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# Demazure Modules and Fusion Products for the Current Algebra $\mathfrak{sl}_{n+1}[t]$

A Dissertation submitted in partial satisfaction
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ABSTRACT OF THE DISSERTATION

Demazure Modules and Fusion Products for the Current Algebra $\mathfrak{sl}_{n+1}[t]$

by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, September 2022
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In 2021, authors Biswal, Chari, Shereen, and Wand showed for the type $A$ current algebra that under suitable conditions on pairs of dominant integral weights $(\nu, \lambda)$ that the fusion product of a local Weyl module with a level 2 Demazure module $D(2,\lambda) * W_{\text{loc}}(\nu)$ is isomorphic to the module $M(\nu, \lambda)$ first introduced by Wand in his 2015 PhD thesis. The work of this thesis is to remove the condition on $(\nu, \lambda)$ by defining new modules $N(\nu, \lambda, \gamma)$ which in the case when $\gamma = 0$ are isomorphic to $M(\nu, \lambda)$. We then construct short exact sequences that show under suitable conditions on $(\nu, \lambda, \gamma)$ that the modules $N(\nu, \lambda, \gamma)$ are isomorphic to the fusion product $W_{\text{loc}}(\nu) * D(2, \lambda) * D(2, \gamma)$. In particular, when $\gamma = 0$, this gives us the desired isomorphism for $M(\nu, \lambda)$ for all pairs $(\nu, \lambda)$. This allows us to define the fusion product of a local Weyl module and a level 2 Demazure module via generators and relations.

# Contents

# Chapter 1

# Introduction

In 1985 Drinfeld [11] and Jimbo [15] independently introduced the notion of quantized enveloping algebras in order to solve the quantum Yang-Baxter equation. The representation theory of quantum affine algebras has been well-studied since, and has many connections to areas in mathematics and physics, such as statistical mechanics, cluster algebras, dynamical systems, the geometry of quiver varieties, and Macdonald polynomials. One particular area of study is the category $\mathcal{F}_q$ of finite-dimensional representations of quantum affine algebras, which even today not much is known about (change wording). For example, we do not have a formula for the dimensions of these modules in general.

One method of studying these representations is by taking the classical ($q \to 1$) limit, which is a finite-dimensional representation of the affine Lie algebra $\hat{\mathfrak{g}}$, to go from a quantum level to a classical level. By restricting and suitably twisting this limit, one obtains the graded limit which is a graded representation for the corresponding current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$; see [4] and [6] for more information on graded limits. Thus finite-dimensional graded representations of the current algebra present an interesting area of research in their own right. In particular we wish to find presentations and graded characters for these modules.

An interesting family of graded representations of the current algebra are the $\mathfrak{g}$-stable Demazure modules. Given a representation of an affine Lie algebra $\hat{\mathfrak{g}}$, a Demazure module is actually a representation of a Borel subalgebra of $\hat{\mathfrak{g}}$ as introduced in [10]. When it is stable under the action of the underlying simple algebra $\mathfrak{g}$, it naturally becomes a graded representation of the corresponding current algebra. These $\mathfrak{g}$-stable Demazure modules are indexed by pairs $(\ell, \lambda) \in \mathbb{N} \times P^+$, where $\ell$ is called the level of the Demazure module. Demazure modules arise as the $q \to 1$ limit of many families of modules of quantum affine algebras. For example, for $\mathfrak{g}$ simply laced, [5][9][13] showed level 1 Demazure modules are the $q \to 1$ limit of irreducible local Weyl modules for the quantum affine algebra first introduced in [7]. In addition, higher level Demazure modules are the $q \to 1$ limit of Kirillov-Reshetikhin modules as shown in [4] and [6].

It is also known about these limits is that the tensor product of the classical limit of quantum affine algebra modules is not isomorphic to the classical limit of their tensor. Instead, the fusion product as introduced in [12] seems to be the appropriate substitute for the tensor product. It is a very challenging problem to find presentations and graded characters for the fusion product of $\mathfrak{g}$-stable Demazure modules. Much work has been done recently in the type $A$ case. Chari and Loktev in [5] solved the problem for the fusion product of level one Demazure modules. For the fusion product of a level one and a level two Demazure module in the type $A$ case, Wand introduced in [17] the module $M(\nu, \lambda)$

indexed by pairs of dominant integral weights and which interpolate between level one and level two Demazure modules. More recently in 2021 Biswal et. al. showed in [1] that for certain restrictions on $(\nu, \lambda)$, the module $M(\nu, \lambda)$ is isomorphic to the fusion product $D(1, \nu) * D(2, \lambda)$.

The work of this thesis is to improve on this result. We start by introducing the module $N(\nu, \lambda, \gamma)$ indexed by triples of dominant integral weights and in the case when $\gamma = 0$ is isomorphic to the module $M(\nu, \lambda)$. Then by constructing short exact sequences and employing a dimension argument, we are able to show that for certain restrictions on $(\nu, \lambda, \gamma)$ the module $N(\nu, \lambda, \gamma)$ is isomorphic to the fusion product $D(1, \nu) * D(2, \lambda) * D(2, \gamma)$. In particular, by letting $\gamma = 0$ we get that $M(\nu, \lambda) \cong D(1, \nu) * D(2, \lambda)$ for every pair $(\nu, \lambda)$, thus producing in general a presentation for the fusion product of a level one and a level two Demazure module via generators and relations. In addition, when $\nu = 0$, we also get under certain restrictions on $(\lambda, \gamma)$ a presentation for the fusion product $D(2, \lambda) * D(2, \gamma)$ of level two Demazure modules.

# Chapter 2

# Background and Notation

This chapter will introduce the notation to be used in the rest of this dissertation, as well as define the fusion product.

## 2.1 Notation

Let $\mathbb{C}$ denote the complex numbers, $\mathbb{Z}$ the integers, $\mathbb{Z}_+$ the nonnegative integers, and $\mathbb{N}$ the positive integers. For $i, j \in \mathbb{Z}$ with $i \leq j$ we will let the set $[i, j] = \{i, i+1, \ldots, j\}$. Given $V, W$ any two complex vector spaces, we will let $V \otimes W$ denote their tensor product over $\mathbb{C}$.

Let $\mathfrak{g}$ be the simple complex Lie algebra $\mathfrak{sl}_{n+1}$, and let $\mathbf{U}(\mathfrak{g})$ be its universal enveloping algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with basis elements $\{h_i : i \in [1, n]\}$. Let $\{\omega_i : i \in [1, n]\}$ be a set of fundamental weights for $(\mathfrak{g}, \mathfrak{h})$, which are defined by setting $\omega_i(h_j) = \delta_{i,j}$ with $\delta_{i,j}$ the Kronecker delta. For notational purposes we let $\omega_0 = \omega_{n+1} = 0$. Let $\{\alpha_i : i \in [1, n]\}$ be a set of simple roots defined by $\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}$. We let

$R^+ = \{\alpha_{i,j} : 1 \le i \le j \le n\}$ be the set of positive roots, where we note

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j = -\omega_{i-1} + \omega_i + \omega_j - \omega_{j+1}.$$

Fix a Chevalley basis $\{x_\alpha^\pm : \alpha \in R^+\} \cup \{h_i : i \in [1,n]\}$ of $\mathfrak{g}$. We have

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{h} = \bigoplus_{i=1}^n \mathbb{C}h_i, \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathbb{C}x_\alpha^\pm.$$

Let $x_{i,j}^\pm = x_{\alpha_{i,j}}^\pm$, and let $x_\beta^\pm = h_\beta = 0$ if $\beta \notin R$.

Let the root lattice $Q$ and the positive root lattice $Q^+$ be the $\mathbb{Z}$-span and $\mathbb{Z}_+$-span of the simple roots, respectively. Similarly define the weight lattice $P$ and positive weight lattice $P^+$ for the fundamental weights. Define a partial order $\le$ on $P$ by $\mu \le \lambda$ if and only if $\lambda - \mu \in Q^+$. Let $P^+(1) = \{\lambda \in P^+ : \lambda(h_i) \le 1 \text{ for all } i \in [1,n]\}$. Equivalently, $P^+(1) = \{\omega_{i_1} + \ldots + \omega_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$. Then for any $\lambda \in P^+$ we define $\lambda_0$ and $\lambda_1$ to be the unique elements of $P^+$ and $P^+(1)$, respectively, such that $\lambda = 2\lambda_0 + \lambda_1$.

Define the support of $\lambda \in P^+$ to be the set

$$\operatorname{supp} \lambda = \{i \in [1,n] : \lambda - \omega_i \in P^+\}.$$

Lastly, let $(\cdot,\cdot)$ denote the symmetric, non-degenerate form on $\mathfrak{h}^*$ induced by the Killing form and normalized so that the square length of a long root is 2.

## 2.2 The current algebra and graded modules

Let $t$ be an indeterminate and $\mathbb{C}[t]$ the polynomial ring with complex coefficients, and let $\mathfrak{a}$ be a complex Lie algebra. We will let $\mathfrak{a}[t]$ denote the current algebra associated to $\mathfrak{a}$, which as a vector space is isomorphic to $\mathfrak{a} \otimes \mathbb{C}[t]$. As a Lie algebra, the bracket of $\mathfrak{a}[t]$

is defined by

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$$

for $x, y \in \mathfrak{a}$, $r, s \in \mathbb{Z}_+$. Let $\mathfrak{a}[t]_+$ be the ideal $\mathfrak{a} \otimes t\mathbb{C}[t]$. We shall identify the Lie algebra $\mathfrak{a}$ with the subspace $\mathfrak{a} \otimes 1$ of the current algebra. We note that $\mathfrak{a}[t]$ inherits a $\mathbb{Z}_+$-grading from the degree grading of $\mathbb{C}[t]$, and this also induces a grading on $\mathbf{U}(\mathfrak{a}[t])$. In particular, an element of the form $(x_1 \otimes t^{r_1}) \cdots (x_k \otimes t^{r_k})$ has grade $r_1 + \ldots + r_k$, for $x_j \in \mathfrak{a}$ and $r_j \in \mathbb{Z}_+$, $j \in [1, k]$.

Since the current algebra is $\mathbb{Z}_+$-graded, we can define the notion of a $\mathbb{Z}$-graded module for $\mathfrak{a}[t]$. In particular, we say a representation $V$ of $\mathfrak{a}[t]$ is $\mathbb{Z}$-graded if we have

$$V = \bigoplus_{r \in \mathbb{Z}} V[r], \quad (x \otimes t^s)V[r] \subset V[r+s], \quad x \in \mathfrak{a}, r \in \mathbb{Z}, s \in \mathbb{Z}_+.$$

We note that for any $s \in \mathbb{Z}$, $V[s]$ is an $\mathfrak{a}$-module. For any $p \in \mathbb{Z}$ let $\tau_p^* V$ be the graded $\mathfrak{a}[t]$-module which is given by shifting all of the grades of $V$ up by $p$ and leaving the action of $\mathfrak{a}[t]$ unchanged. We shall be interested in studying the category of finite-dimensional $\mathbb{Z}$-graded $\mathfrak{g}[t]$-modules whose objects are these modules and whose morphisms are $\mathfrak{g}[t]$-module maps of grade zero.

## 2.3 Local Weyl and Demazure modules

Let $\mathfrak{a}$ be a simple complex finite-dimensional Lie algebra. Given $\lambda \in P^+$, recall the definition of the local Weyl module $W_{\text{loc}}(\lambda)$ to be the cyclic $\mathfrak{a}[t]$-module generated by $w_\lambda$ subject to the relations

$$(x_i^+ \otimes 1)w_\lambda = 0, \quad (h \otimes t^r)w_\lambda = \delta_{r,0}\lambda(h)w_\lambda, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0, \tag{2.3.1}$$

for all $i \in [1,n]$, $h \in \mathfrak{h}$, $r \geq 0$. Since the relations are graded, it follows that $W_{\rm loc}(\lambda)$ is a graded module for the current algebra by setting the grade of $w_\lambda$ to be 0. In [7] it was shown that $W_{\rm loc}(\lambda)$ is finite dimensional. A quick calculation shows that for $v \in W_{\rm loc}(\lambda)$ satisfying

$$(h \otimes t^k)v = 0 \text{ for all } k \in \mathbb{N},\ h \in \mathfrak{h}, \quad (x_\alpha^- \otimes t^r)v = 0 \text{ for some } \alpha \in R^+,\ r \in \mathbb{Z}_+,$$

then $(x_\alpha^- \otimes t^{r+s})v = 0$ for all $s \in \mathbb{N}$, as we have that

$$(x_\alpha^- \otimes t^{r+s})v = -\frac{1}{2}[h_\alpha \otimes t^s, x_\alpha^- \otimes t^r]v = 0.$$

Given $\ell \in \mathbb{N}$, we define the level $\ell$ Demazure module $D(\ell, \lambda)$ to be the quotient of $W_{\rm loc}(\lambda)$ by the relations

$$(x_\alpha^- \otimes t^{s_\alpha})w_\lambda = 0 \ \text{ for every } \alpha \in R^+,$$

$$(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha + 1} w_\lambda = 0 \ \text{ for every } \alpha \in R^+ \text{ with } m_\alpha < d_\alpha \ell,$$

where $s_\alpha, m_\alpha \in \mathbb{Z}_+$ are defined by setting $\lambda(h_\alpha) = d_\alpha \ell(s_\alpha - 1) + m_\alpha$, $0 < m_\alpha \leq d_\alpha \ell$ and $d_\alpha = \frac{2}{(\alpha,\alpha)}$. In particular, if $\mathfrak{a}$ is simply laced, then $d_\alpha = 1$ for all $\alpha \in R^+$. Since the relations are graded, $D(\ell, \lambda)$ is also a graded $\mathfrak{a}[t]$-module where we declare the grade of $w_\lambda$ to be 0. Demazure modules arise from the representation theory of affine Lie algebras, and this particular definition is a result of the work in [9][13][16].

In [9], it was proved that for $\mathfrak{a}$ simply laced and $\lambda \in P^+$, the local Weyl module $W_{\rm loc}(\lambda)$ is isomorphic to the level 1 Demazure module $D(1, \lambda)$. In addition, simpler relations were given for level 1 and level 2 Demazure modules in Proposition 3.4 and Theorem 2 of that paper, which we reformulate here.

**Theorem 2.3.1.** *Let $\lambda \in P^+$, $w_\lambda$ the generator of $W_{\rm loc}(\lambda)$, and suppose $\mathfrak{a}$ is simply laced.*

1. *For all $\alpha \in R^+$,*

$$(x_\alpha^- \otimes t^{\lambda(h_\alpha)})w_\lambda = 0. \tag{2.3.2}$$

   *Equivalently, $D(1,\lambda)$ is the quotient of $W_{\rm loc}(\lambda)$ by the above relations.*

2. *$D(2,\lambda)$ is the quotient of $W_{\rm loc}(\lambda)$ by the relations*

$$(x_\alpha^- \otimes t^{\lceil \lambda(h_\alpha)/2 \rceil})w_\lambda = 0 \text{ for every } \alpha \in R^+. \tag{2.3.3}$$

## 2.4 Garland's Formula and a Useful Lemma

We start by defining the following set of sequences. For $s, r \in \mathbb{Z}_+$, let

$$\mathbf{S}(r,s) = \left\{ (b_p)_{p\geq 0} : b_p \in \mathbb{Z}_+, \ \sum_{p\geq 0} b_p = r, \ \sum_{p \geq 0} p b_p = s \right\}.$$

Notice that $\mathbf{S}(0,s)$ is the empty set if $s > 0$, and also that if $(b_p)_{p\geq 0} \in \mathbf{S}(r,s)$, then $b_p = 0$ for $p > s$. Thus $\mathbf{S}(r,s)$ is a finite set.

Next, given $x \in \mathfrak{g}$ and $s, r \in \mathbb{Z}_+$, define elements $x(r,s) \in \mathbf{U}(\mathfrak{g}[t])$ by

$$x(r,s) = \sum_{(b_p)_{p\geq 0} \in \mathbf{S}(r,s)} (x \otimes 1)^{(b_0)} (x \otimes t)^{(b_1)} \cdots (x \otimes t^s)^{(b_s)}$$

where for any integer $k$ and any $x \in \mathfrak{g}[t]$, we set $x^{(p)} = x^p/p!$.

The following result was first proved in [14], and this current formulation can be found in [9] Lemma 2.3.

**Proposition 2.4.1** (Garland's Formula)**.** *Given $s \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $\alpha \in R^+$, we have that*

$$(x_\alpha^+ \otimes t)^{(s)} (x_\alpha^- \otimes 1)^{(s+r)} - (-1)^s x_\alpha^-(r,s) \in \mathbf{U}(\mathfrak{g}[t])\mathfrak{n}^+[t] \bigoplus \mathbf{U}(\mathfrak{n}^-[t] \oplus \mathfrak{h}[t]_+)\mathfrak{h}[t]_+.$$

An immediate consequence of this proposition is that

$$\Big((x_\alpha^+\otimes t)^{(s)}(x_\alpha^-\otimes 1)^{(s+r)} - (-1)^s x_\alpha^-(r,s)\Big)\, w_\lambda = 0$$

where $\lambda \in P^+$ and $w_\lambda$ is the generator of $W_{\rm loc}(\lambda)$.

We now prove the following corollary of Garland's formula that will be used in later chapters.

**Lemma 2.4.1.** *Let $\lambda \in P^+$ and $\alpha \in R^+$, and let $w_\lambda$ be the generator of $W_{\rm loc}(\lambda)$.*

*(i) If $(x_\alpha^-\otimes 1)^{2r+1}w_\lambda = (x_\alpha^-\otimes t^{r+1})w_\lambda = 0$ for some $r \in \mathbb{Z}_+$, then*

$$(x_\alpha^-\otimes t^{r-1})(x_\alpha^-\otimes t^r)w_\lambda = 0.$$

*(ii) If $(x_\alpha^-\otimes 1)^{2r+2}w_\lambda = (x_\alpha^-\otimes t^{r+1})w_\lambda = 0$ for some $r \in \mathbb{Z}_+$, then*

$$(x_\alpha^-\otimes t^r)^2 w_\lambda = 0.$$

*Proof.* We start by proving (i). By Garland's Formula, we have that

$$(x_\alpha^+\otimes t)^{(2r-1)}(x_\alpha^-\otimes 1)^{(2r+1)}w_\lambda - \sum_{k=0}^{r-1}(x_\alpha^-\otimes t^k)(x_\alpha^-\otimes t^{2r-1-k})w_\lambda = 0.$$

But our assumptions imply that

$$(x_\alpha^+\otimes t)^{(2r-1)}(x_\alpha^-\otimes 1)^{(2r+1)}w_\lambda = 0, \quad (x_\alpha^-\otimes t^k)(x_\alpha^-\otimes t^{2r-1-k})w_\lambda = 0 \text{ for } k \le r-2.$$

Hence the remaining term $(x_\alpha^-\otimes t^{r-1})(x_\alpha^-\otimes t^r)w_\lambda$ is 0 as desired.

