þ} \equiv D + 2h\,\varepsilon + 2\bar{h}\,\varepsilon^*$:

$$
(D + 2h\,\varepsilon + 2\bar{h}\,\varepsilon^*) f_{h,\bar{h}} \to \lambda^{2h+1}\lambda^{*2\bar{h}+1}(D + 2h\,\varepsilon + 2\bar{h}\,\varepsilon^*) f_{h,\bar{h}} \,. \tag{5.42}
$$

The reader should be able to adapt this calculation to show that β and β'^* transform as $\beta \to \lambda\lambda^{*-1}(\beta - \lambda^{-1}\delta\lambda)$ and $\beta'^* \to \lambda\lambda^{*-1}(\beta'^* + \lambda^{*-1}\delta'\lambda^*)$, in which case $\eth \equiv \delta + 2h\,\beta - 2\bar{h}\,\beta'^*$ transforms according to $\eth f_{h,\bar{h}} \to \lambda^{2h+1}\lambda^{*2\bar{h}-1}\eth f_{h,\bar{h}}$. The reader may also wish to insert Eq. (5.20) into the transformation laws for the gauge fields to separate the compact and noncompact factors [cf. Eq. (347) in Sec. 1.8(g) of Chandrasekhar [142]].

We will refer to the operators þ, þ′, ð, and ð′ as "GHP-covariant derivatives," because that is literally what they are.[45]

Something unusual about this gauge theory is that the covariant derivatives themselves carry charge. For a weighted function $f_{h,\bar{h}} \sim (h, \bar{h})$, we have:

$$þ f_{h,\bar{h}} \sim (h + \tfrac{1}{2}, \bar{h} + \tfrac{1}{2}) \ , \ \ þ' f_{h,\bar{h}} \sim (h - \tfrac{1}{2}, \bar{h} - \tfrac{1}{2}) \tag{5.44}$$

$$ð f_{h,\bar{h}} \sim (h + \tfrac{1}{2}, \bar{h} - \tfrac{1}{2}) \ , \ \ ð' f_{h,\bar{h}} \sim (h - \tfrac{1}{2}, \bar{h} + \tfrac{1}{2}) \ . \tag{5.45}$$

This is conveniently summarized as:

$$þ \sim \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \ , \ \ þ' \sim \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right) \ , \ \ ð \sim \left(\tfrac{1}{2}, -\tfrac{1}{2}\right) \ , \ \ ð' \sim \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \ . \tag{5.46}$$

Ordinarily, the covariant derivative of an object transforms as the same representation as the object itself. Not so here.

When calculating in the compacted formalism, it is useful to keep in mind the following relations between the covariant derivatives of l_μ and some of the spin

[45] Literally as in literally, not as in figuratively.

coefficients:[46]

$$\text{þ}l_\mu = \kappa^* m_\mu + \kappa\, m'_\mu\ , \ \ \text{þ}' l_\mu = \tau^* m_\mu + \tau\, m'_\mu\ , \ \ \eth l_\mu = \rho^* m_\mu + \sigma\, m'_\mu\ . \tag{5.47}$$

The other equations like this can be obtained by a combination of complex conjugation, priming, and the Sachs operation. For a handy table of Sachs conjugates, please see Eqs. (4.12.52)-(4.12.55) in *Spinors and Space-time* [12].

5.2.5 Null Cartan equations

The definitions in Eqs. (5.37)-(5.39) are unimpeachable, but it is more practical analytically to take the torsion-free condition [Eq. (5.35)], written in the form

$$de_a = \gamma_{abc}\, e^b \wedge e^c \tag{5.48}$$

[46]To derive Eq. (5.47) from Eq. (5.32), begin with a completely general expansion for a vector. For example, $Dl_\mu = a\, l_\mu + b\, l'_\mu + c^* m_\mu + c\, m'_\mu$ with undetermined coefficients a, b, c [in this case $(Dl_\mu)^* = Dl_\mu$]. Then contract with l^μ, l'^μ, m^μ, and m'^μ, and use the definitions of the spin coefficients in Eqs. (5.37), (5.38), and (5.39) in the form $\gamma_{abc} = e^\mu_a e^\nu_c \nabla_\nu e_{b\mu}$ from Eq. (5.32). (Also, $l^\mu D l_\mu = \frac{1}{2} D(l^\mu l_\mu) = 0$.) Then use the representation $l_\mu \sim (\frac{1}{2}, \frac{1}{2})$ and the definition of þ in Eq. (5.43) to get $\text{þ}l_\mu = \kappa^* m_\mu + \kappa\, m'_\mu$. Repeat the process for $D' l_\mu$ and δl_μ to get the other two equations.

as an operational definition of the spin coefficients. In the Newman-Penrose notation, we have:

$$dl = -2\,\mathrm{Re}(\varepsilon)\, l \wedge l' + 2i\,\mathrm{Im}(\rho)\, m \wedge m' + \Big[\big(\tau - \beta + \beta'^*\big)\; m' \wedge l + \kappa\, m' \wedge l' + c.c. \Big]$$
$$dm = (\beta + \beta'^*)\, m \wedge m' - (\tau - \tau'^*)\, l \wedge l' + \Big[\big(\rho - 2i\,\mathrm{Im}(\varepsilon)\big)\; m \wedge l' + \sigma\, m' \wedge l' + c.c.' \Big] \;. \tag{5.49}$$

The notation "*c.c.′*" indicates that the complex conjugate of the *p*rimed quantities should be taken. The defining equation for dl' can be obtained from that of dl by priming.[47]

We will refer to the expressions in Eq. (5.49) as the *n*ull Cartan equations. By computing the exterior derivatives of the forms in Eq. (5.9), arranging them to match the right-hand sides in Eq. (5.49), and solving the resulting equations, we can compute the spin coefficients for the Kerr-Newman geometry:

$$\kappa = \kappa' = \sigma = \sigma' = \varepsilon = 0\,,\; \varepsilon' = \rho' + \frac{2r - r_+ - r_-}{4|R|^2}\;,$$
$$\rho = \frac{1}{R^*}\;,\quad \rho' = -\frac{\Delta}{2|R|^2}\,\frac{1}{R^*}\;,\quad \tau = \frac{ia\,\sin\theta}{\sqrt{2}\,|R|^2}\;,\quad \tau' = \frac{ia\,\sin\theta}{\sqrt{2}\,(R^*)^2}\;,$$
$$\beta = -\frac{\cot\theta}{2\sqrt{2}\,R}\;,\quad \beta' = -\left(\frac{r}{R^*}\right)\frac{\cot\theta}{2\sqrt{2}\,R^*} + \frac{ia}{2\sqrt{2}}\,\frac{1+\sin^2\theta}{(R^*)^2\sin\theta}\;. \tag{5.50}$$

[47]The defining equation for dm can be obtained from that of dl by applying the Sachs operation. Since we do not use a notation that is manifestly covariant under the Sachs operation (nor should we, since it does not commute with complex conjugation), we provide the equation for dm explicitly.

Note that $|\tau|^2 = |\tau'|^2$, which will be useful later. Also note that a GHP transformation with[48] $\lambda = R^{1/2}$ would set $\tau = \tau' = \frac{ia\sin\theta}{\sqrt{2}R^{1/2}(R^*)^{3/2}}$, but we will continue to work with the asymmetric basis. The products $\tau\tau^*$, $\tau'\tau'^*$, and $\tau\tau'$ have weights $(0,0)$ and would be unaffected by such a transformation.

5.2.6 Null Cartan equations in compacted form

It ought to be possible to express the null Cartan equations in a manifestly GHP-covariant form. There are two equivalent ways to approach this issue.

First, in the group-theoretic spirit of restriction to a subgroup, we can break down the $SO(3,1)$-covariant formulation until we obtain reduced versions of the Cartan equations that retain manifest covariance only under Eq. (5.21). We explore this in Section 5.C.

Second, in the spirit of Yang-Mills theory on curved space-time, we can take the space-time-covariant derivative ∇_μ and tack on the structure of a gauge theory with a local invariance under Eq. (5.21).[49] To do that, we first collect the GHP

[48] We should be careful when distributing square roots over complex numbers. Presumably our glib explanation is fine because $\mathrm{Re}(R) = r > r_+$ and $|a| < M$, while $\mathrm{Im}(R) = a\cos\theta$, so $\mathrm{Re}(R) > |\mathrm{Im}(R)|$.

[49] Consider what would happen if we tried to do this for the full $SO(3,1)$ group: we would define a covariant derivative $(\mathbb{D}_\mu)^a{}_b \equiv \delta^a{}_b \nabla_\mu + (\omega_\mu)^a{}_b$ that acts simultaneously on space-time indices and on tangent-space indices. On a frame field, we have $(\mathbb{D}e)^a_\mu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + (\omega_\mu)^a{}_b e^b_\nu$; given $\gamma_{abc} = e^\mu_a \nabla_c e_{b\mu}$, we just get $(\mathbb{D}e)^a_\mu = 0$. This is standard fare but worth revisiting here in the context of the partially gauge-fixed formalism.

gauge fields into a 1-form

$$\mathrm{b} \equiv \tfrac{1}{2}(-\gamma_{12a} + \gamma_{34a})\, e^a = -l'\varepsilon + l\,\varepsilon' + m'\beta - m\,\beta' \ . \tag{5.51}$$

Then we define the GHP-covariant generalization of ∇_μ as follows:[50]

$$\text{Đ} \equiv e^a \nabla_a + 2h\,\mathrm{b} + 2\bar{h}\,\mathrm{b}^* = -l'\text{þ} - l\,\text{þ}' + m'\eth + m\,\eth' \ . \tag{5.52}$$

This will supply the droids we are looking for. In terms of this exterior derivative, the null Cartan equations in Eq. (5.49) become:[51]

$$\begin{aligned} \text{Đ}l &= (\rho - \rho^*)\, m \wedge m' + \big(\tau\, m' \wedge l + \kappa\, m' \wedge l' + c.c.\big) \ , \\ \text{Đ}m &= -(\tau - \tau'^*)\, l \wedge l' + \big(\rho\, m \wedge l' + \sigma\, m' \wedge l' + c.c.'\big) \ . \end{aligned} \tag{5.53}$$

Unlike the situation in Eq. (5.49), everything in Eq. (5.53) is weighted, with Đ $\sim (0,0)$.

Arranging the equations in this manner distills the essential physics of null congruences from cumbersome technical drudgery like whether the curves are parametrized affinely.[52]

[50] The symbol Đ is the capital version of ð.
[51] The antisymmetrization inherent to the exterior calculus reduces ∇_μ to ∂_μ.
[52] It is entirely possible that the GHP gauge fields express some important physical phenomena that we have not understood.

5.2.7 Refraction, expansion, twist, and shear

The geometrical significance of the weighted spin coefficients κ, ρ, and σ is discussed at length in all of the standard references. To make this paper largely self-contained without excessively rehashing common knowledge, we will provide a streamlined exposition using the null Cartan equations in their compacted form. We will drag the formalism kicking and screaming into the twenty-first century by deriving the dynamical equations for null congruences from an action principle.

We begin by interpreting the exterior derivative of a 1-form.[53] Let A be a 1-form and let C be an arbitrary open curve parametrized by a number u with embedding coordinates $X^\mu(u)$.[54] Forms exist to be integrated over, so let us define an action

$$I[X] \equiv \int_C A(X) = \int_{u_1}^{u_2} du \, \frac{dX^\mu(u)}{du} \, A_\mu(X(u)) \; . \tag{5.54}$$

This is to be thought of as a functional[55] of the classical field variables $X^\mu(u)$. Now we investigate what happens as we deform the curve according to the variation[56]

$$X^\mu(u) \to X^\mu(u) + \blacktriangle^\mu(u) \; . \tag{5.55}$$

[53] The antagonistic reader who considers this review material should kindly point us to a standard reference where this is explained clearly.

[54] This u has no innate relation to "Eddington-like" coordinates for the black hole. We use this letter to match the notation of Ch. 7 in *Spinors and Space-time* [13].

[55] We dislike this pompous word but begrudgingly accept its utility.

[56] We use the symbol $\blacktriangle$ because the usual variational symbol δ denotes a Newman-Penrose derivative, and the symbol Δ has been reserved for the horizon function.

By the usual steps we obtain (let $\dot{X}^\mu \equiv \frac{dX^\mu}{du}$):

$$I[X + \blacktriangle] - I[X] = \int_{u_1}^{u_2} du\, \dot{X}^\nu\, \blacktriangle^\mu\, (\partial_\mu A_\nu - \partial_\nu A_\mu) + \blacktriangle^\mu A_\mu\Big|_{u_1}^{u_2} + O(\blacktriangle^2)\ . \tag{5.56}$$

The antisymmetric derivative thrusts itself upon us. This formula explains why the natural variation of a 1-form is a 2-form and provides a concrete and explicit interpretation of dA without resorting to axiomatic definitions.

To derive evolution equations, we bring the endpoints of the curve close together:[57]

$$u_2 = u_1 + ¢\ . \tag{5.57}$$

Then integrals of the form $\int_{u_1}^{u_2} du\, F(u)$ are to be interpreted as $\int_{u_1}^{u_1+¢} du\, F(u) = 0 + ¢\, F(u_1) + O(¢^2)$, and the boundary term is to be interpreted as:

$$\blacktriangle^\mu A_\mu\Big|_{u_1}^{u_2} = ¢\, \frac{\partial}{\partial u}(\blacktriangle^\mu A_\mu)\Bigg|_{u_1} +\ O(¢^2)\ . \tag{5.58}$$

The action principle we impose is that the leading order term in the double expansion in $\blacktriangle^\mu$ and ¢ is decreed to be zero. With $\frac{\partial}{\partial u} = \dot{X}^\mu \partial_\mu$ and the fact that

[57] The cent is a small parameter.

$\blacktriangle^\mu A_\mu$ is a scalar field, we obtain the evolution equation[58]

$$\dot{X}^\nu \nabla_\nu (\blacktriangle^\mu A_\mu) = \dot{X}^\mu \blacktriangle^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \ . \tag{5.59}$$

The null Cartan equations admit a GHP-covariant formulation, so we may as well pass to a GHP-covariant notation. We therefore replace Eq. (5.59) with[59]

$$\dot{X}^\nu \text{Đ}_\nu (\blacktriangle^\mu A_\mu) = \dot{X}^\mu \blacktriangle^\nu (\text{Đ}_\mu A_\nu - \text{Đ}_\nu A_\mu) \ . \tag{5.60}$$

Now we have a worthy implement. Its input is a 1-form A_μ, a tangent vector $\dot{X}^\mu$, and a deformation vector $\blacktriangle^\mu$. We will be interested in $e$volution along outgoing null curves,[60] so we will feed it a tangent vector

$$\dot{X}^\mu = l^\mu \ . \tag{5.61}$$

To obtain our first evolution equation, we will insert $A_\mu = l_\mu$ into Eq. (5.60). The result is:

$$\text{þ}(\blacktriangle^\mu l_\mu) = \Big(\kappa \, \blacktriangle^\mu m'_\mu + c.c.\Big) \ . \tag{5.62}$$

[58]Everything in this equation is evaluated at $u = u_1$, which we then trivially relabel to just u.
[59]When interpreting the operator $\text{Đ}_\mu = -l'_\mu \, \text{þ} - l_\mu \, \text{þ}' + m'_\mu \, \eth + m_\mu \, \eth'$ in this equation, treat the deformation vector $\blacktriangle^\mu$ as having weight zero.
[60]The resulting dynamical equations in this subsection will be the manifestly GHP-covariant versions of the evolution equations in Ch. 7 of *Spinors and Space-time* [13].

This is actually just $\ddot{X}^\mu + \Gamma^\mu_{\nu\rho}\dot{X}^\mu\dot{X}^\nu = s\dot{X}^\mu + F^\mu$ with $s = 2\,\mathrm{Re}(\varepsilon)$ and $F^\mu = \kappa\, m'^\mu + c.c.$ in an unfamiliar notation.[61] So we learn that $\mathrm{Re}(\varepsilon) = 0$ implies affine parametrization, while $\kappa = 0$ implies the geodesic equation.

The spin coefficient κ therefore parametrizes a "force" on outgoing null curves causing them to bend in the transverse directions. Szekeres has appropriately referred to κ as the *r*efraction of the congruence [151]. An alternative name could be the "transverse acceleration." A colloquial way to phrase the geodesic condition for null curves is that freely falling light does not refract.

Next we supplicate Eq. (5.60) with the offering $A_\mu = m_\mu$ and obediently deliver its sermon:

$$\text{þ}(\blacktriangle^\mu m_\mu) = \rho\, \blacktriangle^\mu m_\mu + \sigma\, \blacktriangle^\mu m'_\mu - (\tau - \tau'^*)\blacktriangle^\mu l_\mu\ . \tag{5.63}$$

[61] If we generalize the one curve C with tangent vector $l^\mu = \dot{X}^\mu$ to a collection of curves with tangent vectors $l^\mu = \frac{\partial}{\partial u}X^\mu(u,\vec{y})$, then we can define a collection of deformation vectors as $\blacktriangle^\mu_I = \frac{\partial}{\partial y^I}X^\mu(u,\vec{y})$. Then, because partial derivatives commute, we have

$$\blacktriangle^\mu_I \partial_\mu l^\nu - l^\mu \partial_\mu \blacktriangle^\nu_I = \frac{\partial}{\partial y^I}\frac{\partial X^\nu}{\partial u} - \frac{\partial}{\partial u}\frac{\partial X^\nu}{\partial y^I} = 0\ .$$

Furthermore, because we assume the absence of torsion, we have

$$\begin{aligned}\blacktriangle^\mu_I \partial_\mu l^\nu - l^\mu \partial_\mu \blacktriangle^\nu_I &= \blacktriangle^\mu_I(\nabla_\mu l^\nu - \Gamma^\nu_{\mu\rho} l^\rho) - l^\mu(\nabla_\mu \blacktriangle^\nu_I - \Gamma^\nu_{\mu\rho}\blacktriangle^\rho_I)\\ &= \blacktriangle^\mu_I \nabla_\mu l^\nu - l^\mu \nabla_\mu \blacktriangle^\nu_I + (\Gamma^\nu_{\mu\rho} - \Gamma^\nu_{\rho\mu}) l^\mu \blacktriangle^\rho_I = \blacktriangle^\mu_I \nabla_\mu l^\nu - D\blacktriangle^\nu_I\ .\end{aligned}$$

Combining this with $\blacktriangle^\mu_I \partial_\mu l^\nu - l^\mu \partial_\mu \blacktriangle^\nu_I = 0$, we get

$$D\blacktriangle^\nu_I = \blacktriangle^\mu_I \nabla_\mu l^\nu\ .$$

Therefore:

$$l_\nu D\blacktriangle^\nu_I = \blacktriangle^\mu_I l_\nu \nabla_\mu l^\nu = \tfrac{1}{2}\blacktriangle^\mu_I \nabla_\mu (l_\nu l^\nu) = 0\ .$$

So we can peel off $\blacktriangle^\mu$ from Eq. (5.62) and arrive at $\text{þ} l_\mu = (\kappa\, m'_\mu + c.c.)$, which is the first relation listed in Eq. (5.47). Putting this into tensor notation provides the equation in the main text. Our analysis in this subsection is the same as that of Ch. 7 in *Spinors and Space-time*, with our "deformation vector" $\blacktriangle^\mu$ being an alternative perspective on their "connecting vector" q^μ. Our analysis in Sec. 5.2.8, however, will deviate substantially from their point of view.

From Eq. (5.62) we know that for geodesics ($\kappa = 0$), the quantity $\blacktriangle^\mu l_\mu$ is GHP-covariantly constant. Since it is constant, let us consider the special case where that constant is zero.

After defining the complex number $z \equiv \blacktriangle^\mu m_\mu$ just to induce visions of complex analysis, we find that Eq. (5.63) becomes

$$þz = \rho\, z + \sigma\, z^* \,. \tag{5.64}$$

In the complex plane of the variable z, the real and imaginary parts of ρ parametrize a dilation and rotation, respectively. Accordingly, $\mathrm{Re}(\rho)$ is called the *e*xpansion and $\mathrm{Im}(\rho)$ is called the *t*wist.

In the parlance of hydrodynamics, any transformation of the plane into itself that preserves area but is *n*ot a rotation is called a "shearing" transformation. In the complex plane, this manifests as a transformation that is not holomorphic. Therefore, the spin coefficient σ is called the *c*omplex shear, or just *s*hear for short.

The primed spin coefficients κ', ρ', and σ' have the analogous interpretations for the ingoing congruence.

5.2.8 Timelike expansion and timelike twist

Every bard recounts legends of refraction, expansion, twist, and shear, but nary a soul tells tales of τ.

We would like to elevate the standing of τ and τ' to match the renown of their colleagues, because these neglected spin coefficients convey the relativistic effects of rotating bodies at least as directly as $\mathrm{Im}(\rho)$ and $\mathrm{Im}(\rho')$ do.

First we relate what we believe to be the original understanding.[62] Return to the progenitor of null dynamics [Eq. (5.60)] with the tangent vector in Eq. (5.61). Because we insist on evolution along *o*utgoing rays, we *c*annot isolate τ by feeding $A_\mu = l_\mu$ or $A_\mu = m_\mu$ into Eq. (5.60); instead we must study τ'.

Priming the first relation in Eq. (5.53) provides the change in l'_μ:

$$Ð l' = (\rho' - \rho'^*)\, m' \wedge m + \big(\tau'\, m \wedge l' + \kappa'\, m \wedge l + c.c.\big)\ . \tag{5.65}$$

The resulting dynamical equation is:

$$þ(\blacktriangle^\mu l'_\mu) = \tau'\, \blacktriangle^\mu m_\mu + c.c.\ . \tag{5.66}$$

[62]Sachs, who pioneered the optical analogy for interpreting the spin coefficients, does not seem to have offered an explanation of τ or τ' in his original paper [152]. Szekeres, in the paper from which we extracted the term "refraction" for κ, calls the spin coefficient τ (denoted by Ω in his paper) the "*a*ngular velocity or *r*otation of the null congruence," but he does not explain why [151]. In a subsequent lecture, Sachs seems to have implicitly recognized this interpretation of τ by also choosing the symbol Ω to denote it, but he does not justify the notation [153].

Our interpretation of this abstruse formula will be guided by lightcone kinematics as articulated by Dirac [154].

In an orthonormal frame, we have a direction of evolution we call time, and we call the rate of change of the other directions with respect to it "velocity." A rotation into null coordinates preserves Poisson brackets and hence also defines a dynamical system within the computational framework of Hamiltonian mechanics. One null direction is chosen as the direction of evolution, and the other null direction is formally interpreted as "spatial."

Therefore, within this formal context, the component $l_\mu \blacktriangle^\mu$ is to be interpreted as "temporal," while the components $l'_\mu \blacktriangle^\mu$, $m_\mu \blacktriangle^\mu$, and $m'_\mu \blacktriangle^\mu$ are to be considered "spatial."

Now we make a simple-minded analogy to ordinary classical mechanics. For circular motion at fixed radius (r = constant) in the equatorial plane ($\theta = \frac{\pi}{2}$), the position vector $\vec{r}$, the velocity vector $\vec{v} \equiv \frac{d}{dt}\vec{r} \equiv \dot{\vec{r}}$, and the angular velocity vector $\vec{\Omega} = \frac{1}{|\vec{r}|^2}\vec{r} \times \vec{v}$ in spherical coordinates (r, θ, φ) are

$$\vec{r} = r\hat{r} \ , \quad \vec{v} = r\dot{\varphi}\,\hat{\varphi} \ , \quad \vec{\Omega} = -\dot{\varphi}\,\hat{\theta} \ . \tag{5.67}$$

A general vector $\vec{\blacktriangle} = \blacktriangle_r \hat{r} + \blacktriangle_\theta \hat{\theta} + \blacktriangle_\varphi \hat{\varphi}$ can therefore equally well be expanded in the (orthogonal but not normalized) basis $\vec{r}, \vec{v}, \vec{\Omega}$, with components:

$$\vec{r} \cdot \vec{\blacktriangle} = \blacktriangle_r \, r \ , \quad \vec{v} \cdot \vec{\blacktriangle} = \blacktriangle_\varphi \, r\dot{\varphi} \ , \quad \vec{\Omega} \cdot \vec{\blacktriangle} = -\blacktriangle_\theta \, \dot{\varphi} \ . \tag{5.68}$$

If (big if) the vector $\vec{\blacktriangle}$ satisfies $\vec{r} \cdot \dot{\vec{\blacktriangle}} = 0$, and if we define the angular velocity as $\omega = \dot{\varphi}$, then the middle expression in Eq. (5.68) becomes

$$\frac{d}{dt} \blacktriangle_r = \omega \, \blacktriangle_\varphi \ . \tag{5.69}$$

Comparing Eq. (5.68) to Eq. (5.66) within the understanding of lightcone kinematics is what gives us the right to call the *p*rimed spin coefficient τ' the angular velocity of the *o*utgoing congruence.[63] The *u*nprimed spin coefficient τ is the angular velocity of the *i*ngoing congruence. This counterintuitive exchange of primed and unprimed notation is an unfortunate consequence of demanding uniformity in Eqs. (5.37) and (5.38).

Now we will offer a complementary point of view based on the Sachs operation of Eq. (5.29).

[63] To go further we could write $\hat{\varphi} = -\sin\varphi \, \hat{x} + \cos\varphi \, \hat{y}$, define a complex basis vector $\hat{m} \equiv \hat{x} + i\hat{y}$, and express $\blacktriangle_\varphi$ in terms of $\hat{m} \cdot \vec{\blacktriangle}$ and its conjugate. But this would belabor the point, which at any rate is just an analogy.

The combinations $\tau \pm \tau'^*$, rather than τ and τ' separately, will appear front and center in the subsequent analysis, so let us consider their meaning and christen them with appropriate names.[64] To do that, we will heed Penrose's advice[65] and apply the Sachs operation to the real and imaginary parts of ρ.[66]

The Sachs conjugates of the expansion and the twist are:[67]

$$
\begin{aligned}
s: \qquad \mathrm{Re}(\rho) &\equiv \tfrac{1}{2}(\rho+\rho^*) \to \tfrac{1}{2}(\tau+\tau'^*) = -\frac{a^2 \sin(2\theta)}{2\sqrt{2}|R|^2 R} \ , \\
\mathrm{Im}(\rho) &\equiv \tfrac{1}{2i}(\rho-\rho^*) \to \tfrac{1}{2i}(\tau-\tau'^*) = \frac{ra\sin\theta}{\sqrt{2}|R|^2 R} \ .
\end{aligned}
\tag{5.70}
$$

Consequently, we will refer to the quantities $\tau + \tau'^*$ and $\tau - \tau'^*$ as the *t*imelike expansion and the *t*imelike twist, with the factors of $\frac{1}{2}$ and $\frac{1}{2i}$ omitted solely because we feel like it.

Note that even though we performed the Sachs operation on spin coefficients associated with l^μ, the result involved both τ and τ'. While this may be jarring at first sight, the GHP covariance of the formalism requires it: the spin coefficients ρ and ρ^* have the same weights and therefore can be added and subtracted at

[64] As we saw but did not emphasize, Eq. (5.63) did contain the combination $\tau - \tau'^*$. But we will wait until Sec. 5.2.14 to study its physical implication.

[65] See Sec. 3 of the original GHP paper [139] or Sec. 4.14 in *Spinors and Space-time* [12].

[66] As an aside, we ruminate briefly on the fact that under the Sachs operation we have $\sigma \to -\kappa$. Evidently skewing l^μ away from optimal curves is the timelike analog of shear. In keeping with the philosophy of this paper, we will content ourselves with this literal account of performing the Sachs operation and leave any potential reveries on why this had to be so for those more inclined toward that sort of homeopathy.

[67] When computing these, take care to remember that the Sachs operation does not commute with complex conjugation.

will, but τ and τ^* transform differently under Eq. (5.21) and therefore cannot be added and subtracted in a GHP-covariant way. It is only τ and τ'^* that can be added and subtracted.

If we had started with the spin coefficients ρ' and ρ'^* associated with l'^μ, then we would have obtained the same quantities as in Eq. (5.70) up to signs and complex conjugation, so there is no double counting of variables.

5.2.9 Kruskal-like coordinates

Now we establish our conventions for the null coordinates suitable for defining quantities at the horizon. Again following Boyer and Lindquist [141], we refer to these coordinates as "Kruskal-like."[68]

These coordinates are expressed conveniently in terms of the surface gravity:[69]

$$\alpha = \frac{r_+ - r_-}{2(r_+^2 + a^2)} \ . \tag{5.71}$$

[68]Historically, as far as we know, it was Boyer and Lindquist [141] and Carter [155] who first explained the maximal analytic extension of the Kerr solution. Our conventions for the Kruskal-like coordinates will largely track those of Chandrasekhar [142], except that we use U and V where he uses $\tan U$ and $\tan V$.

[69]We use the symbol "α" for the surface gravity instead of the more conventional "κ" because the symbol "κ" is already deeply embedded into the spin coefficient formalism.

The null coordinates U and V for the region outside the black hole will be defined as[70]

$$U \equiv -e^{-\alpha u} \ , \quad V \equiv +e^{+\alpha v} \ , \quad u \equiv t - r_* \ , \quad v \equiv t + r_* \ , \quad dr_* \equiv \frac{|R_0|^2}{\Delta} dr \ . \tag{5.72}$$

Note that $U < 0$, which is the standard convention. We choose the integration constant in the tortoise coordinate r_* such that the product of U and V is[71]

$$UV = -\frac{\Delta}{r_+ r_-} \left(\frac{r}{r_-} - 1 \right)^{-k} e^{2\alpha r} \ , \quad k \equiv \frac{r_-^2 + a^2}{r_+^2 + a^2} + 1 \ . \tag{5.73}$$

With these definitions, the coordinate r viewed as an implicitly defined function of U and V retains its desirable property from the spherically symmetric case of depending only on the product UV. In this form, the ratio $\frac{\Delta}{UV}$ is manifestly finite and nonzero at $r = r_+$:

$$c \equiv -\left. \frac{\Delta}{UV} \right|_{r=r_+} = r_+ r_- \left(\frac{r_+}{r_-} - 1 \right)^{k} e^{-2\alpha r_+} \ . \tag{5.74}$$

[70]Our definition of r_* follows Chandrasekhar's and is inspired by the simplified functional form of the metric at the north pole. We mention this to clarify to the beginning student that the choice of $|R_0|^2 = r^2 + a^2$ instead of $|R|^2 = r^2 + a^2 \cos^2\theta$ in the numerator of dr_* is not a typographical error.

[71]Since we always work with $r > r_-$, we have dropped the absolute values that emerge from integrating dr_*. Our coordinates are singular at the inner horizon, and a different set of Kruskal-like coordinates must be established to cross it.

For later convenience, we will also record the result of differentiating both sides of Eq. (5.73):

$$U\partial_U r = V\partial_V r = \frac{\Delta}{\Delta'(r) + \left(2\alpha - \frac{k}{r-r_-}\right)\Delta} \,. \tag{5.75}$$

For any function $F(r)$ that depends only on the radial coordinate, we therefore have:

$$U\partial_U F(r) = V\partial_V F(r) = F'(r)\, U\partial_U r \quad \text{and} \quad U\partial_U r\big|_{r=r_+} = 0 \,. \tag{5.76}$$

We will sometimes use a subscript "+" to label quantities evaluated at the horizon. For instance, $|R_+|^2 \equiv r_+^2 + a^2\cos^2\theta$ and $|R_{0+}|^2 \equiv r_+^2 + a^2$.

Finally, we define the delayed[72] angular coordinate

$$\chi = \varphi - \Omega_H t \,, \quad \Omega_H = \frac{a}{r_+^2 + a^2} \,. \tag{5.77}$$

The physical interpretation of the parameter Ω_H is the angular velocity at the horizon.[73]

[72] It is high time to retire the anachronistic modifier "retarded" in favor of something less derogatory.

[73] We choose the word "at" rather than "of" to emphasize that the reason we may think of the black hole as a rigid rotating body is that any matter that hits the horizon must rotate at the universal value of Ω_H for the angular velocity [156]. As far as we know, Carter was the first to prove [157] and emphasize [107] that the angular velocity is constant over the entire horizon.

5.2.10 A smooth tetrad

Now that we have coordinates suitable for crossing the horizon, we will perform the following GHP transformation to obtain a tetrad that is smooth at the horizon:

$$l_\mu \to \hat{l}_\mu = -U l_\mu \ , \ \ l'_\mu \to \hat{l}'_\mu = -U^{-1} l'_\mu \ , \ \ m_\mu \to \hat{m}_\mu = m_\mu \ . \tag{5.78}$$

This describes the special case

$$\lambda = \lambda^* = (-U)^{1/2} \tag{5.79}$$

of the transformation in Eq. (5.21). A hatted function with weights $(h, \bar{h})$ is related to its unhatted counterpart by

$$\hat{f}_{h,\bar{h}} = (-U)^{h+\bar{h}} f_{h,\bar{h}} \ . \tag{5.80}$$

The spin coefficients ρ and ρ' in the hatted basis,

$$\hat{\rho} = \frac{1}{R^*}(-U) \ \ \text{and} \ \ \hat{\rho}' = \frac{1}{2|R|^2}\left(\frac{\Delta}{UV}\right)\frac{1}{R^*}V \ , \tag{5.81}$$

go to zero at the future horizon ($U = 0$) and the past horizon ($V = 0$) respectively. This property furnishes a local definition of each part of the outer horizon.[74]

Because $\tau \sim (\frac{1}{2}, -\frac{1}{2})$ and $\tau' \sim (-\frac{1}{2}, \frac{1}{2})$, those two spin coefficients are invariant under the rescaling in Eq. (5.78):

$$\hat{\tau} = \tau \ , \quad \hat{\tau}' = \tau' \ . \tag{5.82}$$

After changing coordinates from (t, r, θ, φ) to (U, V, θ, χ) and performing the GHP transformation described in Eq. (5.78), we obtain the following new set of tetrad 1-forms:

$$\begin{aligned}
\hat{l} &= \frac{-1}{2\alpha}\left(1 + \frac{|R|^2}{|R_0|^2} - \Omega_H\, a \sin^2\theta\right) dU - \frac{U}{V}\left(1 - \frac{|R_{0+}|^2}{|R_0|^2}\right)\frac{a^2\sin^2\theta}{r_+ - r_-}\, dV - U a \sin^2\theta\, d\chi \ , \\
\hat{l}' &= \frac{\Delta}{2|R|^2}\left[\frac{1}{2\alpha}\left(1 + \frac{|R|^2}{|R_0|^2} - \Omega_H\, a\sin^2\theta\right)\frac{dV}{UV} + \frac{1}{U^2}\left(1 - \frac{|R_{0+}|^2}{|R_0|^2}\right)\frac{a^2\sin^2\theta}{r_+ - r_-}\, dU - \frac{a\sin^2\theta}{U}\, d\chi\right] , \\
\hat{m} &= \frac{1}{R\sqrt{2}}\left[|R|^2\, d\theta + i|R_0|^2 \sin\theta\, d\chi + \frac{ia\,\sin\theta}{r_+ - r_-}\,\frac{r + r_+}{r - r_-}\,\frac{\Delta}{UV}\,(U\, dV - V\, dU)\right] .
\end{aligned} \tag{5.83}$$

[74] We thank Aaron Zimmerman and Leo Stein for emphasizing this to us. In this paper we will discuss only apparent horizons; in principle there is a nontrivial theorem to prove here, since our perturbed space-time will not be stationary. A wise man once said, "It is important to stress that there is no evidence for the existence of black hole event horizons (as opposed to apparent horizons) in the real world" [158].

