Title
Dynamics of Quantum Solitons

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Publication Date
2022

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
SANTA CRUZ

DYNAMICS OF QUANTUM SOLITONS

A dissertation submitted in partial satisfaction of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

in

PHYSICS

by

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June 2022

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Abstract

Dynamics of Quantum Solitons

by

Daniel Ellis Davies

Solitons are a class of solution to equations that appear in the study of physics, particularly the study of fields and field theory. In the study of quantum field theory, solitons often present themselves as heavy degrees of freedom, and are therefore approached from a semi-classical perspective. Traditionally, the soliton is considered as the background field with which perturbative quantum states are defined with respect to. In this dissertation, we present the position that this approach is flawed. In particular, we suggest that using classical solitons as a starting point of analysis only narrowly covers the scope of the possibilities for solitons to manifest in quantum field theories. We show that there are classical solitons of infinite mass which in the quantum theory have their mass rendered finite. We additionally describe circumstances where quantum effects are large enough to significantly alter the net forces which act upon solitons, compared to the classical case. The conclusion is simple: quantum field theory generically contains a much larger space of dynamically accessible field configurations than classical field theory, and this facilitates the existence of quantum solitons that would not exist in the classical case.

To Colleen,

for the immeasurable value of our time together.

Chapter 1

Introduction

To begin, we shall discuss what is meant by the terms "soliton" or "extended object" within the greater context of field theory. The terms are in a sense interchangeable, and one can make this precise by using effective field theory. In this chapter, we will begin with an overview of classical field theories, with example model theories given that possess and do not possess solitons as solutions to their equations of motion. Familiarity with classical field theory is assumed.

Then, we shall discuss, in not as many examples, how vibrational modes occur in the vicinity of solitons. We will emphasize the conditions that these vibrations must obey to be considered stable. They are of course the result of perturbations to the equations of motion describing solitons, and therefore relevant to the study of classical scattering between solitons and waves, or solitons and other solitons. The discussion will generalize nicely to the quantum theory.

Before a rigorous approach to the quantum theory of solitons and extended objects is considered, their relevance to particular quantum field theory models of interest will be examined. These models will be the quantum descendants of the classical field theory models discussed prior, but emphasis will be placed on the particular details of the quantum theory which make the study of their solitons especially interesting and productive.

Finally in this chapter, we discuss the deceptively simple matter of particles in quantum field theory, how to construct them as states, and their relation to the classical wave solutions to the equations of motion. This discussion will make clear the difficulties in formalizing an equivalent construction for the soliton in quantum theory, and motivate the investigation in the further chapters. We will then discuss the formalism of the quantum effective action, and how it might be used to study solitons.

1.1 Solitons in Classical Physics

In short, a soliton is a non-dissipative solution to the Euler-Lagrange equations of motion for some field theory model, and especially one which is physically confined to a region of space. The term dates to an observation of the behavior of water waves propagating in a rectangular channel that boats passed through[1]. The speed of water waves (specifically those galled “gravity waves”) that propagate on the surface of a body of water depends on the frequency of the wave, as well on geometric properties of the channel. For gravity waves in an unbounded, infinitely deep fluid, the velocity of wave propagation is $U = \sqrt{g/4k}$, with g the gravitational acceleration and k the wave number[2]. Thus any wave pulse that is localized in space, and therefore contains waves of all values k, will at some later time separate and diffuse as the waves of smaller k ‘outrun’ the waves of larger k, deforming the wave packet. Hence ocean waves are diffusive: they spread their energy out over time.

When the depth of the waterway h is not approximately infinity ($hk >> 1$) the wave speed formula changes. In fact for long gravity waves, those with $hk << 1$, the wave speed becomes exactly $U = \sqrt{gh}$. Thus for sufficiently long wavelength gravity waves, the wave speed is identical for all k and a wave pulse maintains its shape as it traverses a channel. The full formula, derived in *Fluid Mechanics* by Landau and Lifshitz[2], is

$$U = \sqrt{\frac{g}{4k \tanh kh}} \left(\tanh kh + \frac{kh}{\cosh^2 kh} \right) \tag{1.1}$$

In this case of water waves of small k flowing in a shallow channel, the wave does not begin to widen and dissipate (except perhaps by the action of viscous forces). It appears solitary, unlike ocean waves at sea, or in the harbor. Together from “solitary” and the Latin suffix “-ion” denoting action or condition, the word *soliton* is born: that which has condition of being solitary. In other words, a lone wave or disturbance.

New meaning has been added to the term since its coinage in 1834. Pulses of light for example, travel with a speed independent of wave number in linear media, and are thus localized and non-dissipative. We do not call these solitons. A soliton has a secondary characteristic, in that it is a stable thing to small perturbations in shape[3]. There is no reactive force on light pulses in linear media that will bring the pulse back to some characteristic width if it is disturbed. For the water wave there are: surface tension and gravitational potential. In what follows we shall investigate these non-dissapative waves in a relativistic context, and take special care to illustrate that they have a *preferred size.*

1.1.1 The Kink

From this point forward we will restrict the discussion to the discussion of Lagrangian field theories. Of particular importance is the theory of the single scalar field, denoted φ. The possible Lagrangian density comes in the form

$$\mathcal{L} = \frac{1}{2}\left(\partial\varphi\right)^2 - V(\varphi) \tag{1.2}$$

where the potential $V(\varphi)$ is an even function of φ, and in more than three spacetime dimensions will be restricted to the form of a quartic polynomial. In lower dimensions this is unnecessary, e.g. in two spacetime dimensions any form of $V(\varphi)$ is physically meaningful in the quantum theory, so long as a global minima exists. These details are cursory, so we will restrict ourselves to the so-called φ^4 model presently:

$$V(\varphi) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4 + \frac{\mu^4}{4\lambda} \tag{1.3}$$

In two spacetime dimensions, the equation of motion for this field is

$$\partial_t^2\varphi - \partial_x^2\varphi - \mu^2\varphi + \lambda\varphi^3 = 0 \tag{1.4}$$

The trivial, stable, zero-energy "vacuum" solution is $\varphi = \pm\sqrt{\mu^2/\lambda}$. Perturbations from this vacuum spate may be labeled η (i.e. $\eta = \phi \mp \mu/\sqrt{\lambda}$), whose equation of motion become

$$\partial_t^2\eta - \partial_x^2\eta + 2\mu^2\eta \pm 3\mu\sqrt{\lambda}\eta^2 + \lambda\eta^3 = 0 \tag{1.5}$$

Assuming the amplitude of η is small compared to $\mu/\sqrt{\lambda}$, the last two terms may be neglected in a first order approximation. The result is the solution for a massive wave, for example if $A << \mu$ we have

$$\eta = \frac{A}{\sqrt{\lambda}}e^{i(kx-\omega t)} \tag{1.6}$$

$$\omega = \sqrt{k^2 + 2\mu^2} \tag{1.7}$$

Perturbations from this pure wave solution can be studied. Suppose a wave of amplitude $B/\sqrt{\lambda}$, frequency ν and wave number q is overlaid on top of solution (1.6). Supposing $B << A << \mu$, the leading order spectral condition (linear in the amplitude B) can be derived:

$$\nu = \sqrt{q^2 - 6\mu A} \tag{1.8}$$

Clearly, if the wavelength of the second order perturbation is sufficiently long, ν may become imaginary and the amplitude B grows exponentially over time. This is actually not totally accurate, as we have discovered a problem with perturbation theory known as the *secular divergence*. Such problems also arise in next-to-leading-order approximations of the motion of a simple pendulum[4]. The pendulum motion does not grow over time, nor does the amplitude of the disturbance η. Secular divergences may be nullified by assuming the values ω and k to be functions of μ and λ. For more information the reader is referred to José and Saletan's text *Classical Dynamics: A Contemporary Approach*[4].

The point is that all pertubations of the vacuum solution will be sensitive to the fact that there lies another minima to the potential. Large enough amplitude waves may even push the field φ from one minima to another in a particular region. When this happens, we form what are known as "kinks." They are the first solitons we shall encounter. The singular kink, which transitions from vacuum solution $-\mu/\sqrt{\lambda}$ at $x = -\infty$ to $+\mu/\sqrt{\lambda}$ at $x = +\infty$, is given by the formula

$$\varphi_K(x,t) = \frac{\mu}{\sqrt{\lambda}} \tanh\left(\frac{\mu(x-x_0)}{\sqrt{2}}\right) \tag{1.9}$$

which is a static, finite energy configuration, localized at $x = x_0$, with characteristic size $\sim \mu^{-1}$. It satisfies equation (1.4). The total energy of this configuration is

$$M_{\text{Kink}} = \frac{\mu^3\sqrt{8}}{3\lambda} \tag{1.10}$$

The Anti-Kink, which is just the configuration multiplied by -1, has the same mass. The solution for a moving (Anti-)Kink may be achieved via Lorentz transformation. About 92% of the total energy is stored within a distance $\sqrt{2}\mu^{-1}$ of the position x_0.

We see that parametrically, the Kink width is of the same order as the inverse mass of the wave solutions η described previously, however the kink mass is inversely related to the quartic coupling constant λ. This is a generic property of solitons, as we will see with the next few examples. Naively, this means that for large enough λ (here λ has the same dimensions as μ^2, so we really mean to say $\mu^2/\lambda << 1$), the lowest energy degrees of freedom in this model will be comprised of kink and anti-kink states, rather than plane waves. Ultimately, this is due to the potential barrier between the degenerate vacuua being very low. Of course we have not given the multi-kink solution, nor described the forces between kinks and anti-kinks at any distance. The equations of motion are not linear, so we cannot just superimpose the solutions. However if the kink anti-kink separation is very large compared to μ^{-1}, the superimposed solution becomes approximately true and the resulting potential energy between the solitons drops off quite fast. Therefore at large distances, the forces between kinks and anti-kinks are very weak. They are also attractive[5]. Precisely what happens when a kink and anti-kink scatter off one another is highly dependent upon the kinematics. There is no simple description for Kink dynamics in the φ^4 model; it is an ongoing area of research, driven largely by advances in simulation[6].

1.1.2 The Global Vortex/String

The Vortex/String soliton is a solution to the equations of motion for a pair of scalar fields in three/four spacetime dimensions. The string solution has translational invariance along the axis of the string; the planar cross section of the string is in fact just the vortex solution one dimension lower. For this reason we restrict the discussion to the vortex case, and point out that the only difference between the two cases are the units of the constants μ and λ that appear in the potential. The formula for the vortex mass becomes, in the four dimensional case, the tension of the string, or mass per unit length.

The Lagrangian density is best expressed in terms of a complex scalar field $\Phi = (\varphi_1 + i\varphi_2)/\sqrt{2}$:

$$\mathcal{L} = \partial\Phi^\dagger\partial\Phi + \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 - \frac{\mu^4}{4\lambda} \tag{1.11}$$

As before, the trivial vacuum solution is given by the condition $\Phi^\dagger\Phi = \mu^2/2\lambda$. The key difference besides factors of 2, is that now there is an infinite degeneracy, described by a parameter α. All solutions of the form $\Phi = \mu e^{i\alpha}/\sqrt{2\lambda}$ minimize the potential, and have zero energy density everywhere. α parametrizes the unitary group $U(1)$, and the selection of a particular value of α is a topic of spontaneous symmetry breaking. The $U(1)$ parameter is a constant: it cannot vary from place to place, else energy is added to the system by the derivative terms. Those higher energy excitations are massless, described by localized perturbations in the parameter α: we call these Goldstone bosons.

The vortex is characterized by a non-trivial configuration for α which now depends on the angular coordinate θ. If we suppose $\alpha = n\theta$ for some integer n called the *winding number*, and that as the radial coordinate r tends to infinity, the vacuum solution is satisfied asymptotically, then we have a stable configuration. More specifically, consider the ansatz

$$\Phi(r,\theta) = \frac{\mu}{\sqrt{2\lambda}} e^{in\theta} \left(1 - \frac{a}{r} - \frac{b}{r^2} - \frac{c}{r^3} + \cdots\right) \tag{1.12}$$

we can, order by order in powers of r^{-1}, determine the coefficients $a, b, c, \ldots$. In short, the function in the parenthesis (1.12), which we call $f(r)$, is to order $1/r^2$:

$$f(r) \sim 1 - \frac{n^2}{2\mu^2 r^2} + \mathcal{O}\left(\frac{n^4}{\mu^4 r^4}\right) \tag{1.13}$$

This is an asymptotic solution. There is no exact formula for all r, and at $r = 0$ the field must be pushed up the top of the potential barrier: $\Phi(0) = 0$. Hence there is an energy stored near the origin, with decreasing energy density away from the vortex "core" (assumed to be of size μ^{-1}). Here we have a problematic development. The total energy contains a term of the form

$$M_{\text{Vortex}} = \int_0^\infty drr \frac{\pi n^2 \mu^2 f(r)^2}{\lambda r^2} + \cdots \tag{1.14}$$

which is divergent. If the integral is cut off at some large radius R, the total vortex mass would scale like

$$M_{\text{Vortex}} = \frac{n^2 \mu^2}{\lambda} \ln(R) + \text{Finite} \tag{1.15}$$

Similarly, the global string tension is divergent. The long range profile of the field may be altered if a vortex or string of opposite winding number is introduced at some distance away R. In general if the total winding number of a collection of vortices is null, the total energy will be finite. The interaction potentials between vortices of various winding numbers will be complicated. At least for large distances compared to μ^{-1}, we should expect a logarithmic confining potential between a vortex/anti-vortex pair, and the total energy to be finite. Over time the pair would attract and ultimately annihilate. We may generally conclude that for this classical field theory, the ansatz (1.12) corresponds to an infinite energy state, and thus is prohibited.

In fact this occurs even if $f(r) = 1$ for some finite r. No configuration of this form is known which satisfies the equations of motion for f, but regardless of this fact the term in (1.14) still diverges, even if a solution exists. It is worth noting that, just as the kink, the vortex mass (even the finite parts) will scale like λ^{-1}, but the overall shape of the soliton depends only on the characteristic distance scale μ^{-1}.

We may upgrade the $U(1)$ symmetry to a gauge symmetry, i.e. we introduce a vector field A^μ that couples to Φ via

$$\mathcal{L}_{\text{gauge}} = \Phi^\dagger \Phi A_\mu A^\mu - \frac{1}{4g^2} F^2 \tag{1.16}$$

with F the usual field strength tensor. The lowest energy configuration for A^μ is one which F vanishes. We call the field "pure gauge" in this limit. It is given by the derivative of the group element $e^{in\theta}$:

$$A_\mu = e^{-in\theta} \partial_\mu e^{in\theta} = \frac{n}{r} \delta_{\theta,\mu} \tag{1.17}$$

The contribution to the total energy of the first term in (1.16) is therefore exactly opposite the contribution from the term in equation (1.14). The total mass of this "local" vortex or string (known as the Nielson-Olesen vortex), is therefore finite. The Goldstone Bosons are removed from the spectrum by the gauge freedom of the field A^μ. Fluctuations in A_μ interact with the vortex in nearly the same way as fluctuations in α.