Part (ii) is proved similarly. Again by Garland's Formula, we have that

$$(x_\alpha^+\otimes t)^{(2r)}(x_\alpha^-\otimes 1)^{(2r+2)}w_\lambda - \left(\sum_{k=0}^{r-1}(x_\alpha^-\otimes t^k)(x_\alpha^-\otimes t^{2r-k})w_\lambda\right) - \frac{1}{2}(x_\alpha^-\otimes t^r)^2 w_\lambda = 0.$$

But our assumptions imply that

$$(x_\alpha^+ \otimes t)^{(2r)}(x_\alpha^- \otimes 1)^{(2r+2)}w_\lambda = 0, \quad (x_\alpha^- \otimes t^k)(x_\alpha^- \otimes t^{2r-k})w_\lambda = 0 \text{ for } k \leq r-1.$$

Therefore, $(x_\alpha^- \otimes t^r)^2 w_\lambda = 0$, completing the proof. □

## 2.5 The Fusion Product

Given a finite-dimensional cyclic $\mathfrak{g}[t]$-module $V$ with generator $v$, we define for every $r \in \mathbb{Z}_+$ the set

$$F^r V = \left( \bigoplus_{0 \leq s \leq r} \mathbf{U}(\mathfrak{g}[t])[s] \right) v.$$

Then $F^r V$ is a $\mathfrak{g}$-submodule of $V$ and defines a $\mathfrak{g}$-module filtration

$$0 \subset F^0 V \subset F^1 V \subset \cdots \subset F^p V = V,$$

for some $p \in \mathbb{Z}_+$. The associated graded vector space

$$\operatorname{gr} V := \bigoplus_{i=0}^{p} F^i V / F^{i-1} V, \quad F^{-1} V = 0,$$

acquires a natural structure as a cyclic $\mathfrak{g}[t]$-module with the action

$$(x \otimes t^m)\overline{w} = \overline{(x \otimes t^m)w}, \quad x \in \mathfrak{g},\ m \in \mathbb{Z}_+,\ w \in V,\ \overline{w} \in F^i V / F^{i-1} V.$$

In addition, $\operatorname{gr} V \cong V$ as $\mathfrak{g}$-modules and is generated by $\overline{v}$.

We now define the fusion product first introduced in [12]. First, given $\mathfrak{g}[t]$-module $V$ and $z \in \mathbb{C}$, let $V^z$ be the $\mathfrak{g}[t]$-module with underlying vector space $V$ and with twisted action given by

$$(x \otimes t^r)w = (x \otimes (t+z)^r)w, \quad x \in \mathfrak{g},\ r \in \mathbb{Z}_+,\ w \in V.$$

Next, let $V_1, \dots, V_s$ be finite-dimensional cyclic $\mathfrak{g}[t]$-modules generated by $v_1, \dots, v_s$, respectively, and let $z_1, \dots, z_s$ be distinct complex numbers. In [12] it was shown that the tensor product $V_1^{z_1} \otimes \cdots \otimes V_s^{z_s}$ is cyclic and generated by $v_1 \otimes \cdots \otimes v_s$. Then define the fusion product

$$V_1^{z_1} * \cdots * V_s^{z_s} := \operatorname{gr}\left(V_1^{z_1} \otimes \cdots \otimes V_s^{z_s}\right).$$

We denote the image of $v_1 \otimes \cdots \otimes v_s$ in the fusion product by $v_1 * \cdots * v_s$. It is conjectured that under suitable conditions the fusion product is independent of the choice of the complex numbers. For ease of notation we shall suppress the dependence on the complex numbers and write $V_1 * \cdots * V_s$ for $V_1^{z_1} * \cdots * V_s^{z_s}$. In addition, we note that

$$\dim V_1 * \cdots * V_s = (\dim V_1) \cdots (\dim V_s)$$

which lets us use dimension arguments when using fusion products. However, we do not know the defining relations for fusion products in general.

In [8] a certain factorization of Demazure modules for the type $A$ case was shown, which we state here.

**Theorem 2.5.1.** *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Let $\lambda \in P^+$, and let $\lambda = \ell(\sum_{j=1}^{k} \lambda^j) + \lambda^0$ for $\lambda^j \in P^+$ for $0 \le j \le k$ and $k, \ell \in \mathbb{N}$. Then there exists an isomorphism of $\mathfrak{g}[t]$-modules*

$$D(\ell, \lambda) \cong D(\ell, \lambda^0) * D(\ell, \ell\lambda^1) * \cdots * D(\ell, \ell\lambda^k).$$

# Chapter 3

# Compatible Triples and the Main Theorem

In this chapter we define the modules of study, discuss their connections with the fusion product and Demazure modules, and state the main results of this thesis.

## 3.1 Modules $M(\nu, \lambda)$ and $N(\nu, \lambda, \gamma)$

First introduced in [17], the modules $M(\nu, \lambda)$ are parametrized by pairs of dominant integral weights $(\nu, \lambda)$ and interpolate between level 1 and level 2 Demazure modules. Formally, given $\lambda, \nu \in P^+$, define the module $M(\nu, \lambda)$ as the quotient of $W_{\text{loc}}(\nu + \lambda)$ by the submodule generated by

$$\{(x_\alpha^- \otimes t^{\nu(h_\alpha)+\lceil \lambda(h_\alpha)/2 \rceil})w_{\nu+\lambda} : \alpha \in R^+\}.$$

Let $w_{\nu,\lambda}$ denote the image of $w_{\nu+\lambda}$ under this quotient. Since these relations are graded, $M(\nu,\lambda)$ is a graded representation of the current algebra by setting the grade of $w_{\nu,\lambda}$ to be 0. By Theorem 2.3.1, we see that

$$M(\nu,0) \cong W_{\mathrm{loc}}(\nu), \quad M(0,\lambda) \cong D(2,\lambda).$$

For $\mu \in P^+ \setminus \{0\}$ let

$$\max \mu = \max\{\, i : \mu(h_i) > 0 \,\}, \quad \min \mu = \min\{\, i : \mu(h_i) > 0 \,\}$$

and for $\mu = 0$ set $\max \mu = 0$, $\min \mu = n+1$. Then we say a pair $(\nu,\lambda)$ of dominant integral weights is *compatible* if either

- $\lambda_1 = 0$, or
- $\lambda_1 \neq 0$, in which case we must have $\nu_0 = \omega_i$ for some $i \in [0,n]$ and $\max \nu_1 < \min \lambda_1$. If in addition $i \geq 1$ (equivalently $\nu_0 \neq 0$) then we require that $i < \min \lambda_1 - 1$ and $\nu_1(h_i) = \nu_1(h_{i+1}) = 0$.

In [1] the following connection between the modules $M(\nu,\lambda)$ and fusion products of Demazure modules was shown.

**Theorem 3.1.1.** *Let $(\nu,\lambda) \in P^+ \times P^+$ be a compatible pair. Then there exists an isomorphism of $\mathfrak{g}[t]$-modules*

$$M(\nu,\lambda) \cong W_{\mathrm{loc}}(\nu) * D(2,\lambda).$$

The goal of this dissertation is to expand this isomorphism to all pairs of dominant integral weights. To do so, we introduce the following generalization of the modules above.

Given $\nu, \lambda, \gamma \in P^+$, define module $N(\nu, \lambda, \gamma)$ to be the quotient of $W_{\text{loc}}(\nu + \lambda + \gamma)$ by the relations

$$(x_\alpha^- \otimes t^{\nu(h_\alpha)+\lceil \lambda(h_\alpha)/2 \rceil + \lceil \gamma(h_\alpha)/2 \rceil}) w_{\nu+\lambda+\gamma} = 0, \quad \alpha \in R^+. \tag{3.1.1}$$

Let $w_{\nu,\lambda,\gamma}$ denote the image of $w_{\nu+\lambda+\gamma}$ under this quotient. Note that since these relations are graded, $N(\nu, \lambda, \gamma)$ becomes a graded representation of the current algebra by setting the grade of $w_{\nu,\lambda,\gamma}$ to be 0. An inspection of the defining relations and Theorem 2.3.1 imply

$$N(\nu, \lambda, 0) \cong N(\nu, 0, \lambda) \cong M(\nu, \lambda),$$

$$N(\nu, \lambda, \gamma) \cong N(\nu, \gamma, \lambda),$$

$$N(\nu, 0, 0) \cong W_{\text{loc}}(\nu), \tag{3.1.2}$$

$$N(0, \lambda, 0) \cong N(0, 0, \lambda) \cong D(2, \lambda). \tag{3.1.3}$$

In addition, if $\gamma_0 - \mu \in P^+$ for some $\mu \in P^+$, then

$$N(\nu, \lambda, \gamma) \cong N(\nu, \lambda + 2\mu, \gamma - 2\mu). \tag{3.1.4}$$

The following proposition collects some properties of these modules.

**Proposition 3.1.1.** *Let $\nu, \lambda \in P^+$. Then the following hold:*

*(i)* $N(\omega_i, 0, 0) \cong_{\mathfrak{g}[t]} N(0, \omega_i, 0) \cong_{\mathfrak{g}[t]} N(0, 0, \omega_i) \cong_{\mathfrak{g}} V(\omega_i)$ *for every* $i \in [1, n]$.

*(ii)* $N(0, \omega_i + \omega_j, 0) \cong_{\mathfrak{g}[t]} N(0, 0, \omega_i + \omega_j) \cong_{\mathfrak{g}} V(\omega_i + \omega_j)$ *for every* $i, j \in [1, n]$

*(iii)* $\dim N(\lambda + \nu, 0, 0) = \dim N(\nu, 0, 0) \dim N(\lambda, 0, 0)$.

*(iv) For all* $\mu \in P^+$ *such that* $\lambda_0 - \mu \in P^+$,

$$\dim N(0, \lambda, 0) = \dim N(0, \lambda - 2\mu, 0) \dim N(0, 2\mu, 0).$$

Parts (i) and (ii) follow from inspecting the defining relations of $N(\nu,\lambda,\gamma)$. Part (iii) was proved in [5], while part (iv) follows from Theorem 2.5.1.

We finish this section by proving the following result about $N(\nu,\lambda,\gamma)$.

**Theorem 3.1.2.** *There exists a surjection of* $\mathfrak{g}[t]$*-modules*

$$N(\nu,\lambda,\gamma) \to W_{\text{loc}}(\nu) * D(2,\lambda) * D(2,\gamma) \to 0.$$

*Proof.* Let $w$, $w_\nu$, $w_\lambda$, and $w_\gamma$ be the generators of $N(\nu,\lambda,\gamma)$, $W_{\text{loc}}(\nu)$, $D(2,\lambda)$, and $D(2,\gamma)$, respectively, and let $z_1$, $z_2$, and $z_3$ be the parameters of their fusion product. Define map $\phi\colon N(\nu,\lambda,\gamma) \to W_{\text{loc}}(\nu) * D(2,\lambda) * D(2,\gamma)$ to be the map sending $w$ to $w_\nu * w_\lambda * w_\gamma$. It remains to show that $\phi$ is well-defined. For this, it is enough to show that relations (2.3.1) and (3.1.1) in $N(\nu,\lambda,\gamma)$ hold for $w_\nu * w_\lambda * w_\gamma$. For the proof, we work with element $w_\nu \otimes w_\lambda \otimes w_\gamma$ in the tensor product, since for any $u \in \mathbf{U}(\mathfrak{sl}_{n+1}[t])$, by definition

$$u(w_\nu * w_\lambda * w_\gamma) = u\overline{(w_\nu \otimes w_\lambda \otimes w_\gamma)} = \overline{u(w_\nu \otimes w_\lambda \otimes w_\gamma)}.$$

First, note that for $i \in [1,n]$, we have that $(x_i^+ \otimes 1)(w_\nu \otimes w_\lambda \otimes w_\gamma)$ is equal to

$$((x_i^+ \otimes 1)w_\nu) \otimes w_\lambda \otimes w_\gamma + w_\nu \otimes ((x_i^+ \otimes 1)w_\lambda) \otimes w_\gamma + w_\nu \otimes w_\lambda \otimes ((x_i^+ \otimes 1)w_\gamma) = 0.$$

Thus $(x_i^+ \otimes 1)(w_\nu * w_\lambda * w_\gamma) = 0$.

Next for $h \in \mathfrak{h}$, $r \geq 0$, we have that $(h \otimes t^r)(w_\nu \otimes w_\lambda \otimes w_\gamma)$ is equal to

$$((h \otimes t^r)w_\nu) \otimes w_\lambda \otimes w_\gamma + w_\nu \otimes ((h \otimes t^r)w_\lambda) \otimes w_\gamma + w_\nu \otimes w_\lambda \otimes ((h \otimes t^r)w_\gamma).$$

Note that for $\mu \in \{\nu,\lambda,\gamma\}$, from the twisted action we have that

$$(h \otimes t^r)w_\mu = (h \otimes (t+z_i)^r)w_\mu = z_i^r \mu(h) w_\mu$$

where $(i,\mu) \in \{(1,\nu),(2,\lambda),(3,\gamma)\}$. Thus

$$(h \otimes t^r)(w_\nu \otimes w_\lambda \otimes w_\gamma) = (z_1^r \nu(h) + z_2^r \lambda(h) + z_3^r \gamma(h)) w_\nu \otimes w_\lambda \otimes w_\gamma$$

This implies that if $r = 0$,

$$(h \otimes t^r)(w_\nu * w_\lambda * w_\gamma) = (\nu + \lambda + \gamma)(h)(w_\nu * w_\lambda * w_\gamma)$$

and if $r > 0$, then $(h \otimes t^r) w_\nu * w_\lambda * w_\gamma$ has both grade 0 and grade $r$ and therefore must equal 0. Thus

$$(h \otimes t^r)(w_\nu * w_\lambda * w_\gamma) = \delta_{r,0}(\nu + \lambda + \gamma)(h)(w_\nu * w_\lambda * w_\gamma).$$

Thirdly, for $i \in [1,n]$, we have that $(x_i^- \otimes 1)^{(\nu+\lambda+\gamma)(h_i)}(w_\nu \otimes w_\lambda \otimes w_\gamma)$ is equal to

$$((x_i^- \otimes 1)^{(\nu+\lambda+\gamma)(h_i)} w_\nu) \otimes w_\lambda \otimes w_\gamma + w_\nu \otimes ((x_i^- \otimes 1)^{(\nu+\lambda+\gamma)(h_i)} w_\lambda) \otimes w_\gamma$$
$$+ w_\nu \otimes w_\lambda \otimes ((x_i^- \otimes 1)^{(\nu+\lambda+\gamma)(h_i)} w_\gamma).$$

This is 0 since $(x_i^- \otimes 1)^{(\nu+\lambda+\gamma)(h_i)} w_\mu = 0$ for $(i,\mu) \in \{(1,\nu),(2,\lambda),(3,\gamma)\}$. Thus

$$(x_i^- \otimes 1)^{(\nu+\lambda+\gamma)(h_i)}(w_\nu * w_\lambda * w_\gamma) = 0,$$

proving relation (2.3.1).

To show (3.1.1), we first note that

$$(x_\alpha^- \otimes (t - z_1)^{\nu(h_\alpha)} (t - z_2)^{\lceil \lambda(h_\alpha)/2 \rceil} (t - z_3)^{\lceil \gamma(h_\alpha)/2 \rceil})(w_\nu \otimes w_\lambda \otimes w_\gamma)$$
$$= (x_\alpha^- \otimes t^{\nu(h_\alpha) + \lceil \lambda(h_\alpha)/2 \rceil + \lceil \gamma(h_\alpha)/2 \rceil})(w_\nu \otimes w_\lambda \otimes w_\gamma) + (x_\alpha^- \otimes f(t))(w_\nu \otimes w_\lambda \otimes w_\gamma)$$

for some $f \in \mathbb{C}[t]$ with $\deg f(t) < \nu(h_\alpha) + \lceil \lambda(h_\alpha)/2 \rceil + \lceil \gamma(h_\alpha)/2 \rceil$. Since

$$(x_\alpha^- \otimes f(t))(w_\nu * w_\lambda * w_\gamma) = 0$$

in the $(\nu(h_\alpha) + \lceil\lambda(h_\alpha)/2\rceil + \lceil\gamma(h_\alpha)/2\rceil)$-graded piece of $W_{\mathrm{loc}}(\nu) * D(2,\lambda) * D(2,\gamma)$, this implies that

$$(x_\alpha^- \otimes t^{\nu(h_\alpha)+\lceil\lambda(h_\alpha)/2\rceil+\lceil\gamma(h_\alpha)/2\rceil})(w_\nu * w_\lambda * w_\gamma)$$
$$= (x_\alpha^- \otimes (t-z_1)^{\nu(h_\alpha)}(t-z_2)^{\lceil\lambda(h_\alpha)/2\rceil}(t-z_3)^{\lceil\gamma(h_\alpha)/2\rceil})(w_\nu * w_\lambda * w_\gamma). \quad (3.1.5)$$

However, we calculate that

$$(x_\alpha^- \otimes (t-z_1)^{\nu(h_\alpha)}(t-z_2)^{\lceil\lambda(h_\alpha)/2\rceil}(t-z_3)^{\lceil\gamma(h_\alpha)/2\rceil})(w_\nu \otimes w_\lambda \otimes w_\gamma)$$
$$= ((x_\alpha^- \otimes t^{\nu(h_\alpha)}(t+z_1-z_2)^{\lceil\lambda(h_\alpha)/2\rceil}(t+z_1-z_3)^{\lceil\gamma(h_\alpha)/2\rceil})w_\nu) \otimes w_\lambda \otimes w_\gamma$$
$$+ w_\nu \otimes ((x_\alpha^- \otimes (t+z_2-z_1)^{\nu(h_\alpha)}t^{\lceil\lambda(h_\alpha)/2\rceil}(t+z_2-z_3)^{\lceil\gamma(h_\alpha)/2\rceil})w_\lambda) \otimes w_\gamma$$
$$+ w_\nu \otimes w_\lambda \otimes ((x_\alpha^- \otimes (t+z_3-z_1)^{\nu(h_\alpha)}(t+z_3-z_2)^{\lceil\lambda(h_\alpha)/2\rceil}t^{\lceil\gamma(h_\alpha)/2\rceil})w_\gamma)$$

which is 0 by (2.3.3) and (2.3.2). Thus (3.1.5) is equal to 0 and relation (3.1.1) holds, whence map $\phi$ is well-defined. □

We immediately get the following corollary.

**Corollary 3.1.1.** *For* $(\nu,\lambda,\gamma) \in P^+ \times P^+ \times P^+$,

$$\dim N(\nu,\lambda,\gamma) \geq \dim N(\nu,0,0) \dim N(0,\lambda,0) \dim N(0,0,\gamma).$$

*Proof.* This follows from Theorem 3.1.2 and isomorphisms (3.1.2) and (3.1.3), as well as the fact that

$$\dim W_{\mathrm{loc}}(\nu) * D(2,\lambda) * D(2,\gamma) = \dim W_{\mathrm{loc}}(\nu) \dim D(2,\lambda) \dim D(2,\gamma).$$

□

## 3.2 Compatible triples and the Main Theorem

Given $\nu, \lambda, \gamma \in P^+$, we say the triple $(\nu, \lambda, \gamma)$ is *compatible* if $\max \lambda_1 < \min \gamma_1$. Define a partial order on compatible triples by letting $(\tilde{\nu}, \tilde{\lambda}, \tilde{\gamma}) < (\nu, \lambda, \gamma)$ if either:

- $\nu + \lambda + \gamma - \tilde{\nu} - \tilde{\lambda} - \tilde{\gamma} \in Q^+ \setminus \{0\}$,
- $\gamma = \gamma' = 0$, $\tilde{\nu} + \tilde{\lambda} = \nu + \lambda$, and $\nu - \tilde{\nu} \in P^+ \setminus \{0\}$, or
- $\nu = \tilde{\nu}$, $\gamma \neq 0$, $\tilde{\gamma} = 0$, and $\tilde{\lambda} = \lambda + \gamma$,

and imposing transitivity. We claim that the triples $\{(0, \omega_i, 0) : i \in [0, n]\}$ are minimal with respect to this order. To see this, take a compatible triple $(\nu, \lambda, \gamma)$. If $\gamma \neq 0$ then $(\nu, \lambda + \gamma, 0)$ is compatible and less than $(\nu, \lambda, \gamma)$. If $\gamma = 0$ and $\nu(h_j) \geq 1$ for some $j \in [1, n]$ then $(\nu - \omega_j, \lambda + \omega_j, 0)$ is compatible and less than $(\nu, \lambda, 0)$. Lastly, if $\nu = \gamma = 0$ with $\lambda \neq 0$, pick $i \in [0, n]$ such that $\lambda - \omega_i \in Q^+ \setminus \{0\}$. Then $(0, \omega_i, 0) < (0, \lambda, 0)$ and hence the claim is proved.