The corresponding directional derivatives are:[75]

$$\hat{D} = -2\alpha\,|R_0|^2 \frac{UV}{\Delta}\,\partial_V - a\,\frac{U}{\Delta}\left(1 - \frac{|R_0|^2}{a}\Omega_H\right)\,\partial_\chi\,,$$
$$\hat{D}' = \frac{\Delta}{2|R|^2}\left[2\alpha\,\frac{|R_0|^2}{\Delta}\,\partial_U - \frac{a}{U\Delta}\left(1 - \frac{|R_0|^2}{a}\Omega_H\right)\,\partial_\chi\right]\,,$$
$$\hat{\delta} = \frac{1}{R\sqrt{2}}\left[\partial_\theta + \frac{i}{\sin\theta}\left(1 - \Omega_H\,a\sin^2\theta\right)\,\partial_\chi + i\alpha\,a\sin\theta\,(-U\,\partial_U + V\,\partial_V)\right]\,. \quad (5.84)$$

We will refer to the forms in Eq. (5.83) and the vectors in Eq. (5.84) as composing the "horizon tetrad." The reader may verify[76] term by term that each component of the 1-forms in Eq. (5.83) and of the vectors in Eq. (5.84) is finite[77] at $U = 0$ for fixed V, and at $V = 0$ for fixed U.

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[75]Before the reader objects that the whole point of defining GHP-covariant derivatives in Eq. (5.43) was that the ordinary NP derivatives do not transform as weighted quantities, we should clarify that *given* $\hat{l}^\mu$ and $\hat{l}'^\mu$ from Eq. (5.78), we *define* $\hat{D} \equiv \hat{l}^\mu\partial_\mu$ and $\hat{D}' \equiv \hat{l}'^\mu\partial_\mu$ to obtain the expressions in Eq. (5.84). For a function $f_{h,\bar{h}} \sim (h,\bar{h})$, it is indeed true that $\hat{D}\hat{f}_{h,\bar{h}} \neq -UDf_{h,\bar{h}}$ and $\hat{D}'\hat{f}_{h,\bar{h}} \neq -U^{-1}D'f_{h,\bar{h}}$. But, by construction, we do have $\hat{\text{þ}}\hat{f}_{h,\bar{h}} = -U\,\text{þ}f_{h,\bar{h}}$ and $\hat{\text{þ}}'\hat{f}_{h,\bar{h}} = -U^{-1}\text{þ}'f_{h,\bar{h}}$.

[76]Some of the terms are manifestly finite as written, while others can be put into a form that is manifestly finite after trading factors of $r - r_+$ for factors of U or V using Eq. (5.73). Note that $1 - \frac{|R_0|^2}{a}\Omega_H = -\frac{r+r_+}{r_+^2+a^2}(r - r_+) = -\left(\frac{r+r_+}{r-r_-}\right)\frac{\Delta}{r_+^2+a^2}$. Therefore, we have $\frac{a}{\Delta}\left(1 - \frac{|R_0|^2}{a}\Omega_H\right) = -\left(\frac{r+r_+}{r-r_-}\right)\Omega_H$, which is manifestly finite and nonzero at $r = r_+$.

[77]By "finite" we mean "not infinite." Often the term "finite" is also taken to mean "nonzero," as in "not infinitesimal," but we do not accept this usage. It is important to notice when, in addition to being finite, some of the components actually do go to zero. For example, the timelike Killing vector of the Kerr-Newman space-time is:

$$\mathscr{V} \equiv \partial_t + \Omega_H\partial_\varphi = \alpha\,(-U\partial_U + V\partial_V)\,. \quad (5.85)$$

It is this vector whose components go to zero at $U = V = 0$. Note that it becomes spacelike as $r \to \infty$ [159].

5.2.11 Curvature scalars

The curvature scalars are the components of the Riemann tensor contracted with the elements of the null tetrad. We will set the stage with Cartan's definition of curvature.

The first equation of structure [Eq. (5.30)] linked the covariant derivative of the frame to the torsion. Besides the frame, we also have the connection. Like moths to a flame we are led to Cartan's second equation of structure, linking the covariant derivative of the connection to the curvature. This is expressed by defining the curvature 2-form:[78]

$$\Omega^a{}_b \equiv \mathcal{D}^a{}_c\,\omega^c{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\ . \tag{5.89}$$

[78] Now that we have introduced curvature, let us embellish an earlier point about torsion. (This is standard in the supergravity books but seems largely disparaged otherwise.) Begin with the action

$$S_{\text{gravity}}[e,\omega] = -\tfrac{1}{16\pi G}\int e\wedge e\wedge *\,\Omega = \tfrac{1}{16\pi G}\int d^4x\ \det(e)\,(e^\mu_a e^\nu_b - e^\nu_a e^\mu_b)(\partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu\,\omega_{\nu c}{}^b)\ . \tag{5.86}$$

For each classical field, define a response (the symbol ∂ denotes a variational derivative in these expressions):

$$\mathscr{E}^\mu_a \equiv \frac{2}{\det(e)}\frac{\partial S_{\text{gravity}}}{\partial e^a_\mu}\ ,\quad \mathscr{S}^\mu_{ab} \equiv \frac{2}{\det(e)}\frac{\partial S_{\text{gravity}}}{\partial \omega^{ab}_\mu}\ . \tag{5.87}$$

On contracting with frame fields, we obtain:

$$\mathscr{E}^\mu_a e^{a\nu} = -2(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R)\ ,\quad \mathscr{S}^\mu_{ab}e_{c\mu} = \gamma_{acb} - \gamma_{bca} - \lambda_{acb} + (\lambda_{ad}{}^d - \gamma_{ad}{}^d)\,\eta_{bc} - (\lambda_{bd}{}^d - \gamma_{bd}{}^d)\,\eta_{ac}\ . \tag{5.88}$$

The first expression is the familiar Einstein tensor, but with a Ricci tensor $R^{\mu\nu}$ that is not yet symmetric. It is the second expression for which we wrote this footnote. In a universe without matter fields, the equations of motion are $\mathscr{E}^\mu_a = 0$ and $\mathscr{S}^\mu_{ab} = 0$. Contracting the equation $\mathscr{S}^\mu_{ab}e_{c\mu} = 0$ with η^{bc} results in $\gamma_{ad}{}^d = \lambda_{ad}{}^d$. Inserting this back into $\mathscr{S}^\mu_{ab}e_{c\mu} = 0$ then provides $\lambda_{abc} = \gamma_{abc} - \gamma_{cba}$, the relation we obtained back in Eq. (5.31) from the torsion-free condition. Substituting this into $\mathscr{E}^\mu_a e^{a\nu} = 0$ then recovers Einstein's equation in vacuum, and with it ordinary general relativity.

This is completely analogous to the Yang-Mills field strength $F^a{}_b \equiv (\delta^a{}_c d + A^a{}_c) \wedge A^c{}_b = dA^a{}_b + A^a{}_c \wedge A^c{}_b$.[79]

We will need the Riemann tensor in the tangent space:

$$R_{abcd} \equiv (\Omega_{\mu\nu})_{ab}\, e_c^\mu e_d^\nu \ . \tag{5.90}$$

This quantity can be split into its trace-free part C_{abcd} (from the Weyl tensor $C_{\mu\nu\rho\sigma}$) and its partial-trace part R_{ab} (from the Ricci tensor $R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}$). [See Eq. (5.276) for our convention.]

Enter, Newman and Penrose: The 10 degrees of freedom in C_{abcd} are to be collected into 5 complex combinations $\Psi_0, \Psi_1, \Psi_2, \Psi_3$, and Ψ_4, which are called the *W*eyl scalars [see Eq. (5.282)].

Of the 10 degrees of freedom in R_{ab}, nine are to be assembled into a 3-by-3 hermitian matrix $\{\Phi_{\alpha\beta}\}_{\alpha,\beta=0,1,2}$ whose components are called the *R*icci scalars [see Eq. (5.283)].

[79] We remind the reader that $\omega^a{}_b$ transforms as a *c*onnection, not as a matter field in the adjoint representation. It is the field strength $\Omega^a{}_b$ that transforms as the adjoint representation.

The remaining degree of freedom is to be denoted[80] Π. We will call it the *E*instein-Hilbert curvature,[81] since it is proportional to the Lagrangian of general relativity (without the $\sqrt{-\det g}$) [see Eq. (5.284)].

These quantities can be defined operationally in the compacted version of the null Cartan formalism as follows. Just as we collected the GHP derivatives into a weighted exterior derivative [recall Eq. (5.52)], we collect the weighted spin coefficients associated with l_μ into a 1-form:[82]

$$\varpi \equiv \gamma_{31a}\, e^a = -\kappa\, l' - \tau\, l + \sigma\, m' + \rho\, m \sim (1,0)\ . \tag{5.91}$$

[80]The original NP notation for Π was Λ, and this notation was retained in the GHP paper. Later, Penrose and Rindler generalized the formalism to accommodate a nonnormalized spinor dyad and introduced a symbol Π ∝ Λ to account for this discrepancy in normalization [12]. But the symbol Λ is already seared into our brains as denoting the cosmological constant, so we refuse to appropriate it for another common symbol in general relativity. (Our graduate educations have also imbued the symbol Λ with the meaning of a cutoff in effective field theory.) Even if we were to introduce spinors we would normalize the dyad. So we adopt the alternative symbol Π and merrily elide any distinction based on the ambiguous normalization of a quantity that is of no relevance to this paper. The reader may ask why we should even agonize over this since Π is just proportional to $\mathscr{R} \equiv g^{\mu\nu} R_{\mu\nu}$. Good question.

[81]The usual terminology is "scalar curvature," but the risk of confusion that would arise from fastidiously distinguishing between "curvature scalars" and "the scalar curvature" is too substantial to be worth the effort.

[82]Penrose and Rindler use the notation S for our ϖ (see Ch. 7 of *Spinors and Space-time* [13]). We chose the symbol ϖ because it looks somewhat like ω (the spin connection) and is outlandish enough that no one seems to have reserved it for anything.

The weighted exterior derivative of this object defines the *c*ompacted curvature 2-form:[83]

$$\leo \equiv \text{Đ}\varpi \ . \tag{5.92}$$

The following notation for the components of $\leo$ in the tangent-space basis of 2-forms is customary:[84]

$$\begin{aligned}\leo \equiv\, &(\Psi_1 + \Phi_{01})\, l' \wedge l + (-\Psi_1 + \Phi_{01})\, m' \wedge m \\ &+ \Psi_0\, l' \wedge m' + \Phi_{00}\, l' \wedge m + \Phi_{02}\, m' \wedge l + (\Psi_2 + 2\Pi)\, m \wedge l \ .\end{aligned} \tag{5.93}$$

[83] Among the few papers that use the compacted formalism only a small fraction favor its Cartan formulation, so we take some liberty in defining nonstandard notation. As mentioned in an earlier footnote, a diamond in the rough is the exposition by Harnett [148]. For the GHP curvature 2-form, he uses the symbol Ω, presumably because it is a standard symbol for the curvature 2-form in differential geometry. This custom led us to choose the notation $\Omega^a{}_b$ in Eq. (5.89). But we wanted to distinguish the quantity defined in Eq. (5.92) from the $SO(3,1)$-covariant curvature. Aiming to do so without severing the association, we searched for a symbol visually distinct from but still evocative of Ω. Between this consideration and our gratitude to Leo Stein for innumerable discussions of differential geometry, the Zodiac symbol $\leo$ was fixed uniquely.

[84] To distinguish the Ψs from the Φs without resorting to index notation we should compute their behavior under conformal transformations. We will not review this here. Instead we refer the interested reader to Sec. 5.6 and 6.8 in *Spinors and Space-time* [12].

By explicitly calculating the right-hand side of Eq. (5.92) using the definitions of Đ and ϖ from Eqs. (5.52) and (5.91), we obtain the "GHP equations":

$$\Phi_{01} + \Psi_1 = þ'\kappa - þ\tau - (\tau - \tau'^*)\rho - (\tau^* - \tau')\sigma \,, \tag{5.94}$$

$$\Phi_{01} - \Psi_1 = ð'\sigma - ð\rho - (\rho - \rho^*)\tau + (\rho' - \rho'^*)\kappa \,, \tag{5.95}$$

$$\Psi_0 = -þ\sigma + ð\kappa - (\rho + \rho^*)\sigma + (\tau + \tau'^*)\kappa \,, \tag{5.96}$$

$$\Phi_{00} = -þ\rho + ð'\kappa - \rho^2 + \tau'\kappa + \tau\kappa^* - |\sigma|^2 \,, \tag{5.97}$$

$$\Phi_{02} = þ'\sigma - ð\tau - \tau^2 - \kappa\kappa'^* + \rho'\sigma + \rho\sigma'^* \,, \tag{5.98}$$

$$\Psi_2 + 2\Pi = þ'\rho - ð'\tau + \rho\rho'^* + \sigma\sigma' - |\tau|^2 - \kappa\kappa' \,. \tag{5.99}$$

Eqs. (5.94) and (5.95) can be solved for Φ_{01} and Ψ_1. Eqs. (5.96), (5.97), and (5.98) define Ψ_0, Φ_{00}, and Φ_{02}, respectively. If we prime these five equations, we will obtain definitions of $\Phi_{21} \equiv \Phi'_{01}$, $\Psi_3 \equiv \Psi'_1$, $\Psi_4 \equiv \Psi'_0$, $\Phi_{22} \equiv \Phi'_{00}$, and $\Phi_{20} \equiv \Phi'_{02}$.

This leaves us with one equation (which is actually self-prime) but three as yet undefined curvature scalars: Ψ_2, Φ_{11}, and Π.

A salient insight in the GHP paper is that the combinations $\Psi_2 - \Pi \pm \Phi_{11}$ are defined by commutators of covariant derivatives and hence have innate geometrical meaning. In our Cartan rendition of the GHP symphony, these commutator equations manifest as the following operator[85] acting on an arbitrary function $f_{h,\bar{h}} \sim (h, \bar{h})$:

$$\text{♊} \equiv \text{ÐÐ} + 2h\, \varpi \wedge \varpi' + 2\bar{h}\, (\varpi \wedge \varpi')^* \, . \tag{5.100}$$

We refer to this operator as the *c*ompacted commutator 2-form.

In Yang-Mills theory, the composition of covariant derivatives leaves in its wake a matrix of 2-forms without any derivatives, namely the curvature. Similarly, the operator in Eq. (5.100) leaves behind a collection of curvature scalars. The standard notation for the components of the compacted commutator 2-form in the usual basis is:

$$\begin{aligned} \text{♊} \equiv & - \left[2h\, (\Psi_2 - \Pi + \Phi_{11}) + 2\bar{h}\, (\Psi_2^* - \Pi + \Phi_{11})\right]\, l \wedge l' \\ & + \left[2h\, (\Psi_2 - \Pi - \Phi_{11}) - 2\bar{h}\, (\Psi_2^* - \Pi - \Phi_{11})\right]\, m \wedge m' \\ & - (2h\, \Psi_1 + 2\bar{h}\, \Phi_{01})\, m' \wedge l' - (2h\, \Phi_{01}^* + 2\bar{h}\, \Psi_1^*)\, m \wedge l' \\ & + (2h\, \Psi_3 + 2\bar{h}\, \Phi_{21})\, m \wedge l + (2h\, \Phi_{21}^* + 2\bar{h}\, \Psi_3^*)\, m' \wedge l \, . \end{aligned} \tag{5.101}$$

The coefficients of $l \wedge l'$ and $m \wedge m'$ provide the two additional independent equations required to define Ψ_2, Π, and Φ_{11}.

[85] Once we have opened the Pandora's box of Zodiac symbols, we may as well ladle out another one. We chose the symbol for Gemini because this operator includes two GHP derivatives, whereas the operator in Eq. (5.93) contains only one.

The operational definitions in this subsection completely specify the curvature in the spin coefficient formalism.

Those encountering this material for the first time might want to work through Sec. 5.B for an equivalent but more traditional derivation of Eqs. (5.94)-(5.99). The reader who wants to know where in the depths of Cartan the 2-forms in Eqs. (5.93) and (5.100) come from should consult the Appendix in Sec. 5.C.

5.2.12 Gravitational compass and Petrov classification

Szekeres conjured an elegant theoretical apparatus called the *g*ravitational compass to interpret the Weyl scalars [160].[86] Following his insight, we will say that Ψ_2 describes a Coulomb field, Ψ_4 describes a transverse outgoing wave, and Ψ_3 describes a longitudinal outgoing wave. The primed quantities, $\Psi_0 \equiv \Psi'_4$ and $\Psi_1 \equiv \Psi'_3$, describe the corresponding ingoing waves.[87]

The Weyl scalars are not gauge invariant: a local $SO(3,1)$ transformation $e^a \to O^a{}_b\, e^b$ results in $\Psi_\alpha \to \sum_{\beta=0}^{4} Q_{\alpha\beta}\Psi_\beta$ for some matrix $Q_{\alpha\beta}$. (We will see an example in Sec. 5.4.4.) We can ask how many Ψ_α can be simultaneously gauged away, and we can classify space-times based on the answer. This is Chandrasekhar's [142] account of the Petrov classification [162] of solutions to the vacuum Einstein

[86] Since concrete examples always help, we recommend the book on gravitational waves by Griffiths [161].

[87] The Coulomb component is self-prime. We might also suggest an alternative notation to make Szekeres's interpretation manifest: $\Psi_\perp \equiv \Psi_4$, $\Psi_\parallel \equiv \Psi_3$, $\Psi_C \equiv \Psi_2$, $\Psi'_\perp \equiv \Psi_0$, and $\Psi'_\parallel \equiv \Psi_1$.

equation.[88] A desire to elucidate the physics behind each Petrov type is what drove Szekeres to engineer the gravitational compass.

We will only study two Petrov types: Type D, in which all of the Weyl scalars besides Ψ_2 can be gauged away, and Type II, in which all of the Weyl scalars besides Ψ_2 and Ψ_4 can be gauged away.[89] Extending the standard terminology slightly beyond its ordinary usage, we will define a *p*rincipal frame as any tetrad basis in which as many Weyl scalars as possible for a given geometry are gauged away.

The Kerr-Newman black hole is Type D, and its nonzero Weyl scalar is

$$\Psi_2 = -\frac{1}{(R^*)^3}\left(M - \frac{Q^2}{R}\right) . \tag{5.102}$$

It is useful to note that the physical parameters of the black hole are related to the arithmetic and geometric means of the two horizons:

$$M = \tfrac{1}{2}(r_+ + r_-) \ , \quad ||(a,Q)|| \equiv (a^2 + Q^2)^{1/2} = (r_+ r_-)^{1/2} \ . \tag{5.103}$$

[88]In the spirit of the Newman-Penrose formalism we should banish any reference to curvature *t*ensors and formulate our understanding in terms of curvature *s*calars. This is the benefit of Chandrasekhar's presentation. We refer the reader to Sec. 1.9(b) of his book for details. A more mathematically elegant but physically opaque treatment requires spinors. See Ch. 8 of *Spinors and Space-time* [13].

[89]For a Type II space-time, we can rotate the tetrad to trade a nonzero Ψ_4 for a nonzero Ψ_3. This resolves the superficial discrepancy between Chandrasekhar's [142] and Penrose and Rindler's descriptions [13]. Szekeres [160] and Griffiths [161] use the terminology of Penrose and Rindler.

We do not know whether this curious fact has any deeper significance.

The charged nature of the Kerr-Newman space-time means that it is not a vacuum solution. So while the algebraic structure of the Weyl tensor is still important, it is no longer the whole story.[90] In the spin coefficient formalism, the additional information about the curvature is encapsulated by the following nonzero Ricci scalar:

$$\Phi_{11} = \frac{Q^2}{2|R|^4} \, . \tag{5.104}$$

The background electromagnetic field required to support this solution will be given shortly, after a brief interlude about the energy-momentum tensor.

5.2.13 Energy-momentum scalars

In the traditional presentation of the spin coefficient formalism, the Ricci scalars are considered a stand-in for the contribution of the energy-momentum tensor by means of Einstein's equation. We suggest that the matter should be treated with more deference.

Since the Ricci scalars are defined from the Ricci tensor itself and not its trace-reversed form,[91] we will define "energy-momentum scalars" from the *t*race-reversed

[90] Someone equipped with only a black box that spits out the Petrov type would not be able to distinguish the Kerr geometry (which is a vacuum solution) from the Kerr-Newman geometry (which requires a Maxwell field).

[91] This distinction is immaterial in the spin coefficient formalism since the tetrad is null. We only say this in case the reader wishes to apply the terminology to an orthonormal tetrad or to a tetrad that is, for example, null in the (x^0, x^3)-plane and spacelike in the (x^1, x^2)-plane.

energy-momentum tensor,

$$\mathcal{T}_{\mu\nu} \equiv T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, g^{\rho\sigma} T_{\rho\sigma} \; . \tag{5.105}$$

From these, we define:[92]

$$\begin{aligned}
&t_{00} \equiv 4\pi\mathcal{T}_{11}\;,\quad t_{01} \equiv 4\pi\mathcal{T}_{13}\;,\quad t_{02} \equiv 4\pi\mathcal{T}_{33}\;,\quad t_{22} \equiv 4\pi\mathcal{T}_{22}\;,\quad t_{21} \equiv 4\pi\mathcal{T}_{24}\;,\\
&t_{20} \equiv 4\pi\mathcal{T}_{44}\;,\quad t_{10} \equiv 4\pi\mathcal{T}_{14}\;,\quad t_{12} \equiv 4\pi\mathcal{T}_{23}\;,\quad t_{11} \equiv 2\pi(\mathcal{T}_{12} + \mathcal{T}_{34})\;,\\
&t_{\Pi} \equiv \tfrac{2\pi}{3}(\mathcal{T}_{12} - \mathcal{T}_{34}) = -\tfrac{\pi}{3} g^{\mu\nu}\mathcal{T}_{\mu\nu} = +\tfrac{2\pi}{3} g^{\mu\nu} T_{\mu\nu} \; .
\end{aligned} \tag{5.106}$$

In this notation, Einstein's equation reads[93]

$$\Phi_{\alpha\beta} = t_{\alpha\beta} \; , \tag{5.107}$$

where $\alpha, \beta \in \{0, 1, 2\}$. For the Kerr-Newman solution, the only nonzero entry is t_{11}, which can be expressed in terms of a complex number ϕ_1 called a Maxwell

[92]This absurd notation is chosen to match that of the Ricci scalars [see Eq. (5.283)].

[93]This corresponds to the tensorial equation

$$R_{\mu\nu} = +8\pi(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\rho\sigma} T_{\rho\sigma}) \; ,$$

chosen to match Eq. (11) in Sec. VI.5 of Zee [140] upon lowering the indices.

scalar:[94]

$$t_{11} = |\phi_1|^2 \ , \ \ \phi_1 = \frac{Q}{\sqrt{2}\,(R^*)^2} \ . \tag{5.108}$$

The purpose of this interlude is to emphasize that the equation $t_{11} = |\phi_1|^2$ is the statement $T_{\mu\nu} = F_{\mu\rho}F_\nu^{\ \rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$ in the tangent space, and the equation $\Phi_{11} = t_{11}$ is Einstein's equation in the tangent space. The equation $\Phi_{11} = |\phi_1|^2$ is a combination of those two steps, not either one by itself.

Specifying the curvature scalars in Eqs. (5.102) and (5.104) and the Maxwell scalar in Eq. (5.108) completes the description of the Kerr-Newman space-time in the spin coefficient formalism and technically concludes our review of the pertinent background material. But there is one further aspect of the geometry that we would like to explore before moving on to the perturbation.

5.2.14 Spacelike and timelike curvatures

We wish to discuss the curvatures of two-dimensional submanifolds in the Kerr-Newman space-time.[95] Following Penrose and Rindler [13],[96] we define the

[94] The Maxwell scalars are defined as the components of the electromagnetic curvature contracted with the vectors of the null tetrad:

$$\phi_0 \equiv F_{\mu\nu}\, l^\mu m^\nu \ , \ \ \phi_1 \equiv \tfrac{1}{2}F_{\mu\nu}\,(l^\mu l'^\nu + m'^\mu m^\nu) \ , \ \ \phi_2 \equiv F_{\mu\nu}\, m'^\mu l'^\nu \ .$$

[95] To some extent this harks back to Smarr's embedding of the Kerr-Newman horizon into E^3 [163]. The reader may also be interested in the relatively recent work by Gibbons, Herdeiro, and Rebelo [164], and by Engman and Cordero-Soto [165].

[96] Penrose and Rindler define $\mathcal{K}$ explicitly and refer to it as the "complex curvature." They do not define $\mathcal{K}_s$ explicitly, but as we said earlier, they do instruct the reader to perform the Sachs transformation to adapt their work on spacelike submanifolds to timelike submanifolds.

*s*pacelike curvature $\mathcal{K}$ and the *t*imelike curvature $\mathcal{K}_s$:

$$\mathcal{K} \equiv \sigma\sigma' - \Psi_2 - \rho\rho' + \Phi_{11} + \Pi \ , \quad \mathcal{K}_s \equiv -\kappa\kappa' - \Psi_2 + \tau\tau' - \Phi_{11} + \Pi \ . \tag{5.109}$$

These quantities appear as natural geometrical objects in the form of the right-hand side of a commutator of covariant derivatives[97] as follows:

$$\begin{aligned} \text{ÐÐ} &= l' \wedge l\,([\text{þ}, \text{þ}'] + (\tau^* - \tau')\eth + (\tau - \tau'^*)\eth') + m' \wedge m\,([\eth, \eth'] + (\rho - \rho^*)\text{þ}' - (\rho' - \rho'^*)\text{þ}) + ... \\ &= l' \wedge l\,(2h\,\mathcal{K}_s + 2\bar{h}\,\mathcal{K}_s^*) + m' \wedge m\,(-2h\,\mathcal{K} + 2\bar{h}\,\mathcal{K}^*) + ... \ , \end{aligned} \tag{5.110}$$

where the ellipses stand for the coefficients of $m \wedge l$, etc., that can be ascertained from Eq. (5.101).

The real and imaginary parts of $\mathcal{K}$ are, up to a conventional factor of 2, the ordinary notions of intrinsic and extrinsic curvatures in Riemannian[98] geometry. While it may also be useful to define the real and imaginary parts of $\mathcal{K}_s$, they cannot in general be interpreted as "intrinsic" or "extrinsic" quantities, because the vectors l^μ and l'^μ do not form a surface.[99] Let us explain as follows.

[97] Note that in Eq. (5.110) it is not just commutators but commutators plus terms linear in covariant derivatives that define curvatures. This is because the covariant derivatives themselves carry charges under the gauge group. [Recall Eq. (5.46).]

[98] Since we are talking about spacelike subspaces, the terminology "Riemannian" as opposed to "Lorentzian" (or the more general "pseudo-Riemannian") is the correct choice.

[99] We thank Leo Stein for emphasizing this to us.

Work with the hatted tetrad basis[100] and consider the commutators of GHP-covariant derivatives acting on a function of weight $(0,0)$ that depends only on the coordinates (θ,χ) on the horizon:[101]

$$[\hat{þ},\hat{þ}'] = (\hat{\tau}-\hat{\tau}'^{*})\hat{ð}' + (\hat{\tau}^{*}-\hat{\tau}')\hat{ð} \tag{5.111}$$

$$[\hat{þ},\hat{ð}] = \hat{\rho}^{*}\hat{ð} + \hat{\sigma}\hat{ð}' - \hat{\tau}'^{*}\hat{þ} - \hat{\kappa}\hat{þ}' \tag{5.112}$$

$$[\hat{þ}',\hat{ð}'] = \hat{\rho}'^{*}\hat{ð}' + \hat{\sigma}'\hat{ð} - \hat{\tau}^{*}\hat{þ}' - \hat{\kappa}'\hat{þ} \tag{5.113}$$

$$[\hat{ð},\hat{ð}'] = (\hat{\rho}-\hat{\rho}^{*})\hat{þ}' - (\hat{\rho}'-\hat{\rho}'^{*})\hat{þ} \tag{5.114}$$

Consider what happens at the origin of Kruskal-like coordinates ($U = V = 0$), starting from Eq. (5.112) and working down the list, ultimately returning back to Eq. (5.111).

Since we have aligned the tetrad with shear-free null geodesics, we have $\hat{\kappa} = \hat{\sigma} = 0$; we also know that $\hat{\rho} = 0$ at $U = 0$. Furthermore, we have parametrized the outgoing null geodesics such that[102] $\hat{\varepsilon} = 0$, and hence $\hat{þ} = \hat{D}$. So all we have left

[100] We will reject the pedagogical abomination of working in the hatted basis but omitting the hats "for convenience." Our equations should be read as written: unhatted quantities are to be computed in the asymptotic basis, and hatted quantities are to be computed in the horizon basis. If the horizon basis did not also include a GHP rescaling, then both sets of operators would be related purely by a change of coordinates, in which case there would be no need to distinguish them.

[101] The general expressions for the commutators acting on an arbitrary function of the coordinates that transforms as an arbitrary representation of the GHP group are given in Eqs. (5.311), (5.312), (5.314), and (5.315).

[102] As is clear from the discussion below Eq. (5.62), requiring the geodesic to be parametrized affinely only means setting $\mathrm{Re}(\varepsilon) = 0$; the meaning of $\mathrm{Im}(\varepsilon)$ is more obscure. We thank Jan Willem Dalhuisen for reminding us of this while assessing an early draft of our paper, and we refer the interested reader to the discussion surrounding Eq. (7.1.20) in *Spinors and Space-time* [13].

of Eq. (5.112) is $[\hat{þ}, \hat{ð}] = -\hat{\tau}'^{*}\hat{D}$. We have seen that τ' is emphatically not zero in the rotating background, but from Eq. (5.84) we have

$$\hat{D}\Big|_{U=V=0} = -2\alpha\, |R_{0+}|^2 \left.\frac{UV}{\Delta}\right|_{r=r_+} \partial_V \; . \tag{5.115}$$

Acting on a function $f(\theta, \chi)$, this gives zero. Hence for the situation we are describing, we have $[\hat{þ}, \hat{ð}] = 0$. Along similar lines, we will also find $[\hat{þ}', \hat{ð}'] = 0$.

Finally, because once again we know that $\hat{\rho} = \hat{\rho}' = 0$ at $U = V = 0$, we find $[\hat{ð}, \hat{ð}'] = 0$. This tells us that the vector fields m^μ and m'^μ form a surface. The origin of Kruskal-like coordinates is accordingly called the "bifurcation surface" of the space-time.[103] Physically it is the surface of the black hole revealing itself to us in a smooth coordinate system, as seen from someone sitting outside at radial coordinate $r > r_+$.

Whatever its meaning, $\mathrm{Im}(\varepsilon)$ can be adjusted by means of a little group transformation (i.e., $m \to e^{i\vartheta(x)}m$ with l and l' held fixed). Furthermore, when we set $\varepsilon = 0$, it is not obvious that $\hat{\varepsilon} = 0$ as well: the transformation law has to be worked out. See Eq. (5.40).

[103] Presumably this terminology comes from the literal bifurcation of Kruskal-like coordinates at $U = V = 0$.

For the Kerr-Newman space-time, the real and imaginary parts of the spacelike curvature (at arbitrary r) are[104]

$$\mathrm{Re}(\mathcal{K}) = \frac{1}{2|R|^6}\Big\{r^2(r^2+a^2) + \big[(r_+^2+a^2) - 4(r_+r+a^2) - (r^2-r_+^2)\big]\,a^2\cos^2\theta \\ +4\,\alpha\,(r_+^2+a^2)(r-r_+)\,a^2\cos^2\theta\Big\}\ , \tag{5.116}$$

and

$$\mathrm{Im}(\mathcal{K}) = \frac{a\cos\theta}{|R|^6}\big\{(r-r_+)\,r_+\,r + (2a^2+r^2)\,r - r_+a^2\cos^2\theta \\ +\alpha(r_+^2+a^2)\,\big[(2r_+-r)\,r + a^2\cos^2\theta\big]\Big\}\ . \tag{5.117}$$

Having expounded the spacelike curvature, let us come full circle by going back to Eq. (5.111) and seeing what we can conclude about the timelike curvature.

We notice immediately that $[\hat{\text{þ}}, \hat{\text{þ}}']$ depends on $\hat{ð}$ and $\hat{ð}'$. So if we try to move points around in the submanifold spanned by l^μ and l'^μ, we will inevitably end up

[104]We have not found any substantial simplification of these expressions, so let us just make the following comments. First, we have traded all dependence on the inner horizon r_- for the surface gravity α, and we have organized $\mathrm{Re}(\mathcal{K})$ and $\mathrm{Im}(\mathcal{K})$ into terms that do depend on α and terms that do not. Notice, for example, that at $r = r_+$ the dependence on α drops out of $\mathrm{Re}(\mathcal{K})$. This is perhaps a manifestation of the fact that, as we have just discussed, $\mathrm{Re}(\mathcal{K})\big|_{r=r_+}$ should describe the intrinsic curvature of the horizon, which should depend only on quantities intrinsic to the surface at $r = r_+$ and not on parameters such as r_- (and hence α) that depend on its embedding in the larger $3+1$ dimensional space-time. Second, note that under a flip in the direction of rotation of the black hole, namely $a \to -a$, $\mathrm{Re}(\mathcal{K})$ is even while $\mathrm{Im}(\mathcal{K})$ is odd.

moving in the spatial directions to which $\mathrm{Re}(m^\mu)$ and $\mathrm{Im}(m^\mu)$ are tangent: the planes will not "match up" as we attempt to traverse a closed path.

