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Applications of Toric Geometry to Geometric Representation Theory

by

Qiao Zhou

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

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in the

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of the

University of California, Berkeley

Committee in charge:

Professor David Nadler, Chair

Professor Denis Auroux

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Summer 2017

Applications of Toric Geometry to Geometric Representation Theory

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Qiao Zhou

Abstract

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Qiao Zhou

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor David Nadler, Chair

We study the algebraic geometry and combinatorics of the affine Grassmannian and affine flag variety, which are infinite-dimensional analogs of the ordinary Grassmannian and flag variety. In particular, we analyze the intersections of Iwahori orbits and semi-infinite orbits in the affine Grassmannian and affine flag variety. These intersections have interesting geometric and topological properties, and are related to representation theory.

Moreover, we study the central degeneration (the degeneration that shows up in local models of Shimura varieties and Gaitsgory's central sheaves) of semi-infinite orbits, Mirković-Vilonen (MV) Cycles, and Iwahori orbits in the affine Grassmannian of type A, by considering their moment polytopes. We describe the special fiber limits of semi-infinite orbits in the affine Grassmannian by studying the action of a global group scheme. Moreover, we give some bounds for the number of irreducible components for the special fiber limits of Iwahori orbits and MV cycles in the affine Grassmannian. Our results are connected to Gaitsgory's central sheaves, affine Schubert calculus and affine Deligne-Lusztig varieties in number theory.

To my family

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Chapter 1

Introduction

1.1 Overview

One of the most important problems in geometric representation theory is the Geometric Langlands correspondence [5, 6, 26, 27]. The goal is to use geometric techniques to tackle questions in the Langlands program in number theory. The affine Grassmannian and affine flag variety are naturally motivated by the Geometric Langlands correspondence. They are infinite-dimensional analogs of \mathbb{P}^n , and also have applications to Schubert calculus [22], quantum physics [18], and mathematical biology [4].

In mathematics, there is a very interesting theme of studying abstract and complicated objects via concrete, combinatorial and visual ways. In this thesis, I contribute to the understanding of the explicit algebraic geometry and combinatorics of the affine Grassmannian and affine flag variety. The tools I use include Lie theory and toric geometry. First, I analyze the geometric properties of some intersections of orbits in the affine Grassmannian and affine flag variety. These intersections have interesting representation theoretic interpretations. Then I study the central degeneration [7, 34, 41], an important and abstract phenomenon in geometric representation theory and number theory, in terms of combinatorial objects like convex polytopes. My results are connected to affine Schubert calculus in combinatorial algebraic geometry and affine Deligne-Lusztig varieties in number theory. In the future, I aspire to build more bridges between geometric representation theory and combinatorial algebraic geometry. Results obtained in this thesis have also appeared in my recent paper [40].

Below is a sample of my results in this thesis.

The limits in the affine flag variety of a family of spaces in the affine Grassmannian under the central degeneration could be understood in terms of certain convex polytopes. For example, in 1.1, the big trapezoid represents a space S in the affine Grassmannian. Let \tilde{S} denote the limit of S in the affine flag variety. The five small convex polytopes of different colors in the big trapezoid represent different irreducible components of \tilde{S} .

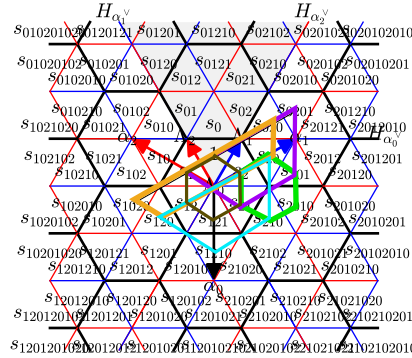


Figure 1.1: An Example of Decomposition of Polytopes

1.2 Thesis Storyline

1.2.1 Background

Let G be a connected reductive group over a field $k = \mathbb{C}$. Fix a Borel subgroup B and a maximal torus $T \subset B$. Let W denote the Weyl group of G , and let $X^*(T)$, $X_*(T)$ and Λ denote the weight, coweight and root lattice respectively. Let \mathcal{K} denote the local field of Laurent series $k((t))$, \mathcal{O} denote its ring of integers $k[[t]]$, $\mathbb{D} = \text{Spec}(\mathcal{O})$ denote the formal disc, and $\mathbb{D}^* = \text{Spec}(\mathcal{K})$ denote the punctured formal disc.

Let $G(\mathcal{K})$ be the corresponding group over the field \mathcal{K} . It is also called the loop group. Let $G(\mathcal{O})$ denote the corresponding group over \mathcal{O} . It is a maximal compact subgroup of $G(\mathcal{K})$, and is called the formal arc group. Let $I \subset G(\mathcal{O})$ denote the Iwahori subgroup, which is the preimage of $B \subset G$ under the map $ev_0 : G(\mathcal{O}) \rightarrow G$ that evaluates t at 0.

When $G = GL_n$, $G(\mathcal{K})$ is the group of invertible matrices with entries in \mathcal{K} ; $G(\mathcal{O})$ is the group of invertible matrices with entries in \mathcal{O} ; the Iwahori subgroup I is the subgroup of $G(\mathcal{O})$ whose entries below the diagonal lie in $t\mathcal{O}$. Let $I_m, m \in \mathbb{N}$ be the subgroup of I whose diagonal entries are 1, and whose off-diagonal entries lie in $t^m\mathcal{O}$.

We can form the two quotients $Gr = G(\mathcal{K})/G(\mathcal{O})$, which is called the affine Grassmannian, and $Fl = G(\mathcal{K})/I$, which is called the affine flag variety. The affine flag variety is a nontrivial G/B bundle over the affine Grassmannian. Let Fl_C denote the global affine flag variety over a curve C where the general fibers are isomorphic to $Gr \times G/B$, and the special fiber is isomorphic to Fl .

Let $U = N(\mathcal{K})$, $U^- = N^-(\mathcal{K})$, and $U_w = wUw^{-1}$. The orbits of these groups in the affine Grassmannian or the affine flag variety are called semi-infinite orbits, and are indexed by the T -fixed points. The T -fixed points in the affine Grassmannian are indexed by elements in the coweight lattice $X_*(T)$ of G . In the affine flag variety, T -fixed points are indexed by the affine Weyl group W_{aff} , which is isomorphic to $X_*(T) \rtimes W$.

1.2.2 Main Results and Future Projects

Consider the global affine flag variety over \mathbb{A}^1 . Each of its general fiber over $\mathbb{A}^1 \setminus \{0\}$ is the direct product $Gr \times G/B$; its special fiber over $\{0\}$ is isomorphic to the affine flag variety Fl . Let S be a subscheme of $Gr \times \{\text{id}\}$ in the general fiber that is invariant under $T \subset G$ and $\text{Aut}(\mathbb{D})$, the automorphism group of \mathbb{D} . For example, S could be a $G(\mathcal{O})$ orbit, an MV cycle, an Iwahori orbit, a semi-infinite orbit, an orbit of $T(\mathcal{O})$, etc. We would like to understand the special fiber limit \tilde{S} of S in the affine flag variety. This is an integral family over a curve, and is therefore flat. More precisely, $\tilde{S} = \overline{(S \times \{\text{id}\}) \times (\mathbb{A}^1 \setminus \{0\})} \cap Fl$ in $Fl_{\mathbb{A}^1}$. In other words, \tilde{S} is the closure of the flat family of schemes isomorphic to S over $\mathbb{A}^1 \setminus \{0\}$ restricted to the special fiber over $\{0\}$ in $Fl_{\mathbb{A}^1}$.

The central degeneration is well-studied in the case when S is a $G(\mathcal{O})$ orbit [7, 9, 41]. In this thesis we focus on the cases when S is an orbit of $U_w = wNw^{-1}(\mathcal{K})$, an MV cycle, or an Iwahori orbit in the affine Grassmannian.

1.2.2.1

In the case of the central degeneration of $G(\mathcal{O})$ orbits and Gaitsgory's central sheaves, we have a global group scheme where the general fibers are isomorphic to $G(\mathcal{O})$. In the case of U_w orbits, we similarly have a global group scheme $U_{w, \text{glob}}$ acting on the global affine flag variety where each fiber is isomorphic to $U_w \subset G(\mathcal{K})$. As a result, we have the following theorem:

Theorem 1. *In type A case, the special fiber limit of the closed orbit of U_w , S_w^μ , is the corresponding closed orbit of U_w , $S_w^{(\mu, e)}$ in the affine flag variety. Here μ is a coweight and (μ, e) is a translation element in W_{aff} .*

Therefore, this central degeneration preserves the semi-infinite/periodic Bruhat order, but not the usual Bruhat order. Semi-infinite orbits in the affine flag variety that correspond to non-translation affine Weyl group elements are not the limit of any semi-infinite orbit in the affine Grassmannian.

Let Φ_1 denote the nearby cycles functor for Gaitsgory's central sheaves $\mathcal{Z}(\mathcal{P}_I(Fl_G))$ and Φ_2 be the weight functor for the geometric Satake correspondence. Let Φ_3 be a similar cohomology functor on the category of Iwahori equivariant perverse sheaves on the affine flag variety of G . Then the following diagram commutes.

Conjecture 1. *Let Φ_1 denote the nearby cycles functor for Gaitsgory's central sheaves $\mathcal{Z}(\mathcal{P}_I(Fl_G))$ and Φ_2 be the weight functor for the geometric Satake correspondence. Let Φ_3 be a similar cohomology functor on the category of Iwahori equivariant perverse sheaves on the affine flag variety of G . Then the following diagram commutes.*

$$\begin{array}{ccc}
 \mathcal{P}_{G(\mathcal{O})}(Gr_G) & \xrightarrow{\Phi_1} & \mathcal{Z}(\mathcal{P}_I(Fl_G)) \\
 & \searrow \Phi_2 & \swarrow \Phi_3 \\
 & \mathcal{P}_{T(\mathcal{O})}(Gr_T) \cong Rep(Gr^\vee) &
 \end{array}$$

1.2.2.2

We consider the central degeneration of the orbits of generalized Iwahori subgroups I_w , where $I_w = ev_0^{-1}(wBw^{-1})$, $w \in W$. The collection of Iwahori orbits forms a basis of the (co)homology and equivariant (co)homology of the affine Grassmannian and affine flag variety. A related topological story is explained in [22, 32].

Theorem 2. *In type A case, the special fiber limits of these generalized Iwahori orbits are not invariant under the action of the original Iwahori group I_w . Instead they are invariant under the action of the group $J, I_2 \subset J \subset I_1$, where J is the group that has 1's along the diagonal, $t(\mathcal{O})$ above the diagonal, and $t^2\mathcal{O}$ below the diagonal.*

We also estimate the number of irreducible components in the special fiber limits of (generalized) Iwahori orbits and MV cycles.

Theorem 3. *Let S be a generalized Iwahori orbit I_w^λ or an MV cycle in the general fiber affine Grassmannian. Let P denote the T -equivariant moment polytope of S , $\{\mu_i\}$ denote the finite set of coweights that correspond to the vertices of P . Let \tilde{S} denote the limit of S in the special fiber. The moment polytope of \tilde{S} , \tilde{P} , is the convex polytope with vertices indexed by $\{(\mu_i, e) \in W_{aff}\}$.*

Then in type A the number of irreducible components of \tilde{S} is bounded below by the number of vertices of P , and is bounded above by the number of convex polytopes contained in \tilde{P} satisfying some extra conditions.

We conjecture that the number of irreducible components in the special fiber limits of Iwahori orbits are closely related to affine Stanley coefficients [22, 21], which show up in [25].

Conjecture 2. *The nearby cycles functor on the global affine flag variety for G induces the same map on cohomology as the map $f : \Omega K \hookrightarrow LK \rightarrow LK/T \cong \Omega K \times K/T$, where K is a maximal compact subgroup of G .*

The special fiber limit of the generalized Iwahori orbit I_w^λ is a union of orbits of different subgroups of I_w . The number of irreducible components in the special fiber limit is a sum of affine Stanley coefficients.

We would like to discuss further the central degeneration of MV Cycles in the affine Grassmannian.

By [16], each MV cycle S is equal to a GGMS stratum in the affine Grassmannian $\cap_{w \in W} S_w^{\mu_w}$. The theorem below compares the special fiber limit of an MV cycle with some other related GGMS strata in the affine flag variety.

Theorem 4. *The special fiber limit of S , \tilde{S} , and the intersection of corresponding closed semi-infinite orbits in Fl , $S' = \bigcap_{w \in W} S_w^{(\mu_w, e)}$, have the same dimension.*

Therefore, the number of irreducible components in \tilde{S} is bounded below by the number of vertices of the corresponding MV polytope, and bounded above by the number of irreducible components in S' .

1.2.2.3

The special fiber limits of MV cycles are contained in the intersections of certain Iwahori orbits and U^- -orbits in the affine flag variety. Such intersections are in turn related to affine Deligne-Lusztig varieties. We start by studying the intersections of Iwahori orbits and U^- orbits in the affine Grassmannian, which corresponds to representations of Demazure modules. Then we proceed to discuss some dimension bounds for the intersections of Iwahori orbits and U^- orbits in the affine flag variety.

Theorem 5. *Let G be a connected reductive algebraic group. Let λ and μ be coweights of G , and let λ_{dom} be the dominant coweight associated to λ . Let $\tilde{W} = W/W_I$ denote the quotient of the finite Weyl group associated to the partial flag variety $G/P_{\lambda_{dom}}$. Let $\lambda = w \cdot \lambda_{dom}$ for a unique $w \in \tilde{W}$. Let X_w denote the Schubert variety for $w \in \tilde{W}$.*

The intersection of the U^- orbit $S_{w_0}^\mu$ with the Iwahori orbit I^λ is equidimensional and of dimension

$$\text{height}(\lambda_{dom} + \mu) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)$$

when $\lambda \leq \mu \leq \lambda_{dom}$, and is \emptyset otherwise.

For readers who like examples and diagrams, at the end we explicitly describe all the irreducible components of the special fiber limits of some MV cycles and Iwahori orbits in the $G = SL_3$ case.

1.2.2.4

Apart from proving the conjectures above, there are a few other new directions that we would like to pursue after this project.

It would be very useful to generalize results in this paper to algebraic groups of other types. At the moment, very few examples in other types for the central degeneration are known.

A natural new direction would be to apply some of the moment polytopes techniques used in this paper to find sharper bounds for the dimensions of affine Deligne-Lusztig varieties. We would like to first generalize the work of Kamnitzer [16] and develop a theory of moment polytopes for the generalized MV cycles in the affine flag variety. Note that these moment polytopes are special cases of the alcoved polytopes studied in [23, 24].

In [7], Gaitsgory constructed some central sheaves in $P_I(Fl)$ by considering the nearby cycles of $P_{G(\mathcal{O})}(Gr)$. In this paper we discovered that the central degeneration behaves well

with respect to semi-infinite orbits. Therefore we would like to apply the nearby cycles functor for the global affine flag variety to the category of U or U^- equivariant perverse sheaves on the affine Grassmannian.

1.2.3 Organization

The layout of this thesis is as follows.

Chapter 1 offers an introductory overview and Chapter 2 consists of background material. Chapter 3 introduces the relevant concepts from toric geometry, and presents some results related to the dimensions of some schemes and their moment polytopes. Chapter 4 discusses some results related to affine Deligne-Lusztig varieties, by presenting some results on the intersections of Iwahori orbits and semi-infinite orbits. Chapters 5 - 7 discuss various aspects of the central degeneration.

