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SHINTANI'S METHOD: ZETA VALUES AND STARK UNITS

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Abstract

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We prove a formula relating Dedekind zeta functions associated to a number field k to certain Shintani zeta functions, whose analytic properties and values at non-positive integers have been well studied by Takuro Shintani. This allows us to compute explicit formulas for Dedekind zeta functions, partial zeta functions and certain L-series and their derivatives evaluated at non-positive integers. We relate the explicitly given value of the derivative of partial zeta functions at $s = 0$ to those predicted by abelian Stark's conjecture. Though this conjecture remains open, we are able to write down explicit formulas for the absolute values of the conjectured Stark units.

The main ingredient in these formulas is an explicit proof of Shintani's unit theorem for number fields of arbitrary signature. This says that the totally positive units of a number field k has a fundamental domain given by a signed union of polyhedral cones in the Minkowski space of the field. Existence of such domains was known to Shintani. In the case k is a totally real field, Colmez, Diaz y Diaz–Friedman and Charollois-Dasgupta-Greenberg were able to construct such domains and give their generators explicitly. We give an explicit construction of such domains for number fields of arbitrary signature with an exact formula for the domain. Moreover, our construction is cohomological, allowing for future cohomological applications of Shintani's method as in the work of Charollois–Dasgupta–Greenberg.

This construction allows us to write Dedekind zeta functions and partial zeta functions in terms of certain analytic zeta functions defined over polyhedral cones (Shintani zeta functions). Thus we are able to translate questions about special values of Dedekind zeta functions to those about special values of Shintani zeta, whose values at non-positive integers are given by closed finite expressions due to work of Shintani.

To Mom and Dad

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Chapter 1

Introduction

It has been seen over the years that zeta functions encode a great deal of information about the objects that define them. The analytic class number formula shows that the Dedekind zeta function associated to a number field k relates many of the most important invariants of the number field to the residue of the zeta function at $s = 1$, and factorization formulas for such zeta functions lead to more refined conjectures about their special values. Hecke proposed the study of a Kronecker limit formula for general algebraic number fields as an approach to Hilbert's twelfth problem [Hec59], and the theory of complex multiplication has led Harold Stark to make precise conjectures relating L-values and abelian extensions [Sta80].

Hecke's and Stark's programs suggest that it may be possible to discover analytic formulas for special values of L-functions that would be suitably natural to explicitly generate abelian extensions. Takuro Shintani's work on Hecke's program led to a series of explicit analytic formulas for zeta values and L-values as well as an elementary proof of the celebrated Klingen-Siegel theorem on the rationality of zeta values for totally real number fields at non-positive integers. The cornerstone of Shintani's work is his unit theorem, which states that for a number field k, a finite index subgroup of the totally positive units of k acting on the Minkowski space of the field has a fundamental domain given by rational polyhedral cones in the Minkowski space of the field [Shi76a]. In the case that k is a totally real field, Colmez [Col88], Diaz y Diaz–Friedman [DF13] and Charollois–Dasgupta–Greenberg [CDG16] gave constructive proofs which determined explicitly the generators of these polyhedral cones. These results also modernized Shintani's theorem by removing the finite index condition and in [CDG16] phrasing the result in terms of a pairing of algebraic homology and cohomology classes associated to the unit group U.

The goal of this thesis is to give a constructive proof of Shintani's unit theorem for number fields of arbitrary signature, namely those with complex places. This is the main challenge and the entirety of the second chapter. A key point of our proof is to consider the action of the totally positive units U of the number field k on a restricted Minkowski space $(\mathbf{R}_{>0})^{r_1} \times (\mathbf{C}^\times)^{r_2}$. This space can be identified with a quotient of a classifying space for U, which allows us to pass from the algebraic (co)homology to simplicial (co)homology and construct topological cocycles analogous to the algebraic cocycles defined in [CDG16]. We analyze a relevant pairing in simplicial cohomology and observe through a limiting process that we may reduce to the totally real case. The equality of our pairing with that of [CDG16] concludes our proof.

The starting point of computing zeta values for us is to consider the partial zeta functions associated to an ideal class c

$$\zeta_k(s, c) = \sum_{\mathfrak{b} \in c} \frac{1}{N\mathfrak{b}^s}.$$

Using a classic trick, we can rewrite this series as a summation over a lattice. Let $\mathfrak{a}$ be an ideal in the inverse class of c. Then for $\mathfrak{b} \in c$ we have that $\mathfrak{ab} = (x)$ is a principal ideal generated by an element $x \in \mathfrak{a}$. This element x is well defined up to units of the number field and each $x \in \mathfrak{a}$ defines an ideal $\mathfrak{b} \in c$. If we embed k in its Minkowski space, we can observe that $\zeta_k(s, c)$ can be rewritten as a summation over the lattice $\mathfrak{a}$ in Minkowski space modulo the action of the unit group U.

$$\zeta_k(s, c) = N\mathfrak{a}^s \sum_{x \in \mathfrak{a}/U} \frac{1}{(x_1 \cdots x_n)^s}.$$

By expressing a fundamental domain for the action of U on the Minkowski space in terms of polyhedral cones, Shintani showed that one can express this lattice summation as a sum of a certain function over a square lattice. This simplification allowed Shintani to prove a number of results on meromorphic continuation and formulas for special values. Having extended Shintani's result on fundamental domains, we apply these same results to compute special values of partial zeta functions $\zeta_k(s, c)$ as well as Hecke L-series and ultimately to get a formula for the absolute values of conjectured Stark units.

Chapter 2

Shintani's Method

This chapter gives a cohomological and explicit proof of Shintani's unit theorem for number fields of arbitrary signature. In the first two sections we maintain and introduce the same notation and key results, as illustrated in the identically named chapter of [CDG16], on cone generating functions and their properties.

We use geometric techniques to prove the existence of a special fundamental domain for the action of a discrete group, one given by rational polyhedral cones. This result is the key idea behind this thesis and is motivated by similar work of Takuro Shintani [Shi76a], whose constructive proof in the totally real case led to many interesting formulas for zeta values.

We adapt the algebraic (co)homological reinterpretations of Shintani's method used in [Col89] and [CDG16] in a geometric way to extend it to handle number fields with complex embeddings. We do this by noticing certain symmetries of the domain in the complex embeddings, allowing us to ultimately reduce to the totally real case. The cohomological method proves convenient for the proof and serves a dual purpose of allowing for future applications as in [CDG16].

2.1 Colmez perturbation

Consider linearly independent vectors $v_1, \ldots, v_n \in \mathbf{R}^k \times \mathbf{C}^l$. The open cone (or simpicial cone) generated by the v_i is the set

$$C(v_1, \ldots, v_n) = \mathbf{R}_{>0}v_1 + \mathbf{R}_{>0}v_2 + \cdots + \mathbf{R}_{>0}v_n. \tag{2.1.1}$$

Denote the characteristic function of this cone by $1_{C(v_1,\ldots,v_n)}$. When $n = 0$, we define $C(\emptyset) = \{0\}$. Let $\mathcal{K}_{\mathbf{R}}$ denote the abelian group of functions $\mathbf{R}^k \times \mathbf{C}^l \to \mathbf{Z}$ generated by characteristic functions of such open cones.

Let $V = \mathrm{span}_{\mathbf{R}}(v_1, \ldots, v_n)$ and $Q = (q_1, \ldots, q_n) \in V$ an auxiliary vector. We define a region $C_Q(v_1, \ldots, v_n)$ and its characteristic function $c_Q(v_1, \ldots, v_n)$ as follows. If the vectors v_i are linearly dependent then $c_Q(v_1, \ldots, v_n) = 0$, otherwise we impose the further condition that Q is not in the $\mathbf{R}$-linear span of any subset of $n-1$ of the v_i. Define $C_Q(v_1, \ldots, v_n)$ to be the disjoint union of $C(v_1, \ldots, v_n)$ and a subset of its boundary faces (of all dimensions, including 0) as follows. An element is included if translation by a small positive multiple of Q sends that element into $C(v_1, \ldots, v_n)$. That is,

$$c_Q(v_1, \ldots, v_n)(w) = \begin{cases} \lim_{\epsilon \to 0^+} 1_{C(v_1,\ldots,v_n)}(w + \epsilon Q) & \text{if } v_i \text{ linearly independent} \\ 0 & \text{otherwise} \end{cases} \tag{2.1.2}$$

equivalently, if $w = \sum_{i=1}^n w_i v_i$ then

$$c_Q(v_1, \ldots, v_n)(w) = \begin{cases} 1 & \text{if all } w_i \geq 0 \text{ and } w_i = 0 \implies q_i > 0 \text{ for } i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{2.1.3}$$

Let $I \subset \{1, \ldots, n\}$ and $C_I = C(v_i \ : \ i \in I)$. Say $d = |I|$, then C_I can be thought of as being contained in the intersection of the $n-d$ hyperplanes determined by $v_j^* = 0$, for $j \in \bar{I} = \{1, \ldots, n\} - I$. Here $\{v_i^*\}$ is the dual basis of $\{v_i\}$. Each hyperplane divides the complement of $v_i^* = 0$ into a half-space containing the cone $C(v_1, \ldots, v_n)$ and the half-space not containing the cone (as an inequality, $\langle w, v_j^* \rangle > 0$ or < 0). The boundary face C_I is included in $C_Q(v_1, \ldots, v_n)$ if and only if $\langle Q, v_j^* \rangle > 0$ for all $j \in \bar{I}$. That is, each boundary face C_I carries a weight equal to 1 or 0, which determines whether or not a boundary face is included in $C_Q(v_1, \ldots, v_n)$ or not. This weight given by

$$\mathrm{weight}(C_I) = \prod_{j \in \bar{I}} \frac{1 + \mathrm{sign}(Q\sigma^{-t})_j}{2}. \tag{2.1.4}$$

where σ is an invertible $(k+l) \times (k+l)$ matrix whose first n columns are the vectors v_i and σ^{-t} its inverse transpose. Using this, the characteristic function $c_Q(v_1, \ldots, v_n)$

may be written as a weighted sum of characteristic functions 1_{C_I} of open cones.

2.2 Cocycle relation

In this section we show that the characteristic functions $c_Q(v_1, \dots, v_n)$ satisfy a certain cocycle relation, this result is fundamental to the construction of an explicit cohomology class in the next section.

Let $v_1, \dots, v_n \in \mathbf{R}^k \times \mathbf{C}^l$ be linearly independent vectors with $n \geq 1$. A set of the form

$$L = \mathbf{R}v_1 + \mathbf{R}_{>0}v_2 + \cdots + \mathbf{R}_{>0}v_n \tag{2.2.1}$$

is called a *wedge*. The characteristic function 1_L of L is an element of $\mathcal{K}_{\mathbf{R}}$ since

$$L = C(v_1, \dots, v_n) \sqcup C(v_2, \dots, v_n) \sqcup C(-v_1, v_2, \dots, v_n).$$

Let $\mathcal{L}_{\mathbf{R}} \subset \mathcal{K}_{\mathbf{R}}$ be the subgroup generated by functions 1_L for all wedges L.

The following theorem was proved for cones in $\mathbf{R}^k$ in [Hil07] for the general position case without Q-perturbation and [CDG16] in general, the same argument applies directly since, for the purposes of the theorem, $\mathbf{R}^k \times \mathbf{C}^l$ can just be thought of as a $k+2l$-dimensional real vector space. We give a proof for completeness.

Proposition 2.2.1. *Let $n \geq 1$, and let $v_0, \dots, v_n \in \mathbf{R}^k \times \mathbf{C}^l$ be nonzero vectors spanning a subspace V of dimension at most n. Let $Q \in V$ be a vector not contained in the span of any subset of $n-1$ of the v_i. Let B denote a fixed ordered basis of V and define for each i the orientation*

$$O_B(\hat{v}_i) := O_B(v_0, \dots, \hat{v}_i, \dots, v_n) = \operatorname{sign}\det(v_0, \dots, \hat{v}_i, \dots, v_n) \in \{0, \pm 1\},$$

where the written matrix gives the representation of the vectors v_j in terms of the basis B, for $j \neq i$. Then

$$\sum_{i=0}^{n} (-1)^i O_B(\hat{v}_i) c_Q(v_0, \dots, \hat{v}_i, \dots, v_n) \equiv 0 \pmod{\mathcal{L}_{\mathbf{R}}}. \tag{2.2.2}$$

Proof. First note that since $c_Q(v_0, \dots, \hat{v}_i, \dots, v_n) = 0$ whenever $\{v_0, \dots, \hat{v}_i, \dots, v_n\}$ have a linear dependence, the left hand side of (2.2.2) is trivially zero by definition of c_Q unless $\dim V = n$.

We assume $\dim V = n$ and separate the remainder of the proof into two cases:

Case 1: The v_i are in general position, i.e. any subset of $\{v_0, \ldots, v_n\}$ of size n spans V. By the assumptions on Q, we also get that for $\epsilon > 0$ small enough, the set $\{v_0, \ldots, v_n, w + \epsilon Q\}$ is in general position for each $w \in V$.