Therefore, the space transverse to the horizon does not form a surface, and there is no meaningful sense in which the corresponding timelike curvature can be split into parts "intrinsic" and "extrinsic" to it. That is all we set out to establish.

We have discussed this at length because it provides a concrete mathematical effect associated with the timelike twist defined in Eq. (5.70). This is how the physical phenomenon of frame dragging is encoded in the method of spin coefficients.

The timelike curvature for the Kerr-Newman geometry is:[105]

$$\mathcal{K}_s = \frac{-2(3r_+ - 2r)rr_+ + 2iar_+(4r - r_+)\cos\theta + a^2[5r - 2r_+ - (2r_+ - r)\cos(2\theta)] + 2ia^3\cos^3\theta}{4R^*|R|^4} \\ + \alpha\,(r_+^2 + a^2)\,\frac{(3r_+ - r)r - ia(2r - r_+)\cos\theta + a^2\cos^2\theta}{R^*|R|^4}\,. \tag{5.118}$$

Having traipsed through the background geometry, we are now ready to perturb it.

[105]To convince the reader that we have expended a modicum of effort to distill an unintelligible output from Mathematica, we note that we have organized this expression into an $O(\alpha^0)$ term and an $O(\alpha)$ term, and in each of those we have organized the numerators in powers of a. We display this insipid formula because we have not seen it anywhere, and, who knows, someone out there might find it helpful.

5.3 Shifted tetrad and Kerr-Schild form

Let us begin with the unhatted tetrad that defines the Kerr-Newman background. We introduce a completely general function $S(t,r,\theta,\varphi)$ and define the *s*hifted tetrad as follows:

$$\tilde{l} \equiv l \ , \quad \tilde{l}' \equiv l' + S\, l \ , \quad \tilde{m} \equiv m \ . \tag{5.119}$$

It cannot be emphasized enough that the meaning of $\tilde{l}'$ in components is

$$\tilde{l}' = \tilde{l}'_\mu \, dx^\mu = (l'_\mu + S l_\mu)\, dx^\mu \ , \tag{5.120}$$

not $\tilde{l}'_\mu \, d\tilde{x}^\mu$ for some shifted coordinate basis $d\tilde{x}^\mu$. Otherwise the shift would describe a change of coordinates, not a physical perturbation.

Just as the ordinary metric is defined as $ds^2 = -2ll' + 2mm'$, we define the shifted metric as

$$d\tilde{s}^2 \equiv -2\tilde{l}\tilde{l}' + 2\tilde{m}\tilde{m}' = ds^2 - 2Sll \ . \tag{5.121}$$

Since we have chosen l^μ to be tangent to a class of shear-free null geodesics of the unshifted metric, the shifted metric is of the generalized Kerr-Schild form, as defined by Taub [137]. If we turn off the angular momentum and the charge and choose the particular ansatz[106]

$$S = \frac{\Delta}{2r^2}\frac{U}{V}\,\delta(U)\,f(\theta,\varphi) \qquad (a = Q = 0) \tag{5.122}$$

then we will reproduce exactly the Dray-'t Hooft metric.[107] If we turn off the angular momentum but leave the charge nonzero and use the same functional form for the ansatz, we will reproduce exactly the metric of Alonso-Zamorano and Sfetsos.

5.3.1 From Reissner-Nordström to Kerr-Newman

To generalize to the Kerr-Newman background, we will scrutinize the factors that appear in the ansatz.

First, by revisiting our conventions for the unshifted tetrad and staring at the definition of the shifted one, we conclude that the factor $\frac{\Delta}{2r^2}$ is there to compensate for the asymmetric normalization[108] of l_μ relative to l'_μ. So the generalization of

[106] Just to be clear, in this equation we mean $\Delta = r(r-2M)$, which is the $a = Q = 0$ limit of the more general definition for the Kerr-Newman horizon function in Eq. (5.7). This is why it is convenient to work with Δ instead of $r - 2M$ for the Schwarzschild solution.

[107] When comparing this to Dray-'t Hooft [29] and to Sfetsos [35], note that we are working in their "hatted" coordinate system (which has nothing to do with our hatted tetrad basis). We also make the trivial modification of using capital letters instead of lowercase letters for the Kruskal-like coordinates, because we prefer to reserve lowercase letters for the "Eddington-like" combinations $t \pm r_*$.

[108] It might be worth considering a symmetric normalization in which both null forms are given an overall factor of $\left(\frac{\Delta}{2r^2}\right)^{1/2}$. We have not seriously explored this possibility for two reasons. First, to make our work accessible to beginners, we elected to parallel many of the conventions in Chandrasekhar's account of the Schwarzschild and Kerr solutions in Newman-Penrose form [142].

this factor to the rotating case is clear:

$$\frac{\Delta}{2r^2} \to \frac{\Delta}{2|R|^2} \ . \tag{5.123}$$

Second, we have defined the Kruskal-like coordinates so that they mimic the coordinates in the nonrotating case: the future horizon is still at $U = 0$ and the radial function r depends only on the product UV. So we might hope that the factor $\frac{U}{V}\,\delta(U)$ could remain unmodified.

Third, we recognize that the function $f(\theta, \varphi)$ is defined only at the origin of Kruskal-like coordinates ($U = V = 0$). Extrapolating to the Kerr-Newman space-time should therefore entail the generalization

$$(\theta, \varphi) \to (\theta, \chi) \ . \tag{5.124}$$

This cross-examination of the Dray-'t Hooft solution coupled with the clear geometrical underpinning of the Newman-Penrose formalism led us to the conviction that the perturbed Kerr-Newman geometry should be described by the shifted tetrad in Eq. (5.119) with the following ansatz:

$$S = \frac{\Delta}{2|R|^2}\,\frac{U}{V}\,\delta(U)\,f(\theta, \chi) \ . \tag{5.125}$$

Second, as a practical matter both for calculation by hand and for symbolic manipulation in Mathematica, we found it inconvenient to deal with square roots of those functions.

We will call S the "shift function," and we will call $f(\theta,\chi)$ the "horizon field." When we calculate the curvature scalars, we will work directly with the rescaled tetrad in Eq. (5.84), thereby enlisting the rescaled shift function

$$\hat{S} = (-U)^{-2} S = \frac{1}{2|R|^2} \frac{\Delta}{UV}\, \delta(U) f(\theta,\chi) \; . \tag{5.126}$$

Like everything else in the hatted tetrad basis, this shift function is finite at the horizon.

By comparing the GHP representations $l'_\mu \sim (-\frac{1}{2},-\frac{1}{2})$ and $l_\mu \sim (+\frac{1}{2},+\frac{1}{2})$ in the context of Eq. (5.119), we deduce that the shift function must transform as:

$$S \sim (-1,-1) \; . \tag{5.127}$$

When interpreting the formulas Eqs. (5.125) and (5.126) in the GHP formalism, we assign the horizon field $f(\theta,\chi)$ the weights of the shift function:

$$f(\theta,\chi) \sim (-1,-1) \; . \tag{5.128}$$

The remaining factors are to be treated as ordinary functions, not physical degrees of freedom, and are therefore assigned weights $(0,0)$.

By explicit calculation, we will indeed find that the ansatz in Eq. (5.125) results in a shifted Ricci tensor of the form

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + R_{UU}^{\text{shift}}\, \delta_\mu^U \delta_\nu^U \tag{5.129}$$

and therefore correctly generalizes the Dray-'t Hooft metric to a rotating background.

5.3.2 Preliminary commentary

Before focusing on the explicit result for R_{UU}, we wish to preview an amazing property of many generalized Kerr-Schild space-times: the Ricci tensor for the shifted metric will depend only *l*inearly on the shift function S, provided that the unshifted tetrad is aligned with the shear-free null geodesics ($\kappa = \sigma = \kappa' = \sigma' = 0$) and that the unshifted Φ_{00} is zero.

This will hold *b*efore specializing to our ansatz for S in terms of f: the non-linearity drops out for a completely general shift function. This crucial property was in fact noticed by Taub [137] and by Alonso and Zamorano [123]; a related but even more general property was recently explored by Harte [166].

We will proceed step by step through the spin coefficient formalism to understand why this happens. A practical reason is to provide explicit formulas for the spin coefficients and curvature scalars of the most general Kerr-Schild metric in terms of the spin coefficients and curvature scalars of the background. For the spin coefficients we maintain full generality in the background, but for the curvature scalars we will restrict to shear-free geodesic congruences.

5.3.3 Shifted spin coefficients

By shifting both sides of the Cartan equations and solving them, or by shifting the definition of each spin coefficient directly,[109] we can express the shifted spin coefficients in terms of their unshifted values. We will derive the system of equations obtained from shifting the null Cartan equations in Eq. (5.49).

We start with the simplest case, the equation for dl. First, tilde[110] every term:

$$d\tilde{l} = -2\,\mathrm{Re}(\tilde{\varepsilon})\,\tilde{l}\wedge\tilde{l}' + 2i\,\mathrm{Im}(\tilde{\rho})\,\tilde{m}\wedge\tilde{m}' + \left[\left(\tilde{\tau} - \tilde{\beta} + \tilde{\beta}'^{*}\right)\tilde{m}'\wedge\tilde{l} + \tilde{\kappa}\,\tilde{m}'\wedge\tilde{l}' + c.c.\right]. \tag{5.130}$$

Note that, in keeping with our advisory remark below the original definition of the shifted tetrad [Eq. (5.119)], we do not tilde the exterior derivative operator.

[109] This requires first solving the matrix inversion problem in Eq. (5.11) for the shifted vectors. In the present situation this is simple, but we focus on the Cartan equations to conceivably enable a more general perturbation. Shifting the spin coefficients directly must be done using Chandrasekhar's "λ-symbols" [see Eq. (5.31)]. Otherwise we would have to establish the shifted space-time-covariant derivative $\tilde{\nabla}_\mu$, defined such that $\tilde{\nabla}_\mu \tilde{g}_{\nu\rho} \equiv 0$, before adorning each symbol with its ceremonial tilde.

[110] American physicists commonly transform the nouns "prime" and "hat" into verbs ("to prime" and "to hat") and adjectives ("primed"/"unprimed" and "hatted"/"unhatted"). Here we affirm this "age-old custom" of functional shift [167] by transforming the noun "tilde" into the verb "to tilde" and the adjectives "tilded"/"untilded."

Begin with the right-hand side. By plugging in the definition of the shifted tetrad in terms of the unshifted tetrad and using the 1-form property $l \wedge l = 0$, we find:

$$d\tilde{l} = -2\,\mathrm{Re}(\tilde{\varepsilon})\, l \wedge l' + 2i\,\mathrm{Im}(\tilde{\rho})\, m \wedge m' + \left[\left(\tilde{\tau} - \tilde{\beta} + \tilde{\beta}'^{*} + \tilde{\kappa}\, S \right) m' \wedge l + \tilde{\kappa}\, m' \wedge l' + c.c. \right] . \tag{5.131}$$

Since $\tilde{l} = l$, we have $d\tilde{l} = dl$, so the left-hand side can be replaced with the nontilded version of Eq. (5.130). The four basis 2-forms $l \wedge l'$, $m \wedge m'$, $m' \wedge l$, and $m' \wedge l'$ are linearly independent, so we can match their coefficients on both sides to obtain the first set of shifted spin coefficient equations:

$$\mathrm{Re}(\tilde{\varepsilon}) = \mathrm{Re}(\varepsilon)\ , \quad \mathrm{Im}(\tilde{\rho}) = \mathrm{Im}(\rho)\ , \quad \tilde{\tau} - \tilde{\beta} + \tilde{\beta}'^{*} + \tilde{\kappa}\, S = \tau - \beta + \beta'^{*}\ , \quad \tilde{\kappa} = \kappa\ . \tag{5.132}$$

Next, we repeat the procedure for dl', starting with its shifted version:

$$d\tilde{l}' = -2\,\mathrm{Re}(\tilde{\varepsilon}')\, \tilde{l}' \wedge \tilde{l} + 2i\,\mathrm{Im}(\tilde{\rho}')\, \tilde{m}' \wedge \tilde{m} + \left[\left(\tilde{\tau}' - \tilde{\beta}' + \tilde{\beta}^{*} \right) \tilde{m} \wedge \tilde{l}' + \tilde{\kappa}'\, \tilde{m} \wedge \tilde{l} + c.c. \right] . \tag{5.133}$$

The steps for the right-hand side of the equation parallel those for dl, resulting in:

$$
\begin{aligned}
d\tilde{l}' &= -2\,\mathrm{Re}(\tilde{\varepsilon}')\, l' \wedge l + 2i\,\mathrm{Im}(\tilde{\rho}')\, m' \wedge m + \\
&+ \left\{ \left(\tilde{\tau}' - \tilde{\beta}' + \tilde{\beta}^*\right)\, m \wedge l' + \left[\tilde{\kappa}' + \left(\tilde{\tau}' - \tilde{\beta}' + \tilde{\beta}^*\right) S\right] m \wedge l + c.c. \right\} \ . \qquad (5.134)
\end{aligned}
$$

But this time, since $l' = l' + Sl$, the steps for the left-hand side are more complicated. Not only do we require the nontilded equations for both dl' and dl, we also require the exterior derivative of the shift function:[111]

$$
dS \equiv dx^\mu\, \partial_\mu S = e^a\, \partial_a S = -l\, D'S - l' DS + m\, \delta' S + m' \delta S \ . \qquad (5.135)
$$

Then we have, for the left-hand side of Eq. (5.133)

$$
\begin{aligned}
d(l' + Sl) = &\left[-2\,\mathrm{Re}(\varepsilon') + 2\,\mathrm{Re}(\varepsilon) S - (DS)\right]\, l' \wedge l + \left[2i\,\mathrm{Im}(\rho') - 2i\,\mathrm{Im}(\rho) S\right]\, m' \wedge m \\
&+ \left\{ \left(\tau' - \beta' + \beta^* + \kappa^* S\right)\, m \wedge l' + \left[\kappa' + \left(\tau^* - \beta^* + \beta'\right) S + (\delta' S)\right]\, m \wedge l + c.c. \right\} .
\end{aligned}
\qquad (5.136)
$$

[111] Note that because this involves the 1-forms e^a, our expression for dS differs by an overall sign relative to what one would obtain using the opposite metric signature.

Matching the coefficients of the basis 2-forms provides the second set of shifted spin coefficient equations:

$$\mathrm{Re}(\tilde{\varepsilon}') = \mathrm{Re}(\varepsilon') - \mathrm{Re}(\varepsilon)S + \tfrac{1}{2}DS \ , \ \ \mathrm{Im}(\tilde{\rho}') = \mathrm{Im}(\rho') - \mathrm{Im}(\rho)S \ ,$$
$$\tilde{\tau}' - \tilde{\beta}' + \tilde{\beta}^* = \tau' - \beta' + \beta^* + \kappa^* S \ , \ \ \tilde{\kappa}' + (\tilde{\tau}' - \tilde{\beta}' + \tilde{\beta}^*)S = \kappa' + (\tau^* - \beta^* + \beta')S + \delta' S \ . \tag{5.137}$$

Before moving to the final set of equations, it is helpful to take stock of where we are. We have already solved directly for $\mathrm{Re}(\tilde{\varepsilon})$, $\mathrm{Im}(\tilde{\rho})$, and $\tilde{\kappa}$, and may thereby observe that they remain unshifted. We have also solved for $\mathrm{Re}(\tilde{\varepsilon}')$ and $\mathrm{Im}(\tilde{\rho}')$. By plugging the third equation in Eq. (5.137) into the fourth one, we obtain the shifted κ':

$$\tilde{\kappa}' = \kappa' + \left(\delta' - 2\beta^* + 2\beta'\right) S + (\tau^* - \tau')S - \kappa^* S^2 \tag{5.138}$$

Recall that $S \sim (-1,-1)$ and that the GHP-covariant version of δ' is $\eth' = \delta' - 2h\beta' + 2h\beta^*$. As expected from GHP covariance, the NP derivatives and gauge fields appear in just the right combination to form a covariant derivative:

$$\tilde{\kappa}' = \kappa' + \eth' S + (\tau^* - \tau')S - \kappa^* S^2 \ . \tag{5.139}$$

On the other hand, the terms involving D, ε, and ε^* in $\mathrm{Re}(\tilde{\varepsilon}')$ do not collect themselves into a GHP-covariant combination. But that too is expected: while

$\tilde{\kappa}'$ is a weighted quantity, $\tilde{\varepsilon}'$ is not. By solving the matrix inversion problem in Eq. (5.11) for the shifted tetrad, we obtain the shifted NP derivatives:

$$\tilde{D} = D \;, \quad \tilde{D}' = D' - S\,D \;, \quad \tilde{\delta} = \delta \;. \tag{5.140}$$

We will see that $\tilde{\varepsilon}'$ does in fact combine with $\tilde{D}'$ to form a shifted $\tilde{\text{þ}}'$ that can be expressed completely in terms of GHP-covariant quantities. But to prove that, we will need to solve for the shifted $\text{Im}(\tilde{\varepsilon}')$, and for that we will need to study dm.

Placing tildes on each term of the defining equation for dm in Eq. (5.49), we have:

$$\begin{aligned} d\tilde{m} = {} & (\tilde{\beta} + \tilde{\beta}'^{*})\,\tilde{m} \wedge \tilde{m}' - (\tilde{\tau} - \tilde{\tau}'^{*})\,\tilde{l} \wedge \tilde{l}' + \big(\tilde{\rho} - 2i\,\text{Im}(\tilde{\varepsilon})\big)\;\tilde{m} \wedge \tilde{l}' + \tilde{\sigma}\,\tilde{m}' \wedge \tilde{l}' \\ & + \big(\tilde{\rho}' + 2i\,\text{Im}(\tilde{\varepsilon}')\big)\;\tilde{m} \wedge \tilde{l} + \tilde{\sigma}'^{*}\tilde{m}' \wedge \tilde{l} \end{aligned} \tag{5.141}$$

$$\begin{aligned} = {} & (\tilde{\beta} + \tilde{\beta}'^{*})\,m \wedge m' - (\tilde{\tau} - \tilde{\tau}'^{*})\,l \wedge l' + (\tilde{\rho} - 2i\,\text{Im}(\tilde{\varepsilon}))\,m \wedge l' + \tilde{\sigma}\,m' \wedge l' \\ & + \Big[\tilde{\rho}' + 2i\,\text{Im}(\tilde{\varepsilon}') + \big(\tilde{\rho} - 2i\,\text{Im}(\tilde{\varepsilon})\big)\,S\Big]\;m \wedge l + \big(\tilde{\sigma}'^{*} + \tilde{\sigma}S\big)\;m' \wedge l \;. \end{aligned} \tag{5.142}$$

Since $d\tilde{m} = dm$, we can equate the coefficients on the right-hand side of Eq. (5.142) with the nontilded version of Eq. (5.141). This gives us the third set of shifted spin coefficient equations:

$$\tilde{\beta} + \tilde{\beta}'^* = \beta + \beta'^* \ , \quad \tilde{\tau} - \tilde{\tau}'^* = \tau - \tau'^* \ , \quad \tilde{\rho} - 2i\,\mathrm{Im}(\tilde{\varepsilon}) = \rho - 2i\,\mathrm{Im}(\varepsilon) \ ,$$

$$\tilde{\rho}' + 2i\,\mathrm{Im}(\tilde{\varepsilon}') + (\tilde{\rho} - 2i\,\mathrm{Im}(\tilde{\varepsilon}))S = \rho' + 2i\,\mathrm{Im}(\varepsilon') \ , \quad \tilde{\sigma}'^* + \tilde{\sigma}S = \sigma'^* \ . \tag{5.143}$$

This completes the set of equations required to solve for all of the shifted spin coefficients.

By solving Eqs. (5.132), (5.137), and (5.143), we learn that the weighted spin coefficients and gauge field associated with l_μ do not receive corrections:

$$\tilde{\kappa} = \kappa \ , \quad \tilde{\tau} = \tau \ , \quad \tilde{\sigma} = \sigma \ , \quad \tilde{\rho} = \rho \ , \quad \tilde{\varepsilon} = \varepsilon \ . \tag{5.144}$$

While it should not be surprising that κ, σ, ρ, and ε do not receive corrections, it may be unexpected that τ does not shift. It turns out that τ' also remains unshifted:

$$\tilde{\tau}' = \tau' \ . \tag{5.145}$$

So the timelike expansion $\tau + \tau'^*$ and the timelike twist $\tau - \tau'^*$ remain unshifted.

Let us emphasize again that at this stage we are dealing with the most general Kerr-Schild metric and have not assumed anything about the tetrad besides the original definition of the shift [Eq. (5.119)]. In particular, we have not yet required the geodesic conditions $\kappa = \kappa' = 0$ or the shear-free conditions $\sigma = \sigma' = 0$.

The weighted spin coefficients and gauge field associated with l'_μ do receive corrections:

$$\tilde{\kappa}' = \kappa' + \Big[ð' + \big(\tau^* - \tau'\big) \Big] S - \kappa^* S^2 \,,$$
$$\tilde{\sigma}' = \sigma' - \sigma^* S \,, \ \ \tilde{\rho}' = \rho' - \rho\, S \,,$$
$$\tilde{\varepsilon}' = \varepsilon' - \varepsilon^* S + \tfrac{1}{2} DS - i\,\mathrm{Im}(\rho)\, S \,. \qquad (5.146)$$

In general, the transverse gauge fields also receive corrections:

$$\tilde{\beta} = \beta + \tfrac{1}{2}\kappa\, S \,, \ \ \tilde{\beta}' = \beta' - \tfrac{1}{2}\kappa^* S \,. \qquad (5.147)$$

Now we will establish our first crucial property of generalized Kerr-Schild metrics: if we align l^μ to be tangent to a geodesic of the background geometry, namely if $\kappa = 0$, then not only do the formulas simplify considerably, but all nonlinearity in the shift function drops out of the spin coefficients.

This already means that the curvature tensor in the tangent space, which is the quantity from which the curvature scalars are defined, will depend on the shift function only to a power no higher than two:

$$\tilde{R}_{abcd} = R_{abcd} + S\, R^{(1)}_{abcd} + S^2\, R^{(2)}_{abcd} \,. \qquad (5.148)$$

Furthermore, if the geodesic to which l^μ is aligned can also be taken shear-free, namely $\sigma = 0$, then we get $\tilde{\sigma}' = \sigma$ as well. Finally, if we also choose l'^μ to be tangent to a geodesic of the background, then $t$he only spin coefficients that receive corrections are:[112]

$$\tilde{\kappa}' = \Big[\eth' + \big(\tau^* - \tau'\big)\Big] S \ , \quad \tilde{\rho}' = \rho' - \rho\, S \ , \quad \tilde{\varepsilon}' = \varepsilon' - \varepsilon^* S + \tfrac{1}{2} DS - i\,\mathrm{Im}(\rho)\, S \ . \quad (5.149)$$

In this case, the only GHP-covariant derivative that shifts is $\text{þ}'$. The shifted version acting on a function $f_{h,\bar{h}} \sim (h, \bar{h})$ is:

$$\tilde{\text{þ}}' f_{h,\bar{h}} = \big[\text{þ}' - S\text{þ} - (h + \bar{h})(\text{þ}S) + 2i(h - \bar{h})\mathrm{Im}(\rho) S\big]\, f_{h,\bar{h}} \ . \quad (5.150)$$

This vindicates the discussion below Eq. (5.139) and completes our derivation of the shifted spin coefficients.

Dray and 't Hooft described two physical effects of the shift on the trajectories of test particles that might cross it: a translation toward the future horizon, and a refraction effect [29]. In the spin coefficient formalism, these are described by the shifted versions of ρ' and κ'. We will now explain this assertion. The reader may also wish to consult the analysis of the Dray-'t Hooft geometry by Matzner [34].

[112] In this formula we have left ε nonzero to treat ε and ε' more symmetrically.

5.3.4 Shifted horizon

Let us work in the hatted basis. As we discussed back in Sec. 5.2.10, the future horizon of this geometry can be defined locally as the subspace of Kruskal-like coordinates on which the expansion of the outgoing congruence vanishes ($\hat{\rho} = 0$). Similarly, the past horizon is the subspace on which the expansion of the ingoing congruence vanishes ($\hat{\rho}' = 0$).

Recalling the unshifted values of $\hat{\rho}$ and $\hat{\rho}'$ in Eq. (5.81) and the shift described in Eq. (5.149), we find that the coordinate V receives a correction while the coordinate U does not:

$$\begin{aligned}
&\tilde{\hat{\rho}} = \hat{\rho} \implies \tilde{U} = U \ , \\
&\tilde{\hat{\rho}}' = \hat{\rho}' - \hat{\rho}\,\hat{S} = \left(1 + \frac{U}{V}\,\delta(U)\,f(\theta,\chi)\right)\hat{\rho}' \equiv \left[\frac{1}{2|R|^2}\left(\frac{\Delta}{UV}\right)\frac{1}{R^*}\right]\tilde{V} \\
&\qquad \implies \tilde{V} - V = U\,\delta(U)\,f(\theta,\chi) \ . \qquad (5.151)
\end{aligned}$$

This last expression is subtle, because it states that smooth functions of U will experience no coordinate shift; it is only functions that go as $\frac{1}{U}$ near $U = 0$ that will experience a discontinuity in the coordinate.

But that is correct: if we interpret Eq. (5.151) as a differential equation in U in the vicinity of $U = 0$, as in $\frac{d(\tilde{V}-V)}{dU} = \lim_{U\to 0}\frac{\tilde{V}-V}{U} = \delta(U)\,f(\theta,\chi)$, then integration gives

$$\tilde{V} = V + \Theta(U)\, f(\theta, \chi) \; . \tag{5.152}$$

This is the shift as described by Dray and 't Hooft and by Sfetsos.

5.3.5 Refraction

We explained back in Sec. 5.2.7 that κ and κ' describe the refraction of light rays. Consequently, the result that κ' becomes nonzero after the shift speaks for itself. But it is also interesting to ask what happens when we realign the ingoing null curves with the corrected geodesics. We will content ourselves with a relatively qualitative discussion of this issue.

Begin with the following rotation[113] of the null tetrad:

$$\begin{aligned} & l \to l^{\circ} \equiv l \; , \\ & m \to m^{\circ} \equiv m + \mathscr{A} l \; , \\ & l' \to l'^{\circ} \equiv l' + \mathscr{A}^* m + \mathscr{A}\, m' + |\mathscr{A}|^2 l \; , \end{aligned} \tag{5.153}$$

where $\mathscr{A}$ is an arbitrary nonzero complex function. By comparing the representations of the GHP group under which l and m transform, namely $l \sim (\frac{1}{2}, \frac{1}{2})$ and

[113]This is called a "rotation of class I." An analogous rotation that leaves l' invariant is called a "rotation of class II." An alternative name for the GHP transformation is "a rotation of class III."

$m \sim (\frac{1}{2}, -\frac{1}{2})$, we conclude that the function $\mathscr{A}$ should be assigned the weights $(0, -1)$, and its conjugate the weights $\mathscr{A}^* \sim (-1, 0)$ (we remind the reader that complex conjugation exchanges $h \leftrightarrow \bar{h}$).

Under the transformation in Eq. (5.153), the weighted spin coefficients associated with l transform as:[114]

$$\kappa \to \kappa^\circ = \kappa\ , \quad \sigma \to \sigma^\circ = \sigma + \mathscr{A}\,\kappa\ ,$$
$$\rho \to \rho^\circ = \rho + \mathscr{A}^*\,\kappa\ , \quad \tau \to \tau^\circ = \tau + \mathscr{A}\,\rho + \mathscr{A}^*\sigma + |\mathscr{A}|^2\kappa\ . \tag{5.154}$$

The weighted spin coefficients associated with l' transform as:

$$\kappa' \to \kappa'^\circ = \kappa' + \mathscr{A}\,\sigma' + \mathscr{A}^*\rho' + |\mathscr{A}|^2\tau' + (\mathscr{A}^*)^2\left(|\mathscr{A}|^2\kappa - \tau - \mathscr{A}^*\sigma + \mathscr{A}\rho\right)$$
$$+ (\text{þ}' + \mathscr{A}^*\eth + \mathscr{A}\,\eth' + |\mathscr{A}|^2\text{þ})\mathscr{A}^*\ , \tag{5.155}$$

$$\sigma' \to \sigma'^\circ = \sigma' + \mathscr{A}^*\tau' - (\mathscr{A}^*)^2\rho - (\mathscr{A}^*)^3\kappa - \eth'\mathscr{A}^* - \mathscr{A}^*\text{þ}\mathscr{A}^*\ , \tag{5.156}$$

$$\rho' \to \rho'^\circ = \rho' + \mathscr{A}\tau' - (\mathscr{A}^*)^2\sigma - \mathscr{A}(\mathscr{A}^*)^2\kappa - \eth\mathscr{A}^* - \mathscr{A}\text{þ}\mathscr{A}^*\ , \tag{5.157}$$

$$\tau' \to \tau'^\circ = \tau' - (\mathscr{A}^*)^2\kappa - \text{þ}\mathscr{A}^*\ . \tag{5.158}$$

[114] These transformation laws are common knowledge, but they are typically presented in the original Newman-Penrose notation, which makes them unsuitable for civilized prose. We reproduce them here mainly for our own convenience, as well as for the beginning student who should not be required to brandish safety glasses to witness them.

The GHP gauge fields transform as:

$$\varepsilon \to \varepsilon^{\circ} = \varepsilon + \mathscr{A}^*\kappa \; , \;\; \beta \to \overset{\circ}{\beta} = \beta + \mathscr{A}\varepsilon + \mathscr{A}^*\sigma + |\mathscr{A}|^2\kappa \; ,$$

$$\varepsilon' \to \varepsilon'^{\circ} = \varepsilon' + \mathscr{A}\beta' - \mathscr{A}^*(\beta + \tau) - |\mathscr{A}|^2(\rho + \varepsilon) - (\mathscr{A}^*)^2\sigma - |\mathscr{A}|^2\mathscr{A}^*\kappa \; ,$$

$$\beta' \to \beta'^{\circ} = \beta' - \mathscr{A}^*(\rho + \varepsilon) - (\mathscr{A}^*)^2\kappa \; . \tag{5.159}$$

We will be concerned primarily with the spin coefficients κ' and σ'. We would like to apply the transformation defined in Eq. (5.153) to the *s*hifted geometry, which entails putting tildes on *b*oth sides of Eqs. (5.155) and (5.156). For the shifted Kerr-Newman space-time, the rotated κ' is:

$$\begin{aligned} \tilde{\kappa}'^{\circ} = \tilde{\kappa}' + \mathscr{A}^*\tilde{\rho}' - (\mathscr{A}^*)^2\tau + |\mathscr{A}|^2\tau' + |\mathscr{A}|^2\mathscr{A}^*\rho \\ + (\tilde{þ}' + \mathscr{A}^*ð + \mathscr{A}ð' + |\mathscr{A}|^2 þ)\mathscr{A}^* \; . \end{aligned} \tag{5.160}$$

Because of the representation $\mathscr{A}^* \sim (-1, 0)$, we have for the shifted $þ'$:

$$\tilde{þ}'\mathscr{A}^* = \left[þ' - Sþ + (þS) + 2i\,\mathrm{Im}(\rho)S\right]\mathscr{A}^* \; . \tag{5.161}$$

After this list of technical statements, we come to the main point: realigning l' with the ingoing geodesics of the shifted space-time requires solving the condition

$$\tilde{\kappa}'^{\circ} \equiv 0 \tag{5.162}$$

for the function $\mathscr{A}$. We will not attempt this, which is why we cautioned that our discussion will be relatively qualitative. But were we to solve Eq. (5.162), we would find that the shifted ingoing geodesics to which $\tilde{l}'^{\circ}$ is now tangent would exhibit nonzero shear:

$$\tilde{\sigma}'^{\circ} = -(\eth' - \tau')\mathscr{A}^{*} - \mathscr{A}^{*}(\text{þ} + \rho)\mathscr{A}^{*} \ . \tag{5.163}$$

Apparently the advent of shear along the ingoing geodesic[115] congruence is the manifestation of the Dray-'t Hooft refraction effect in the spin coefficient formalism.

[115] We will not use Penrose's preferred word "geodetic" for the adjectival form of the noun "geodesic." We understand that an author who would introduce an independent symbol for each type of hodge duality in four dimensions might also insist on this, but it sounds too academic for our taste. [See Eq. (4.13.16) in *Spinors and Space-time*.] The word "geodesic" can function perfectly as both an adjective and a noun without sacrificing one iota of clarity.

5.4 Petrov classification for the Kerr-Newman shockwave

We will first shift the Weyl scalar $\Psi_4 \sim (-2, 0)$, or more conveniently its complex conjugate $\Psi_4^* \sim (0, -2)$. Since this is just our opening act, we will focus mainly on the results. Intricate computational details will be reserved for the main event, the shifted Ricci scalars.

5.4.1 Shifted Ψ_4 and physical interpretation

Aligning the background tetrad with shear-free geodesic congruences, but assuming an arbitrary shift function S, we find:

$$\tilde{\Psi}_4^* = ðð S + 2\tau\, ð S \; . \tag{5.164}$$

To obtain this expression we used the complex conjugate and the prime of Eq. (5.98) in the forms $ð'\tau^* = -\tau^{*2}$ and $ð'\tau' = -\tau'^2$, which hold when $\Phi_{02} = 0$.

To specialize to the shockwave, hat everything and insert the ansatz of Eq. (5.126) for the shift function. Since the calculation is laborious, it is advantageous to first enumerate the types of terms that could appear.