Chapter 2

Affine Grassmannian, Affine Flag Variety, and their Global Counterparts

2.1 Affine Grassmannian and Affine Flag Variety

2.1.1 Loop Groups and Loop Algebra

Let G be a connected, reductive group over the field k , k being \mathbb{C} or \mathbb{F}_q . Fix a Borel subgroup B and a maximal torus $T \subset B$. Let \mathcal{K} denote the local field of Laurent series $k((t))$, and \mathcal{O} denote the ring of integers $k[[t]]$. $G(\mathcal{K})$ is the group scheme that is also called the loop group, as it is the group of analytic maps $\mathbb{C}^* \rightarrow G$. $G(\mathcal{O}) \subset G(\mathcal{K})$ is a maximal compact subgroup of $G(\mathcal{K})$, and is called the formal arc group, as it is the group of analytic maps $\mathbb{C}^* \rightarrow G$ that can be extended to $0 \in \mathbb{C}$. There is a map $ev_0 : G(\mathcal{O}) \rightarrow G$ by evaluating at $t = 0$. Let the Iwahori subgroup I denote the pre-image of B under ev_0 . Similarly, let I_w denote the pre-image of $B_w = wBw^{-1}$ under ev_0 .

Now let $\mathfrak{g}_{\mathbb{C}}$ denote a complexified Lie algebra of G . The complexified Lie algebra of the loop group LG is the loop algebra $L\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}} \cdot z^k$. There is a rotation action of \mathbb{C}^* on the loops, and we obtain a semidirect product $\mathbb{C}^* \tilde{\times} LG$. The complexified Lie algebra of $\mathbb{C}^* \tilde{\times} LG$ decomposes as

$$(\mathbb{C} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus (\bigoplus_{k \neq 0} \mathfrak{t}_{\mathbb{C}} z^k) \oplus_{(k, \alpha)} \mathfrak{g}_{\alpha} z^k$$

according to the characters of $\mathbb{C}^* \times T$.

The affine Weyl group $W_{aff} = N_G(\tilde{T})/\tilde{T}$ is isomorphic to a semi-direct product of the coweight lattice and the finite Weyl group. It acts transitively on the set of alcoves when G is simply connected, e.g. $G = SL_n(k)$.

A root subgroup U_{α} is the one-parameter group $\exp(\eta \cdot e_{\alpha})$, where α is an affine root and η is the parameter. For any w in the finite Weyl group W , the group $U_w = (wNw^{-1})(\mathcal{K})$ is an infinite product of root subgroups in the loop group.

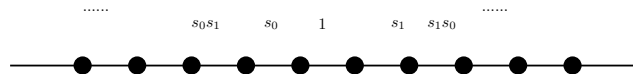


Figure 2.1: Affine A_1 Alcoves

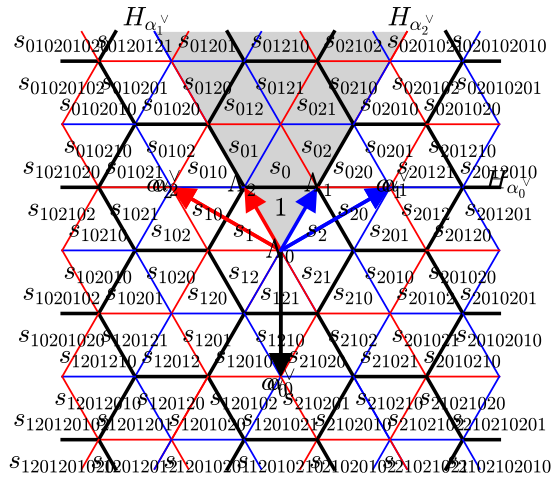


Figure 2.2: Affine A_2 Alcoves

2.1 is a diagram for the affine A_1 root system with alcoves labelled by elements in the affine Weyl group.

2.2 is a diagram for the affine A_2 root system with alcoves labelled by elements in the affine Weyl group.

For further details, see [33].

2.1.2 Affine Grassmannian and Affine Flag Variety

The affine Grassmannian is the ind-scheme $G(\mathcal{K})/G(\mathcal{O})$. There is a sequence of finite type projective schemes Gr^i , $i \in \mathbb{N}$ and closed immersions $Gr^i \hookrightarrow Gr^{i+1}$, such that $Gr(S) = \lim Hom(S, Gr^i)$.

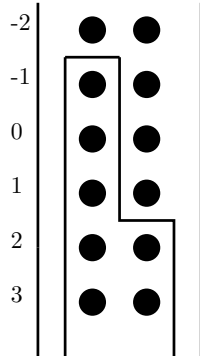


Figure 2.3: The lattice $L = \mathcal{O} \cdot t^{-1}e_1 \oplus \mathcal{O} \cdot e_2$ for $n = 2$

In type A when $G = GL_n(k)$, the affine Grassmannian $Gr = G(\mathcal{K})/G(\mathcal{O})$ is isomorphic to the moduli space below:

$Gr = \{L \subset \mathcal{K}^n \mid L \text{ is a rank } n \mathcal{O}\text{-module, } \exists N \gg 0 \text{ s.t. } t^N L_0 \subset L \subset t^{-N} L_0\}$, where each L is called a lattice. L_0 is the standard lattice \mathcal{O}^n .

When $G = SL_n(k)$, then the affine Grassmannian for G is isomorphic to the moduli spaces of lattices with zero relative dimension. This is related to the fact that matrices in SL_n have determinant one.

Each lattice L could be written as a direct sum $L = \mathcal{O}v_1 \oplus \cdots \oplus \mathcal{O}v_n$, where $\{v_1, \dots, v_n\}$ is a \mathcal{K} -basis of \mathcal{K}^n . By choosing the standard basis $\{e_1, e_2, \dots, e_n\}$ of \mathbb{C}^n , we could represent the lattices in some pictures. Note that $\mathcal{B} = \{t^m \cdot e_i \mid m \in \mathbb{Z}, i = 1, 2, \dots, n\}$ form a \mathbb{C} -basis of \mathbb{K}^n , in the sense appropriate to a topological vector space with the t -adic topology. For example, when $n = 2$, if $L = \mathcal{O} \cdot t^{-1}e_1 \oplus \mathcal{O} \cdot e_2$, L could be represented as 2.3.

In the lattice pictures, each dot represents an element in \mathcal{B} . The straight lines are used to indicate a basis \mathcal{B}_L of a the lattice L , which is a rank $n \mathcal{O}$ -module. Each element of \mathcal{B}_L is expressed as a linear combination of elements in \mathcal{B} , the standard basis.

We could consider the projective schemes Gr^i as all the lattices L such that $t^i L_0 \subset L \subset t^{-i} L_0$. Then Gr^i is a union of ordinary Grassmannians in $V = (t^{-i} L_0)/(t^i L_0)$ with the extra condition $tS \subset S$ for any subspace S in these ordinary Grassmannians.

The (complete) affine flag variety Fl is the quotient $G(\mathcal{K})/I$. In type A, there is also a lattice picture for Fl . It is the space of complete flags of lattices $L = (L_1 \supset \cdots \supset L_n)$. Each L_i is a lattice such that $L_n \supset tL_1$ and $\dim(L_j/L_{j+1}) = 1$.

For further details on the lattice picture of affine Grassmannian and affine flag variety of type A, please see [29].

There are interpretations of the affine Grassmannian and affine flag variety in terms of principal G -bundles, for an algebraic group G of any type, not just type A.

The affine Grassmannian Gr is a functor that associates each scheme S the set of pairs $\{\mathcal{P}, \phi\}$, where \mathcal{P} is an S -family of G -bundles over the formal disc \mathbb{D} , $\phi : \mathcal{P}|_{\mathbb{D}^*} \rightarrow \mathcal{P}^0|_{\mathbb{D}^*}$ is a trivialization of \mathcal{P} on \mathbb{D}^* . The (complete) affine flag variety Fl associates to each scheme S

all the data in Gr plus a B -reduction of the principal G -bundle \mathcal{P} at $\{0\} \in \mathbb{D}$. There is a natural projection map $Fl \rightarrow Gr$ with fibers being isomorphic to the ordinary flag variety G/B .

2.1.3 Semi-infinite Orbits

Semi-infinite orbits S_w^γ in the affine Grassmannian or the affine flag variety are the orbits of the groups $N_w(\mathcal{K})$. They are infinite-dimensional and are indexed by γ in the coweight lattice or the affine Weyl group.

The Gelfand-Goresky-Macpherson-Serganova (GGMS) strata [16] on the affine Grassmannian or the affine flag variety are the closures of intersections of some (open) semi-infinite orbits. More precisely, given any collection $\alpha. = (\alpha_w)_{w \in W}$ of elements in the coweight lattice or the affine Weyl group W_{aff} , we can form the GGMS stratum

$$A(\alpha.) = \overline{\bigcap_{w \in W} S_w^{\alpha_w}}.$$

The moment polytope of the closure of a nonempty GGMS stratum $A(\alpha.)$ is the convex hull of the set of coweights $\{\alpha.\}$.

2.1.4 Iwahori Orbits

Topologically orbits of the Iwahori subgroup correspond to Schubert classes in the (equivariant) homology and cohomology of the affine Grassmannian and affine flag variety [22]. For the rest of this subsection, we explain a bit more about the geometric properties of Iwahori orbits in the affine Grassmannian and affine flag variety.

Let λ denote a coweight of G and λ_{dom} denote the dominant coweight associated to λ . In the affine Grassmannian, the G -orbit of t^λ is the partial flag variety G/P_λ . P_λ denotes the parabolic subgroup of G with a Levi factor associated to the roots α such that $\lambda(\alpha) = 0$.

Each $G(\mathcal{O})$ -orbit Gr^λ in the affine Grassmannian Gr is a vector bundle over a partial flag variety G/P_λ . The vector bundle projection map is given by

$$Gr^\lambda \cong G(\mathcal{O}) \cdot t^\lambda \xrightarrow{ev_0} G/P_{\lambda_{dom}} \cong G \cdot t^\lambda.$$

The fibers are isomorphic to $I_1 \cdot t^{\lambda_{dom}}$ as vector spaces. Here I_1 is the subgroup of I that is the pre-image of the identity element under the map ev_0 . The dimension of each fiber is $2 \cdot ht(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}})$.

Let W denote the Weyl group of G . The partial flag variety G/P_λ has a cell decomposition indexed by elements of the coset $\tilde{W} = W/W_I$, where W_I is the subgroup generated by permutations of the simple roots associated to P_λ . Let X_w and $\overline{X_w}$ denote the open and closed Schubert cell corresponding to $w \in \tilde{W}$. For a unique w in the coset \tilde{W} , $\lambda = w \cdot \lambda_{dom}$. Then the Iwahori orbit I^λ in the affine Grassmannian Gr is the pre-image of the open Schubert cell X_w under the map ev_0 above, and is a vector bundle over X_w .

The dimension of the Iwahori orbit in the affine Grassmannian I^λ is $\dim(X_w \subseteq G/P_{\lambda_{dom}}) + 2 \cdot \text{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}})$.

Example 1. In the case of $G = SL_2(\mathbb{C})$, G/P_λ is \mathbb{P}^1 for $\lambda \neq 0$ and is a point for $\lambda = 0$.

When $\lambda \neq 0$, if λ is dominant, the Iwahori orbit is just a vector bundle over a point with dimension $2 \cdot \text{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}})$; if λ is anti-dominant, the Iwahori orbit is a vector bundle over the dense open cell in \mathbb{P}^1 , and has dimension $2 \cdot \text{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}}) + 1$.

In the affine flag variety, the Iwahori orbits are affine cells that are indexed by elements in the affine Weyl group W_{aff} . The dimension of each Iwahori orbit is given by the length of the corresponding affine Weyl group element $\gamma \in W_{aff}$.

2.1.5 MV Cycles and Generalized MV Cycles

Mirković-Vilonen cycles (MV cycles) are very interesting algebraic cycles in the $G(\mathcal{O})$ orbits of the affine Grassmannian.

In [30], each MV cycle is defined as an irreducible component of the closed intersection of a $G(\mathcal{O})$ orbit Gr^λ and a U^- orbit $S_{w_0}^\mu$, where λ is a dominant coweight and μ is a coweight that belongs to the convex hull of the set $\{w \cdot \lambda | w \in W\}$. In [1], MV cycles for $\overline{Gr^\lambda}$, relative to N , are the irreducible components of $\overline{Gr^\lambda} \cap S_e^\mu$. Equivalently they are those irreducible components of $\overline{S_e^\mu} \cap \overline{S_{w_0}^\lambda}$ contained in $\overline{Gr^\lambda}$. They gave a canonical basis of the highest weight representations of the Langlands dual group [30]. Each MV cycle could be defined as the closure of a GGMS stratum $A(\mu.)$ with some extra conditions on the coweights $\mu.$ involved [1, 16, 30]. MV polytopes are the T -equivariant moment polytopes of MV cycles, T being the maximal torus in G [1].

In the affine flag variety, a Generalized MV Cycle is an irreducible component in a nonempty intersection of an Iwahori orbit and a U^- orbit [31].

2.2 Central Degeneration

2.2.1 Global Versions of Affine Grassmannian and Affine Flag Variety

Now let X be a smooth curve. We would like to define the global analogs of the affine Grassmannian and the affine flag variety, as explained in [7].

The global affine Grassmannian over a curve X is the functor $Gr_X(S) = \{(y, \mathcal{P}, \phi) | y \text{ is a point on } X, \mathcal{P} \text{ is an } S\text{-family of principal } G\text{-bundles on } X, \text{ and } \phi \text{ is a trivialization of } \mathcal{P} \text{ on } X \setminus \{y\}\}$.

The global affine flag variety Fl_X over X is constructed as follows: Let x be a distinguished point on X . $Fl_X(S) = \{(y, \mathcal{P}, \phi, \zeta) | y \text{ is a point on } X, \mathcal{P} \text{ is an } S\text{-family of principal } G\text{-bundle on } X, \phi \text{ is a trivialization of } \mathcal{P} \text{ on } X \setminus \{y\}, \text{ and } \zeta \text{ is a } B\text{-reduction of the principal } G\text{-bundle at } x \in X\}$.

We have canonical isomorphisms $Fl_{\{y\} \in X \setminus \{x\}} \cong Gr_{\{y\}} \times G/B$ and $Fl_{\{x\}} \cong Fl$.

Specializing to the curve \mathbb{A}^1 , we can rewrite the definition as below.

$Fl_{\mathbb{A}^1} = \{(\epsilon \in \mathbb{A}^1, a \in G(k[t, t^{-1}])/I_\epsilon)\}$, where I_ϵ is the pre-image of the Borel subgroup B under the map $G(k[t]) \rightarrow G$ by evaluating at $t = \epsilon \in \mathbb{A}^1$. Topologically I_ϵ are algebraic maps $\mathbb{A}^1 \rightarrow G$ such that $\{\epsilon\} \hookrightarrow B$. Each fiber $Fl_{\mathbb{A}^1}|_\epsilon = G(k[t, t^{-1}])/I_\epsilon$ has a map to the affine Grassmannian $Gr = G(k[t, t^{-1}])/G(k[t])$. When $\epsilon \neq 0$, $Fl_{\mathbb{A}^1}|_\epsilon$ has a map to G/B by evaluating at $t = \epsilon$, so it is isomorphic to $Gr \times G/B$. When $\epsilon = 0$, the fiber $G(k[t, t^{-1}])/I_0$ is isomorphic to the affine flag variety.

We have the following commutative diagram:

$$\begin{array}{ccccccc}
 Gr \times G/B & \hookrightarrow & Gr \times G/B \times (\mathbb{A}^1 \setminus \{0\}) & \hookrightarrow & Fl_{\mathbb{A}^1} & \longleftarrow & Fl \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \{\epsilon \neq 0\} & \hookrightarrow & \mathbb{A}^1 \setminus \{0\} & \hookrightarrow & \mathbb{A}^1 & \longleftarrow & \{0\}
 \end{array}$$

There is also a lattice picture of $Fl_{\mathbb{A}^1}$ of type A. $Fl_{\mathbb{A}^1}$ is the moduli space of triples (ϵ, L, f) , where $\epsilon \in \mathbb{A}^1$, L is a lattice in the affine Grassmannian of type A, and f is a flag in the quotient $L/(t - \epsilon)L$. When $\epsilon = 0$, we recover the lattice picture of the affine flag variety in type A.