The following is the main result of this thesis.

**Theorem 3.2.1.** *Let $(\nu, \lambda, \gamma) \in P^+ \times P^+ \times P^+$ be a compatible triple. Then there exists an isomorphism of $\mathfrak{g}[t]$-modules*

$$N(\nu, \lambda, \gamma) \cong W_{\text{loc}}(\nu) * D(2, \lambda) * D(2, \gamma).$$

An immediate corollary of this theorem is the following.

**Corollary 3.2.1.** *For every $\nu, \lambda \in P^+$, there exists isomorphism of $\mathfrak{g}[t]$-modules*

$$M(\nu, \lambda) \cong W_{\text{loc}}(\nu) * D(2, \lambda).$$

This follows by noting that $(\nu, \lambda, 0)$ is a compatible triple and $N(\nu, \lambda, 0) \cong M(\nu, \lambda)$.

## 3.3 Short Exact Sequences

The proof of Theorem 3.2.1 requires the establishment of certain short exact sequences. We next define some notation to be used in the statement of these short exact sequences.

Given $\lambda \in P^+$ and $1 \leq p \leq q \leq n$, define $\lambda(p,q) \in P^+$ by $\lambda(p,q)(h_j) = \lambda(h_j)$ if $p \leq j \leq q$ and 0 otherwise. In addition, if $\lambda \in P^+$ with $\lambda_1 \neq 0$, define $'\lambda = \lambda - \omega_p + \omega_{p+1}$ where $p = \min \lambda_1$, and define $\lambda' = \lambda - \omega_q + \omega_{q-1}$ where $q = \max \lambda_1$.

Next, given $m \in [1,n]$ and $\lambda \in P^+$, let $m_\diamond$ be minimal such that $m_\diamond > m$ with $m_\diamond \in \operatorname{supp} \lambda_1$, where we let $m_\diamond = n+1$ if $m > \max \lambda_1$. Similarly define $m_\bullet$ to be maximal such that $m_\bullet < m$ with $m_\bullet \in \operatorname{supp} \lambda_1$, where we let $m_\bullet = 0$ if $m < \min \lambda_1$.

Given $\nu, \lambda \in P^+$ and $m \in [1,n]$ with $\nu - \omega_m \in P^+$, set

$$U_{m,\epsilon}(\nu,\lambda) = N(\nu - \omega_m, \quad 2\lambda_0 + \lambda_1(1, m - \delta_{\epsilon,-1})', \quad '\lambda_1(m + \delta_{\epsilon,1}, n)), \quad \epsilon \in \{-1,0,1\},$$

and let $K_m(\nu,\lambda)$ be equal to

$$\tau^*_{(\nu+\lambda_0)(h_m)} U_{m,0}(\nu,\lambda) \bigoplus (1 - \delta_{m,\min \lambda_1})(1 - \delta_{m,\max \lambda_1}) \tau^*_{(\nu+\lambda_0)(h_{m_\bullet,m_\diamond})+1} U_{m,0}(\nu, \lambda + \omega_m)$$

if $m \in \operatorname{supp} \lambda_1$ and otherwise equal to

$$[\![m > \min \lambda_1]\!] \tau^*_{(\nu+\lambda_0)(h_{m_\bullet,m})} U_{m,-1}(\nu, \lambda + \omega_m) \bigoplus [\![m < \max \lambda_1]\!] \tau^*_{(\nu+\lambda_0)(h_{m,m_\diamond})} U_{m,1}(\nu, \lambda + \omega_m)$$

where $[\![\,\cdot\,]\!]$ is the Iverson bracket defined by $[\![P]\!] = 1$ if $P$ is true and 0 if $P$ is false.

The remainder of this thesis will focus on proving the following theorem, which in the process will prove Theorem 3.2.1.

**Theorem 3.3.1.** *Let $(\nu, \lambda, \gamma)$ be a compatible triple of dominant integral weights.*

*(a) Suppose $\gamma \neq 0$. Then there exists a short exact sequence of $\mathfrak{g}[t]$-modules*

$$0 \to (1-\delta_{\lambda_1,0})(1-\delta_{\gamma_1,0})\tau^*_{(\nu+\lambda_0+\gamma_0)(h_{p,q})+1}N(\nu,\ \lambda',\ {}'\gamma) \xrightarrow{\psi^-} N(\nu,\lambda,\gamma)$$
$$\xrightarrow{\psi^+} N(\nu,\lambda+\gamma,0) \to 0$$

*where $p = \max\lambda_1$, $q = \min\gamma_1$.*

*(b) Suppose that $\gamma = 0$ and $\nu - \omega_m \in P^+$ for some $m \in [1,n]$. There exists a short exact sequence of $\mathfrak{g}[t]$-modules*

$$0 \to K_m(\nu,\lambda) \xrightarrow{\varphi^-} N(\nu,\lambda,0) \xrightarrow{\varphi^+} N(\nu-\omega_m,\lambda+\omega_m,0) \to 0.$$

# Chapter 4

# The Map $\varphi^+$ and Its Kernel

We start by defining some notation to be used in the rest of this chapter. Given $\lambda \in P^+$ and $m \in [1, n]$ define $\beta_{\lambda,m},\ \eta_{\lambda,m} \in R^+ \cup \{0\}$ as follows:

$$\beta_{\lambda,m} = \begin{cases} \alpha_m & \text{if } m \in \operatorname{supp} \lambda_1 \\ \alpha_{m,m_\diamond} & \text{if } m \notin \operatorname{supp} \lambda_1,\ m < \max \lambda_1 \\ 0 & \text{otherwise} \end{cases},$$

$$\eta_{\lambda,m} = \begin{cases} \alpha_{m_\bullet,m_\diamond} & \text{if } m \in \operatorname{supp} \lambda_1,\ \min \lambda_1 < m < \max \lambda_1 \\ \alpha_{m_\bullet,m} & \text{if } m \notin \operatorname{supp} \lambda_1,\ m > \min \lambda_1 \\ 0 & \text{otherwise} \end{cases}.$$

Given $\nu \in P^+$ with $m \in \operatorname{supp} \nu$, for each $\alpha \in R^+$ define integers

$$a_\alpha(\nu, \lambda) = \nu(h_\alpha) + \lceil \lambda(h_\alpha)/2 \rceil,$$

$$b_\alpha(\nu, \lambda, m) = (\nu - \omega_m)(h_\alpha) + \lceil (\lambda + \omega_m)(h_\alpha)/2 \rceil.$$

## 4.1 Statement of the Proposition

The following proposition will be the subject of the remainder of this chapter.

**Proposition 4.1.1.** *Let* $\nu, \lambda \in P^+$ *with* $m \in \operatorname{supp} \nu$ *for some* $m \in [1, n]$. *Let* $w$ *denote the generator of* $N(\nu, \lambda, 0)$, *and set*

$$\beta := \beta_{\lambda,m}, \quad \eta := \eta_{\lambda,m}, \quad b_\beta := b_\beta(\nu, \lambda, m), \quad b_\eta := b_\eta(\nu, \lambda, m).$$

*Then there exists a surjective map of* $\mathfrak{sl}_{n+1}[t]$*-modules*

$$\varphi^+ : N(\nu, \lambda, 0) \to N(\nu - \omega_m, \lambda + \omega_m, 0)$$

*sending generator to generator whose kernel is generated by*

$$(x_\beta^- \otimes t^{b_\beta})w \text{ and } (x_\eta^- \otimes t^{b_\eta})w.$$

*In addition, the generators of the kernel are highest weight vectors.*

In the rest of this chapter, for each $\alpha \in R^+$ we will let

$$a_\alpha := a_\alpha(\nu, \lambda), \quad b_\alpha := b_\alpha(\nu, \lambda, m).$$

Then by (3.1.1), $N(\nu, \lambda, 0)$ and $N(\nu - \omega_m, \lambda + \omega_m, 0)$ are both quotients of $W_{\text{loc}}(\nu + \lambda)$, with the additional relations

$$(x_\alpha^- \otimes t^{a_\alpha})w_{\nu,\lambda,0} = 0 \quad \text{and} \quad (x_\alpha^- \otimes t^{b_\alpha})w_{\nu-\omega_m,\lambda+\omega_m,0} = 0,$$

respectively, for every $\alpha \in R^+$. Thus, existence of map $\varphi^+$ follows from the fact that $a_\alpha \geq b_\alpha$ for all $\alpha$. In particular, the kernel is generated by

$$(x_\alpha^- \otimes t^{b_\alpha})w \text{ for all } \alpha \in R^+ \text{ such that } a_\alpha > b_\alpha.$$

In addition, if $\lambda_1 = 0$, then $a_\alpha = b_\alpha$ for all $\alpha$ and $\varphi^+$ is an isomorphism as desired, since $\beta = \eta = 0$ in this case. In the following sections, we prove in the remaining cases that the generators of the kernel of $\varphi^+$ can be reduced to those listed in the proposition, and that these generators are highest weight.

## 4.2 Highest weight vectors

Before we continue with proving the proposition, we start by proving the following useful lemma.

**Lemma 4.2.1.** *Let $\nu \in P^+$ and $\alpha \in R^+$ such that $\nu - \alpha \in P^+$. Let $s \in \mathbb{N}$ such that $(x_\alpha^- \otimes t^{s+1})w_\nu = 0$. Additionally, suppose that $(x_{\alpha-\alpha_j}^- \otimes t^s)w_\nu = 0$ for any $j \in [1, n]$ such that $\alpha - \alpha_j \in R^+$. Then $(x_\alpha^- \otimes t^s)w_\nu$ is highest weight; that is, there exists a map of $\mathfrak{sl}_{n+1}[t]$-modules*

$$\tau_s^* W_{\text{loc}}(\nu - \alpha) \to W_{\text{loc}}(\nu) \to 0$$

*sending $w_{\nu-\alpha}$ to $(x_\alpha^- \otimes t^s)w_\nu$.*

*Proof.* It is sufficient to show that $(x_\alpha^- \otimes t^s)w_\nu$ satisfies relations (2.3.1) in $W_{\text{loc}}(\nu - \alpha)$. Indeed, note that for $h \in \mathfrak{h}$, $r \geq 0$, we have that

$$\begin{aligned}(h \otimes t^r)(x_\alpha^- \otimes t^s)w_\nu &= -\alpha(h)(x_\alpha^- \otimes t^{s+r})w_\nu + \delta_{r,0}\nu(h_\alpha)(x_\alpha^- \otimes t^s)w_\nu \\ &= \delta_{r,0}(\nu - \alpha)(h_\alpha)(x_\alpha^- \otimes t^s)w_\nu\end{aligned}$$

since by our assumption $(x_\alpha^- \otimes t^{s+r})w_\nu = 0$ if $r > 0$.

Next, let $j \in [1, n]$, and note by (2.3.1) that $(x_j^+ \otimes 1)w_\nu = 0$. Thus

$$(x_j^+ \otimes 1)(x_\alpha^- \otimes t^s)w_\nu = ([x_j^+, x_\alpha^-] \otimes t^s)w_\nu. \qquad (4.2.1)$$

Thus we wish to show that (4.2.1) is 0.

If $[x_j^+, x_\alpha^-] = 0$, then this follows trivially.

If $\alpha = \alpha_j$, then (4.2.1) is equal to $(h_j \otimes t^s)w_\nu$ which is 0 since $s > 0$.

If $\alpha - \alpha_j \in R^+$, then (4.2.1) is equal to $\pm(x_{\alpha-\alpha_j}^- \otimes t^s)w_\nu$ which is 0 by assumption.

Finally,

$$(x_j^- \otimes 1)^{(\nu-\alpha)(h_j)+1}(x_\alpha^- \otimes t^s)w_\nu = 0$$

by a standard application of Lie theory.

Therefore, this map is well-defined, and we may conclude that $(x_\alpha^- \otimes t^s)w_\nu$ is highest weight. □

## 4.3 Proof of Proposition 4.1.1

This section will be devoted to the proof of the proposition, which we split into several cases.

### 4.3.1 The kernel when $\lambda_1 \neq 0$ with $m \leq \min \lambda_1$ or $m \geq \max \lambda_1$

Assume that $m \leq \min \lambda_1$, as the case when $m \geq \max \lambda_1$ is similar. Let $p = \lambda_1$. Then $\beta = \alpha_{m,p}$ and $\eta = 0$. Thus we wish to show that

$$(x_\alpha^- \otimes t^{b_\alpha})w \in U(\mathfrak{g}[t])(x_{m,p}^- \otimes t^{b_{\alpha_{m,p}}})w \text{ for all } \alpha \in R^+.$$

Note that if $\alpha$ is either in the span of $\{\alpha_j : j < m\}$ or in the span of $\{\alpha_j : j > m\}$, then $\omega_m(h_\alpha) = 0$ and $a_\alpha = b_\alpha$, implying in this case that $(x_\alpha^- \otimes t^{b_\alpha})w = 0$ by (3.1.1). Similarly, if $\alpha$ is in the span of $\{\alpha_j : j < p\}$ then $\lambda_1(h_\alpha) = 0$, implying $a_\alpha = b_\alpha$ and that $(x_\alpha^- \otimes t^{b_\alpha})w = 0$ by (3.1.1).

Thus we may assume $\alpha = \beta + \alpha_{m,p} + \mu$ for $\beta$ in the span of $\{\alpha_j : j < m\}$ and $\mu$ in the span of $\{\alpha_j : j > p\}$. If $\beta \neq 0$, note that $b_\alpha = b_\beta + b_{\alpha_{m,p}+\mu}$ since $\lambda_1(h_\beta) = 0$. Then since $b_\beta = a_\beta$ since $\omega_m(h_\beta) = 0$, we have that

$$(x_\alpha^- \otimes t^{b_\alpha})w = [x_{\alpha_{m,p}+\mu}^- \otimes t^{b_{\alpha_{m,p}+\mu}}, x_\beta^- \otimes t^{a_\beta}]w = -(x_\beta^- \otimes t^{a_\beta})(x_{\alpha_{m,p}+\mu}^- \otimes t^{b_{\alpha_{m,p}+\mu}})w.$$

Thus we may assume that $\beta = 0$. But if $\mu \neq 0$, then similarly to before, $b_\alpha = b_{\alpha_{m,p}} + a_\mu$ since $\lambda_1(h_\mu) = \omega_m(h_\mu) = 0$. Therefore

$$(x_\alpha^- \otimes t^{b_\alpha})w = [x_\mu^- \otimes t^{a_\mu}, x_{m,p}^- \otimes t^{b_{\alpha_{m,p}}}]w = (x_\mu^- \otimes t^{a_\mu})(x_{m,p}^- \otimes t^{b_{\alpha_{m,p}}})w$$

showing $(x_{m,p}^- \otimes t^{(\nu+\lambda_0)(h_{m,p})})w$ generates the kernel as desired.

Finally, since (3.1.1) implies that

$$(x_{m,p}^- \otimes t^{(\nu+\lambda_0)(h_{m,p})+1})w = 0$$

and, if $m < p$, then

$$(x_{m+1,p}^- \otimes t^{(\nu+\lambda_0)(h_{m,p})})w = (x_{m,p-1}^- \otimes t^{(\nu+\lambda_0)(h_{m,p})})w = 0,$$

since $(\nu+\lambda_0)(h_m) \geq 1$, we conclude by Lemma 4.2.1 that $(x_{m,p}^- \otimes t^{(\nu+\lambda_0)(h_{m,p})+1})w$ is highest weight.

### 4.3.2 The kernel when $m \in \operatorname{supp}\lambda_1$ with $\min\lambda_1 < m < \max\lambda_1$.

Let $m \in \operatorname{supp}\lambda_1$ with $\min\lambda_1 < m < \max\lambda_1$. Then $\beta = \alpha_m$ and $\eta = \alpha_{m_\bullet,m_\diamond}$. Hence we wish to show that $(x_\alpha^- \otimes t^{b_\alpha})w$ is in the module generated by $(x_m^- \otimes t^{b_{\alpha_m}})w$ and $(x_{m_\bullet,m_\diamond}^- \otimes t^{b_{\alpha_{m_\bullet,m_\diamond}}})w$ for all $\alpha \in R^+$. Note that $b_{\alpha_m} = (\nu + \lambda_0)(h_m)$ and $b_{\alpha_{m_\bullet,m_\diamond}} = (\nu + \lambda_0)(h_{m_\bullet,m_\diamond}) + 1$.

Note that if $\alpha$ is either in the span of $\{\alpha_j : j < m\}$ or in the span of $\{\alpha_j : j > m\}$, then $\omega_m(h_\alpha) = 0$ and so $a_\alpha = b_\alpha$. Thus $(x_\alpha^- \otimes t^{b_\alpha})w = 0$ by (3.1.1). So suppose $\alpha = \beta + \alpha_m + \mu$ for $\beta$ in the span of $\{\alpha_j : j < m\}$ and $\mu$ in the span of $\{\alpha_j : j > m\}$. Note that if $(\lambda_1 + \omega_m)(h_\alpha)$ is odd, then

$$\lceil (\lambda_1 + \omega_m)(h_\alpha)/2 \rceil = \omega_m(h_\alpha) + \lceil \lambda_1(h_\alpha)/2 \rceil$$

and so $a_\alpha = b_\alpha$, implying $(x_\alpha^- \otimes t^{b_\alpha})w = 0$ by (3.1.1). So further assume that $(\lambda_1 + \omega_m)(h_\alpha)$ is even. Then $(\lambda_1 + \omega_m)(h_\beta)$ and $(\lambda_1 + \omega_m)(h_\mu)$ are either both even or both odd, since $(\lambda_1 + \omega_m)(h_m) = 2$.

First, suppose $(\lambda_1 + \omega_m)(h_\beta)$ and $(\lambda_1 + \omega_m)(h_\mu)$ are both even. If $\beta \neq 0$, then this implies

$$\lceil (\lambda_1 + \omega_m)(h_\alpha)/2 \rceil = \lceil (\lambda_1 + \omega_m)(h_\beta)/2 \rceil + \lceil (\lambda_1 + \omega_m)(h_{\alpha_m + \mu})/2 \rceil$$

and so $b_\alpha = b_\beta + b_{\alpha_m + \mu}$. Furthermore since $\omega_m(h_\beta) = 0$, $b_\beta = a_\beta$, and so

$$(x_\alpha^- \otimes t^{b_\alpha})w = [x_{\alpha_m+\mu}^- \otimes t^{b_{\alpha_m+\mu}}, x_\beta^- \otimes t^{a_\beta}]w = -(x_\beta^- \otimes t^{a_\beta})(x_{\alpha_m+\mu}^- \otimes t^{b_{\alpha_m+\mu}})w.$$

Thus we may assume that $\beta = 0$. If $\mu \neq 0$, then similarly as before,

$$(x_\alpha^- \otimes t^{b_\alpha})w = (x_\mu^- \otimes t^{a_\mu})(x_m^- \otimes t^{b_{\alpha_m}})w$$

because $b_\alpha = b_{\alpha_m} + b_\mu$ since

$$\lceil (\lambda_1 + \omega_m)(h_{\alpha_m+\mu})/2 \rceil = \lceil (\lambda_1 + \omega_m)(h_m)/2 \rceil + \lceil (\lambda_1 + \omega_m)(h_\mu)/2 \rceil$$

and $b_\mu = a_\mu$ since $\omega_m(h_\mu) = 0$. Thus when $(\lambda_1 + \omega_m)(h_\beta)$ and $(\lambda_1 + \omega_m)(h_\mu)$ are both even, $(x_\alpha^- \otimes t^{b_\alpha})w$ is in the module generated by $(x_m^- \otimes t^{b_{\alpha_m}})w$ as desired.