Recall that the horizon field $f(\theta, \chi)$ has weights $(-1, -1)$. Since Ψ_4^* has weights $(0, -2)$, we will have to find operators of weights $(1, -1)$. Fortunately, the list of such operators that are nonzero at the Kerr-Newman horizon is surprisingly short:

$$\eth\eth \, ; \quad \tau\eth \, , \quad \tau'^{*}\eth \, ; \quad \tau^2 \, , \quad \tau'^{*\,2} \, , \quad \tau\tau'^{*} \, . \tag{5.165}$$

In principle we would also need $\eth\tau$ and $\eth\tau'^{*}$, but again when $\Phi_{02} = 0$ those can be traded for $-\tau^2$ and $-\tau'^{*2}$. So the result must have the form

$$\hat{\hat{\Psi}}_4^{*} = k_0 \, \delta(U) \left[\eth\eth f + (k_1 \, \tau + k_2 \, \tau'^{*})\eth f + (k_3 \, \tau^2 + k_4 \, \tau'^{*\,2} + k_5 \, \tau\tau'^{*}) f \right] \tag{5.166}$$

for some functions $k_i(\theta)$ that will depend on the parameters r_+, a, and α. Whether by hand or by machine we ultimately find:

$$\begin{aligned} &k_0 = -\tfrac{c}{2|R|^2} \, , \quad k_1 = \tfrac{\alpha|R|^2}{r} \, , \quad k_2 = -2\left(1 + \tfrac{\alpha|R|^2}{2r}\right) \, , \quad k_3 = \left(\tfrac{\alpha|R|^2}{2r}\right)^2 \, , \\ &k_4 = 1 + \left(1 + \tfrac{\alpha|R|^2}{2r}\right)^2 \, , \quad k_5 = -\tfrac{\alpha|R|^2}{r}\left(1 + \tfrac{\alpha|R|^2}{2r}\right) \, . \end{aligned} \tag{5.167}$$

On the way to obtaining this expression, we necessarily encounter terms involving $\partial_U\delta(U)$ and $\partial_U^2\delta(U)$.[116] We interpret them according to the distributional edict of integrating by parts against an arbitrary smooth test function $\mathcal{F}(U)$:

$$\int dU \, \mathcal{F}(U) \, \partial_U{}^n \delta(U) = \int dU \, (-1)^n \, \partial_U{}^n \mathcal{F}(U) \, \delta(U) \, . \tag{5.168}$$

[116] We also stumble upon the gargantuan notational implosion "$\eth\delta(U) = \delta\delta(U)$."

It should also be understood, as required by the overall factor $\delta(U)$, that all instances of r in Eq. (5.167) actually denote r_+. Let us also note that numerically we have

$$\tau' = -\frac{R}{R^*}\tau^* \,, \tag{5.169}$$

so it is possible to shuffle terms among the coefficients k_3, k_4, and k_5. The particular form shown in Eq. (5.167) is what we exhumed upon performing the rituals to be disclosed in Sec. 5.6.

Recalling the gravitational compass from Sec. 5.2.12, we interpret Eqs. (5.166) and (5.167) as describing a transverse "outgoing" gravitational wave stuck to the horizon.[117]

5.4.2 Static limit

It is worth taking a moment to consider the static limit,[118] $a \to 0$, in which case only the $\eth\eth f$ term in Eq. (5.166) survives.

As far as we know, the Weyl scalars for the shifted Reissner-Nordström geometry have not been calculated explicitly, so we will unpack the definitions of the

[117] We cannot help calling the reader's attention to the following famous quotation: "Now, here, you see, it takes all the running you can do, to keep in the same place." This is originally from *Through the Looking-Glass* by Lewis Carroll, but we first encountered its application to the horizon of a black hole from the textbook on the Kerr geometry by O'Neill [168]. Apparently the credit for this cultural appropriation should be attributed to the compilation *Analog Essays on Science* edited by Stanley Schmidt, but we have not read it.

[118] Some relativists call the surface on which g_{tt} goes to zero the "static limit." We prefer the name "stationary limit surface." In this paper we will use the terminology "static limit" to mean "the limit in which the black hole stops rotating," or more precisely, "$a \to 0$."

GHP derivatives at the horizon. Remembering that $f \sim (-1,-1)$ and therefore $ðf \sim (-\frac{1}{2},-\frac{3}{2})$, and that in the static limit we have $\beta = \beta' = \beta^* = \beta'^*$, we find:

$$ðð f|_{a=0} = \delta\delta f - 2\beta\,\delta f\ . \tag{5.170}$$

We could then mechanically unravel the Newman-Penrose derivatives and perhaps even use the Dray-'t Hooft constraint equation to simplify the result further, but we will not.

5.4.3 Shifted Ψ_3 and Petrov type

Having discussed Ψ_4 in some detail, let us turn to Ψ_3. Our generalized Kerr-Schild metric (with $\kappa = \sigma = \kappa' = \sigma'$) appears to generate this Weyl scalar:[119]

$$\tilde{\Psi}_3^* = \Psi_3^* + \tfrac{1}{4}(þð + ðþ)S + \tfrac{1}{4}(2\tau - \tau'^*)þS + \tfrac{1}{4}(2\rho - 5\rho^*)ðS - \tfrac{1}{2}[(2\tau - \tau'^*)\rho^* + ð\rho^*]S\ . \tag{5.171}$$

By placing hats on everything and specializing to the particular form of our hatted shift function [see Eq. (5.126)], we find that each term in Eq. (5.171) actually goes

[119]To arrive at this formula we wrote $þð = \frac{1}{2}(þð + ðþ) + \frac{1}{2}[þ, ð]$ and then traded the commutator for curvature scalars and spin coefficients using Eq. (5.314).

to zero at $U = 0$ for fixed nonzero V:

$$\hat{\hat{\Psi}}_3^* = 0 \; . \tag{5.172}$$

Since the unshifted geometry already had a nonzero Ψ_2, we arrive at the result:

$$\hat{\hat{\Psi}}_0 = \hat{\hat{\Psi}}_1 = \hat{\hat{\Psi}}_3 = 0 \; , \quad \hat{\hat{\Psi}}_2 \neq 0 \; , \quad \hat{\hat{\Psi}}_4 \neq 0 \; . \tag{5.173}$$

This describes a solution of Type II in the Petrov classification.[120] To quote Szekeres, "it can be viewed as a Coulomb field with an outgoing wave component superimposed" [160].

[120] For another explicit example of a Type II solution, see Sec. 19.1 of Griffiths [161].

5.4.4 Newman-Penrose rotation of the shifted Weyl scalars

Under the Newman-Penrose rotation in Eq. (5.153), the Weyl scalars transform as:

$$\begin{aligned}
&\Psi_0 \to \mathring{\Psi}_0 = \Psi_0 \ , \\
&\Psi_1 \to \mathring{\Psi}_1 = \Psi_1 + \mathscr{A}^*\Psi_0 \ , \\
&\Psi_2 \to \mathring{\Psi}_2 = \Psi_2 + 2\mathscr{A}^*\Psi_1 + (\mathscr{A}^*)^2\Psi_0 \ , \\
&\Psi_3 \to \mathring{\Psi}_3 = \Psi_3 + 3\mathscr{A}^*\Psi_2 + 3(\mathscr{A}^*)^2\Psi_1 + (\mathscr{A}^*)^3\Psi_0 \ , \\
&\Psi_4 \to \mathring{\Psi}_4 = \Psi_4 + 4\mathscr{A}^*\Psi_3 + 6(\mathscr{A}^*)^2\Psi_2 + 4(\mathscr{A}^*)^3\Psi_1 + (\mathscr{A}^*)^4\Psi_0 \ .
\end{aligned} \tag{5.174}$$

Therefore, were we to realign the ingoing principal null vector so that it would be tangent to an ingoing null geodesic, we would generate a nonzero Ψ_3 while keeping Ψ_2 and Ψ_4 nonzero.

So we have a choice: the vector $\tilde{l}'^{\mu}$ can either be principal or it can be geodesic, but it cannot be both.

5.4.5 Curvatures of submanifolds

We will calculate the shifted spacelike and timelike curvatures by shifting both sides of the GHP commutator equations [see Eq. (5.313)]. The shifted spacelike curvature for the generalized Kerr-Schild metric is:

$$\tilde{\mathcal{K}} = \mathcal{K} + \mathrm{Im}(\rho)\left[2\,\mathrm{Im}(\rho)\,S + i\,(\text{þ}S)\right] \ . \tag{5.175}$$

So the intrinsic curvature receives a correction $2[\mathrm{Im}(\rho)]^2 S$, while the extrinsic curvature receives a correction $\mathrm{Im}(\rho)\text{þ}S$. When we turn off the angular momentum, the twist goes to zero and the spacelike curvature remains unshifted.

For generic angular momenta, if we hat everything and specialize to the shockwave ansatz, we will find that both correction terms in Eq. (5.175) go to zero.

For the timelike curvature, we have:

$$\tilde{\mathcal{K}_s} = \mathcal{K}_s - \tfrac{1}{2}(\text{þ}^2 S) + i\,\text{þ}\left[\mathrm{Im}(\rho)S\right] \ . \tag{5.176}$$

When we turn off the angular momentum and set $\varepsilon = 0$, we recover the simple formula[121] $\tilde{\mathcal{K}_s}\Big|_{a=0} = \mathcal{K}_s - \frac{1}{2}D^2 S$. Again, for generic angular momenta, if we hat everything we will find that the corrections go to zero.

Curiously enough, the gravitational shockwave does not alter the curvatures of submanifolds in the Kerr-Newman geometry.

[121] It is perhaps interesting to note that in this case the correction term is real. In the unshifted spherically symmetric case (that is, for the Reissner-Nordström or Schwarzschild background), the intrinsic curvature $\mathrm{Re}(\mathcal{K})$ happens to equal $\mathrm{Re}(\mathcal{K}_s)$. So one effect of the shift is to break that equality.

5.4.6 Shifted Ψ_2

Having computed the shifted timelike and spacelike curvatures, we can readily compute the correction to the Weyl scalar of weight zero:

$$\tilde{\Psi}_2 = \Psi_2 + \tfrac{1}{6}\text{þ}^2 S - \tfrac{1}{3}(\rho - \rho^*)\text{þ}S + \tfrac{1}{3}\left[(3\rho - 2\rho^*)\rho + \Phi_{00}\right] S \, . \tag{5.177}$$

To arrive at this expression, we used the relation[122]

$$\Phi_{00} = -(\text{þ}\rho + \rho^2) \qquad (\text{if } \kappa = \sigma = 0) \tag{5.178}$$

along with the fact that $\Phi^*_{00} = \Phi_{00}$. Just as we found for the shifted Ψ_3, we find upon disbursing hats and plugging in the ansatz of Eq. (5.126) that the correction to Ψ_2 is actually zero. So while the generalized Kerr-Schild form results in a shift of Ψ_2, that correction vanishes upon specialization to the actual shockwave geometry.

Appealing again to the gravitational compass [160], we say that the Coulomb field remains unchanged by the presence of a massless particle on the future horizon.

[122]For the experts in general relativity, we point out that this is Raychaudhuri's equation for null shear-free geodesic congruences. When $\Phi_{00} = 0$, it tells us that $\text{þ}\rho = -\rho^2$. Given the interpretation of $\text{Re}(\rho)$ from Eq. (5.63), we recognize this as the focusing theorem.

5.5 Shifted Ricci scalars

Show time. We will first describe the shifted Ricci scalars for the generalized Kerr-Schild geometry under the assumption $\kappa = \kappa' = \sigma = \sigma'$ for the background, and then we will specialize to the shockwave.

5.5.1 Ricci scalar of weight $(-1, -1)$: absence of nonlinearity

As a result of the general shift procedure defined by Eq. (5.119), three of the Ricci scalars will become nonzero. Of these, the apple of our eye will be $\Phi_{22} \sim (-1, -1)$.

This quantity is defined by priming the definition of Φ_{00} in Eq. (5.97):

$$\Phi_{22} = -\left(\text{þ}'\rho' + \rho'^2\right) + \eth\kappa' + \tau\kappa' + \tau'\kappa'^* - |\sigma'|^2 \, . \tag{5.179}$$

Using our expressions for the shifted ρ' from Eq. (5.149) and the shifted þ$'$ from Eq. (5.150), and using the weight $h = \bar{h} = -\frac{1}{2}$ for ρ' [recall Eq. (5.38)], we find:

$$\begin{aligned} \tilde{\text{þ}}'\tilde{\rho}' &= \text{þ}'\rho' + (\rho'\text{þ} - \rho\text{þ}')S - (\text{þ}'\rho + \text{þ}\rho')S + (\text{þ}\rho)S^2 \, , \\ \tilde{\rho}'^2 &= \rho'^2 - 2\rho\rho' S + \rho^2 S^2 \, . \end{aligned} \tag{5.180}$$

It is worth keeping in mind the formula for Φ_{00} under the shear-free geodesic assumption [Eq. (5.178)].

Next recall Eq. (5.99), specialized to $\sigma\sigma' = \kappa\kappa' = 0$:

$$\Psi_2 + 2\Pi = -(\eth'\tau + |\tau|^2) + \text{þ}'\rho + \rho'^*\rho \tag{5.181}$$

$$= -(\eth\tau' + |\tau'|^2) + \text{þ}\rho' + \rho^*\rho' \, . \tag{5.182}$$

The reader who *is* paying attention has every right to be confused by the second equality: indeed it turns out that the combination of derivatives and products of spin coefficients in Eq. (5.181) is in fact equal to its primed version in Eq. (5.182). This must be so, since both $\Psi_2 = C_{1342}$ and $\Pi = \frac{1}{12}(-R_{12} + R_{34})$ are self-prime.

It is a matter of some discretion as to which variables we want to keep and which variables we want to trade away. We are guided by comparison with the static limit, which suggests we should express as much as possible in terms of τ and τ' and their derivatives. So we will use Eqs. (5.181) and (5.182) to evict $\text{þ}'\rho$ and $\text{þ}\rho'$ from Eq. (5.180).

Using our shifted κ' from Eq. (5.149), we find:

$$\eth\tilde{\kappa}' = \eth\eth' S + (\tau^* - \tau')\eth S + (\eth\tau^* - \eth\tau')S \, ,$$

$$\tau\tilde{\kappa}' + \tau'\tilde{\kappa}'^* = (\tau\eth' + \tau'\eth)S + (|\tau|^2 - |\tau'|^2)S \, . \tag{5.183}$$

The aggregate of Eqs. (5.180)-Eq. (5.183) then supplies the preliminary expression:

$$\tilde{\Phi}_{22} = \Phi_{22} + (\rho \text{þ}' - \rho' \text{þ})S + \Phi_{00} S^2 + ðð' S + \tau ð' S + \tau^* ð S$$
$$+ \left[2\rho\rho' - (\rho\rho'^* + \rho^* \rho') + ð'\tau + ð\tau^* + 2|\tau|^2 + 2(\Psi_2 + 2\Pi)\right] S \, . \qquad (5.184)$$

It is now clear that when we begin with a background for which $\Phi_{00} = 0$, $a$ll nonlinear dependence on the perturbation drops out of the curvature scalars. Terms of $O(S^2)$ could not possibly show up elsewhere, because the only curvature scalar with the appropriate weight to include a $p$roduct of shifted quantities (in this case þ$'$ and ρ') is Φ_{22}.

Let us reflect on the simple conditions required for this result: $\kappa = \sigma = \kappa' = \sigma' = 0$ and $\Phi_{00} = 0$. Newman in particular has long emphasized the importance of shear-free geodesic congruences [169], and our calculation exemplifies another miraculous phenomenon associated with them.

To make sense of Eq. (5.184) we will rewrite it in a manifestly real form:[123]

$$\begin{aligned}\tilde{\Phi}_{22} = \mathrm{Re}(\tilde{\Phi}_{22}) = \Phi_{22} + \mathrm{Re}\left[(\rho þ' - \rho' þ)S\right] + \Phi_{00}S^2 \\ + \tfrac{1}{2}\left[ðð' + ð'ð + (\tau+\tau'^*)ð' + (\tau^*+\tau')ð + (\tau-\tau'^*)ð' + (\tau^*-\tau')ð\right] S \\ + \Big\{(\rho-\rho^*)(\rho'-\rho'^*) + (ð'\tau + c.c.) + 2|\tau|^2 + 2\left[\mathrm{Re}(\Psi_2) + 2\Pi\right]\Big\} S\ .\end{aligned} \tag{5.185}$$

Experts in the GHP formalism should recognize the combination $ðð' + (\tau+\tau'^*)ð' +$ *c.c.* as composing part of the generalized Laplacian. Those experiencing the notation for the first time should grab a coffee and work through Sec. 5.D. We will have much more to say about this shortly, but first let us study the remaining curvature scalars.

5.5.2 Other Ricci scalars

The Ricci scalar of weight $(-1, 0)$ receives a correction from the general shift:

$$\tilde{\Phi}_{21} = \Phi_{21} + \tfrac{1}{4}(þð' + ð'þ)S + \tfrac{1}{4}(2\tau^* - \tau')þS + \tfrac{1}{4}(3\rho - 2\rho^*)ð'S + \tfrac{1}{2}(\tau'\rho - 2\tau^*\rho^* + ð'\rho)S\ . \tag{5.186}$$

[123] Or, to mutter a hex from days gone by, we use the "eliminant relation" $\mathrm{Im}(\Phi_{22}) = 0$.

The steps leading to this expression parallel closely those that led to $\tilde{\Psi}_3^*$, surprising nobody: the combinations $\Phi_{21} \pm \Psi_3$ materialize naturally in the Cartan formalism as the primed versions of Eqs. (5.94) and (5.95).

For the Kerr-Newman background, we have $\Phi_{21} = 0$. Once we hat everything and specialize to Eq. (5.126), we will find that each correction term on the right-hand side of Eq. (5.186) will go to zero.

Next, we have the Ricci scalar of weight $(0, 0)$:

$$\tilde{\Phi}_{11} = \Phi_{11} + \tfrac{1}{4}\text{þ}^2 S + \tfrac{1}{2}\left[\Phi_{00} - (\rho - \rho^*)^2\right] S \,. \tag{5.187}$$

When we hat both sides and plug in the ansatz, we will find that $\hat{\tilde{\Phi}}_{11} = \hat{\Phi}_{11} = \Phi_{11}$, the unshifted value for the Kerr-Newman background.

The Einstein-Hilbert curvature also appears to become nonzero as a result of the shift:

$$\tilde{\Pi} = \Pi + \tfrac{1}{6}\left\{\tfrac{1}{2}\text{þ}^2 S - (\rho + \rho^*)\text{þ} S + |\rho|^2 S\right\} \,. \tag{5.188}$$

But we know that Π is proportional to the Lagrangian of general relativity, so its first order variation must comport with the standard formula[124]

$$S_{\mathrm{GR}}[g+h] - S_{\mathrm{GR}}[g] = \tfrac{1}{2}\int d^4x\,(-\det g)^{1/2}\,T^{\mu\nu}h_{\mu\nu} + O(h^2)\;. \tag{5.189}$$

The shift ansatz in Eq. (5.119) effects the metric variation

$$h_{\mu\nu} = -2S\,l_\mu l_\nu\;. \tag{5.190}$$

So varying the action with respect to the shift function S will result in a quantity proportional to $T^{\mu\nu}l_\mu l_\nu = \mathcal{T}^{\mu\nu}l_\mu l_\nu = 4\pi\,t_{00}$.[125]

Because the only nonzero energy-momentum scalar for the background metric is $t_{11} \propto (l_\mu l'_\nu + l'_\mu l_\nu + m_\mu m'_\nu + m'_\mu m_\nu)\mathcal{T}^{\mu\nu}$, we know that $t_{00} = 0$ and thereby expect the $O(S)$ term in Eq. (5.189) to equal zero.[126]

The nonzero $O(S)$ term in Eq. (5.188) might invite consternation, but we have to remember that in our entire treatment of general relativity we have been cavalierly ignoring possible boundary terms in the action. So all we require is that

[124] The action for general relativity in our notation is $S_{\mathrm{GR}} = -\frac{3}{2\pi}\int d^4x\,(-\det g)^{1/2}\,\Pi$. We use $h_{\mu\nu}$ for the variation of the metric since, as promised, we forsake the traditional notation $\delta g_{\mu\nu}$ to avoid a possible mixup with the NP derivative $\delta \equiv m^\mu\nabla_\mu$. As we have already discussed, the $O(h^2)$ terms and higher in Π vanish identically.

[125] This is as good a place as any to remark that, as noticed by Taub [137], the unperturbed l_μ is also null with respect to the perturbed metric: $\tilde{g}_{\mu\nu}l^\mu l^\nu = (g_{\mu\nu} + 2Sl_\mu l_\nu)l^\mu l^\nu = g_{\mu\nu}l^\mu l^\nu + 2S(l_\mu l^\mu)^2 = 0$.

[126] We thank Alexei Kitaev for proposing this check on our work.

the $O(h)$ term in Eq. (5.189) should be zero, not necessarily that the shift in Π itself should be zero.

In the Kerr-Newman background, we have[127]

$$(-\det g)^{1/2} = i\,\varepsilon^{\mu\nu\rho\sigma} l_\mu l'_\nu m_\rho m'_\sigma = |R|^2 \sin\theta\ . \tag{5.191}$$

After integrating by parts, dropping total derivatives, and using the explicit forms of D and ρ for the Kerr-Newman background [recall Eqs. (5.13) and (5.50) respectively], we indeed obtain

$$\int d^4x\, (-\det g)^{1/2}\, \tilde{\Pi} = 0\ . \tag{5.192}$$

This completes our account of the shifted curvature scalars for the generalized Kerr-Schild geometry. (The Ricci scalars not explicitly enumerated in this section do not shift.) Now we will specialize the shifted Φ_{22} to the shockwave ansatz.

[127]The tetrad form of $(-\det g)^{1/2}$ makes clear that it does not receive a correction under the shift of Eq. (5.119).

5.6 Derivatives of the shift

The space-time Laplacian $\nabla^2 = \nabla_\mu \nabla^\mu$ finds refuge in the compacted spin coefficient formalism within a more general operator $\Box$ that can act covariantly on arbitrarily-weighted functions. (See Sec. 5.D for a derivation.)

This operator readily splits into a "parallel" part $\Box_\parallel$ associated with the outgoing and ingoing null directions, and a "perpendicular" part $\Box_\perp$ associated with the transverse plane:

$$\Box = -\Box_\parallel + \Box_\perp \, , \tag{5.193}$$

where

$$\Box_\parallel \equiv [\text{þ} + 2\,\mathrm{Re}(\rho)]\text{þ}' \; + \; ' \, , \;\; \Box_\perp \equiv [\eth + (\tau + \tau'^*)]\eth' \; + \; ' \, . \tag{5.194}$$

The operator $\Box_\perp$ will be called the "transverse box." Evaluating its action on the shift function is the most technically cumbersome aspect of computing $\tilde{\Phi}_{22}$.

We will do our best to show how the sausage is made without belaboring mindless algebra. By the time this paper is published, we will release a companion Mathematica notebook to facilitate the inspired reader's pilgrimage across the shockwave.

5.6.1 Key facts

To set up the calculation we will first collect some useful formulas.

From what may seem like a lifetime ago, we recall that $U\partial_U r = V\partial_V r$ (which can be traced back to the relation $-U\partial_U + V\partial_V = \frac{1}{\alpha}\partial_t$). Therefore, acting on a weight-$(0,0)$ function $F(r)$, we have:

$$ðF(r) = 0\ . \tag{5.195}$$

This is our first key fact.

Next we recall the explicit formulas for the timelike expansion and the timelike twist in the Kerr-Newman geometry [Eq. (5.70)]. They will compose our basic mnemonic for making sense of complicated algebraic expressions: the trigonometric functions $\sin(2\theta)$ and $\sin\theta$ should evoke $\tau + \tau'^*$ and $\tau - \tau'^*$ respectively.

We will use this to establish additional useful formulas. Treating $\delta(U)$ as having weight $(0,0)$ and recalling the Newman-Penrose derivatives in Kruskal-like coordinates [Eq. (5.84)], we find:

$$ð\delta(U) = -\alpha\,\frac{ia\sin\theta}{R\sqrt{2}}\,U\partial_U\delta(U) = -\,\frac{\alpha|R|^2}{2r}\,(\tau-\tau'^*)\,U\partial_U\delta(U)\ . \tag{5.196}$$

This is our second key fact.

Finally, we must remember that although functions of r can be treated as constants, the generalized radial function $R = r + ia\cos\theta$ is also a function of θ. Treating this too as a function of weight $(0,0)$, we compute the following:

$$ð\left(\frac{1}{|R|^2}\right) = -\frac{1}{(|R|^2)^2}\,ð\big(|R|^2\big) = +\frac{a^2\sin(2\theta)}{\sqrt{2}\,R|R|^4} = -\frac{1}{|R|^2}(\tau+\tau'^{*})\ . \tag{5.197}$$

This is our third key fact.

5.6.2 Integration by parts

We described back in Eq. (5.168) the standard integration by parts procedure that defines the delta function. Here it will be useful to study two special cases of that formula.

First consider a distribution $\mathcal{O}(U)\,U\partial_U\delta(U)$ (where the conditions on $\mathcal{O}(U)$ will be specified shortly), and integrate it against a test function $\mathcal{F}(U)$ that falls off quickly enough to merit dropping the boundary term:

$$\int dU\,\mathcal{O}(U)\,U\partial_U\delta(U)\,\mathcal{F}(U) = -\int dU\ \big[\mathcal{O}(U)\mathcal{F}(U) + U\partial_U(\mathcal{O}(U)\mathcal{F}(U))\big]\,\delta(U)\ . \tag{5.198}$$

If $\partial_U(\mathcal{O}(U)\mathcal{F}(U)) \sim U^n$ with $n \geq 0$ near $U = 0$, then the second term evaluates to zero. We then obtain the following distributional equality:

$$\mathcal{O}(U)\, U\partial_U\delta(U) = -\mathcal{O}(U)\,\delta(U)\ . \tag{5.199}$$

Along similar lines, we will obtain a second distributional equality:

$$\mathcal{O}(U)\, U\partial_U(U\partial_U\delta(U)) = +\mathcal{O}(U)\,\delta(U)\ . \tag{5.200}$$

Equipped with the key facts in Eqs. (5.195)-(5.197) and the above distributional equalities, we are ready to face the transverse box.

5.6.3 First-derivative terms

We warm up with a first-derivative term. Specializing to the shockwave ansatz in Eq. (5.126) and applying our key facts, we obtain the preliminary expression

$$\begin{aligned}
ð\hat{S} &= \tfrac{1}{2}\tfrac{\Delta}{UV}\left[ð(\tfrac{1}{|R|^2})\delta(U)f(\theta,\chi) + \tfrac{1}{|R|^2}ð\delta(U)f(\theta,\chi) + \tfrac{1}{|R|^2}\delta(U)ðf(\theta,\chi)\right] \\
&= \tfrac{1}{2|R|^2}\tfrac{\Delta}{UV}\left\{\delta(U)\left[ð - (\tau+\tau'^*)\right]f(\theta,\chi) - \tfrac{\alpha|R|^2}{2r}(\tau-\tau'^*)\,U\partial_U\delta(U)\,f(\theta,\chi)\right\}\ .
\end{aligned} \tag{5.201}$$

Before integrating by parts against a test function, we need to multiply by $\tau^* + \tau'$ to obtain the term $(\tau^* {+} \tau')\eth\hat{S}$ that appears in the transverse box.[128]

Note that since $|\tau|^2 = |\tau'|^2$ for the Kerr-Newman space-time, we have

$$(\tau^* {+} \tau')(\tau {-} \tau'^*) = 2i\,\mathrm{Im}(\tau\tau') \,. \tag{5.202}$$

Using this and the distributional equality in Eq. (5.199), we obtain [also recall $c \equiv -\frac{\Delta}{UV}\big|_{r=r_+}$ from Eq. (5.74)]

$$(\tau^* {+} \tau')\,\eth\hat{S} = -\frac{c}{2|R|^2}\,\delta(U)\left\{(\tau^* {+} \tau')\,\eth - |\tau {+} \tau'^*|^2 + i\,\tfrac{\alpha|R|^2}{r}\,\mathrm{Im}(\tau\tau')\right\} f(\theta,\chi)\,. \tag{5.203}$$

5.6.4 Second-derivative terms

Returning to Eq. (5.201), we act with $\eth'$ (and skip a few steps now that the method is presumably clear) to obtain

$$\begin{aligned}\eth'\eth\hat{S} = \tfrac{1}{2|R|^2}\tfrac{\Delta}{UV}\Big\{&\delta(U)\,\eth'\eth f - \Big[(\tau {+} \tau'^*)\,\delta(U) + \tfrac{\alpha|R|^2}{2r}(\tau {-} \tau'^*)\,U\partial_U\delta(U)\Big]\,\eth' f \\ &- \Big[(\tau^* {+} \tau')\,\delta(U) + \tfrac{\alpha|R|^2}{2r}(\tau^* {-} \tau')\,U\partial_U\delta(U)\Big]\,\eth f + \mathscr{C}\,f\Big\}\,,\end{aligned} \tag{5.204}$$

[128] Since $\tau^* + \tau'$ depends on U and V only through $r = r(UV)$, and since we have already said such functions can be treated as constants with respect to $U\partial_U$ for our calculation, it does not matter in this particular instance whether we integrate by parts before or after multiplying by $\tau^* + \tau'$.

where

$$\mathscr{C} = |\tau+\tau'^*|^2\,\delta(U) - i\tfrac{\alpha|R|^2}{r}\mathrm{Im}(\tau\tau')\,U\partial_U\delta(U) - \Big(\delta(U) + \tfrac{\alpha|R|^2}{2r}\,U\partial_U\delta(U)\Big)\,\eth'\tau$$
$$- \Big(\delta(U) - \tfrac{\alpha|R|^2}{2r}\,U\partial_U\delta(U)\Big)\,(\eth\tau')^* + \Big(\tfrac{\alpha|R|^2}{2r}\Big)^2 |\tau-\tau'^*|^2\,U\partial_U\left[U\partial_U\delta(U)\right]\;. \tag{5.205}$$

Note that in Eq. (5.204) the coefficient of $\eth' f$ is the complex conjugate of the coefficient of $\eth f$. This did not have to be so, because we are computing $\eth'\eth\hat{S}$ right now, not $\eth'\eth\hat{S} + c.c.$, and in general $\eth'\eth \neq \eth\eth'$.

This quantity $\eth'\eth\hat{S}$ will be integrated directly against a test function (because it appears directly in the transverse box, which in turn appears directly in $\hat{\hat{\Phi}}_{22}$), so we can use the distributional equalities in Eqs. (5.199) and (5.200), loosely expressed as $U\partial_U\delta(U) \to -\delta(U)$ and $U\partial_U[U\partial_U\delta(U)] \to +\delta(U)$. Applying these to Eq. (5.204), we obtain:

$$\eth'\eth\hat{S} = -\frac{c}{2|R|^2}\,\delta(U)\Big\{\eth'\eth - \Big[(\tau+\tau'^*) - \tfrac{\alpha|R|^2}{2r}(\tau-\tau'^*)\Big]\,\eth' - \Big[(\tau^*+\tau') - \tfrac{\alpha|R|^2}{2r}(\tau^*-\tau')\Big]\,\eth$$
$$+|\tau+\tau'^*|^2 + i\tfrac{\alpha|R|^2}{r}\mathrm{Im}(\tau\tau') - \Big(1-\tfrac{\alpha|R|^2}{2r}\Big)\,\eth'\tau - \Big(1+\tfrac{\alpha|R|^2}{2r}\Big)\,(\eth\tau')^* + \Big(\tfrac{\alpha|R|^2}{2r}\Big)^2 |\tau-\tau'^*|^2\Big\} f(\theta,\chi)\,. \tag{5.206}$$

5.6.5 Transverse box

Now we can finish the job. Returning to the first-derivative term in Eq. (5.201) and adding its complex conjugate, we obtain:

$$(\tau^* + \tau')\eth\hat{S} + c.c. = -\frac{c}{2|R|^2}\,\delta(U)\left\{(\tau+\tau'^*)\eth' + (\tau^*+\tau')\eth - 2|\tau+\tau'^*|^2\right\} f(\theta,\chi)\ . \tag{5.207}$$

Next we obtain the anticommutator of GHP derivatives by taking Eq. (5.204) plus its complex conjugate:

$$\eth'\eth\hat{S} + c.c. = -\frac{c}{2|R|^2}\,\delta(U)\Bigg\{(\eth'\eth + \eth\eth') + \left[-\left(2(\tau+\tau'^*) - \tfrac{\alpha|R|^2}{r}(\tau-\tau'^*)\right)\eth' + c.c.\right] \\ +2|\tau+\tau'^*|^2 - \left(1-\tfrac{\alpha|R|^2}{2r}\right)(\eth'\tau + c.c.) - \left(1+\tfrac{\alpha|R|^2}{2r}\right)(\eth\tau' + c.c.) + 2\left(\tfrac{\alpha|R|^2}{2r}\right)^2|\tau-\tau'^*|^2\Bigg\} f(\theta,\chi)\ . \tag{5.208}$$

We then add Eqs. (5.207) and (5.208) to obtain the transverse box. Notice that, for reasons morally unbeknownst to us, the $|\tau+\tau'^*|^2$ term will cancel out. It also turns out, by explicit computation, that for the Kerr-Newman geometry we have

$$\eth\tau' = \eth'\tau\ . \tag{5.209}$$

There is probably a good reason for this, but it escapes us.[129] At any rate, it implies that the α-dependent parts of the coefficients of $\eth'\tau + c.c.$ and $\eth\tau' + c.c.$ drop out.

Therefore, the transverse box acting on the shift function, expressed in terms of GHP derivatives at the horizon, simplifies to:

$$\Box_\perp \hat{S} = -\frac{c}{2|R|^2}\,\delta(U)\Bigg\{(\eth'\eth + \eth\eth') + \Big[-\Big((\tau+\tau'^*) - \tfrac{\alpha|R|^2}{r}(\tau-\tau'^*)\Big)\,\eth' + c.c.\Big] \\ -2(\eth'\tau + c.c.) + 2\left(\tfrac{\alpha|R|^2}{2r}\right)^2 |\tau-\tau'^*|^2\Bigg\}\, f(\theta,\chi)\ . \tag{5.210}$$

This completes the most arduous part of the calculation. It bears repeating that all quantities in Eq. (5.210) are understood to be evaluated at $r = r_+$, as mandated by the overall delta function.