Let $x \in V$ be a nonzero vector and suppose $x \in C_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)$ for some i. Say $x = \sum_{i=0}^n a_i v_i$ for some real constants $a_i \geq 0$, not all zero. Let λ_i be nonzero real constants such that $\sum_{i=0}^n \lambda_i v_i = 0$, these are well defined up to simultaneous scaling. Then we get that

$$x = \sum_{j \neq i} \left(a_j - \frac{\lambda_j a_i}{\lambda_i} \right) v_j$$

By definition, $x \in C_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)$ if and only if the constants $\left(a_j - \frac{\lambda_j a_i}{\lambda_i}\right) \geq 0$ for all $j \neq i$. Note first that if $\operatorname{sign}(\lambda_i) = \operatorname{sign}(\lambda_j)$, then $a_j - \frac{\lambda_j a_i}{\lambda_i} > 0$ implies $a_i - \frac{\lambda_i a_j}{\lambda_j} < 0$. So in particular, if λ_i all have the same sign, then the interiors of the cones $C_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)$ are disjoint for $i \neq j$, which by (2.1.2) implies $C_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)$ are disjoint for $i \neq j$.

Moreover, if λ_i are all the same sign, we can say *which* cone x must lie in. Consider the indexed multi-set $A = \{\frac{a_j}{\lambda_j}\}_{j=0}^n$. Without loss assume λ_j are all positive then take i_0 to be the index of the least element of A (if it is unique), then it is clear the inequality $a_j - \frac{\lambda_j a_{i_0}}{\lambda_{i_0}} > 0$ holds for all $j \neq i_0$. If more than one least element exists then x is a boundary point so instead consider $x + \epsilon Q$ which will lie in the interior of a unique cone $C_Q(v_0, \ldots, \hat{v}_{i_0}, \ldots, v_n)$ for $\epsilon > 0$ small enough.

Now assume λ_i not all the same sign, so if $x \in C_Q(v_0, \ldots, \hat{v}_{i_0}, \ldots, v_n)$, then $x \notin C_Q(v_0, \ldots, v_j, \ldots, v_n)$ for all j having $\operatorname{sign}(\lambda_{i_0}) = \operatorname{sign}(\lambda_j)$. Since this sign condition partitions the set of cones which x may belong to, x lies in at most two cones. Let $\tilde{A} = \{(j, \frac{a_j}{\lambda_j}) \ : \ \operatorname{sign}(\lambda_j) = -\operatorname{sign}(\lambda_{i_0})\}$, if $\operatorname{sign}(\lambda_{i_0}) = +$ (resp. $-$), let j_0 be the index such that $\frac{a_j}{\lambda_j}$ is maximal (resp. minimal). We may assume the index is unique since otherwise we may replace x with $x + \epsilon Q$.

We show that $x \in C_Q(v_0, \ldots, \hat{v}_{j_0}, \ldots, v_n)$, equivalently $a_j - \frac{\lambda_j a_{j_0}}{\lambda_{j_0}} \geq 0$ for all j. For j such that $\operatorname{sign}(\lambda_j) = \operatorname{sign}(\lambda_{j_0})$ this follows from the choice of j_0, for j such that $\operatorname{sign}(\lambda_j) = -\operatorname{sign}(\lambda_{j_0})$ we have that

$$a_j - \frac{\lambda_j a_{j_0}}{\lambda_{j_0}} > a_j - \frac{\lambda_j \lambda_{j_0} a_{i_0}}{\lambda_{j_0} \lambda_{i_0}} = a_j - \frac{\lambda_j a_{i_0}}{\lambda_{i_0}} > 0.$$

So, we get that x is contained in either 0 or 2 of the cones $C_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)$, in the case that λ_i are not all the same sign.

$$\sum_{i=0}^{n}(-1)^i O_B(\hat{v}_i) c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)$$
$$= \begin{cases} (-1)^{i_0} O_B(\hat{v}_{i_0}) & \text{if the } \lambda_i \text{ the same sign} \\ (-1)^{i_0} O_B(\hat{v}_{i_0}) + (-1)^{j_0} O_B(\hat{v}_{j_0}) & \text{if } x \text{ lies in two cones} \\ 0 & \text{if } x \text{ lies in 0 cones} \end{cases}$$

To conclude this case, we show that in fact these values are independent of the choice of x and hence i_0 and j_0, and are in fact constant on V. We give an alternative expression for $O_B(\hat{v}_i)$ using an auxiliary index, which without loss we choose as the 0^{th} index.

$$\begin{aligned} (-1)^i O_B(\hat{v}_i) =& (-1)^i \operatorname{sign} \det(v_0, \ldots, \hat{v}_i, \ldots, v_n) \\ =& (-1)^i \operatorname{sign} \det(\sum_{j=1}^{n} -\frac{\lambda_j}{\lambda_0} v_j, v_1, \ldots, \hat{v}_i, \ldots, v_n) \\ =& (-1)^i \operatorname{sign} \det(-\frac{\lambda_i}{\lambda_0} v_i, v_1, \ldots, \hat{v}_i, \ldots, v_n) \\ =& \operatorname{sign} \det(v_1, \ldots, -\frac{\lambda_i}{\lambda_0} v_i, \ldots, v_n) \\ =& \operatorname{sign}(-\frac{\lambda_j}{\lambda_0}) \operatorname{sign} \det(v_1, \ldots, v_i, \ldots, v_n) \end{aligned}$$

So whenever λ_i all have the same sign, $(-1)^i O_B(\hat{v}_i)$ is independent of the choice of i, moreover $(-1)^{i_0} O_B(\hat{v}_{i_0}) + (-1)^{j_0} O_B(\hat{v}_{j_0}) = 0$ since $\operatorname{sign}(\lambda_{i_0}) = -\operatorname{sign}(\lambda_{j_0})$. So we get that the left hand side of (2.2.2) is constant on V and equal to

$$\sum_{i=0}^{n}(-1)^i O_B(\hat{v}_i) c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n) = \begin{cases} (-1)^i O_B(\hat{v}_i) & \text{if all } \lambda_i \text{ the same sign} \\ 0 & \text{otherwise} \end{cases}.$$

The characteristic function on V lies in $\mathcal{L}_{\mathbf{R}}$ so this gives the desired result.

Case 2: The v_i are not in general position. Without loss, we assume that $v_0, \ldots, v_{n-1}$ are linearly dependent. Let V' denote the $(n-1)$-dimensional space spanned by these n vectors. Denote by $\pi' : V \to V'$ and $\pi : V \to \mathbf{R}$ the projections according to the direct sum decomposition $V = V' \oplus \mathbf{R}v_n$. We claim that for $i = 0, \ldots, n-1$ and $w \in V$, we have

$$c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)(w) = c_{\pi'(Q)}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1})(\pi'(w)) \cdot g_Q(w), \quad (2.2.3)$$

where

$$g_Q(w) = \begin{cases} 1 & \begin{array}{c} \text{if } \pi(w) = v_n^*(w) \geq 0 \\ \text{and} \\ \pi(w) = v_n^*(w) = 0 \implies \pi(Q) = v_n^*(Q) > 0 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

Note that if $v_0, \ldots, \hat{v}_i, \ldots, v_n$ are linearly dependent, then by the direct sum decomposition on V, we must also have that $v_0, \ldots, \hat{v}_i, \ldots, v_{n-1}$ are linearly dependent. In this case, both sides of (2.2.3) are zero. Furthermore, $\pi'(Q) \in V'$ cannot lie in any subset of $n-2$ vectors of $\{v_0, \ldots, v_{n-1}\}$, or else Q would lie in a subset of $n-1$ of the original vectors $\{v_0, \ldots, v_n\}$. This ensures that the equation (2.2.3) is defined.

In the non-degenerate case, it's clear that if $v_0, \ldots, \hat{v}_i, \ldots, v_n$ are linearly indepdnedent, then $v_0, \ldots, \hat{v}_i, \ldots, v_{n-1}$ must also be linearly independent. Computing the left hand side of (2.2.3), by definition we have

$$\begin{aligned} &c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)(w) \\ &\qquad = \begin{cases} 1 & \text{if } v_j^*(w) \geq 0 \text{ and } v_j^*(w) = 0 \implies v_j^*(Q) > 0 \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for the right hand side of (2.2.3), we have

$$\begin{aligned} &c_{\pi'(Q)}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1})(\pi'(w)) \cdot g_Q(w) \\ &\qquad = \begin{cases} 1 & \begin{array}{c} \text{if } v_j^*(\pi'(w)) \geq 0, \text{ for } j \neq i, n \text{ and } v_n^*(w) \geq 0 \\ v_j^*(\pi'(w)) = 0 \implies v_j^*(\pi'(Q)) > 0 \text{ for } j \neq i, n \\ v_n^*(w) = 0 \implies v_n^*(Q) > 0 \end{array} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

note however, if we let B' be a basis of V whose last element is the vector v_n and B'' be a basis of V' consisting of the image of the first $n-1$ vectors of B under π', then by the construction of π' and π, we have that $v_j^*(\pi'(w)) = v_j^*(w)$ and $v_j^*(\pi'(Q)) = v_j^*(Q)$ for $1 \leq j \leq n-1$, so we can see that these conditions are equivalent and so (2.2.3) holds.

What remains is to compute the left hand side of (2.2.2) in this case. Direct computation shows that

$$O_B(\hat{v}_i) = O_B(B') \cdot O_{B'}(\hat{v}_i).$$

Using this fact and that $c_Q(v_0, \ldots, v_{n-1}) = 0$, since $v_0, \ldots, v_{n-1})$ are linearly dependent, we compute that

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i O_B(\hat{v}_i) c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)(w) \\
&\qquad\qquad = O_B(B') \sum_{i=0}^{n-1} (-1)^i O_{B'}(\hat{v}_i) c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)(w) \\
&\qquad\qquad = O_B(B') l_Q(w) g_Q(w),
\end{aligned}$$

where

$$l_Q(w) = \sum_{i=0}^{n-1} (-1)^i O_{B'}(\hat{v}_i) c_{\pi'(Q)}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1})(\pi'(w)).$$

By our choice of B' and B'', we get that

$$O_{B'}(\hat{v}_i) = O_{B''}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1}).$$

Thus l_Q can be written as

$$l_Q(w) = \sum_{i=0}^{n-1} (-1)^i O_{B''}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1}) c_{\pi'(Q)}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1})(\pi'(w)).$$

this is precisely the form for which we can use the inductive hypothesis to conclude that $l_Q \in \mathcal{L}_{\mathbf{R}}(V')$. It's straightforward to check that $l_Q g_Q \in \mathcal{L}_{\mathbf{R}}(V)$ as desired. The base case follows from the general position argument.

$\square$

2.3 Fundamental domain

Consider the region $(\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$, which forms a group under componentwise multiplication. Let $D = \{(x, z) \in (\mathbf{R}_{>0})^k \times (\mathbf{C})^\times)^l \ : \ x_1 x_2 \cdots x_k |z_1|^2 |z_2|^2 \cdots |z_l|^2 = 1\}$. Let $U \subset D$ be a discrete subgroup that is free of rank $k + l - 1$. We wish to determine explicitly a fundamental domain for the action of U on this region in terms of an ordered basis of $U = \langle u_1, \ldots, u_{k+l-1} \rangle$.

Our goal is to construct an explicit fundamental domain for the action of U on $(\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$ in terms of cones generated by explicit elements of U. We choose for our perturbation vector the coordinate basis vector $e_k = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$. Since the first k components of each $u \in U$ are strictly positive, the vector e_k is not contained in any cone generated by $(k + 2l - 1)$ of the elements of U, hence $0 \notin C_{e_n}(v_1, \ldots, v_n)$ for $v_i \in U$. Note that the action of U preserves the ray $\mathbf{R}_{>0} e_k$.

First let us try to get an idea of how a fundamental domain for the action U may behave. Let $(x, z) \in D$, consider the classic logarithmic map

$$\phi : D \to H \subset \mathbf{R}^{k+l} \tag{2.3.1}$$

$$(x, z) \mapsto (\log |x_1|, \ldots, \log |x_k|, 2 \log |z_1|, \ldots, 2 \log |z_l|).$$

The image of ϕ lands in the trace zero hyperplane H in $\mathbf{R}^{k+l}$, it is also a group homomorphism, taking the multiplicative structure of D to the additive structure of H where $u \in D$ acts by translation by $\phi(u)$.

Since ϕ is proper, in particular continuous, the action of U on H (via ϕ) is discrete and free of rank $k + l - 1 = \dim_{\mathbf{R}}(H)$.

In H, the fundamental domain for the action of U is given by a fundamental polygon. Properness of ϕ insures that the inverse of this fundamental polygon will give a compact fundamental domain in D. In general, we can expect such a domain to be 'curved' in some sense. We want to construct a similar 'polygon' in D (not H) and show that cones defined by rays through this polygon is also a fundamental domain for the action of U.