Lastly assume $(\lambda_1+\omega_m)(h_\beta)$ and $(\lambda_1+\omega_m)(h_\gamma)$ are both odd. Then $\omega_{m_\bullet}(h_\beta)$ and $\omega_{m_\diamond}(h_\gamma)$ are both nonzero. Thus $\alpha = \tilde{\beta} + \alpha_{m_\bullet,m_\diamond} + \tilde{\gamma}$ where $\tilde{\beta} = \beta - \alpha_{m_\bullet,m-1}$ and $\tilde{\gamma} = \gamma - \alpha_{m+1,m_\diamond}$ are positive roots, and $(\lambda_1+\omega_m)(h_{\tilde{\beta}})$ and $(\lambda_1+\omega_m)(h_{\tilde{\gamma}})$ are both even. We then proceed as above. If $\tilde{\beta} \neq 0$, then this implies that

$$\lceil(\lambda_1+\omega_m)(h_\alpha)/2\rceil = \lceil(\lambda_1+\omega_m)(h_{\tilde{\beta}})/2\rceil + \lceil(\lambda_1+\omega_m)(h_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}})/2\rceil.$$

Thus $b_\alpha = b_{\tilde{\beta}} + b_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}}$. Furthermore since $\omega_m(h_{\tilde{\beta}}) = 0$, $b_{\tilde{\beta}} = a_{\tilde{\beta}}$, and so

$$(x_\alpha^- \otimes t^{b_\alpha})w = [x^-_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}} \otimes t^{b_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}}}, x^-_{\tilde{\beta}} \otimes t^{a_{\tilde{\beta}}}]w = -(x^-_{\tilde{\beta}} \otimes t^{M_{\tilde{\beta}}})(x^-_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}} \otimes t^{M'_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}}})w.$$

Therefore we may assume that $\tilde{\beta} = 0$. Then if $\tilde{\gamma} \neq 0$, we have that

$$\lceil(\lambda_1+\omega_m)(h_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}})/2\rceil = \lceil(\lambda_1+\omega_m)(h_{m_\bullet,m_\diamond})/2\rceil + \lceil(\lambda_1+\omega_m)(h_{\tilde{\gamma}})/2\rceil$$

and that $\omega_m(h_{\tilde{\gamma}}) = 0$. Therefore so $b_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}} = b_{m_\bullet,m_\diamond} + a_{\tilde{\gamma}}$. Thus we have that

$$(x^-_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}} \otimes t^{b_{\alpha_{m_\bullet,m_\diamond}+\tilde{\gamma}}})w = [x^-_{\tilde{\gamma}} \otimes t^{a_{\tilde{\gamma}}}, x^-_{m_\bullet,m_\diamond} \otimes t^{b_{m_\bullet,m_\diamond}}]w = (x^-_{\tilde{\gamma}} \otimes t^{M_{\tilde{\gamma}}})(x^-_{m_\bullet,m_\diamond} \otimes t^{M'_{m_\bullet,m_\diamond}})w.$$

Thus we have shown that $(x_\alpha^- \otimes t^{b_\alpha})w$ is in the module generated by $(x_m^- \otimes t^{(\nu+\lambda)(h_m)})w$ and $(x^-_{m_\bullet,m_\diamond} \otimes t^{(\nu+\lambda)(h_{m_\bullet,m_\diamond})+1})w$, and so these elements generate the kernel of $\varphi^+$.

We conclude by showing these elements are highest weight. But since we see by (3.1.1) that

$$(x_m^- \otimes t^{(\nu+\lambda)(h_m)+1})w = 0$$

$$(x^-_{m_\bullet,m_\diamond} \otimes t^{(\nu+\lambda)(h_{m_\bullet,m_\diamond})+2})w = 0$$

$$(x^-_{m_\bullet+1,m_\diamond} \otimes t^{(\nu+\lambda)(h_{m_\bullet,m_\diamond})+1})w = 0$$

$$(x^-_{m_\bullet,m_\diamond-1} \otimes t^{(\nu+\lambda)(h_{m_\bullet,m_\diamond})+1})w = 0$$

then this follows by Lemma 4.2.1.

### 4.3.3 The kernel when $\min \lambda_1 < m < \max \lambda_1$ with $m \notin \operatorname{supp} \lambda_1$

Assume that $m \notin \operatorname{supp} \lambda_1$ with $\min \lambda_1 < m < \max \lambda_1$. Then $\beta = \alpha_{m_\bullet, m}$ and $\eta = \alpha_{m, m_\diamond}$. Hence it suffices to show that $(x_\alpha^- \otimes t^{b_\alpha})w$ is in the module generated by $(x_{m_\bullet,m}^- \otimes t^{b_{m_\bullet,m}})w$ and $(x_{m,m_\diamond}^- \otimes t^{b_{m,m_\diamond}})w$ for all $\alpha \in R^+$. We note $b_{m_\bullet,m} = (\nu + \lambda_0)(h_{m_\bullet,m})$ and $b_{m,m_\diamond} = (\nu + \lambda_0)(h_{m,m_\diamond})$.

First note that if $\alpha$ is in either the span of $\{\alpha_j : j < m\}$ or the span of $\{\alpha_j : j > m\}$, then $\omega_m(h_\alpha) = 0$, and so $(x_\alpha^- \otimes t^{b_\alpha})w = 0$ by (3.1.1) as $b_\alpha = a_\alpha$.

So from now on assume $\omega_m(h_\alpha) \neq 0$, and write $\alpha = \beta + \alpha_m + \mu$ for $\beta$ in the span of $\{\alpha_j : j < m\}$ and $\mu$ in the span of $\{\alpha_j : j > m\}$. If $\beta = 0$ or $\omega_{m_\bullet}(h_\beta) = 0$, and $\mu = 0$ or $\omega_{m_\diamond}(h_\mu) = 0$, then note that

$$(\omega_m + \lambda_1)(h_\alpha) = (\omega_m + \lambda_1)(h_m) = 1.$$

Thus $b_\alpha = a_\alpha$ and so $(x_\alpha^- \otimes t^{b_\alpha})w = 0$ by (3.1.1).

Next assume that $\omega_{m_\bullet}(h_\beta) \neq 0$ but either $\mu = 0$ or $\omega_{m_\diamond}(h_\mu) = 0$. If $\mu \neq 0$, note that

$$(\omega_m + \lambda_1)(h_\alpha) = (\omega_m + \lambda_1)(h_{\beta+\alpha_m}).$$

Therefore $b_\alpha = b_{\beta+\alpha_m} + b_\mu$. But since $\omega_m(h_\mu) = 0$, this implies $b_\mu = a_\mu$, and so

$$(x_\alpha^- \otimes t^{b_\alpha})w = [x_\mu^- \otimes t^{a_\mu}, x_{\beta+\alpha_m}^- \otimes t^{b_{\beta+\alpha_m}}]w = (x_\mu^- \otimes t^{M_\mu})(x_{\beta+\alpha_m}^- \otimes t^{b_{\beta+\alpha_m}})w.$$

Thus in this case we may assume that $\mu = 0$. So let $\alpha = \tilde{\beta} + \alpha_{m_\bullet,m}$ where $\tilde{\beta} = \beta - \alpha_{m_\bullet,m-1}$

is a positive root. If $\tilde\beta \neq 0$ then since $(\omega_m + \lambda_1)(h_{m_\bullet,m}) = 2$, we have

$$\lceil(\omega_m + \lambda_1)(h_{\tilde\beta+\alpha_{m_\bullet,m}})/2\rceil = 1 + \lceil(\omega_m + \lambda_1)(h_{\tilde\beta})/2\rceil.$$

Hence $b_{\tilde\beta+\alpha_{m_\bullet,m}} = b_{\tilde\beta} + b_{m_\bullet,m} = a_{\tilde\beta} + b_{m_\bullet,m}$ Then since $b_{\tilde\beta} = a_{\tilde\beta}$ since $\omega_m(h_{\tilde\beta}) = 0$, we conclude that

$$(x^-_{\tilde\beta+\alpha_{m_\bullet,m}} \otimes t^{b_{\tilde\beta+\alpha_{m_\bullet,m}}})w = [x^-_{m_\bullet,m} \otimes t^{b_{m_\bullet,m}}, x^-_{\tilde\beta} \otimes t^{a_{\tilde\beta}}]w = -(x^-_{\tilde\beta} \otimes t^{a_{\tilde\beta}})(x^-_{m_\bullet,m} \otimes t^{b_{m_\bullet,m}})w$$

as desired.

Likewise, if $\omega_{m_\diamond}(h_\mu) \neq 0$ but either $\beta = 0$ or $\omega_{m_\bullet}(h_\beta) = 0$, then a similar computation shows that $(x^-_\alpha \otimes t^{b_\alpha})w$ is in the module generated by $(x^-_{m,m_\diamond} \otimes t^{b_{m,m_\diamond}})w$.

It remains to show the case when $\omega_{m_\bullet}(h_\beta)$ and $\omega_{m_\diamond}(h_\mu)$ are both nonzero. If $(\omega_m + \lambda_1)(h_\alpha)$ is odd, then

$$\lceil(\omega_m + \lambda_1)(h_\alpha)/2\rceil = \omega_m(h_\alpha) + \lceil\lambda_1(h_\alpha)/2\rceil$$

and so $b_\alpha = a_\alpha$. Thus (3.1.1) implies $(x^-_\alpha \otimes t^{b_\alpha})w = 0$. So assume $(\omega_m + \lambda_1)(h_\alpha)$ is even. Then since $(\omega_m + \lambda_1)(h_m) = 1$, this implies either $(\omega_m + \lambda_1)(h_\beta)$ is even and $(\omega_m + \lambda_1)(h_\mu)$ is odd or vice versa. If $(\omega_m + \lambda_1)(h_\beta)$ is even then this implies that

$$\lceil(\omega_m + \lambda_1)(h_\alpha)/2\rceil = \lceil(\omega_m + \lambda_1)(h_\beta)/2\rceil + \lceil(\omega_m + \lambda_1)(h_{\alpha_m+\mu})/2\rceil.$$

Hence $b_\alpha = b_\beta + b_{\alpha_m+\mu}$. In addition $b_\beta = a_\beta$ since $\omega_m(h_\beta) = 0$. Therefore,

$$(x^-_\alpha \otimes t^{b_\alpha})w = [x^-_{\alpha_m+\mu} \otimes t^{b_{\alpha_m+\mu}}, x^-_\beta \otimes t^{a_\beta}]w = -(x^-_\beta \otimes t^{a_\beta})(x^-_{\alpha_m+\mu} \otimes t^{b_{\alpha_m+\mu}})w.$$

But we already showed above that $(x^-_{\alpha_m+\mu} \otimes t^{b_{\alpha_m+\mu}})w$ is in the module generated by $(x^-_{m,m_\diamond} \otimes t^{b_{m,m_\diamond}})w$.

Likewise, if $(\omega_m + \lambda_1)(h_\mu)$ is even and $(\omega_m + \lambda_1)(h_\beta)$ is odd, then a similar computation shows that $(x_\alpha^- \otimes t^{b_\alpha})w$ is in the module generated by $(x_{m_\bullet,m}^- \otimes t^{b_{m_\bullet,m}})w$. Therefore for every $\alpha \in R^+$, we have shown that $(x_\alpha^- \otimes t^{b_\alpha})w$ is in the module generated by $(x_{m_\bullet,m}^- \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m})})w$ and $(x_{m,m_\diamond}^- \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})w$ as desired.

It now remains to show that these generators are highest weight. But since by (3.1.1), we have that

$$(x_m^- \otimes t^{(\nu+\lambda)(h_{m_\bullet,m})+1})w = 0$$

$$(x_{m_\bullet,m_\diamond}^- \otimes t^{(\nu+\lambda)(h_{m,m_\diamond})+1})w = 0$$

$$(x_{m_\bullet+1,m_\diamond}^- \otimes t^{(\nu+\lambda)(h_{m_\bullet,m})})w = 0$$

$$(x_{m_\bullet,m_\diamond-1}^- \otimes t^{(\nu+\lambda)(h_{m,m_\diamond})})w = 0$$

then this follows by Lemma 4.2.1.

# Chapter 5

# The Leftmost Map $\varphi^-$

We now define some notation to be used in the rest of this section. Given $\lambda \in P^+$ and $m \in [1, n]$, define $\mathbf{P}_m(\lambda) \subset [1, n] \times [1, n]$ as follows:

$$\mathbf{P}_m(\lambda) = \begin{cases} \{(m, m)\} & \text{if } m \in \operatorname{supp} \lambda_1 \text{ with } m = \min \lambda_1 \text{ or } m = \max \lambda_1, \\ \{(m, m), (m_\bullet, m_\diamond)\} & \text{if } m \in \operatorname{supp} \lambda_1 \text{ with } \min \lambda_1 < m < \max \lambda_1, \\ \{(m, m_\diamond)\} & \text{if } m \notin \operatorname{supp} \lambda_1 \text{ with } m < \min \lambda_1, \\ \{(m_\bullet, m)\} & \text{if } m \notin \operatorname{supp} \lambda_1 \text{ with } m > \max \lambda_1, \\ \{(m_\bullet, m), (m, m_\diamond)\} & \text{if } m \notin \operatorname{supp} \lambda_1 \text{ with } \min \lambda_1 < m < \max \lambda_1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Given $i, m, j \in [1, n]$, define integer

$$\varepsilon(i, m, j) = (1 - \delta_{i,m})(1 - \delta_{m,j}).$$

In particular, for $i \leq m \leq j$, $\varepsilon(i, m, j) = 0$ unless $i < m < j$, in which case $\varepsilon(i, m, j) = 1$.

Let $\nu, \lambda \in P^+$ and $m \in \operatorname{supp} \nu$ with $(i,j) \in \mathbf{P}_m(\lambda)$. Then for each $\alpha \in R^+$ define integers

$$a_\alpha(\nu, \lambda) = \nu(h_\alpha) + \lceil \lambda(h_\alpha)/2 \rceil,$$

$$b_\alpha(\nu, \lambda, m) = (\nu - \omega_m)(h_\alpha) + \lceil (\lambda + \omega_m)(h_\alpha)/2 \rceil,$$

$$\begin{aligned} c_\alpha(\nu, \lambda, i, m, j) &= (\nu + (\varepsilon(i,m,j) - 1)\omega_m + \lambda_0)(h_\alpha) \\ &\quad + \lceil ( \ (\delta_{\lambda_1(h_m),0}\omega_m + \lambda_1)(1,i) \ )' \ (h_\alpha)/2 \rceil \\ &\quad + \lceil \ '( \ (\delta_{\lambda_1(h_m),0}\omega_m + \lambda_1)(j,n) \ ) \ (h_\alpha)/2 \rceil. \end{aligned}$$

## 5.1 The Main Proposition

In this section, our goal is to show the existence of map $\varphi^-$. The main result needed for that is the following proposition:

**Proposition 5.1.1.** *Let $\nu, \lambda \in P^+$, and let $m \in [1,n]$.*

*If $m \in \operatorname{supp} \lambda_1$, then there exists map of $\mathfrak{sl}_{n+1}[t]$-modules*

$$\varphi_1^- : \tau^*_{(\nu+\lambda_0)(h_m)} U_{m,0}(\nu, \lambda) \to N(\nu, \lambda, 0)$$

*sending generator to $(x_m^- \otimes t^{(\nu+\lambda_0)(h_m)})w$. Furthermore, if $m \in \operatorname{supp} \lambda_1$ and $\min \lambda_1 < m < \max \lambda_1$, then then there exists map of $\mathfrak{sl}_{n+1}[t]$-modules*

$$\varphi_2^- : \tau^*_{(\nu+\lambda_0)(h_{m_\bullet, m_\diamond})+1} U_{m,0}(\nu, \lambda + \omega_m) \to N(\nu, \lambda, 0)$$

*sending generator to $(x^-_{m_\bullet, m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet, m_\diamond})+1})w$.*

*If $m \notin \operatorname{supp} \lambda_1$ and $m > \min \lambda_1$, then there exists map of $\mathfrak{sl}_{n+1}[t]$-modules*

$$\varphi_L^- : \tau^*_{(\nu+\lambda_0)(h_{m_\bullet, m})} U_{m,-1}(\nu, \lambda + \omega_m) \to N(\nu, \lambda, 0)$$

*sending generator to* $(x^-_{m_\bullet,m} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m})})w$.

*If* $m \notin \operatorname{supp}\lambda_1$ *and* $m < \max\lambda_1$, *then there exists map of* $\mathfrak{sl}_{n+1}[t]$*-modules*

$$\varphi^-_R \colon \tau^*_{(\nu+\lambda_0)(h_{m,m_\diamond})} U_{m,1}(\nu, \lambda + \omega_m) \to N(\nu, \lambda, 0)$$

*sending generator to* $(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})w$.

The rest of this section will consist of the proof of this proposition, whose organization will be as follows. First, we introduce an important lemma, and then assuming the lemma prove this proposition. The remainder of the section will consist of proving the lemma.

## 5.2 Lemma and Proof of proposition

We state the following lemma, which we will use to prove Proposition 5.1.1.

**Lemma 5.2.1.** *Let* $\nu, \lambda \in P^+$ *and* $m \in \operatorname{supp}\nu$. *Let* $(i,j) \in \mathbf{P}_m(\lambda)$. *Let* $w$ *denote the generator of* $N(\nu, \lambda, 0)$, *and set*

$$c_\alpha := c_\alpha(\nu, \lambda, i, m, j) \text{ for each } \alpha \in R^+, \quad b_{\alpha_{i,j}} := b_{\alpha_{i,j}}(\nu, \lambda, m).$$

*Then for all* $\alpha \in R^+$,

$$(x^-_\alpha \otimes t^{c_\alpha})(x^-_{i,j} \otimes t^{b_{\alpha_{i,j}}})w = 0. \tag{5.2.1}$$

Assuming this lemma, we may now prove the main proposition.

*Proof of Proposition 5.1.1.* Suppose first that $m \in \operatorname{supp}\lambda_1$. Note

$$(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w = (x^-_\beta \otimes t^{b_\beta})w$$

as in the notation of Proposition 4.1.1. Since

$$U_{m,0}(\nu,\lambda) \cong N(\nu-\omega_m,\ 2\lambda_0+\lambda_1(1,m)'\ ,\ '\lambda_1(m,n))$$

and $(x_m^- \otimes t^{(\nu+\lambda_0)(h_m)})w$ is highest weight by Proposition 4.1.1 of weight

$$\nu+\lambda-\alpha_m = \nu-\omega_m+2\lambda_0+\lambda_1(1,m)'+'\lambda_1(m,n),$$

then by (3.1.1) the map $\varphi_1^-$ exists if and only if

$$(x_\alpha^- \otimes t^{(\nu-\omega_m+\lambda_0)(h_\alpha)+\lceil(\lambda_1(1,m)')(h_\alpha)/2\rceil+\lceil('\lambda_1(m,n))(h_\alpha)/2\rceil})(x_m^- \otimes t^{(\nu+\lambda_0)(h_m)})w = 0$$

for every $\alpha \in R^+$. But this follows from Lemma 5.2.1 since

$$(\nu-\omega_m+\lambda_0)(h_\alpha)+\lceil(\lambda_1(1,m)')(h_\alpha)/2\rceil+\lceil('\lambda_1(m,n))(h_\alpha)/2\rceil$$

is equal to $c_\alpha(\nu,\lambda,m,m,m)$ for every $\alpha \in R^+$ and

$$(\nu+\lambda_0)(h_m) = b_{\alpha_m}(\nu,\lambda,m)$$

where we note $(m,m) \in \mathbf{P}_m(\lambda)$.