5.6.6 Laplacian on the squashed sphere

As a matter of principle, we could leave the result for $\Box_\perp \hat{S}$ in the form of Eq. (5.210). But readers familiar with the Dray-'t Hooft solution will recall that the second-derivative terms collect themselves into the Laplacian on the sphere,

[129] This can probably be traced back to the GHP transformation that sets τ' numerically equal to τ [recall the discussion below Eq. (5.50)]. That transformation also sets the θ-components of m^μ and m'^μ equal, whereas beforehand they were complex conjugates. Because τ and τ' depend only on r and θ (not on t or φ), and because $\eth\tau'$ and $\eth'\tau$ are invariant under GHP transformations, we should perhaps expect to recover Eq. (5.209). But the more honest way to describe our thought process is: we blindly computed $\eth\tau'$ and $\eth'\tau$ using the original tetrad basis, found that they were equal, then tried to understand why. This footnote is the result of that introspection coupled with a conviction not to lie to our readers or to ourselves.

and they might wonder how the Laplacian on the *s*quashed sphere will present itself here. Since the two-dimensional (henceforth "2d") spatial Laplacian is not by itself a GHP-covariant operator, we will have to break GHP covariance to express $\Box_\perp \hat{S}$ in terms of it.

We are happy to indulge such readers. Our shift function S and our horizon field $f(\theta,\chi)$ have GHP weight $(-1,-1)$. In general, a weighted function $f_{h,h}(\theta,\chi) \sim (h,h)$ has "spin weight" $s \equiv h - \bar{h} = h - h = 0$. The shockwave ansatz has $h = -1$, but with minimal additional exertion we can understand the situation for $s = 0$ but arbitrary h.[130]

By explicit computation on a function $f_{h,h}(\theta,\chi)$ of the Kruskal-like angular coordinates only, we find that the following combination of NP derivatives and GHP gauge fields reproduces the Laplacian on the squashed sphere:[131]

---

[130]Since complex conjugation exchanges h and $\bar{h}$, only functions with $s = 0$ can be taken real. We therefore assume $f^*_{h,h} = f_{h,h}$ for simplicity.

[131]Given that: (1) we have usurped the symbol "$\Box$" to represent the *g*eneralized (3+1)-dimensional Laplacian; (2) the ordinary (3+1)-dimensional Laplacian is itself typically denoted either by $\Box$ or by $\nabla^2 = \nabla_\mu \nabla^\mu$; (3) introducing a vector arrow to define an operator "$\vec{\nabla}^2$" would be readily misconstrued as denoting a 3-dimensional operator $\sum_{i=1}^3 \nabla_i \nabla^i$; and (4) the mathematician's preferred symbol "Δ" already denotes the horizon function, we have officially exhausted our notational reserves. We considered introducing a symbol $\bigcirc \equiv \nabla^2_{\mathrm{2d}}$ since, after all, what better symbol to complement a box than a circle, especially if we intend to evoke an association with a sphere (squashed or otherwise). But in the interest of smooth sailing toward intelligibility instead of a turbulent odyssey into the fever dreams of madmen, we will just stamp the clunky subscript "2d" onto the usual ∇^2 to denote the Laplacian on a two-dimensional spatial surface. For the Kerr-Newman horizon, the 2d space of interest is the squashed sphere. The line element for a squashed sphere of radius r is (recall $|R|^2 = r^2 + a^2\cos^2\theta$ and $|R_0|^2 = r^2 + a^2$):

$$ds^2 = |R|^2 d\theta^2 + \frac{|R_0|^4}{|R|^2}\sin^2\theta \, d\chi^2 \equiv h_{ij}\, dx^i\, dx^j \,. \tag{5.211}$$

$$\delta'\delta + \delta\delta' - (\beta + \beta'^*)\delta' - (\beta' + \beta^*)\delta = \nabla^2_{2\mathrm{d}} \,. \tag{5.213}$$

So unpacking the GHP derivatives according to their original definitions back in Eq. (5.43) provides the desired expression:

$$(\eth'\eth + \eth\eth') f_{h,h}(\theta,\chi) = \Big\{ \nabla^2_{2\mathrm{d}} + \left[4h(\beta - \beta'^*)\delta' + c.c. \right] \\ + 2h \left[(\delta'\beta - \delta\beta' + c.c.) + 2(|\beta'|^2 - |\beta|^2) + 4h|\beta - \beta'^*|^2 \right] \Big\} f_{h,h}(\theta,\chi) \,. \tag{5.214}$$

That is how our coveted 2d spatial Laplacian manifests in our story. Its tragedy is that while we may find temporary solace in a familiar face, this yearning for camaraderie cost us the guidance of GHP covariance, without which we are hopelessly lost.

The Laplacian $\nabla^2_{2\mathrm{d}} \equiv \frac{1}{\sqrt{\det h}} \partial_i (\sqrt{\det h}\, h^{ij} \partial_j)$ obtained from this metric is:

$$\nabla^2_{2\mathrm{d}} = \frac{1}{|R|^2} \left[\partial^2_\theta + \left(\frac{|R_0|^2 + a^2 \sin^2\theta}{|R|^2} \right) \cot\theta\, \partial_\theta + \frac{|R|^4}{|R_0|^4} \frac{1}{\sin^2\theta} \partial^2_\chi \right] . \tag{5.212}$$

Those are the only facts about the squashed sphere that we need for now.

5.7 Ricci tensor

The trace reversed Ricci tensor, being necessary to the gravitational field of a localized Source, the propensity of a massless particle to generate Curvature, shall now be realized.

5.7.1 Relation to curvature scalars

We emerge from the chrysalis of the tangent space by translating the usual prescription $R_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} R_{ab}$ into the Newman-Penrose notation:

$$\tfrac{1}{2} R_{\mu\nu} = l'_\mu l'_\nu \, \Phi_{00} + l_\mu l_\nu \, \Phi_{22} + \Big[m'_\mu m'_\nu \, \Phi_{02} - (l'_\mu m'_\nu + m'_\mu l'_\nu) \, \Phi_{01} - (l_\mu m_\nu + m_\mu l_\nu) \, \Phi_{21} + c.c. \Big] + (l_\mu l'_\nu + l'_\mu l_\nu + m_\mu m'_\nu + m'_\mu m_\nu) \, \Phi_{11} - 3(-l_\mu l'_\nu - l'_\mu l_\nu + m_\mu m'_\nu + m'_\mu m_\nu) \, \Pi \, . \tag{5.215}$$

To evaluate the right-hand side, we first need to tilde everything (to calculate shifted quantities), and then we need to hat everything (to work in the horizon basis).

We will specialize directly to the shockwave ansatz, so the only Ricci scalar that will receive a shift is Φ_{22}. Meanwhile, the unshifted geometry has only a nonzero Φ_{11}. Therefore, we have for the full (i.e., including the unshifted part)

Ricci tensor:

$$
\begin{aligned}
\tfrac{1}{2}\tilde{R}_{\mu\nu} &= \hat{\tilde{l}}_\mu \hat{\tilde{l}}_\nu \, \hat{\tilde{\Phi}}_{22} + (\hat{\tilde{l}}_\mu \hat{\tilde{l}}'_\nu + \hat{\tilde{l}}'_\mu \hat{\tilde{l}}_\nu + \hat{\tilde{m}}_\mu \hat{\tilde{m}}'_\nu + \hat{\tilde{m}}'_\mu \hat{\tilde{m}}_\nu) \, \hat{\tilde{\Phi}}_{11} \\
&= \hat{l}_\mu \hat{l}_\nu \, \hat{\tilde{\Phi}}_{22} + (\hat{l}_\mu \hat{\tilde{l}}'_\nu + \hat{\tilde{l}}'_\mu \hat{l}_\nu + m_\mu m'_\nu + m'_\mu m_\nu) \, \Phi_{11} \ .
\end{aligned}
\tag{5.216}
$$

In the second line we have removed the tildes for quantities that equal their unshifted counterparts, and we have removed the hats on quantities that do not get rescaled by factors of U when passing from the asymptotic tetrad to the horizon one. Also, note that there is no need to place a hat on the Ricci *t*ensor, because by construction it is invariant under GHP transformations of the tetrad.

Recalling from Eq. (5.119) the premise that launched this travail in the first place, we isolate the part of the Ricci tensor that results from the shift:

$$
R^{\text{shift}}_{\mu\nu} = 2 \, \hat{l}_\mu \hat{l}_\nu \, (\hat{\tilde{\Phi}}_{22} + 2\hat{S} \, \Phi_{11}) \ . \tag{5.217}
$$

Returning to our explicit expressions for the tetrad 1-forms in Eq. (5.83), we find

$$
\hat{l}\Big|_{U=0} = \frac{|R_+|^2}{\alpha |R_{0+}|^2} \, dU \ . \tag{5.218}
$$

So we learn first of all that $R^{\text{shift}}_{\mu\nu} = R^{\text{shift}}_{UU} \, \delta_\mu^{\ U} \delta_\nu^{\ U}$, as promised.

5.7.2 Relation to energy-momentum scalars

Meanwhile, the trace-reversed energy-momentum tensor also admits an expansion analogous to Eq. (5.215):

$$\frac{1}{4\pi}\mathcal{T}_{\mu\nu} = l'_\mu l'_\nu\, t_{00} + l_\mu l_\nu\, t_{22} + \Big[m'_\mu m'_\nu\, t_{02} - (l'_\mu m'_\nu + m'_\mu l'_\nu)\, t_{01} - (l_\mu m_\nu + m_\mu l_\nu)\, t_{21} + c.c.\Big]$$
$$+ (l_\mu l'_\nu + l'_\mu l_\nu + m_\mu m'_\nu + m'_\mu m_\nu)\, t_{11} - 3(-l_\mu l'_\nu - l'_\mu l_\nu + m_\mu m'_\nu + m'_\mu m_\nu)\, t_\Pi \;. \tag{5.219}$$

Anticipating the required energy-momentum tensor term by term, we conclude:

$$\mathcal{T}^{\text{shift}}_{\mu\nu} = 4\pi\, \hat{l}_\mu \hat{l}_\nu (\hat{\tilde{t}}_{22} + 2\hat{S}\, t_{11}) \;. \tag{5.220}$$

Given that the background Einstein equation is $\Phi_{11} = t_{11}$ (by construction), all we have to do is to introduce a t_{22} such that

$$\hat{\tilde{\Phi}}_{22} = \hat{\tilde{t}}_{22} \;. \tag{5.221}$$

The whole point of this tale is that the correction to the left-hand side can be interpreted as the backreaction from a massless particle on the future horizon, so that is what will populate the right-hand side. Before investigating this matter (pun intended), we will first complete our calculation of $\hat{\tilde{\Phi}}_{22}$.

5.7.3 Final result for Φ_{22}

Returning to our earlier calculation of $\eth\hat{S}$ [Eq. (5.201)], multiplying by $\tau^*-\tau'$, integrating by parts, and adding the complex conjugate, we obtain the remaining first-derivative terms:

$$(\tau-\tau'^*)\eth'\hat{S} + c.c. = -\frac{c}{2|R|^2}\,\delta(U)\left\{(\tau-\tau'^*)\eth' + (\tau^*-\tau')\eth + \tfrac{\alpha|R|^2}{r}|\tau-\tau'^*|^2\right\} f(\theta,\chi)\ . \tag{5.222}$$

Next, we return to our general expression for the shifted Φ_{22} in Eq. (5.185), hat it, and recognize that $\hat{\rho}'\,\hat{\text{þ}}\hat{S}$ and $(\hat{\rho}-\hat{\rho}^*)(\hat{\rho}'-\hat{\rho}'^*)$ go to zero at $U=0$.

But $\hat{\rho}\,\hat{\text{þ}}'\hat{S}$ is more subtle, since hidden inside $\hat{D}'$ is a partial derivative with respect to U. Applying Eq. (5.199), we obtain

$$\mathrm{Re}(\hat{\rho}\hat{D}'\hat{S}) = \alpha r_+\frac{|R_{0+}|^2}{|R_+|^4}\hat{S} = -\tfrac{1}{2}\left[\eth\tau + \eth'\tau^* + 2|\tau|^2 + 2\,\mathrm{Re}(\Psi_2)\right]\Big|_{r=r_+}\hat{S}\ . \tag{5.223}$$

Because $\hat{\rho}|_{U=0} = 0$, the terms involving $\hat{\varepsilon}'$ and $\hat{\varepsilon}'^*$ drop out, leaving us with $\mathrm{Re}(\hat{\rho}\hat{\text{þ}}'\hat{S}) = \mathrm{Re}(\hat{\rho}\hat{D}'\hat{S}) = -\mathrm{Re}(\hat{\text{þ}}'\hat{\rho})\hat{S}$. To obtain that last equality we used Eq. (5.99). This makes clear that $\mathrm{Re}(\hat{\rho}\hat{\text{þ}}'\hat{S})$ equals its definition by integration by parts also at the level of GHP derivatives.

Putting all this together (and using $\Phi_{00} = \Pi = 0$), our shifted Φ_{22} reduces to the relatively compact form:

$$\hat{\hat{\Phi}}_{22} = \tfrac{1}{2}\left\{\Box_\perp + (\tau - \tau'^*)\eth' + (\tau^* - \tau')\eth + (\eth'\tau + c.c.) + 2|\tau|^2 + 2\,\mathrm{Re}(\Psi_2)\right\}\hat{S}\ . \tag{5.224}$$

Using our result for $\Box_\perp \hat{S}$ in Eq. (5.210) and the relation $4|\tau|^2 = |\tau+\tau'^*|^2 + |\tau-\tau'^*|^2$, we finally obtain the beautiful, exquisite, magical expression

$$\hat{\hat{\Phi}}_{22} = -\frac{c}{4|R|^2}\,\delta(U)\,\mathscr{D}f(\theta,\chi)\ , \tag{5.225}$$

where the differential operator $\mathscr{D}$ is

$$\begin{aligned}\mathscr{D} = \eth'\eth + \eth\eth' + \left[-(\tau+\tau'^*) + \left(1+\tfrac{\alpha|R|^2}{r}\right)(\tau-\tau'^*)\right]\eth' + \left[-(\tau^*+\tau') + \left(1+\tfrac{\alpha|R|^2}{r}\right)(\tau^*-\tau')\right]\eth \\ + 2\,\mathrm{Re}(\Psi_2) - (\eth'\tau + \eth\tau^*) + \tfrac{1}{2}|\tau+\tau'^*|^2 + \tfrac{1}{2}\left(1+\tfrac{\alpha|R|^2}{r}\right)^2|\tau-\tau'^*|^2\ .\end{aligned} \tag{5.226}$$

This is our final result.

It is expressed in terms of quantities that have innate geometrical significance, in that each operator has a definite GHP weight. When $a = 0$, we obtain[132]

$$\mathscr{D}\big|_{a=0} = \eth'\eth + \eth'\eth + 2\,\mathrm{Re}(\Psi_2)\ . \tag{5.227}$$

[132] At $r = r_+$, we have $\mathrm{Re}(\Psi_2)\big|_{a=0} = -\frac{\alpha}{r_+}$.

As could be anticipated from the Type-D character of the background, we see that it is part of the Weyl tensor, $\mathrm{Re}(\Psi_2)$, not the intrinsic curvature, $\mathrm{Re}(\mathcal{K})$, that appears most naturally in the GHP-covariant form of the shifted Φ_{22} for generic values of the angular momentum.

On the other hand, the intrinsic curvature presents itself when we trade the GHP-covariant derivatives for the 2d Laplacian plus its associated ejecta. We first expand $\eth f = [\delta + 2(-1)\beta - 2(-1)\beta'^{*}]f$ and specialize Eq. (5.214) to $h = -1$. Then we shuffle the terms around using numerical relations like[133]

$$\beta' - \beta^* = \tau' \qquad \text{(Kerr-Newman)} \tag{5.228}$$

and

$$|\beta'|^2 - |\beta|^2 = \frac{a^2}{2|R|^4} \qquad \text{(Kerr-Newman)}\ . \tag{5.229}$$

In this way we obtain the following alternative form for Eq. (5.226):

$$\begin{aligned}\mathscr{D} = \nabla^2_{\text{2d}} &+ \tfrac{1-\alpha R}{r}(\tau - \tau'^*)R^*\,\delta' + \tfrac{1-\alpha R^*}{r}(\tau^* - \tau')R\,\delta \\ &+ 2r\alpha\left\{-2\left(\frac{|R|^2}{|R_0|^2}\mathrm{Re}(\mathcal{K}) + \frac{2a^2}{|R|^4}\right) + |\tau + \tau'^*|^2 + \left[1 + r\alpha\left(\frac{|R|^2}{2r^2}\right)^2\right]|\tau - \tau'^*|^2\right\}\ .\end{aligned} \tag{5.230}$$

[133]It is possible that these relations embody some hidden meaning. But, for example, the two sides of Eq. (5.228) do not transform in the same way under Eq. (5.21), so we hesitate to dig deeper.

We will refer to the coefficient of $f(\theta,\chi)$ in $\hat{\tilde{\Phi}}_{22}$, encapsulated by the term in Eq. (5.230) without any derivatives, as the "mass term." It is organized in terms of the intrinsic curvature at the horizon,[134]

$$\mathrm{Re}(\mathcal{K})\big|_{r=r_+} = \frac{|R_{0+}|^2}{2|R_+|^6}(r_+^2 - 3a^2\cos^2\theta) \; , \tag{5.231}$$

and quantities that are proportional to some power of the angular momentum. Expressed in this way, the mass term reeks of Kaluza-Klein, but we will leave that for another day.[135] At any rate, the practical advantage of this form is that it shows clearly which terms go to zero as $a \to 0$ and which terms do not.

Explicitly, when $a = 0$ (but $Q \neq 0$), we recover the known answer in the spherically symmetric case:[136]

$$\hat{\tilde{\Phi}}_{22}\bigg|_{a=0} = -\frac{c}{4r_+^2}\,\delta(U)\left(\nabla_{\mathrm{2d}}^2 - \frac{2\alpha}{r_+}\right) f(\theta,\varphi) \; . \tag{5.232}$$

While the geometrical significance of the mass term in Eq. (5.230) eludes us, the apparent fact that we can extract from it an overall factor of α does have

[134]By explicit computation, we find $\mathrm{Re}(\mathcal{K})\big|_{r=r_+} = \frac{1}{4}h^{ij}R_{ij}^{(\mathrm{2d})}$, where $R_{ij}^{(\mathrm{2d})}$ is the Ricci tensor defined by the metric h_{ij} on the squashed sphere [see Eq. (5.211)].

[135]For example, $(\det h)^{1/2}/\,(-\det g)^{1/2}\Big|_{r=r_+} = |R_{0+}|^2/|R_+|^2$. That is probably not a coincidence.

[136]At $r = r_+$, we have $\mathrm{Re}(\mathcal{K})\big|_{a=0} = \frac{1}{2r_+^2}$. Also, when $a = 0$ the delayed angle χ becomes the ordinary azimuthal angle φ.

physical significance.[137] In the extremal limit, which in this case corresponds to $a^2 + Q^2 = M^2$ and hence $r_- = r_+$, the surface gravity α goes to zero (as usual), and the entire mass term vanishes.

As far as we know, the first to point this out in the spherically symmetric situation was Sfetsos, who interpreted it as a breakdown of the solution [35]. The effect was recently revisited by Leichenauer in the context of entanglement between the conformal field theories dual to the asymptotically-AdS generalization of the Reissner-Nordström black hole [170]. In the context of shockwave scattering, the vanishing of the mass term in the operator $\mathscr{D}$ is what Maldacena and Stanford call the "βJ enhancement" of the scattering amplitude [117].

But let us not get ahead of ourselves. In this paper we are concerned exclusively with the single-shockwave geometry and its interpretation within general relativity. The sun will rise tomorrow, and we will have another opportunity to traverse that wormhole.

[137]We thank Douglas Stanford for explaining this to us.

5.8 Energy-momentum tensor

The localized source suitable for our shifted curvature tensor must be described by an energy-momentum scalar[138]

$$\hat{\tilde{t}}_{22} = \xi\, \delta(U)\, \delta^2(\vec{x}_\perp) \tag{5.233}$$

for some constant ξ, with all other energy-momentum scalars being zero. (The notation is such that $\vec{x}_\perp$ points to a fixed location on the squashed sphere; we will be more explicit later on.) Only in that case will Einstein's equation in the form of Eq. (5.221) reduce to a Green's function equation at the horizon:

$$\mathscr{D} f(\theta, \chi) = -4c^{-1} |R_+|^2\, \xi\, \delta^2(\vec{x}_\perp)\ . \tag{5.234}$$

Harmony with the ancient canons requires only that we announce the desired energy-momentum tensor and promptly conclude our inquiry. But, as pronounced back in Sec. 5.2.13, we find this negligence unbecoming.

This paper is formulated exclusively within the computational framework of classical field theory. Classical field theory is, as the name suggests, a theory of classical fields. It is not a theory of particles. If we wish to interpret the source of

[138]More generally, we could have $\hat{\tilde{t}}_{22} = \xi\, \delta(U)\, F(\vec{x}_\perp)$ for some profile $F(\vec{x}_\perp)$ in the angular directions.

gravitational field described by Eq. (5.233) as the gravitational backreaction from a *m*assless particle, then we must define what we mean by "massless particle" in the language of classical field theory.

The reader who could not possibly care less may skip to Sec. 5.9.

5.8.1 Laplacian at the Kerr-Newman horizon

We begin with some comments about the Laplacian. The equation of motion for a free, massless,[139] real scalar field $H \sim (0,0)$ is

$$(-\Box_{\|} + \Box_{\perp})H = 0 \ . \tag{5.235}$$

Our goal is to construct a solution to this equation in the form of a wave packet such that, when we send the width of the wave packet to zero, the field disappears but leaves behind a fixed nonzero energy-momentum concentrated arbitrarily close to the line $U = 0$.

It seems imbued in the collective consciousness that a scalar field decomposes into ingoing and outgoing plane waves near the horizon. This is certainly correct in the "Eddington-like" coordinates (u, v, θ, χ). The inquisitive student might ask whether it remains true in the Kruskal-like coordinates (U, V, θ, χ). No earlier

[139]Because we are doing classical field theory, the terminology here requires clarification. In the Klein-Gordon equation $(\nabla^2 - \ell^{-2})H = 0$, the "mass" parameter ℓ^{-1} should be thought of as a correlation length. There is no quantization in the picture, and no intrinsic interpretation of ℓ^{-1} as the mass of any particle.

than last year did a notable article go out of its way to address this point in the spherically symmetric case [171], so there is reason enough to fixate on it here.[140]

As always we will let the equations speak for themselves. We first state the parallel and perpendicular box operators acting on a smooth field configuration with weight (0,0) at $U = 0$ for arbitrary fixed nonzero V.

[140] The reader might also be interested in the relatively recent work by Castro, Lapan, Maloney, and Rodriguez [172].

5.8.2 Laplacian and spheroidal harmonics

The parallel box acting on a smooth field configuration $H \sim (0,0)$ is[141]

$$\Box_{\parallel}\big|_{U=0} H = \left[\frac{(2\alpha |R_{0+}|^2)^2}{|R_+|^2} c^{-1}\, \partial_U \partial_V - \frac{2r_+}{|R_+|^2} \left(\alpha + \frac{a}{|R_{0+}|^2}\, \partial_\chi \right) V \partial_V \right] H \,. \tag{5.240}$$

Note that there is a term proportional to $V\partial_V$, which only disappears if we further restrict to the bifurcation surface at $U = V = 0$. It remains present in the limit $a \to 0$.

[141] Establishing the correct functional form for the GHP-covariant derivatives provides a good exercise in the compacted formalism: given ε and ε' from Eq. (5.50), what are $\hat{\varepsilon}$ and $\hat{\varepsilon}'$? The devoted GHP acolyte will recall that ε and ε' transform as gauge fields, not as matter fields. (We use this terminology to evoke a parallel that *is* meant to be taken literally, at least as far as the mathematics are concerned.) In particular, for $\lambda = \lambda^* = U^{1/2}$, we have:

$$\hat{\varepsilon} = \lambda\lambda^*(\varepsilon - \lambda^{-1} D\lambda) = -U(\varepsilon - (-U)^{-1/2} D(-U)^{1/2}) = -U\varepsilon - \tfrac{1}{2} U^{-1} \hat{D} U = 0 \,, \tag{5.236}$$

and

$$\hat{\varepsilon}' = \lambda^{-1}\lambda^{*-1}(\varepsilon' - \lambda D' \lambda^{-1}) = -U^{-1}\left(\varepsilon' - \tfrac{1}{2}\hat{D}' U\right) = -U^{-1}\left(\varepsilon' - \alpha \frac{|R_0|^2}{2|R|^2}\right) \,. \tag{5.237}$$

So indeed we were correct in setting $\hat{\text{þ}} = \hat{D}$ way back in Sec. 5.2.14. Using the explicit functional form for ε' for the Kerr-Newman solution, which we reorganize in terms of α,

$$\varepsilon' = \rho' + \frac{r - r_+}{2|R|^2} + \alpha \frac{|R_0|^2}{2|R|^2} \,, \tag{5.238}$$

we find the hatted gauge field associated with the ingoing null trajectories:

$$\hat{\varepsilon}' = \hat{\rho}' - \frac{r - r_+}{2|R|^2 U} \,. \tag{5.239}$$

Like everything else in the hatted basis, it is finite at $U = 0$, since $\frac{r-r_+}{U}\Big|_{U=0} = \frac{\Delta}{UV}\, \frac{V}{r-r_-}\Big|_{U=0} = -\,c\, \frac{V}{2\alpha|R_{0+}|^2}$.

The perpendicular box is:

$$\Box_{\perp}\big|_{U=0} H = \frac{1}{|R_+|^2}\left[\mathscr{L}^2 + \alpha^2\, a^2 \sin^2\theta\, V\partial_V(V\partial_V) + 2\alpha\, a\,\frac{|R_+|^2}{|R_{0+}|^2} V\partial_V\partial_\chi\right] H\ , \tag{5.241}$$

where we have defined the angular operator

$$\mathscr{L}^2 \equiv |R|^2\nabla^2_{\rm 2d} - \frac{a^2\sin(2\theta)}{|R|^2}\partial_\theta = \partial_\theta^2 + \cot\theta\,\partial_\theta + \frac{|R|^4}{|R_0|^4}\frac{1}{\sin^2\theta}\partial_\chi^2\ . \tag{5.242}$$

The eigenfunctions of this operator can be written as $Z_{\ell m}(\theta)\, e^{im\chi}$, in which case $Z_{\ell m}(\theta)$ are the oblate spheroidal harmonics. This merits some discussion, since the traditional separation of variables for the wave equation in the Kerr background [155, 173] is performed in Boyer-Lindquist coordinates:

$$H_{\omega\ell m}(t, r, \theta, \varphi) = e^{-i\omega t} e^{im\varphi} R_{\omega\ell m}(r) Z_{\ell m}(\theta)\ . \tag{5.243}$$

The equation that Brill et al. [173] call the "flat-space angular spheroidal equation" is (with the mass term set to zero)

$$\left\{\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta}\right)\right\} Z_{\ell m}(\theta) = -\lambda_{\ell m} Z_{\ell m}(\theta)\ . \tag{5.244}$$

If we restrict the discussion to those modes for which

$$\omega = m\Omega_H \tag{5.245}$$

then we are considering a dependence on t and φ only in the combination $\chi = t - \Omega_H \varphi$:

$$H_{\omega\ell m}(t, r, \theta, \varphi) = H_{\omega\ell m}(r, \theta, \chi) = e^{im\chi} R_{\omega\ell m}(r) Z_{\ell m}(\theta) \ . \tag{5.246}$$

In that case, we have:

$$a^2\omega^2 \cos^2\theta - \frac{m^2}{\sin^2\theta} = -\frac{|R_+|^4}{|R_{0+}|^4} \frac{m^2}{\sin^2\theta} - \frac{2r_+^2 + a^2}{|R_{0+}|^4} a^2 m^2 \ . \tag{5.247}$$

The discrepancy between the quantity on the left and the coefficient of ∂_χ^2 in Eq. (5.242) is just a constant. Therefore, Eq. (5.244) restricted to modes satisfying Eq. (5.245) becomes

$$\mathscr{L}^2 Z_{\ell m}(\theta) = -\left(\lambda_{\ell m} - \frac{2r_+^2 + a^2}{|R_{0+}|^4} a^2 m^2\right) Z_{\ell m}(\theta) \equiv -\Lambda_{\ell m} Z_{\ell m}(\theta) \ . \tag{5.248}$$

So the ordinary spheroidal harmonics $Z_{\ell m}(\theta)$ are also eigenfunctions of $\mathscr{L}^2$. Note that even though they depend only on θ, they are labeled by both ℓ and m.[142]

5.8.3 Purely outgoing field configurations

An ansatz of the form $H(U, \theta, \chi)$ will satisfy $\partial_V H = 0$ and hence $\Box_{\|}\big|_{U=0} H = 0$. But that means the Klein-Gordon equation in Eq. (5.235) will simply reduce to

$$\mathscr{L}^2\Big|_{U=0} H(U, \theta, \chi) = 0 \; . \tag{5.250}$$

So a solution that depends only on U must also have no dependence on θ or χ. It is of no practical consequence to us that the solution does not depend on the azimuthal angle χ; but if we wish to localize the particle to a fixed longitude, then the field configuration must include modes at all values of ℓ.[143]

An alternative way to express the same thing is to proceed systematically with separation of variables. For simplicity, consider the Laplacian at the bifurcation surface, and input the ansatz $H(U, V, \theta, \chi) = X(U, V)\, Z_{\ell m}(\theta)\, e^{im\chi}$. Then the

[142] In the static limit, we recover the usual spherical operator and its eigenvalues:

$$\mathscr{L}^2\Big|_{a=0} = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\varphi^2 \; , \quad \Lambda_{\ell m}\big|_{a=0} = \ell(\ell+1) \; . \tag{5.249}$$

[143] Assuming a normalization such that $\int_0^\pi d\theta\, \sin\theta\, Z_{\ell m}(\theta)\, Z_{\ell' m'}(\theta)^* = \delta_{\ell\ell'}\delta_{mm'}$, the decomposition $\frac{1}{\sin\theta}\delta(\theta) = \sum_{\ell,m} C_{\ell m}\, Z_{\ell m}(\theta)$ implies $C_{\ell m} = \int_0^\pi d\theta\, Z_{\ell m}(\theta)^*\, \delta(\theta) = Z_{\ell m}(0)^*$, which are just some ℓ- and m-dependent numbers.

function $X(U,V)$ would have to satisfy

$$(\partial_U \partial_V + \mu_{\ell m}^2) X(U,V) = 0 \; , \tag{5.251}$$

with $\mu_{\ell m}^2 \equiv \Lambda_{\ell m}\, c\, (2\alpha |R_{0+}|^2)^{-2}$. As we all learned in kindergarten, this equation is only chiral when $\mu_{\ell m}^2 = 0$, in which case $X(U,V) = X_L(U) + X_R(V)$ for some functions $X_{L,R}$. But this happens only when $\ell = m = 0$. Undoubtedly there exists some grammatically correct sequence of words to rationalize this observation, but we will not venture to ascertain it.

If we rewrite the Laplacian terms of ∂_u and ∂_v instead of ∂_U and ∂_V, then the angular operators will be suppressed near $U = 0$, and the discussion will proceed along the lines of that given by Kiem, Verlinde, and Verlinde for the spherically symmetric case [124].

So if we wish to construct a wave packet whose energy-momentum distribution is localized at the future horizon *and* at a fixed location on the squashed sphere, then we must perform the mode decomposition in Eddington-like coordinates.

The following field configuration is fit for our purpose:

$$H(U,\theta) = N^{1/2}\, e^{-\frac{r_+^2}{4\sigma^2}\cos^2\theta}\, e^{-\frac{1}{4\sigma^2}(u-u_0)^2} \cos(Pu) \; . \tag{5.252}$$

Here u_0 is a constant that we will eventually take to ∞, and σ describes a width that we will eventually take to 0. We insert the nonstandard factor $4\sigma^2$ instead of $2\sigma^2$ because we want the square of H, not H itself, to become a delta function. The normalization factor $N^{1/2}$ is chosen in just the right way to leave behind an energy-momentum P such that

$$\mathcal{T}_{UU} = P\,\delta(U)\,\delta^2(\vec{x}_\perp)\ , \tag{5.253}$$

where $\vec{x}_\perp = \vec{x} - \vec{x}_E$, and $\vec{x}_E$ points to an arbitrary point on the equator.[144] Readers content with this meager account of the notation should feel free to proceed, while those looking for more details should work through Sec. 5.E.

5.8.4 Energy-momentum scalars

The trace-reversed energy-momentum tensor for a massless complex scalar field $\mathcal{H}$ is:

$$\mathcal{T}_{\mu\nu} \equiv \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\rho\sigma} T_{\mu\nu}\right) = \partial_\mu \mathcal{H}^*\, \partial_\nu \mathcal{H} + c.c.\ . \tag{5.254}$$

[144] The equator is more temperate than the north pole.

The energy-momentum scalars [recall our definition in Eq. (5.106)] for this classical matter field therefore take an especially simple form:

$$
\begin{aligned}
&t_{00} = 8\pi |D\mathcal{H}|^2\ , \quad t_{01} = 8\pi\,\mathrm{Re}(D\mathcal{H}^*\delta\mathcal{H})\ , \quad t_{02} = 8\pi\,\mathrm{Re}(\delta\mathcal{H}^*\delta\mathcal{H})\ ,\\
&t_{22} = 8\pi |D'\mathcal{H}|^2\ , \quad t_{21} = 8\pi\,\mathrm{Re}(D'\mathcal{H}^*\delta'\mathcal{H})\ , \quad t_{20} = 8\pi\,\mathrm{Re}(\delta'\mathcal{H}^*\delta'\mathcal{H})\ ,\\
&t_{10} = 8\pi\,\mathrm{Re}(D\mathcal{H}^*\delta'\mathcal{H})\ , \quad t_{12} = 8\pi\,\mathrm{Re}(D'\mathcal{H}^*\delta\mathcal{H})\ , \quad t_{11} = 4\pi\,\mathrm{Re}(D\mathcal{H}^*D'\mathcal{H} + \delta\mathcal{H}^*\delta'\mathcal{H})\ ,\\
&t_{\Pi} = -\frac{\pi}{3}g^{\mu\nu}\mathcal{T}_{\mu\nu} = -\frac{2\pi}{3}\,\nabla_\mu\mathcal{H}^*\,\nabla^\mu\mathcal{H} = \frac{4\pi}{3}\,\mathrm{Re}(D\mathcal{H}^*D'\mathcal{H} - \delta\mathcal{H}^*\delta\mathcal{H})\ . \qquad (5.255)
\end{aligned}
$$

Just as the energy-momentum scalars for a $U(1)$ gauge field can be expressed as products of complex numbers called Maxwell scalars, the energy-momentum scalars for a massless complex scalar field are expressed as products of what we will call *K*lein-Gordon scalars[145]

$$
s_a \equiv \sqrt{2}\, e_a^{\mu}\nabla_\mu\mathcal{H} = \sqrt{2}\,(D\mathcal{H}, D'\mathcal{H}, \delta\mathcal{H}, \delta'\mathcal{H}) \qquad (5.256)
$$

and their complex conjugates.