1.1.3 The Schwarzschild Black Hole

The final example we will give of classical solitons is of the gravitational variety. Special distinction should be made in this case. By a gravitational soliton, we mean a stable, localized configuration of the gravitation fields (i.e. metric tensor components) which is not supported by external sources, in this case meaning components of some matter stress-energy tensor. This is not a universal definition: we adopt it here only to make a special point that the gravitational field equations admit unique solutions which are stable and inhomogeneous. The Gravitational (Einstein) field equations in the absence of matter are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{1.18}$$

The solution known as the Schwarzschild metric can be read off from the line element. For radial coordinate r greater than some free parameter r_s:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega_2^2 \tag{1.19}$$

For an observer in the region $r > r_s$ it appears as though a curvature singularity appears at $r = 0$, but since there are no timelike paths from the interior of the horizon to the exterior, an external observer may never verify the validity of this metric for $r < r_s$. We do not mean to invoke a discussion of the black hole paradox here, only to illustrate that the Schwarzschild solution is static and inhomogeneous, characterized by a scale r_s which is a free parameter. There is an important distinction between astrophysical black holes, which formed from the collapse of a body (in which the horizon becomes physical at some moment in the collapse, but not prior), and primordial black holes. The primordial black hole may be thought as a relic of some gravitational phase transition in the early universe, forming as a result of some unknown quantum gravitational dynamics, and not due to the collapse of an astrophysical object composed of matter. For the astrophysical black hole, the parameter r_s is related to the total rest energy plus gravitational potential energy (here labeled as M) of the object which collapsed:

$$r_s = 2GM \tag{1.20}$$

There is a Hamiltonian formulation in gravitational physics in which one may assign a "mass" to gravitation fields that are asymptotically flat[7]. This is called the ADM mass, and for the Schwarzschild solution it is given by the formula (1.20). It is a philosophical question, at least in the classical Einstein gravitational theory, whether one wants to distinguish these two objects in one's mind. Certainly there are no classical measurements which can distinguish the two black holes, and one can only be certain of the astrophysical origin of a black hole if one witnesses it emerge from a collapsing star. However in the studies of quantum gravity, the distinction becomes a well-founded one, rooted in solutions to quantum equations of motion. The notion of a singularity-free, matter-free black hole being formed in an early phase transition is within the realm of possibility, motivating the study of gravitational solitons.

1.1.4 Derrick's Theorem

Besides relying on brute force calculations to solve non-linear field equations, is there any way to predict whether stable soliton configurations exist, *a priori*? In fact there is, and while the approach has its limitations, it is quite informative and may be applied generally as long as the field theory model admits a Hamiltonian description. We shall discuss the procedure and its results now, which are collectively referred to as Derrick's Theorem[8].

Suppose the existence of a Hamiltonian density for the field or set of fields Φ, A_μ, etc. that may be separated into a sum of distinct terms, each with their own tensor components. We'll label the number of these components as i, and the density as

$$\mathcal{H}[\Phi] = \mathcal{H}_0 + \mathcal{H}_1 + \cdots + \mathcal{H}_i + \cdots \tag{1.21}$$

For example, the following terms would have $i = 2$.

$$\partial_\mu \Phi \partial^\mu \Phi \quad m^2 A_\mu A^\mu \quad g^2 \Phi^2 \partial_\mu A^\mu \tag{1.22}$$

And the following for $i = 4$.

$$F_{\mu\nu} F^{\mu\nu} \quad \Phi \partial^2 \Phi \quad F_{\mu\nu} A^\mu \partial^\nu \Phi \tag{1.23}$$

The index i counts the total number of derivatives plus the spin of the field variables. A scalar potential would have $i = 0$. It is clear that the index must be even, since the Hamiltonian density is a scalar variable: all spacetime indices must be contracted. Calculating the total energy, we have (in d+1 dimensional spacetime)

$$E = \int d^d x \mathcal{H} = E_0 + E_1 + \cdots + E_i + \cdots \tag{1.24}$$

We suppose E is a functional of some inhomogeneous solutions to the field equations. For simplicity, we write this explicitly for a scalar field $\Phi(\vec{x})$. It stands to reason that if Φ is a stable solution with some intrinsic scale μ^{-1}, it has lower energy than the same solution with intrinsic scale $\lambda\mu^{-1}$, for λ a real number in the vicinity of unity. Stability is reflected in the precise statement that E is a local minima with respect to λ at $\lambda = 1$, or by replacing $\Phi(\vec{x}) \to \Phi(\lambda\vec{x})$:

$$\frac{d}{d\lambda} E[\lambda]\Big|_{\lambda=1} = 0 \tag{1.25}$$

$$\frac{d^2}{d\lambda^2} E[\lambda]\Big|_{\lambda=1} \geq 0 \tag{1.26}$$

At first glance it seems one would need to know the actual soliton solution Φ to apply this reasoning, but this is not accurate. We can, after the rescaling of the solution to Φ, make a coordinate change of the spatial variables: $\vec{x} \to \lambda^{-1}\vec{x}'$. It is then clear that the dependence on λ is constrained by the index i (called the "engineering dimension" and the space dimension d. From the definition of the engineering dimension we have the exact result

$$E[\lambda] = \lambda^{-d} E_0 + \lambda^{-d+1} E_1 + \cdots + \lambda^{-d+i} E_i + \cdots \tag{1.27}$$

The matter of stability is reduced to a condition upon the signs of the various terms E_i and the relative difference $i - d$ for each term. It should be emphasized that this approach will fail if $E[\lambda]$ is not finite. Let us see how this works for the kink. We note first of all that there are only two terms in the energy density.

$$\mathcal{H}_0 = V(\varphi) \geq 0 \tag{1.28}$$

$$\mathcal{H}_2 = \frac{1}{2} (\partial_x \varphi)^2 \geq 0 \tag{1.29}$$

The solution is assumed to be stable so the time derivatives are omitted. It is not hard to generalize to a case where the field is periodic, and therefore include the frequency in the conditions (1.25) and (1.26). Here we find

$$E[\lambda] = \lambda^{-1} E_0 + \lambda E_2 \tag{1.30}$$

The stability conditions are satisfied if $E_0 = E_2 > 0$, and hence we may conclude static, stable, inhomogeneous solution to the field equations are not forbidden. Determining the form of the solution is generally a more difficult task. The set of conditions described here is not guaranteed to support the existence of soliton configurations, though if they cannot be satisfied, solitons are guaranteed to be absent from the model.

We may ask ourselves, does the φ^4 theory have such solutions in higher dimensions? The answer is yes, since the kink may be promoted to planar domain walls of transverse dimension $d - 1$, but these solutions have infinite energy (they have infinite area). How about compact domain walls? Considering $d \geq 2$, we find

$$E[\lambda] = \lambda^{2-d} E_2 + \lambda^{-d} E_0 \tag{1.31}$$

has a minima only at $\lambda = 0$, indicating an absolute instability, in this case due to contraction of the domain wall to a point. Generally speaking, we can conclude that for scalar field theories, stability of inhomogeneous field configurations requires there be at least as many derivatives in the action as the number of spatial dimensions, d. The critical case, where the 'highest derivative' term has $i = d$, is like the case of the vortex. Special considerations are to be made, especially regarding the finiteness of the terms in the series (1.24). As we saw for the global vortex, E_2 was infinite, but not in the case of the Nielson-Olesen vortex. The main conclusion of Derrick's theorem is this: soliton solutions may be found only if there are energy costs which scale with positive and negative powers of λ. For fields with scalar potential energies, this requires having at least as many derivatives as the number of spatial dimensions. Let's apply this analysis to a four-derivative model in $d = 3$ dimensional space. Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{2M^2}\left(\partial^2\varphi\right)^2 + \frac{1}{2}\left(\partial\varphi\right)^2 \tag{1.32}$$

The Hamiltonian density (dropping the time derivatives) is

$$\mathcal{H} = \frac{1}{2M^2}\left(\nabla^2\varphi\right)^2 + \frac{1}{2}\left(\nabla\varphi\right)^2 \tag{1.33}$$

Clearly the engineering dimensions of the terms are 4 and 2, respectively. Therefore in $d = 3$ we should expect, from (1.25) the condition $E_4 = E_2$ to be met. Since both terms are positive this is possible. Condition (1.26) is met automatically. Thus a stable soliton exists, even though there is no potential term. The equation of motion for a radially symmetric soliton is

$$-\varphi^{''''} - \frac{4}{r}\varphi^{'''} - M^2\left(\varphi^{''} + \frac{2}{r}\varphi^{'}\right) = 0 \tag{1.34}$$

which admits a family of solutions (finite as $r \to \infty$):

$$\varphi(r) = \frac{ke^{-Mr}}{r} + \frac{c}{M^2 r} \tag{1.35}$$

The total energy of this configuration is

$$E = \frac{2\pi}{M^2}\left(k^2M^3 + kc\right) \tag{1.36}$$

If $c = 0$ this is just the solution $K_{1/2}(Mr)/\sqrt{r}$, with $K_{1/2}(Mr)$ the modified Bessel function. For c fixed and non-zero, k may be varied to minimize the total energy. The actual solution has no physical meaning, but is evidence of the strength of Derrick's theorem. We were motivated by the conclusion of scaling arguments to investigate the equations of motion further, and easily discovered the solution.

The real power of Derrick's theorem is with regards to non-linear equations of motion. Scale arguments are immune to the complexities of differential equation solving, so we know a solution exists to be discovered far before one is actually found. For example, the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_x\psi\partial_t\psi + (\partial_x\psi)^3 - \frac{1}{2}\left(\partial_x^2\psi\right)^2 \tag{1.37}$$

yields an equation of motion which is not easily solved. With the variable definition $\phi = \partial_x\psi$, the equation of motion for ϕ is called the Korteweg–De Vries (KdV) equation, which describes the gravity wave solitons that propagate in shallow water, as were discussed at the beginning of the chapter[9]. The conditions for stability are

$$\int dx\left(\frac{3}{2}\left(\partial_x^2\psi\right)^2 - 2\left(\partial_x\psi\right)^3\right) = 0 \tag{1.38}$$

$$\int dx\left(3\left(\partial_x^2\psi\right)^2 - 2\left(\partial_x^3\psi\right)\right) > 0 \tag{1.39}$$

If the first condition is satisfied, the second follows automatically. A thorough analysis of the equations of motion and integrals of motion reveal that the first condition may be satisfied when specific spectral conditions are met, relating the mass and total energy of the soliton. Most importantly, one may not rule out the existence of solitons from these equations using Derrick's approach. Thus it is easy to motivate the search for a localized solution by merely applying Derrick's theorem. Derivation of the actual solution is more challenging, but we now know that a solution exists, and satisfies certain conditions that narrow the scope of the problem significantly.

1.2 Soliton Oscillations

We have discussed a few circumstances where field equations admit static localized solutions. In reality, these solitons are never truly alone, but accompanied by waves and other solitons which interfere with the motion. Interactions between these degrees of freedom may excite vibrations of the soliton, and it is important to study these vibration, their spectra, and stability.

1.2.1 Amplitude Fluctuations & Bound States

The first of two kinds of fluctuation is the amplitude fluctuation. By this we mean vibrational modes which change the absolute value of the field from the soliton solution, in such a way that the fluctuations are superimposed upon the original static configuration. For simplicity we will restrict this discussion to the kink, as the resulting equations have been solved exactly. We write the field variable as

$$\varphi(x,t) = \varphi_K(x,t) + \eta(x,t) \tag{1.40}$$

with η the fluctuations. Inserting (1.40) into (1.2) and (1.3) we find

$$\mathcal{L}[\eta] = \frac{1}{2}(\partial\eta)^2 - \frac{1}{2}m^2\eta^2 - U(x)\eta^2 - \frac{3\lambda}{4}\varphi_K\eta^3 - \frac{\lambda}{4}\eta^4 + \text{constant} \tag{1.41}$$

where $m^2 = 2\mu^2$ and the potential term $U(x)$ is given by

$$U(x) = -\frac{3m^2}{4}\text{sech}^2\left(\frac{\mu(x-x_0)}{\sqrt{2}}\right) \tag{1.42}$$

In other words, far from the kink, η is the same as in equation (1.5). However, near $x = x_0$, the equations of motion are modified dramatically. The cubic term changes sign at x_0, and the linearized equations of motion for η become

$$\left(\partial^2 + m^2 - \frac{3}{2}m^2\text{sech}^2\left(\frac{\mu(x-x_0)}{\sqrt{2}}\right)\right)\eta = 0 \tag{1.43}$$

Assuming definite energy solutions $\eta = \eta(x)e^{i\omega t}$ we find there are two classes of solution. Those with $\omega^2 > m^2$ which we shall call asymptotically free, and those with $m^2 > \omega^2 \geq 0$ which are called bound states. ω must be real otherwise the solution is unstable, causing $\eta(x)$ to grow in magnitude comparable to $\varphi_K(x)$. For this equation of motion, there are no such solutions. The bound states are easy to find, and there are two of them.

$$\eta_0(x) = \sqrt{\frac{3m}{8}}\text{sech}^2\left(\frac{m(x-x_0)}{2}\right) \tag{1.44}$$

$$\eta_1(x) = \sqrt{\frac{15m}{8}}\tanh\left(\frac{m(x-x_0)}{2}\right)\text{sech}^2\left(\frac{m(x-x_0)}{2}\right) \tag{1.45}$$

The squared frequencies are $\omega_1^2 = 3m^2/4$ and $\omega_0^2 = 0$. The eigenfunction (1.44) is called a "zero mode" and is a massless state associated with the translational symmetry breaking of the collective coordinate x_0. It is easy to check that $\eta_0 \propto \partial_{x_0}\varphi_K$. Both bound states are normalized to unity. Perturbative corrections to the equations of motion, which are order $\sqrt{\lambda}$ and λ, will modify the formulae here and set bounds on the size of these fluctuations relative to the other internal parameters. They will not destabilize the kink, but will however result in the production of asymptotically free states if the initial amplitude of either bound state is too large. The presence of bound states affects the overall shape of the soliton, much in the same way that perturbations from the simple pendulum could be associated with comparatively small corrections to the pendulum period.

The remaining amplitude oscillations, the free states, are solutions to the linearized equations of motion with $\omega > m$. These are investigated much in the same manner that electromagnetic waves scattering off a localized potential were studied. There is an appropriate asymptotic expansion in terms of plane waves with fixed momentum, and the mixing of different waves via a phase angle. This phase angle, as in the case of electromagentic wave scattering, is derived by matching conditions near the soliton core, and is not a trivial process. We currently have no interest in these results, except to say that they are used to construct a scattering matrix between free states and solitons in the quantum theory.

1.2.2 Collective Coordinate Excitations

The second kind of fluctuation a soliton may experience is a variation of its collective coordinates. These are the variables, like x_0 for the kink, which describe the unconstrained properties of a soliton. The location, orientation, phase, and quantities of this sort are included in the list of collective coordinates, and are associated with translationa, rotational, and various other global symmetries. For a string-like object, the coordinates would include a parametrization of the string core's embedding in the larger space. These coordinates are generally functions of time, especially if the soliton is moving under the influence of external or internal forces.

We may also assume other parameters, such as the thickness/width of a soliton can be varied. For example, from Derrick's theorem, we know that a change in scale of the soliton (changing the width of the kink from μ^{-1} to $\lambda\mu^{-1}$, for example) increases the total energy for λ slightly above or below unity. The concavity of this potential means these kinds of excitations are massive, and we ought to study them further. For the kink, an ansatz for this behavior is

$$\varphi_K(x,t) = \frac{\mu}{\sqrt{\lambda}} \tanh\left(\frac{(x-x_0)}{a(t)}\right) \tag{1.46}$$

We may derive an effective Lagrangian for the new degree of freedom a by substituting this expression into (1.2) and integrating over x. The result is

$$L[a,\dot{a}] = \left(\gamma\frac{\dot{a}^2}{a} - \frac{2}{3a} - \frac{m^2 a}{6}\right) \times \frac{m^2}{2\lambda} \tag{1.47}$$

where $\gamma = (\pi^2 - 6)/18$. Casting the solution into the form $a(t) = 2m^{-1}(1 + \delta(t))$, we find the equation of motion for $\delta(t)$ to be

$$\ddot{\delta} - \frac{\dot{\delta}^2}{2(1+\delta)} + \frac{\gamma^{-1}m^2}{12}\left(1 + \delta - \frac{1}{1+\delta}\right) = 0 \tag{1.48}$$

Assuming δ is small, we linearize and find at first order the simple harmonic oscillator with frequency $\omega = m/\sqrt{6\gamma} \approx m \times 0.88$, which is only about 1.5% larger than the frequency ω_1 of the excited bound state (1.45). The actual total energy of the oscillations is however (for δ_0 the amplitude of oscillation):

$$E_\delta = M_{\text{Kink}} \times \delta_0^2 \left(\gamma^2 + \frac{1}{4}\right) \approx M_{\text{Kink}} \times \frac{\delta_0^2}{3} \tag{1.49}$$

For δ_0 sufficiently small ($\delta_0^2 \sim \lambda/m^2$), the energy contained in these oscillations is comparable to that contained in the (normalized) excited bound state η_1. Given the closeness of their frequencies, it is a reasonable assumption that these modes would couple together quite strongly, exchanging energy efficiently and significantly complicating the dynamics of an excited kink. This conclusion extends to higher dimensional versions as well, where one might expect excited states bound to φ^4 domain walls to mix readily with propagating waves in the 'thickness' parameter of the wall.