2.2.2 Central Degeneration

The Iwahori affine Hecke algebra H_I as a vector space consists of compactly supported bi- I -invariant functions $G(\hat{\mathcal{K}}) \rightarrow \overline{\mathbb{Q}_l}$, with the algebra structure given by convolution of functions. The spherical affine Hecke algebra H_{sph} as a vector space consists of compactly supported bi- $G(\hat{\mathcal{O}})$ -invariant functions with the algebra structure also given by convolution. The center of the Iwahori Hecke algebra $Z(H_I)$ is isomorphic to H_{sph} . In [7] Gaitsgory constructed a map from H_{sph} to $Z(H_I)$ geometrically, through the nearby cycles functor from $P_{G_{\mathcal{O}}}(Gr)$ to $P_I(Fl)$.

We aim to understand this construction more explicitly by examining how some interesting torus invariant spaces in the affine Grassmannian degenerate.

Example 2. *An interesting basic example was worked out for $G = GL_2(k)$ or $PGL_2(k)$ in [7].*

Consider the closed minuscule $G(\mathcal{O})$ -orbit $Y_0 = \{ \text{lattices } L \text{ which are contained in the standard lattice } \mathcal{O} \oplus \mathcal{O} \text{ with } \dim(\mathcal{O} \oplus \mathcal{O} \setminus L) = 1 \}$. By construction, Y_0 is isomorphic to \mathbb{P}_1 .

The special fiber limit of $Y_0 \times \{e\}$ in the affine flag variety Fl is isomorphic to a union of two \mathbb{P}^1 s intersecting at a point. Locally, this is the family $\{xy = a | a \in k\}$.

Later on we are going to show that all \mathbb{P}^1 s invariant under the extended torus would degenerate in this way (when the underlying curve is \mathbb{A}^1).

2.2.3 Degenerations of $G(\mathcal{O})$ -Orbits

In [41], it was proved that the special fiber limit of a $G(\mathcal{O})$ orbit Gr^λ in the affine Grassmannian is the union of Iwahori orbits I^α , where α is a λ -admissible element in the affine

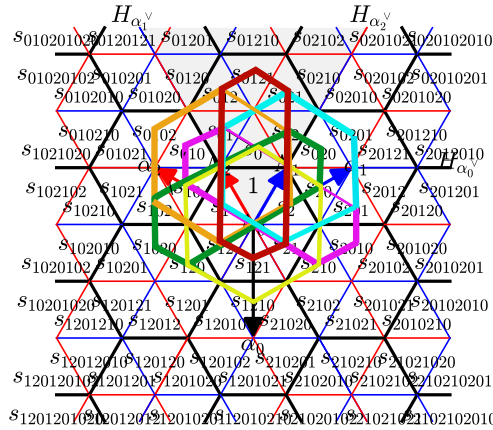


Figure 2.4: The central degeneration of the $G(\mathcal{O})$ -orbit $Gr^{\alpha+\beta}$ for $G = SL_3$ illustrated as convex polytopes. The big hexagon represents the moment polytope of $Gr^{\alpha+\beta}$, and the 6 smaller interior convex polytopes represent the moment polytopes of the $|S_3| = 6$ irreducible components in the limit.

Weyl group. The term λ -admissible means that there exists a w in the finite Weyl group such that $\alpha \leq w \cdot \lambda$ in the usual Bruhat order.

The number of irreducible components in the special fiber limit of a $G(\mathcal{O})$ orbit Gr^λ is equal to the size of the quotient of the finite Weyl group $\tilde{W} = W/W_I$ for the (partial) flag variety G/P_λ . When λ is regular, \tilde{W} the Weyl group W . Pictorially, the number of irreducible components in the special fiber limit is equal to the number of vertices in the T -equivariant moment polytope of Gr^λ .

Example 3. 2.4 illustrates the the T -equivariant moment polytopes of the W -many irreducible components in the limit of the $G(\mathcal{O})$ -orbit $Gr^{\alpha+\beta}$ for $G = SL_3$.

By [7], there is a global group scheme \mathcal{G} that acts on our family. It maps a scheme S to the set of pairs $\{(y, \Phi_y)\}$, where $y \in \mathbb{A}^1$, and Φ_y is the group of jets of the maps $f_y : \mathbb{A}^1 \times S \rightarrow G$ such that $f_y(\{0\} \times S) \in B$. By taking Taylor expansions at 0, we obtain a map $\mathcal{G} \rightarrow I$; by taking Taylor expansions at a general point ϵ , we obtain a map $\mathcal{G} \rightarrow G(\mathcal{O})$.

When $G = GL_n$ and λ is a minuscule coweight, Gr^λ is an ordinary Grassmannian. There are explicit equations of this central degeneration of Gr^λ of type A in [9]. These equations are obtained from studying the following moduli functor [9, 14, 13].

Consider a functor M_μ such that for any ring R $M_\mu(R)$ is the set of $L = (L_0, \dots, L_{n-1})$ where L_0, \dots, L_{n-1} are $R[t]$ submodules of $R[t]^n/t^d \cdot R[t]^n$, d being a positive integer, satisfying the following properties

- as R -modules L_0, \dots, L_{n-1} are locally direct factors of corank $nd - r$ in $R[t]^n/t^d \cdot R[t]^n$.

- $\gamma(L_i) \subset L_{i+1}(\text{mod } n)$, where γ is the matrix

$$\begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ t + \epsilon & 0 & \dots & 0 \end{bmatrix}$$

When $d = 1$, [9] presents the case of degenerations of $G(\mathcal{O})$ orbits in the minuscule case, which are exactly ordinary Grassmannians $G(\mathbb{C}^n, r)$.

Chapter 3

Toric Geometry

3.1 Moment Polytopes

3.1.1 Torus Action

The maximal torus T in G acts on the entire family $Fl_{\mathbb{A}^1}$. Its action changes the trivializations of principal G bundles away from a point on the curve, as well as the B reductions at 0. This torus action preserves each individual fiber $G[t, t^{-1}]/I_\epsilon$.

Note that the rotation torus \mathbb{C}^* scales the base curve \mathbb{A}^1 and moves one fiber to another. Therefore the action of the rotation torus does not preserve individual fibers.

3.1.2 Moment Polytopes for the Affine Grassmannian and the Affine Flag Variety

Let \mathcal{L} be an ample line bundle on the global affine flag variety Fl_C , where C is a curve. Let $\Gamma(Fl_C, \mathcal{L}^*)$ be the vector space of global sections of \mathcal{L}^* . Then Fl_C embeds in the projective space $\mathbb{P}(V)$, where $V = \Gamma(Fl_C, \mathcal{L}^*)$, by mapping $x \in Fl_C$ to the point determined by the line in V dual to the hyperplane $\{s \in \Gamma(Fl_C, \mathcal{L}) \mid s(x) = 0\}$.

The moment map Φ , for the action of the torus $T \subset G$, is a map from $Fl_C \hookrightarrow \mathbb{P}(V)$ to $\mathfrak{t}_{\mathbb{R}}^*$, which is isomorphic to $\mathfrak{t}_{\mathbb{R}}$. When we restrict Φ to individual fibers in Fl_C , we get a map from $Gr \times G/B$ or Fl to $\mathfrak{t}_{\mathbb{R}}^*$.

As defined in section 3.2 of [23], the alcove lattice is the infinite graph whose vertices correspond to alcoves, and edges correspond to pairs of alcoves that are separated by a wall. For example, in the A_2 case, we get a hexagonal lattice.

By [39], with appropriate choices, the moment map image of each T -fixed point $x_\gamma, \gamma = (\lambda, w) \in W_{aff}$ in Fl_C is the vertex of the alcove lattice that corresponds to $\gamma = (\lambda, w)$, where λ is a coweight, and $w \in W$.

There is a canonical embedding of the weight lattice into $\mathfrak{t}_{\mathbb{R}}^*$, by differentiating a map $T \rightarrow \mathbb{C}^*$ at the identity. If X is a one-dimensional torus orbit, then $\Phi(X)$ is a line segment

that points in a root direction in this embedded weight lattice and joins the images of the two T -fixed end points of X . The moment polytope of a T -invariant projective scheme is the convex hull of the images of the T -fixed points in this embedded weight lattice in $\mathfrak{t}_{\mathbb{R}}^*$.

Throughout this paper, in a T -projective scheme S , we call the T -fixed points whose moment map images are the vertices of the moment polytope extremal T -fixed points. All the T -fixed points in S that are not extremal T -fixed points are called internal T -fixed points.

3.2 Flat Families

All the families of schemes that we are interested in this paper are flat. This is because each family of schemes \mathcal{F} in the global affine flag variety Fl_C over a curve C is integral, and therefore the morphism $\mathcal{F} \rightarrow C$ is flat.

A flat family of schemes is called T -equivariant if a torus T acts on each fiber and its action is preserved by the degeneration. Then the central degeneration is T -equivariant and flat, where T is the maximal torus in G .

Lemma 1. *Consider a T -equivariant flat family of projective schemes, Let S be a T -invariant projective scheme S in the general fiber, and let \tilde{S} denote its limit in the special fiber. Then the T -equivariant moment polytope of \tilde{S} coincides with that of S .*

Proof. For a T -equivariant flat family of projective varieties, the multi-graded Hilbert polynomial is constant. Then the Duistermaat-Heckman measure on \mathfrak{t}^* , being the leading order behavior of the multi-graded Hilbert polynomial, also stays constant. The moment polytopes, which is the support of the Duistermaat-Heckman measure on \mathfrak{t}^* , is constant as well.

For more details, see [15] as well as [12].

□

Corollary 1. *The central degeneration is T -equivariant. Therefore, the moment polytopes of a flat family of schemes in the global affine flag variety stay the same. In other words, the moment polytope for the special fiber limit is the convex hull of the limits of the vertices of the moment polytope for the general fiber.*

The lemma below is related to flat degenerations of toric varieties. It was proved in [37] and Theorem 6.6.18 in [28].

Lemma 2. *Consider a flat degeneration of toric varieties for a torus $T = (\mathbb{C}^*)^d$. Then the moment polytope of the special fiber is a regular subdivision of the moment polytope of the general fiber. Equivalently, this degeneration is characterized by a rational polyhedral complex in \mathbb{R}^d .*

There are two other related facts that we would like to recall.

Lemma 3. *Consider a flat family of schemes over a curve. Let S_1 and S_2 be two closed subschemes in the general fibers (which are isomorphic). Then the special fiber limit of the intersection of S_1 and S_2 , $S_1 \cap S_2$, is contained in the intersection of the special fiber limit of S_1 and the special fiber limit of S_2 .*

Lemma 4. *Consider a flat family of projective schemes in which the general fibers are irreducible of dimension d , then the special fiber is (set-theoretically) equi-dimensional of dimension d . In other words, each irreducible component of the special fiber has the same dimension d .*

Proof. This follows from Chapter 3, Corollary 9.6 of [15]. □

3.3 Dimensions and Polytopes

Now let's try to get more geometric information from the moment polytopes of certain T -invariant schemes in the affine Grassmannian or the affine flag variety. This will be useful for dimension estimations later.

Theorem 6. *Let S be an Iwahori orbit in the affine Grassmannian or the affine flag variety, with T -equivariant moment polytope P . The dimension of S equals to the size of the finite set $(v + \Lambda) \cap P$, where v is any vertex of P and Λ is the (affine) root lattice. This is independent of the choice of the vertex v .*

Proof. Pick any extremal T -fixed point p of S . Let v denote its moment map image, which is a vertex of the moment polytope of S . There is an open dense neighborhood O_p that is a product of root subgroup orbits $\prod_{i \in I} U_{\gamma_i} \cdot p$. Therefore, all the T -fixed points indexed by $s_{\gamma_i} \cdot v$ also lie in the moment polytope P . Since O_p is open in S , the dimension of S is the same as the size of the set I above. This is exactly the size of $v + \Lambda \cap P$ because S is invariant under $\text{Aut}(\mathbb{D})$. □

We have a similar theorem for MV cycles in the affine Grassmannian.

Theorem 7. *The dimension of an MV cycle of type A equals to the size of the finite set $(v + \Lambda) \cap P$, where v is any vertex of the corresponding MV polytope P and Λ is the (affine) root lattice. This is independent of the choice of the vertex v .*

Example 4. *Consider the case of $G = SL_2$. There is only one simple root. Each intersection $S_e^\lambda \cap S_{w_0}^\mu$ only has one irreducible component, where λ is an anti-dominant coweight, and μ is a coweight.*

By direct computation, we see that the size of the set $(v + \Lambda) \cap P$, where v is one of the two vertices of the MV polytope P , equals to $\text{height}(\mu - \lambda)$, which is the dimension of the corresponding MV cycle.

Example 5. By [1, 16], there is a finite set of prime MV polytopes whose Minkowski sums generate all the MV polytopes. In [1], all the prime MV polytopes for $G = SL_3$ were worked out.

Theorem 7 is true for the MV cycles that correspond to the SL_3 prime MV polytopes. Therefore, it is true for all the MV cycles in the SL_3 case.

Proof. This is true for $G = SL_2$. Let's prove this for $G = SL_n, n > 2$.

Suppose it is true for SL_{n-1} . We will prove the SL_n case by induction.

Let S be an MV cycle for SL_n that is an irreducible component in $S_e^\lambda \cap S_{w_0}^\mu$. Recall $\dim(S) = \text{height}(\mu - \lambda)$. Let P be its MV polytope. We could restrict S to an MV cycle, and P to an MV polytope for an embedded $SL_m \hookrightarrow SL_n, m \leq n$. If we write $\mu - \lambda$ as a sum of simple coroots, we see that $\dim(S)$ is a sum of the dimensions of the restrictions of S to all the different copies of $SL_2 \hookrightarrow SL_n$ corresponding to different simple coroots.

Let v be any vertex of P . Restrict P to an MV polytope P' containing v as a vertex, for a copy of $SL_{n-1} \hookrightarrow SL_n$. The highest and lowest vertices of P' are the restrictions of μ and λ respectively. Now let $\mu - \lambda = \alpha' + m\alpha_n$, where α' is a sum of simple coroots for this copy of SL_{n-1} and α_n is the additional simple coroot. Then the highest or lowest vertex of P' would have m more affine root directions contained in P , each of them is a multiple of α_n . Due to the geometry of convex polytopes in the weight lattice, there will also be m additional affine root directions from v contained in P that involves α_n .

Furthermore, the dimension of S is greater than the dimension of its restriction to this copy of SL_{n-1} by m . Therefore, the number of affine root directions from v contained in P equals to the dimension of S .

We could construct the restriction maps above by using the restriction map q_J introduced by Braverman-Gaitsgory [2] and further discussed in [17].

□

Now let's look at the moment polytopes of a few T -invariant projective schemes in the affine Grassmannian for $G = SL_3$. Things work completely analogously for T -invariant schemes in the affine flag variety.

Example 6. Consider the $G(\mathcal{O})$ orbit $Gr^{\alpha+\beta}$ in the affine Grassmannian with moment polytope P . This is also an Iwahori orbit and an MV cycle. We can calculate its dimension by calculating the size of the set $(\alpha + \beta + \Lambda) \cap P$, as shown in 3.1.

Example 7. Now consider the G orbit containing the T -fixed point $t^{\alpha+\beta}$ in the affine Grassmannian with moment polytope P . It is three-dimensional. On the other hand, the size of the set $(\alpha + \beta + \Lambda) \cap P$ is four. This is because G -orbits in the affine Grassmannian or the affine flag variety are not $\text{Aut}(\mathbb{D})$ invariant. For more details, see 3.2.