Definition 2.3.1. *A* ***signed fundamental domain*** *for the action of U on $(\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$ is by definition a formal linear combination*

$$D = \sum_i a_i C_i$$

of open cones with $a_i \in \mathbf{Z}$ such that

$$\sum_{u \in U} a_i \mathbf{1}_{C_i}(u * x) = 1$$

for all $x \in (\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$. We call $\mathbf{1}_D := \sum_i a_i \mathbf{1}_{C_i} \in \mathcal{K}$ the characteristic function of D.

Define the orientation

$$w_u := \operatorname{sign}\det(\log(|(u_i)_j|))_{i,j=1}^{k+l-1}) = \pm 1,$$

where $(u_i)_j$ is the j^{th} component in $(\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$, for $j > k$ this have imaginary parts. For $\sigma \in S_{k+l2-1}$ and $u_1, \dots, u_{k+2l-1} \in U$ define

$$v_{i,\sigma} = u_{\sigma(1)} \cdots u_{\sigma(i-1)} \in U, \quad i = 1, \dots, k+2l$$

(by convention $v_{1,\sigma} = (1, 1, \dots, 1)$ for all σ). Define the sign

$$w_\sigma = (-1)^{k+2l-1} w_u \operatorname{sign}(\sigma) \operatorname{sign}(\det(v_{i,\sigma})_{i=1}^{k+2l}) \in \{0, \pm 1\}.$$

Where $(v_{i,\sigma})_{i=1}^{k+2l}$ is the matrix with column vectors $v_{i,\sigma}$ with separated real and imaginary coordinates, that is if $\theta_{i,k+j}$ is the argument of the j^{th} complex component of $v_{i,\sigma}$ then the $k+2j-1^{\text{st}}$ and $k+2j^{\text{th}}$ real coordinates are $\cos(\theta_{i,k+j})|(v_{i,\sigma})_{k+j}|$ and $\sin(\theta_{i,k+j})|(v_{i,\sigma})_{k+j}|$ respectively and are denoted $v_{i,\sigma}^{(k+2j-1)}$ and $v_{i,\sigma}^{(k+2j)}$ respectively. For $1 \leq j \leq k$, $v_{i,\sigma}^{(j)} = (v_{i,\sigma})_j$.

In the case when $l = 0$ and $k = n$, the following result was proved in [DF13] using topological degree theory and [CDG16] extending results of Colmez [Col88].

Theorem 2.3.1. *(*Diaz y Diaz-Friedman, [DF13], Theorem 1*). If $l = 0$, $k = n$ and $U = \langle u_1, \dots, u_{n-1} \rangle$ is discrete free abelian group of rank $n-1$ in $(\mathbf{R}_{>0})^n$, then the formal linear combination*

$$\sum_{\sigma \in S_{n-1}} w_\sigma C_{e_n}(v_{1,\sigma}, \dots, v_{n,\sigma})$$

is a signed fundamental domain for the action of U on $(\mathbf{R}_{>0})^n$.

Our aim is to prove an analogous result for U of rank $k+l-1$ and $n = k+2l$. The following analysis suggests an outline for a way to reduce our problem to the known case of Theorem 2.3.1.

As discussed in the introduction of this section, the log map ϕ given in (2.3.1) maps D onto a hyperplane H where the induced action of U on H is given simply by translation by $\phi(u)$.

A fundamental domain for the action of U on H can be given by a parallelotope P spanned by vectors $\phi(u_i)$, with the appropriate closure considerations. Since ϕ is U-equivariant, we get that $\phi^{-1}(P)$ is a fundamental domain for the action of U on D.

We notice that since ϕ is a homomorphism and $\log|e^{i\theta}| = 0$, $\phi^{-1}(P)$ is invariant under multiplication by $e^{i\theta}$ in its complex components. This means that, in particular, $\phi^{-1}(P)$ is completely determined by the absolute values of the coordinates of u_i and does not depend on the argument of $(u_i)_j$. Let $\tilde{u}_i = (|(u_i)_1|, \ldots, |(u_i)_k|, |(u_i)_{k+1}|, \ldots, |(u_i)_{k+l}|)$ in $(\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$ and

$$\tilde{U} = \langle \tilde{u}_1, \ldots, \tilde{u}_{k+l} \rangle$$

The above discussion shows that $\phi^{-1}(P)$ is a fundamental domain for the action of both U and $\tilde{U}$. Considering each complex coordinate via the isomorphism $\mathbf{C}^\times \cong \mathbf{R}_{>0} \times S^1$ and $(\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$ as $(\mathbf{R}_{>0})^{k+l} \times (S^1)^l$, we see that $\tilde{U}$ only acts non-trivially on the $(\mathbf{R}_{>0})^{k+l}$ component of the space. The case done by [CDG16] and [DF13] applies and their results tells us that the formal linear combination

$$\sum_{\sigma \in S_{k+l-1}} \tilde{w}_\sigma c_{e_n}(\tilde{v}_{1,\sigma}, \ldots, \tilde{v}_{k+l,\sigma}) \tag{2.3.2}$$

is the characteristic function of a fundamental domain for the action of $\tilde{U}$ on $(\mathbf{R}_{>0})^{k+l}$. Where $\tilde{v}_{i,\sigma}$ and $\tilde{w}_\sigma$ are

$$\tilde{v}_{i,\sigma} = \tilde{u}_{\sigma(1)} \cdots \tilde{u}_{\sigma(i-1)}$$

$$\tilde{w}_\sigma = (-1)^{k+l} w_u \operatorname{sign}(\sigma) \operatorname{sign}(\det(\tilde{v}_{i,\sigma})_{i=1}^{k+l}) \in \{0, \pm 1\},$$

with $(\tilde{v}_{i,\sigma})_{i=1}^{k+l}$ is the reduced square matrix having removed the rows corresponding to the l imaginary parts of the complex components, which are all zero for $|v_{i,\sigma}|$.

Since U and $\tilde{U}$ act identically on the $(\mathbf{R}_{>0})^{k+l}$ component, this is also a signed fundamental domain for the action of U on $(\mathbf{R}_{>0})^{k+l}$. This analysis suggests that a possible proof would be to incorporate the extra $(S^1)^l$ component into the space and fundamental domain while maintaining the description in terms of a signed union of simplicial cones.

A key point of the proof in [CDG16] is the construction of a group cohomology class $[\phi_{\tilde{U}}] \in H^{k+l-1}(\tilde{U}, \mathcal{K}_{\mathbf{R}})$ given as follows. Given $v_1, \ldots, v_n \in \tilde{U}$, then

$$\phi_{\tilde{U}}(v_1, \ldots, v_n) = \operatorname{sign}(\det(v_i)_{i=1}^n) c_{e_{k+l}}(v_1, \ldots, v_n) \in \mathcal{K}_{\mathbf{R}}. \tag{2.3.3}$$

To prove our generalization, we construct an analogous class to $\phi_{\tilde{U}}$ in Betti cohomology and apply their techniques to define a candidate for a signed fundamental domain. By observing certain symmetries we show that the characteristic function on our candidate domain is equivalent to equation 2.3.2.

First let us recall some facts about simplices. Considering a signed fundamental domain as a sort of *signed triangulation* of a fundamental domain, we recall a fundamental result in algebraic topology which gives an explicit triangulation of the product of two spaces in terms of the triangulations of the spaces. The **Eilenberg-Zilber map** or **Eilenberg-MacLane map** or **shuffle map** [Dol80] is a natural transformation on chain complexes defined on a pair of simplices

$$\sigma : \Delta_p \to X$$

$$\tau : \Delta_q \to Y.$$

Define a linear map

$$\eta^\mu : \Delta_{p+q} \to \Delta_p$$

$$e_i \mapsto e_j \quad \text{if} \quad \mu_j \le i < \mu_{j+1},$$

where e_i are the vertices of Δ, and $\mu_0 = 0, \mu_{p+1} = p + q + 1$.

A (p, q)-shuffle (μ, ν) is a pair of disjoint sets of integers

$$1 \le \mu_1 < \mu_2 < \ldots < \mu_p \le p + q, \quad 1 \le \nu_1 < \nu_2 < \cdots < \nu_q \le p + q$$

between 1 and $p + q$, and $\operatorname{sign}(\mu, \nu)$ is the associated sign of the permutation of the integers $(1, \ldots, p + q)$.

The Eilenberg-Zilber map is defined as

$$\nabla_{p,q} : S_pX \otimes S_qY \to S_{p+q}(X \times Y)$$

$$\sigma \otimes \tau \mapsto \sum_{(\mu,\nu)} \operatorname{sign}(\mu, \nu)(\sigma \circ \eta^\mu, \tau \circ \eta^\nu) \tag{2.3.4}$$

where the sum is over all (p,q) shuffles. The differential on a shuffle product of simplices is defined as

$$\partial(\sigma \otimes \tau) = \partial\sigma \otimes \tau + (-1)^{\deg \sigma}\sigma \otimes \partial\tau.$$

Given simpices $a = [a_0, \ldots, a_p]$ and $b = [b_0, \ldots, b_q]$ in terms of the images of their vertices, $\Delta_{p,q}(a \otimes b)$ is the $(p+q)$-simplex whose i^{th} vertex in the product space is given by $(a_{\eta^\mu(i)}, b_{\eta^\nu(i)})$. This map, in particular the fact that it commutes with the boundary map, is one of the key ingredients in the proof of the original Künneth formula.

Remark 2.3.1. *The (p,q)-shuffle group is isomorphic to $S_{p+q}/S_p \times S_q$, where S_p acts on the first p letters and S_q acts on the last q letters. This notion can be generalized to $(n_1, n_2, \ldots, n_k)$-shuffles, where $\sum n_i = n$ and $k \leq n$, in the obvious way.*

Let $\xi_{j,d} = (1, \ldots, 1, e^{2\pi i/d}, 1, \ldots, 1) \in D$, where $e^{2\pi i/d}$ is in the $k + j^{\text{th}}$ coordinate. The set D acts naturally on cones $C(v_1, \ldots, v_n)$ by $\xi C(v_1, \ldots, v_n) = C(\xi v_1, \ldots, \xi v_n)$, this translates naturally to an action on characteristic functions by $\xi c_Q(v_1, \ldots, v_n)(x) = c_Q(v_1, \ldots, v_n)(\xi^{-1}x)$. Fix $\{u_1, \ldots, u_{k+l-1}\}$ to be the generators of U and have $\{\tilde{u}_1, \ldots, \tilde{u}_{k+l}\}$ generate $\tilde{U}$. For $\sigma \in S_{n-1}$ define

$$v_{\sigma(i)} = \tilde{u}_{\sigma(i)} \qquad \text{for } \sigma(i) < k + l$$

$$v_{\sigma(i)} = \xi_{\sigma(i)-(k+l-1)} \qquad \text{for } \sigma(i) \geq k + j.$$

$$v_{i,\sigma} = v_{\sigma(1)} \cdots v_{\sigma(n-1)}.$$

We claim the following

Theorem 2.3.2. *For $d > 2$, the formal linear combination*

$$\sum_{0 \leq e_j < d} \left(\prod_{j=1}^{l} \xi_{j,d}^{e_j} \right) \sum_{\sigma \in S_{n-1}} w_\sigma C_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma})$$

is a signed fundamental domain for the action of U on $(\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$, and is independent of d, i.e.

$$\sum_{u \in U} \sum_{0 \leq e_j < d} \left(\prod_{j=1}^{l} \xi_{j,d}^{e_j} \right) \sum_{\sigma \in S_{n-1}} w_\sigma c_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma})(ux) = 1 \tag{2.3.5}$$

for all $x \in (\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$. Here $\xi C_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma}) = C_{e_n}(\xi v_{1,\sigma}, \ldots, \xi v_{n,\sigma})$.

Proof. For ease of notation we will denote by X the space $(\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$, by $\tilde{X}$ its universal cover, $X/\tilde{U}$ the topological quotient by $\tilde{U}$ and $\pi = \pi_1(X/\tilde{U})$.