We could consider these quantities to have definite GHP weights if we assign the scalar field $\mathcal{H}$ the representation $(0,0)$ under the GHP group (since physical

[145] We employ the tried and true tactic in physics education of italicizing something to mask its pointlessness.

matter fields should not depend on the choice of tetrad), in which case all of the NP derivatives in Eqs. (5.255) and (5.256) can be replaced with GHP derivatives.

We have treated the complex case for generality; now we will reduce the complex field to a real field by setting

$$\mathcal{H} = \tfrac{1}{\sqrt{2}} H \ . \tag{5.257}$$

Let us calculate the Klein-Gordon scalars for the field configuration in Eq. (5.252). To do that, we will rewrite the asymptotic tetrad[146] of Eq. (5.13) in Eddington-like coordinates (u, v, θ, χ):

$$D = \frac{2|R_0|^2}{\Delta}\partial_v + \frac{a}{\Delta}\left(1 - \frac{|R_0|^2}{|R_{0+}|^2}\right)\partial_\chi \ , \quad D' = \frac{\Delta}{2|R|^2}\left[\frac{2|R_0|^2}{\Delta}\partial_u + \frac{a}{\Delta}\left(1 - \frac{|R_0|^2}{|R_{0+}|^2}\right)\partial_\chi\right] \ ,$$
$$\delta = \frac{1}{R\sqrt{2}}\left[\partial_\theta + \frac{i}{\sin\theta}\left(1 - a\Omega_H \sin^2\theta\right)\partial_\chi + ia\,\sin\theta\,(\partial_u + \partial_v)\right] \ . \tag{5.258}$$

Clearly $DH = 0$, while

$$D'H = \frac{|R_0|^2}{|R|^2}\partial_u H \approx -P\frac{|R_0|^2}{|R|^2} N^{1/2}\, e^{-\frac{r_+^2}{4\sigma^2}\cos^2\theta} e^{-\frac{1}{4\sigma^2}(u-u_0)^2} \sin(Pu) \ . \tag{5.259}$$

[146]Hatting at this stage would be premature because the field configuration in Eq. (5.252) is naturally expressed in terms of u, not U.

The purpose of the wave packet construction is to localize the field in position space (i.e., $\frac{|u-u_0|}{\sigma} \ll 1$), in which case it is delocalized in momentum space (i.e., $\frac{\sigma_P}{P} \ll 1$ with $\sigma_P \equiv \sigma^{-1}$). The approximation symbol in Eq. (5.259) denotes that we have dropped a term proportional to $\frac{u-u_0}{\sigma^2 P}$, in line with this aim.

Meanwhile, we also have

$$\delta H = \frac{1}{R\sqrt{2}} \left[\frac{r_+^2}{4\sigma^2} \sin(2\theta)\, H + ia\, \sin\theta \frac{|R|^2}{|R_0|^2} D'H \right] \approx \frac{ia}{\sqrt{2}} \frac{R^*}{|R_0|^2} D'H \,. \quad (5.260)$$

Here we have expanded around $\theta = \frac{\pi}{2}$ (or, equivalently, defined $\epsilon \equiv \theta - \frac{\pi}{2}$ and expanded around $\epsilon = 0$) and assumed localization in longitude in the form $\frac{|\theta - \frac{\pi}{2}|}{\sigma / r_+} \ll 1$.

At this stage the potentially nonzero energy-momentum scalars are

$$t_{22} = 4\pi\, (D'H)^2,\; t_{02} = 4\pi\, \mathrm{Re}[(\delta H)^2],\; t_{21} = 4\pi\, \mathrm{Re}(D'H\delta H)\;,\;\; t_{11} = -3t_{\Pi} = 2\pi\, |\delta H|^2, \quad (5.261)$$

along with those related by conjugation.

5.8.5 The delta function limit

We fix the normalization factor by demanding Eq. (5.233). This condition is in terms of the *h*atted tetrad. The relations between the hatted and unhatted versions of the pertinent energy-momentum scalars are:

$$\hat{t}_{22} = U^{-2} t_{22} \ , \ \ \hat{t}_{02} = t_{02} \ , \ \ \hat{t}_{21} = -U^{-1} t_{21} \ , \ \ \hat{t}_{11} = t_{11} \ , \ \ \hat{t}_{\Pi} = t_{\Pi} \ . \tag{5.262}$$

Suppose we calculate $\hat{t}_{22}$ and get a finite nonzero result. (Spoiler alert: we will.) Then Eq. (5.259) will imply that all of the energy-momentum scalars in Eq. (5.261) are proportional to each other. Because this holds in the *u*nhatted basis, the GHP scaling law in Eq. (5.262) implies

$$\hat{t}_{21} = -U^{-1} t_{21} \propto -U^{-1} t_{22} = -U^{+1}\, \hat{t}_{22} \ , \ \ \hat{t}_{02} = t_{02} \propto t_{22} = U^{+2}\, \hat{t}_{22} \ , \tag{5.263}$$

and similarly $\hat{t}_{11} = 3\hat{t}_{\Pi} \propto U^2\, t_{22}$. Since we require $\hat{t}_{22} = \xi\, \delta(U)\, \delta^2(\vec{x}_\perp)$, we have

$$\hat{t}_{21} \propto U\, \delta(U)\, \delta^2(\vec{x}_\perp) \ , \ \ \hat{t}_{02} \propto U^2\, \delta(U)\, \delta^2(\vec{x}_\perp) \ , \tag{5.264}$$

and $\hat{t}_{11} = -3\hat{t}_{\Pi} \propto U^2\, \delta(U)\, \delta^2(\vec{x}_\perp)$. Acting on smooth test functions $\mathcal{F}(U)$, these are all zero. That is good, because we wanted to satisfy Eq. (5.221).

Having breathed that sigh of relief, we return to Eq. (5.259) and square it, applying judiciously the machinations of Sec. 5.E.5 to obtain the result:

$$\hat{t}_{22} \approx NP^2 \left(\frac{r_+^2 + a^2}{r_+^2} \right)^3 (2\pi)^3 \sigma^2 \alpha r_+ |U_0|^{-1}\, \delta(U - U_0)\, \delta^2(\vec{x}_\perp) \ . \tag{5.265}$$

Setting this equal to $\xi\,\delta(U-U_0)\,\delta^2(\vec{x}_\perp)$ finally fixes the normalization factor of the field configuration in Eq. (5.252):

$$N^{1/2} = \frac{1}{\left(2\pi\,\frac{r_+^2+a^2}{r_+^2}\right)^{3/2} P\sigma} \left(\frac{|U_0|}{\alpha r_+}\xi\right)^{1/2} . \tag{5.266}$$

Readers may well wonder whether there is any insight to be gleaned from Eq. (5.266) or the steps that led to it. If so they are in good company. Notwithstanding the relative dryness of this part of the calculation, both sides of Einstein's equation have now been given the attention they deserve, and our journey is complete.

5.9 Discussion

Inspired by 't Hooft's S-matrix approach to quantum gravity and Kitaev's recent revival thereof, we have generalized the Dray-'t Hooft gravitational shockwave to the Kerr-Newman black hole using the method of spin coefficients.[147]

[147] We hope we have convinced the reader that the spin coefficient formalism is useful for something and is worth studying for its own sake. By and large, we encourage the interested reader to spend at least a solid month diving into *Spinors and Space-time*. (One of us reminisces about the pastoral serenity of the Netherlands, and the other renounces the cantankerous metropolis of Manhattan.) The only major point of contention between us pertains to the abstract index notation. We were amused to learn that there is a subject in algebraic geometry called "abstract nonsense" – one of us considers this sufficient precedent to refer to the abstract index notation as "concrete nonsense."

In doing so we have stressed the solution's generalized Kerr-Schild form [137] and showed that when the null tetrad for the unshifted geometry is aligned with shear-free geodesics, and when the unshifted weight-$(1,1)$ Ricci scalar is zero,[148] the shifted curvature scalars depend only linearly on the perturbation. We have also crusaded for the spin coefficients τ and τ' and interpreted the shockwave solution according to Szekeres's gravitational compass [160].

We have not attempted to solve the differential equation $\mathscr{D} f \propto \delta^2(\vec{x}_\perp)$. The operator $\mathscr{D}$ is analytic near $a = 0$, so we should be able to perturb around the integral formula of Dray and 't Hooft [29]. Along those lines, it may be of some interest to derive an integral formula in terms of spheroidal harmonics, but we would probably have to resort to numerics for anything beyond a rudimentary understanding.[149]

An obvious direction for future research is to perturb other backgrounds by shifting the tetrad. For example, Taub showed that the Kerr-Vaidya metric is of the generalized Kerr-Schild form [137]. Our introduction suggests that we should generalize to the Kerr-AdS space-time, since that might lead to precise statements about chaos in a putative dual field theory.[150]

[148]If we had shifted l instead of l', then the required curvature condition would be that the weight-$(-1,-1)$ Ricci scalar should be zero.

[149]Dray and 't Hooft themselves "have not attempted to perform the integration explicitly" for their result [29]. Sfetsos, for his part, did elaborate somewhat on his solutions in Appendix D of his paper [35].

[150]We thank Nick Hunter-Jones for this motivational speech. The voracious reader committed to preempting our next move could compute the out-of-time-ordered correlators and thereby

We will conclude with a pedantic remark about the effective action for the horizon field $f(\theta, \chi)$, which is the classical field variable in our solution.

In general, when we have a classical equation of motion, it is natural to ask what variational principle could lead to it. Since the Ricci tensor is linear in $f(\theta, \chi)$, our equation of motion is linear in the field, so our experience in field theory might lead us to expect an action quadratic in the field.

But this is not so: the Lagrangian of general relativity is proportional to the Einstein-Hilbert curvature Π, which we have already seen is *l*inear in f. All terms of $O(f^2)$ and higher vanish identically, and, as required by the background Einstein equation, the integral of the $O(f)$ term is zero, leaving no effective action at all for the horizon field.

What to make of this? Procedurally, there is another way to vary an action to get an equation of motion: vary with respect to a different field. If the "equation of motion" is actually a constraint equation, then it should be implemented in the calculus of variations with a Lagrange multiplier.

Consider a path integral over all classical fields $f(\theta, \chi)$ that satisfy $\mathscr{D} f = 0$:[151]

$$\mathcal{Z} \equiv \int \mathcal{D}f \; \delta\left(\mathscr{D} f\right) = \int \mathcal{D}f \, \mathcal{D}f' \; e^{i\int d^2x\, f' \mathscr{D} f} \; . \tag{5.267}$$

reevaluate the Kerr/CFT conjecture beyond eigenstate thermalization. Lord knows anyone who actually worked through our paper deserves some recompense.

[151] For the sake of argument we are only considering the gravitational part of the action. More generally there should be an f-independent function on the right-hand side of the constraint.

We have used the Fourier representation of the delta function and thereby introduced[152] the classical field variable f', which serves as a Lagrange multiplier for the constraint $\mathscr{D} f = 0$.

The argument of the exponential in Eq. (5.267) is exactly 't Hooft's effective action [30]. This straightforward interpretation of the constraint equation for the horizon field provides a path-integral sense in which the two shockwaves are canonically conjugate variables.

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[152]The prime is part of the name of the field and does not denote any kind of derivative.

5.A Signature change

In this section, we systematically track down the minus signs that arise when changing the metric signature from the "mostly minus" convention in Chandrasekhar [142] and Penrose and Rindler [12] to the "mostly plus" convention in Zee [140].[153]

In this section, barred quantities will denote those in the old signature $(+---)$, while unbarred quantities will denote those in the new signature $(-+++)$.[154]

5.A.1 Basic assumptions

We should first state outright that we will flip the sign of the space-time metric *and* the metric in the tangent space. Consequently, our first fundamental assumption of signature change is:

$$g_{\mu\nu} \equiv \zeta\, \bar{g}_{\mu\nu} \ , \quad \eta_{ab} \equiv \zeta\, \bar{\eta}_{ab} \ , \quad \zeta \equiv -1 \ . \tag{5.268}$$

The unbarred frame fields e^a_μ are defined by $g_{\mu\nu} = \eta_{ab}\, e^a_\mu e^b_\nu$, and the barred frame fields $\bar{e}^a_\mu$ are defined by $\bar{g}_{\mu\nu} = \bar{\eta}_{ab}\, \bar{e}^a_\mu \bar{e}^b_\nu$. The inverse statements are $\eta_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b$ and $\bar{\eta}_{ab} = \bar{g}_{\mu\nu} \bar{e}^\mu_a \bar{e}^\nu_b$. So, defining the quantities $e_{a\mu} \equiv \eta_{ab}\, e^b_\mu$ and $\bar{e}_{a\mu} \equiv \bar{\eta}_{ab}\, \bar{e}^b_\mu$, we

[153] One of us is in fact credited for effectuating that sign change, so the duty to write this section falls squarely on our shoulders.

[154] We work with an off-diagonal metric, so the scrupulous way to express the signature is: "Let $\{\lambda_\alpha\}^4_{\alpha=1}$ be the eigenvalues of η_{ab}. Then the old signature is the convention $\sum^4_{\alpha=1} \lambda_\alpha = -2$, while our signature is the convention $\sum^4_{\alpha=1} \lambda_\alpha = +2$."

obtain from Eq. (5.268) the requirements:

$$e^a_\mu e_{a\nu} = \zeta\, \bar{e}^a_\mu \bar{e}_{a\nu} \ , \quad e^\mu_a e_{b\mu} = \zeta\, \bar{e}^\mu_a \bar{e}_{b\mu} \ . \tag{5.269}$$

We now have a choice as to which quantities will flip sign. In comparison to the older references, our null *v*ectors $(l^\mu, l'^\mu, m^\mu, m'^\mu)$ will not receive a sign flip, in which case our null *f*orms $(l_\mu, l'_\mu, m_\mu, m'_\mu) \equiv (g_{\mu\nu}\, l^\nu, g_{\mu\nu}\, l'^\nu, g_{\mu\nu}\, m^\nu, g_{\mu\nu}\, m'^\nu)$ will.

It requires attention to infer from this the appropriate index structure, because the basis is null instead of orthonormal. The correct translation will form our second fundamental assumption of signature change:

$$e^\mu_a \equiv \bar{e}^\mu_a \ , \quad e^a_\mu \equiv \bar{e}^a_\mu \ . \tag{5.270}$$

That is, we ordain that *n*either e^μ_a nor e^a_μ flips sign. What does flip sign is the quantity with both indices lowered:

$$e_{a\mu} = \zeta\, \bar{e}_{a\mu} \ . \tag{5.271}$$

5.A.2 Spin coefficients flip sign

If we insert the above definitions into the torsion-free condition in Eq. (5.35), we will find that the spin connection *w*ith one index up and one index down does not flip sign:

$$\omega^a{}_b = \bar{\omega}^a{}_b \, . \tag{5.272}$$

This means that ω_{ab} *d*oes flip sign. Unpacking the hidden 1-form index according to our conventions gives $\omega_{ab} = (\omega_\mu)_{ab}\, dx^\mu$. Since $dx^\mu = d\bar{x}^\mu$ (no lowering or raising of indices is involved), we conclude that $(\omega_\mu)_{ab} = \zeta\, (\bar{\omega}_\mu)_{ab}$. Therefore, recalling the definition $\gamma_{abc} \equiv (\omega_\mu)_{ab}\, e_c^\mu$ from Eq. (5.32), we find that *t*he spin coefficients flip sign:

$$\gamma_{abc} = \zeta\, \bar{\gamma}_{abc} \, . \tag{5.273}$$

Meanwhile, because of Eq. (5.271), *t*he null Cartan equations look the same in either signature. So Eq. (5.49) looks exactly the same as Eq. (4.13.44) in *Spinors and Space-time* [12].

5.A.3 Curvature scalars do not flip sign

Next we engage with the $SO(3,1)$ field strength, as defined by Eq. (5.89): $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$. Since $\omega^a{}_b$ does not flip sign, neither does $\Omega^a{}_b$:

$$\Omega^a{}_b = \bar{\Omega}^a{}_b \, . \tag{5.274}$$

This means that the quantity Ω_{ab} *d*oes flip sign. As an unavoidable consequence, the Riemann tensor in the tangent space with all indices down, $R_{abcd} \equiv$

$(\Omega_{\mu\nu})_{ab}\, e_c^{\mu} e_d^{\nu}$, flips sign:

$$R_{abcd} = \zeta\, \bar{R}_{abcd}\ . \tag{5.275}$$

It is misleading to simply assert that the Newman-Penrose equations remain fixed upon changing the metric signature, as if it were to follow like night the day.

Besides the sign from Eq. (5.275), there is also an overall sign convention in the $d$efinition of the curvature scalars, irrespective of the choice of metric signature. For example, Penrose and Rindler allegedly define $\Psi_0 \equiv +C_{1313}$ [see their Eq. (4.11.9)] while Chandrasekhar defines $\Psi_0 \equiv -C_{1313}$ [see his Eq. (294) on p. 43], even though both books use the $(+---)$ signature and ostensibly proclaim identical Newman-Penrose equations. We will return to this precipice of insanity in a moment.

From the quantity R_{abcd}, the Weyl tensor in the tangent space with all indices down is defined as:

$$C_{abcd} \equiv R_{abcd} + \tfrac{1}{2}\,(\eta_{ad}\, R_{bc} + \eta_{bc}\, R_{ad} - \eta_{ac}\, R_{bd} - \eta_{bd}\, R_{ac}) + \tfrac{1}{6}\,(\eta_{ac}\,\eta_{bd} - \eta_{ad}\,\eta_{bc})\ \eta^{ef} R_{ef}\ . \tag{5.276}$$

This quantity flips sign under a change of signature:

$$C_{abcd} = \zeta\, \bar{C}_{abcd}\ . \tag{5.277}$$

Since this would be obnoxious to track, we might be tempted to insert an extra sign in the definition of the Weyl scalars to obviate it. But we will not. The reason is that, as we forewarned a moment ago, there is an additional sign that comes up; apparently our conventions up to this point result in an automatic cancellation of this additional sign with respect to the GHP equations in the original paper and in *Spinors and Space-time*. [See Eq. (5.282) for our definitions.]

Meanwhile, since $R^a{}_{bcd} = \eta^{ae} R_{ebcd}$ and $R_{abcd} = \zeta\, \bar{R}_{abcd}$, the Ricci tensor in the tangent space does *n*ot flip sign:

$$R_{ab} \equiv R^c{}_{acb} = \bar{R}^c{}_{acb} \equiv \bar{R}_{ab}\ . \tag{5.278}$$

There is yet another overall sign convention, this one pertaining to the Ricci scalars; and this time, Penrose and Rindler [see their Eq. (4.11.10)] and Chandrasekhar [see his Eq. (300) on p. 44] use the *s*ame convention. With a sigh heard 'round the world, we decree that this mutually consistent sign is the *l*ess convenient choice for our rendition of the formalism, and we hereby excommunicate it from the definition of the Ricci scalars [see Eq. (5.283)].

The Einstein-Hilbert curvature contains one additional sign relative to the components in Eq. (5.278):

$$\eta^{ab} R_{ab} = \zeta\, \bar{\eta}^{ab} \bar{R}_{ab}\ . \tag{5.279}$$

At this point we have a choice: we could accept an extra sign flip, but only at the cost of flipping the sign of Π in all of the Newman-Penrose equations. One of the authors insisted on maintaining the sanctity of these equations, and the other author shrugged in resigned indifference. We therefore define

$$\Pi \equiv -\frac{1}{24}\eta^{ab}R_{ab} = -\zeta\frac{1}{24}\bar{\eta}^{ab}\bar{R}_{ab} = +\bar{\Pi}\ . \tag{5.280}$$

Like night the day.

5.A.4 Extra sign in GHP derivatives

A final important potential source of sign confusion is in the GHP-covariant derivatives defined in Eq. (5.43). While the spin coefficients flip sign under a change of metric signature, the NP derivatives do not. This means there must be an extra sign in the definition of the GHP derivatives, so that the sign flips from the spin coefficients cancel out.

To track this down, consider the definition of the GHP derivatives in the $(+---)$ signature:[155]

$$\begin{aligned} \bar{\text{þ}} &\equiv \bar{D} - 2h\,\bar{\varepsilon} - 2\bar{h}\,\bar{\varepsilon}^* \;, \quad \bar{\eth} \equiv \bar{\delta} - 2h\,\bar{\beta} + 2\bar{h}\,\bar{\beta}'^* \;, \\ \bar{\text{þ}}' &\equiv \bar{D}' + 2h\,\bar{\varepsilon}' + 2\bar{h}\,\bar{\varepsilon}'^* \;, \quad \bar{\eth}' \equiv \bar{\delta}' + 2h\,\bar{\beta}' - 2\bar{h}\,\bar{\beta}^* \;. \end{aligned} \tag{5.281}$$

If we follow the steps of the calculation in Eq. (5.40) (which is in a footnote, in case the reader missed it the first time), we see that a certain crucial sign emerges as a result of whether $l^\mu l'_\mu = -m^\mu m'_\mu$ is $+1$ or -1. It is this sign that determines the extra signs in Eq. (5.281) relative to those in Eq. (5.43).

5.B Curvature scalars in the compacted formalism

In this section, we collect and discuss the explicit formulas for the curvature scalars. While this is standard material on its face, we will take some care to emphasize our preferred formatting of "curvature scalar on the left, derivatives and products of spin coefficients on the right." This rearranging of what are typically called the "Newman-Penrose equations" is just, for the most part, "solving" the Newman-Penrose equations for each curvature scalar,[156] although we have not

[155] We remind the reader that the bar on $\bar{h}$ is part of the name of the weight and therefore has nothing to do with the other barred notation of this section.

[156] In this form the NP equations seem to be the same as the null Gauss-Codazzi equations of Vega, Poisson, and Massey [174].

seen our method for computing the spacelike and timelike curvatures in any of the standard references.

We do this to emphasize that the spin coefficient formalism is just a convenient way of organizing differential geometry. The Newman-Penrose equations should be read as *d*efinitions of the Weyl scalars and of the Ricci scalars, consistent with the philosophy of the spin coefficient formalism that once we specify a tetrad we should never again see a tensor index.

Properly interpreted, the Newman-Penrose equations in no way involve any field equations. They are purely geometrical definitions of the curvature, equivalent to the individual components of Cartan's second structure equation; they are not equations of motion to be solved.[157]

The nongravitational fields enter only when we either write down the most general solution to the gravitational Bianchi identities (which correspond to a covariant conservation law for the most general two-index symmetric tensor), or when we explicitly introduce other fields into the action and define the energy-

[157] To beat a dead horse with another dead horse, let us elaborate further. The NP equations describe only "free" gravity (i.e., the Weyl tensor) and the left-hand side of the trace-reversed Einstein equation (i.e., the Ricci tensor), not the energy-momentum distribution on the right-hand side. While the Weyl tensor and the Ricci tensor describe locally independent degrees of freedom, they are related globally by the gravitational analog of Gauss's law (i.e., the Bianchi identities). This relation constitutes an entire discipline by itself, so we will simply defer to two authoritative references: the introduction to Szekeres's interpretation of the Weyl scalars [160], and the discussion surrounding Eq. (4.10.12) in *Spinors and Space-time* [12], which explains that "the *d*erivative of T_{ab} may be regarded as a source for the gravitational spinor field Ψ_{ABCD}."

momentum tensor as the variational derivative of their action with respect to the metric.

Without further ado, we begin with the obligatory formulas that are regurgitated in every textbook on the subject, starting with the Weyl scalars and their weights:

$$\begin{aligned}&\Psi_0 \equiv C_{1313} \sim (2,0)\ , \quad \Psi_1 \equiv C_{1312} \sim (1,0)\ , \quad \Psi_2 \equiv C_{1342} \sim (0,0)\ , \\ &\Psi_3 \equiv \Psi_1' = C_{2421} \sim (-1,0)\ , \quad \Psi_4 \equiv \Psi_0' = C_{2424} \sim (-2,0)\ . \end{aligned} \tag{5.282}$$

These five complex numbers account for the 10 real degrees of freedom in the Weyl tensor.

The Ricci scalars are defined as[158]

$$\Phi_{00} \equiv \tfrac{1}{2}R_{11} \sim (1,1)\ , \quad \Phi_{01} \equiv \tfrac{1}{2}R_{13} \sim (1,0)\ , \quad \Phi_{02} \equiv \tfrac{1}{2}R_{33} \sim (1,-1)\ ,$$
$$\Phi_{22} \equiv \Phi'_{00} = \tfrac{1}{2}R_{22} \sim (-1,-1)\ , \quad \Phi_{21} \equiv \Phi'_{01} = \tfrac{1}{2}R_{24} \sim (-1,0)\ ,$$
$$\Phi_{20} \equiv \Phi'_{02} = \tfrac{1}{2}R_{44} = \Phi^*_{02} \sim (-1,1)\ , \quad \Phi_{10} \equiv \Phi^*_{01} = \tfrac{1}{2}R_{14} \sim (0,1)\ ,$$
$$\Phi_{12} \equiv \Phi^*_{21} = \tfrac{1}{2}R_{23} \sim (0,-1)\ , \quad \Phi_{11} \equiv \tfrac{1}{4}(R_{12}+R_{34}) \sim (0,0)\ . \tag{5.283}$$

In total, the collection of fields $\Phi_{\alpha\beta}$ with $\alpha, \beta = 0,1,2$ can be considered a Hermitian 3-by-3 matrix, which accounts for 9 of the real degrees of freedom in the Ricci tensor. The last degree of freedom is defined by the weight $(0,0)$ combination of

[158]The following unrepentant diatribe should be attributed solely to one of the authors and for a personal touch will eschew the royal "we." The author Y. B. begins: While I reluctantly accept most of the Newman-Penrose hieroglyphics, I must disavow the notation for the Ricci scalars. A far better notation would be, for example, to label these components by their weights: say $\Phi_{(h,\bar{h})}$ where h and $\bar{h}$ take the appropriate values $+1, 0, -1$. Or just work with $R_{ab} = e^\mu_a e^\nu_b R_{\mu\nu}$, untainted by fanciful obscurity. A novel and foreign notation should only be introduced when it provides useful information or affords some efficient means of calculation. The Newman-Penrose notation for the Ricci scalars does neither. (Even if I were to introduce spinors the notation would be uncalled for: labeling the $\Phi_{\alpha\beta}$ by their $d$yad components, as is also standard, poses no threat to pedagogical decency.) An apologist might defend this atrocity on the grounds that Einstein's equation with an electromagnetic field takes the form $\Phi_{\alpha\beta} = \phi_\alpha \phi^*_\beta$. Well done: a bad notation on one side of an equation can match an equally bad notation on the other side. Stop the presses: the entire discipline of differential geometry should be tailored exclusively to the special case of a $U(1)$ gauge field. Never mind that a Maxwell field is but one type of classical field that could couple to gravity. Never mind that $t$his entire paper is about the gravitational backreaction from a massless scalar. This flimsy justification might satisfy a radio astronomer, but it holds no sway over a quantum field theorist. Furthermore, $l$abeling the components by their weights would achieve the same thing. But instead I am told to remember that the Maxwell scalars are labeled by $1-h \in \{0,1,2\}$ instead of $h \in \{+1,0,-1\}$, and that the Ricci scalars follow suit. There is no good reason to compel a bright eyed, bushy tailed student to decipher such arcane scripture. Since my primary goal in this paper is to explain a new result to existing practitioners in the field, I have not seriously attempted to overhaul a convention that has been standard since before the passage of the Civil Rights Act. Instead, I resign myself to leaving this footnote for the consideration of future generations who might be more willing to deal with it.

tetrad vectors orthogonal to the combination that defines Φ_{11}. We refer to this remaining degree of freedom as the Einstein-Hilbert curvature:

$$\Pi \equiv \tfrac{1}{12}(R_{12} - R_{34}) = -\tfrac{1}{24} R_{\mu\nu}\, g^{\mu\nu} \ . \tag{5.284}$$

As emphasized in the main text, we prefer to privilege the 1-forms over the vectors and therefore define everything in the Cartan formalism. To continue this line of reasoning, we recall the definition of the curvature 2-form from Eq. (5.89) as the field strength of the $SO(3,1)$ gauge group:

$$(\Omega_{\mu\nu})^a{}_b = \partial_\mu (\omega_\nu)^a{}_b + (\omega_\mu)^a{}_c\, (\omega_\nu)^c{}_b - (\mu \leftrightarrow \nu) \ . \tag{5.285}$$

Then we lower the index a and contract the base-space 2-form indices with the vectors of the null tetrad:

$$R_{abcd} \equiv (\Omega_{\mu\nu})_{ab}\, e_c^{\mu} e_d^{\nu} \ . \tag{5.286}$$

The goal now is to write this in terms of the γ_{abc} that we defined in Eq. (5.32). Because we believe, heaven forbid, that it should be possible for a reader to actually learn something from a research paper, we will show this explicitly.

First, since $\gamma_{abc} \equiv (\omega_\mu)_{ab}\, e_c^{\ \mu}$, we have $(\omega_\mu)_{ab} = \gamma_{abc}\, e^c_{\ \mu}$. Therefore:

$$\partial_\mu(\omega_\nu)_{ab} = \partial_\mu(\gamma_{abc}\, e^c_{\ \nu}) = \partial_\mu\gamma_{abc}\, e^c_{\ \nu} + \gamma_{abc}\, \partial_\mu e^c_{\ \nu}\ . \tag{5.287}$$

Contracting with $e_c^{\ \mu} e_d^{\ \nu}$ and relabeling some indices gives:

$$\partial_\mu(\omega_\nu)_{ab}\, e_c^{\ \mu} e_d^{\ \nu} = e_c^{\ \mu}\partial_\mu\gamma_{abe}\, \delta^e_d + \gamma_{abe}\, e_c^{\ \mu}\, e_d^{\ \nu}\, \partial_\mu e^e_{\ \nu} = e_c^{\ \mu}\nabla_\mu\gamma_{abd} + \gamma_{abe}\, e_c^{\ \mu} e_d^{\ \nu}\, \partial_\mu e^e_{\ \nu}\ . \tag{5.288}$$

That last equality is one reason we show each step: since γ_{abc} carries no *t*ensor indices, we can replace the partial derivative acting on it by a covariant derivative. This is what will give us the right to claim that we get Newman-Penrose directional derivatives in the resulting expressions.

Next, recalling the definition $\lambda_{abc} \equiv -e_a^{\ \mu} e_c^{\ \nu}\partial_\mu e_{b\nu} - (\mu \leftrightarrow \nu) = \gamma_{abc} - \gamma_{cba}$ from an earlier footnote [see Eq. (5.31)], and defining $e_a^{\ \mu}\nabla_\mu \equiv \nabla_a$, we find:[159]

$$\left[\partial_\mu(\omega_\nu)_{ab} - (\mu \leftrightarrow \nu)\right]\, e_c^{\ \mu} e_d^{\ \nu} = \nabla_c\gamma_{abd} - \nabla_d\gamma_{abc} - \gamma_{abe}(\gamma_c{}^e{}_d - \gamma_d{}^e{}_c)\ . \tag{5.289}$$

Notice that this expression is manifestly odd under $a \leftrightarrow b$ (since $\gamma_{abc} = -\gamma_{bac}$), and that it is manifestly odd under $c \leftrightarrow d$ (because it was defined from a 2-form in the base space). Further note, however, that it is *n*ot manifestly invariant

[159]Notice that some of the terms that are $O(\gamma^2)$ come from the $d\omega$ part of $\Omega^a{}_b$. So as far as the curvature is concerned, it is not a completely trivial matter to go from the spin connection to the spin coefficients.

under the exchange $ab \leftrightarrow cd$. It may or may not be $s$ecretly invariant under that exchange after the use of some identities (it is), but the formula as written is not $m$anifestly invariant under that exchange.

Having said that, let us return to Eqs. (5.285) and (5.286) and transcribe the ω^2 terms into spin coefficient notation:

$$\left[(\omega_\mu)_{ae}\,(\omega_\nu)^e{}_b - (\mu \leftrightarrow \nu)\right]\, e_c^\mu e_d^\nu = \gamma_{aec}\,\gamma^e{}_{bd} \;-\; \gamma_{aed}\,\gamma^e{}_{bc}\;. \tag{5.290}$$

Combining the results of Eqs. (5.289) and (5.290), we find the Riemann tensor in the tangent space in terms of the spin coefficients:

$$R_{abcd} = X_{abcd} + Y_{abcd} + Z_{abcd}\;, \tag{5.291}$$

where:

$$X_{abcd} \equiv \nabla_c\gamma_{abd} - \nabla_d\gamma_{abc}\;,\quad Y_{abcd} \equiv -\gamma_{abe}\big(\gamma_c{}^e{}_d - \gamma_d{}^e{}_c\big)\;,\quad Z_{abcd} \equiv \gamma_{aec}\,\gamma^e{}_{bd} \;-\; \gamma_{aed}\,\gamma^e{}_{bc}\;. \tag{5.292}$$

Because of the 2-form origin of the indices c and d, the quantity Z_{abcd} is manifestly odd under $c \leftrightarrow d$. However, unlike X_{abcd} and Y_{abcd}, it is not so obvious from Eq. (5.292) as written whether $Z_{abcd} = -Z_{bacd}$. Let us check:

$$Z_{bacd} = \gamma_{bec}\gamma^{e}{}_{ad} - \gamma_{bed}\gamma^{e}{}_{ac} = -\gamma_{ebc}\gamma^{e}{}_{ad} + \gamma_{ebd}\gamma^{e}{}_{ac}$$
$$= -\gamma^{e}{}_{bc}\gamma_{ead} + \gamma^{e}{}_{bd}\gamma_{eac} = +\gamma^{e}{}_{bc}\gamma_{aed} - \gamma^{e}{}_{bd}\gamma_{aec}$$
$$= -\gamma^{e}{}_{bd}\gamma_{aec} + \gamma^{e}{}_{bc}\gamma_{aed} = -(\gamma^{e}{}_{bd}\gamma_{aec} - \gamma^{e}{}_{bc}\gamma_{aed}) = -Z_{abcd}\,. \tag{5.293}$$

Since all we had to do was relabel some indices and use the known antisymmetry property $\gamma_{abc} = -\gamma_{bac}$, we are inclined to conclude that the formula for Z_{abcd} as written is fairly construed as being manifestly odd under $a \leftrightarrow b$. But the real test is whether a straightforward transcription of Z_{abcd} and Z_{bacd} into Newman-Penrose notation will result in the same formulas up to an overall sign. It turns out that it will, so we will in fact consider R_{abcd} as written to be manifestly odd under $a \leftrightarrow b$ and manifestly odd under $c \leftrightarrow d$.