Example 8. Consider the closure of a generic T -orbit in the $G(\mathcal{O})$ orbit $Gr^{\alpha+\beta}$ in the affine Grassmannian with moment polytope P . This T -orbit is only two-dimensional, while the size of the set $(\alpha + \beta + \Lambda) \cap P$ is four. This is because different root-directions in the

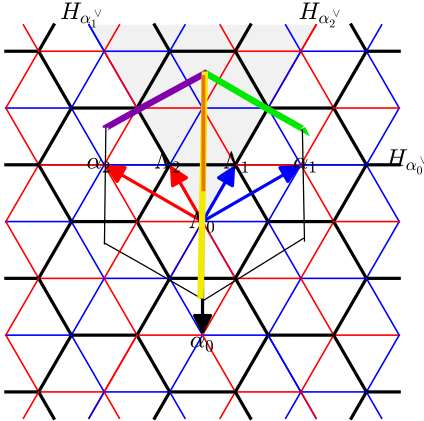


Figure 3.1: This is the moment polytope P of $Gr^{\alpha+\beta}$. The four lines of different colors that link the vertex $\alpha + \beta$ to four other elements in W_{aff} represent the orbits of four different root subgroups of $G(\mathcal{K})$. This corresponds to the dimension of $Gr^{\alpha+\beta}$.

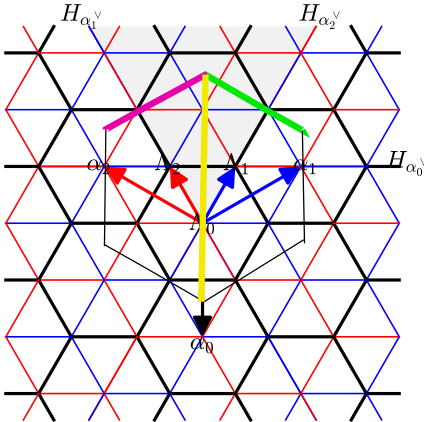


Figure 3.2: This is the moment polytope P of the three-dimensional ordinary flag variety containing the T -fixed point indexed by $\alpha + \beta$. The three lines of different colors that link the vertex $\alpha + \beta$ to three other points represent the orbits of three different root subgroups of G .

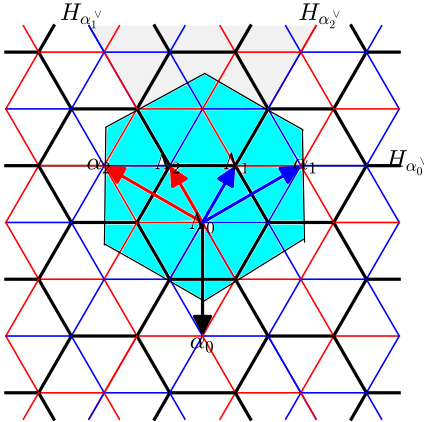


Figure 3.3: This is the moment polytope P of a generic T -orbit in $Gr^{\alpha+\beta}$. The number of orbits of root subgroups does not immediately indicate the dimension in this case.

same T -orbit may not have independent parameters. Another issue is that a T -orbit, just like a G -orbit, is not $\text{Aut}(\mathbb{D})$ -invariant. See 3.3 for illustration.

Finally, we can get an upper bound on the dimension of a T -invariant scheme from its moment polytope.

Lemma 5. *Let P denote the T -equivariant moment polytope of a T -invariant scheme S in the affine Grassmannian or the affine flag variety. Let v denote any vertex of P . Then the size of the set $(v + \Lambda) \cap P$ is greater than or equal to the dimension of S .*

Chapter 4

Results Related to Affine Deligne-Lusztig Varieties

Affine Deligne-Lusztig variety [35] is a generalization Deligne and Lusztig's classical construction [3]. It is very important to the study of Shimura varieties in number theory. In [11], the question of non-emptiness and dimension of affine Deligne-Lusztig varieties in the affine flag variety is reduced to questions related to the generalized MV cycles. Therefore, some geometric properties of the intersections of Iwahori orbits and semi-infinite orbits in the affine Grassmannian have interpretations in terms of representation theory and number theory.

We will first prove that the intersections of Iwahori orbits and U^- orbits in the affine Grassmannian are equi-dimensional, and give an explicit dimension formula. Then we will proceed to discuss some dimension bounds for the intersections of Iwahori orbits and U^- orbits in the affine flag variety.

4.1 Intersections of the Iwahori and U^- Orbits in the Affine Grassmannian

Let's first consider the intersection of the I -orbits with the U^- -orbits in the affine Grassmannian. This is the same as the intersections of the Iwahori orbits with the open MV cycles in the $G(\mathcal{O})$ orbits.

Theorem 8. *Let G be a connected reductive algebraic group of any type. Let λ and μ be coweights of G , and let λ_{dom} be the dominant coweight associated to λ . Let $\tilde{W} = W/W_I$ denote the quotient of the finite Weyl group associated to the partial flag variety $G/P_{\lambda_{dom}}$. Let $\lambda = w \cdot \lambda_{dom}$ for a unique $w \in \tilde{W}$, and X_w be the Schubert variety for $w \in W_I$.*

The intersection of the U^- orbit $S_{w_0}^\mu$ with the Iwahori orbit I^λ is equidimensional and of dimension

$$\text{height}(\lambda_{dom} + \mu) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)$$

when $\lambda \leq \mu \leq \lambda_{dom}$, and is \emptyset otherwise.

Proof. The group $G(\mathcal{O})$ acts transitively on its orbit Gr^λ . Consider the two subgroups I and $U_{\mathcal{O}}^- = U^- \cap G(\mathcal{O})$. Given a T -fixed point t^γ , there is an inclusion $U_{\mathcal{O}}^- \cdot t^\gamma \hookrightarrow S_{w_0}^\gamma \cap Gr^\gamma = U^- \cdot t^\gamma \cap Gr^\gamma$.

First consider the case when $\mu = \lambda_{dom}$.

Claim 1: We have the equalities $S_{w_0}^{\lambda_{dom}} \cap Gr^\lambda = U_{\mathcal{O}}^- \cdot t^{\lambda_{dom}} \cap Gr^\lambda = J^- \cdot t^{\lambda_{dom}}$. Here $J^- = ev_0^{-1}(N^-)$, where N^- is the unipotent radical of the opposite Borel B^- in G . These equalities hold only at a dominant coweight. This intersection is the pre-image of the open opposite Schubert cell X_e^- under the map ev_0 , and is a vector bundle over X_e^- .

Claim 2: We have the equality $S_{w_0}^{\lambda_{dom}} \cap I^\lambda = J^- \cdot t^{\lambda_{dom}} \cap I^\lambda$. This intersection is the pre-image of the open Richardson variety $X_e^- \cap X_w$ under the map ev_0 . The open Richardson variety $X_e^- \cap X_w$ has dimension $l(w)$, $w \in \tilde{W}$ and is dense in X_w . Therefore, $S_{w_0}^{\lambda_{dom}} \cap I^\lambda$ is dense in the Iwahori orbit I^λ and only has one component.

Then it follows that $S_{w_0}^\mu \cap I^\lambda$ is dense in I^λ with only one irreducible component. The dimension of the intersection is the same as the dimension of the Iwahori orbit I^λ itself, namely $2 \cdot \text{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)$.

On the other hand, let's consider the case $\mu = \lambda$.

Claim 3: $S_{w_0}^\lambda \cap Gr^\lambda = U_{\mathcal{O}}^- \cdot t^\lambda \cap Gr^\lambda$.

Proof of Claim 3:

U^- could be written as a product of some root subgroups U_a^- , a being an affine root of the form $-\alpha + j\delta$, for some $\alpha \in R_{Re}^+$.

If $j \geq 0$, U_a^- is a subgroup of $U_{\mathcal{O}}^-$;

If $j < 0$, then $U_a^- \cdot t^\lambda \cap Gr^\lambda$ equals to the T -fixed point t^λ .

Therefore $S_{w_0}^\lambda \cap I^\lambda = U_{\mathcal{O}}^- \cdot t^\lambda \cap I^\lambda$.

We know that $G(\mathcal{O})$ acts on Gr^λ transitively. Therefore, the tangent space at t^λ in Gr^λ is isomorphic to the following quotient of the loop subalgebra $L^+\mathfrak{g} = \bigoplus_{k \geq 0} \mathfrak{g}t^k$, $L^+\mathfrak{g}/Z$, where $Z = \{a \in L_{\mathfrak{g}}^+ | a \cdot t^\lambda = t^\lambda \cdot h \text{ for some } h \in L_{\mathfrak{g}}^+\}$.

There is a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{t} \oplus (\bigoplus_{\alpha} (\mathfrak{e}_{\alpha} \oplus \mathfrak{e}_{-\alpha}))$, where α ranges over all the positive roots.

The tangent space at t^λ in Gr^λ generated by the orbits of the Iwahori subgroup I is $\mathfrak{t} \oplus (\bigoplus_{\alpha} (\mathfrak{e}_{\alpha})) \oplus_{k > 0} \mathfrak{g}t^k/Z$. The tangent space at t^λ generated by the orbits of the subgroup $U_{\mathcal{O}}^-$ is $\bigoplus_{\alpha, k \geq 0} \mathfrak{e}_{-\alpha} \cdot t^k/Z$. As a result, the Iwahori and U^- orbits generate the tangent space at t^λ in the $G(\mathcal{O})$ orbit Gr^λ .

Therefore the intersection $I^\lambda \cap S_{w_0}^\lambda$ is transverse. By a transversality theorem in [19], it is equi-dimensional and each component has dimension $(\text{height}(\lambda_{dom} + \lambda)) + (2 \cdot \text{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)) - 2 \cdot \text{height}(\lambda_{dom}) = \text{height}(\lambda_{dom} + \lambda) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)$.

Now we know the theorem holds for the extreme cases $\mu = \lambda_{dom}$ and $\mu = \lambda$. Let's consider coweights μ such that $\lambda < \mu < \lambda_{dom}$.

In the affine Grassmannian Gr , $\overline{S_{w_0}^\gamma} = \cup_{\eta \leq \gamma} S_{w_0}^\eta$. Given a projective embedding of Gr , for each U^- orbit $S_{w_0}^\mu$, its boundary is given by a hyperplane section H_μ .

This means that given two coweights μ_1 and μ_2 such that $\text{height}(\mu_2) = \text{height}(\mu_1) - 1$, and given any irreducible component C_2 of $I^\lambda \cap S_{w_0}^{\mu_2}$, there is an irreducible component C_1 of $I^\lambda \cap S_{w_0}^{\mu_1}$ such that $C_2 \cap H_\mu$ is dense in C_1 . Dimension of C_1 is bigger than or equal to $\dim(C_2) - 1$, as C_1 is cut out by a hyperplane.

The difference of $\dim(S_{w_0}^{\lambda_{dom}} \cap I^\lambda)$ and $\dim(S_{w_0}^\lambda \cap I^\lambda)$ is exactly $\text{height}(\lambda_{dom} - \lambda)$. For $\lambda < \mu \leq \lambda_{dom}$, whenever the height of μ decreases by 1, the dimension of the intersection $S_{w_0}^\mu \cap I^\lambda$ has to also decrease by 1.

As a result, we know that for $\lambda \leq \mu \leq \lambda_{dom}$, the intersections $S_{w_0}^\mu \cap I^\lambda$ are equidimensional and we have the dimension formula as stated in the theorem. □

Remark 1. *In [36], it was shown the number of top-dimensional irreducible components in the intersections of Iwahori orbits and U^- orbits in the affine Grassmannian is equal to the dimensions of different weight spaces in some Demazure modules for the Langlands dual group. The main original contribution here is the proof for equi-dimensionality. Also, the methods used to derive a dimension formula are different.*

4.2 Intersections of Iwahori Orbits with U^- Orbits in the Affine Flag Variety

Now we are going to discuss some dimension estimates for the intersections of Iwahori orbits and U^- orbits in the affine flag variety of type A.

Lemma 6. *Let $L_{1,\cdot} = (L_0, L_1, \dots, L_{n-1})$ and $L_{2,\cdot} = (L'_0, L'_1, \dots, L'_{n-1})$ be two sequences of coordinate lattices in the affine flag variety of type A. Each L_i and L'_i corresponds to a coweight μ_i and μ'_i for GL_n . The sequences of lattices $L_{1,\cdot}$ and $L_{2,\cdot}$ correspond to two T -fixed points indexed by $w_1, w_2 \in W_{aff}$.*

Then the intersection $I^{w_1} \cap S_{w_0}^{w_2} \neq \emptyset$ only if $I^{\mu_i} \cap S_{w_0}^{\mu'_i} \neq \emptyset$ in the i -th affine Grassmannian for all $i = 0, \dots, n - 1$.

If the intersection is indeed nonempty, then this intersection is contained in the G/B -bundle above the intersection $I^{\mu_0} \cap S_{w_0}^{\mu'_0}$ in the 0-th affine Grassmannian Gr_0 .

Proof. The affine flag variety of type A has natural projection maps $p_i, i \in \{0, 1, \dots, n - 1\}$ to the n copies of affine Grassmannians. The lattices in these different copies of affine Grassmannians have different relative positions. If $I^{w_1} \cap S_{w_0}^{w_2} \neq \emptyset$, then its image under the projection map p_i should be nonempty too.

For the last statement, I^{w_1} and $S_{w_0}^{w_2}$ lie in the G/B bundle above I^{μ_0} and $S_{w_0}^{\mu'_0}$ respectively. Therefore the intersection $I^{w_1} \cap S_{w_0}^{w_2}$ lies in the G/B bundle above $I^{\mu_0} \cap S_{w_0}^{\mu'_0}$. □

Corollary 2. *Let $w_1 = (w \cdot \lambda_{dom}, w')$, $w_2 = (\mu, w'')$ be two elements in the affine Weyl group for G . Let \tilde{W} denote the quotient of Weyl group associated with the partial flag variety $G/P_{\lambda_{dom}}$, $w', w'' \in W, w \in \tilde{W}$, λ_{dom} be a dominant coweight, and μ be a coweight.*

The dimension of the intersection of the U^- orbit $S_{w_0}^{w_2}$ with the Iwahori orbit I^{w_1} is bounded above by

$$\text{height}(\lambda_{dom} + \mu) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w) + \dim(G/B)$$

Proof. By Lemma 6, the intersection of an Iwahori orbit and a U^- orbit lie in the G/B bundle above the projection of the intersection to the 0-th affine Grassmannian Gr_0 . Therefore, the dimension of the intersection $I^{w_1} \cap S_{w_0}^{w_2}$ in the affine flag variety is less than or equal to the sum of the dimension of the intersection in Gr_0 and the dimension of the fiber G/B . \square

Below is a sharper bound on the dimensions of the intersection of an Iwahori orbit and a U^- orbit in the affine flag variety.

Theorem 9. *Let $w_1 = (w \cdot \lambda_{dom}, w')$, $w_2 = (\mu, w'') \in W_{aff}$, \tilde{W} denote the quotient of Weyl group associated with the partial flag variety $G/P_{\lambda_{dom}}$, $w', w'' \in W, w \in \tilde{W}$, λ_{dom} be a dominant coweight and μ be a coweight.*

The dimension of $I^{w_1} \cap S_{w_0}^{w_2}$ in the affine flag variety for G is less than or equal to

$$\begin{aligned} & \text{height}(\lambda_{dom} + \mu) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w) + \\ & \begin{cases} \dim(G/B) - l(w') & \text{for } w' = w'' \in W \\ l(w') - l(w'') & \text{for } w' > w'' \in W \end{cases} \end{aligned}$$

Proof. The intersection $I^{w_1} \cap S_{w_0}^{w_2}$ is contained in the G/B -bundle above the intersection $I^{L_0} \cap S_{w_0}^{L'_0}$ in the 0-th affine Grassmannian. The dimension of $I^{L_0} \cap S_{w_0}^{L'_0}$ is $\text{height}(\lambda_{dom} + \mu) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)$.