Let $\tau^{n-1} : \Delta_{n-1} \to X$ be a singular $(n-1)$-simplex in X and $[\tau_0, \dots, \tau_{n-1}]$ its vertices. A key point of the proof is the construction of the class $[\phi] \in H^{n-1}(X/\tilde{U}, \mathcal{K})$ in Betti cohomology, where $\mathcal{K}$ comes with a nontrivial action by $\tilde{U} \subset \pi_1(X/\tilde{U})$ given by

$$(uf)(x) = f(u^{-1}x). \tag{2.3.6}$$

Recall that given a Singular chain complex $C_*(X)$, the group $H^{n-1}(X/\tilde{U}, \mathcal{K})$ may be computed via the cochain complex [Hat02]

$$Hom_\pi(C_*(\tilde{X}), \mathcal{K}) \cong Hom_\pi(C_*(X), \mathcal{K}).$$

Let us denote by $\Delta(a_1, \dots, a_n)$ a literal simplex in $X = (\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$ given by the set

$$\Delta(a_1, \dots, a_n) = \left\{ \sum_i \alpha_i a_i \ : \ \alpha_i \in \mathbf{R}_{\geq 0} \text{ and } \sum_i \alpha_i = 1 \right\}.$$

Given $\tau \in C_{n-1}(X)$, let

$$\phi(\tau) = \frac{\det(\tau_i)_{i=1}^n}{|\det(\tau_i)_{i=1}^n|} c_{e_n}(\tau_1, \dots, \tau_n) \in \mathcal{K}. \tag{2.3.7}$$

A priori, this is not defined for τ where $\mathrm{Arg}(\tau_1^{(j)}), \dots, \mathrm{Arg}(\tau_n^{(j)})$ have a linear dependence by positive real coefficients (in particular if $\mathrm{Arg}(\tau_i^{(j)}) = -\mathrm{Arg}(\tau_{i'}^{(j)})$), since then $\Delta(\tau_1, \dots, \tau_n)$ may have points with coordinates equal to zero which would not lie in X. The fact that ϕ satisfies the cocycle property (Proposition 2.2.1)

$$\partial_n \phi(\tau) = \sum_{i=0}^n (-1)^i \phi([\tau_0, \dots, \hat{\tau}_i, \dots, \tau_n]) = 0$$

ensures that the function $\phi(\tau)$ is invariant upon subdivision of τ. Subdividing τ allows us to make certain that the arguments $\mathrm{Arg}(\tau_1^{(j)}), \dots, \mathrm{Arg}(\tau_n^{(j)})$ lie in a connected interval $(a_j, a_j + \pi)$ for each j, which is sufficient to make the class $[\phi]$ well defined.

Note that $\phi(\Delta(a_1, \dots, a_n))$ is simply a signed characteristic function of the cone generated by $a_1, \dots, a_n$, or equivalently a signed characteristic function of the set of rays passing through $\Delta(a_1, \dots, a_n)$.

We define an explicit element $\alpha \in H_{n-1}(X/\tilde{U}, \mathbf{Z}) \cong \mathbf{Z}$ in terms of simplices in $C_n(X)$ modulo the action of $\tilde{U}$. We give these in terms of the basis $\tilde{u}_1, \dots, \tilde{u}_{k+l-1}$

of $\tilde{U}$ together with $\xi_{1,d}, \ldots, \xi_{l,d}$, where we recall that

$$\xi_{j,d} = (1, \ldots, 1, e^{2\pi i/d}, 1, \ldots, 1)$$

where $e^{2\pi i/d}$ is in the $k + j^{\text{th}}$ component. We let α be the class represented by the singular simplex

$$\alpha_d = (-1)^{n-1} w_u \sum_{0 \leq e_j < d} \left(\prod_{j=1}^{l} \xi_{j,d}^{e_j} \right) \sum_{\sigma \in S_{n-1}} \operatorname{sign}(\sigma) \Delta(v_{1,\sigma}, \ldots, v_{n,\sigma}) \in C_{n-1}(X), \tag{2.3.8}$$

$$v_{i,\sigma} = v_{\sigma(1)} \cdots v_{\sigma(i-1)}$$

$$v_j = \tilde{u}_j \text{ for } j < k + l \text{ and } v_j = \xi_{j-(k+l-1)} \text{ for } k + l \leq j < k + 2l,$$

and $\xi_{j,d}$ is acting by componentwise multiplication. This is only defined for $d > 2$. Note that

$$\sum_{\sigma \in S_{n-1}} \operatorname{sign}(\sigma) \Delta(v_{1,\sigma}, \ldots, v_{n,\sigma}) \in C_{n-1}(X)$$

is nothing but the $(1, 1, \ldots, 1)$-shuffle product $\Delta(1, \tilde{u}_1) \otimes \cdots \otimes \Delta(1, \tilde{u}_{k+l-1}) \otimes \Delta(1, \xi_1) \otimes \cdots \otimes \Delta(1, \xi_l)$ in X^{n-1} pushed forward to X via componentwise multiplication.

$$m : X^{n-1} \to X$$

$$(x_1, \ldots, x_{n-1}) \mapsto x_1 \cdots x_{n-1}.$$

It can be checked that componentwise multiplication sends shuffle products of literal simplices to literal simplices.

We show that α_d is a cycle, i.e. $\partial \alpha_d = 0$. First we recall that

$$\partial(\sigma \otimes \tau) = \partial\sigma \otimes \tau + (-1)^{\deg \sigma} \sigma \otimes \partial\tau.$$

So then

$$\begin{aligned} &\partial(\Delta(1, \tilde{u}_1) \otimes \cdots \otimes \Delta(1, \tilde{u}_{k+l-1}) \otimes \Delta(1, \xi_{1,d}) \otimes \cdots \otimes \Delta(1, \xi_{l,d})) \\ &= \left(\sum_{i=1}^{k+l-1} \cdots \otimes (-1)^{i+1} \partial\Delta(1, \tilde{u}_i) \otimes \cdots \right) + \left(\sum_{j=1}^{l} \cdots \otimes (-1)^{j+k+l} \partial\Delta(1, \xi_{j,d}) \otimes \cdots \right). \end{aligned}$$

Since the points $1 = (1, \ldots, 1)$ and $\tilde{u}_i$ are identified in $X/\tilde{U}$, we get that $\partial\Delta(1, \tilde{u}_i) = \Delta(1) - \Delta(\tilde{u}_i) = 0$. On the other hand $\partial\Delta(1, \xi_j) = \Delta(1) - \Delta(\xi_j)$ is not identified, but α_d is defined in terms of the all of the $\xi_{j,d}$ translates of such shuffles, so we get that

$$\begin{aligned}\partial\alpha_d &= \cdots \otimes \left((-1)^{n-1} w_u \sum_{j=1}^{l} (-1)^{j+k+l} \sum_{0 \le e_{j'} < d} \prod_{j'=1}^{l} \xi_{j',d}^{e_{j'}} \partial(\Delta(1, \xi_{j,d})) \right) \otimes \cdots \\ &= \cdots \otimes \left((-1)^{n-1} w_u \sum_{j=1}^{l} (-1)^{j+k+l} \sum_{\substack{0 \le e_{j'} < d \\ j \neq j'}} \prod_{\substack{j'=1 \\ j' \neq j}}^{l} \xi_{j',d}^{e_{j'}} \sum_{e_j=0}^{d-1} \Delta(\xi_{j,d}^{e_j}) - \Delta(\xi_{j,d}^{e_j+1}) \right) \otimes \cdots .\end{aligned}$$

Here we see notice we have the telescoping sum $\sum_{e_j=0}^{d-1} \Delta(\xi_{j,d}^{e_j}) - \Delta(\xi_{j,d}^{e_j+1})) = \Delta(1) - \Delta(\xi_{j,d}^{d})$, but since $\xi_{j,d} = (1, \ldots, 1, e^{2\pi i/d}, 1, \ldots, 1)$, we get that $\xi_{j,d}^{d} = 1 = (1, \ldots, 1)$. This shows that $\partial\alpha = 0$.

Now show that the class represented by α_d is independent of d so long as $d > 2$. This fact is critical to the remainder of the proof and is the key ingredient that will allow us to reduce to the case treated by [CDG16].

Again we use the fact that

$$\sum_{\sigma \in S_{n-1}} \operatorname{sign}(\sigma)\Delta(v_{1,\sigma}, \ldots, v_{n,\sigma}) \in C_{n-1}(X)$$

is the $(1, 1, \ldots, 1)$-shuffle product $\Delta(1, \tilde{u}_1) \otimes \cdots \otimes \Delta(1, \tilde{u}_{k+l-1}) \otimes \Delta(1, \xi_{1,d}) \otimes \cdots \otimes \Delta(1, \xi_{l,d})$. We show $\alpha_{pd} - \alpha_d$ is a boundary by induction on l, the number of complex components in $X = (\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$.

Base case ($l = 1$): Let $\Delta(\tilde{U}) = \Delta(1, \tilde{u}_1) \otimes \cdots \otimes \Delta(1, \tilde{u}_{k+l} - 1)$, then we have that

$$
\begin{aligned}
\alpha_{pd}-\alpha_d = (-1)^{n-1}w_u &\left\{\left(\sum_{0\le e_j<pd}\xi_{1,pd}^{e_j}m_*(\Delta(\tilde{U})\otimes\Delta(1,\xi_{1,pd}))\right)\right.\\
&\left.-\left(\sum_{0\le e_j<d}\xi_{1,d}^{e_j}m_*(\Delta(\tilde{U})\otimes\Delta(1,\xi_{1,d}))\right)\right\}\\
= (-1)^{n-1}w_u &\left\{\left(\sum_{0\le e_j<pd}m_*(\Delta(\tilde{U})\otimes\Delta(\xi_{1,pd}^{e_j},\xi_{1,pd}^{e_j+1}))\right)\right.\\
&\left.-\left(\sum_{0\le e_j<d}m_*(\Delta(\tilde{U})\otimes\Delta(\xi_{1,pd}^{pe_j},\xi_{1,pd}^{pe_j+p}))\right)\right\}\\
= (-1)^{n-1}w_u &\sum_{0\le e_j<d}m_*\left\{\Delta(\tilde{U})\otimes(-\Delta(\xi_{1,pd}^{pe_j},\xi_{1,pd}^{pe_j+p})\right.\\
&\left.+\sum_{0\le k<p}\Delta(\xi_{1,pd}^{pe_j+k},\xi_{1,pd}^{pe_j+k+1}))\right\}.
\end{aligned}
$$

Now we notice that

$$
-\Delta(\xi_{1,pd}^{pe_j},\xi_{1,pd}^{pe_j+p})+\sum_{0\le k<p}\Delta(\xi_{1,pd}^{pe_j+k},\xi_{1,pd}^{pe_j+k+1})
$$

is a contractible loop, hence a boundary we'll call $\partial c_{pd,d}$. Moreover, the previous fact that $\partial\alpha_d=0$ shows that $\Delta(\tilde{U})$ is a cycle, so $\alpha_{pd}-\alpha_d$ is the shuffle product of a cycle and a boundary

$$
\alpha_{pd}-\alpha = C\cdot m_*(\Delta(\tilde{U})\otimes\partial c_{pd,d})).
$$

Which implies that it is itself a boundary

$$
\begin{aligned}
\partial(C\cdot m_*(\Delta(\tilde{U})))\otimes c_{pd,d})) =& C\cdot m_*(\partial(\Delta(\tilde{U})\otimes c_{pd,d}))\\
=& C\cdot m_*(\partial(\Delta(\tilde{U}))\otimes c_{pd,d})\\
&+(-1)^{k+l-1}\Delta(\tilde{U})\otimes\partial c_{pd,d})\\
=& C\cdot m_*((-1)^{k+l-1}\Delta(\tilde{U}\otimes\partial c_{pd,d}))\\
=& (-1)^{k+l-1}(\alpha_{pd}-\alpha_d).
\end{aligned}
$$

Inductive step: Let

$$
\Delta_{pd}=\Delta(1,\tilde{u}_1)\otimes\cdots\otimes\Delta(1,\tilde{u}_{k+l-1})\otimes\Delta(1,\xi_{1,pd})\otimes\cdots\Delta(1,\xi_{l-1,pd}),
$$

note that α_{pd} and $m_*(\sum_{0\le e_l<pd}\xi_{l,pd}^{e_l}\Delta_{pd}\otimes\Delta(1,\xi_{l,pd}))$ differ by a sign not depending on p or d. The inductive assumption is that

$$\Delta_{pd}-\Delta_d$$

is a boundary. By an identical argument as in the base case (and using the fact that $\Delta_{pd}=\Delta_d+\partial\mu$), one can show that

$$\sum_{0\le e_l<pd}\xi_{l,pd}^{e_l}\Delta_{pd}\otimes\Delta(1,\xi_{l,pd})-\sum_{0\le e_l<d}\xi_{l,d}^{e_l}\Delta_d\otimes\Delta(1,\xi_{l,d})$$

is a boundary. This shows that $\alpha_{pd}-\alpha$ is the shuffle product of two boundaries, this is sufficient to show that $\alpha_{pd}-\alpha_d$ is a boundary:

If $\alpha_{pd}-\alpha_d=b\otimes c$ with $b=\partial\mu$ and $c=\partial\nu$ then

$$\partial(b\otimes\nu)=\partial b\otimes\nu+(-1)^{|b|}b\otimes\partial\nu=(-1)^{|b|}b\otimes c=(-1)^{|b|}(\alpha_{pd}-\alpha_d)$$

$$\partial(\mu\otimes c)=\partial\mu\otimes c+(-1)^{|\mu|}\mu\otimes\partial c=b\otimes c=\alpha_{pd}-\alpha_d.$$

This makes the cycle class $[\alpha]$ well defined independent of d, so long as $d>2$. The construction of this class and the class ϕ are the critical ingredients of this proof.