The curvature in the form of Eq. (5.292) is the geometrical quantity from which the Newman-Penrose equations are derived: the left-hand side is a curvature quantity, and the right-hand side comprises derivatives and products of spin coefficients. The resulting formulas will not be manifestly invariant under the interchange symmetry $ab \leftrightarrow cd$, and this is the origin of many seemingly mysterious identities in the spin coefficient formalism.

5.B.1 Weyl scalars with nonzero weight

We will derive the expression for Ψ_0 as an example, and then we will simply state the result for Ψ_1. (Then $\Psi_3 \equiv \Psi_1'$ and $\Psi_4 \equiv \Psi_0'$ will be defined by priming.)

From Eq. (5.282), we are instructed to focus on the component C_{1313}. From the relation between C_{abcd} and R_{abcd} [recall Eq. (5.276)], and from the purely off-diagonal nature of the tangent space metric [recall Eq. (5.3)], we conclude that we only have to calculate R_{1313}. Recalling the three sets of terms defined in Eq. (5.292), we begin:

$$X_{1313} = \nabla_1\gamma_{133} - \nabla_3\gamma_{131} = -\nabla_1\gamma_{313} + \nabla_3\gamma_{311} = -D\sigma + \delta\kappa \,. \tag{5.294}$$

Since everything must collect itself into GHP-covariant combinations, we should expect terms like $\varepsilon\sigma$ and $\beta\kappa$ to show up. Indeed they will:

$$\begin{aligned} Y_{1313} &= -\gamma_{13e}(\gamma_1{}^e{}_3 - \gamma_3{}^e{}_1) \\ &= +\gamma_{131}(\gamma_{123} - \gamma_{321}) + \gamma_{132}(\gamma_{113} - \gamma_{311}) - \gamma_{133}(\gamma_{143} - \gamma_{341}) - \gamma_{134}(\gamma_{133} - \gamma_{331}) \\ &= -\gamma_{311}(-\gamma_{213} - \gamma^*_{421}) - \gamma_{312}(0 - \gamma_{311}) + \gamma_{313}(-\gamma_{413} - \gamma_{341}) + \gamma_{314}(-\gamma_{313} - 0) \\ &= -\kappa[-(\beta - \beta'^*) - \tau'^*] - \tau(-\kappa) + \sigma[-\rho^* - (\varepsilon - \varepsilon^*)] + \rho(-\sigma) \\ &= (\beta - \beta'^*)\kappa + \tau'^*\kappa + \tau\kappa - \rho^*\sigma - (\varepsilon - \varepsilon^*)\sigma - \rho\,\sigma \,. \end{aligned} \tag{5.295}$$

But there must be more terms, because $\kappa \sim (\frac{3}{2}, \frac{1}{2})$ and $\sigma \sim (\frac{3}{2}, -\frac{1}{2})$, and therefore the number of terms involving β'^* and ε^* has to differ from the number of terms involving β and ε. Our expectation is vindicated by the final set of terms:

$$
\begin{aligned}
Z_{1313} = \gamma_{1e1}\,\gamma^{e}{}_{33} - \gamma_{1e3}\,\gamma^{e}{}_{31} &= -\gamma_{111}\gamma_{233} - \gamma_{121}\gamma_{133} + \gamma_{131}\gamma_{433} + \gamma_{141}\gamma_{333} \\
&+ \gamma_{113}\gamma_{231} + \gamma_{123}\gamma_{131} - \gamma_{133}\gamma_{431} - \gamma_{143}\gamma_{331} \\
&= 0 - (-1)^2\gamma_{211}\gamma_{313} + (-1)^2\gamma_{311}\gamma_{343} + 0 \\
&+ 0 + (-1)^2\gamma_{213}\gamma_{311} - (-1)^2\gamma_{313}\gamma_{341} - 0 \\
&= -(\varepsilon + \varepsilon^*)\sigma + \kappa(\beta + \beta'^*) + (\beta - \beta'^*)\kappa - \sigma(\varepsilon - \varepsilon^*) \\
&= -2\varepsilon\sigma + 2\beta\kappa\ . \qquad (5.296)
\end{aligned}
$$

The total tally in $Y_{1313} + Z_{1313}$ provides $(-3\varepsilon + \varepsilon^*)\sigma$ and $(3\beta - \beta'^*)\kappa$. Meanwhile, recalling $\text{þ} \equiv D + 2h\varepsilon + 2\bar{h}\varepsilon^*$ and $\eth \equiv \delta + 2h\beta - 2\bar{h}\beta'^*$, we have

$$
\text{þ}\sigma = (D + 3\varepsilon - \varepsilon^*)\sigma\ , \quad \eth\kappa = (\delta + 3\beta - \beta'^*)\kappa\ , \qquad (5.297)
$$

so indeed the result obediently collects itself into a GHP-covariant form. Packaging the result into the sum $C_{1313} = R_{1313} = X_{1313} + Y_{1313} + Z_{1313}$, we find:

$$\Psi_0 \equiv C_{1313} = -\left[\text{þ} + (\rho + \rho^*)\right]\sigma + \left[\eth + (\tau + \tau'^*)\right]\kappa. \tag{5.298}$$

To compare this explicitly with Eq. (2.23) in the original GHP paper [139] or with Eq. (4.12.32)(b) in *Spinors and Space-time* [12], let us relate it to its "barred" version in the opposite metric signature according to the rules in Sec. 5.A:

$$\Psi_0 = +\bar{\text{þ}}\bar{\sigma} - (\bar{\rho} + \bar{\rho}^*)\bar{\sigma} - \bar{\eth}\bar{\kappa} + (\bar{\tau} + \bar{\tau}'^*)\bar{\kappa} \equiv \bar{\Psi}_0 \,. \tag{5.299}$$

The final equality is written as a *d*efinition, because we define $\bar{\Psi}_0$ by taking Penrose and Rindler's Eq. (4.12.32)(b) *a*s written and moving all the products of spin coefficients to the same side as the derivatives of spin coefficients. So our Weyl scalars will be *n*umerically identical to those in *Spinors and Space-time*, despite our use of the opposite metric signature.

Having toiled through this dreary but essential derivation of Ψ_0, we will content ourselves with just the statement of Ψ_1:

$$\Psi_1 = \tfrac{1}{2}\left\{\zeta\left[\text{þ}\tau - \text{þ}'\kappa - \eth\rho + \eth'\sigma\right] - (\tau - \tau'^*)\rho - (\tau^* - \tau')\sigma + (\rho - \rho^*)\tau - (\rho' - \rho'^*)\kappa\right\} \,. \tag{5.300}$$

To use our conventions, plug in $\zeta = -1$. To use Penrose and Rindler's conventions, plug in $\zeta = +1$.[160]

As a final remark, let us recall that the Weyl scalars as defined have weights $(h, 0)$ with $h = +2, +1, 0, -1, -2$, for $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$, respectively. (We will postpone discussion of Ψ_2 until after studying the Ricci scalars of nonzero weight.) The Weyl scalars with weights of the form $(0, \bar{h})$ are defined by complex conjugation and are not traditionally given independent names.

5.B.2 Ricci scalars with nonzero weight

Since we encountered a litany of irritating signs in the derivation of Ψ_0, we feel obliged to repeat the process for at least one Ricci scalar.

The most important Ricci scalar for our shockwave solution is $\Phi_{22} \equiv \Phi'_{00}$, which we define from Φ_{00} by priming. For this reason, we will derive the expression for Φ_{00} in GHP notation explicitly. We begin with

$$R_{11} \equiv R^a{}_{1a1} = R^3{}_{131} + R^4{}_{141} = R_{4131} + R_{3141} \ . \tag{5.301}$$

[160] Our use of ζ in Eq. (5.300) differs from our use of ζ in Sec. 5.A. There are only so many symbols.

Since complex conjugation exchanges the labels $3 \leftrightarrow 4$ and leaves the labels 1 and 2 fixed, we have $R_{4131} = R^*_{3141}$ and therefore

$$\Phi_{00} \equiv \tfrac{1}{2} R_{11} = \mathrm{Re}(R_{3141}) \; . \tag{5.302}$$

Returning once more to Eq. (5.292), we compute the three sets of terms that compose R_{3141}. The first of these is:

$$X_{3141} = \nabla_4 \gamma_{311} - \nabla_1 \gamma_{314} = \delta' \kappa - D\rho \; . \tag{5.303}$$

So far so good. The next is:

$$\begin{aligned} Y_{3141} &= -\gamma_{31e}(\gamma_4{}^e{}_1 - \gamma_1{}^e{}_4) \\ &= +\gamma_{311}(\gamma_{421} - \gamma_{124}) + \gamma_{312}(\gamma_{411} - \gamma_{114}) - \gamma_{313}(\gamma_{441} - \gamma_{144}) - \gamma_{314}(\gamma_{431} - \gamma_{134}) \\ &= +\gamma_{311}(\gamma_{421} + \gamma_{214}) + \gamma_{312}(\gamma^*_{311} - 0) - \gamma_{313}(0 + \gamma_{414}) - \gamma_{314}(-\gamma_{341} + \gamma_{314}) \\ &= \kappa[\tau' + (\beta^* - \beta')] + \tau\kappa^* - \sigma\sigma^* - \rho[-(\varepsilon - \varepsilon^*) + \rho] \\ &= \tau'\kappa + \tau\kappa^* + (\beta^* - \beta')\kappa - |\sigma|^2 - \rho^2 + (\varepsilon - \varepsilon^*)\rho \; . \end{aligned} \tag{5.304}$$

Once again, this looks promising, but we know there will be more terms because, for example, $\rho \sim (\frac{1}{2}, \frac{1}{2})$ not $(-\frac{1}{2}, +\frac{1}{2})$. Proceeding to the last set, we find:

$$
\begin{aligned}
Z_{3141} &= \gamma_{3e4}\gamma^{e}{}_{11} - \gamma_{3e1}\gamma^{e}{}_{14} \\
&= -\gamma_{314}\gamma_{211} - \gamma_{324}\gamma_{111} + \gamma_{334}\gamma_{411} + \gamma_{344}\gamma_{311} \\
&+ \gamma_{311}\gamma_{214} + \gamma_{321}\gamma_{114} - \gamma_{331}\gamma_{414} - \gamma_{341}\gamma_{314} \\
&= -\gamma_{314}\gamma_{211} - 0 + 0 - \gamma_{434}\gamma_{311} \\
&+ \gamma_{311}\gamma_{214} + 0 - 0 - \gamma_{341}\gamma_{314} \\
&= -\rho(\varepsilon + \varepsilon^*) - (\beta^* + \beta')\kappa + \kappa(\beta^* - \beta') - (\varepsilon - \varepsilon^*)\rho \\
&= -2\varepsilon\rho - 2\beta'\kappa \; . \qquad (5.305)
\end{aligned}
$$

The sum $Y_{3141} + Z_{3141}$ therefore bequeaths unto us the combination $-(\varepsilon + \varepsilon^*)\rho + (-3\beta' + \beta^*)\kappa$, which is exactly the inheritance we deserve as scholars of the GHP formalism. Therefore:

$$R_{3141} = -\text{þ}\rho - \rho^2 - |\sigma|^2 + \eth'\kappa + \tau'\kappa + \tau\,\kappa^* \; . \qquad (5.306)$$

While the straightforward definition of the tangent space Ricci tensor results in $R_{11} = 2\,\mathrm{Re}(R_{3141})$, it turns out that R_{3141} is actually real all by itself.[161] Therefore, we have the following formula for the weight $(1,1)$ Ricci scalar:

$$\Phi_{00} = -\text{þ}\rho - \rho^2 - |\sigma|^2 + \eth'\kappa + \tau'\kappa + \tau\,\kappa^* \,. \tag{5.307}$$

To compare this with the analogous formula as presented by Penrose and Rindler, we again apply our rules for changing signature:

$$\Phi_{00} = +\bar{\text{þ}}\bar{\rho} - \bar{\rho}^2 - |\bar{\sigma}|^2 - \bar{\eth}'\bar{\kappa} + \bar{\tau}'\bar{\kappa} + \bar{\tau}\,\bar{\kappa}^* \equiv \bar{\Phi}_{00} \,. \tag{5.308}$$

To understand this last definition [which becomes identical to Eq. (4.12.32)(a) in *Spinors and Space-time* upon moving the products of spin coefficients to the same side as the derivatives of spin coefficients], we remind the reader of our discussion immediately after Eq. (5.278). Thus, our Ricci scalars are also numerically identical to those of Penrose and Rindler.

The Sachs conjugate of Φ_{00} is the weight $(1,-1)$ Ricci scalar:

$$\Phi_{02} = -\eth\tau + \text{þ}'\sigma + \rho'\sigma + \sigma'^*\rho - \tau^2 - \kappa\kappa'^* \,. \tag{5.309}$$

[161]This is an example of the kind of mysterious relation we alluded to earlier, which results from a lack of manifest invariance under exchanging the first and second pairs of indices in the symbol R_{abcd}.

As we mentioned in the main text, the condition $\eth\tau = -\tau^2$ when $\Phi_{02} = 0$ is the timelike analog of the condition $þ\rho = -\rho^2$ when $\Phi_{00} = 0$. (In both cases we assumed $\kappa = \kappa' = \sigma = \sigma' = 0$.)

Next, we have the weight $(1, 0)$ Ricci scalar:

$$\Phi_{01} = \tfrac{1}{2}\Big\{-\big(þ\tau - þ'\kappa - \eth\rho - \eth'\sigma\big) - (\tau - \tau'^*)\rho - (\tau^* - \tau')\sigma - (\rho - \rho^*)\tau + (\rho' - \rho'^*)\kappa\Big\}\ . \tag{5.310}$$

We have nothing enlightening to say about this quantity. One observation is that when $\kappa = \sigma = 0$ it becomes $\Phi_{01} = \frac{1}{2}\left[-(þ\tau - \eth\rho) - (\tau - \tau'^*)\rho - (\rho - \rho^*)\tau\right]$, which exhibits front and center a dependence on both the twist and its Sachs conjugate.

The remaining Ricci scalars of nonzero weight can be defined by priming and conjugating the definitions already listed: $\Phi_{22} = \Phi'_{00}$, $\Phi_{21} = \Phi'_{01}$, $\Phi_{10} = \Phi^*_{01}$, $\Phi_{12} = \Phi^*_{21}$, and $\Phi_{20} = \Phi^*_{02}$.

5.B.3 Spacelike and timelike curvatures

In Eq. (5.109) we defined the spacelike curvature $\mathcal{K}$ and the timelike curvature $\mathcal{K}_s$. They appear on the right-hand side of two of the commutators of GHP derivatives:

$$[ð, ð'] = -(\rho - \rho^*)þ' + (\rho' - \rho'^*)þ - 2h\,\mathcal{K} + 2\bar{h}\,\mathcal{K}^* \,, \tag{5.311}$$

$$[þ, þ'] = -(\tau - \tau'^*)ð' - (\tau^* - \tau')ð + 2h\,\mathcal{K}_s + 2\bar{h}\,\mathcal{K}_s^* \,. \tag{5.312}$$

As a practical matter, we can use these equations to solve for these complex curvatures by acting on a test function ξ_h with weights $(h, 0)$. When acting on such a test function, we have:

$$\begin{aligned} \mathcal{K}\,\xi_h &= -\frac{1}{2h}\left([ð, ð'] + (\rho - \rho^*)þ' - (\rho' - \rho'^*)þ\right)\xi_h \,, \\ \mathcal{K}_s\,\xi_h &= \frac{1}{2h}\left([þ, þ'] + (\tau - \tau'^*)ð' + (\tau^* - \tau')ð\right)\xi_h \,. \end{aligned} \tag{5.313}$$

This is how we actually calculate these curvatures in Mathematica, and how we calculate their shifted versions in Eqs. (5.175) and (5.176). In principle we could go further and simply set $h = 1$, but leaving h arbitrary provides a good check on the algebra since it must cancel out for the formalism to be self-consistent.

For the convenience of the reader, we also write down the following "mixed" commutator of GHP derivatives:

$$[þ, ð] = -\left(\rho^* ð + \sigma ð' - \tau'^* þ - \kappa þ'\right) - 2h\,(\rho'\kappa - \tau'\sigma + \Psi_1) - 2\bar{h}\,(\sigma'^*\kappa^* - \rho^*\tau'^* + \Phi_{01}) \,. \tag{5.314}$$

The commutator $[þ', ð']$ can be defined from Eq. (5.314) by priming (recall that priming flips the signs of the weights):

$$[þ', ð'] = -\left(\rho'^* ð' + \sigma' ð - \tau^* þ' - \kappa' þ\right) + 2h\left(\rho\kappa' - \tau\sigma' + \Psi_3\right) + 2\bar{h}\left(\sigma^*\kappa'^* - \rho'^*\tau^* + \Phi_{21}\right) . \tag{5.315}$$

Since þ and $þ'$ are real while $ð' = ð^*$, the commutators $[þ, ð']$ and $[þ', ð]$ can be obtained from Eqs. (5.314) and (5.315) by complex conjugation (recall for the umpteenth time that complex conjugation exchanges $h \leftrightarrow \bar{h}$).

5.B.4 Curvature scalars with weight zero

Once we calculate the spacelike and timelike curvatures, we can express the curvature scalars of weight-$(0, 0)$ in terms of them along with derivatives and products of spin coefficients.

The Weyl scalar of weight $(0, 0)$ is:[162]

$$\Psi_2 = \tfrac{1}{3}\left[\rho\rho'^* - |\tau|^2 + þ'\rho - ð'\tau - (\mathcal{K} + \mathcal{K}_s) - \kappa\kappa' + \tau\tau' + \sigma\sigma' - \rho\rho'\right] . \tag{5.316}$$

[162]The quantity $\rho\rho'^* - |\tau|^2 - þ'\rho + ð'\tau$ is actually self-prime, although not manifestly so. As per our previous comments, this comes from $C_{abcd} = C_{cdab}$.

The Ricci scalar of weight $(0,0)$ is:[163]

$$\Phi_{11} = \tfrac{1}{2}\left(\mathcal{K} - \mathcal{K}_s - \kappa\kappa' + \tau\tau' - \sigma\sigma' + \rho\rho'\right) \,. \tag{5.317}$$

Finally, the Einstein-Hilbert curvature is:

$$\Pi = -\tfrac{1}{6}\left[2\left(\rho\rho'^{*} - |\tau|^2 + \text{þ}'\rho - \eth'\tau\right) + \mathcal{K} + \mathcal{K}_s - \kappa\kappa' - \tau\tau' + \sigma\sigma' + \rho\rho'\right] \,. \tag{5.318}$$

This completes our discussion of the curvature in the method of spin coefficients.

To form a mathematically self-consistent gravitational field, the curvature scalars must satisfy the Bianchi identities. Unlike the situation for rest of the formalism, we feel that these identities are explained with sufficient clarity in the standard texts, and we will not discuss them here.

5.C Reduction of $SO(3,1)$ to the GHP group

In this section, we show how the compacted null Cartan equations in Eq. (5.53), the compacted curvature 2-form in Eq. (5.93), and the obscure-looking operator in Eq. (5.100) are embedded within the $SO(3,1)$-covariant formalism.

[163] Reality of the left-hand side requires that $\mathrm{Im}(\mathcal{K} - \mathcal{K}_s) = \mathrm{Im}(\kappa\kappa' - \tau\tau' + \sigma\sigma' - \rho\rho')$.

5.C.1 Reduction of the spin connection

First define the following 4-by-4 antisymmetric matrix $\mathscr{W}_{ab}$:

$$\mathscr{W}_{ab} \equiv \begin{pmatrix} 0 & W \\ -W^T & 0 \end{pmatrix} , \quad W \equiv \begin{pmatrix} \varpi & \varpi^* \\ \varpi'^* & \varpi' \end{pmatrix} . \tag{5.319}$$

Acting on the 1-form frame fields[164] $e^a = \begin{pmatrix} -l' \\ -l \\ m' \\ m \end{pmatrix}$, we find:

$$\mathscr{W}_{ab} \wedge e^b = \begin{pmatrix} W \wedge \begin{pmatrix} m' \\ m \end{pmatrix} \\ W^T \wedge \begin{pmatrix} l' \\ l \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \varpi \wedge m' + \varpi^* \wedge m \\ \varpi'^* \wedge m' + \varpi' \wedge m \\ \varpi \wedge l' + \varpi'^* \wedge l \\ \varpi^* \wedge l' + \varpi' \wedge l \end{pmatrix} . \tag{5.320}$$

Next define an improved exterior derivative

$$\text{đ} \equiv d + 2h\,\mathrm{b} + 2\bar{h}\,\mathrm{b}^* \quad \text{with} \quad \mathrm{b} \equiv -\varepsilon\, l' + \varepsilon'\, l + \beta\, m' - \beta'\, m\,. \tag{5.321}$$

[164]Now that we regress to matrix multiplication, we express the collection of frame fields as a column vector.

Behold by explicit computation that the torsion 2-form

$$\Theta^a \equiv \mathcal{D}^a{}_b\, e^b \tag{5.322}$$

can be equivalently expressed as

$$\Theta_a = \text{đ} e_a - \mathscr{W}_{ab} \wedge e^b \ . \tag{5.323}$$

The torsion-free condition therefore implies

$$\text{đ} l = \varpi \wedge m' + c.c. \ , \quad \text{đ} m = \varpi \wedge l' + c.c.' \tag{5.324}$$

and their primed verisons, which are the null Cartan equations as they appear in Eq. (5.53).

5.C.2 Reduction of the curvature 2-form

Now let us study the curvature forms. They will emerge most readily from the first gravitational Bianchi identity[165]

$$\mathcal{D}^a{}_b\, \Theta^b = \Omega^a{}_b \wedge e^b \ . \tag{5.325}$$

[165]The second gravitational Bianchi identity is $(\mathcal{D}\Omega)^a{}_b \equiv d\Omega^a{}_b + \omega^a{}_c \wedge \Omega^c{}_b - \Omega^a{}_c \wedge \omega^c{}_b = 0$. Our terminology follows the exposition on p. 285 of Nakahara [175].

From Eq. (5.323), we get:

$$\mathcal{D}^a{}_b \Theta^b = (\delta^a{}_c đđ - đ\mathscr{W}^a_{\ c} + \mathscr{W}^a_{\ b} \wedge \mathscr{W}^b_{\ c}) \wedge e^c \ . \tag{5.326}$$

By invoking Eq. (5.325) we deduce that the operator on the right-hand side of Eq. (5.326) is the curvature 2-form.

From the block decomposition of $\mathscr{W}_{ab}$ in Eq. (5.319), we first observe that the middle term on the right-hand side of Eq. (5.326) is purely off-diagonal:

$$đ\mathscr{W}_{ab} = \begin{pmatrix} 0 & đW \\ -đW^T & 0 \end{pmatrix} \ . \tag{5.327}$$

Up to priming and complex conjugation, the entire content of this off-diagonal block is encapsulated by the quantity

$$đ\varpi \ . \tag{5.328}$$

Next, for notational convenience, we express the Newman-Penrose metric η_{ab} in terms of a 2-by-2 Pauli matrix:

$$\eta_{ab} = \begin{pmatrix} -\Sigma & 0 \\ 0 & +\Sigma \end{pmatrix} \ , \quad \Sigma \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{5.329}$$

Then, given $\mathscr{W}_{ab}$ from Eq. (5.319), we write $\mathscr{W}^{ab}$ as:

$$\mathscr{W}^{ab} = \eta^{ac}\mathscr{W}_{cd}\eta^{db} = \begin{pmatrix} -\Sigma & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} 0 & W \\ -W^T & 0 \end{pmatrix} \begin{pmatrix} -\Sigma & 0 \\ 0 & \Sigma \end{pmatrix} = \begin{pmatrix} 0 & -\Sigma W \Sigma \\ \Sigma W^T \Sigma & 0 \end{pmatrix} . \tag{5.330}$$

Therefore:

$$\mathscr{W}_{ab} \wedge \mathscr{W}^{bc} = \begin{pmatrix} W\Sigma W^T\Sigma & 0 \\ 0 & W^T\Sigma W\Sigma \end{pmatrix} . \tag{5.331}$$

Unlike the matrix in Eq. (5.327), this matrix is diagonal. Evidently the curvature 2-form $\Omega^a{}_b$ splits into an off-diagonal part whose entries are defined from $đ\varpi$ and a diagonal operator

$$(\delta_a{}^c đđ + \mathscr{W}_{ab} \wedge \mathscr{W}^{bc}) \wedge e_c = \begin{pmatrix} (Iđđ + W \wedge \Sigma W^T\Sigma) \wedge \begin{pmatrix} l \\ l' \end{pmatrix} \\ (Iđđ + W^T \wedge \Sigma W \Sigma) \wedge \begin{pmatrix} m \\ m' \end{pmatrix} \end{pmatrix} . \tag{5.332}$$

Using the explicit form of the matrix W from Eq. (5.319), we have

$$\Sigma W^T \Sigma = \begin{pmatrix} \varpi' & \varpi^* \\ \varpi'^* & \varpi \end{pmatrix} , \quad \Sigma W \Sigma = \begin{pmatrix} \varpi' & \varpi'^* \\ \varpi^* & \varpi \end{pmatrix} . \tag{5.333}$$

Therefore, we first find:[166]

$$W \wedge \Sigma W^T \Sigma = \begin{pmatrix} \varpi \wedge \varpi' + (\varpi \wedge \varpi')^* & 0 \\ 0 & -(\varpi \wedge \varpi' + (\varpi \wedge \varpi')^*) \end{pmatrix} . \tag{5.334}$$

This implies:

$$(I đđ + W \wedge \Sigma W^T \Sigma) \wedge \begin{pmatrix} l \\ l' \end{pmatrix} = \begin{pmatrix} [đđ + \varpi \wedge \varpi' + (\varpi \wedge \varpi')^*] l \\ [đđ - \varpi \wedge \varpi' - (\varpi \wedge \varpi')^*] l' \end{pmatrix} . \tag{5.335}$$

Next, we have:[167]

$$W^T \wedge \Sigma W \Sigma = \begin{pmatrix} \varpi \wedge \varpi' - (\varpi \wedge \varpi')^* & 0 \\ 0 & -\varpi \wedge \varpi' + (\varpi \wedge \varpi')^* \end{pmatrix} . \tag{5.336}$$

Therefore:

$$(I đđ + W^T \wedge \Sigma W \Sigma) \wedge \begin{pmatrix} m \\ m' \end{pmatrix} = \begin{pmatrix} [đđ + \varpi \wedge \varpi' - (\varpi \wedge \varpi')^*] m \\ [đđ - \varpi \wedge \varpi' + (\varpi \wedge \varpi')^*] m' \end{pmatrix} . \tag{5.337}$$

[166] The off-diagonal parts are zero because $\varpi \wedge \varpi^* + \varpi^* \wedge \varpi = 0$ by antisymmetry of the wedge product.

[167] Again, the off-diagonal terms are zero because $\varpi \wedge \varpi'^* + \varpi'^* \wedge \varpi = 0$ by antisymmetry of the wedge product.

The right-hand sides of Eqs. (5.335) and (5.337) can be summarized as:

$$[\text{đđ} + 2h\,\varpi \wedge \varpi' + 2\bar{h}\,(\varpi \wedge \varpi')^*]\,e_a\,. \tag{5.338}$$

This is the germ from which Eq. (5.100) sprouts. Why its natural soil is a Bianchi identity instead of a structure equation remains unclear to us.

5.C.3 Extension to tensor calculus

To go from exterior calculus in the tangent space to tensor calculus on the base space (and to match the traditional notation), we need to extend the formalism so that it can act not only on scalar fields but also on vector and tensor fields.

This entails passing from the ordinary exterior derivative

$$d \equiv dx^\mu \partial_\mu = e^a e_a^\mu \partial_\mu = -l'\,l^\mu \partial_\mu - l\,l'^\mu \partial_\mu + m'\,m^\mu \partial_\mu + m\,m'^\mu \partial_\mu \tag{5.339}$$

to its space-time-covariant generalization:

$$\nabla \equiv dx^\mu \nabla_\mu = e^a\,e_a^\mu \nabla_\mu = -l'\,D - l\,D + m'\,\delta + m\,\delta'\,. \tag{5.340}$$

To reproduce the GHP formalism as presented in *Spinors and Space-time* [12], we generalize the operator

$$\text{đ} = d + 2h\,\mathrm{b} + 2\bar{h}\,\mathrm{b}^* \tag{5.341}$$

to its space-time-covariant version:

$$Ð \equiv \nabla + 2h\,\mathrm{b} + 2\bar{h}\,\mathrm{b}^* = -l'\,\text{þ} - l\,\text{þ}' + m'\,\eth + m\,\eth' \,. \tag{5.342}$$

Then it is appropriate to define the GHP curvature 2-form as

$$\Omega \equiv Ð\varpi \,, \tag{5.343}$$

which generalizes the expression $đ\varpi$ outside the exterior calculus. The diagonal operator in Eq. (5.338) should be generalized to

$$\amalg \equiv ÐÐ + 2h\,\varpi \wedge \varpi' + 2\bar{h}\,(\varpi \wedge \varpi')^* \,. \tag{5.344}$$

This is the quantity that encodes the weight-$(0,0)$ combinations of curvature scalars in Eq. (5.101). The operator ÐÐ itself encodes $\mathcal{K}$ and $\mathcal{K}_s$, as we explained in Eq. (5.110).

Let us clarify the distinction between Eqs. (5.344) and (5.338).

Inserting đ from Eq. (5.341) into Eq. (5.338) would beget critters like $D'\varepsilon + D\varepsilon'$ and $\delta'\beta + \delta\beta'$.[168] They compose the ugliest of the NP equations, those that do not admit a GHP-covariant formulation.

[168]On scalar fields we can trade ∂_μ for ∇_μ and thereby obtain NP derivatives.

That is fine but not what we want: we are after the GHP commutator equations, not the corresponding NP equations. Those commutator equations are what Eq. (5.344) provides.

5.D Laplacian in compacted form

The subject of this paper is the gravitational field of a massless spinless particle, and the equation of motion for a scalar field involves the Laplacian. If we are to champion the spin coefficient formalism in earnest, then we ought to express the operator

$$\nabla^2 \equiv \nabla_\mu \nabla^\mu = \frac{1}{|\det g|^{1/2}} \partial_\mu \left(|\det g|^{1/2} g^{\mu\nu} \partial_\nu \right) \tag{5.345}$$

in terms of GHP derivatives. Because we have only encountered Eqs. (5.193) and (5.194) in the literature as a cryptic passing remark and in a form that is not manifestly self-prime,[169] we will derive them explicitly.

Let H be a real scalar field. Since the Laplacian involves second derivatives and since the metric in the spin coefficient formalism is off-diagonal, we begin by

[169] See Eq. (4.12.45) in *Spinors and Space-time* [12].

computing the following two compositions:

$$D'DH = l'^{\mu}\nabla_{\mu}(l^{\nu}\nabla_{\nu}H) = l'^{\mu}l^{\nu}\nabla_{\mu}\nabla_{\nu}H + l'^{\mu}(\nabla_{\mu}l^{\nu})\nabla_{\nu}H \ ,$$

$$\delta'\delta H = m'^{\mu}\nabla_{\mu}(m^{\nu}\nabla_{\nu}H) = m'^{\mu}m^{\nu}\nabla_{\mu}\nabla_{\nu}H + m'^{\mu}(\nabla_{\mu}m^{\nu})\nabla_{\nu}H \ . \tag{5.346}$$

From Eq. (5.47), we have $D'l^{\nu} = (\varepsilon' + \varepsilon'^{*})l^{\nu} + \tau^{*}m^{\nu} + \tau\, m'^{\nu}$. Performing the Sachs operation on this gives $\delta'm^{\nu} = (\beta' + \beta^{*})m^{\nu} + \rho'^{*}l^{\nu} + \rho\, l'^{\nu}$. Therefore, adding the primed versions to the brew, we find:

$$l'^{\mu}(\nabla_{\mu}l^{\nu})\nabla_{\nu}H \ + \ ' \ = \ (\varepsilon{+}\varepsilon^{*})D'H + (\tau{+}\tau'^{*})\delta'H \ + \ ' \ ,$$

$$m'^{\mu}(\nabla_{\mu}m^{\nu})\nabla_{\nu}H \ + \ ' \ = \ (\beta{+}\beta'^{*})\delta'H + (\rho{+}\rho^{*})D'H \ + \ ' \ . \tag{5.347}$$

With $g^{\mu\nu} = -l'^{\mu}l^{\nu} + m'^{\mu}m^{\nu} \ + \ '$, we have:

$$-D'DH + \delta'\delta H \ + \ ' = g^{\mu\nu}\,\nabla_{\mu}\nabla_{\nu}H + \big[-(\varepsilon{+}\varepsilon^{*})\,D'H + (\rho{+}\rho^{*})\,D'H$$

$$+(\beta{+}\beta'^{*})\,\delta'H - (\tau{+}\tau'^{*})\,\delta'H \ + \ '\,\big] \ . \tag{5.348}$$

If we cared only about NP derivatives we would be done, but this would leave the expression too crabbed for sensible advocacy. As Penrose and Rindler instruct, if we think of H as possessing the weight $(0,0)$, then we have:

$$\begin{aligned}
&þH = DH \sim (+\tfrac{1}{2},+\tfrac{1}{2}) \implies þ'þH = [D' - (\varepsilon'+\varepsilon'^*)]DH \ , \\
&þ'H = D'H \sim (-\tfrac{1}{2},-\tfrac{1}{2}) \implies þþ'H = [D - (\varepsilon+\varepsilon^*)]D'H \ , \\
&ðH = \delta H \sim (+\tfrac{1}{2},-\tfrac{1}{2}) \implies ð'ðH = [\delta' - (\beta'+\beta^*)]\delta H \ , \\
&ð'H = \delta' H \sim (-\tfrac{1}{2},+\tfrac{1}{2}) \implies ðð'H = [\delta - (\beta+\beta'^*)]\delta' H \ . \qquad (5.349)
\end{aligned}$$

So if we define a generalized Laplacian

$$\Box \equiv -\left[þ + 2\,\mathrm{Re}(\rho)\right] þ' + \left[ð + (\tau + \tau'^*)\right] ð' \ + \ ' \qquad (5.350)$$

then, when acting on GHP-invariant functions $H \sim (0,0)$, we conclude

$$\Box H = \nabla^2 H \qquad \text{(weight zero only)} \ . \qquad (5.351)$$

When acting on a more general weighted function $f_{h,\bar{h}} \sim (h,\bar{h})$, the generalized Laplacian in Eq. (5.350) contains extra terms beyond the ordinary Laplacian.