In the G/B bundle above each point in $I^{L_0} \cap S_{w_0}^{L'_0}$, a U^- orbit is the same as the B^- orbit containing w_2 , and an I orbit is a subset of the product of the B^- orbit and B orbit containing w_1 .

When $w_1 = w_2$, the intersection in the G/B fiber above any point is a subset of the B^- orbit containing the T -fixed point indexed by $w_1 = w_2$, whose dimension is given by $\dim(G/B) - l(w_1)$. When $w_1 > w_2$, the intersection in the G/B fiber above any point is a subset of the Richardson variety $X_{w_1} \cap X_{w_2}^-$, whose dimension is $l(w_1) - l(w_2)$. When $w_1 < w_2$, the intersection is empty. \square

So far we have studied the dimensions intersections of the Iwahori orbits and U^- orbits in the affine flag variety by focusing on the fact that the affine flag variety is a G/B bundle over the affine Grassmannian.

On the other hand, we could also think more about the picture of alcoves. In previous sections, we developed some techniques of calculating/estimating the dimensions of certain projective schemes in the affine Grassmannian or the affine flag variety by looking at T -equivariant moment polytopes.

Conjecture 3. *Let A be the closure of an irreducible component in the intersection of an Iwahori orbit and a U^- orbit in the affine flag variety $I^{w_1} \cap S_{w_0}^{w_2}$. Let P denote the T -equivariant moment polytope of A . Then the dimension of A is the same as the size of the set $(v + \Lambda) \cap P$, where v is any vertex of P . This is independent of the choice of the vertex v of P .*

As a next step, we hope to employ more moment polytope techniques to study general affine Deligne-Lusztig varieties.

Chapter 5

Central Degeneration of the Basic Building Blocks

5.1 Degenerations of One-Dimensional Subschemes

We would like to start by considering the degenerations of one-dimensional subvarieties in the affine Grassmannian, with the base curve being \mathbb{A}^1 .

5.1.1 Degenerations of Extended-torus Invariant \mathbb{P}^1 s

In the affine Grassmannian and the affine flag variety, there are discretely many \mathbb{P}^1 s that are invariant under the action of the extended torus, as expressed in the lemma below [8].

Lemma 7. *Given an affine root α , we have a one parameter subgroup U_α in the loop group LG generated by the exponential of e_α in the loop algebra $L\mathfrak{g}$.*

In Gr and Fl , every $T \times \mathbb{C}^$ -invariant \mathbb{P}^1 is the closure of one orbit of U_α , for some affine root α . In particular, there is a discrete number of one-dimensional $T \times \mathbb{C}^*$ orbits in Gr and Fl .*

There is an explicit description of the T -fixed points in an orbit of a root subgroup U_α [8].

Lemma 8. *Let p be a T -fixed point in the affine Grassmannian or the affine flag variety, $\alpha = \alpha_0 + k\delta$ be an affine root, where α_0 is a root of G , $k \in \mathbb{Z}$, and δ is the imaginary root.*

Let s_α be the simple reflection in the affine Weyl group for the hyperplane H_α , where $H_\alpha = \{\beta \mid \langle \beta, \alpha \rangle = k\}$. Then

$$\lim_{\eta \rightarrow \infty} \exp(\eta \cdot e_\alpha) \cdot \gamma = s_\alpha \cdot \gamma.$$

In other words, there is a unique one-dimensional U_α orbit whose closure is a \mathbb{P}^1 connecting p and the T -fixed point indexed by $s_\alpha \cdot \gamma$.

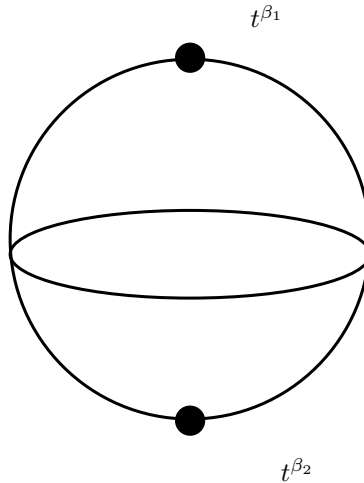


Figure 5.1: An orbit of a root subgroup in the affine Grassmannian that connects the two T -fixed points t^{β_1} and t^{β_2} .

We also know how each T -fixed point degenerate, as expressed in the lemma below [41].

Lemma 9. *Under the central degeneration, the image of the T -fixed point indexed by $\beta \in X_*(T)$ in the affine Grassmannian is the T -fixed point indexed by $(\beta, e) \in W_{aff}$ in the affine flag variety.*

Now we would like to explain the central degenerations of all the \mathbb{P}^1 s that are invariant under the extended torus.

Theorem 10. *As $\epsilon \rightarrow 0$, the limit of any $T \times \mathbb{C}^*$ -invariant $\mathbb{P}^1 \subset Gr \times \{id\}$ in the special fiber is two copies of \mathbb{P}^1 intersecting at one T -fixed point.*

More explicitly, consider the $T \times \mathbb{C}^$ -invariant \mathbb{P}^1 connecting the two T -fixed points t^{β_1} and t^{β_2} , where $\beta_1 > \beta_2$ as coweights. This is an orbit of U_α , where $\alpha = \beta_1 - \beta_2 = \alpha_0 + k\delta$ is an affine root.*

The special fiber limit of this \mathbb{P}^1 are the following two $T \times \mathbb{C}^$ -invariant \mathbb{P}^1 s intersecting at a point:*

- (1) *The \mathbb{P}^1 connecting the fixed points (β_1, e) and (β_1, s_{α_0}) ;*
- (2) *The \mathbb{P}^1 connecting the fixed points $(\beta_1, s_{\alpha_0}) = s_\alpha \cdot (\beta_2, e)$ and (β_2, e) .*

Proof. Consider the $T \times \mathbb{C}^*$ -invariant \mathbb{P}^1 connecting the two T -fixed points t^{β_1} and t^{β_2} in the affine Grassmannian, where $\beta_1 > \beta_2$. See 5.1 for the picture.

In the moduli interpretation for the central degeneration, when $\epsilon \neq 0$, this a \mathbb{P}^1 -family of principal G -bundles on \mathbb{A}^1 together with a trivialization away from $\{\epsilon\}$, times the choice of the standard B -reduction at the point 0. A trivialization of a G -bundle away from $\{\epsilon\}$ is the same as a section of the bundle on $\mathbb{A}^1 \setminus \{\epsilon\}$. We could allow different poles at ϵ .

When $\epsilon = 0$, we still have this \mathbb{P}^1 family of G -bundles with trivializations away from $\{\epsilon\}$, and we also have additional choices of standard B -reductions at $\epsilon = 0$. The difference is that when ϵ is 0, the B -reduction is no longer applied to the trivial G -bundle.

More explicitly, in the special fiber limit we have the following two \mathbb{P}^1 s intersecting at a point.

(1) The \mathbb{P}^1 connecting the T -fixed points (β_1, s_{α_0}) and (β_2, e) ; this represents the original family of trivializations away from $\epsilon = 0$. This \mathbb{P}^1 in the affine flag variety is the closure of an one-dimensional orbit of the same root subgroup U_{α}^- mentioned above, where $\alpha = \beta_2 - \beta_1$.

It is natural to also consider the \mathbb{P}^1 connecting the T -fixed points (β_1, e) and (β_2, s_{α_0}) . However, since the T -fixed point (β_2, s_{α_0}) does not lie in the line segment connecting (β_1, e) and (β_2, e) in the T -equivariant moment polytope, this \mathbb{P}^1 is not in the special fiber limit by Corollary 1.

(2) The \mathbb{P}^1 connecting the fixed points (β_1, e) and (β_1, s_{α_0}) . By Lemma 9, the limit of the T -fixed point indexed by β is the T -fixed point indexed by (β, e) . This extra \mathbb{P}^1 represents the extra choices of standard B -reductions at 0 for different nontrivial G -bundles near $0 \in \mathbb{A}^1$.

These two different copies of \mathbb{P}^1 intersect at the T -fixed point (β_1, s_{α_0}) .

No other T -fixed points in the affine flag variety should be in the special fiber limit. Consider a T -fixed point p indexed by $(\gamma, w) \in W_{aff}$. If $\gamma \neq \beta_1$ or β_2 , the p will not be in the closure of the limit because those t^γ is not in the closure of the original family to begin with. If $\gamma = \beta_1$ or β_2 but $w \neq e$ or s_{α_0} , then p is not in the special fiber limit by Corollary 1. This is because in the moment map image, p does not lie on the same line as the three T -fixed points indicated above. □

Now let's look at a few examples of the degenerations of \mathbb{P}^1 s that are invariant under the extended torus.

Example 9. Let $G = SL_2(\mathbb{C})$. Consider the $T \times \mathbb{C}^*$ -invariant \mathbb{P}^1 that connects the T -fixed points t^α and $t^{-\alpha}$ in the affine Grassmannian.

In 5.2, the upper diagram illustrates the \mathbb{P}^1 connecting t^α and $t^{-\alpha}$ in the affine Grassmannian. The lower diagram illustrates its limit in the affine flag variety, which consists of three T -fixed points and two \mathbb{P}^1 s. The three T -fixed points are indexed by $(\alpha, e), (\alpha, \sigma), (-\alpha, e) \in W_{aff}$, σ being the nontrivial element in the finite Weyl group S_2 .

Example 10. Let $G = SL_3(\mathbb{C})$. Let α and β denote the two positive simple coroots for $G = SL_3$. Consider the $T \times \mathbb{C}^*$ -invariant \mathbb{P}^1 that connects the T -fixed points $t^{\alpha+\beta}$ and t^e in the affine Grassmannian.

In 5.3, the upper diagram illustrates the \mathbb{P}^1 in the affine Grassmannian. The two T -fixed points are indexed by $\alpha + \beta$ and e respectively.

The lower diagram illustrates its limit in the affine flag variety, which consists of three T -fixed points and the two $T \times \mathbb{C}^*$ -invariant \mathbb{P}^1 s connecting them. The three T -fixed points are indexed by $(\alpha + \beta, e), (\alpha + \beta, w_0), (0, e) \in W_{aff}$.

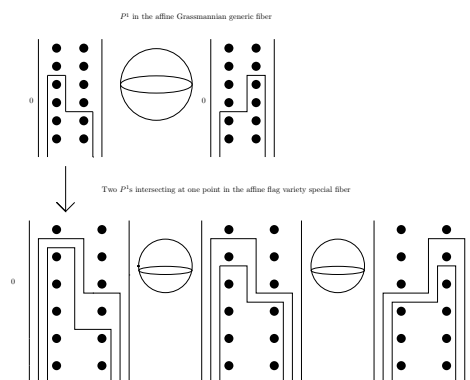


Figure 5.2: Central Degeneration of a $T \times \mathbb{C}^*$ -invariant \mathbb{P}^1 in the affine Grassmannian of $G = SL_2(\mathbb{C})$.

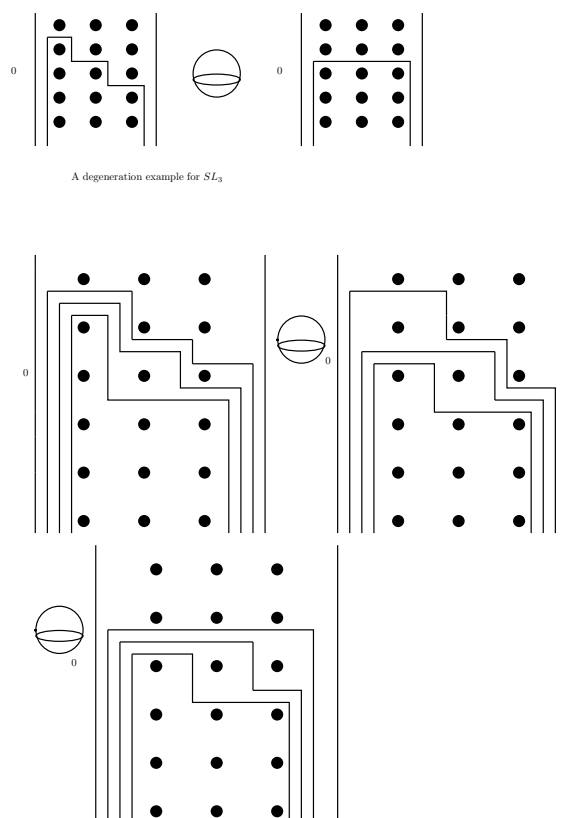


Figure 5.3: Central Degeneration of a $T \times \mathbb{C}^*$ -invariant \mathbb{P}^1 in the affine Grassmannian of $G = SL_3(\mathbb{C})$.

5.1.2 Degenerations of Other One-dimensional Subschemes

The extended-torus invariant \mathbb{P}^1 s studied in the previous subsection are very special. In general, the affine Grassmannian is a union of infinitely many T -orbits, T being the maximal torus in G .

The degeneration of a one-dimensional subscheme S in a generic T -orbit is more complicated. Since this degeneration is T -equivariant, the special fiber limit of S would still be a one-dimensional subscheme in a union of T -orbits. However, there could be a lot more T -fixed points and irreducible components in the special fiber limit.

Example 11. $G = SL_2(\mathbb{C})$. In this case the maximal torus $T \subset G$ is one-dimensional.

There are infinitely many orbits of the maximal torus $T \subset G$ connecting t^α and $t^{-\alpha}$ in the affine Grassmannian, α being a positive coroot. The limit of a generic closed T -orbit has more than two irreducible components. This is because we can choose the local coordinates of the original T -orbit so that its special fiber limit is arbitrarily close to any T -fixed point in between (α, e) and $(-\alpha, e)$ in the T -equivariant moment polytope.

Therefore, the special fiber limits of a generic closed T -orbit has more than three T -fixed points and more complicated geometric properties.

5.2 Degenerations of a Finite Product of Orbits of Root Subgroups

After discussing the degenerations of orbits of a single root subgroup, now let's consider things related to finitely many root subgroups. We would like to understand the degenerations of the closures of a product of finitely many root subgroup orbits which contain the same T -fixed point t^μ .

Theorem 11 below describes an important irreducible component in the special fiber limit. This is the only irreducible component that contains the T -fixed point indexed by (μ, e) , which is the limit of the T -fixed point indexed by μ in Gr .

Theorem 11. *Let $\overline{O}_\mu \subset Gr$ be the closure of a finite product of orbits of root subgroups $\prod_i U_{\alpha_i} \cdot t^\mu$. Here each $\alpha_i = \alpha_{0,i} + k_i\delta$ is an affine root. Let $\widetilde{\overline{O}}_\mu$ denote the special fiber limit of \overline{O}_μ . Then one of the irreducible components for $\widetilde{\overline{O}}_\mu$ is the product $\prod_i U_{\alpha'_i} \cdot t^{(\mu,e)}$. Here $\alpha'_i = \alpha_i$ when $\alpha_{0,i}$ is a positive root, and $\alpha'_i = \alpha_i + \delta$ when $\alpha_{0,i}$ is a negative root.*

Proof. From the lattice picture of this degeneration, we know that the special fiber limit of a product of m root subgroup orbits containing t^μ would have an m -dimensional neighborhood of $t^{(\mu,e)}$ that is invariant under $T \subset G$, and is also a product of root subgroup orbits containing $t^{(\mu,e)}$.

We need to treat the cases of positive α_0 and negative α_0 a bit differently. This is basically because this process is a flat T -equivariant degeneration, so the shapes of moment polytopes stay the same by Corollary 1.