Next suppose $m \in \operatorname{supp}\lambda_1$ with $\min\lambda_1 < m < \max\lambda_1$. Note

$$(x_{m_\bullet,m_\diamond}^- \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m_\diamond})+1})w = (x_\eta^- \otimes t^{b_\eta})w$$

as in the notation of Proposition 4.1.1. Since

$$U_{m,0}(\nu,\lambda+\omega_m) \cong N(\nu-\omega_m,\ 2\lambda_0+2\omega_m+\lambda_1(1,m-1)'\ ,\ '\lambda_1(m+1,n))$$

and $(x_{m_\bullet,m_\diamond}^- \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m_\diamond})+1})w$ is highest weight by Proposition 4.1.1 of weight

$$\nu+\lambda-\alpha_{m_\bullet,m_\diamond} = \nu+2\lambda_0+\omega_m+\lambda_1(1,m-1)'+'\lambda_1(m+1,n),$$

then by (3.1.1) the map $\varphi_2^-$ exists if and only if

$$(x_\alpha^- \otimes t^{(\nu+\lambda_0)(h_\alpha)+\lceil(\lambda_1(1,m-1)')(h_\alpha)/2\rceil+\lceil('\lambda_1(m+1,n))(h_\alpha)/2\rceil})(x_{m_\bullet,m_\diamond}^- \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m_\diamond})+1}) = 0$$

for every $\alpha \in R^+$. But this follows from Lemma 5.2.1 since

$$(\nu+\lambda_0)(h_\alpha) + \lceil(\lambda_1(1,m-1)')(h_\alpha)/2\rceil + \lceil('\lambda_1(m+1,n))(h_\alpha)/2\rceil = c_\alpha(\nu,\lambda,m_\bullet,m,m_\diamond)$$

for every $\alpha \in R^+$,

$$(\nu+\lambda_0)(h_{m_\bullet,m_\diamond}) + 1 = b_{\alpha_{m_\bullet,m_\diamond}}(\nu,\lambda,m)$$

where we note $(m_\bullet, m_\diamond) \in \mathbf{P}_m(\lambda)$.

Next we assume $m \notin \operatorname{supp}\lambda_1$ and $m > \min\lambda_1$. Note

$$(x_{m_\bullet,m}^- \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m})})w = (x_\eta^- \otimes t^{b_\eta})w$$

as in the notation of Proposition 4.1.1. Since

$$U_{m,-1}(\nu,\lambda+\omega_m) \cong N(\nu-\omega_m,\ 2\lambda_0+\lambda_1(1,m-1)'\ ,\ \omega_{m+1}+\lambda_1(m+1,n))$$

and $(x_{m_\bullet,m}^- \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m})})w$ is highest weight by Proposition 4.1.1 of weight

$$\nu+\lambda-\alpha_{m_\bullet,m} = \nu+2\lambda_0+\lambda_1(1,m-1)'-\omega_m+\omega_{m+1}+\lambda_1(m+1,n),$$

then by (3.1.1) the map $\varphi_{\mathrm{L}}^-$ exists if and only if

$$(x_\alpha^- \otimes t^{(\nu-\omega_m+\lambda_0)(h_\alpha)+\lceil(\lambda_1(1,m-1)')(h_\alpha)/2\rceil+\lceil(\omega_{m+1}+\lambda_1(m+1,n))(h_\alpha)/2\rceil})(x_{m_\bullet,m}^- \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m})}) = 0$$

for every $\alpha \in R^+$. But this follows from Lemma 5.2.1 since

$$(\nu-\omega_m+\lambda_0)(h_\alpha) + \lceil(\lambda_1(1,m-1)')(h_\alpha)/2\rceil + \lceil(\omega_{m+1}+\lambda_1(m+1,n))(h_\alpha)/2\rceil$$

$$= c_\alpha(\nu,\lambda,m_\bullet,m,m) \text{ for every } \alpha \in R^+,$$

$$(\nu+\lambda_0)(h_{m_\bullet,m}) = b_{\alpha_{m_\bullet,m}}(\nu,\lambda,m)$$

where we note $(m_\bullet, m) \in \mathbf{P}_m(\lambda)$.

Last suppose that $m \notin \operatorname{supp}\lambda_1$ and $m < \max\lambda_1$. Note

$$(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})w = (x^-_\beta \otimes t^{b_\beta})w$$

as in the notation of Proposition 4.1.1. Since

$$U_{m,1}(\nu, \lambda + \omega_m) \cong N(\nu - \omega_m,\ 2\lambda_0 + \lambda_1(1, m-1) + \omega_{m-1}\ ,\ {}'\lambda_1(m+1, n))$$

and $(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})w$ is highest weight by Proposition 4.1.1 of weight

$$\nu + \lambda - \alpha_{m,m_\diamond} = \nu + 2\lambda_0 + \lambda_1(1, m-1) + \omega_{m-1} - \omega_m +' \lambda_1(m+1, n),$$

then by (3.1.1) the map $\varphi^-_{\mathrm{R}}$ exists if and only if

$$(x^-_\alpha \otimes t^{(\nu-\omega_m+\lambda_0)(h_\alpha)+\lceil(\lambda_1(1,m-1)+\omega_{m-1})(h_\alpha)/2\rceil+\lceil({}'\lambda_1(m+1,n))(h_\alpha)/2\rceil})(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})}) = 0$$

for every $\alpha \in R^+$. But this follows from Lemma 5.2.1 since

$$(\nu - \omega_m + \lambda_0)(h_\alpha) + \lceil(\lambda_1(1, m-1) + \omega_{m-1})(h_\alpha)/2\rceil + \lceil({}'\lambda_1(m+1, n))(h_\alpha)/2\rceil$$
$$= c_\alpha(\nu, \lambda, m, m, m_\diamond) \text{ for every } \alpha \in R^+,$$

$$(\nu + \lambda_0)(h_{m,m_\diamond}) = b_{\alpha_{m,m_\diamond}}(\nu, \lambda, m)$$

where we note $(m, m_\diamond) \in \mathbf{P}_m(\lambda)$.

Thus the existence of all four maps has been shown.

□

## 5.3 Proof of Lemma 5.2.1

We finish the section with a proof of the main lemma. For the rest of this section, we will let

$$a_\alpha := a_\alpha(\nu, \lambda) \quad \text{for every } \alpha \in R^+.$$

Then (3.1.1) in $N(\nu, \lambda, 0)$ can be written as

$$(x_\alpha^- \otimes t^{a_\alpha})w = 0 \text{ for every } \alpha \in R^+. \tag{5.3.1}$$

In addition, let

$$\varepsilon := \varepsilon(i, m, j).$$

### 5.3.1 The case when $\alpha = \alpha_{i,j}$

**Lemma 5.3.1.** *Relation* (5.2.1) *holds for* $\alpha = \alpha_{i,j}$.

*Proof.* A simple calculation shows that

$$c_{\alpha_{i,j}} = (\nu + \lambda_0)(h_{i,j}) - 1 + \varepsilon,$$

$$b_{\alpha_{i,j}} = (\nu + \lambda_0)(h_{i,j}) - 1 + \lceil \lambda_1(h_{i,j})/2 + 1/2 \rceil = (\nu + \lambda_0)(h_{i,j}) + \varepsilon.$$

Thus to show (5.2.1) for $\alpha = \alpha_{i,j}$, we must show that

$$(x_{i,j}^- \otimes t^{(\nu+\lambda_0)(h_{i,j})+\varepsilon-1})(x_{i,j}^- \otimes t^{(\nu+\lambda_0)(h_{i,j})+\varepsilon})w = 0. \tag{5.3.2}$$

Let $r = (\nu + \lambda_0)(h_{i,j}) + \varepsilon$. Then

$$2r + 1 = 2(\nu + \lambda_0)(h_{i,j}) + 2\varepsilon + 1 = (2\nu + \lambda)(h_{i,j}) \geq (\nu + \lambda)(h_{i,j}) + 1$$

since $\nu(h_{i,j}) \geq 1$. By (2.3.1), $(x_{i,j}^-)^{(\nu+\lambda)(h_{i,j})+1}w = 0$. Since $w$ is highest weight, this implies that $(x_{i,j}^-)^{2r+1}w = 0$.

Second, note that by (3.1.1), we have that

$$(x_{i,j}^- \otimes t^{r+1})w = (x_{i,j}^- \otimes t^{a_{\alpha_{i,j}}})w = 0.$$

Hence by Lemma 2.4.1, we can conclude that relation (5.3.2) is true, finishing the proof. □

### 5.3.2 The case when $\alpha = \beta + \alpha_{i-1}$ or $\alpha = \alpha_{j+1} + \mu$

**Lemma 5.3.2.** *Relation* (5.2.1) *holds for* $\alpha = \beta + \alpha_{i-1}$ *or* $\alpha = \alpha_{j+1} + \mu$ *for* $\beta$ *in the span of* $\{\alpha_\ell : \ell < i-1\}$ *and* $\mu$ *in the span of* $\{\alpha_\ell : \ell > j+1\}$.

*Proof.* We prove the case when $\alpha = \beta + \alpha_{i-1}$, as the case when $\alpha = \alpha_{j+1} + \mu$ is similar. First, note that if $i-1 = 0$, there is nothing to show. So let $i-1 > 0$. Then since $[x_{\beta+\alpha_{i-1}}^-, x_{i,j}^-] = -x_{\beta+\alpha_{i-1,j}}^-$, the left hand side of (5.2.1) is equal to

$$-(x_{\beta+\alpha_{i-1,j}}^- \otimes t^{c_{\beta+\alpha_{i-1}}+b_{\alpha_{i,j}}})w + (x_{i,j}^- \otimes t^{b_{\alpha_{i,j}}})(x_{\beta+\alpha_{i-1}}^- \otimes t^{c_{\beta+\alpha_{i-1}}})w.$$

We begin by showing the first term in the sum is zero. But this follows from the fact that

$$c_{\beta+\alpha_{i-1}} + b_{\alpha_{i,j}} = (\nu + \lambda_0)(h_{\beta+\alpha_{i-1,j}}) + \lceil \lambda_1(h_{\beta+\alpha_{i-1}})/2 + 1/2 \rceil + \varepsilon = a_{\beta+\alpha_{i-1,j}}.$$

It now suffices to show that $(x_{\beta+\alpha_{i-1}}^- \otimes t^{c_{\beta+\alpha_{i-1}}})w = 0$. But note that

$$c_{\beta+\alpha_{i-1}} = (\nu + \lambda_0)(h_{\beta+\alpha_{i-1}}) + \lceil \lambda_1(h_{\beta+\alpha_{i-1}})/2 + 1/2 \rceil \geq a_{\beta+\alpha_{i-1}}$$

finishing this lemma.

□

### 5.3.3 A Useful Lemma

The next lemma will be used in proving the following lemma.

**Lemma 5.3.3.** *Suppose $i \neq m$ or $j \neq m$. Then*

$$(x_m^- \otimes t^{c_{\alpha_m} - [\![ m \in \operatorname{supp} \lambda_1 ]\!]})(x_m^- \otimes t^{c_{\alpha_m}})w = 0.$$

*Proof.* Note that

$$c_\alpha = (\nu + \lambda_0)(h_m) - 1 + \varepsilon.$$

First, suppose $m \in \operatorname{supp} \lambda_1$. Then we wish to show that

$$(x_m^- \otimes t^{(\nu+\lambda_0)(h_m)-1})(x_m^- \otimes t^{(\nu+\lambda_0)(h_m)})w = 0.$$

But by Lemma 2.4.1(i), this follows from the fact that

$$(x^- \otimes 1)^{2(\nu+\lambda_0)(h_m)+1} w = 0 \text{ and } (x^- \otimes t)^{(\nu+\lambda_0)(h_m)+1} w = 0$$

by (2.3.1) and (3.1.1) in $N(\nu, \lambda, 0)$, respectively, where we note that

$$2(\nu + \lambda_0)(h_m) + 1 = (2\nu + \lambda)(h_m) \geq (\nu + \lambda)(h_m) + 1$$

since $\nu(h_m) \geq 1$.

Second, suppose $m \notin \operatorname{supp} \lambda_1$. Then we wish to show that

$$(x_m^- \otimes t^{(\nu+\lambda_0)(h_m)-1})^2 w = 0.$$

But by Lemma 2.4.1(i), this follows from the fact that

$$(x^- \otimes 1)^{2(\nu+\lambda_0)(h_m)} w = 0 \text{ and } (x^- \otimes t)^{(\nu+\lambda_0)(h_m)} w = 0$$

by (2.3.1) and (3.1.1) in $N(\nu, \lambda, 0)$, respectively, where we note that

$$2(\nu + \lambda_0)(h_m) = (2\nu + \lambda)(h_m) \geq (\nu + \lambda)(h_m) + 1$$

since $\nu(h_m) \geq 1$. □

### 5.3.4 The case when $\alpha = \beta + \alpha_m + \mu$

**Lemma 5.3.4.** *Relation* (5.2.1) *holds for* $\alpha = \beta + \alpha_m + \mu$ *for* $\beta$ *in the span of* $\{\alpha_\ell : i < \ell < m\}$ *and* $\mu$ *in the span of* $\{\alpha_\ell : m < \ell < j\}$.

*Proof.* By Lemma 5.3.1 this is already shown for when $i = j = m$ with $m \in \operatorname{supp} \lambda_1$. So assume that $m \neq i$ or $m \neq j$. Note that

$$c_{\beta+\alpha_m+\mu} = (\nu + \lambda_0)(h_{\beta+\alpha_m+\mu}) + \varepsilon - 1 = a_\beta + c_{\alpha_m} + a_\mu.$$

Thus if $\mu \neq 0$, then $x^-_{\beta+\alpha_m+\mu} = [x^-_\mu, x^-_{\beta+\alpha_m}]$ and $[x^-_\mu, x^-_{i,j}] = 0$, and so (5.2.1) is equivalent to the relation

$$(x^-_\mu \otimes t^{a_\mu})(x^-_{\beta+\alpha_m} \otimes t^{a_\beta + c_{\alpha_m}})(x^-_{i,j} \otimes t^{b_{\alpha_{i,j}}})w = 0.$$

Thus it suffices to show

$$(x^-_{\beta+\alpha_m} \otimes t^{a_\beta + c_{\alpha_m}})(x^-_{i,j} \otimes t^{b_{\alpha_{i,j}}})w \tag{5.3.3}$$

is equal to 0. Similarly, since $x^-_{\beta+\alpha_m} = [x^-_m, x^-_\beta]$ and $[x^-_\beta, x^-_{i,j}] = 0$ when $\beta \neq 0$, (5.3.3) is equal to

$$-(x^-_\beta \otimes t^{a_\beta})(x^-_m \otimes t^{c_{\alpha_m}})(x^-_{i,j} \otimes t^{b_{\alpha_{i,j}}})w.$$

Thus, it suffices to show that

$$(x^-_m \otimes t^{c_{\alpha_m}})(x^-_{i,j} \otimes t^{b_{\alpha_{i,j}}})w = 0. \tag{5.3.4}$$

We proceed in cases.

First, suppose that $i < m < j$ with $m \in \operatorname{supp} \lambda_1$. Then $i = m_\bullet$ and $j = m_\diamond$. Then since $[x^-_m, x^-_{m_\bullet, m_\diamond}] = 0$, we wish to show that

$$(x^-_{m_\bullet, m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet, m_\diamond})+1})(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w = 0. \tag{5.3.5}$$

But since $x^-_{m_\bullet,m_\diamond} = [x^-_{m+1,m_\diamond}, x^-_{m_\bullet,m}]$, the left hand side of (5.3.5) is equal to

$$[x^-_{m+1,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m+1,m_\diamond})+1}, x^-_{m_\bullet,m} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m})}](x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w \tag{5.3.6}$$

Note that

$$(x^-_{m+1,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m+1,m_\diamond})+1})(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w$$

is equal to

$$(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})+1})w + (x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})(x^-_{m+1,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m+1,m_\diamond})+1})w.$$

But this is equal to 0 since

$$a_{\alpha_{m,m_\diamond}} = (\nu+\lambda_0)(h_{m,m_\diamond}) + 1,$$

$$a_{\alpha_{m+1,m_\diamond}} = (\nu+\lambda_0)(h_{m+1,m_\diamond}) + 1.$$

This combined with (5.3.6) implies that it is sufficient to show that

$$(x^-_{m_\bullet,m} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m})})(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w = 0. \tag{5.3.7}$$

Again, since $x^-_{m_\bullet,m} = [x^-_m, x^-_{m_\bullet,m-1}]$, the left hand side of (5.3.7) is equal to

$$[x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1}, x^-_{m_\bullet,m-1} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m-1})+1}](x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w.$$

A similar computation shows that

$$(x^-_{m_\bullet,m-1} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m-1})+1})(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w = 0.$$

Therefore, we may further reduce the problem to showing that

$$(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1})(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w = 0.$$

But since

$$c_{\alpha_m} = (\nu + \lambda_0)(h_m),$$

this follows by Lemma 5.3.3.

Next we assume instead that $m \notin \operatorname{supp} \lambda_1$ with $i = m < j = m_\diamond$. Then relation (5.3.4) is equivalent to showing that

$$(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1})w = 0. \tag{5.3.8}$$

Since $x^-_{m,m_\diamond} = [x^-_{m+1,m_\diamond}, x^-_m]$, then the left hand side of (5.3.8) is equal to

$$[x^-_{m+1,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m+1,m_\diamond})+1}, x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1}](x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1})w. \tag{5.3.9}$$

Note that

$$(x^-_{m+1,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m+1,m_\diamond})+1})(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1})w = (x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})w$$

since

$$(x^-_{m+1,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m+1,m_\diamond})+1})w = 0 \text{ and } [x^-_{m+1,m_\diamond}, x^-_m] = x^-_{m,m_\diamond}.$$

Thus (5.3.9) is equal to

$$\begin{aligned}(x^-_{m+1,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m+1,m_\diamond})+1})(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1})^2 w \\ - (x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1})(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})w.\end{aligned} \tag{5.3.10}$$

But since

$$c_{\alpha_m} = (\nu + \lambda_0)(h_m) - 1,$$

Lemma 5.3.3 implies that the first term of (5.3.10) is equal to 0. Therefore, we have that

$$\begin{aligned}(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1})w \\ = -(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)-1})(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})w\end{aligned}$$

But since $[x^-_{m,m_\diamond}, x^-_m] = 0$, we get (5.3.8) as desired.

A similar computation holds for when $m \notin \operatorname{supp} \lambda_1$ with $i = m_\bullet < m = j$, finishing the proof of the lemma.