The generalized Laplacian is written in terms of GHP-covariant operators and is therefore itself GHP-covariant. This makes it the operator we should use. Notice the presence of the timelike expansion, which appears as the Sachs counterpart to the term involving $\mathrm{Re}(\rho)$.

5.E Massless particle in classical field theory

In this section, we explain our point-particle limit within classical field theory. We begin with flat space-time.[170]

The equation of motion for a real scalar field H is

$$(\partial_t^2 - \sum_{i=1}^{3} \partial_i^2)H = 0\ . \tag{5.352}$$

Its general solution is typically expressed as an arbitrary linear combination of positive frequency solutions:

$$H(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}} \left(a(\vec{p})\, e^{-i\omega_{\vec{p}}t + i\vec{p}\cdot\vec{x}} + c.c. \right)\ , \quad \omega_{\vec{p}} = |\vec{p}|\ . \tag{5.353}$$

Because this is *c*lassical field theory, the quantity denoted by $\vec{p}$ describes a wave vector, not a momentum. We should refer to $\vec{p}$-space as "wave vector space" or "Fourier space," but we will persist with the letter p and the patois of quantum mechanics.

[170]This section is a bit too Jacksonian (the electrodynamics book, not the 7th President) for our palette, but once in a blue moon we can swallow our pride and dabble in vector notation. We thank Justin Wilson for detailed notes on this derivation.

Our goal is to choose a specific functional form for $a(\vec{p})$ such that the resulting trace-reversed energy-momentum tensor will have the form

$$\mathcal{T}_{\mu\nu} = P\,\delta^u_\mu\,\delta^u_\nu\,\delta(u)\,\delta^2(\vec{x}_\perp)\ , \tag{5.354}$$

where $E \equiv \int du\, d^2x_\perp T_{uu} = P$ is the energy of the field configuration, and

$$u \equiv t - \hat{n}\cdot\vec{x}\ ,\ \ \vec{x}_\perp \equiv \vec{x} - (\hat{n}\cdot\vec{x})\,\hat{n} \tag{5.355}$$

for some unit vector $\hat{n}$.[171]

5.E.1 Momentum space distribution

We will try a momentum-space distribution peaked around a fixed momentum $\vec{P} = P\hat{n}$ (with $P \equiv |\vec{P}| > 0$):

$$a(\vec{p}) = \omega_{\vec{p}}\,N_P^{1/2}\,e^{-\frac{1}{2\sigma_P^2}|\vec{p}-\vec{P}|^2}\ . \tag{5.356}$$

The factor $\omega_{\vec{p}}$ was chosen to cancel the standard normalization in Eq. (5.353). The factor $N_P^{1/2}$ is a numerical factor [to be fixed eventually by matching to Eq. (5.354)] that depends only on P but not on $\vec{p}$. The quantity σ_P is the width in momentum

[171] We could also define $v \equiv t + \hat{n}\cdot\vec{x}$, but we will not need it.

space: we are looking for a distribution that is $l$ocalized in space and is therefore $d$elocalized in momentum space. In practice, this will entail the assumption

$$\frac{\sigma_P}{P} \ll 1 \, . \tag{5.357}$$

Plug Eq. (5.356) into Eq. (5.353) and change integration variable to $\vec{k} \equiv \vec{p} - \vec{P}$:

$$H(t, \vec{x}) = \tfrac{1}{2} N_P^{1/2} \int \frac{d^3k}{(2\pi)^3} \, e^{-\frac{1}{2\sigma_P^2}|\vec{k}|^2 - i|\vec{k}+\vec{P}|t + i(\vec{k}+\vec{P})\cdot\vec{x}} \, . \tag{5.358}$$

Next, use Eq. (5.357) as follows:

$$|\vec{k} + \vec{P}| = \left(|\vec{k}|^2 + 2P\,\hat{n}\cdot\vec{k} + P^2\right)^{1/2} \approx P\left(1 + \frac{\hat{n}\cdot\vec{k}}{P}\right) \, . \tag{5.359}$$

The expansion is controlled because the factor $e^{-\frac{1}{2\sigma_P^2}|\vec{k}|^2}$ effectively restricts $|\vec{k}|$ to the range $[0, \sigma_P]$. Therefore,

$$H(t, \vec{x}) \approx \tfrac{1}{2} N_P^{1/2} \, e^{-iP(t - \hat{n}\cdot x)} \int \frac{d^3k}{(2\pi)^3} \, e^{-\frac{1}{2\sigma_P^2} F(\vec{k})} + c.c. \, , \tag{5.360}$$

where

$$F(\vec{k}) = |\vec{k}|^2 - 2i\sigma_P^2(\vec{x} - \hat{n}t)\cdot\vec{k} = |\vec{k} - i\sigma_P^2(\vec{x} - \hat{n}t)|^2 + \sigma_P^4|\vec{x} - \hat{n}t|^2 \, . \tag{5.361}$$

With the usual contour integration prescription, we can evaluate the Gaussian integrals:

$$H(t, \vec{x}) = \tfrac{1}{2} N_P^{1/2}\, e^{-iP(t-\hat{n}\cdot\vec{x})} \left(\frac{\sigma_P^2}{2\pi}\right)^{3/2} e^{-\frac{1}{2}\sigma_P^2|\vec{x}-\hat{n}t|^2} + c.c. \,. \tag{5.362}$$

With $u = t - \hat{n}\cdot\vec{x}$, we have $u^2 = t^2 + (\hat{n}\cdot\vec{x})^2 - 2\,t\,\hat{n}\cdot\vec{x}$, and therefore

$$|\vec{x} - \hat{n}t|^2 = |\vec{x}|^2 + t^2 - 2\,t\,\hat{n}\cdot\vec{x} = |\vec{x}|^2 - (\hat{n}\cdot\vec{x})^2 + u^2 = |\vec{x}_\perp|^2 + u^2 \,. \tag{5.363}$$

So the field configuration that results from the momentum space distribution in Eq. (5.356) is $H(t, \vec{x}) \approx H(u, \vec{x}_\perp)$, where

$$H(u, \vec{x}_\perp) = N_P^{1/2} \left(\frac{\sigma_P^2}{2\pi}\right)^{3/2} e^{-\frac{1}{2}\sigma_P^2 u^2}\, e^{-\frac{1}{2}\sigma_P^2|\vec{x}_\perp|^2} \cos(Pu) \,. \tag{5.364}$$

5.E.2 Delta functions and T_{uu}

Our version of the delta function will be defined by a properly normalized Gaussian whose width goes to zero:

$$\delta(u) = \lim_{\epsilon\to 0} \delta_\epsilon(u) \,, \;\; \delta_\epsilon(u) \equiv \frac{1}{(2\pi\epsilon^2)^{1/2}}\, e^{-\frac{1}{2\epsilon^2}u^2} \,. \tag{5.365}$$

Remember that our goal is to obtain $\mathcal{T}_{uu} \sim \delta(u)$, not $H \sim \delta(u)$. This requirement will fix the relation between ϵ and σ_P and specify how to normalize the field configuration in Eq. (5.364). Since $\mathcal{T}_{\mu\nu} = \partial_\mu H \partial_\nu H$, we first compute the partial derivative with respect to u:

$$\partial_u H = N_P^{1/2} \left(\frac{\sigma_P^2}{2\pi} \right)^{3/2} e^{-\frac{1}{2}\sigma_P^2 u^2} e^{-\frac{1}{2}\sigma_P^2 |\vec{x}_\perp|^2} \left[-\sigma_P^2 u \cos(Pu) + P \sin(Pu) \right] . \quad (5.366)$$

We will again use the assumption in Eq. (5.357), this time in the form

$$\left(\frac{\sigma_P}{P} \right) \left(\frac{u}{\sigma_X} \right) \ll 1 \ , \quad \sigma_X \equiv \sigma_P^{-1} \ . \quad (5.367)$$

As a consequence of being delocalized in momentum ($P \gg \sigma_P$), the field configuration is localized about its average position ($u \ll \sigma_X \equiv \sigma_X$). So we can drop the $\cos(Pu)$ term in Eq. (5.366), leaving us with

$$\partial_u H = N_P^{1/2} \left(\frac{\sigma_P^2}{2\pi} \right)^{3/2} e^{-\frac{1}{2}\sigma_P^2 u^2} e^{-\frac{1}{2}\sigma_P^2 |\vec{x}_\perp|^2} P \sin(Pu) \ . \quad (5.368)$$

Writing $\sin^2(Pu) = \frac{1}{2}[1 - \cos(2Pu)]$, we have

$$(\partial_u H)^2 = \tfrac{1}{2} P^2 N_P \left(\frac{\sigma_P^2}{2\pi} \right)^3 e^{-\sigma_P^2 u^2} e^{-\sigma_P^2 |\vec{x}_\perp|^2} [1 - \cos(2Pu)] \ . \quad (5.369)$$

Like our Ricci tensor, our energy-momentum tensor is distribution valued, so we define it by its action on arbitrary smooth test functions. Against such a function of u, with P arbitrarily large, the wildly oscillating term $\cos(2Pu)$ can be dropped.

Furthermore, comparing with Eq. (5.365), we find

$$\sigma_P^2 = \frac{1}{2\epsilon^2} \ , \tag{5.370}$$

and hence

$$e^{-\sigma_P^2 u^2} = (2\pi\epsilon^2)^{1/2}\, \delta_\epsilon(u) = \left(\frac{\pi}{\sigma_P^2}\right)^{1/2} \delta_\epsilon(u) \ . \tag{5.371}$$

Similarly,

$$e^{-\sigma_P^2 |\vec{x}_\perp|^2} = \left(\frac{\pi}{\sigma_P^2}\right) \delta_\epsilon^2(\vec{x}_\perp) \ . \tag{5.372}$$

Therefore, we have found the trace-reversed energy-momentum tensor we were looking for:

$$\mathcal{T}_{uu} = (\partial_u H)^2 = 2^{-4} P^2 N_P \left(\frac{\sigma_P^2}{\pi}\right)^{3/2} \delta_\epsilon(u)\, \delta_\epsilon^2(\vec{x}_\perp) \equiv P\, \delta_\epsilon(u)\, \delta_\epsilon^2(\vec{x}_\perp) \ . \tag{5.373}$$

This fixes the normalization factor:

$$N_P = \frac{2^4 \pi^{3/2}}{\sigma_P^3\, P} \ . \tag{5.374}$$

The Fourier coefficient in Eq. (5.356) with the momentum-space delocalization assumption in Eq. (5.357), the normalization factor in Eq. (5.374), and the coarse-grained delta function in Eq. (5.365) together define what we mean by a "massless particle" in classical field theory.

5.E.3 T_{ui} and T_{ij}

Next we will show that $\mathcal{T}_{ui}$ and $\mathcal{T}_{ij}$ are parametrically suppressed with respect to $\mathcal{T}_{uu}$. Let us return to Eq. (5.364) and compute a partial derivative with respect to $x^i_\perp$:

$$\partial_i H = -\sigma_P^2 N_P^{1/2} \left(\frac{\sigma_P^2}{2\pi}\right)^{3/2} e^{-\frac{1}{2}\sigma_P^2 u^2} e^{-\frac{1}{2}\sigma_P^2 |\vec{x}_\perp|^2} \cos(Pu)\, x_{\perp i} \,. \tag{5.375}$$

Therefore,

$$\begin{aligned} \mathcal{T}_{ui} = \partial_u H \partial_i H &= -P\sigma_P^2 N_P \left(\frac{\sigma_P^2}{2\pi}\right)^{3} e^{-\sigma_P^2 u^2} e^{-\sigma_P^2 |\vec{x}_\perp|^2} \sin(Pu)\cos(Pu) \\ &= -\sigma_P^2\, \delta_\epsilon(u)\, \delta_\epsilon^2(\vec{x}_\perp) \sin(2Pu) \,. \end{aligned} \tag{5.376}$$

This is a wildly oscillating function that averages to zero against smooth test functions. Finally, we have

$$\begin{aligned}\mathcal{T}_{ij} = \partial_i H \partial_j H &= -\sigma_P^4 N_P \left(\frac{\sigma_P^2}{2\pi}\right)^3 e^{-\sigma_P^2 u^2} e^{-\sigma_P^2 |\vec{x}_\perp|^2} \cos^2(Pu)\, x_{\perp i}\, x_{\perp j} \\ &= -\frac{\sigma_P^4}{P}\, \delta_\epsilon(u)\, \delta_\epsilon^2(\vec{x}_\perp)\, [1 + \cos(2Pu)]\, x_{\perp i}\, x_{\perp j} \; . \end{aligned} \tag{5.377}$$

The oscillatory $\cos(2Pu)$ term can be dropped, as usual. Meanwhile, the magnitude of $\mathcal{T}_{ij}$ is suppressed relative to that of $\mathcal{T}_{uu}$ in Eq. (5.373) by the dimensionless ratio [recall $\sigma_X \equiv \sigma_P^{-1}$]

$$\left(\frac{\sigma_P}{P}\right)^2 \left(\frac{|\vec{x}_\perp|}{\sigma_X}\right)^2 \ll 1 \; . \tag{5.378}$$

So for the field configuration in Eq. (5.364), we indeed have $\mathcal{T}_{\mu\nu} \approx \delta^u_\mu \delta^u_\nu \, \mathcal{T}_{uu}$.

5.E.4 Physical meaning of limit

Let us reflect on what happened. We chose the field configuration in Eq. (5.364) so that it would describe, roughly speaking, the square root of a delta function, not an actual delta function. Our definition of the regularized delta function in Eq. (5.365) and the requirement $\mathcal{T}_{uu} \propto \delta(u)\, \delta^2(\vec{x}_\perp)$ from Eq. (5.373) imply that our field configuration is:

$$
\begin{aligned}
H(u,\vec{x}_\perp) &= \frac{2^{1/2}}{\pi^{3/4}P^{1/2}}\,\sigma_P^{3/2}\,e^{-\frac{1}{2}\sigma_P^2u^2}e^{-\frac{1}{2}\sigma_P^2|\vec{x}_\perp|^2}\cos(Pu) \\
&= \frac{2^2\pi^{3/4}}{P^{1/2}}\,\sigma_P^{9/2}\,\delta_{\sigma_P}(u)\,\delta^2_{\sigma_P}(\vec{x}_\perp)\cos(Pu)\;. \qquad (5.379)
\end{aligned}
$$

In the limit $\sigma_P \to 0$, we have $H(u,\vec{x}_\perp) \to 0$. Meanwhile, $(\partial_u H)^2 \to C\,\delta(u)\,\delta^2(\vec{x}_\perp)$ with C a fixed constant. So the limit we are describing is one in which we take the field away and leave behind only a localized source of energy-momentum.

This is a sensible point-particle limit of classical field theory.

5.E.5 Spherically symmetric curved space-time

As explained in the main text, the applicable mode decomposition occurs in the Eddington-like coordinate $u = t - r_*$, not in the Kruskal-like coordinate $U = -e^{-\alpha u}$. For the Schwarzschild or Reissner-Nordström background, we would use the following series expansion:

$$
H(U,\theta,\varphi) \equiv \sum_{l,m}\int_{-\infty}^{\infty}\frac{d\omega}{4\pi}\,a_{lm}(\omega)\,Y_{lm}(\theta,\varphi)\,e^{-i\omega u} + c.c. \qquad (5.380)
$$

For H to be real, the Fourier coefficients must satisfy $a_{lm}(-\omega)^* = a_{lm}(\omega)$. The choice of normalization is convenient but not standard. Using the usual orthogonality property $\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\; Y_{lm}(\theta,\varphi)Y_{l'm'}(\theta,\varphi)^* = \delta_{ll'}\delta_{mm'}$, imposing an upper cutoff on the integration over u, and requiring only that $H(U, \theta, \varphi) = H(U, \theta)$, we find

$$a_{lm}(\omega) = \delta_{m0} \int_{-\infty}^{\Lambda} du \int_0^{\pi} d\theta \, \sin\theta \, Y_{l0}(\theta)^* \, H(U, \theta) \; . \tag{5.381}$$

The question now is what distribution to assert for $H(U, \theta)$. The choice of wave packet in u about a value u_0 follows exactly the construction in flat space-time, except with $u = t - r_*$. The angular delta function, however, requires a few words.

We might like to localize a particle to the north pole of a sphere of radius r, namely to $\vec{x}_N = r\hat{z}$. But the usual spherical coordinates are singular there. To avoid trudging through an obstacle course of coordinate singularities, we will instead localize a particle to the equator:

$$\vec{x}_E = r(\cos\varphi \, \hat{x} + \sin\varphi \, \hat{y}) \; . \tag{5.382}$$

A general vector in spherical coordinates is $\vec{x} = r \, \sin\theta(\cos\varphi \, \hat{x} + \sin\varphi \, \hat{y}) + r\cos\theta \, \hat{z}$. The generalization of $\vec{x}_\perp$ is therefore

$$\vec{x}_\perp = \vec{x} - \vec{x}_N = r \left[-(1 - \sin\theta)(\cos\varphi \, \hat{x} + \sin\varphi \, \hat{y}) + \cos\theta \, \hat{z} \right] \; . \tag{5.383}$$

We will work with the coordinate $\mu \equiv \cos\theta$. A regularized delta function in the angular direction will then take the form

$$\delta^2_\sigma(\vec{x}_\perp) = C\, e^{-\frac{|\vec{x}_\perp|^2}{2\sigma^2}} \,, \tag{5.384}$$

with the overall factor C to be fixed by the normalization condition

$$1 \equiv \int d^2x_\perp\, \delta^2_\sigma(\vec{x}_\perp) = 2\pi C \int_{-1}^{1} d\mu\, e^{-\frac{|\vec{x}_\perp|^2}{2\sigma^2}} \,. \tag{5.385}$$

Implicit everywhere is the limit $\sigma \to 0$, so we can expand near $\theta = \frac{\pi}{2}$. Define $\epsilon \equiv \theta - \frac{\pi}{2}$. Then we have[172]

$$|\vec{x}_\perp|^2 = 2r^2(1 - \sin\theta) = 2r^2\,(1 - \cos\epsilon) = r^2\epsilon^2 + O(\epsilon^4)\,. \tag{5.386}$$

The twin prong of this approximation is that as far as the limits of integration are concerned, ± 1 may as well be taken to $\pm\infty$. Therefore:

$$1 \approx 2\pi C \int_{-\infty}^{\infty} d\epsilon\, e^{-\frac{r^2}{2\sigma^2}\epsilon^2} = 2\pi C \left(\frac{2\pi\sigma^2}{r^2}\right)^{1/2} \implies C = \frac{r}{(2\pi)^{3/2}\sigma}\,. \tag{5.387}$$

[172] The fact that the a priori dependence on φ has disappeared justifies our assumption that $H(U, \theta, \varphi) = H(U, \theta)$.

So the wave packet we will take for the field configuration is

$$H(U,\theta) = N^{1/2}\, e^{-\frac{1}{2\sigma^2}(u-u_0)^2}\, e^{-\frac{r^2}{2\sigma^2}(\theta-\frac{\pi}{2})^2}\, \cos(Pu)\ , \tag{5.388}$$

with P interpreted as the momentum of the wave packet. The reader could now attempt to put this into Eq. (5.381) and calculate the Fourier coefficients explicitly, at least in some approximation. We will not.

With the understanding of the angular delta function provided above, a calculation completely analogous to the steps in flat space-time would result in a trace-reversed energy-momentum tensor of the form $\mathcal{T}_{\mu\nu} = \delta^u_\mu \delta^u_\nu\, \mathcal{T}_{uu}$, where

$$\mathcal{T}_{uu} = P_0\, \delta(u-u_0)\ . \tag{5.389}$$

The final wrinkle is that we want $\mathcal{T}_{UU} \propto \delta(U)$, not $\mathcal{T}_{uu} \propto \delta(u)$. By the usual rules for transforming coordinates, we have:

$$\mathcal{T}_{uu} = (\partial_u U)^2 T_{UU} = \alpha^2 U^2 T_{UU}\ . \tag{5.390}$$

Therefore,

$$T_{UU} = \frac{P_0}{\alpha^2 U^2}\, \delta(u-u_0)\ . \tag{5.391}$$

Because $U = -e^{-\alpha u}$, we know that

$$\delta(U - U_0) = \sum_{\{u_0\}} \frac{1}{|U'(u_0)|}\,\delta(u - u_0)\ ,\quad U(u_0) \equiv U_0\ . \tag{5.392}$$

The required condition is $u_0 = \infty$. Because $U'(u) = -\alpha U$, we have $\lim_{u_0\to\infty}\delta(u - u_0) = \alpha|U|\delta(U)$, and the correct limit is to take $P_0 \to 0$ and $u_0 \to \infty$ such that $\lim_{U\to 0}\frac{P_0}{|U|}$ is held fixed. Therefore, defining

$$P \equiv \frac{P_0}{\alpha^2|U|} \tag{5.393}$$

we obtain the desired expression

$$T_{UU} = \lim_{u_0\to\infty}\frac{P}{|U|}\,\delta(u - u_0) = P\,\delta(U)\ . \tag{5.394}$$

5.E.6 Rotating space-time

The only real complication relative to the spherically symmetric case is how to generalize Eq. (5.383). Every student is familiar with the embedding of a sphere of radius r into 3 Euclidean dimensions by the parametrization $\vec{x} = r[\sin\theta\,(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}) + \cos\theta\,\hat{z}]$. What is the correct generalization for the squashed sphere?

Let $\mu \equiv \cos\theta$ and posit the following embedding of a 2d surface with coordinates (μ, χ) into an auxiliary 3d Euclidean space with coordinates (X, Y, Z) [163]:

$$X = F(\mu)\cos\chi \ , \quad Y = F(\mu)\sin\chi \ , \quad Z = G(\mu) \ . \tag{5.395}$$

The induced 2d metric on the surface is then

$$ds^2_{\rm 2d} = dX^2 + dY^2 + dZ^2 \,\big|_{\rm Eq.\ (5.395)} = (F'^2 + G'^2)\, d\mu^2 + F^2\, d\chi^2 \ . \tag{5.396}$$

With $d\mu^2 = \sin^2\theta\, d\theta^2 = (1-\mu^2)\, d\theta^2$, we match this induced metric to that on the squashed sphere of radius r [see Eq. (5.211)] to obtain[173]

$$F'(\mu)^2 + G'(\mu)^2 = \frac{r^2 + a^2\mu^2}{1-\mu^2} \ , \quad F(\mu)^2 = \frac{(r^2+a^2)^2}{r^2+a^2\mu^2}(1-\mu^2) \ . \tag{5.397}$$

For the parametrization in Eq. (5.395), the equator is defined such that $Z = G(0) = 0$. Therefore:

$$\vec{x}_E = F(0)(\cos\chi\,\hat{x} + \sin\chi\,\hat{y}) \ , \quad F(0) = \frac{r^2+a^2}{r} \ . \tag{5.398}$$

[173] Let us check the static limit. When $a \to 0$, we get $F(\mu)^2 \to r^2(1-\mu^2) = r^2\sin^2\theta \implies F(\mu) = r(1-\mu)^{1/2} = r\sin\theta$ (taking the positive root), which is correct. In that case $F'(\mu) = -r\mu(1-\mu^2)^{-1/2} = -2r\cot\theta$, and $G'(\mu) = r \implies G(\mu) = r\mu = r\cos\theta$ (setting the integration constant to zero), which is also correct.

A general location on the squashed sphere is $\vec{x} = X\hat{x}+Y\hat{y}+Z\hat{z}$. The generalization of Eq. (5.383) is then

$$\vec{x}_\perp = -(F(0) - F(\mu))(\cos\chi\,\hat{x} + \sin\chi\,\hat{y}) + G(\mu)\,\hat{z}\ . \tag{5.399}$$

The magnitude squared of this vector is:

$$|\vec{x}_\perp|^2 = (F(0) - F(\mu))^2 + G(\mu)^2\ . \tag{5.400}$$

In the vicinity of $\epsilon \equiv \theta - \frac{\pi}{2} = 0$, we have:

$$F(\mu) = F(0) - \frac{(r^2+a^2)^2}{2r^3}\,\epsilon^2 + O(\epsilon^3)\ ,\quad G(\mu) = -r\epsilon + O(\epsilon^3)\ . \tag{5.401}$$

Therefore $|\vec{x}_\perp|^2 = r^2\epsilon^2 + O(\epsilon^4)$, and

$$e^{-\frac{1}{2\sigma^2}|\vec{x}_\perp|^2} \approx e^{-\frac{r^2}{2\sigma^2}\epsilon^2}\ . \tag{5.402}$$

Curiously, the angular momentum parameter a dropped out in this approximation. The determinant of the metric on the squashed sphere is $|R_0|^4$, so the normalization factor $\mathcal{N}$ of the regularized delta function,

$$\delta^2_\sigma(\vec{x}_\perp) \equiv \mathcal{N}\,e^{-\frac{1}{2\sigma^2}|\vec{x}_\perp|^2}\ , \tag{5.403}$$

is set by

$$\begin{aligned} 1 &\equiv (r^2+a^2)\int_0^{2\pi} d\chi \int_{-1}^{1} d\mu\, \delta_\sigma^2(\vec{x}_\perp) \approx (r^2+a^2)\, 2\pi\mathcal{N} \int_{-\infty}^{\infty} d\epsilon\, e^{-\frac{r^2}{2\sigma^2}\epsilon^2} \\ &= (r^2+a^2)\, 2\pi\mathcal{N} \left(\frac{2\pi\sigma^2}{r^2}\right)^{1/2} \implies \mathcal{N} = \frac{r}{(2\pi)^{3/2}(r^2+a^2)\,\sigma}\,. \end{aligned} \tag{5.404}$$

So the replacement $r \to \frac{r^2+a^2}{r}$ in the normalization is all we need to generalize the spherically symmetric case. Because it is somewhat jarring to see a function of θ that is not manifestly periodic, and because any function $F(\epsilon) = \epsilon^2 + O(\epsilon^3)$ would do as far as the exponential is concerned, we could replace ϵ^2 with $\sin^2\epsilon = \cos^2\theta = \mu^2$ in Eq. (5.388).

This completes our chronicle of the field configuration in Eq. (5.252).

Chapter 6

Conclusion

This research project can be traced back to the work of Bouwmeester, and Irvine [42] who studied Rañada's solution and outlined an experimental implementation. Their work on an experimental realization has been extended recently in [176, 177]. Since that original paper, the Bouwmeester group has expanded its study of topologically non-trivial classical fields into plasmas as well as electromagnetic and gravitational fields, leading to over a dozen theses [1, 2, 21, 22, 178–194] and publications [3, 14, 6, 97, 195, 7, 196–198].

We have shown by direct construction that linearized gravitational fields can possess the same topologically non-trivial features as plasma solitons. Typically electromagnetism is only thought of as a gauge theory in a quantum mechanical context associated with magnetic monopoles or Aharonov-Bohm phase factors. However, in plasma, the gauge field A appears directly in the definition of magnetic helicity density via the Chern-Simons three form current $J_{CS} = A \wedge dA$. The integral of this 3-form gives a topological charge which is conserved in ideal MHD since $dJ_{CS} = 0$.

In the context of general electromagnetism we may introduce connections A and C such that $dA = F$ and $dC = *F$. A dual bundle approach allows us to compute topological currents and topological charges associated with the individual electric and magnetic fields. Thus, we have two Chern-Simons currents[1]

$$J^B_{CS} = A \wedge F = (\vec{A} \cdot \vec{B})\mathrm{Vol}_{\mathbb{R}^3} + \left[(\vec{A} \times \vec{E}) \lrcorner \mathrm{Vol}_{\mathbb{R}^3}\right] \wedge dt$$

$$J^E_{CS} = C \wedge *F = (\vec{C} \cdot \vec{E})\mathrm{Vol}_{\mathbb{R}^3} - \left[(\vec{C} \times \vec{B}) \lrcorner \mathrm{Vol}_{\mathbb{R}^3}\right] \wedge dt$$

and two topological charges

$$Q^B_{CS} = \int_{\mathbb{R}^3} J^B_{CS}$$

$$Q^E_{CS} = \int_{\mathbb{R}^3} J^B_{CS}.$$

[1] Typically, the inhomogeneous Maxwell equation is $d * F = *J$, where $*J$ is the 1-form 4-current dual to the 3-form $d * F$. For simplicity and to maximize clarity as to what terms in an action functional represent currents, we adopt the convention that currents associated with 1-form gauge fields are 3-forms. In 3-form notation, the electric current 3-form is $J = \rho \mathrm{Vol}_{\mathbb{R}^3} - (\vec{j} \lrcorner \mathrm{Vol}_{\mathbb{R}^3}) \wedge dt$.

The two currents yield two continuity equations

$$dJ^B_{CS} = F \wedge F = -2(\vec{E} \cdot \vec{B})\mathrm{Vol}_{\mathbb{M}^4} = \left[\partial_t(\vec{A} \cdot \vec{B}) - \vec{\nabla} \cdot (\vec{A} \times \vec{E})\right] \mathrm{Vol}_{\mathbb{M}^4}$$

$$dJ^E_{CS} = *F \wedge *F = +2(\vec{E} \cdot \vec{B})\mathrm{Vol}_{\mathbb{M}^4} = \left[\partial_t(\vec{C} \cdot \vec{E}) + \vec{\nabla} \cdot (\vec{C} \times \vec{B})\right] \mathrm{Vol}_{\mathbb{M}^4}.$$

For a finite volume V this yields

$$\partial_t Q^B_{CS} = -2\int_V (\vec{E} \cdot \vec{B})\mathrm{Vol}_{\mathbb{R}^3} + \int_{\partial V} (\vec{A} \times \vec{E}) \cdot \hat{n}\, ds$$

$$\partial_t Q^E_{CS} = +2\int_V (\vec{E} \cdot \vec{B})\mathrm{Vol}_{\mathbb{R}^3} - \int_{\partial V} (\vec{C} \times \vec{B}) \cdot \hat{n}\, ds.$$

If $V = \mathbb{R}^3$ then $\partial_t Q^B_{CS} = -\partial_t Q^E_{CS}$ and the electric and magnetic fields exchange topology. For null fields $\vec{E} \cdot \vec{B} = 0$, and thus Q^E_{CS} and Q^B_{CS} are constants, and the topology of both the electric and magnetic fields are constant under time evolution. The total topological current is given by

$$J^{EM}_{CS} = \tfrac{1}{2}(J^B_{CS} + J^E_{CS}) \tag{6.1}$$

whence,

$$dJ^{EM}_{CS} = \tfrac{1}{2}(F \wedge F + *F \wedge *F) \tag{6.2}$$

$$= 0, \tag{6.3}$$

implies $Q_{CS}^{EM} = \int J_{CS}^{EM}$ is conserved.

We saw the effects of these formulas play out in the plots of Chapters 2-4. In Ch. 4, we saw linearized gravitational fields with similar topology and topology exchange.

Differential geometry is defined using the tangent bundle to a manifold. The tangent bundle is a vector bundle. On the other hand, a Yang-Mills field is defined on a principal bundle. The difference between the two bundles types is the fiber's nature. A vector bundle has vector space fibers, and a principal bundle possesses group manifold fibers. Luckily, every vector bundle has a principal frame bundle where the group manifold associated with each point is the group of transformations that carries one frame into any other frame at the same point. In this way, we can view Cartan's method of moving frames as a way of doing differential geometry on the frame bundle of a manifold. We have seen in Ch. 5 that the Riemann curvature 2-form $\Omega^a{}_b$ is the Yang-Mills curvature of the local $so(1,3)$ spin connection $\omega^a{}_b$. Everything we have seen thus far leads us to believe that one should be able to construct a theory of gravitational helicity by considering the local $so(1,3)$ Chern-Simons form $tr(\omega \wedge \Omega + \frac{2}{3}\omega \wedge \omega \wedge \omega)$ since the integral of this 3-form is proportional to a topological invariant of the manifold called the Euler character.

Deser first demonstrated [199] that a one-parameter family of global duality transformation which rotated electric and magnetic fields into each other was a symmetry of the Maxwell action[2]. Later Henneaux and Teitelboim [201] showed that the vacuum linearized Einstein-Hilbert action possesses a similar symmetry which globally rotates the GEM fields E_{ij} and B_{ij} into each other. Their equation V.5 yields the spin-2 generalization of Eq. (6.1) and the charge they obtain from integrating their current is conserved precisely when $E_{ij}B^{ij} = 0$. These results were also obtained by Aghapour, *et al.* [202, 203].

The GEM fields provide a kinetic description of how the curvature of spacetime interacts with materials moving through it. Specifically, wave equations for EM fields in a gravitational background show how we might construct gravitational wave detectors that can probe all the degrees of freedom of the curvature tensor. Note that this formalism is exact for gravitational radiation far from the sources. We hope that these concepts will one day lead to humanity observing the cosmos with gravitational eyes.

[2]The astute reader will notice the relation to S-duality in string theory. For a review, we point the reader towards *Gravity and Strings* [200].

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