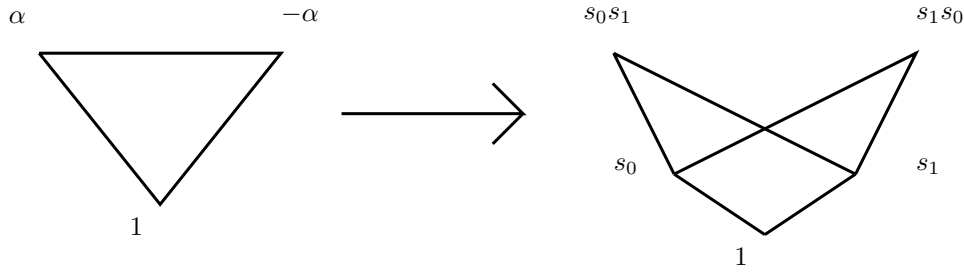


Figure 5.4: Central Degeneration of $Gr^\alpha, G = SL_2(\mathbb{C})$ as illustrated in terms of $T \times \mathbb{C}^*$ -equivariant moment polytopes.

More explicitly, by Lemma 8, in the affine flag variety, the orbit of the root subgroup $U_\alpha, \alpha = \alpha_0 + k\delta$ connects a translation T -fixed point (μ', e) to another T -fixed point (μ'', s_{α_0}) .

When α_0 is positive, $\mu' < \mu''$ and (μ'', s_{α_0}) is contained in the moment polytope line connecting (μ, e) and (μ'', e) .

When α_0 is negative, $\mu' > \mu''$ and (μ'', s_{α_0}) is not contained in the moment polytope connecting (μ, e) and (μ'', e) . On the other hand, the orbit $U_{\alpha+\delta} \cdot t^{(\mu, e)}$ is contained in the special fiber limit.

□

Example 12. Consider the case of the closed orbit of a single root subgroup U_α . This \mathbb{P}^1 is a union $U_{-\alpha} \cdot t^{\beta_1} \cup U_\alpha \cdot t^{\beta_2}$.

The limit contains a union of two \mathbb{P}^1 s intersecting at a point p : $U_{-\alpha+\delta} \cdot t^{(\beta_1, e)} \cup U_\alpha \cdot t^{(\beta_2, e)} \cup \{p\}$.

Example 13. This example is about the limit of the closure of a product of orbits of two root subgroups containing the same T -fixed point.

Consider the $G(\mathcal{O})$ orbit Gr^α for $G = SL_2$. It is a union of the products of root subgroup orbits $U_\alpha \cdot t^{-\alpha} \times U_{\alpha+\delta} \cdot t^{-\alpha}, U_{-\alpha} \cdot t^{-\alpha} \times U_{-\alpha+\delta} \cdot t^{-\alpha}$, and the T -fixed point e in the affine Grassmannian.

Its special fiber limit consists of two irreducible components. The closed Iwahori orbit $s_0 s_1$ is the closure of $U_{\alpha+\delta} \cdot t^{(-\alpha, e)} \times U_{\alpha+2\delta} \cdot t^{(-\alpha, e)}$. The closed Iwahori orbit $s_1 s_0$ is the closure of $U_{-\alpha} \cdot t^{(\alpha, e)} \times U_{-\alpha+\delta} \cdot t^{(\alpha, e)}$.

This agrees with Theorem 11.

This degeneration is illustrated in the transformation of $T \times \mathbb{C}^*$ -equivariant moment polytopes (not the T -equivariant moment polytopes) in 5.4.

Chapter 6

Central Degeneration of Semi-infinite Orbits

Let $U = N(\mathcal{K})$, $U_w = wUw^{-1}$ for some $w \in W$, $U^- = w_0Uw_0^{-1}$. Each group U_w is an infinite product of root-subgroups. A semi-infinite orbit S_w^γ is an orbit of U_w for some $w \in W$. Orbits of U_w in Gr or Fl are indexed by T -fixed points.

6.1 Closure Relations of Different Semi-infinite Orbits

We describe the closure relations of semi-infinite orbits in the affine Grassmannian and the affine flag variety. We will focus on the orbits of U^- , but the closure relations of orbits of other U_w could be worked out in completely similar ways.

The description in the alcove picture [20] works for all types of G . The description in the lattice picture was discovered by the author, and only works for type A.

Lemma 10 (Alcove Picture). *The closure relation of U^- orbits (indexed by elements in the affine Weyl group) for SL_2 is given below:*

$$\cdots < s_1s_0s_1 < s_1s_0 < s_1 < 1 < s_0 < s_0s_1 < s_0s_1s_0 < \cdots .$$

For general SL_n , an affine root determines a copy of the affine symmetric group \tilde{S}_2 , which is the affine Weyl group for SL_2 . Pictorially this generates a line in the diagram of alcoves.

For each $w \in W_{aff}$, $w' \leq w$ iff w' is contained in a closed cone containing w that is generated by affine roots $\alpha = \alpha_0 + k\delta$, where α_0 is a negative ordinary root.

This would also be true for other types of G .

Lemma 11 (Lattice Picture for type A). *Let $G = SL_n(k)$ or $GL_n(k)$, k being an algebraically closed field. Let $L = (L_0 \supset L_1 \supset \cdots \supset L_{n-1} \supset t \cdot L_0)$ and $L' = (L'_0 \supset L'_1 \supset \cdots \supset L'_{n-1} \supset t \cdot L'_0)$ be two sequences of coordinate lattices. L, L' represent elements $\gamma, \gamma' \in W_{aff}$, which indexes all the T -fixed points in the affine flag variety. Similarly, L_i, L'_i represent coweights $\eta_i, \eta'_i \in X_*(T)$, which indexes all the T -fixed points in the affine Grassmannian.*

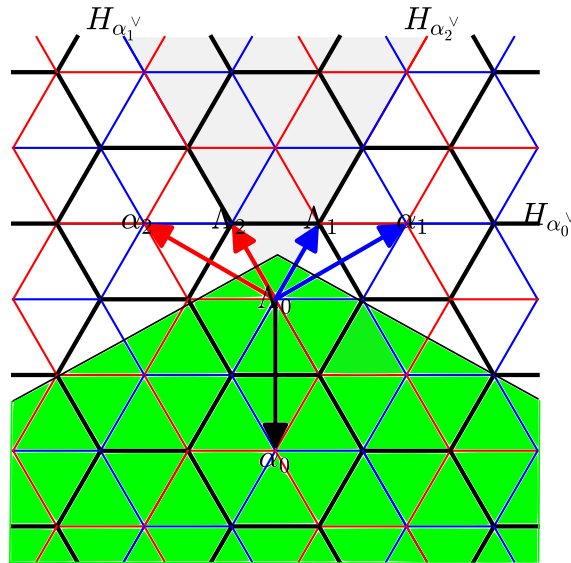


Figure 6.1: The moment map image of an infinite-dimensional U^- orbit in the affine flag variety for $G = SL_3$.

Then $\gamma \leq \gamma'$ in the ordering relations for U^- orbits in the affine flag variety if and only if $\eta_i \leq \eta'_i$ in the coweight lattice $\forall i = 0, \dots, n - 1$.

6.1 is the moment map image of the U^- orbit containing the T -fixed point indexed by $\text{id} \in W_{\text{aff}}$ in the affine flag variety for $G = SL_3$.

6.2 Degenerations

Theorem 12. *Given any closed $U_w = wUw^{-1}$ orbit S_w^μ in the affine Grassmannian of type A , its special fiber limit is the corresponding closed U_w orbit $S_w^{(\mu, e)}$ in the affine flag variety. Here μ is an element of the coweight lattice, and (μ, e) is a translation element of the affine Weyl group.*

Proof. We would like to show that the limit of the closure of the U^- orbit $\overline{S_{w_0}^\mu}$ in Gr , $\Phi(\overline{S_{w_0}^\mu})$, is the closed U^- orbit $S_{w_0}^{(\mu, e)}$ in Fl . Similar techniques should work for the orbits of other U_w .

Note that U^- is an infinite product of root subgroups corresponding to affine roots of the form $\alpha_0 + k\delta$, α_0 being a negative root.

First note that $\Phi(\overline{S_{w_0}^\mu})$ and $S_{w_0}^{(\mu, e)}$ contain the same T -fixed points and have the same moment map images, as this degeneration is flat and T -equivariant.

The closed U^- orbit $\overline{S_{w_0}^\mu}$ in Gr is the union of $S_{w_0}^{\mu'}$, where $\mu' \leq \mu$. By Theorem 11, we know that the limit of the closure of a finite product of root subgroups orbits containing

the T -fixed point $\mu \in X_*(T)$ has the closure of a finite product of root subgroups orbits containing the T -fixed point $(\mu, e) \in W_{aff}$ as one irreducible component. Taking an infinite limit of this theorem, we know that $\Phi(\overline{S_{w_0}^\mu})$ contains the closure of $S_{w_0}^{(\mu, e)}$.

Therefore $\overline{S_{w_0}^{(\mu, e)}} \subseteq \Phi(\overline{S_{w_0}^\mu})$.

On the other hand, suppose there is a point $p \in \Phi(\overline{S_{w_0}^\mu})$ such that p is not contained in $\overline{S_{w_0}^{(\mu, e)}}$. Let O_p denote the T -orbit containing p . Since $\overline{S_{w_0}^{(\mu, e)}}$ is closed, $\overline{S_{w_0}^{(\mu, e)}} \cap O_p = \emptyset$.

This degeneration is T -equivariant, so $\overline{O_p}$ lies in the special fiber limit of the closure of a T -orbit O in $\overline{S_{w_0}^\mu} \subset Gr$. There is a U^- orbit $S_{w_0}^\theta$, $\theta \leq \mu \in X_*(T)$, so that $O \subset S_{w_0}^\theta$. The moment polytope of $\overline{O_p}$ is a convex polytope in a regular subdivision of the moment polytope of \overline{O} . This implies that there is a T -fixed point $q \in \overline{O_p}$ indexed by $\beta \in W_{aff}$ such that $\beta \leq (\theta, e)$ (according to the semi-infinite Bruhat order), and the moment map image of the O_p is contained in the moment map image of the U^- orbit containing β , $S_{w_0}^\beta$.

Since q is contained in $\overline{O_p}$, O_p must be contained in the $S_{w_0}^\beta$, which is then contained in $\overline{S_{w_0}^{(\mu, e)}}$. We have arrived at a contradiction.

Therefore, $\Phi(\overline{S_{w_0}^\mu}) \subseteq \overline{S_{w_0}^{(\mu, e)}}$.

The argument for the orbits of wUw^{-1} for some other finite Weyl group element w is completely analogous. □

There is a more structural way to look at the degenerations of orbits of U_w . Just like in the case of $G(\mathcal{O})$ orbits [7], we construct a global version of U_w that acts on our family.

Lemma 12. *There exists a global version of $U_w = N_w(\mathcal{K})$ over \mathbb{A}^1 , U_{w, \mathbb{A}^1} , defined as follows.*

U_{w, \mathbb{A}^1} maps a scheme S to the set

$\{p \times S \in \mathbb{A}^1 \times S, \text{ the jet group of algebraic maps } f_p : \mathbb{A}^1 \times S \rightarrow G \text{ s.t. } f_p(\hat{p} \setminus \{p\} \times S) \subseteq N_w\}$.

Here $\hat{p} \setminus \{p\}$ is isomorphic to the formal punctured disc for each point $p \in C$.

This global group scheme acts on our family in the global affine flag variety $Fl_{\mathbb{A}^1}$.

Let Φ_1 denote the nearby cycles functor for Gaitsgory's central sheaves: $\Phi_1 : \mathcal{P}_{G(\mathcal{O})}(Gr_G) \rightarrow \mathcal{Z}(\mathcal{P}_I(Fl_G))$. Let Φ_2 denote the weight functor from $\mathcal{P}_{G(\mathcal{O})}(Gr_G)$ to $\mathcal{P}_{T(\mathcal{O})}(Gr_T)$, which is isomorphic to the tensor category of highest weight representations of $G^\vee, Rep(Gr^\vee)$. Let Φ_3 denote a similar cohomology functor on $\mathcal{P}_I(Fl_G)$. The conjecture of Xinwen Zhu below connects Gaitsgory's central sheaves to geometric Satake correspondence. We expect to use Theorem 12 and Lemma 12 above to prove it.

Conjecture 4. *The following diagram commutes:*

$$\begin{array}{ccc}
 \mathcal{P}_{G(\mathcal{O})}(Gr_G) & \xrightarrow{\Phi_1} & \mathcal{Z}(\mathcal{P}_I(Fl_G)) \\
 & \searrow \Phi_2 & \swarrow \Phi_3 \\
 & \mathcal{P}_{T(\mathcal{O})}(Gr_T) \cong Rep(Gr^\vee) &
 \end{array}$$

Chapter 7

Central Degeneration of Iwahori Orbits and MV Cycles in Type A

7.1 Central Degeneration of Iwahori Orbits and MV Cycles: SL_2 Case

When $G = SL_2$, MV cycles and Iwahori orbits are a bit simpler than the general case. Each MV cycle is just the closure of $Gr^\lambda \cap S_{w_0}^\mu$ or $S_e^{-\lambda} \cap S_{w_0}^\mu$, where λ is a dominant coweight and μ is a coweight. For Iwahori orbits I^γ in the affine Grassmannian, $\gamma \in X_*(T)$, there are basically two types: I^γ is the $G(\mathcal{O})$ orbit $Gr^{-\gamma}$ if γ is anti-dominant; it is the closure of a vector space (vector bundle over a point) if γ is dominant. Each closed Iwahori orbit is also an MV cycle in this very special case.

Theorem 13. *Let $G = SL_2$. Let S be a d -dimensional MV cycle. Note that $S = \overline{Gr^\lambda \cap S_{w_0}^\mu}$ and $\overline{S_e^{-\lambda} \cap S_{w_0}^\mu}$, where λ is a dominant coweight and μ is a coweight.*

The special fiber limit of S , \tilde{S} , is a union of open intersections of Iwahori orbits I^{w_1} and U^- orbits $S_{w_0}^{w_2}$ such that $w_1 \in W_{aff}$ is λ -admissible, and $w_2 \leq \mu$ according to the semi-infinite Bruhat order for U^- and is also λ -admissible. Each such intersection is an (open) generalized MV cycle.

Moreover, \tilde{S} consists of $2d + 1$ T -fixed points and $2(d - k)$ k -dimensional (open) generalized MV cycle for $k > 0$. In particular there are two top-dimensional (open) generalized MV cycle whose closures give rise to the two irreducible components.

In addition, \tilde{S} is also equal to the intersection of two closed semi-infinite orbits in the affine flag variety, $\overline{S_e^{(-\lambda, e)}} \cap \overline{S_{w_0}^{(\mu, e)}}$. One of its irreducible components contains the T -fixed point $(-\lambda, e)$, and is contained in $\overline{S_e^{(-\lambda, e)}}$. The other irreducible component contains the T -fixed point (μ, e) , and is contained in $\overline{S_{w_0}^{(\mu, e)}}$.

Proof. The special fiber limit of an MV cycle S , \tilde{S} , is contained in the intersection of the special fiber limits of $\overline{Gr^\lambda}$ and $\overline{S_{w_0}^\mu}$, which we already understand. Therefore, we need to

consider the intersections of Iwahori orbits I^{w_1} and U^- orbits $S_{w_0}^{w_2}$ with the extra conditions $w_1 \in W_{aff}$ is λ -admissible, and $w_2 \leq \mu$ in the semi-infinite Bruhat order and is also λ -admissible.

Through the technique of alcove walks in [31], we could directly compute the dimensions of all the relevant intersections of Iwahori orbits and U^- orbits and get explicit dimension formula for SL_2 . There are only two relevant intersections of Iwahori orbits with U^- orbits that are of the dimension d . From the lattice picture of this degeneration, we know that these two top-dimensional intersections must be in the special fiber limit of an MV cycle.