Now we see that as $d\to\infty$, the loop

$$\sum_{e_i=0}^{d-1}\xi_{i,d}^{e_i}\Delta(1,\xi_{i,d})$$

approaches the unit circle $S_i^1=\{(1,\dots,1,e^{2\pi\theta},1,\dots,1)\ :\ 0\le\theta<1\}$ in the $k+i^{\text{th}}$ component of $X=(\mathbf{R}_{>0})^k\times(\mathbf{C}^\times)^l$. So as $d\to\infty$, we get that α_d approaches

$$\lim_{d\to\infty}\alpha_d=(-1)^{n-1}w_u\sum_{\sigma\in S_{n-1}}\operatorname{sign}(\sigma)m_*(\Delta(1,\tilde{u}_1)\otimes\cdots\otimes\Delta(1,\tilde{u}_{k+l-1})\otimes S_1^1\otimes\cdots\otimes S_l^1).$$

Where S_j^1 represents the counter-clockwise oriented unit circle in the j^{th} component of X. Since the class $[\alpha_d]=[\alpha]$ remains unchanged with d, we see that also $\lim_{d\to\infty}[\alpha_d]=[\alpha]$.

The image of $([\phi],\alpha_d)$ under the cap product

$$H^{n-1}(X/\tilde{U},\mathcal{K})\times H_{n-1}(X/\tilde{U},\mathbf{Z})\to H_0(X/\tilde{U},\mathcal{K})=\mathcal{K}_{\tilde{U}}$$

is the image of the function

$$\sum_{0\le e_j<d}\left(\prod_{j=1}^{l}\xi_{j,d}^{e_j}\right)\sum_{\sigma\in S_{n-1}}w_\sigma c_{e_n}(v_{1,\sigma},\dots,v_{n,\sigma})$$

in $\mathcal{K}_{\tilde{U}}$. Where

$$w_\sigma = (-1)^{n-1} w_u \text{sign}(\sigma) \frac{\det(v_{i,\sigma})_{i=1}^n}{|\det(v_{i,\sigma})_{i=1}^n|}.$$

We can observe that for $\sigma = 1$ we have

$$(-1)^{l^2} \det(v_{i,1})_{i=1}^n \tag{2.3.9}$$

$$= \left| \begin{array}{c|ccc} & v_{k+l,\sigma}^{(1)} & \cdots & v_{k+l,\sigma}^{(1)} \\ & \vdots & \cdots & \vdots \\ (v_{i,\sigma}^{(j)})_{i=1,j=1}^{k+l} & v_{k+l,\sigma}^{(k)} & \cdots & v_{k+l,\sigma}^{(k)} \\ & \cos(2\pi/d) v_{k+l,\sigma}^{(k+1)} & \cdots & \cos(2\pi/d) v_{k+l,\sigma}^{(k+1)} \\ & & \ddots & \vdots \\ & v_{k+l,\sigma}^{(k+l)} & & \cos(2\pi/d) v_{k+l,\sigma}^{(k+l)} \\ \hline & \sin(2\pi/d) v_{k+l,\sigma}^{(k+1)} & \cdots & \sin(2\pi/d) v_{k+l,\sigma}^{(k+l)} \\ \mathbf{0}_{l\times(k+l)} & & \ddots & \vdots \\ & 0 & & \sin(2\pi/d) v_{k+l,\sigma}^{(k+l)} \end{array} \right| \tag{2.3.10}$$

We say that two permutations $\sigma, \sigma' \in S_{n-1}$ give the same ordering on $I \subset \{1, \ldots, n-1\}$ if for all non-equal $i, j \in I$ we have $\sigma(i) < \sigma(j)$ if and only if $\sigma'(i) < \sigma'(j)$.

We show that $\text{sign}(\sigma) \det(v_{i,\sigma})_{i=1}^n$ does not depend the image of the letters $\{k+l, \ldots, k+2l-1\}$ and depends only on the order in which σ permutes $\tilde{u_i}$ (that is, the letters $\{1, \ldots, k+l-1\}$)(Lemma 2.4.1, proven in the next section). So we get that $w_\sigma = w_{\sigma'}$ for σ, σ' giving the same ordering of $\{1, \ldots, k+l-1\}$.

For each σ, let $\tilde{\sigma}$ denote the representative (of permutations giving the same ordering of the letters $\{1, \ldots, k+l-1\}$ as σ) for which $\tilde{\sigma}(i) < k+l$ for all $i < k+l$. Define

$$w'_\sigma = (-1)^{l^2} \text{sign}(\tilde{\sigma}) \frac{\det(v_{i,\tilde{\sigma}})_{i=1}^{k+l}}{|\det(v_{i,\tilde{\sigma}})_{i=1}^{k+l}|}$$

where $\det(v_{i,\tilde{\sigma}})_{i=1}^{k+l}$ denotes the determinant of just the upper left block of the matrix given in equation 2.3.9, whose rows are a reordering of $(v_{i,\tilde{\sigma}})_{i=1}^n$. We see that for the representative σ fixing $\{k+l, \ldots, k+2l-1\}$

$$w_\sigma = \left(\frac{\sin(2\pi/d)}{|\sin(2\pi/d)|} \right)^l w'_\sigma = w'_\sigma.$$

Combining these facts we get that

$$([\phi], \alpha_d) = \sum_{0 \le e_j < d} \left(\prod_{j=1}^{l} \xi_{j,d}^{e_j} \right) \sum_{\sigma \in S_{n-1}} w'_\sigma c_{e_n}(v_{1,\sigma}, \dots, v_{n,\sigma}).$$

Now we again use the fact that α_d (and hence $([\phi], \alpha_d)$) is independent of d. Recall that for $f \in \mathcal{K}$ and $\xi \in X$, we have $(\xi f)(x) = f(\xi^{-1}x)$. So in particular, since the points in the cone $C_{e_n}(v_{1,\sigma}, \dots, v_{n,\sigma})$ have arguments in the range $[0, 2\pi/d]$, that the value

$$\left[\sum_{0 \le e_j < d} \left(\prod_{j=1}^{l} \xi_{j,d}^{e_j} \right) \sum_{\sigma \in S_{n-1}} w'_\sigma c_{e_n}(v_{1,\sigma}, \dots, v_{n,\sigma}) \right](x) \tag{2.3.11}$$

only depends on the argument of $x \in X = (\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$ in its complex coordinates up to $2\pi/d$. So as $d \to \infty$, we get that $C(v_{1,\sigma}, \dots, v_{n,\sigma})$ approaches $C(v_{1,\tilde{\sigma}}, \dots, v_{k+l,\tilde{\sigma}})$, where $\tilde{\sigma}$ is the representative fixing $\{k+l, \dots, k+2l-1\}$ with ordering of $\{1, \dots, k+l-1\}$ identical to σ. The cones $C(v_{1,\sigma_i}, \dots, v_{n,\sigma_i})$ are disjoint for σ_1,σ_2 giving the same ordering of $\{1, \dots, k+l-1\}$(Lemma 2.4.2, proven in the next section), the many-to-one reduction of $\sigma \mapsto \tilde{\sigma}$ can not incur a multiplicative factor. At any given point in the limit $d \to \infty$, the point x can lie in at most one cone $C(v_{1,\sigma}, \dots, v_{n,\sigma})$ being reduced to $C(v_{1,\tilde{\sigma}}, \dots, v_{k+l,\tilde{\sigma}})$. Thus the value of (2.3.11) is independent of the arguments of x.

Let $\mathcal{J}$ denote the group of functions $X \to \mathbf{Z}$, these are endowed with a natural action of $\tilde{U}$ (the same action as 2.3.6). Denote by $\Sigma_{\tilde{U}} : \mathcal{K}_{\tilde{U}} \to \mathcal{J}^{\tilde{U}}$ the map

$$(\Sigma_{\tilde{U}} f)(x) = \sum_{u \in \tilde{U}} f(ux). \tag{2.3.12}$$

This sum is finite for any given $f \in \mathcal{K}_{\tilde{U}}$. Applying this to $([\phi], \alpha)$ with $d \to \infty$ we get that

$$(\sum_{\tilde{U}}([\phi], \alpha))(x) = \sum_{u \in \tilde{U}} \sum_{\tilde{\sigma} \in S_{k+l-1}} w'_{\tilde{\sigma}} c_{e_n}(v_{1,\tilde{\sigma}}, \dots, v_{k+l,\tilde{\sigma}})(u|x|)$$

where $|x| = (x_1, \dots, x_k, |x_{k+1}|, \dots, |x_{k+l}|)$.

We now see that we have completely reduced this to the totally real case treated in [CDG16] given in equation (2.3.2), which told us that

$$\sum_{u \in \tilde{U}} \sum_{\tilde{\sigma} \in S_{k+l-1}} w'_{\tilde{\sigma}} c_{e_n}(v_{1,\tilde{\sigma}}, \dots, v_{k+l-1,\tilde{\sigma}})(u|x|) = (-1)^{l^2+l} = 1$$

for all $x \in X$. Hence, $(\sum_{\tilde{U}}([\phi], \alpha))$ is the constant function on X equal to 1, and since the fundamental domains for U and $\tilde{U}$ coincide, this proves the result. $\square$

2.4 Sign properties of w_σ and cones

Here we prove necessary technical results for the proof of Theorem 2.3.2. We say that two permutations $\sigma, \sigma' \in S_{n-1}$ give the same ordering on $I \subset \{1, \dots, n-1\}$ if for all non-equal $i, j \in I$ we have $\sigma(i) < \sigma(j)$ if and only if $\sigma'(i) < \sigma'(j)$.

In the previous section we defined a certain sign

$$w_\sigma = (-1)^{n-1}\mathrm{sign}(\sigma)\frac{\det(v_{i,\sigma})_{i=1}^n}{|\det(v_{i,\sigma})_{i=1}^n|}.$$

The following result shows when two such signs w_σ, $w_{\sigma'}$ may be equivalent for two permutation $\sigma, \sigma' \in S_{n-1}$.

Lemma 2.4.1. *The value* $\mathrm{sign}(\sigma)\det(v_{i,\sigma})_{i=1}^n$ *does not depend on the image of the letters* $\{k+l, \dots, k+2l-1\}$ *under the permutation* σ *and only depends on the order in which* σ *permutes the letters* $\{1, \dots, k+l-1\}$.

Proof. It suffices to show $\mathrm{sign}(\sigma)\det(v_{i,\sigma}) = \mathrm{sign}(\sigma')\det(v_{i,\sigma'})$ for σ, σ' differing by the simple transposition $(m\ m+1)$ which doesn't change the order of $(\sigma(1), \dots, \sigma(k+l-1))$ or $(\sigma'(1), \dots, \sigma'(k+l-1))$.

Say $\sigma = \sigma'(m\ m+1)$, we must be in one of two cases: $\sigma(m)$ and $\sigma(m+1)$ are both strictly greater than $k+l$, or (without loss) $\sigma(m) > k+l$ and $\sigma(m+1) \leq k+l$.

Let

$$S_\tau(N) = \{\tau(j)\ :\ j \leq N\},$$

then we can see that $S_\sigma(N) = S_{\sigma'}(N)$ for $N < m$ and $N \geq m+1$. This implies that $v_{j,\sigma} = v_{j,\sigma'}$ for $j < m+1$ and $j \geq m+2$, so the matrices $(v_{i,\sigma})_{i=1}^n$ and $(v_{i,\sigma'})_{i=1}^n$ differ only in $(m+1)^{\mathrm{st}}$ column.