We emphasize that in classical field theory, it is not just the lone soliton configurations which are present and stable, but also field configurations of solitons and other physical excitations in the field. These excitations interact with and even re-shape the soliton, in a way consistent with their relative amplitudes and total energies. The mathematics is complicated, and except in a few cases, must be approached in a systematic, perturbative way.

1.3 Particles in Quantum Theory

In quantum theory, the amplitude of an excitation is replaced with the concept of occupation number, as the excitations themselves are promoted to "particle states." Particle number is however an uncertain quantity, and so the physical characteristics of soliton configurations in quantum theory are effected not only by the real particles with which it interacts, but also by virtual particles. This is the point of view taken within the context of renormalized perturbation theory.

1.3.1 The Axiomatic Construction of Single Particle States

The construction of single particle states in quantum field theory begins with the construction of a zero particle state, the vacuum. We will denote this state $|\Omega\rangle$ and simply state that it exists in a Hilbert Space. We define a four-momentum operator $\hat{P}_\mu$ with which the vacuum is an eigenstate. We must have

$$\hat{P}_0 |\Omega\rangle = 0 \tag{1.50}$$

$$\hat{P}^2 |\Omega\rangle = 0 \tag{1.51}$$

from which we deduce that the vaccuum is both the lowest energy state of the Hilbert space, and it is translationally invariant. The vacuum is also fully Poincaré invariant. For any other state of the Hilbert space, the expectation values of $\hat{P}_\mu$ are real values which lie within the forward light cone.

The particle states are constructed from the field operators, which we will generically denote as $\hat{\varphi}$. These fields are operator-valued distributions, which means the following. Every field operator has an associated distribution, or test function, f. This function and all its derivatives are integrable on the entirety of spacetime, and therefore admits a Fourier transform. For a collection of such operators forming a polynomial basis, one may associate with each product of operators a unique distribution called a Wightman function[10]:

$$\langle\Omega|\,\hat{\varphi}_{f_1}(x_1)\hat{\varphi}_{f_2}(x_2)\cdots\hat{\varphi}_{f_1}(x_n)\,|\Omega\rangle = W_{f_1 f_2\cdots f_n}(x_1, x_2, \cdots, x_n) \tag{1.52}$$

The collection of all Wightman functions W signifies *all* of the local observables of the quantum field theory. They are used to construct scattering matrices, correlation functions, energy densities, etc. The physical meaning of the distributions f is that of the profile of a wave-packet for the particle state(s) $\hat{\varphi}_f$ is meant to excite from the vacuum. In practice, infinite plane wave distributions are used, which while significantly reducing the computational complexity and making the subject approachable for new graduate students, introduces normalization and integrability issues.

To construct a single particle state with mass m and momentum $\vec{p}$, we require only that the Fourier transform $\tilde{f}$ has support on the mass shell, and define the state as $|\vec{p}, m\rangle \equiv \hat{\varphi}_f(x)\,|\Omega\rangle$, and note the following spectral conditions.

$$\hat{P}_i\,|\vec{p}, m\rangle = p_i\,|\vec{p}, m\rangle \tag{1.53}$$

$$\hat{P}^2\,|\vec{p}, m\rangle = m^2\,|\vec{p}, m\rangle \tag{1.54}$$

It is sufficient therefore to restrict the choices of f to those which satisfy the classical equations of motion for the field φ. As we have seen, this is almost certainly done in the linearized regime, and non-linear terms are treated perturbatively. In this linearized regime, multi-particle states may be constructed by simply multiplying together different operators as in (1.52), so long as the distributions f_i satisfy the correct spectral conditions[11].

1.3.2 Interactions

Actual field theories do not have linear equations of motion, usually. Therefore the true single particle states are only approximately given by the previous construction. In the event that the non-linear field equations may be solved exactly, of which there are only a few known cases in two dimensions, the formulation may proceed as described. If however this is not the case, complications arise.

Classically, we were justified in using perturbation theory to solve non-linear equations of motion only if the amplitude were small. Here, the wave amplitude has been discarded as a feature, and the particle number has taken its place. When particle number density becomes significant (for example when the wave packet profiles

f_1 and f_2 coincide during a scattering event), a systematic perturbative approximation must be taken which incorporates the non-linearity/interaction terms in the equations of motion. Indeed, even for a single particle propagating freely, the interaction term is not zero and modifies the properties of the particle. We alluded to this in section 1.1.1 when discussing secular divergences. Perturbative calculations of Wightman functions also lead to divergent quantities, and the process of removing those quantities by altering the underlying parameters (such as the mass) of the particle is known as perturbative renormalization theory.

Strictly speaking, the classical equations of motion are not satisfied exactly in quantum theory; they are generally dependencies on the field at multiple locations in spacetime, as we will see in section1.4.2. The renormalization procedure alters the parameters of the Lagrangian, shifting ground states in field space and the mass of excitations. In quantum theory, we seek to compute the effective action or effective potential of a set of fields, then solve the new equations of motion of the system. The result is that generally, the classical equations of motion are typically not satisfied by the quantum states. In practice, that one would like to do is assume a form of the underlying field:

$$\varphi = \varphi_{\mathrm{cl}} + \delta\varphi \tag{1.55}$$

The classical piece, φ_{cl} is assumed to be a function of spacetime variables, and the fluctuations $\delta\varphi$ are promoted to quantum operators. Through some mechanism for calculation of observables, perhaps Wightman functions, one would compute an effective action for $\varphi_{\rm cl}$, in a fashion with the spirit of our approach to calculating $L[a, \dot{a}]$ in the collective coordinate discussion, section 1.2.2. The effective equations of motion for the classical field would then be derived, and solved. This approach can only be done using perturbation theory, and is fairly challenging. We will discuss the approach and examine the first few terms in the series for this dissertation, ultimately drawing the conclusion that there are circumstances where these quantum effects stabilize field configurations which would not otherwise be stable in the classical theory. In other words, quantum effects, often in the strongly interacting regime, should be expected to violate the conclusions of Derrick's theorem. For this approach we will use the path integral, as it is uniquely set up to compute such effective actions.

1.4 Effective Actions for Solitons

In the path integral formulation of quantum field theory, it is made explicit that field configurations which do not solve the classical equations of motion are relevant. The path integral, or generating functional, for the field φ is expressed

$$\mathcal{Z} = \int \mathcal{D}\varphi \exp\left(i \int d^d x \mathcal{L}[\varphi] \right) \tag{1.56}$$

where the integration over the field variable is a functional integration. Strictly speaking, this is not a well-constructed mathematical object. Almost nothing is finite, and new ideas must be introduced to derive consistent physical quantities from this generating functional. The aim described above is to make the replacement (1.55) and compute the resulting integral over $\delta\varphi$. We use the result to define the effective action $W[\varphi_{\rm cl}]$.

$$W[\varphi_{\rm cl}] \equiv -i \ln \int \mathcal{D}\delta\varphi \exp\left(i \int d^d x \mathcal{L}[\varphi_{\rm cl} + \delta\varphi]\right) \tag{1.57}$$

In the case that φ is a free, non-interacting field, W is the action of a free, non-interacting field, plus an infinite constant. In the case of interactions, perturbation theory instructs us to compute an infinite number of additional terms for W, which are ordered roughly by the number of internal loops that appear in the Feynman diagrams. The terms in this effective action are generally integrals of the aforementioned Wightman distributions, or more often, correlation functions for plane-wave states. There is no general result, as the details depend largely on the form of $\mathcal{L}$, the number of spacetime dimensions, etc.

1.4.1 The Functional Determinant

In chapter 3, where we calculate these quantum forces for a few explicit examples, the first term in the series is called the "functional determinant." Since it appears first there, we will discuss it now. Suppose that $\varphi_{\rm cl}$ actually obeys the classical field equations for φ. Then in the expansion of $\mathcal{L}[\varphi_{\rm cl} + \delta\varphi]$, the term linear in $\delta\varphi$ will vanish. The resulting Lagrangian for $\delta\varphi$ is therefore

$$\mathcal{L}[\delta\varphi] = -\frac{1}{2}\delta\varphi\left(\partial^2 + V''(\varphi_{\rm cl})\right)\delta\varphi + \text{interactions} \tag{1.58}$$

In the imaginary time formalism, $\tau = it$ and the path integral becomes (dropping the interaction terms and those independent of $\delta\varphi$)

$$\mathcal{Z}_\delta = \int \mathcal{D}\delta\varphi \exp\left(-\frac{1}{2}\int d^d x\ \delta\varphi\Delta\delta\varphi\right) \tag{1.59}$$

where Δ is the operator

$$\Delta = -\partial_\tau^2 - \nabla_{(d-1)}^2 + V_{\mathrm{cl}}'' \tag{1.60}$$

We will find other forms of the so-called "fluctuation operator" Δ for each Lagrangian and field expansion. The evaluation of (1.59) involves an expansion of $\delta\varphi$ into an eigenbasis of Δ, then preforming the resulting Gaussian integral after moving the exponent to a Fourier representation of the spacetime integral. If the eigenbasis were finite dimensional, the result would be

$$\mathcal{Z}_\delta \propto \det{}^{-1/2}\Delta \tag{1.61}$$

Hence the name "functional determinant." The effective action therefore contains a term

$$W[\varphi_{\mathrm{cl}}] \supseteq -\frac{1}{2}\mathrm{Tr}\ln\Delta \tag{1.62}$$

However, the eigenbasis of Δ is usually infinite dimensional, so if we are to use these results we must be specific about what is meant by the trace of the log of an operator. So long as the operator is elliptic[12], we may construct a quantity known as the Zeta function of the operator, $\zeta_\Delta(s)$:

$$\zeta_\Delta(s) = \sum_\lambda \lambda^{-s} \tag{1.63}$$

where the sum is taken over all eigenvalues of Δ, with complex parameter s. If the spectrum of Δ is continuous, the sum is replaced with an integral. The zeta function is a meromorphic function, and the sum only explicitly converges for $\mathrm{Re}(s)$ larger than some value. To evaluate W, we need the derivative at $s = 0$:

$$W[\varphi_{\mathrm{cl}}] = \frac{1}{2}\zeta_\Delta'(0) \tag{1.64}$$

The meromorphic continuation of ζ is achieved by constructing a quantity called the Heat kernel of Δ, which is composed of eigenfunctions ψ_λ of Δ. The details of this computation, and the formal discussion of the topic of elliptic operators in field theory can be found in the text *Anomalies in Quantum Field Theory* by Bertlmann[12]. One defines the functional form of the Zeta function of Δ as the following.

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\sigma \sigma^{s-1} \int d^d x \sum_\lambda \psi_\lambda^\dagger(x) \psi_\lambda(x) e^{-\sigma\lambda} \tag{1.65}$$

If $\psi_\lambda(x)$ are all normalized functions, this equation reduces to (1.63) immediately. Often this is not the case (e.g. plane waves), and the sum over λ needs to be preformed first. At $\sigma = 0$ the summation diverges, and this reflects the ultraviolet divergences present in the free-space (φ_{cl} = constant) vacuum loop diagrams for $\delta\varphi$. Generally, the lower bound on the σ integral would be cut off and the usual renormalization group procedure would be employed. On the other hand, there might be a physical cutoff for the field $\delta\varphi$ and in that case the lower bound on the integral is real.

Another approach to evaluating (1.62) and (1.63) is available if there is translational symmetry in the τ direction (the field configuration φ_{cl} is static). We call k the wave number of ψ in the τ direction and write $\lambda = k^2 + \omega_n^2$, where ω_n are the eigenvalues of the time-independent Schrödinger equation in the spatial directions. The Zeta function becomes

$$\zeta_\Delta(s) = \sum_n \int \frac{dk}{2\pi} \left(k^2 + \omega_n^2\right)^{-s} = \sum_n \frac{\omega_n^{1-2s}\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \tag{1.66}$$

or by inserting into (1.64)

$$W = \int d\tau \frac{1}{2} \sum_n \omega_n \tag{1.67}$$

After moving back to real time and isolating the contribution to the total energy at a given instant, this is the famed Casimir energy, the summation of energy states as if they were equally occupied. It is obviously divergent, but one may preform meromorphic continuation (1.66) towards $s = 0$ and preform any pole subtractions (renormalization) at the origin if necessary. In deriving the right hand side of (1.66) we made the change of variables $u = k\omega_n$, ultimately casting the integral as a B function. This redefinition is not valid if zero modes ($\omega_n = 0$) are present in the spectrum. These modes must be considered separately.

In the case that $\varphi_{\rm cl}$ obeys the classical equations of motion and is static, this Casimir energy, and the resulting force, is the first order quantum correction to dynamics of the soliton. Parametrically, it is of order λ^0, with λ the coupling constant of the microscopic physics. The classical soliton stresses are of order λ^{-1}, so at weak coupling this effect is small. Further, perturbative corrections to W will be higher order in λ, and so comparably weaker. For λ order 1 or larger, at strong coupling, one can generally not trust the perturbative expansion at all. However, if one found that the Casimir force acted oppositely the classical forces, it would suggest that quantum forces have the potential to stabilize an otherwise unstable configuration, or even destabilize one which was classically stable. In other words, a violation of the conclusion of Derrick's theorem.

1.4.2 The Source Terms

If Derrick's theorem is to be violated, the linear term in the expansion of $\mathcal{L}[\varphi_{\mathrm{cl}} + \delta\varphi]$ will generally not be zero. The quantum effects manifested in the effective action generally force this to be true of all measured configurations of the field, but ordinarily the deviations from zero are assumed to be small, at least of order λ.This is taken to mean that at small coupling, the classical equations of motion are good descriptions of the quantum system. However this assumption precludes the event that these deviations are large and stable. A rigorous approach should assume this is not the case, and that the linear term cannot be omitted in any perturbative approach.

In fact we may justify this by considering the effect which small quantum forces may have on the dynamics of φ_{cl}. If initially the classical equations of motion are satisfied, the Casimir force (and others which we have not calculated) will certainly deviate this linear term from zero. The terms can therefore not be left as zero in any perturbative expansion. We must investigate the stability of these deviations from classical motion, and we will begin by supposing a static inhomogeneous classical field exists. The deviation from zero of the linear term is not zero, and we shall refer to it as a "source" and give it the label J. For the φ^4 theory we have

$$J[\varphi_{\mathrm{cl}}] = -\nabla^2\varphi_{\mathrm{cl}} + \mu^2\varphi_{\mathrm{cl}} - \lambda\varphi_{\mathrm{cl}}^3 \tag{1.68}$$

It should be noted that parametrically, $J \sim \lambda^{-1/2}$, just as the field φ_{cl} itself. The first contribution to the effective action will be $\mathcal{O}(J^2) \sim \lambda^{-1}$ which is the same size as the classical contributions. There is however an infinite series in powers of J that all are non vanishing and of the same parametric size. If one assumed that J vanished or at least scaled by increasing powers of λ, one would arrive at the conclusion that it could not possibly account for forces significant in comparison to those of the classical limit. This should be stated again for emphasis. If one assumes J to be parametrically small, then they are guaranteeing that no perturbative expansion will support a contrary conclusion. We therefore assume the form (1.68) and the resulting contributions to the effective action may be as large as possible, and look for evidence of a solution which exists in this framework. If no solution can be found, the assumption of J to be small is valid. We will quickly observe that we may not generally exclude such possibilities.