□

### 5.3.5 The case when $\alpha = \alpha_i + \beta$ or $\alpha = \mu + \alpha_j$

**Lemma 5.3.5.** *Relation* (5.2.1) *holds for $\alpha = \alpha_i + \beta$ for $\beta$ in the span of $\{\alpha_\ell : i < \ell < m\}$; and when $\alpha = \mu + \alpha_j$ for $\mu$ in the span of $\{\alpha_\ell : m < \ell < j\}$.*

*Proof.* We prove the case when $\alpha = \alpha_i + \beta$, as the computations when $\alpha = \mu + \alpha_j$ are similar. Note that if $i = m$ then by Lemma 5.3.4 we are done. So assume $i < m$; then $i = m_\bullet$ with either $m \in \operatorname{supp} \lambda_1$ or $m \notin \operatorname{supp} \lambda_1$. In either case, note that if $\beta \neq 0$, then

$$c_{\alpha_{m_\bullet}+\beta} = (\nu + \lambda_0)(h_{\alpha_{m_\bullet}+\beta}) = a_\beta + c_{\alpha_{m_\bullet}}.$$

Then since $x^-_{\alpha_{m_\bullet}+\beta} = [x^-_\beta, x^-_{m_\bullet}]$, (5.2.1) is equivalent to

$$[x^-_\beta \otimes t^{a_\beta}, x^-_{m_\bullet} \otimes t^{c_{\alpha_{m_\bullet}}}](x^-_{m_\bullet,j} \otimes t^{b_{\alpha_{m_\bullet,j}}})w = 0.$$

Since $[x^-_\beta, x^-_{m_\bullet,j}] = 0$, (5.3.1) implies that it is sufficient to show that

$$(x^-_{m_\bullet} \otimes t^{c_{\alpha_{m_\bullet}}})(x^-_{m_\bullet,j} \otimes t^{b_{\alpha_{m_\bullet,j}}})w = 0. \tag{5.3.11}$$

Note that

$$b_{\alpha_{m_\bullet,j}} = (\nu + \lambda_0)(h_{m_\bullet,j}) + \varepsilon = c_{\alpha_{m_\bullet}} + a_{\alpha_{m_\bullet+1,j}}.$$

Since $x^-_{m_\bullet,j} = [x^-_{m_\bullet+1,j}, x^-_{m_\bullet}]$, (5.3.1) implies that the left hand side of (5.3.11) is equal to

$$(x^-_{m_\bullet} \otimes t^{c_{\alpha_{m_\bullet}}})(x^-_{m_\bullet+1,j} \otimes t^{a_{\alpha_{m_\bullet+1,j}}})(x^-_{m_\bullet} \otimes t^{c_{\alpha_{m_\bullet}}})w. \tag{5.3.12}$$

Again since $[x^-_{m_\bullet}, x^-_{m_\bullet+1,j}] = -x^-_{m_\bullet,j}$, it follows that (5.3.12) is equal to

$$-(x^-_{m_\bullet,j} \otimes t^{b_{\alpha_{m_\bullet,j}}})(x^-_{m_\bullet} \otimes t^{c_{\alpha_{m_\bullet}}})w + (x^-_{m_\bullet+1,j} \otimes t^{a_{\alpha_{m_\bullet+1,j}}})(x^-_{m_\bullet} \otimes t^{c_{\alpha_{m_\bullet}}})^2 w. \tag{5.3.13}$$

We claim that the second term in this sum is equal to 0. Indeed, since

$$(x^-_{m_\bullet} \otimes 1)^{(\nu+2\lambda_0)(h_{m_\bullet})+2} w = 0 \text{ and } (x^-_{m_\bullet} \otimes t^{(\nu+\lambda_0)+1})w = 0$$

by (2.3.1) and (3.1.1) in $N(\nu, \lambda, 0)$, respectively, where we note that

$$2(\nu + \lambda_0)(h_{m_\bullet}) + 2 \geq (\nu + 2\lambda_0)(h_{m_\bullet}) + 2$$

since $\nu(h_{m_\bullet}) \geq 0$, $(x^-_{m_\bullet} \otimes t^{c_{\alpha_{m_\bullet}}})^2 w$ is 0 by Lemma 2.4.1(ii).

Therefore, since (5.3.11) is equal to (5.3.13), by rearranging terms we get (5.3.11) is equal to zero whence the lemma is proven.

□

### 5.3.6 The General Case

We now proceed to proving the main lemma. First suppose $\alpha \in R^+$ such that $\omega_\ell(h_\alpha) = 0$ for $\ell \in \{i-1, i, m, j, j+1\}$. Then (5.2.1) follows from the fact that $c_\alpha = a_\alpha$ and $[x^-_\alpha, x^-_{i,j}] = 0$ for such $\alpha$.

From now on we may assume that $\omega_\ell(h_\alpha) \neq 0$ for some $\ell \in \{i-1, i, m, j, j+1\}$. So first suppose that $\omega_{i-1}(h_\alpha) \neq 0$. Then we may also assume that $\omega_i(h_\alpha) \neq 0$ as otherwise by Lemma 5.3.2 we are done. So let $\alpha = \beta + \alpha_{i-1,i} + \mu$ for $\beta$ in the span of $\{\alpha_\ell : \ell < i-1\}$ and $\mu$ in the span of $\{\alpha_\ell : \ell > i\}$. Note that

$$c_{\beta+\alpha_{i-1,i}+\mu} - c_{\alpha_i+\mu} = (\nu + \lambda_0)(h_{\beta+\alpha_{i-1}}) + \lceil \lambda_1(h_{\beta+\alpha_{i-1}})/2 + 1/2 \rceil = c_{\beta+\alpha_{i-1}}.$$

This then implies that (5.2.1) is equivalent to showing

$$[x^-_{\alpha_i+\mu} \otimes t^{c_{\alpha_i+\mu}}, x^-_{\beta+\alpha_{i-1}} \otimes t^{c_{\beta+\alpha_{i-1}}}](x^-_{i,j} \otimes t^{b_{\alpha_{i,j}}})w = 0.$$

By Lemma 5.3.2 we have that

$$(x^-_{\beta+\alpha_{i-1}} \otimes t^{c_{\beta+\alpha_{i-1}}})(x^-_{i,j} \otimes t^{b_{\alpha_{i,j}}})w = 0.$$

Therefore we see it is sufficient to show (5.2.1) for $\alpha \in R^+$ such that $\omega_{i-1}(h_\alpha) = 0$. A similar calculation lets us also assume that $\omega_{j+1}(h_\alpha) = 0$.

So from now on assume that $\omega_{i-1}(h_\alpha) = \omega_{j+1}(h_\alpha) = 0$, and that $\omega_\ell(h_\alpha) \neq 0$ for some $\ell \in \{i, m, j\}$. Then by Lemma 5.3.1, Lemma 5.3.4, and Lemma 5.3.5 we see that we may further assume that $i = m_\bullet < m < m_\diamond = j$, and that either $\alpha = \alpha_{m_\bullet,m} + \mu$ for $\mu$ in the span of $\{\alpha_\ell : m < \ell < m_\diamond\}$ or $\alpha = \beta + \alpha_{m,m_\diamond}$ for $\beta$ in the span of $\{\alpha_\ell : m_\bullet < \ell < m\}$. We shall show the case when $\alpha = \alpha_{m_\bullet,m} + \mu$, as the proof for $\alpha = \beta + \alpha_{m,m_\diamond}$ is similar. We have

$$c_{\alpha_{m_\bullet,m}+\mu} = (\nu + \lambda_0)(h_{\alpha_{m_\bullet,m}+\mu}) = c_{\alpha_{m_\bullet}} + c_{\alpha_{m_\bullet+1,m}}.$$

Therefore the left hand side of (5.2.1) is equal to

$$[x^-_{\alpha_{m_\bullet+1,m}+\mu} \otimes t^{c_{\alpha_{m_\bullet+1,m}+\mu}}, x^-_{m_\bullet} \otimes t^{c_{\alpha_{m_\bullet}}}](x^-_{i,j} \otimes t^{b_{\alpha_{i,j}}})w,$$

which is equal to 0 by Lemma 5.3.4 and Lemma 5.3.5. Therefore we have shown that (5.2.1) holds for all $\alpha \in R^+$. □

# Chapter 6

# The Maps $\psi^+$ and $\psi^-$

We define the following notation to be used in the rest of this chapter.

Given $\lambda, \gamma \in P^+$ with $\max \lambda_1 < \min \gamma_1$, define $\beta_{\lambda,\gamma} \in R^+ \cup \{0\}$ by

$$\beta_{\lambda,\gamma} = \begin{cases} \alpha_{\max \lambda_1, \min \gamma_1} & \text{if } \lambda_1, \gamma_1 \neq 0, \\ \\ 0 & \text{otherwise} \end{cases}.$$

Furthermore, given $\nu \in P^+$, for each $\alpha \in R^+$ define integers

$$a_\alpha(\nu, \lambda, \gamma) = \nu(h_\alpha) + \lceil \lambda(h_\alpha)/2 \rceil + \lceil \gamma(h_\alpha)/2 \rceil,$$

$$b_\alpha(\nu, \lambda, \gamma) = \nu(h_\alpha) + \lceil (\lambda + \gamma)(h_\alpha)/2 \rceil.$$

If $\lambda_1, \gamma_1 \neq 0$, for each $\alpha \in R^+$ define integer

$$c_\alpha(\nu, \lambda, \gamma) = \nu(h_\alpha) + \lceil \ \lambda' \ (h_\alpha)/2 \rceil + \lceil \ ' \gamma \ (h_\alpha)/2 \rceil.$$

## 6.1 The map $\psi^+$

We prove the following proposition to begin the proof of the other short exact sequence in Theorem 3.3.1.

**Proposition 6.1.1.** *Let $(\nu, \lambda, \gamma) \in P^+ \times P^+ \times P^+$ be compatible. Let $w$ denote the generator of $N(\nu, \lambda, \gamma)$, and set*

$$\beta := \beta_{\lambda,\gamma}, \quad b_\beta := b_\beta(\nu, \lambda, \gamma).$$

*Then there exists a surjective map of $\mathfrak{sl}_{n+1}[t]$-modules*

$$\psi^+ \colon N(\nu, \lambda, \gamma) \to N(\nu, \lambda + \gamma, 0) \to 0$$

*sending generator to generator, whose kernel is generated by $(x_\beta^- \otimes t^{b_\beta})w$. In addition, this element is highest weight.*

*Proof.* For each $\alpha \in R^+$ we will let

$$a_\alpha := a_\alpha(\nu, \lambda), \quad b_\alpha := b_\alpha(\nu, \lambda)$$

Note that $N(\nu, \lambda, \gamma)$ and $N(\nu, \lambda + \gamma, 0)$ are quotients of $W_{\mathrm{loc}}(\nu + \lambda + \gamma)$ by the additional relations

$$(x_\alpha^- \otimes t^{a_\alpha})w_{\nu,\lambda,\gamma} = 0 \text{ and } (x_\alpha^- \otimes t^{a_\alpha})w_{\nu,\lambda+\gamma,0} = 0,$$

respectively, for every $\alpha \in R^+$. Thus, existence of the map $\psi^+$ follows from the fact that $a_\alpha \geq b_\alpha$ for all $\alpha$. In particular, the kernel of $\psi^+$ is generated by elements of the form

$$(x_\alpha^- \otimes t^{b_\alpha})w \text{ for all } \alpha \text{ such that } a_\alpha > b_\alpha. \tag{6.1.1}$$

In addition, if either $\lambda_1 = 0$ or $\gamma_1 = 0$, then $a_\alpha = b_\alpha$ for all $\alpha$ and so $\psi^+$ is an isomorphism as desired, since in this case $\beta = 0$.

So now assume that both $\lambda_1, \gamma_1 \neq 0$. Let $p = \max \lambda_1$ and $q = \min \gamma_1$. Then we wish to show that all elements of the form (6.1.1) are in the module generated by

$$(x_{p,q}^- \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q})+1})w$$

and that this element is highest weight.

First, suppose that $\alpha \in R^+$ such that $\omega_j(h_\alpha) = 0$ for $j \leq p$. Then $a_\alpha = b_\alpha$ since $\lambda_1(h_\alpha) = 0$, and thus $(x_\alpha^- \otimes t^{b_\alpha})w = 0$. Similarly, if $\alpha \in R^+$ with $\omega_j(h_\alpha) = 0$ for $j \geq q$, then $a_\alpha = b_\alpha$ since $\gamma_1(h_\alpha) = 0$, and so $(x_\alpha^- \otimes t^{b_\alpha})w = 0$.

So now assume that $\alpha \in R^+$ such that $\omega_j(h_\alpha) \neq 0$ for $p \leq j \leq q$. Write $\alpha = \beta + \alpha_{p,q} + \mu$ for $\beta$ in the span of $\{\alpha_j : j < p\}$ and $\mu$ in the span of $\{\alpha_j : j > q\}$. Then note that

$$b_\alpha = (\nu + \lambda_0 + \gamma_0)(h_\alpha) + \lceil(\lambda_1(h_\beta) + 2 + \gamma_1(h_\mu))/2\rceil.$$

If at most one of $\lambda_1(h_\beta)$ and $\gamma_1(h_\mu)$ is odd, then this implies that

$$b_\alpha = (\nu + \lambda_0 + \gamma_0)(h_\alpha) + 1 + \lceil\lambda_1(h_\beta)/2\rceil + \lceil\gamma_1(h_\mu)/2\rceil = a_\beta + b_{\alpha_{p,q}} + a_\mu.$$

This implies that

$$(x_\alpha^- \otimes t^{b_\alpha})w = [x_\mu^- \otimes t^{a_\mu}, [x_{p,q}^- \otimes t^{b_{\alpha_{p,q}}}, x_\beta^- \otimes t^{a_\beta}]]w = -(x_\mu^- \otimes t^{a_\mu})(x_\beta^- \otimes t^{a_\beta})(x_{p,q}^- \otimes t^{b_{\alpha_{p,q}}})w.$$

If instead both $\lambda_1(h_\beta)$ and $\gamma_1(h_\mu)$ are odd, then both $\lambda_1(h_\beta) + 1$ and $\gamma_1(h_\mu) + 1$ are even. Thus

$$\begin{aligned} b_\alpha &= (\nu + \lambda_0 + \gamma_0)(h_\alpha) + \lceil(\lambda_1(h_\beta) + 1)/2\rceil + \lceil(\gamma_1(h_\mu) + 1)/2\rceil \\ &= (\nu + \lambda_0 + \gamma_0)(h_\alpha) + \lceil\lambda_1(h_{\beta+\alpha_p})/2\rceil + \lceil\gamma_1(h_{\alpha_q+\mu})/2\rceil \\ &= a_{\beta+\alpha_p} + a_{\alpha_{p+1,q-1}} + a_{\alpha_q+\mu} \end{aligned}$$

where we let $a_{\alpha_{p+1,q-1}} = 0$ if $p+1 > q-1$. This implies that

$$(x_\alpha^- \otimes t^{b_\alpha})w = \begin{cases} [x^-_{\alpha_q+\mu} \otimes t^{a_{\alpha_q+\mu}}, x^-_{\beta+\alpha_p} \otimes t^{a_{\beta+\alpha_p}}]w & \text{if } p+1 > q-1, \\ [x^-_{\alpha_q+\mu} \otimes t^{a_{\alpha_q+\mu}}, [x^-_{p+1,q-1} \otimes t^{a_{\alpha_{p+1,q-1}}}, x^-_{\beta+\alpha_p} \otimes t^{a_{\beta+\alpha_p}}]]w & \text{otherwise} \end{cases}$$

which in either case is equal to 0.

Thus $(x_\alpha^- \otimes t^{b_\alpha})w$ is in the module generated by $(x^-_{p,q} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q})+1})w$ for all $\alpha \in R^+$, and so this element generates the kernel of $\psi^+$.

We lastly conclude by Lemma 4.2.1 that $(x^-_{p,q} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q})+1})w$ is highest weight, since

$$(x^-_{p,q} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q})+2})w = 0,$$

$$(x^-_{p+1,q} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p+1,q})+1})w = 0,$$

$$(x^-_{p,q-1} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q-1})+1})w = 0.$$

□

## 6.2 The map $\psi^-$

**Proposition 6.2.1.** *Let $(\nu, \lambda, \gamma)$ be a compatible triple, and let $w$ denote the generator of $N(\nu, \lambda, \gamma)$. If $\gamma_1, \lambda_1 \neq 0$, then there exists map of $\mathfrak{sl}_{n+1}[t]$-modules*

$$\psi^- \colon \tau^*_{b_{\alpha_{p,q}}} N(\nu,\ \lambda',\ {}'\gamma) \to N(\nu, \lambda, \gamma)$$

*sending generator to $(x^-_{p,q} \otimes t^{b_{\alpha_{p,q}}})w$, where*

$$p = \max \lambda_1,\ q = \min \gamma_1,\ \text{and } b_{\alpha_{p,q}} := b_{\alpha_{p,q}}(\nu, \lambda, \gamma).$$

*Proof.* By Proposition 6.1.1 $(x_{p,q}^- \otimes t^{b_{\alpha_{p,q}}})w$ is a highest weight vector, with weight

$$\nu + \lambda + \gamma - \alpha_{p,q} = \nu + \lambda' + {}'\gamma.$$

Thus $\tau^*_{b_{\alpha_{p,q}}} N(\nu,\ \lambda',\ {}'\gamma)$ and $\mathbf{U}(\mathfrak{sl}_{n+1}[t])(x_{p,q}^- \otimes t^{b_{\alpha_{p,q}}})w$ are both quotients of $W_{\text{loc}}(\nu+\lambda'+{}'\gamma)$. Therefore, to prove that $\psi^-$ is well-defined, by relation (3.1.1) in $N(\nu,\ \lambda',\ {}'\gamma)$ it suffices to show that

$$(x_\alpha^- \otimes t^{c_\alpha})(x_{p,q}^- \otimes t^{b_{\alpha_{p,q}}})w = 0 \tag{6.2.1}$$

for every $\alpha \in R^+$, where we let

$$c_\alpha := c_\alpha(\nu, \lambda, \gamma).$$

We also define

$$a_\alpha := a_\alpha(\nu, \lambda, \gamma)$$

for every $\alpha \in R^+$, and note that $N(\nu, \lambda, \gamma)$ is the quotient of $W_{\text{loc}}(\nu + \lambda + \gamma)$ by the relation

$$(x_\alpha^- \otimes t^{a_\alpha})w_{\nu+\lambda+\gamma} = 0 \text{ for all } \alpha \in R^+. \tag{6.2.2}$$

First suppose that $\omega_\ell(h_\alpha) = 0$ for $\ell \in \{p-1, p, q, q+1\}$. Then (6.2.1) follows from the fact that $c_\alpha = a_\alpha$ and $[x_\alpha^-, x_{p,q}^-] = 0$ for such $\alpha$.

Next suppose that $\alpha = \beta + \alpha_{p-1}$ for $\beta$ in the span of $\{\alpha_\ell : \ell < p-1\}$ and $p > 1$. Then since $[x_{\beta+\alpha_{p-1}}^-, x_{p,q}^-] = -x_{\beta+\alpha_{p-1,q}}^-$, the left hand side of (5.2.1) is equal to

$$-(x_{\beta+\alpha_{p-1,q}}^- \otimes t^{c_{\beta+\alpha_{p-1}}+b_{\alpha_{p,q}}})w + (x_{p,q}^- \otimes t^{b_{\alpha_{p,q}}})(x_{\beta+\alpha_{p-1}}^- \otimes t^{c_{\beta+\alpha_{p-1}}})w.$$

Note the first term in this sum is zero since

$$c_{\beta+\alpha_{p-1}} + b_{\alpha_{p,q}} = (\nu + \lambda_0 + \gamma_0)(h_{\beta+\alpha_{p-1,q}}) + \lceil \lambda_1(h_{\beta+\alpha_{p-1}})/2 + 1/2 \rceil + 1 = a_{\beta+\alpha_{p-1,q}}.$$

The second term in this sum is also zero since

$$c_{\beta+\alpha_{p-1}} = (\nu + \lambda_0 + \gamma_0)(h_{\beta+\alpha_{p-1}}) + \lceil \lambda_1(h_{\beta+\alpha_{p-1}})/2 + 1/2 \rceil \geq a_{\beta+\alpha_{p-1}}$$

showing (6.2.1) in this case.

We omit the case when $\alpha = \alpha_{q+1} + \mu$ for $\mu$ in the span of $\{\alpha_\ell : \ell > q+1\}$ and $q < n$ as the computation is similar.

Next suppose that $\alpha = \beta + \alpha_{p-1,p} + \mu$ for $\beta$ in the span of $\{\alpha_\ell : \ell < p-1\}$, $\mu$ in the span of $\{\alpha_\ell : \ell > p\}$, and $p > 1$. We calculate that

$$c_{\beta+\alpha_{p-1,p}+\mu} = (\nu+\lambda_0+\gamma_0)(h_{\beta+\alpha_{p-1,p}+\mu}) + \lceil\ \lambda_1'(h_{\beta+\alpha_{p-1}})/2\ \rceil + \lceil\ '\gamma_1(h_\mu)/2\ \rceil = c_{\beta+\alpha_{p-1}} + c_{\alpha_p+\mu}$$

Then since $x^-_{\beta+\alpha_{p-1,p}+\mu} = [x^-_{\alpha_p+\mu}, x^-_{\beta+\alpha_{p-1}}]$, we see that (6.2.1) is equivalent to showing

$$[x^-_{\alpha_p+\mu} \otimes t^{c_{\alpha_p+\mu}}, x^-_{\beta+\alpha_{p-1}} \otimes t^{c_{\beta+\alpha_{p-1}}}](x^-_{p,q} \otimes t^{b_{\alpha_{p,q}}})w = 0.$$

But since we have already shown that

$$(x^-_{\beta+\alpha_{p-1}} \otimes t^{c_{\beta+\alpha_{p-1}}})(x^-_{p,q} \otimes t^{b_{\alpha_{p,q}}})w = 0,$$

we see that it suffices to show the case when $\omega_{p-1}(h_\alpha) = 0$. A similar argument lets us also assume that $\omega_{q+1}(h_\alpha) = 0$.