Case 1: Assume $\sigma(m)$ and $\sigma(m+1)$ are both strictly greater than $k+l$ (without loss assume $\sigma(m) < \sigma(m+1)$), then the relevant columns of $(v_{i,\sigma})_{i=1}^n$ are

$$(v_{i,\sigma})_{i=1}^n = \begin{pmatrix} \vdots & \vdots & \vdots \\ v_{m,\sigma}^{(\sigma(m)-l)} & v_{m,\sigma}^{(\sigma(m)-l)} \cos(2\pi/d) & v_{m,\sigma}^{(\sigma(m)-l)} \cos(2\pi/d) \\ \vdots & \vdots & \vdots \\ v_{m,\sigma}^{(\sigma(m+1)-l)} & v_{m,\sigma}^{(\sigma(m+1)-l)} & v_{m,\sigma}^{(\sigma(m+1)-l)} \cos(2\pi/d) \\ \vdots & \vdots & \vdots \\ v_{m,\sigma}^{(\sigma(m))} & v_{m,\sigma}^{(\sigma(m))} \sin(2\pi/d) & v_{m,\sigma}^{(\sigma(m))} \sin(2\pi/d) \\ \vdots & \vdots & \vdots \\ v_{m,\sigma}^{(\sigma(m+1))} & v_{m,\sigma}^{(\sigma(m+1))} & v_{m,\sigma}^{(\sigma(m+1))} \sin(2\pi/d) \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

These columns are identical except in these entries, subtracting the $(m+2)^{\text{nd}}$ column from the $(m+1)^{\text{st}}$ column gives us

$$\det(v_{i,\sigma})_{i=1}^n = \\ \begin{vmatrix} \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m)-l)} & \vdots & v_{m,\sigma}^{(\sigma(m)-l)} \cos(2\pi/d) \\ \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m+1)-l)} & v_{m,\sigma}^{(\sigma(m+1)-l)}(1-\cos(2\pi/d)) & v_{m,\sigma}^{(\sigma(m+1)-l)} \cos(2\pi/d) \\ \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m))} & \vdots & v_{m,\sigma}^{(\sigma(m))} \sin(2\pi/d) \\ \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m+1))} & v_{m,\sigma}^{(\sigma(m+1))}(1-\sin(2\pi/d)) & v_{m,\sigma}^{(\sigma(m+1))} \sin(2\pi/d) \\ \vdots & 0 & \vdots \end{vmatrix}.$$

Subtracting the m^{th} column from the $(m+2)^{\text{th}}$ column gives us

$$\det(v_{i,\sigma})_{i=1}^{n} =$$
$$\begin{vmatrix} \vdots & 0 & 0 \\ v_{m,\sigma}^{(\sigma(m)-l)} & \vdots & v_{m,\sigma}^{(\sigma(m)-l)}(\cos(2\pi/d)-1) \\ \vdots & 0 & 0 \\ v_{m,\sigma}^{(\sigma(m+1)-l)} & v_{m,\sigma}^{(\sigma(m+1)-l)}(1-\cos(2\pi/d)) & v_{m,\sigma}^{(\sigma(m+1)-l)}(\cos(2\pi/d)-1) \\ \vdots & 0 & 0 \\ v_{m,\sigma}^{(\sigma(m))} & \vdots & v_{m,\sigma}^{(\sigma(m))}(\sin(2\pi/d)-1) \\ \vdots & 0 & 0 \\ v_{m,\sigma}^{(\sigma(m+1))} & v_{m,\sigma}^{(\sigma(m+1))}(1-\sin(2\pi/d)) & v_{m,\sigma}^{(\sigma(m+1))}(\sin(2\pi/d)-1) \\ \vdots & 0 & 0 \end{vmatrix}.$$

Adding the $(m+1)^{\text{st}}$ column to the $(m+2)^{\text{nd}}$ column gives us

$$\det(v_{i,\sigma})_{i=1}^{n} =$$
$$\begin{vmatrix} \vdots & 0 & 0 \\ v_{m,\sigma}^{(\sigma(m)-l)} & \vdots & v_{m,\sigma}^{(\sigma(m)-l)}(\cos(2\pi/d)-1) \\ \vdots & 0 & 0 \\ v_{m,\sigma}^{(\sigma(m+1)-l)} & v_{m,\sigma}^{(\sigma(m+1)-l)}(1-\cos(2\pi/d)) & \vdots \\ \vdots & 0 & 0 \\ v_{m,\sigma}^{(\sigma(m))} & \vdots & v_{m,\sigma}^{(\sigma(m))}(\sin(2\pi/d)-1) \\ \vdots & 0 & 0 \\ v_{m,\sigma}^{(\sigma(m+1))} & v_{m,\sigma}^{(\sigma(m+1))}(1-\sin(2\pi/d)) & \vdots \\ \vdots & 0 & 0 \end{vmatrix}.$$

An identical computation shows that for $\sigma' = \sigma(m\ m+1)$ we get that

$$
\det(v_{i,\sigma'})_{i=1}^n =
\begin{vmatrix}
\vdots & 0 & 0 \\
v_{m,\sigma}^{(\sigma(m)-l)} & v_{m,\sigma}^{(\sigma(m+1)-l)}(1-\cos(2\pi/d)) & \vdots \\
\vdots & 0 & 0 \\
v_{m,\sigma}^{(\sigma(m+1)-l)} & \vdots & v_{m,\sigma}^{(\sigma(m)-l)}(\cos(2\pi/d)-1) \\
\vdots & 0 & 0 \\
v_{m,\sigma}^{(\sigma(m))} & v_{m,\sigma}^{(\sigma(m+1))}(1-\sin(2\pi/d)) & \vdots \\
\vdots & 0 & 0 \\
v_{m,\sigma}^{(\sigma(m+1))} & \vdots & v_{m,\sigma}^{(\sigma(m))}(\sin(2\pi/d)-1) \\
\vdots & 0 & 0
\end{vmatrix}.
$$

So it's clear now that by swapping and negating the $(m+1)^{\text{st}}$ and $(m+2)^{\text{nd}}$ columns of the previous matrix, we get that $\det(v_{i,\sigma'})_{i=1}^n = -\det(v_{i,\sigma})$, hence $\operatorname{sign}(\sigma)\det(v_{i,\sigma}) = \operatorname{sign}(\sigma')\det(v_{i,\sigma'})$.

Case 2: Assume now that $\sigma(m) > k+l$ and $\sigma(m+1) \le k+l$. As before there are only three relevant columns

$$
(v_{i,\sigma})_{i=1}^n =
\begin{pmatrix}
\vdots & \vdots & \vdots \\
v_{m,\sigma}^{(\sigma(m)-l)} & v_{m,\sigma}^{(\sigma(m)-l)}\cos(2\pi/d) & v_{\sigma(m+1)}^{(\sigma(m)-l)} v_{m,\sigma}^{(\sigma(m)-l)}\cos(2\pi/d) \\
\vdots & \vdots & \vdots \\
v_{m,\sigma}^{(\sigma(m+1)-l)} & v_{m,\sigma}^{(\sigma(m+1)-l)} & v_{\sigma(m+1)}^{(\sigma(m+1)-l)} v_{m,\sigma}^{(\sigma(m+1)-l)} \\
\vdots & \vdots & \vdots \\
v_{m,\sigma}^{(\sigma(m))} & v_{m,\sigma}^{(\sigma(m))}\sin(2\pi/d) & v_{\sigma(m)}^{(\sigma(m))} v_{m,\sigma}^{(\sigma(m))}\sin(2\pi/d) \\
\vdots & \vdots & \vdots \\
v_{m,\sigma}^{(\sigma(m+1))} & v_{m,\sigma}^{(\sigma(m+1))} & v_{\sigma(m+1)}^{(\sigma(m)-l)} v_{m,\sigma}^{(\sigma(m+1))} \\
\vdots & \vdots & \vdots
\end{pmatrix}.
$$

Subtracting the m^{th} column from the $(m+1)^{\text{st}}$ column gives us

$$\det(v_{i,\sigma})_{i=1}^n =$$

$$\begin{vmatrix} \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m)-l)} & v_{m,\sigma}^{(\sigma(m)-l)}(\cos(2\pi/d)-1) & v_{\sigma(m+1)}^{(\sigma(m)-l)} v_{m,\sigma}^{(\sigma(m)-l)} \cos(2\pi/d) \\ \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m+1)-l)} & \vdots & v_{\sigma(m+1)}^{(\sigma(m+1)-l)} v_{m,\sigma}^{(\sigma(m+1)-l)} \\ \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m))} & v_{m,\sigma}^{(\sigma(m))}(\sin(2\pi/d)-1) & v_{\sigma(m)}^{(\sigma(m))} v_{m,\sigma}^{(\sigma(m))} \sin(2\pi/d) \\ \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m+1))} & \vdots & v_{\sigma(m+1)}^{(\sigma(m)-l)} v_{m,\sigma}^{(\sigma(m+1))} \\ \vdots & 0 & \vdots \end{vmatrix}.$$

On the other hand, for $\sigma' = \sigma(m\ m+1)$ we have

$$(v_{i,\sigma'})_{i=1}^n = \begin{pmatrix} \vdots & \vdots & \vdots \\ v_{m,\sigma}^{(\sigma(m)-l)} & v_{m+1,\sigma}^{(\sigma(m)-l)} & v_{m+1,\sigma}^{(\sigma(m)-l)} \cos(2\pi/d) \\ \vdots & \vdots & \vdots \\ v_{m,\sigma}^{(\sigma(m+1)-l)} & v_{m+1,\sigma}^{(\sigma(m+1)-l)} & v_{m+1,\sigma}^{(\sigma(m+1)-l)} \\ \vdots & \vdots & \vdots \\ v_{m,\sigma}^{(\sigma(m))} & v_{m+1,\sigma}^{(\sigma(m))} & v_{m+1,\sigma}^{(\sigma(m))} \sin(2\pi/d) \\ \vdots & \vdots & \vdots \\ v_{m,\sigma}^{(\sigma(m+1))} & v_{m+1,\sigma}^{(\sigma(m+1))} & v_{m+1,\sigma}^{(\sigma(m+1))} \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Subtracting the $(m+2)^{\text{nd}}$ column from the $(m+1)^{\text{st}}$ column gives us

$$\det(v_{i,\sigma'})_{i=1}^n = \begin{vmatrix} \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m)-l)} & v_{m+1,\sigma}^{(\sigma(m)-l)}(1-\cos(2\pi/d)) & v_{m+1,\sigma}^{(\sigma(m)-l)}\cos(2\pi/d) \\ \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m+1)-l)} & \vdots & v_{m+1,\sigma}^{(\sigma(m+1)-l)} \\ \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m))} & v_{m+1,\sigma}^{(\sigma(m))}(1-\sin(2\pi/d)) & v_{m+1,\sigma}^{(\sigma(m))}\sin(2\pi/d) \\ \vdots & 0 & \vdots \\ v_{m,\sigma}^{(\sigma(m+1))} & \vdots & v_{m+1,\sigma}^{(\sigma(m+1))} \\ \vdots & 0 & \vdots \end{vmatrix}.$$

Again we see clearly that negating the $(m+1)^{\text{st}}$ column gives us that $\det(v_{i,\sigma})_{i=1}^n = -\det(v_{i,\sigma'})_{i=1}^n$, hence $\operatorname{sign}(\sigma)\det(v_{i,\sigma}) = \operatorname{sign}(\sigma')\det(v_{i,\sigma'})$.

□

Lemma 2.4.2. *The cones $C(v_{1,\sigma},\ldots,v_{n,\sigma})$ and $C(v_{1,\sigma'},\ldots,v_{n,\sigma'})$ are disjoint for unequal σ,σ' giving the same ordering of $\{1,\ldots,k+l-1\}$.*

Proof. For this proof we fix positive integer $d > 2$ and use the notation $\xi_j = \xi_{j,d} = (1,\ldots,1,e^{2\pi i/d},1,\ldots,1)$, where $e^{2\pi i/d}$ is in the $k+j^{\text{th}}$ component of $X = (\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$. For the First we recall from Lemma 2.4.1 that

$$\operatorname{sign}(\sigma)\det(v_{i,\sigma})_{i=1}^n = \operatorname{sign}(\sigma')\det(v_{i,\sigma'})_{i=1}^n.$$

Where $(v_{i,\sigma})_{i=1}^{k+2l}$ is the matrix with column vectors $v_{i,\sigma}$ with separated real and imaginary coordinates (as per discussion following Def. 2.3.1). Lemma 2.4.1 implies that the set of vectors $\{v_{i,\sigma} : 1 \le i \le n\}$ is linearly independent over $\mathbf{R}$ if and only if $\{v_{i,\sigma'} : 1 \le i \le n\}$ is linearly independent over $\mathbf{R}$. Without loss we assume both sets are linearly independent (otherwise both cones are empty by definition).

Let R_d be a linear transformation acting on $X = (\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$ which rotates (clockwise) by π/d in every complex component of X. Since the arguments of the complex components of $v_{i,\sigma}$ and $v_{i,\sigma'}$ all lie in $\{0, 2\pi/d\}$, the arguments of $R_d v_{i,\sigma}$

must lie in $\{-\pi/d, \pi/d\}$. Suppose $x \in C(v_{1,\sigma}, \ldots, v_{n,\sigma}) \cap C(v_{1,\sigma'}, \ldots, v_{n,\sigma'})$ for σ,σ' giving the same ordering of $\{1, \ldots, k+l-1\}$. Then

$$x = \sum_{i=1}^{n} a_i v_{i,\sigma} = \sum_{i=1}^{n} b_i v_{i,\sigma'}$$

for some $a_i, b_i > 0$. Applying R_d to x and taking the real parts of each component we get that

$$\sum_{i=1}^{n} a_j \Re(R_d v_{j,\sigma}) = \sum_{i=1}^{n} b_j \Re(R_d v_{j,\sigma'}).$$

In terms of the components of $X = (\mathbf{R}_{>0})^k \times (\mathbf{C}^\times)^l$, taking the real part of $R_d v_{i,\sigma}$ we get

$$\Re(R_d v_{i,\sigma}) = (v_{i,\sigma}^{(1)}, \ldots, v_{i,\sigma}^{(k)}, \cos(\pi/d)|v_{i,\sigma}^{(k+1)}|, \ldots, \cos(\pi/d)|v_{i,\sigma}^{(k+l)}|)$$

where $|\cdot|$ denotes norm $|x+iy| = \sqrt{x^2+y^2}$ in the complex components. Recall that

$$v_{i,\sigma} = v_{\sigma(1)} \cdots v_{\sigma(i-1)}$$

$$v_j = \tilde{u}_j \text{ for } j < k+l \text{ and } v_j = \xi_{j+1-(k+l)} \text{ for } k+l \le j < k+2l.$$

So

$$|v_{i,\sigma}| = |y_{\sigma(1)}| \cdots |y_{\sigma(i-1)}|.$$

Since $|\xi_{j-(k+l)}| = (1, \ldots, 1)$, we see that there are at most $k+l$ distinct elements $\Re(R_d v_{i,\sigma})$ for a fixed ordering of $\{1, \ldots, k+l-1\}$.