All other such intersections are of lower dimensions, and are contained in the special fiber limit of smaller MV cycles in the closure of S . Therefore, the special fiber limit of S is equal to the intersection of the intersection of the special fiber limits of Gr^λ and $S_{w_0}^\mu$.

Similarly the special fiber limit of S is equal to the intersection of the special fiber limits of $\overline{S_e^{-\alpha}}$ and $\overline{S_{w_0}^\mu}$. By dimension arguments, each nonempty intersection of an open U orbit in $\overline{S_e^{(-\lambda, e)}}$ and a U^- orbit in $\overline{S_{w_0}^{(\mu, e)}}$ coincide with a relevant intersection of an Iwahori orbit and a U^- orbit discussed in the previous paragraphs. Therefore $\tilde{S} = \overline{S_e^{(-\lambda, e)}} \cap \overline{S_{w_0}^{(\mu, e)}}$. □

The above theorem also applies to the degenerations of Iwahori orbits in the affine Grassmannian for $G = SL_2$ as Iwahori orbits are special MV cycles in this case. We can say a bit more about the special fiber limits of Iwahori orbits. In fact, we will also discuss the special fiber limits of orbits of $I_\sigma, \sigma \in \mathbb{Z}_2$, which is the pre-image of the opposite Borel under the map that projects $G(\mathcal{O})$ to G by evaluating at $t = 0$.

Theorem 14. *Let I^γ be a closed Iwahori orbit of dimension d in the affine Grassmannian for SL_2 . Let I_σ^γ be a closed orbit of I_σ . If I^γ or I_σ^γ is a $G(\mathcal{O})$ orbit, then its special fiber limit is a union of γ -admissible Iwahori orbits in the affine flag variety.*

If I^γ is not a $G(\mathcal{O})$ orbit, then γ is dominant. Its special fiber limit consists of two irreducible components. One is the closure of the orbit of the subgroup H containing the T -fixed point indexed by $(\gamma, e) \in W_{aff}$, where H is the subgroup of I generated by $I_2 + (U \cap G(\mathcal{O}))$. The other is the closure of the Iwahori orbit containing the T -fixed point indexed by $(s_0 s_1)^{\frac{d-1}{2}} s_0 \in W_{aff}$.

If I_σ^γ is not a $G(\mathcal{O})$ orbit, then γ is anti-dominant. Its special fiber limit consists of two irreducible components: one is the closed I_1 orbit containing the T -fixed point indexed by $(\gamma, e) \in W_{aff}$, and the other is the closure of the Iwahori orbit containing the T -fixed point indexed by $(s_1 s_0)^{\frac{d-1}{2}} s_1 \in W_{aff}$.

Proof. From the discussion about the degenerations of MV cycles above, we can infer that the special fiber limit of an orbit of I or I_σ is the closure of the two irreducible components containing the two extremal T -fixed points. Each irreducible component is the closure of the orbit of a subgroup of I or I_σ . The exact descriptions of these components are deduced from the moment polytopes. □

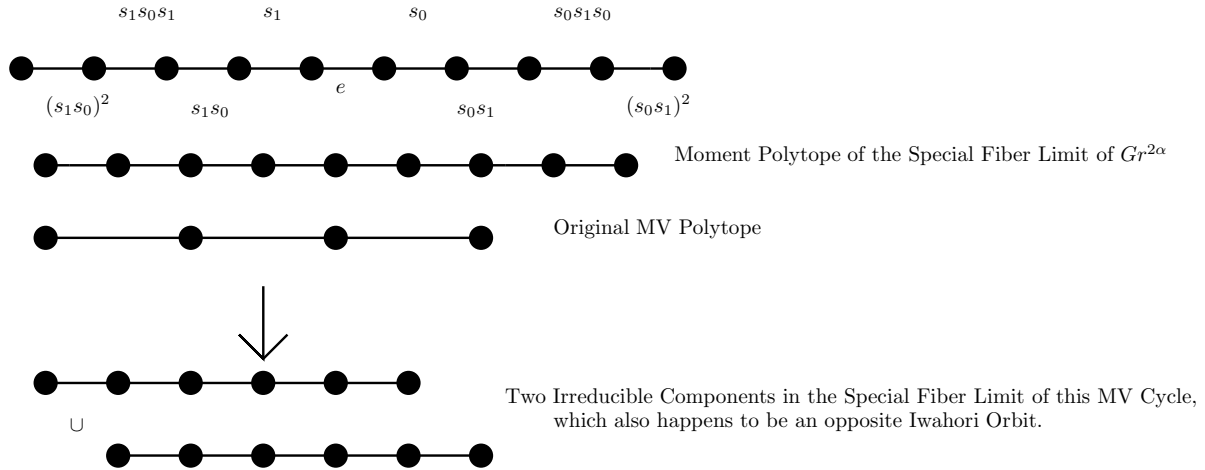


Figure 7.1: Central Degeneration of an MV Cycle/Opposite Iwahori orbit for $G = SL_2$.

Example 14. Consider the MV cycle $\overline{Gr^{2\alpha}} \cap S_{w_0}^\alpha$ for $G = SL_2$. This is also the opposite Iwahori orbit indexed by $-2\alpha \in X_*(T)$.

This degeneration is illustrated in terms of moment polytopes in 7.1.

Note that the special fiber limit of this opposite Iwahori orbit $I_\sigma^{-2\alpha}$ has two irreducible components. One is the orbit of I_1 containing the T -fixed point indexed by $-2\alpha = (s_1s_0)^2$, and the other is the standard Iwahori orbit indexed by $s_1s_0s_1$.

Similarly we can consider the special fiber limit of the Iwahori orbit $I^{2\alpha}$. One irreducible component is the closed orbit of the subgroup H of I . This orbit contains the T -fixed point indexed by $2\alpha = (s_0s_1)^2 \in W_{aff}$. Each matrix in H has the form: $\begin{bmatrix} 1 & \mathcal{O} \\ t^2 + a_3t^3 + \dots & 1 \end{bmatrix}$. The other irreducible component is the Iwahori orbit containing the T -fixed point indexed by $s_0s_1s_0$.

7.2 Central Degeneration of Iwahori Orbits in Type A

Let K be a maximal compact subgroup of G . Topologically, this degeneration could be understood as the sequence of natural maps $\Omega K \hookrightarrow LK \rightarrow LK/T$, where ΩK is the based loop group of K , and LK is the corresponding loop group of K . As a result, there is an $H_T(\Omega K)$ -module structure on $H^T(LK)$. This action with respect to the Schubert bases was studied by [22, 32], and is related to affine Stanley coefficients [21]. These coefficients also show up in Schubert structure constants for Pontryagin product of the cohomology of the ordinary Grassmannian and 3-point Gromov-Witten invariants for the ordinary flag variety [25].

Consider generalized Iwahori subgroups of $G(\mathcal{O})$, $I_w = ev_0^{-1}(wBw^{-1})$, $w \in W$. When w is the identity, we recover our standard Iwahori subgroup of $G(\mathcal{O})$. In this section we are

going to discuss the central degeneration of the orbits of $I_w, w \in W$. This is a geometric analog of the topological story discussed above.

Theorem 15. *Let S be a T -invariant projective scheme S in the affine Grassmannian that satisfies the following condition: each extremal T -fixed point $t^{\mu_{ex}}$ has an open neighborhood $O_{\mu_{ex}}$ that is isomorphic to a product of finitely many orbits of distinct root subgroups containing $t^{\mu_{ex}}$. Let \tilde{S} denote the special fiber limit of S . Then the number of irreducible components of \tilde{S} is bounded below by the number of vertices of the T -equivariant moment polytope P of S .*

Proof. By Theorem 11, the special fiber limit of the closure of each $O_{\mu_{ex}}$ has a component that is also contained an orbit of a finite product of root subgroups containing the T -fixed point (μ, e) .

For two distinct $O_{\mu_{ex}}$, their limit components described above are distinct cells. The T -orbits contained in these cells do not belong to the same irreducible component. Therefore these cells corresponding to different extremal T -fixed points are distinct generators of the top-dimensional cohomology of \tilde{S} . Consequently, the number of irreducible components of \tilde{S} is bounded below by the number of vertices of P . □

Corollary 3. *Let S be an Iwahori orbit or a Borel orbit with T -equivariant moment polytope P . Then the number of irreducible components of the special fiber limit of S is bounded below by the number of vertices in P .*

Proof. Closed Iwahori orbits and closed Borel orbits both satisfy the following property: each extremal T -fixed point $t^{\mu_{ex}}$ has an open neighborhood $O_{\mu_{ex}}$ that is isomorphic to a finite product of distinct root subgroups orbits containing $t^{\mu_{ex}}$. They are both closures of a finite product of root subgroups. □

After getting a lower bound for the number of irreducible components for the special fiber limit of an Iwahori orbit, we would also like to give an upper bound.

Theorem 16. *Let $S = I_w^\lambda$ be a generalized Iwahori orbit, where $w \in W, \lambda \in X_*(T)$. Let P denote the moment polytope of S . Let \tilde{S} denote the special fiber limit of S with moment polytope \tilde{P} . \tilde{P} is the convex hull of the limits of the vertices of P .*

The number of irreducible components for \tilde{S} is bounded above by the number of convex polytopes \tilde{P} that satisfy a few extra conditions. Firstly, the vertices of each polytope corresponds to a T -fixed point of the affine flag variety. Secondly, if two vertices of the polytope lie on the same line, then there is an orbit of a root subgroup that connects these two extremal T -fixed points. Lastly, the size of the finite set $(v + \Lambda) \cap \tilde{P}$ (Λ is the root lattice) at each vertex v is greater than or equal to the dimension of S .

Proof. Different irreducible components of \tilde{S} have distinct moment polytopes. They correspond to distinct polytopes in \tilde{P} .

The moment polytope of any irreducible component of \tilde{S} has to satisfy the first two conditions. The last condition on the moment polytope is a corollary of Lemma 5. □

Conjecture 5. *The number of irreducible components in the special fiber limit of a generalized Iwahori orbit is a sum of affine Stanley coefficients.*

As an analogy to the global version of the group $G(\mathcal{O})$, we consider a global group scheme that acts on a family of Iwahori orbits. More explicitly, the special fiber limit of an Iwahori orbit is invariant under the action of a smaller subgroup J of I_1 , as explained below.

Theorem 17. *Let I_w^λ be an orbit of I_w , $\lambda \in X_*(T)$. In type A the special fiber limit of I_w^λ is invariant under the action of the group $J \subset I_1 \subset I_w$, where J is the group of matrices that has 1 along the diagonal, $t(\mathcal{O})$ above the diagonal, and $t^2\mathcal{O}$ below the diagonal.*

Proof. Consider the global group scheme \mathcal{G} constructed as below.

For each $\epsilon \in \mathbb{A}^1$, the group \mathcal{G}_ϵ is the group of jets of maps

$$\{\Phi_\epsilon(t) : \mathbb{A}_t^1 \rightarrow G \mid \Phi_\epsilon(0) \in B, \Phi_\epsilon(\epsilon) = 1\}.$$

When $\epsilon \neq 0$, \mathcal{G}_ϵ is isomorphic to $I_1 \subset I_w$ when restricted to D_ϵ . When $\epsilon = 0$, \mathcal{G}_0 is isomorphic to the group $J \subset I_1 \subset I_w$ generated by $I_2 + N(t\mathcal{O})$.

In type A, \mathcal{G} is represented by the group of matrices with $1 + (t - \epsilon)\mathcal{O}$ on the diagonal, $(t - \epsilon)\mathcal{O}$ above diagonal, and $t(t - \epsilon)\mathcal{O}$ below diagonal, $\epsilon \in \mathbb{A}^1$. □

Note that in the special fiber limit of $G(\mathbb{O})$ orbits, every irreducible component is the closure of an Iwahori orbit. This is not the case for Iwahori orbits. Every irreducible component in the special fiber limit is a union of possibly infinitely many J orbits.

Conjecture 6. *The nearby cycles functor on the global affine flag variety for G induces the same map on cohomology as the map $f : \Omega K \hookrightarrow LK \rightarrow LK/T \cong \Omega K \times K/T$, where K is a maximal compact subgroup of G .*

The special fiber limit of the generalized Iwahori orbit I_w^λ is a union of orbits of different subgroups of I_w . The number of irreducible components in the special fiber limit is a sum of affine Stanley coefficients.

7.3 Central Degeneration of MV Cycles in Type A

We study the limits of MV cycles under the central degeneration, and some of our results are related to affine Deligne-Lusztig varieties.

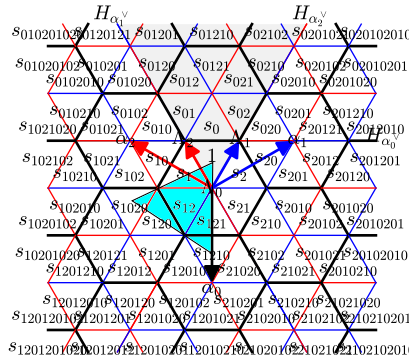


Figure 7.2: The MV polytope for one of the MV cycles in $Gr^{\alpha+\beta} \cap S_{w_0}^e$ for $G = SL_3$.

Lemma 13. *Let S be an MV cycle for an algebraic group G of type A, \tilde{S} be its special fiber limit. Let $\lambda, \mu, \mu_w, w \in W$ be such that $S \subset Gr^\lambda \cap S_{w_0}^\mu$ and $S = \bigcap_{w \in W} S_w^{\mu_w}$.*

Then \tilde{S} is contained in the union of non-empty intersections of Iwahori orbits and U^- orbits $I^{w_1} \cap S_{w_0}^{w_2}$ in the affine flag variety, such that $w_1, w_2 \in W_{aff}$ are both λ -admissible, $w_2 \leq \mu$ in the semi-infinite Bruhat order for U^- . Moreover, \tilde{S} is contained in an intersection of closed semi-infinite orbits in the affine flag variety, $\bigcap_{w \in W} S_w^{(\mu_w, e)}$.

Unlike in the SL_2 case, the limit of an MV cycle for general G in type A does not equal the intersection of the limit of a $G(\mathcal{O})$ orbit and the limit of a closed U^- orbit. One reason is that an MV cycle for general G may only be one of the many irreducible components in the intersection of a $G(\mathcal{O})$ orbit and a U^- orbit in the affine Grassmannian.

Example 15. *Consider one of the MV cycles in $Gr^{\alpha+\beta} \cap S_{w_0}^e$ for $G = SL_3$. This MV cycle is two-dimensional and is isomorphic to \mathbb{P}^2 .*

Now consider the intersection of the special fiber limit of $Gr^{\alpha+\beta}$ and the special fiber limit of $S_{w_0}^e$. Its dimension is strictly bigger than two. In particular, it contains $I^{s_1 s_2 s_1} \cap S_{w_0}^e$, whose closure is the three-dimensional G/B bundle above t^e in the affine Grassmannian.

Therefore, the special fiber limit of this MV cycle is not the same as the intersection of the special fiber limit of $Gr^{\alpha+\beta}$ and the special fiber limit of $S_{w_0}^e$. We could also see this from the moment polytopes.

The moment polytope of the special fiber limit of our MV cycle is illustrated in 7.2.

The moment polytope of the three-dimensional G/B bundle above e mentioned above is shown in 7.3. It does not lie in the moment polytope of the special limit of our MV cycle.

Conjecture 7. *If $Gr^\lambda \cap S_{w_0}^\mu \subset Gr$ has only one irreducible component, the special fiber limit of this MV cycle is the same as the intersection of the special fiber limit of Gr^λ and that of $S_{w_0}^\mu$.*

Now let's discuss the number of irreducible components in the special fiber limits of MV cycles in type A.

Theorem 7, the size of the finite set $(v + \Lambda) \cap P$ is equal to the dimension of S . This is independent of the choice of the vertex v of P . We call each C_v an extremal irreducible component of \tilde{S} .