The term is referred to as a "source" because of the ability to create quanta of the field $\delta\varphi$ from "nothing." If we label a quantum state with non-vanishing J by $|J\rangle$ and time evolve it to infinity, we see that the probability to create a 1-particle state is non-zero.

$$\langle\vec{p}, m|\, U(\infty)\, |J\rangle = \langle\vec{p}, m|\, \mathcal{T} e^{i\int d^dx \delta\varphi J}\, |J\rangle \approx i \int d^dx\, \langle\vec{p}, m|\, \delta\varphi J\, |J\rangle \neq 0 \tag{1.69}$$

Hence any linear term in the action of a field $\delta\varphi$ may act to produce real quanta of that field. Exercise 4.1 of Peskin and Schröder's text asks the reader to show that the expected number of particles created by this mechanism is

$$\bar{N} = \int \frac{d^3p}{(2\pi)^2} \frac{|\tilde{J}(p)|^2}{2E_p} \tag{1.70}$$

and that the particle number follows a Poisson distribution. Here, E_p is the energy of the quanta with momentum $\vec{p}$. The calculation in their text assumes the quanta are free other than their being sourced by the function J, but in general the sum of the Fourier modes in (1.70) is taken over discrete and continuous spectra, including of course the fluctuations which are bound to the background field configuration, as in section 1.2.1.

The first contribution to the effective action W from the source term is

$$W \supseteq -\frac{1}{2} \iint d^d x_1 d^d x_2 J(x_1) \Delta^{-1}(x_1, x_2) J(x_2) \tag{1.71}$$

with Δ^{-1} the Green's function of the fluctuation operator (1.60). If expanded in a plane-wave basis, this double integral reduces to (1.70), with an extra factor of E_p, and a leftover time integral. In other words, this term calculates the expected contribution to the total energy of the system of all the quanta which were generated by the source J. Note that while the energy of said quanta are order λ^0, the expected number of those quanta, and thus the contribution to the effective action/energy is λ^{-1}, as promised.

The next contribution is cubic in J. It is obtained by computing the connected component of the 3-point function, where the fields at the 3 coordinates $x_{1,2,3}$ are weighted by factors $J(x_1)$, etc. These coordinates are thus only taken to be in the regions where J is non-zero, but the internal vertices of the Feynman graphs are to be integrated over all space. The 3-point function, for example, requires an insertion of the cubic term in the action, which is linear in φ_{cl}. This argument extends to any n-point function. In total, we find

$$W[J] = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} \left(\prod_{i=1}^{n} \int d^d x_i J(x_i) \right) G^{(n)}(x_1, \cdots, x_n) \tag{1.72}$$

The n-point function $G^{(n)}$ is defined perturbatively, with the Feynman graphs. If n is odd, there must be an odd number of insertions of the cubic interaction term. J is an odd function of φ_{cl}, as are the cubic terms. The effective action is therefore an even function of φ_{cl}, preserving the original symmetry of the classical action.

The cubic interaction terms are order $\lambda^{1/2}$, and the quadratic terms order λ. The tree level contributions to $G^{(n)}$ are therefore order $\lambda^{n/2-1}$, and thus all terms in (1.72) are order λ^{-1} at tree level. Higher order (loop) corrections are all necessarily higher order in λ, and are thus less important at small λ. The factorial suppression at large n in (1.72) is overcome by the combinatoric factors in the computation of $G^{(n)}$. The comparison of the relative size of terms in (1.72) therefore is a purely geometric consideration, depending on (for example) the volume of spacetime where J is non-zero, the parametric dependence of J on these geometrical factors, and the mass dependence in the computation of $G^{(n)}$ at tree level. We will consider an approach to computation of these terms later in this text, though we emphasize that the ordering of terms by size is situation-dependent.

If we take only the terms which are potentially of order λ^{-1}, we find the complete result is

$$W[\varphi_{\rm cl}] = S[\varphi_{\rm cl}] + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} \left(\prod_{i=1}^{n} \int d^d x_i J(x_i) \right) G^{(n)}(x_1, \cdots x_n) + \mathcal{O}(\lambda^0) \qquad (1.73)$$

This of course is already assuming $\varphi_{\rm cl} \sim \lambda^{-1/2}$, which we have justified, and is expected from classical soliton solutions. In principle, one could preform a variational analysis on this action and derive first order Euler-Lagrange equations of the quantum mechanical system. However this is easier said than done. The functions $G^{(n)}(x_1, \cdots, x_n)$ involve integrals of the Greens function Δ^{-1}, which depends not only on $\varphi_{\rm cl}$, but on *all* derivatives of $\varphi_{\rm cl}$. We will demonstrate this now.

Consider the response of the following function to the variation $\delta\varphi_{\rm cl}(y)$, for some test function f.

$$I(y) = \int d^d x f(x) \Delta^{-1}(x, y) \qquad (1.74)$$

To compute this quantity (symbolically) we will write the Greens function in an integral form

$$I(y) = \int_0^{\infty} d\sigma \exp(-\sigma\Delta) f(y) \qquad (1.75)$$

One can verify that $\Delta I = f$ by passing Δ into the integral, equating the resulting expression to a similar integral with a derivative of σ, and integrating by parts. This expression comes from the same origin as the Zeta function (1.65). For simplicity, we will write the fluctuation operator as $\Delta = -\nabla^2 + U(\varphi_{\rm cl})$, and note that variations of U are easily calculated. We wish to expand the integrand of (1.75) and calculate separately the integral of every term. However, U does not commute with $-\nabla^2$, so many derivatives of U are introduced. The result is found by application of the Zassenhaus formula.

$$e^{-s\Delta} f = e^{-s(-\nabla^2)} \left(1 - \sigma U + \frac{\sigma^2}{2}\left(\nabla^2 U + U^2\right) - \frac{\sigma^3}{6}\left(2\nabla^4 U + U^3\right) + \cdots\right) f \qquad (1.76)$$

The integration over σ may be preformed now, denoting Δ_0^{-1} as the inverse of $-\nabla^2$, Δ_0^{-2} as the inverse of ∇^4, etc, we find

$$I(y) = \int d^d x \left(\Delta_0^{-1}(x,y) - \Delta_0^{-2}(x,y)U(x) + \Delta_0^{-3}(x,y)\left(\nabla^2 U(x) + U^2(x)\right) + \cdots\right) f(x) \qquad (1.77)$$

Not only are there an infinite number of terms in just the simple expression (1.74), they depend on infinitely high numbers of derivatives of φ_{cl}. U is quadratic in φ_{cl}, so other than the very first term in this series, there will be non-vanishing variation with respect to the underlying field and derivatives of the field in every term. In practice, the test function f will be replaced with J, further complicating the matter. The variational calculus of Euler and Lagrange is intractable: we cannot reasonably apply it to the quantum theory in even the most trivial approximations.

We are incapable of applying the variational calculus, and we cannot apply Derrick's approach reliably. The engineering dimensions of Δ_0^{-n} are well known, as are those of ∇^2 and the terms in J. However there are infinite terms and the signs are alternating. In the present configuration, the effective action is hopelessly complicated, and not in a form amenable for the classical approach to the study of solitons. It should be clear that what one means by "quantum corrections" are not necessarily insignificant when compared to the classical pieces. From this perspective, the quantum field theory clearly has the potential to admit solutions that are not present in the classical field theory.

It is unfortunate that no simple analysis exists for the quantum effective actions, so that we may import the techniques from classical physics. However, the actual calculations of terms in (1.67) or (1.73) are not too challenging. For highly symmetric functions, one may simplify these expressions and even discover expansions which may be ordered by powers of the scales involved, or inversely in powers of the distance from the soliton core, for example. We will demonstrate how this computation is approached in chapter 3, especially for spherically symmetric field configurations. Before then, we will discuss a few known cases of objects existing in quantum theories which are not described or not stable in the classical field theory counterparts.

Chapter 2

Known Examples of Stable Quantum Objects

It is not an original idea that quantum mechanics should upset the rules and conclusions regarding stability of classical particles and waves. The non-relativistic theory was practically invented for this purpose. Heisenberg's uncertainty principle and Schrödinger's wave equations defy the classical laws in a broad sense: for example, phenomena such as tunneling destabilize and delocalize particles between potential barriers. Here we will discuss a few examples of field configurations that exist and are stable in quantum theories, but not in the classical versions of those theories.

The first and most important of these examples is the Hydrogen atom (or any atom, for that matter). Second, we will move beyond non-relativistic quantum mechanics, introducing the Dirichlet branes of string theory: objects which are considered solitons. Finally, we return to the discussion of black holes in quantum gravity, but in a field theoretic point of view, instead of the string theory perspective. We will discuss Hawking radiation, which drives evaporation of black holes, and scale anomalies, which have been conjectured to halt the Hawking effect, re-stabilizing the black hole.

2.1 The Hydrogen Atom

The Maxwell equations forbid a stable, localized configuration of point charges. The vector wave equations can be sourced by any changing dipole charge configuration, and the energy carried away by radiation of an orbiting electron would cause the orbital radius to decay to zero very quickly. Assuming the electron orbit is roughly circular, we may use the Larmor formula[13] to describe the adiabatic decay of the electron orbit. The power radiated per unit time is

$$P = \frac{q^4}{96\pi^3\epsilon_0^3 r^4 m_e^2} \tag{2.1}$$

The total energy of an orbiting electron is half the potential energy, or

$$E(r) = -\frac{q^2}{4\pi\epsilon_0 r} \tag{2.2}$$

Taking the derivative of (2.2) and setting it equal to $-P$ we find an equation for the electron radius over time:

$$\dot{r} = -\frac{q^4}{24\pi^2 m_e^2 c^3 \epsilon_0^2 r^2} \tag{2.3}$$

which has the solution

$$r^3(t) = r^3(0) - \frac{q^4}{8\pi^2 m_e^2 c^3 \epsilon_0^2} t \tag{2.4}$$

If the electron begins at a radius of 10^{-9} meters, in about 10^{-6} seconds the radius would drop to zero. For smaller initial radii, like the Bohr radius, more than 10 times smaller, the decay time is closer to 10^{-10} seconds.

Needless to say, there are actually atoms in the Universe, and there is evidence of their formation within the seconds after the Universe cooled below the ionization energy of Hydrogen. The Hydrogen atom exists because quantum mechanics stabilizes the classically dissipative charge/current configuration. It does so because electrons are not point particles, but excitations of an underlying field that is governed by wave equations. The equation, described by Schrödinger, is a linear equation, but possesses solutions that are static, inhomogeneous, and localized. The eigenvalue spectrum for these configurations is discrete and bounded from below, thereby preventing adiabatic decay into lower and lower energy states.

The language used here is deliberate. With the definitions given for a soliton, and the framework developed for studying their dynamics, it is a conscious choice, and an apt one, to describe the bound state configurations of the electron in a proton's potential well as such. The states of total positive energy (ignoring rest mass energy) are asymptotically free states, and are plane waves when measured sufficiently far from the proton. Those other states, described by the orbital wavefunctions, are solitons in the sense that we have been discussing. This is the first example of a soliton which is only stable due to quantum mechanical effects, and it is stabilized in a system with a weak coupling constant (the fine structure constant $\alpha = q^2/4\pi\epsilon_0\hbar c \sim 1/137$).

2.2 Strings & D-Branes

The premise of string theory is that there are one-dimensional objects called strings which exist in the universe and interact with each other. The theory, when quantized, turns out to be a theory of quantum gravity. This is apparent from the string theory action, which comes in many forms. The closed bosonic string is given the Polyakov action:

$$S = \frac{T}{2} \int d^2\sigma \sqrt{-h} h^{ab}(X) g_{\mu\nu} \partial_a X^\mu(\sigma) \partial_b X^\nu(\sigma) \tag{2.5}$$

where the worldsheet of the string has coordinates σ, local worldsheet metric h, and the variables X^μ describe the embedding of the string into the spacetime manifold, which is given metric g. The dimensionful variable T is the tension of the string. Upon solving the equations of motion for the string worldsheet metric components h_{ab} and substituting back into (2.5), we find that $S = TA$, with A the two-dimensional area of the worldsheet.

Hence, the classical dynamics of string theory, at least those of the Polyakov action, dictate that strings should decrease their energy by shrinking whenever possible.

The string clearly interacts with the background metric (or equivalently with a condensate of strings in a particular spin-2 state), so assuming gravitational wave production is a mechanism for an oscillating string to dissipate energy, it will do so.

2.2.1 S-Duality in the Type IIB Theory

A type of duality which appears in string theory but also throughout quantum field theory, is the *S-Duality.* It stands for "strong-weak" duality, and refers to the exchange of large and small coupling constants, which respectively entitle their theories with the strong and weak coupling descriptions[14][15].

Before we discuss examples of S-Duality, we should discuss the case of open strings. These are segments of energy that do not form closed loops like the bosonic strings of the previous section. They have two end points, and therefore require additional description of the dynamics of those end points. Denoting the time and space worldsheet coordinates by $\tau \in \mathbb{R}$ and $\sigma \in [0, 1]$ respectfully, we demand there be some boundary condition on the embedding functions

$$X^\mu(\tau, 0), \quad X^\mu(\tau, 1) \tag{2.6}$$

or their derivatives with respect to worldsheet coordinates. The simple case is the Neumann boundary conditions:

$$\partial_a X^\mu(\tau, 0 \text{ or } 1) = 0 \tag{2.7}$$

This ensures the total momentum and energy of the string remains finite. Such strings may exist on their own, far from other objects. The string theory with Neumann boundary conditions on open strings and supersymmetry in the worldsheet is called the type I superstring theory. Type I also contains closed strings. If the string theory is defined with perturbation theory (the string coupling constant $g_s << 1$), closed strings may even break open or scatter off each other to produce open strings, conserving energy and momentum in the process. The open Neumann string contributes to vacuum fluctuations in the same way that any weakly coupled particles would in quantum field theory.

The open Neumann strings are the asymptotically free states. They have the same spectral properties as closed strings, they exist on their own, they scatter in accordance with perturbation theory, and they're mundane. The open Dirichlet strings are the bound states of string theory. Their ends are fixed in place on the ambient space, or generally to submanifolds of the spacetime that we call Dirichlet Branes, or D-Branes[16]. Dirichlet boundary conditions allow the strings to absorb energy from their surroundings, since the curvature of the end of the string is unconstrained, and they do this through interaction with the D-Branes that they intersect.

The submanifold where the open strings are fixed in place is a dynamical object, and therefore posesses an energy density or tension. The tension of the D-brane scales inversely with the string coupling; we describe the branes with an action, e.g. the

Dirac-Born-Infeld action, for a $p-$dimensional brane:

$$S_p = -T_p \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det\left(G_{ab} + B_{ab} + T^{-1} F_{ab}\right)} \tag{2.8}$$

where Φ, G, B and F represent the condensation of particle string states defined in the perturbative sector[16], and T_p is the tension of the brane.