Let $T_{1,d}$ be the scaling transformation

$$T_{1,d} = \left(\begin{array}{c|c} \mathbf{1}_{k\times k} & \mathbf{0}_{k\times l} \\ \hline \mathbf{0}_{l\times k} & \mathrm{Diag}(\frac{1}{\cos(\pi/d)}) \end{array} \right).$$

We note that $|v_{i,\sigma}| = \Re(T_{1,d}(R_d v_{i,\sigma})) = (|v_{1,\sigma}|, \ldots, |v_{n,\sigma}|)$. Applying $T_{1,d}$ to $R_d x$ and taking the real parts of each component we get

$$\sum_{i=1}^{n} a_i |v_{i,\sigma}| = \sum_{i=1}^{n} b_i |v_{i,\sigma'}| \tag{2.4.1}$$

but since $|v_{i,\sigma}| = |y_{\sigma(1)}| \cdots |y_{\sigma(i-1)}|$ is a product of $\tilde{u}_i$'s, this gives a linear dependence on a subset of $\{v_{j,\sigma} \ : \ 1 \le j \le k+2l,\ \sigma \in S_{n-1}\}$ (those not containing any ξ_i factors). For a fixed ordering of $\{1, \ldots, k+l-1\}$ given by σ (or equivalently σ').

There are exactly $k+l$ distinct elements $|v_{i,\sigma}|$, each of which must occur, the set of which is equivalent for σ and σ' as it only depends on the ordering of $\{1,\dots,k+l-1\}$.

The set $\{|v_{i,\sigma}| \ : \ 1 \le i \le n\}$ (without repetitions) is linearly independent if and only if $\{v_{i,\sigma} \ : \ 1 \le i \le n\}$ is, this can be seen by considering $\det(v_{i,\sigma''})$ for representative σ'' having the feature that $\sigma''(j) \in \{1,\dots,k+l-1\}$ for $j \in \{1,\dots,k+l-1\}$ and applying Lemma 2.4.1.

This implies that for each distinct $|v_{j,\sigma}|$, that

$$\sum_{|v_{i,\sigma}|=|v_{j,\sigma}|} a_i = \sum_{|v_{i,\sigma'}|=|v_{j,\sigma}|} b_i. \tag{2.4.2}$$

Now we consider the imaginary parts. Let $T_{2,d}$ be the scaling transformation

$$T_{2,d} = \left(\begin{array}{c|c} \mathbf{1}_{k\times k} & \mathbf{0}_{k\times l} \\ \hline \mathbf{0}_{l\times k} & \mathrm{Diag}(\frac{1}{\sin(\pi/d)}) \end{array}\right).$$

Applying $T_{2,d}$ to $R_d x$ and taking imaginary projection in the complex components we get

$$\sum_{i=1}^{n} a_i \Im(T_{2,d}R_d v_{i,\sigma}) = \sum_{i=1}^{n} b_i \Im(T_{2,d}R_d v_{i,\sigma'}). \tag{2.4.3}$$

For example, in the case $\sigma = \mathrm{id}$ and $i \le k+l$, then

$$\Im(T_{2,d}R_d v_{i,\mathrm{id}}) = (|v_{i,\mathrm{id}}^{(1)}|,\dots,|v_{i,\mathrm{id}}^{(k)}|, -|v_{i,\mathrm{id}}^{(k+1)}|,\dots,-|v_{i,\mathrm{id}}^{(k+l)}|).$$

In general, these terms are of this form up to change of sign in the complex components. Specifically,

$$\Im(T_{2,d}R_d v_{i,\sigma})^{(k+j)} = |v_{i,\sigma}^{(k+j)}| \text{ if } \sigma^{-1}(k+l+j-1) < i$$

and

$$\Im(T_{2,d}R_d v_{i,\sigma})^{(k+j)} = -|v_{i,\sigma}^{(k+j)}| \text{ if } \sigma^{-1}(k+l+j-1) \ge i.$$

So, looking at the $k+j^{\text{th}}$ component of the equation 2.4.3 gives us

$$\begin{aligned} -\sum_{i=1}^{\sigma^{-1}(k+l+j)} a_i|v_{i,\sigma}^{(k+j)}| + \sum_{i=\sigma^{-1}(k+l+j)+1}^{n} a_i|v_{i,\sigma}^{(k+j)}| \\ = -\sum_{i=1}^{\sigma'^{-1}(k+l+j)} b_i|v_{i,\sigma'}^{(k+j)}| + \sum_{i=\sigma'^{-1}(k+l+j)+1}^{n} b_i|v_{i,\sigma'}^{(k+j)}|, \end{aligned}$$

adding or subtracting the $(k+j)^{\text{th}}$ component of equation 2.4.1 gives us the following equalities

$$\sum_{i=1}^{\sigma^{-1}(k+l+j)} a_i|v_{i,\sigma}^{(k+j)}| = \sum_{i=1}^{\sigma'^{-1}(k+l+j)} b_i|v_{i,\sigma'}^{(k+j)}| \tag{2.4.4}$$

$$\sum_{i=\sigma^{-1}(k+l+j)+1}^{n} a_i|v_{i,\sigma}^{(k+j)}| = \sum_{i=\sigma'^{-1}(k+l+j)+1}^{n} b_i|v_{i,\sigma'}^{(k+j)}|. \tag{2.4.5}$$

From the construction of $v_{i,\sigma} = v_{\sigma(1)} \cdots v_{\sigma(i-1)}$, we see that the collection of indices i such that $|v_{i,\sigma}| = |v_{j,\sigma}|$ is sequential. Meaning, there minimal index $j_{\min,\sigma}$ and maximal index $j_{\max,\sigma}$ such that $|v_{i,\sigma}| = |v_{j,\sigma}|$ if and only if $j_{\min,\sigma} \leq i \leq j_{\max,\sigma}$. Moreover, it must be the case that and $\sigma(j_{\min,\sigma} - 1) < k + l$ and that $\sigma(i-1) \geq k + l$ for $j_{\min,\sigma} < i \leq j_{\max,\sigma}$.

Assume $j_{\min,\sigma} < j_{\max,\sigma}$, then equations 2.4.2 and 2.4.5 give us

$$\sum_{i=j_{\min,\sigma}}^{n} a_i|v_{i,\sigma}| = \sum_{i=\sigma'^{-1}(\sigma(j_{\min,\sigma}-1))+1}^{n} b_i|v_{i,\sigma'}|$$

$$\sum_{i=j_{\min,\sigma}+1}^{n} a_i|v_{i,\sigma}^{(\sigma(j_{\min,\sigma})-l)}| = \sum_{i=\sigma'^{-1}(\sigma(j_{\min,\sigma}))+1}^{n} b_i|v_{i,\sigma'}^{(\sigma(j_{\min,\sigma})-l)}|$$

subtracting the second equation from the $(\sigma(j_{\min,\sigma}) - l)^{\text{th}}$ tells us that the term $a_{j_{\min,\sigma}}|v_{i,\sigma}^{(\sigma(j_{\min,\sigma}))}|$ must be equal to either

$$\sum_{i=\sigma'^{-1}(\sigma(j_{\min,\sigma}-1))+1}^{\sigma'^{-1}(\sigma(j_{\min}))+1} b_i|v_{i,\sigma'}|$$

or

$$-\sum_{i=\sigma'^{-1}(\sigma(j_{\min,\sigma}))+1}^{\sigma'^{-1}(\sigma(j_{\min}-1))+1} b_i|v_{i,\sigma'}|.$$

Seeing as how all of the terms a_i,b_i,$|v_{i,\sigma}^{(k)}|$,$|v_{i,\sigma'}^{(k)}|$ are strictly positive, it must be the former. This implies in particular that

$$\sigma'^{-1}(\sigma(j_{\min,\sigma} - 1)) < \sigma'^{-1}(\sigma(j_{\min,\sigma}))$$

meaning that σ and σ' give the same ordering of $\{\sigma(j_{\min,\sigma} - 1), \sigma(j_{\min,\sigma})\}$, that is, $\sigma(\sigma(j_{\min,\sigma})) > \sigma(\sigma(j_{\min,\sigma} - 1))$ if and only if $\sigma'(\sigma(j_{\min,\sigma})) > \sigma'(\sigma(j_{\min,\sigma} - 1))$.

Analogously, for each $j \in \{j_{\min,\sigma}+1, \ldots, j_{\max,\sigma}\}$, by comparing the equations

$$\sum_{i=j}^{n} a_i |v_{i,\sigma}^{(\sigma(j)-l)}| = \sum_{i=\sigma'^{-1}(\sigma(j))+1}^{n} b_i |v_{i,\sigma'}^{(\sigma(j)-l)}|$$

$$\sum_{i=j+1}^{n} a_i |v_{i,\sigma}^{(\sigma(j)-l)}| = \sum_{i=\sigma'^{-1}(\sigma(j+1))+1}^{n} b_i |v_{i,\sigma'}^{(\sigma(j)-l)}|$$

we see that we must have that σ and σ' give the same ordering of $\{\sigma(j), \sigma(j+1)\}$.

By assumption, we know that σ, σ' give the same ordering of $\{1, \ldots, k+l-1\}$. We know $\sigma(j_{\min,\sigma}) < k+l$ and each element of $\{1, \ldots, k+l-1\}$ occurs as a $\sigma(j_{\min,\sigma})$ for some j. This implies that σ, σ' give the same ordering for all pairs $\{\sigma(j), \sigma(j+1)\}$, this defines an ordering on all letters $\{1, \ldots, k+2l-1\}$ which defines a permutation, so we must have that $\sigma = \sigma'$. $\square$

Chapter 3

Special values of zeta functions

Let k be an algebraic number field of degree $n = r_1 + 2r_2$, with r_1 real embeddings and r_2 complex embeddings, let d_k and $\mathcal{O}_k$ denote the discriminant and ring of integers (resp.) of k . Fix an integral ideal $\mathfrak{f}$ of k and denote by $H_k(\mathfrak{f})$ the narrow ideal classes modulo $\mathfrak{f}$ of k. Define the Dedekind zeta function associated to k as

$$\zeta_k(s) = \sum_{\mathfrak{b} \subset \mathcal{O}_k} \frac{1}{N\mathfrak{b}^s}.$$

For each $c \in H_k(\mathfrak{f})$, the partial zeta function corresponding to c is

$$\zeta_k(s, c) = \sum_{\mathfrak{b} \in c} \frac{1}{N\mathfrak{b}^s},$$

where $N = N_{k/\mathbf{Q}}$, the algebraic norm to the rational field.

In this section we apply our result from Chapter 2 to rewrite $\zeta_k(s, c)$ in terms of certain analytic zeta functions constructed by Takuro Shintani. These functions have been extensively studied by Shintani, who proved a number of key results about them, namely meromorphic continuation and explicit formulas for their values at non-positive integers. We use these facts to derive explicit analytic formulas for $\zeta_k(s, c)$ as well as certain L-series at non-positive integers.

3.1 Shintani zeta functions

Let A be an $n \times r$ matrix *with positive real coefficents*, and x be a $1 \times n$ matrix or row vector, also with positive coefficients. Denote by

$$L_m(z) = \sum_{\alpha=1}^{r} a_{m\alpha} z_\alpha$$

the linear form corresponding to the m^{th} row of A. Shintani defines [Shi76a] a zeta function associated to this data by

$$\zeta(s, A, x) = \sum_{z} \prod_{m=1}^{n} (L_m(z) + x_m)^{-s}, \tag{3.1.1}$$

where summation over z is taken over all r-tuples of nonnegative integers. Shintani shows this function is absolutely convergent if $\Re(s) > r/n$ and that it extends to a meromorphic function on $\mathbf{C}$ (Proposition 1. [Shi76a]).

Let $J : k \to \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$, $x \mapsto (x^{(1)}, \ldots, x^{(n)})$, where $x^{(1)}, \ldots, x^{(n)}$ are the n conjugates of x, be the natural embedding of k into its Minkowski space $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$. By componentwise multiplication, the group of totally positive units $E(k)_+$ acts on $(\mathbf{R}_{>0})^{r_1} \times (\mathbf{C}^\times)^{r_2}$. Our main result from Chapter 2, Theorem 2.3.2, applied to $E(k)_+$ shows that a fundamental domain for the action of $E(k)_+$ on $(\mathbf{R}_{>0})^{r_1} \times (\mathbf{C}^\times)^{r_2}$ can be realized as a signed union of a *finite number* of simplicial cones with generators in a finite extension k'/k containing the absolute values of a generating set of $E(k)_+$ and $e^{2\pi/d}$ for some $d > 2$. In detail, let $\{u_1, \ldots, u_{r_1+r_2-1}\}$ be a set of generators for $E(k)_+$ and let k' be an extension of k containing $e^{2\pi/d}$ for some $d > 2$ and all of the absolute values of $\{u_1, \ldots, u_{r_1+r_2-1}\}$.