Let Q be the moment polytope of an irreducible component C_i in S' , and v' be a vertex of Q . By Lemma 5, the dimension of C_i is less than or equal to the size of the finite set $(v' + \Lambda) \cap Q$. We claim that $|(v' + \Lambda) \cap Q| \leq \dim(S)$.

The claim is true for $G = SL_2$. Let's prove this for $G = SL_3$ first.

By [1], all the MV polytopes for $G = SL_3$ are Minkowski sums of four generators: two lines a_1, a_2 and two equilateral triangles b_1, b_2 . In particular, all the MV polytopes only have three possible basic shapes: hexagon, equilateral triangle and isosceles trapezoid. The angle at each vertex could only be 60 degrees or 120 degrees.

Case 1: the angle of Q at one of its vertices a agrees with the angle of P at least one of its vertices v . In fact, there will be at least two such vertices of P . Then we could translate and reflect Q so that it is contained in the moment polytope of C_v . By comparison, we see that $|(a + \Lambda) \cap Q| \leq |(v + \Lambda) \cap P| = \dim(S)$.

Case 2: the angles of Q at any of its vertices a differ from the angles at all vertices of P . In this case, $\dim(S) > |(a + \Lambda) \cap Q|$. To see this, note that there are only three basic shapes of convex polytopes in the weight lattice for SL_3 . If this MV polytope P has four sides, then its vertices have all the possible angles allowed and we are back to the previous case. If not, the pair (P, Q) must be (triangle, hexagon) or (hexagon, triangle). In both cases, Q would be much smaller than P .

Now consider the case of $SL_n, n > 3$. Note that the set of affine root directions from any vertex of Q is a union of affine root directions corresponding to different copies of $SL_3 \hookrightarrow SL_n$. By [16], a polytope is an MV polytope if and only if all of its 2-faces are MV polytopes. So the restriction of Q to any copy of SL_3 lies in the projection of P which is an MV polytope for SL_3 . We could construct such restriction maps by using the restriction map q_J introduced by Braverman-Gaitsgory [2] and further discussed in [17].

Suppose the number of affine root directions from a vertex of Q contained in Q is greater than that of any vertex of P . Then this must be true for the restriction of P and Q to a copy of $SL_3 \hookrightarrow SL_n$. We have arrived at a contradiction given our knowledge of the SL_3 case. Therefore, the dimension of any irreducible component of S' cannot exceed the dimension of any of the extremal irreducible components of \tilde{S} , which equals $\dim(S)$. □

Now we are ready to give some upper bounds for the number of irreducible components in the special fiber limits of an MV cycle.

Corollary 4. *Let $S = \overline{\bigcap_{w \in W} S_w^{\mu_w}}$ be an MV cycle in the affine Grassmannian.*

The number of irreducible components in the special fiber limit of S , \tilde{S} , is bounded above by the total number of irreducible components in $S' = \overline{\bigcap_{w \in W} S_w^{(\mu_w, e)}}$.

Let P denote the moment polytope of \tilde{S} and S' . Another upper bound for the number of irreducible components of \tilde{S} is given by the number of convex polytopes contained in P that

satisfy two extra conditions. First, if two vertices of the polytope lie on the same line in the embedded weight lattice in $\mathfrak{t}_{\mathbb{R}}^*$, then there is an orbit of a root subgroup that connects these two extremal T -fixed points. Second, the number of root-directions at each vertex is greater than or equal to the dimension of S .

Proof. By Theorem 19, $\dim(S') = \dim(S)$. Therefore we get the first upper bound in the theorem.

For the second upper bound, the moment polytope of every irreducible component in \tilde{S} has to satisfy the conditions specified above. Two distinct irreducible components of \tilde{S} have distinct moment polytopes, as they have different sets of T -fixed points. \square

Now let's illustrate our theorems in some examples.

Example 16. $G(\mathcal{O})$ orbits Gr^λ are special examples of MV cycles. The number of irreducible components equals to the size of the quotient of the Weyl group $\tilde{W} = W/W_I$ associated to G/P_λ . The upper bounds and lower bounds we proved so far agree in this special case.

Example 17. Let $G = GL_n$. Any $G(\mathcal{O})$ orbit Gr^λ for minuscule λ is an ordinary Grassmannian. MV cycles in Gr^λ coincide with ordinary Schubert varieties. The explicit equations for this case are worked out in [9].

In the special case where $Gr^\lambda \cong \mathbb{P}^n$, all the MV cycles are $\mathbb{P}^k, k \leq n$. Their special fiber limits have k irreducible components, one for each T -fixed point in the original \mathbb{P}^k .

By Lemma 13, the special fiber limits of MV cycles are contained in a union of intersections of Iwahori orbits with U^- -orbits in the affine flag variety. On the other hand, intersections of Iwahori orbits and U^- orbits in the affine Grassmannian and affine flag variety are closely related to affine Deligne-Lusztig varieties, as explained in [11] and [10]. We expect to use the limits of central degeneration to get better understanding of affine Deligne Lusztig varieties.

7.4 Main Examples

In these examples, we focus on the case of $G = SL_3(\mathbb{C})$, where α and β are the two simple coroots. We will describe the degenerations of all the MV cycles and generalized Iwahori orbits in the $G(\mathcal{O})$ orbit $Gr^{\alpha+\beta} \subset Gr$.

7.4.1 Examples for Degenerations of MV Cycles

Let's consider all the MV cycles in $Gr^{\alpha+\beta} \cap S_{w_0}^\mu$ for some relevant coweight μ for $G = SL_3$.

(1) When $\mu = -(\alpha + \beta)$, we see one T -fixed point t^μ degenerates to the T -fixed point indexed by (μ, e) in the affine flag variety.

(2) When $\mu = -\alpha$ or $-\beta$, this MV cycle an extended-torus invariant \mathbb{P}^1 . As explained earlier, it degenerates to two \mathbb{P}^1 s intersecting at one T -fixed point.

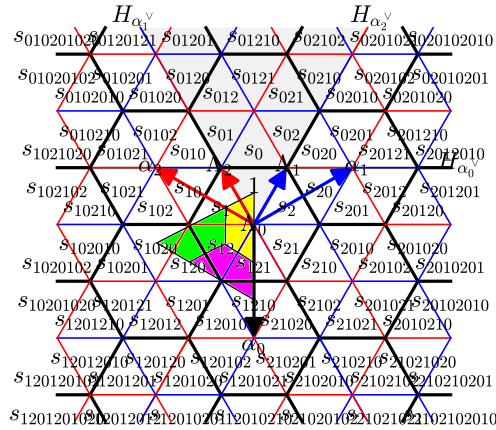


Figure 7.4: Central Degeneration of \mathbb{P}^2 as illustrated in moment polytopes. The big triangle is the original MV polytope. The three interior convex polytopes are the moment polytopes of the three irreducible components in the special fiber limit of this MV cycle.

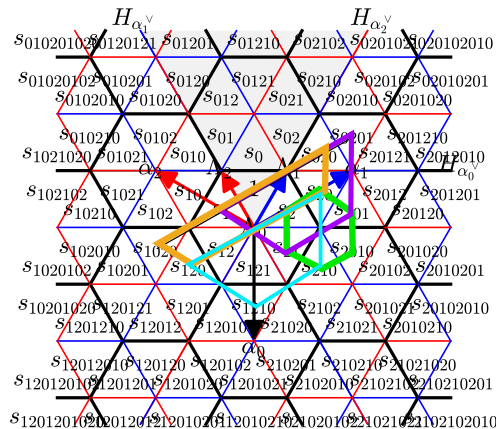


Figure 7.5: The moment polytopes of a three dimensional MV cycle and the four extremal irreducible components in its special fiber limit.

(3) When $\mu = e$, each of the two two-dimensional MV cycles is isomorphic to \mathbb{P}^2 , and degenerates to three irreducible components, as illustrated in 7.4.

Note that each of these MV cycles is isomorphic to the $G(\mathcal{O})$ orbit $G_r^{(1,0,0)} \cong \mathbb{P}^2$ in the affine Grassmannian for GL_3 . The degenerations of $G(\mathcal{O})$ orbits in type A for minuscule coweights are explained in [9].

(4) When $\mu = \alpha$ (or β), the MV cycle is three-dimensional.

The irreducible components in the special limit corresponding to the four vertices of the original MV polytope are shown in 7.5.

There is one additional irreducible component in the intersection of the corresponding

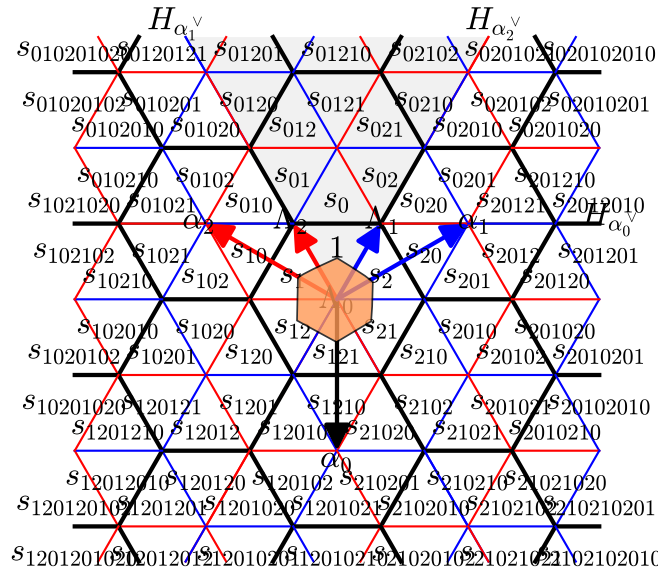


Figure 7.6: The moment polytopes of the three dimensional G/B bundle above e in the affine Grassmannian for $G = SL_3$.

closed semi-infinite orbits in the affine flag variety. This is the three-dimensional G/B bundle above e if we think of the affine flag variety as a G/B bundle over the affine Grassmannian. Its moment polytope is shown in 7.6.

To see that this internal irreducible component with hexagonal moment polytope must be present, we consider the degenerations of all the closed generic T -orbits in this MV cycle. This forms a continuous one-parameter family. The data of a torus equivariant degeneration of a toric variety is given by a regular subdivision of its moment polytope, as well as the polyhedral complex dual to this regular subdivision [37, 28].

We restrict our attention to the possible regular subdivisions of this hexagon under this degeneration of our continuous family of generic toric varieties.

Due to the shapes of the moment polytopes of the extremal irreducible components, there exist generic toric varieties in the original MV cycle whose special fiber limits divide this hexagon in the two ways shown in 7.7 and 7.8. We show the possible regular subdivisions of this hexagon as well as the associated polyhedral complexes.

The toric varieties represented in the subdivisions above coincide with some two-dimensional Borel orbits in the closure of the flag variety G/B above $e \in Gr$. Since these generic toric varieties in the original MV cycle form a continuous family, in the special fiber limit there must be a one-parameter family of generic toric varieties whose moment polytopes are this hexagon. We show this moment polytope with the associated polyhedral complex in 7.9.

There must be a three-dimensional irreducible subscheme that contains this one-parameter family of generic toric varieties in the G/B fiber above the T -fixed point e . Therefore, the G/B bundle above e , which is also a GGMS stratum in the affine flag variety and the closed

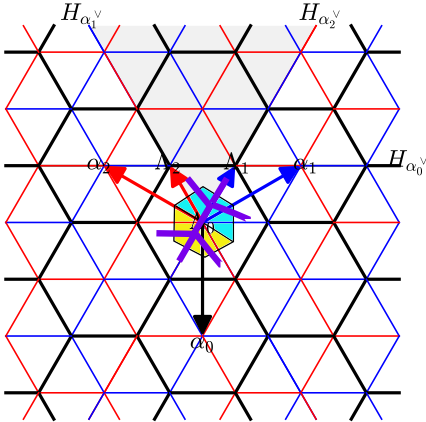


Figure 7.7: Central Degeneration of one generic toric variety in the original MV Cycle.

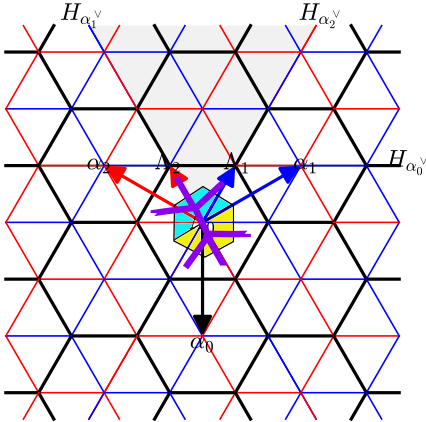


Figure 7.8: Central Degeneration of another generic toric variety in the original MV Cycle.

Iwahori orbit $I^{s_1 s_2 s_1}$, must be an irreducible component of the special fiber limit of this MV cycle.

This observation is related to the fact that the tropicalization map on the moduli space of algebraic curves on a scheme X is continuous. For further details, see [38].

Now all the five irreducible components in the special fiber limit of this MV cycle are illustrated in 7.10.

Finally, When $\mu = \alpha + \beta$, we are back in the picture 7.11 of the degeneration of the entire $G(O)$ orbit.

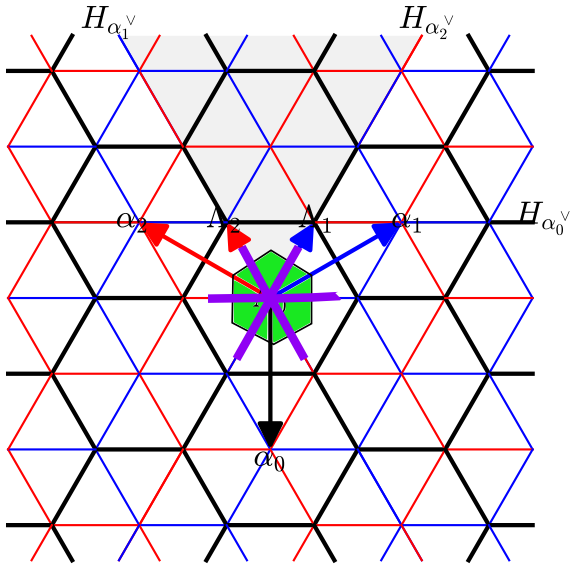


Figure 7.9: The moment polytope of a one-parameter family of toric varieties in the special fiber limit of some toric varieties.

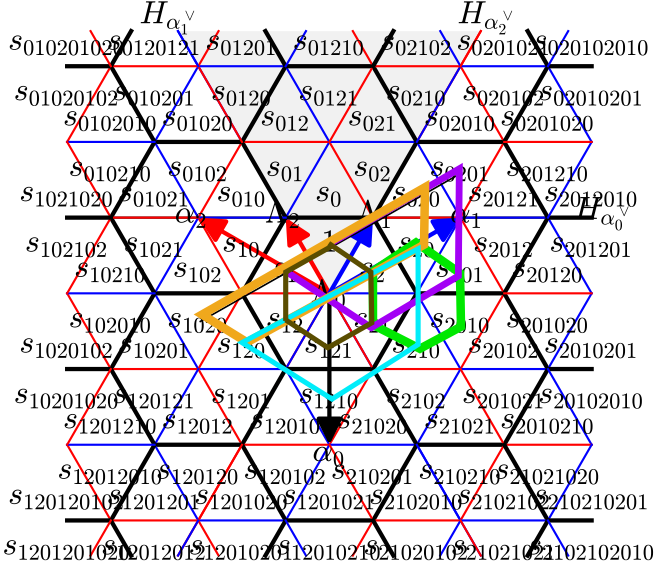


Figure 7.10: Central Degeneration of a three-dimensional MV cycle as illustrated in moment polytopes. The big trapezoid is the MV polytope, and the five smaller convex polytopes of different colors are the moment polytopes of the five different irreducible components in the special fiber limit.

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