There may be coefficients that depend on the geometry of the target space, generated by intermediate open string states connecting the brane to itself, and these coefficients may diverge[14]. Hence the treatment of these submanifolds as dynamical objects and the labeling as "branes" (short for membrane). The inverse dependence of the brane tension on the string coupling is reminiscent of solitons. That there are states (the open strings) which only exist bound to the branes is also very much like a soliton. The Dirichlet strings cannot be described as perturbative fluctuations from the vacuum: one must construct a whole D-Brane with its large tension to see the open strings attached to it. Solitons are also not perturbative fluctuations from the vacuum.

In the type I and II superstring theories, the D-branes can be objects of various dimension, up to 11 (the dimension of the target space of the theory). D1-Branes are heavy string objects, D2-Branes are walls of energy, etc. The physics of D-Branes opens a deep and thought provoking discussion of dualities between the superstring and supergravity theories, but this is of little importance to us presently. The duality we are interested in is the type IIB self-duality, which is an S-Duality. It will suffice to say that there is a coupling constant in the type IIB theory, and there is an equivalence in the spectrum between the cases where the coupling is small and when it is large. As in the case of the T-Duality, this requires a relabeling of degrees of freedom, and in the IIB case, what is swapped is the Branes themselves[17]. In particular, at weak coupling we have light fundamental strings (F-strings) and heavy D1-Branes (D-strings), but at strong coupling these one-dimensional objects reverse their roles. The solitons become light and the perturbative states become heavy. Perturbations from the vacuum are therefore best described at strong coupling with the solitons, the D-strings, instead of the F-strings. There is an $\mathcal{O}(1)$ coupling where the two objects have the same tension. Probably the theory is not well described by any perturbative expansion at this point.

2.3 Black Holes

Our previous description of black holes as solitons gave no mention to stability or fluctuations of the geometry. This is because gravitational waves cannot be emitted from a spherically symmetric black hole, or even a rotating one. The waves are perturbations of a rank-two symmetric tensor, and thus the lowest moment which produces them is an oscillating quadrupole moment. Gravitational waves have been observed from inspiraling black holes which then collided and merged, but never from the solitary Schwarzschild black holes discussed earlier. In fact no radiation has been observed of any type originating at the event horizon, besides that from accretion disks. Quantum field theory predicts that the event horizon of a black hole should be a source of continual radiation however, and this effect ought to (slowly) evaporate black holes. This is known as Hawking radiation, and it draws into question our description of solitary black holes as permanent objects.

2.3.1 Hawking Radiation

The discussion of Hawking radiation begins with the observation that energy is not a quantity which can be defined globally in curved spacetime, except in cases where the spacetime is asymptotically flat [7]. There is no local measurement of energy that separated observers may agree on. Additionally, an observer in a weak gravitational field will observe a time-dilation for observers in a strong gravitational field. An object falling into the event horizon of a black hole never appears to reach the horizon, according to an observer far away, for example. Since the Hamiltonian is the generator of time translations for local observables, and time is relative, so is the quantity of total energy, at least for a localized distribution of matter.

It stands to reason, therefore, that the spectral properties of a Poincaré invariant vacuum state are not valid and consistent everywhere in spacetime, especially if a region exists where the gravitational time dilation diverges (relative to a far away observer). In short, the description of the vacuum near the event horizon of a black hole is not identical to the vacuum described infinitely far away. This may be traced to the failure of the spectral relations 1.53 and 1.54 in spacetimes with non-trivial connections. In fact, the vacuum near the horizon has non-zero overlap with $n-$particle states described infinitely far away:

$$|\Omega\rangle_{\mathrm{BH}} = b_0 |\Omega\rangle_\infty + b_1 |1\rangle_\infty + b_2 |2\rangle_\infty + \cdots \tag{2.9}$$

the coefficients b_i are calculated using the Bogoliubov transformations, and describe a thermal distribution. The temperature of this distribution is proportional to the surface gravity at the black hole[18], and in SI units is

$$T_{\mathrm{BH}} = \frac{\hbar c^3}{8\pi G k_B M} \tag{2.10}$$

Equating to a temperature of about 10^{23} Kelvin for a 1 kg black hole. For astrophysical black holes, the temperature is orders of magnitude lower than the temperature of the CMB. Small black holes are hot, and should emit radiation according to the Stephan-Boltzmann law. For a black hole radiating purely into photons, the radiative power is

$$P = \frac{\hbar c^6}{15360\pi G^2 M^2} \tag{2.11}$$

A 1 kg black hole should completely evaporate in 10^{-16} seconds via this mechanism. The lifetime of a black hole with a mass of 10^{11} kg comes in just under the current age of the Universe. The final moments of an evaporating black hole's life should be very hot, and the death of primordial (formed before galaxies, from non-astrophysical origins) black holes should be accompanied by an explosion of high energy, very heavy particles.

That is, unless the effects of quantum gravity slow and stop the evaporative processes at small mass scales, near and above the Planck scale.

2.3.2 Effective Action and Anomaly

The effective action for a field theory generally contains an infinite number of terms, and these terms can be ordered according to the number of derivatives they contain. We have seen this explicitly in section 1.4. Generally, these terms include not only classical background configurations for the field itself, but also geometric invariants related to the geometry of the target space manifold. The first geometric term supplied by some quantum field theory dynamics may be of the form

$$W \supseteq \Lambda^2 \int d^4x \sqrt{-g} R \tag{2.12}$$

with Λ some high energy scale that describes the microscopic field theory degrees of freedom, and R the Ricci scalar. This is in fact just the Einstein-Hilbert action for General Relativity. In this context, we think of Einstein gravity as an effective theory of some high energy quantum theory that is currently unconstrained in form by experiment.

The next most important terms are the four-derivative geometrical scalars. In a gravitational field sourced by particles with mass m, these terms will be less important than the Einstein-Hilbert action by a factor of $m^2 G \propto m^2/M_{\mathrm{PL}}^2$. Higher derivative terms are further suppressed, except of course in regions with curvature singularities, like the center of a black hole. The four-derivative terms are

$$W \supseteq \int d^4x\sqrt{-g}\left(\alpha R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^2 + \delta C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma}\right) \tag{2.13}$$

with the Riemann, Ricci, and Weyl tensors present, respectively. The coefficients $\alpha, \beta, \gamma, \delta$ are dimensionless, but acquire beta functions due to renormalization. They are not actually independent, but related to each other and the particle content that generates such terms in an effective action, e.g. the number of scalar, fermionic, or vector degrees of freedom in the microscopic theory[19]. These terms appear even in theories which possess conformal symmetry, and will explicitly break the conformal symmetry of the action. They are thus collectively referred to as the "trace anomaly," since conformal symmetry is characterized by a vanishing trace of the stress-energy tensor.

We should investigate the effects of these terms on the Schwarzschild solution, and therefore the temperature of Hawking radiation. We will consider a combination of the coefficients so that the trace anomaly is the Euler density, and call the new coefficient a. The Euler density is

$$E_{(4)} = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{2.14}$$

We now study the equations of motion for the metric perturbatively. Taking the trace of the usual Einstein field equations and inserting the anomalous contribution to the trace of the stress tensor, we find the equations of motion

$$\frac{R}{8\pi G} = aE_{(4)} \tag{2.15}$$

The anomaly coefficient a is small when the number of degrees of freedom in the microscopic theory is small, and large otherwise. Allowing for an $\mathcal{O}(a)$ perturbation to the line element (1.19) we find there is a solution asymptotically at large r:

$$g_{tt} = -1/g_{rr} = 1 - \frac{2GM}{r} + \frac{64\pi G^3 M^2 a}{r^4} + \mathcal{O}\left(\frac{a^2}{r^6}\right) \tag{2.16}$$

Thus the location of the event horizon (according to an observer far from it) is

$$r_s \approx 2GM - \frac{16\pi a}{M} \tag{2.17}$$

The correction to the event horizon radius is small when $M >> M_{\mathrm{PL}}\sqrt{a}$, not a difficult limit to achieve. The corresponding temperature of the horizon is

$$T_{\mathrm{BH}} = \frac{\hbar c^3}{8\pi G k_B M}\left(1 - \frac{2\pi a}{GM^2}\right) \tag{2.18}$$

Suppose there are a great number of degrees of freedom in the physics generating this term in the effective action. a is therefore large. This can be obtained if, for example, there was a weakly interacting scalar field at high energies with a great number of flavors, perhaps in the thousands. We conclude then that for a black hole mass $10 - 100$ times larger than the Planck mass, the evaporative effects halt. Of course near the Planck mass, we cannot omit the higher order derivative terms in the effective action. Perhaps though, if a cold (not evaporating) 1 kg black hole is discovered, one now has a working knowledge of a mechanism which stabilized the black hole. One may also infer there to be a large number of degrees of freedom at the Planck scale.

More realistically, the effect of an anomaly lowering the temperature begs the question: should we trust any of these results for very small black holes? Most would probably say no: a more systematic study of the infinite derivative structure of the effective action is required. Unfortunately, the exact form of the action eludes us, as do the tools with which to study its exact solutions.

In addition to the possibility of halting the Hawking evaporation, there are conjectures that these full quantum mechanical equations of gravity rid us of the singularity at the center of a classical black hole. These are mostly guesses, some more educated than others (based, for example, on the description of a black hole as a multi-brane junction in superstring theory), and ultimately we do not have answers to most questions about the solutions to the field equations of quantum gravity. It is expected that at the smallest of scales, they are fundamentally different than the solutions to the classical Einstein field equations.

Chapter 3

Modifications of Forces by Quantum Effects

We are now prepared to discuss the calculation of quantum forces on the previously detailed solitons. As already noted, we cannot calculate the exact values of any of these forces, except for particular cases and in some limiting behavior, e.g. long range effects order by order in their relevance at large distances. To that end we will begin this chapter by calculating corrections to the energy density of the global vortex/string perturbatively in powers of $1/r$ at distances far from the core of the soliton. The calculation will assume the classical vortex/string configuration (1.13), and evaluate leading order terms in the functional determinant[20]. Then, we shall proceed to the discussion of the functional determinant on spherical geometries, citing past results in odd dimensions, and extending those results to the even, two-dimensional case via computation of the Zeta function[21]. Finally, we will consider the effect of the source terms for a pathological field configuration in the four dimensional φ^3 classical field theory. We will see that quantum corrections cure this disease[22].

3.1 The Mass/Tension of the Global Vortex/String

We will restrict the discussion of this section to the case $d = 4$, i.e. the global string, which when stretched uniformly along the z axis, may be considered an extension global vortex in the x-y plane. The usefulness of studying this soliton in the four-dimensional case is that the quartic coupling constant λ is dimensionless, and the dimension of any term may be read off easily from the powers that appear of the quadratic coupling μ. The Lagrangian density is

$$\mathcal{L} = (\partial\Phi)^2 + \mu^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 + \frac{\mu^4}{4\lambda} \tag{3.1}$$

The field will be expanded around the solution (1.12), and we will consider fluctuations in the phase angle of the field Φ (these are the Goldstone bosons, which we call α):

$$\Phi = \Phi_n e^{i\alpha} = \frac{\mu}{\sqrt{2\lambda}} e^{in\theta + i\alpha(x^\mu)}\left(1 - \frac{n^2}{2\mu^2 r^2} + \mathcal{O}\left(\frac{n^4}{\mu^4 r^4}\right)\right) \tag{3.2}$$

The heavy fluctuations, those in the modulus of the field Φ, are not of any importance to us, as they will generally produce forces that are only relevant on distance scales of μ^{-1}. The field α is massless, and therefore produces long-range forces which will most significantly impact the large r asymptotics of the effective action. The action for α is easily derived:

$$S[\alpha] = -\int d^4x\ \alpha\partial_\mu \left(\Phi_n^\dagger \Phi_n \partial^\mu\right) \alpha \tag{3.3}$$

Upon computing the path integral, we see that the effective action for Φ_n due to the field α is equal to the functional determinant component. This however is only valid up to the scale at which the Goldstone bosons couple with the massive fluctuations in the modulus field. This occurs at a scale $\sim \mu$, so in the following analysis we will consider any high frequency limits to be in reference to this value. According to (1.62), we must have (in the case of $\dot{\Phi}_n = 0$)

$$E_{\text{Cas}}[\Phi_n] \propto W[\Phi_n] \supseteq \frac{1}{2}\text{Tr}\ln\Delta \tag{3.4}$$

where the operator Δ is

$$\Delta = -\frac{1}{r^2}\partial_r \left(r^2 f^2(r)\partial_r\right) \tag{3.5}$$

and $f(r)$ the radial profile of the soliton (1.13). Note that the coordinate r here is the polar radius in the cylindrical coordinate system $\{r, \theta, z\}$, not the spherical radial coordinate. We now introduce the connection ω_μ, covariant derivative D_μ, and potential U:

$$\omega_\mu = \partial_\mu \ln f \tag{3.6}$$

$$D_\mu = \partial_\mu + \omega_\mu \tag{3.7}$$

$$U = \partial^2 \ln f + (\partial \ln f)^2 \tag{3.8}$$

using these definitions, the log of the fluctuation operator may be written

$$\ln\Delta = \ln f^2 + \ln\left(-D^2 + U\right) \tag{3.9}$$

The connection is singular at the origin, as is the potential, but the effective action may only contain geometric invariants associated with these objects, which will be integrable. We will return to the second logarithm in a moment, but first let's evaluate the $\ln f^2$ term.

3.1.1 Another Log Divergence

The contribution to the effective action is the trace of (3.9), which involves an integration over spacetime, as well as an integration over four-dimensional momentum space. This must be the case as the action is to remain dimensionless. The term of interest is not momentum-dependent, so a constant cutoff Λ is to be introduced. From previous discussions it is clear that $\Lambda \sim \mu$. Hence we have

$$\mathrm{Tr} \ln f^2 = \Lambda^4 \int dz \int_0^{2\pi} d\theta \int_0^{\infty} dr r \ln f^2 \tag{3.10}$$

Expanding the integrand at large r and simplifying, we find a familiar result: the integral is divergent.

$$E_{\mathrm{Cas}} \supseteq -\frac{\pi n^2 \Lambda^4}{\mu^2} \int dz \int \frac{dr}{r} + \text{constant} \tag{3.11}$$

Compare this with the contribution to the vortex mass (1.14), which when translated to a classical string tension, becomes

$$T_{\text{string, cl}} = \frac{\pi n^2 \mu^2}{\lambda} \int \frac{dr}{r} \tag{3.12}$$

Both quantities diverge with the logarithm of some large radius R, but have opposite signs. (Note: the integration bounds are assumed to be taken at some finite radius to infinity, since the asymptotic expansion is not valid near the origin. There is no problem at $r = 0$). Therefore, the total divergent contribution to the global string tension is

$$T_{\text{string}} \sim \frac{\pi n^2 \mu^2 \ln(R)}{\lambda} \left(1 - \frac{\lambda \Lambda^4}{\mu^4}\right) + \text{constant} \tag{3.13}$$

We have already made the point that $\Lambda \sim \mu$, as any frequency modes for the field α above this value will couple to the massive excoriations and therefore be suppressed at long distances. Therefore, one expects that for $\lambda \sim 1$ (recall that in $d = 4$ this is a dimensionless quantity) the divergent contribution can be made to change sign or even cancel. Of course what one means by $\lambda \sim 1$ must be put into the context of the renormalization group flow, and is generally scheme-dependent. In $d = 3$ however the parameter λ is in fact not renormalized, and takes on a fixed value that defines the model. μ is renormalized, but then so is the physical Λ. There is actually no reason one cannot take $\Lambda << \mu$. This corresponds to preforming the path integration only for the very lowest momentum modes of α, well below the scales that they begin to couple with the massive excitations. For a sufficiently large λ, it stands to reason that even the lowest of the low frequency modes of the quantum theory may significantly alter the mass/tension of the global vortex/string, and therefore compromise the approximations appropriated from classical physics.