Next suppose that $\alpha = \alpha_{p,q}$. Then (6.2.1) is equivalent to showing that

$$(x^-_{p,q} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q})})(x^-_{p,q} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q})+1})w = 0.$$

But this follows by Lemma 2.4.1 since

$$(x^-_{p,q} \otimes 1)^{(\nu+2\lambda_0+2\gamma_0)(h_{p,q})+3}w = 0 \text{ and } (x^-_{p,q} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q})+2})w = 0$$

by (2.3.1) and (3.1.1) in $N(\nu, \lambda, \gamma)$, respectively.

Next suppose that $\alpha = \alpha_p + \beta$ for $\beta$ in the span of $\{\alpha_\ell : p < \ell < q\}$. Note that if $\beta \neq 0$, then

$$c_{\alpha_p+\beta} = (\nu + \lambda_0 + \gamma_0)(h_{\alpha_p+\beta}) = c_{\alpha_p} + a_\beta.$$

Then since $x^-_{\alpha_p+\beta} = [x^-_\beta, x^-_p]$, (6.2.1) is equivalent to showing that

$$[x^-_\beta \otimes t^{a_\beta}, x^-_p \otimes t^{c_{\alpha_p}}](x^-_{p,q} \otimes t^{b_{\alpha_{p,q}}})w = 0.$$

Since $[x^-_\beta, x^-_{p,q}] = 0$, (6.2.2) implies that it suffices to show that

$$(x^-_p \otimes t^{c_{\alpha_p}})(x^-_{p,q} \otimes t^{b_{\alpha_{p,q}}})w = 0. \tag{6.2.3}$$

We compute that

$$b_{\alpha_{p,q}} = (\nu + \lambda_0 + \gamma_0)(h_{p,q}) + 1 = c_{\alpha_p} + a_{\alpha_{p+1,q}}$$

Since $x^-_{p,q} = [x^-_{p+1,q}, x^-_p]$, (6.2.2) implies that the left hand side of (6.2.3) is equal to

$$(x^-_p \otimes t^{c_{\alpha_p}})(x^-_{p+1,q} \otimes t^{a_{\alpha_{p+1,q}}})(x^-_p \otimes t^{c_{\alpha_p}})w. \tag{6.2.4}$$

Again applying the fact that $= [x^-_p, x^-_{p+1,q}] = -x^-_{p,q}$, we have that (6.2.4) is equal to

$$-(x^-_{p,q} \otimes t^{b_{\alpha_{p,q}}})(x^-_p \otimes t^{c_{\alpha_p}})w + (x^-_{p+1,q} \otimes t^{a_{\alpha_{p+1,q}}})(x^-_p \otimes t^{c_{\alpha_p}})^2 w.$$

We claim the second term of this sum is zero, which then implies (6.2.3) since $[x^-_{p,q}, x^-_p] = 0$. Indeed, note that

$$(x^-_p \otimes 1)^{(\nu+2\lambda_0+2\gamma_0)(h_p)+2} w = 0 \text{ and } (x^-_p \otimes t^{(\nu+\lambda_0+\gamma_0)(h_p)+1})w = 0$$

by (2.3.1) and (3.1.1) in $N(\nu, \lambda, \gamma)$, respectively. Then Lemma 2.4.1(ii) implies that

$$(x^-_p \otimes t^{c_{\alpha_p}})^2 w = 0,$$

finishing this case.

We omit the case when $\alpha = \beta + \alpha_q$ for $\beta$ in the span of $\{\alpha_\ell : p < \ell < q\}$ as the computation is similar.

As we have shown (6.2.1) for every possible $\alpha \in R^+$, this completes the proof of the proposition. $\square$

# Chapter 7

# Proof of Short Exact Sequences

In this chapter we prove Theorem 3.3.1 and Theorem 3.2.1. We start by proving that the sequences in Theorem 3.3.1 are right exact, and then use a dimension argument to show left exactness. We then note that the dimension argument also implies Theorem 3.2.1.

## 7.1 Right exact sequences

Combining the results of the previous two chapters allows us to establish the right exactness of the sequences in Theorem 3.3.1.

**Proposition 7.1.1.** *Let $(\nu, \lambda, \gamma)$ be a compatible triple.*

*(a) Suppose $\gamma \neq 0$. Then there exists a right exact sequence of $\mathfrak{g}[t]$-modules*

$$(1-\delta_{\lambda_1,0})(1-\delta_{\gamma_1,0})\tau^*_{(\nu+\lambda_0+\gamma_0)(h_{p,q})+1}N(\nu,\ \lambda',\ '\gamma) \xrightarrow{\psi^-} N(\nu,\lambda,\gamma) \xrightarrow{\psi^+} N(\nu,\lambda+\gamma,0) \to 0$$

*where $p = \max \lambda_1$, $q = \min \gamma_1$.*

*(b) Suppose that $\gamma = 0$ and $\nu - \omega_m \in P^+$ for some $m \in [1, n]$. There exists a right exact sequence of $\mathfrak{g}[t]$-modules*

$$K_m(\nu, \lambda) \xrightarrow{\varphi^-} N(\nu, \lambda, 0) \xrightarrow{\varphi^+} N(\nu - \omega_m, \lambda + \omega_m, 0) \to 0.$$

*Proof.* We start with the case when $\gamma \neq 0$. Existence of surjective map $\psi^+$ was established in Proposition 6.1.1, with trivial kernel if $\lambda_1 = 0$ or $\gamma_1 = 0$, and otherwise with kernel generated by $(x^-_{p,q} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q})+1})w_{\nu,\lambda,\gamma}$. In the case when $\lambda_1 \neq 0$ and $\gamma_1 \neq 0$, Proposition 6.2.1 established map $\psi^-$ sending generator of $\tau^*_{(\nu+\lambda_0+\gamma_0)(h_{p,q})+1} N(\nu,\ \lambda',\ '\gamma)$ to $(x^-_{p,q} \otimes t^{(\nu+\lambda_0+\gamma_0)(h_{p,q})+1})w_{\nu,\lambda,\gamma}$, establishing the right exact sequence in (a).

Next, suppose $\gamma = 0$, and let $m \in [1, n]$ with $\nu - \omega_m \in P^+$. Existence of map $\varphi^+$ was established in Proposition 4.1.1, with kernel depending on $m$ and $\lambda$. The map $\varphi^-$ can be constructed using Proposition 5.1.1 as follows:

If $\lambda_1 = 0$, then $\ker \varphi^+$ is trivial by Proposition 4.1.1. But also $K_m(\nu, \lambda) = 0$, and so we let $\varphi^- = 0$.

If $m \in \operatorname{supp} \lambda_1$ with $m \in \{\min \lambda_1, \max \lambda_1\}$, then by Proposition 4.1.1 $\ker \varphi^+$ is generated by $(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w_{\nu,\lambda,0}$. But $K_m(\nu, \lambda) \cong \tau^*_{(\nu+\lambda_0)(h_m)} U_{m,0}(\nu, \lambda)$. Thus we let $\varphi^- := \varphi^-_1$ as defined in Proposition 5.1.1.

If $m \in \operatorname{supp} \lambda_1$ with $\min \lambda_1 < m < \max \lambda_1$, then by Proposition 4.1.1 $\ker \varphi^+$ is generated by $(x^-_m \otimes t^{(\nu+\lambda_0)(h_m)})w_{\nu,\lambda,0}$ and $(x^-_{m_\bullet,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m_\diamond})+1})w_{\nu,\lambda,0}$. But $K_m(\nu, \lambda)$ is equal to $\tau^*_{(\nu+\lambda_0)(h_m)} U_{m,0}(\nu, \lambda) \bigoplus \tau^*_{(\nu+\lambda_0)(h_{m_\bullet,m_\diamond})+1} U_{m,0}(\nu, \lambda + \omega_m)$. Thus we let $\varphi^- := \varphi^-_1 \oplus \varphi^-_2$ as defined in Proposition 5.1.1.

If $m \notin \operatorname{supp} \lambda_1$ with $m > \max \lambda_1$, then by Proposition 4.1.1 $\ker \varphi^+$ is generated by $(x^-_{m_\bullet,m} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m})})w_{\nu,\lambda,0}$. But $K_m(\nu, \lambda) \cong \tau^*_{(\nu+\lambda_0)(h_{m_\bullet,m})} U_{m,-1}(\nu, \lambda + \omega_m)$. Thus we let $\varphi^- := \varphi^-_{\mathrm{L}}$ as defined in Proposition 5.1.1.

If $m \notin \operatorname{supp}\lambda_1$ with $m < \min\lambda_1$, then by Proposition 4.1.1 $\ker\varphi^+$ is generated by $(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})w_{\nu,\lambda,0}$. But $K_m(\nu,\lambda) \cong \tau^*_{(\nu+\lambda_0)(h_{m,m_\diamond})}U_{m,1}(\nu,\lambda+\omega_m)$. Thus we let $\varphi^- := \varphi^-_{\mathrm{R}}$ as defined in Proposition 5.1.1.

Lastly if $m \notin \operatorname{supp}\lambda_1$ with $\min\lambda_1 < m < \max\lambda_1$, then by Proposition 4.1.1 $\ker\varphi^+$ is generated by $(x^-_{m_\bullet,m} \otimes t^{(\nu+\lambda_0)(h_{m_\bullet,m})})w_{\nu,\lambda,0}$ and $(x^-_{m,m_\diamond} \otimes t^{(\nu+\lambda_0)(h_{m,m_\diamond})})w_{\nu,\lambda,0}$. But $K_m(\nu,\lambda)$ is equal to $\tau^*_{(\nu+\lambda_0)(h_{m_\bullet,m})}U_{m,-1}(\nu,\lambda+\omega_m) \bigoplus \tau^*_{(\nu+\lambda_0)(h_{m,m_\diamond})}U_{m,1}(\nu,\lambda+\omega_m)$. Thus we let $\varphi^- := \varphi^-_{\mathrm{L}} \oplus \varphi^-_{\mathrm{R}}$ as defined in Proposition 5.1.1.

Thus in all cases we have defined map $\varphi^-$, which by construction gives the right exact sequence in (b).

$\square$

The proposition above immediately implies the following corollary.

**Corollary 7.1.1.** *Let $(\nu,\lambda,\gamma)$ be a compatible triple. If $\gamma \neq 0$ then*

$$\dim N(\nu,\lambda,\gamma) \le \dim N(\nu,\lambda+\gamma,0) + (1-\delta_{\lambda_1,0})(1-\delta_{\gamma_1,0})\dim N(\nu,\ \lambda',\ {}'\gamma). \tag{7.1.1}$$

*If $\gamma = 0$ and $\nu - \omega_m \in P^+$, then*

$$\dim N(\nu,\lambda,\gamma) \le \dim N(\nu-\omega_m,\lambda+\omega_m,0) + \dim K_m(\nu,\lambda). \tag{7.1.2}$$

To prove these sequences are left exact, it suffices to show that that equality holds in the above. We do this by inducting on the partial order on compatible triples of dominant integral weights introduced in section 3.2.

## 7.2 Some Useful Propositions and Lemmas

The following is a consequence of Theorem 4 in [2] and Theorem 1 and Proposition 1.11 in [3], and it will be used in our proof of the main theorem.

**Proposition 7.2.1.** *Let* $\nu, \lambda \in P^+(1) \setminus \{0\}$ *such that* $\max \nu < \min \lambda$. *Then*

$$\dim D(2,\nu) \dim D(2,\lambda) = \dim D(2,\nu+\lambda) + \dim D(2,\ \nu'\ ) \dim D(2,\ '\lambda\ ). \tag{7.2.1}$$

*In addition, if* $\max \nu < m < \min \lambda$, *we have the following equations:*

$$\begin{aligned} \dim D(2,\nu+\omega_m+\lambda) \dim D(2,\omega_m) &= \dim D(2,\nu) \dim D(2,2\omega_m) \dim D(2,\lambda) \\ &\quad + \dim D(2,\nu+\omega_{m-1}) \dim D(2,\omega_{m+1}+\lambda) \end{aligned} \tag{7.2.2}$$

$$\begin{aligned} \dim D(2,\nu+\lambda) \dim D(2,\omega_m) &= \dim D(2,\nu+\omega_m) \dim D(2,\lambda) \\ &\quad + \dim D(2,\ \nu'\ ) \dim D(2,\omega_{m+1}+\lambda) \end{aligned} \tag{7.2.3}$$

In addition, the following result from [1] will be used in the proof of left exactness.

**Lemma 7.2.1.** *Let* $\lambda \in P^+$ *with* $|\operatorname{supp} \lambda_1| \geq 2$, *and let* $p = \min \lambda_1$. *Then*

$$\begin{aligned} \dim N(0,\lambda,0) &= \dim V(\omega_p) \dim N(0,\lambda-\omega_p,0) \\ &\quad - \dim V(\omega_{p-1}) \dim N(0, 2\lambda_0 + {}'\lambda_1(p+1,n), 0) \end{aligned}$$

*Proof.* By Proposition 2.5(i)(b) in [1] there exists a short exact sequence of $\mathfrak{g}[t]$-modules

$$0 \to \tau^*_{\lambda_0(h_{p,p_\diamond})+1} N(\omega_{p-1}, 2\lambda_0 + {}'\lambda_1(p+1,n), 0) \to N(\omega_p, \lambda-\omega_p, 0) \to N(0,\lambda,0) \to 0.$$

Therefore

$$\dim N(0,\lambda,0) = \dim N(\omega_p, \lambda-\omega_p, 0) - \dim N(\omega_{p-1}, 2\lambda_0 + {}'\lambda_1(p+1,n), 0).$$

The lemma then follows by noting that $(\omega_p, \lambda-\omega_p)$ and $(\omega_{p-1}, 2\lambda_0 + {}'\lambda_1(p+1,n))$ are compatible pairs and then applying Proposition 3.1.1(i). $\square$

## 7.3 Proof of Dimensions

We now prove the following proposition, which will complete the proof of our short exact sequences.

**Proposition 7.3.1.** *The inequalities in Corollary 7.1.1 are all equalities. In particular, for* $(\nu, \lambda, \gamma) \in P^+ \times P^+ \times P^+$ *compatible, we have that*

$$\dim N(\nu, \lambda, \gamma) = \dim N(\nu, 0, 0) \dim N(0, \lambda, 0) \dim N(0, 0, \gamma). \tag{7.3.1}$$

We prove this proposition via an induction on the partial order on compatible triples in $P^+ \times P^+ \times P^+$. We note that induction begins, since for minimal elements $(0, \omega_i, 0)$ with $i \in [0, n]$, we have $\dim N(0, 0, 0) = 1$ and so (7.3.1) follows.

For the inductive step, we will fix compatible triple $(\nu, \lambda, \gamma)$ and suppose that (7.3.1) holds for all compatible triples $(\tilde{\nu}, \tilde{\lambda}, \tilde{\gamma}) < (\nu, \lambda, \gamma)$, as well as equality in (7.1.1) for any $(\tilde{\nu}, \tilde{\lambda}, \tilde{\gamma}) < (\nu, \lambda, \gamma)$ with $\tilde{\gamma} \neq 0$ and (7.1.2) for any $(\tilde{\nu}, \tilde{\lambda}, 0) < (\nu, \lambda, \gamma)$ and $\ell \in [1, n]$ with $\tilde{\nu} - \omega_\ell \in P^+$. Before continuing, we recall by Corollary 3.1.1 that for any $\nu, \lambda, \gamma \in P^+$,

$$\begin{aligned} \dim N(\nu, \lambda, \gamma) &\geq \dim N(\nu, 0, 0) \dim N(0, \lambda, 0) \dim N(0, 0, \gamma) \\ &= \dim N(\nu, 0, 0) \dim D(2, \lambda) \dim D(2, \gamma) \end{aligned} \tag{7.3.2}$$

where the equality follows from (3.1.3).

The rest of the proof proceeds by cases on $(\nu, \lambda, \gamma)$, which constitutes the remaining sections of this chapter.

## 7.4 The case when $\gamma \neq 0$

Assume that $\gamma \neq 0$. There are two possibilities for $(\nu, \lambda, \gamma)$ in this case.

If $\lambda_1, \gamma_1 \neq 0$, then $(\nu, \lambda+\gamma, 0)$ and $(\nu, \lambda', {}'\gamma)$ are both less than $(\nu, \lambda, \gamma)$ in our partial order. Also by (7.1.1) we have that

$$\begin{aligned}\dim N(\nu, \lambda, \gamma) &\leq \dim N(\nu, \lambda+\gamma, 0) + \dim N(\nu,\ \lambda',\ {}'\gamma)\\ &= \dim N(\nu, 0, 0)(\dim N(0, \lambda+\gamma, 0) + \dim N(0, \lambda', 0) \dim N(0, 0, {}'\gamma))\\ &= \dim N(\nu, 0, 0) \dim D(2, \lambda) \dim D(2, \gamma)\end{aligned}$$

where the first equality follows from the inductive hypothesis and the second equality follows from (7.2.1).

On the other hand, by (7.3.2) we have that

$$\dim N(\nu, \lambda, \gamma) \geq \dim N(\nu, 0, 0) \dim N(0, \lambda, 0) \dim N(0, 0, \gamma)$$

and therefore equality holds in (7.3.1) and (7.1.1).

Next suppose that $\gamma \neq 0$ but $\gamma_1 = 0$ or $\lambda_1 = 0$. Then $(\nu, \lambda+\gamma, 0) < (\nu, \lambda, \gamma)$ and $N(\nu, \lambda, \gamma) \cong N(\nu, \lambda+\gamma, 0)$. Hence by the inductive hypothesis,

$$\dim N(\nu, \lambda, \gamma) = \dim N(\nu, \lambda+\gamma, 0) = \dim N(\nu, 0, 0) \dim N(0, \lambda+\gamma, 0).$$

But note that by Proposition 3.1.1(iv),

$$\dim N(0, \lambda+\gamma, 0) = \dim D(2, \lambda+\gamma) = \dim D(2, \lambda) \dim D(2, \gamma)$$

since either $\lambda = 2\lambda_0$ or $\gamma = 2\gamma_0$. Therefore equality holds in (7.3.1) and (7.1.1), completing all cases when $\gamma \neq 0$.

## 7.5 The case when $\gamma = 0$

From now on we shall assume $\gamma = 0$. If $\nu = 0$ then (7.3.1) follows trivially, so from now on we shall also assume that $m \in \operatorname{supp} \nu$ for some $m \in [1, n]$. By Corollary 3.1.1, we have that

$$\begin{aligned} \dim N(\nu, \lambda, 0) &\geq \dim N(\nu, 0, 0) \dim N(0, \lambda, 0) \\ &= \dim N(\nu - \omega_m, 0, 0) \dim N(\omega_m, 0, 0) \dim N(0, \lambda, 0). \end{aligned} \tag{7.5.1}$$

where the equality follows from Proposition 3.1.1(iii). On the other hand, by (7.1.2), we have that

$$\begin{aligned} \dim N(\nu, \lambda, 0) &\leq \dim N(\nu - \omega_m, \lambda + \omega_m, 0) + \dim K_m(\nu, \lambda) \\ &= \dim N(\nu - \omega_m, 0, 0) \dim N(0, \lambda + \omega_m, 0) + \dim K_m(\nu, \lambda) \end{aligned} \tag{7.5.2}$$

where the equality follows by the inductive hypothesis, since $(\nu - \omega_m, \lambda + \omega_m, 0) < (\nu, \lambda, 0)$. We thus wish to show that (7.5.1) is equal to (7.5.2).