Recall that for r linearly independent vectors $v_1, \ldots, v_r$ we call all $C(v_1, \ldots, v_r) = \{a_1 v_1 + \cdots + a_r v_r \ : \ a_i > 0\}$ the r-dimensional open simplicial cone with generators $v_1, \ldots, v_r$. With k' and k given as above

Theorem 3.1.1. *There exist a finite set of open simplicial cones $\{C_i \ : \ i \in I\}$ with generators in $\mathcal{O}_{k'} \cap (\mathbf{R}_{>0})^{r_1} \times (\mathbf{C}^\times)^{r_2}$ such that for all $x \in (\mathbf{R}_{>0})^{r_1} \times (\mathbf{C}^\times)^{r_2}$ we have*

$$\sum_{i \in I} \sum_{u \in E(k)_+ \times U_d} s_i 1_{C_i}(u^{-1}x) = 1,$$

where $s_i = \pm 1$ and U_d is a finite group of order d^{r_2}.

Proof. This follows directly from Theorem 2.3.2, the group U_d is generated by elements $\{\xi_1, \ldots, \xi_{r_2}\}$ as given in the proof. We note only that the Q-purturbed indicator function $c_Q(v_1, \ldots, v_r)$ can be decomposed into a finite union of indicator functions on open simplicial cones of varying dimension, in this setting it is preferable to consider a decomposition into *open* simplicial cones. $\square$

For each $i \in I$, denote by $r(i)$ the dimension of C_i, fix a system of generators $v_{i,1}, \ldots, v_{i,r(i)} \in \mathcal{O}_{k'}$ for C_i, then for each $x \in C_j$ we set

$$x = \sum_{\alpha=1}^{r(i)} z(x)_\alpha v_{i,\alpha}.$$

For $x \in C_i \cap k$, if $k' = k$ then the coefficients $z(x)_1, \ldots, z(x)_{r(i)}$ are all rational.

Fix a complete set of representatives $\mathfrak{a}_1, \ldots, \mathfrak{a}_{h_k}$ for *narrow* ideal classes of k. For each $c \in H_k(\mathfrak{f})$, there is a unique representative $\mathfrak{a}_i$ such that c and $\mathfrak{a}_i\mathfrak{f}$ are in the same narrow ideal class of k. For each $i \in I$, set

$$R(c, C_i) = \{x \in C_i \cap (\mathfrak{a}_i\mathfrak{f})^{-1} + 1;\ 0 < z(x)_\alpha \leq 1\ (\alpha = 1, \ldots, r(i))\}. \tag{3.1.2}$$

Where $(\mathfrak{a}_i\mathfrak{f})^{-1} + 1$ is the set of all elements μ such that $\mu - 1 \in (\mathfrak{a}_i\mathfrak{f})^{-1}$. The set $R(c, C_i)$ is always finite. Let A_i be the $n \times r(i)$ matrix whose (α, β)-entry is given by $v_{i,\beta}^{(\alpha)}$. Then we get that

$$\begin{aligned}\zeta_k(s, c) =& \sum_{\mathfrak{b} \in c} \frac{1}{N\mathfrak{b}^s} \\ =& N(\mathfrak{a}_\mathfrak{i}\mathfrak{f})^{-s} \sum_{\mu} \frac{1}{N\mu^{-s}}\end{aligned}$$

where μ is taken over all totally positive numbers $\mu \in k$ satisfying $\mu - 1 \in (\mathfrak{a}_i\mathfrak{f})^{-1}$ and are not associated with each other by the action of the group $E(k)_+$. Applying Theorem 3.1.1 we get

$$\zeta_k(s, c) = N(\mathfrak{a}_i\mathfrak{f})^{-s} \sum_{i \in I} s_i \sum_{\mu \in C_i \cap (\mathfrak{a}_i\mathfrak{f})^{-1} + 1} \frac{1}{N\mu^{-s}}.$$

Each $\mu \in C_i \cap (\mathfrak{a}_i\mathfrak{f})^{-1} + 1$ has a unique expression:

$$\mu = \sum_{k=1}^{r(j)} (x_k + z_k) v_{i,k},$$

for an appropriate $x = (x_1, \ldots, x_{r(i)}) \in R(c, C_i)$ and $r(j)$-tuple $z = (z_1, \ldots, z_{r(i)})$ of non-negative integers. Hence,

$$\sum_{\mu \in C_i \cap (\mathfrak{a}_i \mathfrak{f})^{-1}+1} \frac{1}{N\mu^{-s}} = \sum_{x \in R(c,C_i)} \zeta(s, A_i, x),$$

and we get the following formula for $\zeta_k(s, c)$ in terms of Shintani zeta functions

$$\zeta_k(s, c) = N(\mathfrak{a}_i \mathfrak{f})^{-s} \sum_{i \in I} s_i \sum_{x \in R(c,C_i)} \zeta(s, A_i, x). \tag{3.1.3}$$

So the evaluation of $\zeta_k(s, c)$ is reduced to the evaluation of $\zeta(s, A_j, x)$. At non-positive integers, Shintani showed that one can evaluate $\zeta(s, A_j, x)$ in the following way.

Let $L_i^*(t) = \sum_{\alpha=1}^n a_{\alpha,i} t_\alpha$ be the linear form in n variables corresponding to the i^{th} column of A and let $B_m(A, x)^{(\beta)}/(m!)^n$ be the coefficient of $u^{n(m-1)}(t_1 \cdots t_{\beta-1} t_{\beta+1} \cdots t_n)^{m-1}$ in the Laurent expansion at the origin of the following function in t and u:

$$\left[\exp\left(-u \sum_{\alpha=1}^n t_\alpha x_\alpha\right) \prod_{i=1}^r (1 - \exp(-uL_i^*(t)))^{-1}\right]_{t_\beta=1}.$$

Further, setting $B_m(A, x) = \sum_{\beta=1}^m B_m(A, x)^{(\beta)}/n$, Shintani regarded $B_m(A, x)$ as a generalized Bernoulli polynomial. In Proposition 1 of [Shi76a], he showed that

$$\zeta(1 - m, A, x) = (-1)^{n(m-1)} m^{-n} B_m(A, x) \qquad (\text{ for } m = 1, 2, \ldots). \tag{3.1.4}$$

Combining 3.1.3 and 3.1.4, one obtains

$$\zeta_k(1-m, c) = N(\mathfrak{a}_i \mathfrak{f})^{m-1} \sum_{i \in I} s_i \sum_{x \in R(c,C_i)} (-1)^{n(m-1)} m^{-n} B_m(A_i, x) \quad (\text{ for } m = 1, 2, \ldots).$$

In the case k is a totally real field, this implies a classic theorem of Klingen-Siegel [Kli62] on the rationality of zeta values at non-positive integers. When k is not totally real, these values are known to be zero.

Additionally, Shintani derived the following formula for the derivative of $\zeta(s, A, x)$ at $s = 0$ (Proposition 1' [Shi76a])

$$\frac{d}{ds}\zeta(s, A, x)|_{s=0} = (n\gamma - (2n - 1)\pi i)\zeta(0, A, x) + \sum_{k=1}^n \sum_{l=1}^n I_{kl}(A, x), \tag{3.1.5}$$

where γ is the Euler constant and $I_{kl}(A,x)$ is given by the following formula:

$$I_{kk} = \frac{1}{2\pi i}\int_{I_\epsilon(+\infty)} \prod_{j=1}^{r} \frac{\exp(1-x_j)a_{jk}t}{\exp(a_{jk}t)-1}\frac{\log t}{t}dt,$$

if $l \neq k$,

$$I_{kl} = \frac{1}{(2\pi i)^2 n}\int_{I_\epsilon(+\infty)} \frac{dt}{t}\int_{I_\epsilon(1)} \prod_{j=1}^{r} \frac{\exp\{(1-x_j)t(a_{jl}u+a_{jk})\}}{\exp\{t(a_{jl}u+a_{jk})\}-1}\frac{\log u}{u}du.$$

The paths of integration $I_\epsilon(1)$ and $I_\epsilon(+\infty)$ are defined as follows. For a positive number $\epsilon > 0$, denoted by $I_\epsilon(1)$ (resp. $I_\epsilon(+\infty)$) is the integral path in $\mathbf{C}$ consisting of the interval $[1,\epsilon]$ (resp. $[+\infty,\epsilon]$), counterclockwise circle of radius ϵ around the origin and the interval $[\epsilon,1]$ (resp. $[\epsilon,+\infty]$).

This allows the following formula for the derivative of $\zeta_k(s,c)$ at $s=0$

Theorem 3.1.2.

$$\frac{d}{ds}\zeta(s,c)|_{s=0} = \sum_{i\in I} s_i\Big(\sum_{x\in R(c,C_i)}(n\gamma-(2n-1)\pi i-\log N(\mathfrak{a}_i\mathfrak{f}))B_1(A_i,x) + \sum_{k=1}^{n}\sum_{l=1}^{n} I_{kl}(A,x)\Big).$$

3.2 Abelian Stark's Conjecture

Take k, $\mathfrak{f}$ and $H_k(\mathfrak{f})$ be as in the previous section and let K_G be the class field over k corresponding to the subgroup G of $H_k(\mathfrak{f})$.

Let $\chi \in \hat{G}$ be a character of G, we define the L-function associated to the extension K_G/k, χ and $\mathfrak{f}$ as

$$L_{K_G/k,\mathfrak{f}}(s,\chi) = \sum_{c\in G}\chi(c)\zeta_k(s,c).$$

Applying the previous formulas to the above equation gives us the following formula for L-values, for $m=1,2,\ldots$

$$L_{K_G,\mathfrak{f}}(1-m,\chi) = \sum_{c\in G}\chi(c)N(\mathfrak{a}_i\mathfrak{f})^{m-1}\sum_{i\in I}s_i\sum_{x\in R(c,C_i)}(-1)^{n(m-1)}m^{-n}B_m(A_i,x). \quad (3.2.1)$$

Let $S(\mathfrak{f})$ be the set containing all infinite primes of k and all finite primes dividing $\mathfrak{f}$. We state the following classic results [Lan00] [Das99]

$$\zeta_{K_G,\mathfrak{f}}(s) = \prod_{\chi\in\hat{G}} L_{K_G/k,\mathfrak{f}}(s,\chi).$$

$$\mathrm{ord}_{s=0} L_{K_G,\mathfrak{f}}(s,\chi) = \begin{cases} |\{v \in S(\mathfrak{f}) \ : \ \chi(G_v) = 1\} & \text{if } \chi \neq 1 \\ |S(\mathfrak{f})| - 1 & \text{if } \chi = 1 \end{cases}$$

Assume the following

- $S(\mathfrak{f})$ contains all non-archimedean primes which ramify in K_G
- $S(\mathfrak{f})$ contains at least one place which splits completely in K_G.
- $|S(\mathfrak{f})| \geq 3$.

Under these assumptions, the rank one abelian Stark's conjecture says that

Conjecture 3.2.1 (Stark)**.** *With the notation as above, there exists an $S(\mathfrak{f})$-unit $\epsilon \in \{u \in K_G^\times \ : \ |u|_{w'} = 1 \text{ for all } w' \mid v\}$ such that*

$$\log|\epsilon^c|_w = -e_{K_G}\zeta_k'(0,c) \qquad \text{for all } c \in G,$$

or equivalently

$$L'_{K_G/k,\mathfrak{f}}(0,\chi) = -\frac{1}{e_{K_g}}\sum_{c\in G}\chi(c)\log|\epsilon^c|_w \qquad \text{for all } \chi \in \hat{G}.$$

Additionally, it is asserted that $K_G(\epsilon^{1/e_{K_G}})/k$ is abelian.

Note: ϵ^c denotes the conjugate of ϵ associated to c via the reciprocity map.

Combining this with 3.1.2 gives the following formula for $|\epsilon^c|_w$

Theorem 3.2.1 (Stark unit)**.** *With notation as above, under the assumption of Stark's conjecture in the rank one abelian case, the following is a formula for the absolute value of a Stark unit*

$$|\epsilon^c|_w = \exp\Bigg[-e_{K_G}\sum_{i\in I}s_i\Bigg(\sum_{x\in R(c,C_i)}(n\gamma-(2n-1)\pi i-\log N(\mathfrak{a}_i\mathfrak{f}))B_1(A_i,x) + \sum_{k=1}^{n}\sum_{l=1}^{n}I_{kl}(A,x)\Bigg)\Bigg].$$

In the case that k is totally real, this completely determines the Stark unit. This was known to Shintani and has since been used to construct class fields. In the case when k has complex embeddings, it would be of great interest if one could discover a formula for the argument of these units. [Das99]

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