3.1.2 The Spectral Approach to the Log Divergence

Another equivalent method may be employed to arrive at the result (3.13). This will justify the use of the parameter Λ, and will make contact with methods explored in a pertubative approach to quantum theory. We may consider a spectral approach to computing the trace of $\ln \Delta$. Suppose the eigenvalues of Δ are labeled ξ, and at large distances these are perturbed from their base values by the $1/r$ corrections to the operator. In the four-dimensional Euclidean spacetime we have (still in cylindrical coordinates)

$$\Delta = \left(-\nabla^2_{(4)}\right) - \frac{n^2}{\mu^2 r^2}\left(-\nabla^2_{(4)} + \frac{2}{r}\partial_r\right) + \mathcal{O}\left(\frac{1}{r^4}\partial_r^2\right) \tag{3.14}$$

the eigenvalues ξ are defined by

$$\Delta\psi_\xi = \xi\psi_\xi \tag{3.15}$$

and we may use perturbation theory to compute deviations from the "free space" eigenstates $\psi^{(0)}_{\xi_0}$:

$$-\nabla^2_{(4)}\psi^{(0)}_{\xi_0} = \xi_0\psi^{(0)}_{\xi_0} \tag{3.16}$$

The first order corrections, ξ_1 are given by

$$\xi_1 = -\int d^4x \frac{n^2}{\mu^2 r^2}\psi^{\dagger(0)}_{\xi_0}\left(-\nabla^2_{(4)} + \frac{2}{r}\partial_r\right)\psi^{(0)}_{\xi_0} \tag{3.17}$$

and the Casimir energy by

$$E_{\text{Cas}} \propto \frac{1}{2}\sum_\xi \ln(\xi) \approx \frac{1}{2}\sum_{\xi_0}\ln(\xi_0) + \frac{1}{2}\sum_{\xi_0}\frac{\xi_1}{\xi_0} \tag{3.18}$$

The functions $\psi^{(0)}_{\xi_0}$ are easily derived, as are their eigenvalues ξ_0:

$$\psi^{(0)}_{\xi_0} = J_\ell(kr)\exp\left(i\omega\tau + iqz + i\ell\theta\right) \tag{3.19}$$

$$\xi_0 = q^2 + k^2 + \omega^2 \tag{3.20}$$

It is clear then that the the summations in (3.18) ought to be replaced with integrals over k, q, ω and a summation over integers ℓ, and that the first term on the right hand side of (3.18) is divergent. This does, in fact, correspond to an infinite, homogeneous energy density that is present in all quantum field theories. It is dealt with by renormalization of the constant term in (3.1), but is otherwise not associated with an observable quantity. The second term in (3.18) may now be expanded. We will group the momentum integrals into one four-dimensional integral with measure d^4k.

$$W \supseteq -\sum_{\ell=-\infty}^{\infty} \int \frac{d^4k}{(2\pi)^4} \int d^4x \frac{n^2}{2\mu^2 r^2 (k^2 + q^2 + \omega^2)} J_\ell(kr) \left(k^2 + q^2 + \omega^2 + \frac{2}{r}\partial_r\right) J_\ell(kr) \tag{3.21}$$

We now employ an important identity of Bessel functions of the first kind:

$$\sum_{\ell=-\infty}^{\infty} J_\ell^2(kr) = 1 \tag{3.22}$$

The term with the linear derivative in r thus vanished when the ℓ summation is preformed, and the factors of ξ_0 cancel in the remaining integrand. The result is

$$W \supseteq -\frac{n^2}{2\mu^2} \int \frac{d^4k}{(2\pi)^4} \int d^4x \frac{1}{r^2} \tag{3.23}$$

The momentum integral is clearly some constant cutoff, which we call Λ^4. Going back to the real time formalism, dropping the time integrals and the z integral (being careful with negative signs), we derive the contribution to the tension from the Casimir effect/functional determinant to be exactly as found before.

$$T_{\text{string, Cas}} \supseteq -\frac{\pi n^2 \Lambda^4}{\mu^2} \int \frac{dr}{r} \tag{3.24}$$

3.1.3 Higher Order Terms in the Functional Determinant

We now focus on the remainder of equation (3.9), i.e. those terms involving the covariant derivative D_μ and the potential U. To do so we recall equation (1.65), in a new form:

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\sigma \sigma^{s-1} \int d^d x K_\Delta(\sigma|x, x) \tag{3.25}$$

where $K_\Delta(\sigma|x, y)$ is called the "heat kernel" of Δ, and solves the corresponding heat equation[19][12]

$$\left(\frac{d}{d\sigma} + \Delta\right) K_\Delta(s|x, y) = 0 \tag{3.26}$$

with boundary condition

$$\lim_{\sigma \to 0} K_\Delta(\sigma|x, y) = \delta^{(d)}(x - y) \tag{3.27}$$

We will solve for the heat kernel perturbatively in powers of U, $D_\mu U$, etc. The approximation is known as the Seely-DeWitt expansion, and it involves perturbing the solution to the Green's function for the heat equation in d dimensions:

$$K_\Delta(\sigma|x, y) = \frac{1}{(4\pi\sigma)^{d/2}} \exp\left(-\frac{|x-y|^2}{4\sigma}\right) \sum_{n=0}^{\infty} \sigma^n a_n(x, y) \tag{3.28}$$

Where $a_0(x, y) = 1$, and the remaining terms are recursively derived by solving the differential equation

$$\left(\frac{n}{2} + (x^\mu - y^\mu)\partial_\mu\right) a_n(x, y) + \Delta a_{n-1}(x, y) = 0 \tag{3.29}$$

In the coincidence limit $y \to x$, which is relevant for the functional determinant, the recursion relation reduces to a simple functional form that can be solved explicitly for all n:

$$a_n(x,x) = \frac{(-1)^n}{n!}\Delta^n 1 \tag{3.30}$$

Thus, we find the integrand of (3.25) to be

$$K_\Delta(\sigma|x,x) = \frac{1}{(4\pi\sigma)^{d/2}} e^{-\sigma\Delta} 1 \tag{3.31}$$

It is however easier to analyze the terms as a power series in σ, preforming the integration over σ before the summation over n. One of course encounters divergences when exchanging the order of operations for an expression that is (probably) not finite, but we will ignore these as they are the same infinities that have been dealt with by the theory of perturbative renormalization. Integrating over σ will give divergent term if (as $s = 0$ eventually) $2n \leq d$. Thus we introduce a cutoff Λ^{-2} for the lower bound of the σ integral. Because the field α is massless, there is a large σ divergence for $2n > d$, and thus we cut off the upper bound of the integral with some large distance scale R^2. In principal one must compute every term in the series with these parameters, then re-sum them, and pray to a higher power that the limit exists (and is finite) as $R \to \infty$. We will do no such thing, only remark on the asymptotic form of the functions $a_n(x,x)$. Ultimately the result is (for some coefficients c_n that depend on R, Λ):

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)(4\pi)^{d/2}} \int d^d x \left(c_0 + c_1 \Delta 1 + c_2 \Delta^2 1 + \cdots\right) \tag{3.32}$$

The functions $\Delta 1, \Delta^2 1$, etc. are typically geometric invariants, such as the potential U, derivatives of U, the Ricci scalar, derivatives of the Ricci scalar, the curvature tensor for connection ω, etc. In this case we have explicitly

$$\Delta 1 = U(x) \sim \frac{1}{r^4} \tag{3.33}$$

$$\Delta^2 1 = U^2(x) - D^2 U(x) \sim \frac{1}{r^6} \tag{3.34}$$

and so forth. The c_0 term contains a volume divergence, but c_0 is also proportional to Λ^{4-2s}. It is a typical homogeneous energy density, of the same kind we previously encountered. $c_1 \sim \Lambda^{2-2s}$, but the contribution to the string tension from integrating $U(x)$ is finite. We see that the terms in this expansion all correspond to integrable contributions to the tension of the string, though the coefficients may be cuttoff-dependent and thus renromalizable parameters.

An alternative way to look at these results is the following. If a field is defined at a temperature T, the mass parameter depends on that temperature: $\mu = \mu(T)$. There is a critical temperature T_c near which $\mu^2(T) = a(T - T_c)$ for some constant a. Thus for $T < T_c$ we have a spontaneously broken $U(1)$ symmetry, and above it, a restored/unbroken symmetry. As T vanishes, μ^2 tends to some finite positive constant. Λ is the stand-in for T, and from equation (3.13) we see that there is some intermediate $\mu(T) \sim T_* < T_c$ for which the coefficient in parenthesis changes sign. The expression in (3.13) may also be considered as the leading order long-range potential energy between infinitesimal string segments (of opposite winding number) that are separated by distance R. For $T < T_*$ the strings are pulled together and confined, but for $T > T_*$ they are repulsed, unconfined. At T_c the string core ceases to exist and the strings evaporate. A similar mechanism has been studied before in a thermodynamic context under the name "Berezinskii–Kosterlitz–Thouless (BKT) transition"[23][24], though always in the context of non-relativistic systems in two or lower spatial dimensions. This effect suggests that cosmic global strings may have been deconfined for a long period after their formation, or perhaps still are, depending on the particular value of λ that describes the underlying physics.

3.2 Casimir Forces on Spheres

We turn now to a far simpler computation of the functional determinant. To motivate the discussion, we should review some material relevant to cosmic axion strings. These are global strings like we have already studied, where the axion is the Goldstone boson of a global $U(1)$ symmetry broken at very high energies. There are many reasons a phenomenologist would consider such a system, the most important of which is that it provides a mechanism to solve the Strong CP problem[25]. In this case, a confining gauge theory at low energies has many degenerate vacuum states, and therefore processes exist which tunnel between these vacuua. The tunneling processes are called *instantons*, which are like solitons except they are localized in spacetime, not just space. They occur in an instant, hence the name. The instantons make identification of the many degenerate vacuum states (in e.g. $SU(3)$ gauge theories) indistinguishable, and thus only a relative phase between states is measured. This phase angle is called θ, and it breaks charge-parity (CP) in the gauge theory, when fermions are present, and $\theta \neq 0$. The standard model should have a θ, but it has been measured to be zero, to one part in 10 billion. The anomalous smallness of θ without real explanation is known as the Strong-CP problem, and the axion was proposed to solve this problem.

In the context of Strong-CP, θ is promoted to a dynamical field, and this is the axion, the Goldstone boson of some high energy spontaneously broken $U(1)$ symmetry. The instantons dynamically generate a potential for the axion, forcing it to its minimum at zero, explaining the smallness of θ and creating an anomalous axion mass. Through interactions with the quarks, the axion couples anomalously to the photon, and thus mixes with the photon. This is not important to us. What matters is the instantons cause the axion to develop a small anomalous mass[26], the phase angles of the axion field around the global string (equation (3.2)) are no longer isotropic. Angles away from $\alpha = 0$ develop higher energy density, and therefore a two-dimensional membrane forms (it will have width proportional to the inverse axion mass, but this is very small in comparison with the area formed by the membrane), bounded by the much more dense string core.

All this is to say that if global strings are to be expected and sought out in the cosmos, many models predict that the strings will be accompanied by thin membranes. Here we will consider the case of a circular string of tension T, hosting a disk membrane of tension Σ. If $\Sigma^{1/3} << T^{1/2}$, and the radius is large, most of the energy will be stored in the string, and the lowest energy oscillations of the string-membrane system will be vibrations of the disk which leave the string undeformed. The frequencies of oscillation are the same of a drum head, and are the roots of the ordinary Bessel function of the first kind, i.e. $J_\ell(j_{\ell,n}) = 0$, $n = 1, 2, 3, \cdots$.

$$\omega = \frac{j_{\ell,n}}{a} \tag{3.35}$$

The Casimir energy is therefore some constant divided by the radius of the string/disk a, and it is our goal to determine the sign and size of this constant. We will find it is positive, indicating a repulsive force. The Casimir effect on spherical geometries has been explored for a long time, the main avenue of research being the computation of the zeta functions for summations over the roots of ordinary and spherical Bessel functions. In three spatial dimensions, the force on a sphere from massless modes trapped within is known to be repulsive. As it turns out, the forces and therefore the energies of the even spatial dimensions (on 1-spheres, 3-spheres, etc) is divergent[27]. There are simple poles in the Zeta function at the point corresponding to the Casimir effects. We can see this explicitly from McMahon's expansion[28]. For fixed ℓ and large n, the following form is known.

$$j_{\ell,n} \sim \pi\left(n + \frac{\ell}{2} + \frac{1}{4}\right) - \frac{4\ell^2 - 1}{8\pi\left(n + \frac{\ell}{2} + \frac{1}{4}\right)} - \mathcal{O}\left(\frac{1}{n^3}\right) \tag{3.36}$$

The summation is to be taken over the natural number n, and also over the integer ℓ. It is clear that the first term diverges, but one may regulate the summation with the Riemann zeta function. i.e.

$$\sum_{n=1}^{\infty} n \to \zeta_R(-1) = -\frac{1}{12} \tag{3.37}$$

and so forth. However, the second term in (3.36) is infinity even in this scheme.

$$\sum_{n=1}^{N} \frac{1}{n+c} \sim \ln(N) \rightarrow \zeta_R(1) = \infty \tag{3.38}$$

Logarithmic divergences appear as simple poles in the complex plane. Multi-log divergences are higher order poles. The zeta function for the eigenvalues (3.35) contains only simple poles. In fact, its analytic structure is known for a significant portion of its domain[29]. Let's define this zeta function as the circular zeta function, calling it $\zeta_{\odot}$. Where the sum converges, it is equal to

$$\zeta_{\odot}(s) = \sum_{\ell=-\infty}^{\infty} \sum_{n=1}^{\infty} (j_{\ell,n})^{-s} \tag{3.39}$$

Since from (3.36) is is clear $j_{\ell,n}^{-1} \sim n^{-1}$, it must be that this sum diverges for $\mathrm{Re}(s) \leq 1$. In fact the first pole appears at $s = 2$. The sum over n has been computed for $s = 2, 4, 6, \cdots$ and in the case $s = 2$ we find

$$\zeta_{\odot}(s) = \sum_{\ell=-\infty}^{\infty} \frac{1}{4(\ell+1)} \rightarrow \infty \tag{3.40}$$

We must mention that while the summation over ℓ is taken over all the integers, the ℓ in (3.36) and (3.40) is positive. This is because the roots for J_ℓ and $J_{-\ell}$ are identical, corresponding to the waves which move clockwise and anti-clockwise in the two-dimensional space. For $\ell \neq 0$ there is two-fold degeneracy, but the $\ell = 0$ state is singularly degenerate. In three-dimensions, for example, the degeneracy is $2\ell + 1$. We omit absolute value signs in these expressions because they are unsightly.

The circular zeta function has poles at $s = 2, 1, -1, -3, -5, -7, \cdots$. The pole at $s = 2$ is due to a log divergence of the summation over ℓ, while the remaining poles are due to log divergences in the summation over n. They appear at odd integers because the powers of $1/n$ in (3.36) are only odd. One may calculate the asymptotic form of $j_{\ell,n}$ to any odd positive power and verify the series contains a $1/n$ term. Hence the location of the poles in $\zeta_\odot$ are due to asymptotic spacing of the eigenvalues $j_{\ell,n}$, and the singularity of the Riemann zeta function in (3.38).

The first few residues are known as well[29]. We will only need the first three, as the form of $\zeta_\odot$ we will take is

$$\zeta_\odot(s) = \frac{1}{2(s-2)} - \frac{1}{2(s-1)} - \frac{1}{128(s+1)} + \varphi(s) \tag{3.41}$$

where $\varphi(s)$ is an analytic function in a neighborhood around $s = -1$. The residue at $s = -5$ is positive, and about half the magnitude of the residue at $s = -1$. Since the simple poles in $\zeta_\odot$ are due to distributions of eigenvalues in the high-frequency limit of the spectrum, we are justified in dealing with these logarithmic divergences in the usual way of renormalization. We define the Minimal-Subtraction (MS) renormalized Casimir Energy as

$$E^{\text{ren}}_{\text{Cas}}(a) = \frac{1}{a}\lim_{s\to-1}\left(\zeta_\odot(s) + \frac{1}{128(s+1)}\right) = \frac{1}{a}\left(\frac{1}{12} + \varphi(-1)\right) \tag{3.42}$$

The subtraction of the simple pole at $s = -1$ may be associated with the renormalization group flow of some other parameter in the theory, which depends on a in the same way. For example, the tension T or Σ may be renormalized in a radius-dependent way by this cancellation. This is unimportant, since the physical forces exerted on the string/membrane must be finite and therefore determined by (3.42). We have preformed the first computation of $\varphi(-1)$ and found it to be negative, but smaller in magnitude than 1/12. Hence the renormalized Casimir energy is positive, and the forces repulsive[21]. To preform this calculation, we employed numerical analytic continuation of the function $\varphi(s)$ on a region of the complex plane, which we will now discuss.

3.2.1 Data Generation

Efficient computation of the $j_{\ell,n}$ for small ℓ and n is handled quite well by libraries like Python's SciPy, which utilizes Newton's algorithm for root finding. For large n and small ℓ, the McMahon asymptotic expansion is preferable to root finding. The built-in SciPy root finder for $J_\ell(x)$ uses $x = 0$ as its starting point however, which poses a significant problem for larger values of ℓ. These functions are extremely small near the origin for larger values of ℓ, and do not grow to appreciable values until x is comparable with the first root $j_{\ell,1}$. For example, $J_{200}(x)$ is smaller 10^{-15} until $x \sim 150$. Floating point error in determination of $J_\ell(x)$ throws Newton's method into a loop for starting points too close to the origin. Thus, the SciPy functions are of no use for ℓ any larger than about 200. On our machine, the first failure of these built-in methods occurs at $\ell = 231$, far too small to compute the zeta function with any impressive accuracy. Ultimately, we have efficiently computed roots for $\ell \lesssim 10,000$.

The problem would be alleviated if a better starting position were known than $x = 0$. In fact, the first root of $J_\ell(x)$ is always at least ℓ, but we can do much better than this. The roots of the Bessel functions are known to be interlaced. That is,

$$j_{\ell-1,n} < j_{\ell,n} < j_{\ell-1,n+1} \tag{3.43}$$

so if one had already computed the roots to $J_{\ell-1}$, the work required to find those of J_ℓ is greatly reduced. Since all of the roots are necessary, this is the approach we took. This sequential computation was employed, and using the interlacing property described above, the bisection method was found to be most efficient, converging usually within 20-30 iterations to the threshold accuracy of 10^{-10}.

The number of roots $j_{\ell,n}$ less than some value N scales as $\mathcal{O}(N^2)$, so multiprocessing was applied to cut down on computation time. The computation for $N = 10^5$ was preformed on a 2015-era laptop with a 4-core processor, and completed in about 90 minutes, averaging $\sim$4 ms per root found. For $N = 10^6$ our machine would need to run for an estimated 6 days, prohibitively long.

The result of the computation, shown as a density plot of the $j_{\ell,n}$ for bins of size 1 is shown below in Figure 3.1. The density is consistent asymptotically with the function:

$$\rho(j) \sim \frac{j}{2} - \frac{1}{4} + \text{error} \tag{3.44}$$

The error between this data and the asymptotic form $\rho(j)$ is randomly distributed around zero and seems to grow in absolute value no faster than $\sqrt{j} \times$ constant. Within these relative bounds on the error, the roots are not distributed uniformly.

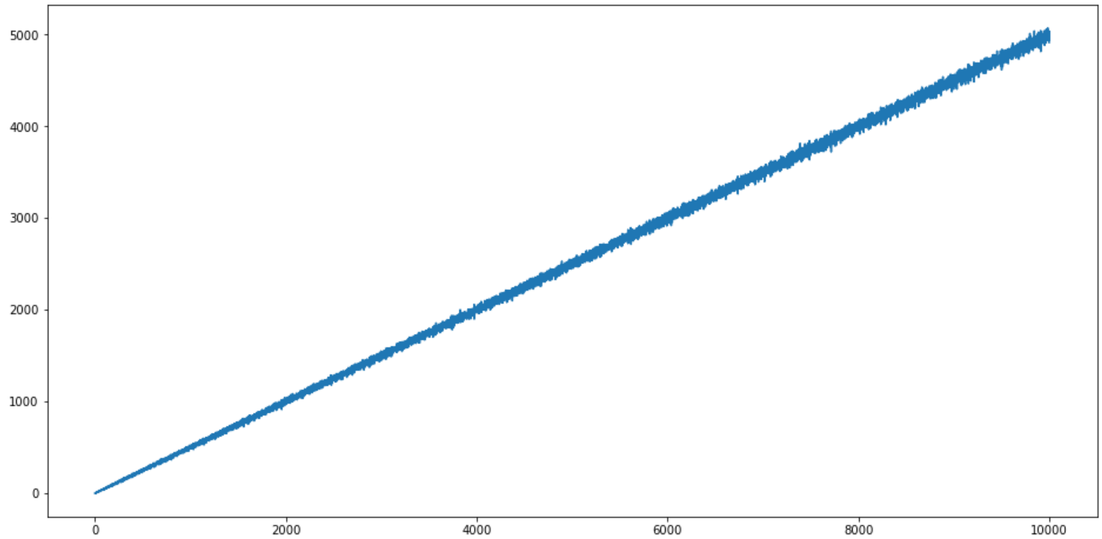

Figure 3.1: Number of Bessel roots $j_{\ell,n}$ per unit interval, including only those less than 10^5. Values for $\ell \neq 0$ have been counted twice, accounting for the 2-fold degeneracy of waves in 2+1 dimensions.

Knowing the asymptotic form of this eigenvalue distribution is important. The summations involved in computing the zeta function converge slowly relative to those of the Riemann zeta function, so the function $\rho(j)$ allows us to approximate the remainder of the infinite sums without actually generating the roots themselves. Particular values of the zeta function have been found analytically[29]. For instance:

$$\zeta_{\odot}(4) = \sum_{\ell=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{j_{\ell,n}^4} = \frac{2\pi^2 - 15}{96} \approx 0.049366758356028 \tag{3.45}$$

Direct computation of this sum, cut off at $j_{\ell,n} \leq 10^5$, with accuracy of each root to 10 decimal places, yields

$$\sum_{j \leq 10^5} \frac{1}{j_{\ell,n}^4} \approx 0.049366755856... \tag{3.46}$$

which is accurate for the first 8 decimal places. This can be improved significantly by simply adding to the sum an integration term:

$$\sum_{j\leq 10^5}\frac{1}{j_{\ell,n}^4}+\int_{10^5}^{\infty}\frac{\rho(j)}{j^4}dj\approx 0.049366758356... \tag{3.47}$$

thereby increasing the accuracy to 12 decimal places. In the next section we characterize how these inaccuracies build when analytic continuation is preformed.

3.2.2 Analytical Continuation

For simplicity, we solved the discrete Cauchy-Riemann equations on a square lattice of spacing ε representing the complex plane. This requires a connected domain of initial values for the zeta function, which were acquired by directly computing the summation described in the previous section, with the integral over $\rho(j)$ to represent the zeros which were not explicitly derived. This procedure does not work well near poles, but since we know where the poles are and their residues, this can be avoided. Strictly speaking, we computed functions u and v:

$$\varphi(s)=\zeta_{\odot}(s)-\frac{1}{2(s-2)}+\frac{1}{2(s-1)}+\frac{1}{128(s+1)}\equiv u+iv \tag{3.48}$$

(which is analytic for $\mathrm{Re}(s)>-3$) then iteratively solved the discrete Cauchy-Riemann equations

$$u_{i,j}=u_{i+2,j}-v_{i+1,j+1}+v_{i+1,j-1} \tag{3.49}$$

$$v_{i,j}=v_{i+2,j}+u_{i+1,j+1}-u_{i+1,j-1} \tag{3.50}$$

where the index i represents steps in the positive real direction of the complex s plane, and the index j represents steps in the positive imaginary direction. These formula easily compute a region to the left of your initial known values, and can be manipulated to compute regions above, below, and to the right. Our procedure was as follows: directly compute the zeta function for a vertical strip of width 2ε and height h (symmetric about the real axis), then apply the Cauchy-Riemann equations to estimate the zeta function on a domain to the left and to the right of this strip. The final domain of computed values spans a 45° rotated square in the complex plane, centered on $s = x_0 + i0$, with corners at $x_0 - h/2 + i0, x_0 + h/2 + i0, x_0 + ih/2, x_0 - ih/2$.

The purpose of computing for larger real values of s was to establish bounds on the failure of this iterative process to accurately compute $\zeta_\odot$. To this end, ε, h, and x_0 had to be chosen wisely. Too many steps (too small ε) resulted in the blowup of floating point errors, and too few steps created errors which we believe represent the failure of coarse-graining of the Cauchy-Riemann equations. x_0 needed to be as close to $s = -1$ as possible without incurring the wrath of the simple pole at $s = 2$ (the summation does not converge well at all for s to the left of $4 + i0$), but not too large lest we run into the problem of step size.

Ultimately, we settled on the parameters $x_0 = 8$, $\varepsilon = 0.36$, $h = 36$. h was chosen to be so large so that we were guaranteed to see a blowup of the algorithm at some definite number of steps. We found this to be a good combination of the described constraints, and this is apparent in the data. In Figure 3.2 you will find the numerically generated plots of $\varphi(s)$ on the real axis.

In Figure 3.2, we see that the function $\varphi(s)$ is negative, but beginning to turn around near $s = -1$. This is consistent with the existence of a simple pole of positive residue at $s = -3$, which is known to be the case. As previously stated, we extracted a value of $\varphi(-1) \approx -0.07594$, and estimate a relative error of about 5% for this value. This estimate is determined by inspecting the relative (or absolute) error at the location $s = 17$, which is equal number of lattice steps to the right of our starting point x_0 as the point of interest, $s = -1$. By $s = 19$, the error build up becomes large, and this is also seen to the left of the graph, close to the pole at $s = -3$. However, due to the pole at this location, we did not feel displaying the calculated values was of any interest, as the values are quite large. At the location $s = 17$, the values of the blue and orange graphs are -0.001706 and -0.001649, respectively. The 5% error bound on $\varphi(-1)$ squarely places the renormalized value of $\zeta_{\odot,\mathrm{ren}}(-1) \approx 0.0074 \pm 0.0038$, which is positive.

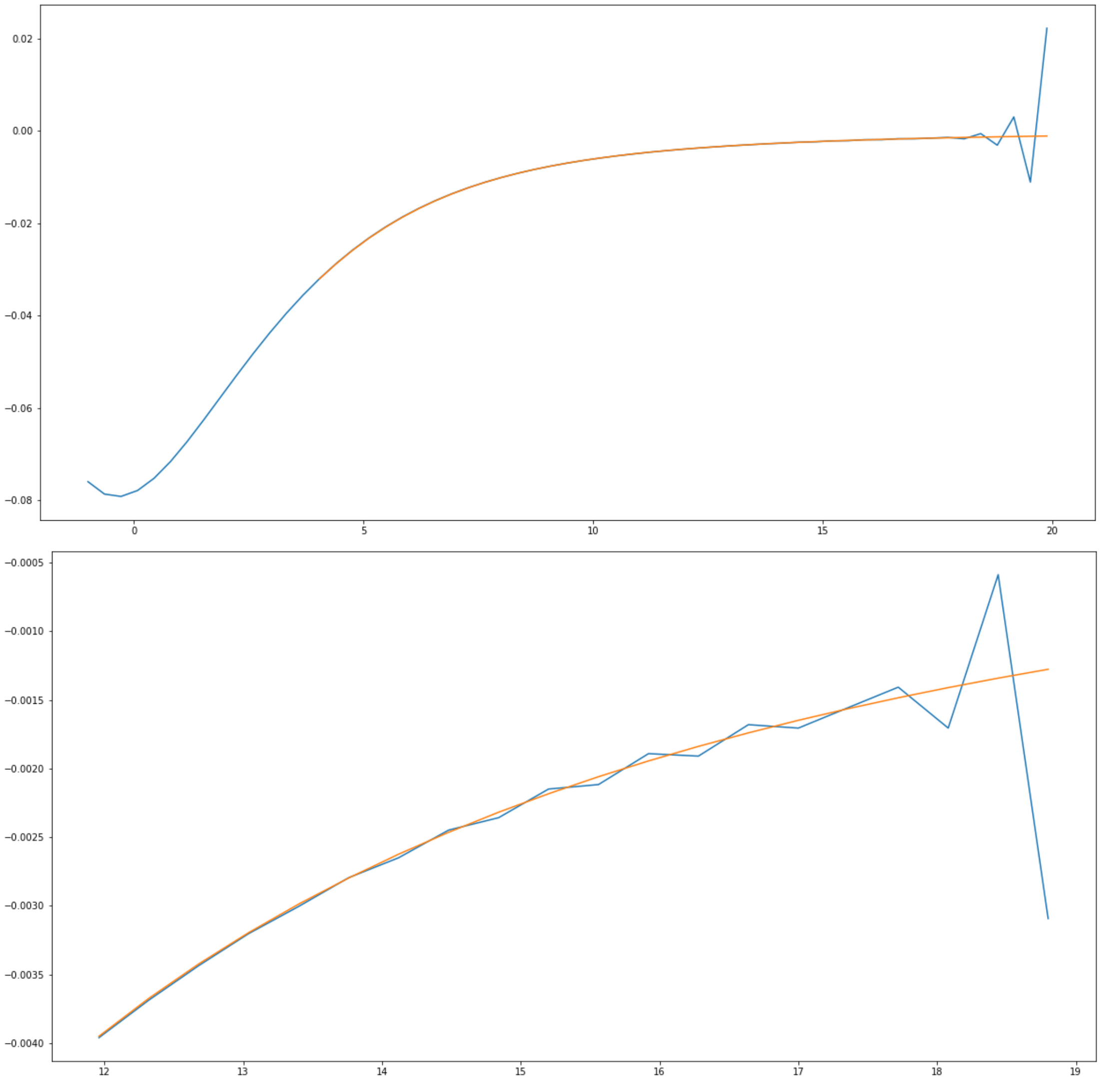

Figure 3.2: In blue, we show the numerically determined (analytically continued) values of $\varphi(s)$ along the real axis, and in orange, the exact determination using the summation described in section 2.1. A portion of the entire domain, $x \in (-1, 20)$ is shown (top) as well as a zoomed-in section $x \in (12, 19)$ used to illustrate the growth of numerical error (bottom).

3.2.3 String/Membrane Stability

If we suppose a circular string is formed, and later its membrane, then we may calculate the total energy in the regime where membrane excitations are allowed but the boundary string is fixed in place. This is the case where the energy stored in the string is much greater than that of the membrane. Since the membrane oscillations have the same energy density as tension, the waves are massless and described by the spectrum of the ordinary Bessel functions J_ℓ. Therefore the total energy is

$$E(a) = 2\pi a T + \pi a^2 \Sigma + \frac{\zeta_{\odot,\mathrm{ren}}(-1)}{a} \tag{3.51}$$

We however assume that the second term is much smaller than the first and third components. In other words

$$a << \frac{2\pi T}{\Sigma} \tag{3.52}$$

The solution is, as before, what we call a_*, and it must be large compared to the inverse of the anomalously generated Axion mass, m_a^{-1}, since this determines the thickness of the membrane. Denote the scale of the $U(1)$ symmetry breaking by M_{SSB} and the coupling constant of this field by λ. The tension Σ scales with m_a like $\Sigma \sim m_a f_a^2$, where f_a is the scale of physics that generates the anomaly (for the QCD Axion, $M_{\mathrm{SSB}} = f_a$, the Axion decay constant)[26]. In principle there is no reason these cannot be different scales for generic Axion models. The hierarchy of scales is therefore:

$$1 << \frac{m_a}{M_{\mathrm{SSB}}}\sqrt{\frac{\lambda\zeta_{\odot,\mathrm{ren}}(-1)}{2\pi}} << 2\left(\frac{M_{\mathrm{SSB}}}{f_a}\right)^2 \tag{3.53}$$

The first inequality is the condition that the thin-membrane approximation is valid at the stable radius, and the second inequality the condition that membrane oscillations are the primary source of the Casimir effect, or that oscillations of the string boundary are kinetically inaccessible. These are the approximations we must take to apply the results of our Zeta function calculation with confidence.

Our thin wall approximation is easily met if λ is parametricaly large compared to the mass ratio of the heavy $U(1)$ quanta to the anomalous Axion mass. λ can be small if $m_a >> M_{\text{SSB}}$, but this is likely prohibited phenomenologically, at least in the case that the Axion is weakly coupled with the anomalous physics. For the QCD Axion, these inequalities clearly cannot be met, and an analysis must be done of the full vibration spectrum of the string+membrane system (much larger radii). This involves determination of the zeta function for the roots of Bessel functions, $j'_{\ell,n}$. We attempted this calculation but progress was slow since the residues and locations of the poles are unknown to this author. We expect the result to be repulsive, as the exchange of Dirichlet to Neumann boundary conditions does not significantly effect the spectral density.

Clearly, when coupling constants are large, quantum effects may be strong enough to counteract classical forces and tensions, stabilizing soliton configurations which would not otherwise be stable. If one only considers the functional determinant contribution, one finds that coupling constants need to be very big for this to occur. However, there are always higher and higher order contributions to consider, and we expect a full analysis of the quantum forces will produce stabilized solitons for much more reasonable values of λ.

3.3 Source Terms in the φ^3 Theory

We will now explore the quantum effects that are dominant at weak coupling. These are the terms generated by the source terms, as discussed in section 1.4.2. In most cases, it is impossible to simplify the calculations involved, since they generally contain infinite derivatives, or alternatively are non-local. However, the scalar φ^3 theory in four spacetime dimensions provides us with a workable scenario, at least in some asymptotic limit. To begin, let's discuss the model Lagrangian and equations of motion. In this theory, the field is restricted to positive values: $\varphi \geq 0$. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{2}\varphi^2 - \frac{g}{3}\varphi^3 - \frac{m^6}{6g^2} \tag{3.54}$$

The Euler-Lagrange equations are

$$\partial^2\varphi - m^2\varphi + g\varphi^2 = 0 \tag{3.55}$$

Derrick's theorem tells us that in three or higher dimensions, we can generally not expect solitons to solve the equations of motion. Supposing we find a radially symmetric solution of the form

$$\varphi(r) = \frac{m^2}{g} f(mr) \tag{3.56}$$

where $f(0) = 0$ and $f(\infty) = 1$. the equations of motion reduce to (for a dimensionless variable $u = mr$ with $f = f(u)$)

$$f'' + \frac{2}{u} f' - f + f^2 = 0 \tag{3.57}$$

from which we can deduce that f must tend to 1 at infinity faster than any inverse power of u. Further analysis shows that the asymptotic behavior of f is approximately given by

$$f(u) \sim 1 - \frac{c_1 \cos(u) + c_2 \sin(u)}{u} + \mathcal{O}\left(u^{-2}\right) \tag{3.58}$$

This field configuration does not however have definite energy. We will explore this now, bounding the total energy of the field configuration from below by a non-oscillating form.

3.3.1 Large Quantum Corrections for a Particular Configuration

Suppose we are given the asymptotic configuration for φ:

$$\varphi(r) = \frac{m^2}{g}\left(1 - \frac{\alpha}{mr} - \frac{\beta}{m^2 r^2} - \frac{\gamma}{m^3 r^3}\cdots\right) \tag{3.59}$$

with some constant $\alpha > 0$, and unspecified corrections of order r^{-4}. Our goal will not be to make a convincing argument that such a configuration solves the equations of motion for the complete effective action detailed in section 1.4.2. We will however demonstrate that the classical contribution to the total energy is compensated significantly by the contributions from the effective action source terms. First, the classical total energy is computed. It in fact diverges, and we will only consider the leading order pieces. This configuration was picked in particular due to the fact

$$-\nabla^2\varphi = \mathcal{O}\left(\frac{1}{r^4}\right) \tag{3.60}$$

The terms of this order are integrable, and we will be concerned only with the divergent contributions to the total energy. So where leading order contributions are concerned, we may completely omit any derivative terms for φ (and consequently, the fluctuation operator will commute with J and V'' up to these smaller subleading corrections). We then have for the classical total energy

$$E_{\rm cl} \sim 4\pi \int^R drr^2 \left(\frac{m^4\beta}{g^2r^2} + \frac{m^3(\alpha\beta - \alpha^3/3)}{g^2r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\right) \tag{3.61}$$

Integrating, we find

$$E_{\rm cl} \sim \frac{4\pi m^4 \beta R}{g^2} + \frac{4\pi m^3}{g^2}\left(\alpha\beta - \frac{\alpha^3}{3}\right)\ln(R) + \text{finite} \tag{3.62}$$

If this were a text on classical physics, we would conclude $\alpha = \beta = 0$ at all times, regardless of the equations of motion. Before drawing such conclusions here we will consider the quantum effects. The source term is given by

$$J(r) = \frac{m^4}{g}\left(-\frac{\alpha}{mr} + \frac{\alpha^2-\beta}{m^2r^2} + \frac{2\alpha\beta - \gamma}{m^3r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\right) \tag{3.63}$$

The fluctuation operator is

$$\Delta = -\nabla^2_{(4)} + m^2 - 2m^2\left(\frac{\alpha}{mr} + \frac{\beta}{m^2r^2} + \frac{\gamma}{m^3r^3} + \cdots\right) \equiv -\nabla^2_{(4)} + m^2 + U(r) \tag{3.64}$$

As previously mentioned, up to the order r^{-4} terms, the algebra of these functions and operators is quite simple. In particular

$$\Delta J = (m^2 + U)J + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{3.65}$$

$$\Delta U = (m^2 + U)U + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{3.66}$$

We may now evaluate the effective action term that is quadratic in J. We use the form discussed in equations (1.74) and (1.75):

$$W \supseteq -\frac{1}{2}\int d^4x J(r) \int_0^\infty d\sigma e^{-\sigma\Delta} J(r) \tag{3.67}$$

To evaluate this expression, one expands the exponential in powers of σ, applying the expressions (3.65) and (3.66) at each order. Alternatively, because $-\nabla^2_{(4)}$ and $U(r)$ "commute" (up to integrable corrections) we may simply split the exponential.

$$e^{-\sigma(-\nabla^2_{(4)}+m^2+U)}J(r) \sim e^{-\sigma(m^2+U)}e^{-\sigma(-\nabla^2_{(4)})}J(r) \tag{3.68}$$

It is clear that the action of the final exponential is (again up to integrable corrections) just the identity operator. Thus (3.67) reduces to a very simple form. The derivatives may be dropped and the σ integral done explicitly. The result is

$$W_{(2)} = -\frac{1}{2}\int d^4x \frac{J^2}{m^2+U} + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{3.69}$$

Returning to real-time formalism, the action and therefore the contribution to the energy of this static configuration may be read off:

$$S_{(2)} = \frac{1}{2}\int dt \int d^3x \frac{J^2}{m^2+U} \tag{3.70}$$

$$E_{(2)} \sim -4\pi \int^R drr^2 \left(\frac{m^4\alpha^2}{2g^2r^2} + \frac{m^3\alpha\beta}{g^2r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\right) \tag{3.71}$$

And after integration:

$$E_{(2)} \sim -\frac{2\pi m^4\alpha^2 R}{g^2} - \frac{4\pi m^3\alpha\beta}{g^2}\ln(R) + \text{finite} \tag{3.72}$$

Inspection of (3.72) and (3.62) yields an exact cancellation of the order R term if $\beta = \alpha^2/2$, and partial cancellation of the $\ln(R)$ terms. We have not finished calculating the terms which contain a $\ln(R)$ divergence, so complete cancellation may yet be possible. It is impressive however that the leading order divergence is already nullified, by considering only the first quantum term in the effective action. Already this proves our point, that the effective action as a functional of the source J can not be neglected from an analysis of the equations of motion for soliton configurations.

Let us go further and investigate the term cubic in J. This will be the final divergent contribution to the total energy of the configuration (3.59), and will contribute a final $\ln(R)$ divergence. All other terms in the effective action (1.72) will be integrable. Confirming this is simply a problem of dimensional analysis. The cubic term is

$$W_{(3)} = \frac{1}{6}\left(\prod_{i=1}^{3}\int d^4x_i J(x_i)\right) G^{(3)}(x_1, x_2, x_3) \tag{3.73}$$

The three-point function at tree level is easily derived. It is

$$G^{(3)}(x_1, x_2, x_3) = -2g \int d^4x_4 \Delta^{-1}(x_1, x_4)\Delta^{-1}(x_2, x_4)\Delta^{-1}(x_3, x_4) \tag{3.74}$$

As before, the calculation involves the integral and its following simplification (again, up to integrable terms):

$$\int d^4x_i \Delta^{-1}(x_i, x_4)J(x_i) \sim \frac{J(x_4)}{m^2 + U(x_4)} \tag{3.75}$$

After returning to real-time and dropping the time integration from the action, we find the divergent contribution to the energy to be

$$E_{(3)} \sim \frac{4\pi g}{3}\int^R drr^2 \left(\frac{J}{m^2 + U}\right)^3 \sim \frac{4\pi m^3 \alpha^3}{3g^2}\ln(R) + \text{finite} \tag{3.76}$$

The $\ln(R)$ divergences from equations (3.62), (3.72), and (3.76) add to zero. Hence the total energy of this configuration is finite if $\beta = \alpha^2/2$, a conclusion that we certainly would not have expected from a study of the classical equations of motion. Of particular importance is that this cancellation happens at weak coupling. We therefore conclude that in quantum field theory, one generically expects to finds a class of solutions to the full quantum equations of motion that are classically inaccessible, regardless of the size of the theory's parameters.

Chapter 4

Conclusion

In chapter 1 were were introduced to the concept of the soliton as a stable, inhomogeneous field configuration. By no means did we cover the details of the many many solitons which have appeared and become relevant in high energy physics and many-body physics, or in fluid dynamics. We discussed important aspects, such as scaling relations, oscillations and bound states, and collective coordinates. This approach, or rather these quantities, are the oft-discussed topics of soliton physics, and they are all classical.

We then considered an approach consistent with the quantum effective field theory, and quickly realized that not only did solitons inherit the difficulties with quantum field theory around the vacuum, but these difficulties were amplified because of the non-linear nature of the soliton equations of motion. In particular, the scaling arguments in Derrick's theorem and the variational methods of Euler and Lagrange are futile in a first principles approach. When considering a soliton that is already known and studied in classical physics, one of course may take the point of view that quantum corrections to the profile are small. In this context, the quantum system of equations are ordered perturbatively and except in circumstances involving tunneling effects, the stability of the classical soliton is not often put under threat by quantum physics.

However there are known cases of solitons in quantum theory that are not satisfactorily explained by the classical physics we associate with the particle/wave content of the model. These cases were explored briefly in chapter 2, and they provide a justification for asking the question: are there soliton solutions to quantum field theory equations that do not exist when the quantum effects are removed? If the answer is yes, then the approach taken in the typical treatment of solitons in quantum theory - that which we have just described as an assumption of small corrections - is too weak of an approach. To answer this question we must begin with the assumption that any such field configuration is largely different from what one expects in classical physics. We then generically must compute an infinite number of integrals to determine if a given configuration is a minima of the energy functional, and methods for completing this task in a finite amount of time presently elude us.

In general, one should expect quantum effects to be strong compared to classical effects when there is a large self-interaction in the theory. This is a regime where most modern tools of quantum theory break down. The various phenomenology that is known when coupling constants become large run the gamut of physics: in some circumstances, strong coupling leads to more symmetry, and total loss of scale; in other cases, it breaks what little symmetry is available and generates new mass scales from nothing. It is not a controversial suggestion that in this regime, classical dynamics of particles, waves, and solitons break down.

Ordinarily, one cannot distill the quantum interactions down to a finite number of computations, even when systems are weakly coupled. It is a rare circumstance when the theorist only need calculate a handful of integrals to arrive at a result, and especially in the case of strong interactions where the infinite number of terms cannot even be arranged by some order of magnitude estimates. One circumstance where this is possible is the case of large distance, or infrared physics, when there are free massless degrees of freedom. In the case we saw of the global vortex/string, the lowest energy degrees of freedom were the Goldstone bosons, and at large distances they are only moderately coupled to the vortex itself. The problem of the infinite classical vortex mass and string tension was due to the long range behavior of the phase degrees of freedom, and so we were inclined to investigate the quantization of those same degrees of freedom to resolve the issue of infinity. The massive quanta could not possibly produce energy densities that fell off with polynomial dependence, as is well known from the spectral properties of massive fields. For this same reason, the interactions between the massive quanta and the Goldstone bosons also should depend on the intrinsic mass scale in a non-polynomial way, so we omit these terms from the calculation.

What is left is the functional determinant, which we saw in chapter 3 could be expressed as a series of energy density contributions which fell off slowly from the core of the vortex/string. The leading order of these corrections was established, and it had the opposite sign and same form of the classical divergent contribution to the mass/tension. Ultimately, we observed the mass/tension could be rendered finite if the coupling constant was of order unity. The functional determinant was also evaluated for the case of a vibrating membrane, and we additionally observed that at very large coupling one may find this force sufficient to balance a very large soliton.

Of course, at large coupling, there are many other effects which may produce comparably strong forces. These effects are all due to physics that is set at an intrinsic mass scale, and as such will be ordered in an expansion around powers of that mass scale and the geometric scales of the soliton. We make no claim in computing these effects, or knowing what their nature should be. The purpose of the functional determinant computations we have presented is to illustrate the potential for even the most simple quantum corrections to upend the (in)stability of a soliton in quantum theory.

At the end of chapter 3, we demonstrated an exact cancellation of the classically divergent component of the soliton mass by quantum corrections. This occurs at any coupling, small or large. The configuration we considered, a radially symmetric field in a φ^3 potential, which existed in an unbroken symmetry phase at the origin, is probably not of physical interest. This was a toy model we developed to illustrate the importance of quantum effects even at weak coupling. Quantum effects are often discarded when they are infinite, but in the case of infrared divergences, or ones we associate with the infinite volume of a system, we should keep these infinities and look for their opposite sign counterparts. In this case we see they cancel exactly, without tuning any parameters of the model.

The question we asked earlier in this section: *are there soliton solutions to quantum field theory equations that do not exist when the quantum effects are removed?* does not have a direct answer. We cannot write down the full equations of motion, let alone solve them. However, an adjacent question: *are there field configurations dynamically accessible in quantum field theory which are not in classical field theory?*, has an answer. It is a resounding yes. This discovery will prove to be a critical component in the study of the dynamics of quantum solitons.

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