The remaining subsections consider all possibilities for $(\nu, \lambda, 0)$ and $m$.

### 7.5.1 The case when $m \in \operatorname{supp} \lambda_1$

Suppose that $m \in \operatorname{supp} \lambda_1$. Then by Proposition 3.1.1(ii) and (iv) we have that

$$\dim N(0, \lambda + \omega_m, 0) = \dim N(0, \lambda - \omega_m, 0) \dim V(2\omega_m). \tag{7.5.3}$$

Additionally, $\dim K_m(\nu, \lambda)$ is equal to

$$\begin{aligned} &(1 - \delta_{m,\min \lambda_1})(1 - \delta_{m,\max \lambda_1}) \dim N(\nu - \omega_m, 2\lambda_0 + 2\omega_m + \lambda_1(1, m-1)', {}'\lambda_1(m+1, n)) \\ &\qquad + \dim N(\nu - \omega_m, 2\lambda_0 + \lambda_1(1, m)', {}'\lambda_1(m, n)). \end{aligned} \tag{7.5.4}$$

Since $(\nu-\omega_m, 2\lambda_0+\lambda_1(1,m)', {}'\lambda_1(m,n)) < (\nu,\lambda,0)$ and, if $\min\lambda_1 < m < \max\lambda_1$, then $(\nu-\omega_m, 2\lambda_0+2\omega_m+\lambda_1(1,m-1)', {}'\lambda_1(m+1,n)) < (\nu,\lambda,0)$, by the inductive hypothesis (7.5.4) is equal to

$$\begin{aligned}(1-\delta_{m,\min\lambda_1})(1-\delta_{m,\max\lambda_1}) \dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0+2\omega_m+\lambda_1(1,m-1)',0)\\ \cdot \dim N(0,0,{}'\lambda_1(m+1,n))\\ + \dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0+\lambda_1(1,m)',0)\dim N(0,0,{}'\lambda_1(m,n)). \quad (7.5.5)\end{aligned}$$

Therefore, by substituting (7.5.3) and (7.5.5) into (7.5.2), we see it is enough to show that (7.5.1) is equal to

$$\begin{aligned}\dim N(\nu-\omega_m,0,0)\dim N(0,\lambda-\omega_m,0)\dim V(2\omega_m)\\ + \dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0+\lambda_1(1,m)',0)\dim N(0,0,{}'\lambda_1(m,n))\\ + (1-\delta_{m,\min\lambda_1})(1-\delta_{m,\max\lambda_1}) \dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0+2\omega_m+\lambda_1(1,m-1)',0)\\ \cdot \dim N(0,0,{}'\lambda_1(m+1,n)). \quad (7.5.6)\end{aligned}$$

**The case when** $m \in \{\min\lambda_1, \max\lambda_1\}$

We will assume that $m=\min\lambda_1$, as the case when $m=\max\lambda_1$ is similar. Then (7.5.6) is equal to

$$\begin{aligned}\dim N(\nu-\omega_m,0,0)\dim N(0,\lambda-\omega_m,0)\dim V(2\omega_m)\\ + \dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0+\lambda_1(1,m)',0)\dim N(0,0,{}'\lambda_1(m,n)). \quad (7.5.7)\end{aligned}$$

By the character theory of $\mathfrak{sl}_{n+1}$, we have that

$$\dim V(2\omega_m) = (\dim V(\omega_m))^2 - \dim V(\omega_{m-1})\dim V(\omega_{m+1}).$$

Therefore (7.5.7) is equal to

$$\dim N(\nu-\omega_m,0,0)\dim N(0,\lambda-\omega_m,0)\big((\dim V(\omega_m))^2-\dim V(\omega_{m-1})\dim V(\omega_{m+1})\big)$$
$$+\dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0+\lambda_1(1,m)',0)\dim N(0,0,{}'\lambda_1(m,n)). \quad (7.5.8)$$

We thus wish to show that (7.5.1) is equal to (7.5.8).

If $|\operatorname{supp}\lambda_1| = 1$, i.e. $\lambda_1 = \omega_m$, an application of Proposition 3.1.1(i) and (iv) shows that

$$\dim N(0,2\lambda_0+\lambda_1(1,m)',0) = \dim N(0,2\lambda_0+\omega_{m-1},0) = \dim N(0,2\lambda_0,0)\dim V(\omega_{m-1}).$$

We also note that

$$N(0,0,{}'\lambda_1(m,n)) = N(0,0,\omega_{m+1}) \cong_{\mathfrak{g}} V(\omega_{m+1}) \quad \text{by Proposition 3.1.1(i).}$$

Thus after substitution and cancellation (7.5.8) is equal to

$$\dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0)(\dim V(\omega_m))^2.$$

But this is equal to (7.5.1) since by Proposition 3.1.1(i) and (iv), we have that

$$\dim N(0,\lambda,0) = \dim V(\omega_m)\dim N(0,2\lambda_0,0).$$

So now assume that $m = \min\lambda_1$ and $|\operatorname{supp}\lambda_1| \geq 2$. By Lemma 7.2.1 we have that (7.5.1) is equal to

$$\dim N(\nu-\omega_m,0,0)(\dim V(\omega_m))^2\dim N(0,\lambda-\omega_m,0)$$
$$-\dim N(\nu-\omega_m,0,0)\dim V(\omega_{m-1})\dim V(\omega_m)\dim N(0,2\lambda_0+{}'\lambda_1(m+1,n),0). \quad (7.5.9)$$

We thus wish to show that (7.5.8) is equal to (7.5.9). By Proposition 3.1.1(i) and (iv) we note that

$$\dim N(0, 2\lambda_0 + \lambda_1(1,m)', 0) \dim N(0, 0, {}'\lambda_1(m,n))$$

$$= \dim N(0, 2\lambda_0 + \omega_{m-1}, 0) \dim N(0, 0, \lambda_1 - \omega_m + \omega_{m+1})$$

$$= \dim V(\omega_{m-1}) \dim N(0, 0, \lambda - \omega_m + \omega_{m+1})$$

Thus after removing the common term of

$$\dim N(\nu - \omega_m, 0, 0)(\dim V(\omega_m))^2 \dim N(0, \lambda - \omega_m, 0)$$

in(7.5.8) and (7.5.9), we see it is sufficient to show that

$$- \dim N(\nu - \omega_m, 0, 0) \dim V(\omega_{m-1}) \dim V(\omega_{m+1}) \dim N(0, \lambda - \omega_m, 0)$$

$$+ \dim N(\nu - \omega_m, 0, 0) \dim V(\omega_{m-1}) \dim N(0, 0, \lambda - \omega_m + \omega_{m+1}) \quad (7.5.10)$$

is equal to

$$- \dim N(\nu - \omega_m, 0, 0) \dim V(\omega_{m-1}) \dim V(\omega_m) \dim N(0, 2\lambda_0 + {}'\lambda_1(m+1, n), 0). \quad (7.5.11)$$

If $|\operatorname{supp} \lambda_1| = 2$, let $\lambda_1 = \omega_m + \omega_p$ for $p > m$. Then after factoring out a common term of $\dim N(\nu - \omega_m, 0, 0) \dim V(\omega_{m-1})$ from (7.5.10) and (7.5.11), we see it suffices to show

$$- \dim V(\omega_{m+1}) \dim N(0, 2\lambda_0 + \omega_p, 0) + \dim N(0, 0, 2\lambda_0 + \omega_{m+1} + \omega_p)$$

$$= - \dim V(\omega_m) \dim N(0, 2\lambda_0 + \omega_{p+1}, 0) \quad (7.5.12)$$

By Proposition 3.1.1(iv) we may factor out a common term of $\dim N(0, 2\lambda_0, 0)$ from (7.5.12), and thus need to prove that

$$- \dim V(\omega_{m+1}) \dim N(0, \omega_p, 0) + \dim N(0, 0, \omega_{m+1} + \omega_p) = - \dim V(\omega_m) \dim N(0, \omega_{p+1}, 0).$$

But Proposition 3.1.1(i) and (ii) and the the character theory of $\mathfrak{sl}_{n+1}$ imply this is true.

Now suppose $|\operatorname{supp}\lambda_1| \geq 3$. Again we wish to show (7.5.10) is equal to (7.5.11). By Proposition 3.1.1(i) and (iii) we have that

$$\dim N(\nu-\omega_m,0,0)\dim V(\omega_{m-1})\dim V(\omega_{m+1})\dim N(0,\lambda-\omega_m,0)$$
$$=\dim N(\nu-\omega_m+\omega_{m-1}+\omega_{m+1},0,0)\dim N(0,\lambda-\omega_m,0). \quad (7.5.13)$$

Since $(\nu-\omega_m+\omega_{m-1}+\omega_{m+1},\lambda-\omega_m,0)<(\nu,\lambda,0)$, the inductive hypothesis implies that (7.5.13) is equal to

$$\dim N(\nu-\omega_m+\omega_{m-1}+\omega_{m+1},\lambda-\omega_m,0). \quad (7.5.14)$$

In addition the inductive hypothesis implies there exists short exact sequence

$$0\to\tau^*_{(\nu+\lambda_0)(h_{m+1,m_\diamond})+1}N(\nu-\omega_m+\omega_{m-1},2\lambda_0+\omega_m,{}'\lambda_1(m+1,n))$$
$$\to N(\nu-\omega_m+\omega_{m-1}+\omega_{m+1},\lambda-\omega_m,0)\to N(\nu-\omega_m+\omega_{m-1},\lambda-\omega_m+\omega_{m+1},0)\to 0.$$

This and (3.1.4) imply that (7.5.14) is equal to

$$\dim N(\nu-\omega_m+\omega_{m-1},\lambda-\omega_m+\omega_{m+1},0)$$
$$+\dim N(\nu-\omega_m+\omega_{m-1},\omega_m,2\lambda_0+{}'\lambda_1(m+1,n)). \quad (7.5.15)$$

Lastly, since $(\nu-\omega_m+\omega_{m-1},\lambda-\omega_m+\omega_{m+1},0)$ and $(\nu-\omega_m+\omega_{m-1},\omega_m,2\lambda_0+{}'\lambda_1(m+1,n))$ are less than $(\nu,\lambda,\gamma)$ in our partial order, the inductive hypothesis and Proposition 3.1.1(i) and (iii) imply that (7.5.15) is equal to

$$\dim N(\nu-\omega_m,0,0)\dim V(\omega_{m-1})\dim N(0,\lambda-\omega_m+\omega_{m+1},0)$$
$$+\dim N(\nu-\omega_m,0,0)\dim V(\omega_{m-1})\dim V(\omega_m)\dim N(0,0,2\lambda_0+{}'\lambda_1(m+1,n)). \quad (7.5.16)$$

Therefore by substituting (7.5.16) for

$$\dim N(\nu - \omega_m, 0, 0) \dim V(\omega_{m-1}) \dim V(\omega_{m+1}) \dim N(0, \lambda - \omega_m, 0)$$

in (7.5.10), we see it is equal to (7.5.11) as desired, finishing the case when $m = \min \lambda_1$.

**The case when** $\min \lambda_1 < m < \max \lambda_1$

Now suppose that $m \in \operatorname{supp} \lambda_1$ with $\min \lambda_1 < m < \max \lambda_1$. Then we wish to show that (7.5.1) is equal to

$$\begin{aligned} &\dim N(\nu - \omega_m, 0, 0) \dim N(0, 2\lambda_0 + \lambda_1(1, m)', 0) \dim N(0, 0, {}'\lambda_1(m, n)) \\ &+ \dim N(\nu - \omega_m, 0, 0) \dim N(0, 2\lambda_0 + 2\omega_m + \lambda_1(1, m-1)', 0) \dim N(0, 0, {}'\lambda_1(m+1, n)) \\ &\qquad + \dim N(\nu - \omega_m, 0, 0) \dim N(0, \lambda - \omega_m, 0) \dim V(2\omega_m). \end{aligned} \quad (7.5.17)$$

Note that by Proposition 3.1.1(iv), every term of (7.5.1) and (7.5.17) is divisible by $\dim N(0, 2\lambda_0, 0)$. Thus by removing a common factor of $\dim N(\nu - \omega_m, 0, 0) \dim N(0, 2\lambda_0, 0)$, we see it is enough to show that

$$\begin{aligned} &\dim N(0, \lambda_1 - \omega_m, 0) \dim V(2\omega_m) + \dim N(0, \lambda_1(1, m)', 0) \dim N(0, 0, {}'\lambda_1(m, n)) \\ &\qquad + \dim N(0, 2\omega_m + \lambda_1(1, m-1)', 0) \dim N(0, 0, {}'\lambda_1(m+1, n)) \end{aligned} \quad (7.5.18)$$

is equal to

$$\dim N(\omega_m, 0, 0) \dim N(0, \lambda_1, 0). \quad (7.5.19)$$

By (7.2.1), $\dim N(0, \lambda_1 - \omega_m, 0)$ is equal to

$$\begin{aligned} &\dim N(0, \lambda_1(1, m-1), 0) \dim N(0, 0, \lambda_1(m+1, n)) \\ &\qquad - \dim N(0, \lambda_1(1, m-1)', 0) \dim N(0, 0, {}'\lambda_1(m+1, n)). \end{aligned} \quad (7.5.20)$$

Since by Proposition 3.1.1(ii) and (iv) we have

$$\dim N(0, 2\omega_m + \lambda_1(1, m-1)', 0) = \dim V(2\omega_m) \dim N(0, \lambda_1(1, m-1)', 0), \tag{7.5.21}$$

substituting (7.5.20) and (7.5.21) into (7.5.18) and cancelling like terms yields the expression

$$\dim V(2\omega_m) \dim N(0, \lambda_1(1, m-1), 0) \dim N(0, 0, \lambda_1(m+1, n)) \\ + \dim N(0, \lambda_1(1, m)', 0) \dim N(0, 0, {}'\lambda_1(m, n)).$$

But this is equal to (7.5.19) by (7.2.2), completing this case.

### 7.5.2 The case when $m \notin \operatorname{supp} \lambda_1$

From now on, we assume that $m \notin \operatorname{supp} \lambda_1$. Then $\dim K_m(\nu, \lambda)$ is equal to

$$[\![m > \min \lambda_1]\!] \dim N(\nu - \omega_m,\ 2\lambda_0 + \lambda_1(1, m-1)',\ \omega_{m+1} + \lambda_1(m+1, n)) \\ + [\![m < \max \lambda_1]\!] \dim N(\nu - \omega_m,\ 2\lambda_0 + \lambda_1(1, m-1) + \omega_{m-1},\ {}'\lambda_1(m+1, n)) \tag{7.5.22}$$

Since $(\nu - \omega_m,\ 2\lambda_0 + \lambda_1(1, m-1)',\ \omega_{m+1} + \lambda_1(m+1, n)) < (\nu, \lambda, \gamma)$ if $m > \min \lambda_1$ and $(\nu - \omega_m,\ 2\lambda_0 + \lambda_1(1, m-1) + \omega_{m-1},\ {}'\lambda_1(m+1, n)) < (\nu, \lambda, \gamma)$ if $m < \max \lambda_1$, by the inductive hypothesis (7.5.22) is equal to

$$[\![m > \min \lambda_1]\!] \dim N(\nu - \omega_m, 0, 0) \dim N(0, 2\lambda_0 + \lambda_1(1, m-1)', 0) \\ \cdot \dim N(0, 0,\ \omega_{m+1} + \lambda_1(m+1, n)) \\ + [\![m < \max \lambda_1]\!] \dim N(\nu - \omega_m, 0, 0) \dim N(0, 2\lambda_0 + \lambda_1(1, m-1) + \omega_{m-1}, 0) \\ \cdot \dim N(0, 0,\ {}'\lambda_1(m+1, n)) \tag{7.5.23}$$

Thus, by substituting (7.5.23) into (7.5.2), we see that it is enough to show that (7.5.1) is equal to

$$\begin{aligned}\dim N(\nu-\omega_m,0,0)\dim N(0,\lambda+\omega_m,0)&\\ +[\![m>\min\lambda_1]\!]\dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0+\lambda_1(1,m-1)',0)&\\ \cdot\dim N(0,0,\,\omega_{m+1}+\lambda_1(m+1,n))&\\ +[\![m<\max\lambda_1]\!]\dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0+\lambda_1(1,m-1)+\omega_{m-1},0)&\\ \cdot\dim N(0,0,\,{}'\lambda_1(m+1,n))&\end{aligned}\tag{7.5.24}$$

If $\lambda_1=0$, then (7.5.24) is equal to

$$\dim N(\nu-\omega_m,0,0)\dim N(0,\lambda+\omega_m,0).$$

But this is equal to (7.5.1) by Proposition 3.1.1(i) and (iv). We proceed to when $\lambda_1\neq 0$.

**The case when $m<\min\lambda_1$ or $m>\max\lambda_1$**

Suppose that $m<\min\lambda_1$. We note that the case when $m>\max\lambda_1$ is similar and omit the computation here. Then (7.5.24) is equal to

$$\begin{aligned}\dim N(\nu-\omega_m,0,0)\dim N(0,2\lambda_0+\omega_{m-1},0)\dim N(0,0,\,{}'\lambda_1)&\\ +\dim N(\nu-\omega_m,0,0)\dim N(0,\lambda+\omega_m,0)&\end{aligned}\tag{7.5.25}$$

An application of Proposition 3.1.1(i) and (iv) implies that

$$\dim N(0,2\lambda_0+\omega_{m-1},0)\dim N(0,0,\,{}'\lambda_1)=\dim V(\omega_{m-1})\dim N(0,0,{}'\lambda)\tag{7.5.26}$$

Substituting (7.5.26) into (7.5.25), we see we wish to show that (7.5.1) is equal to

$$\dim N(\nu - \omega_m, 0, 0) \dim V(\omega_{m-1}) \dim N(0, 0, {}'\lambda) \\ + \dim N(\nu - \omega_m, 0, 0) \dim N(0, \lambda + \omega_m, 0). \quad (7.5.27)$$

But by Lemma 3.1.1(i) and Lemma 7.2.1, $\dim N(\omega_m, 0, 0) \dim N(0, \lambda, 0)$ is equal to

$$\dim N(0, \lambda + \omega_m, 0) + \dim V(\omega_{m-1}) \dim N(0, {}'\lambda).$$

Substituting this into (7.5.1) shows that it equals (7.5.27) as desired.

**The case when** $\min \lambda_1 < m < \max \lambda_1$

Finally, assume that $\min \lambda_1 < m < \max \lambda_1$ with $m \notin \operatorname{supp} \lambda_1$. Then we wish to show that (7.5.1) is equal to

$$\dim N(\nu - \omega_m, 0, 0) \dim N(0, 2\lambda_0 + \lambda_1(1, m-1)', 0) \dim N(0, 0, \omega_{m+1} + \lambda_1(m+1, n)) \\ + \dim N(\nu - \omega_m, 0, 0) \dim N(0, 2\lambda_0 + \lambda_1(1, m-1) + \omega_{m-1}, 0) \dim N(0, 0, {}'\lambda_1(m+1, n)) \\ + \dim N(\nu - \omega_m, 0, 0) \dim N(0, \lambda + \omega_m, 0). \quad (7.5.28)$$

By Proposition 3.1.1(iv), we note that every term of (7.5.1) and (7.5.28) is divisible by $\dim N(0, 2\lambda_0, 0)$. Thus by factoring out a common term of $\dim N(\nu - \omega_m, 0, 0) \dim N(0, 2\lambda_0, 0)$, we see that it suffices to show that

$$\dim N(0, \lambda_1 + \omega_m, 0) + \dim N(0, \lambda_1(1, m-1)', 0) \dim N(0, 0, \omega_{m+1} + \lambda_1(m+1, n)) \\ + \dim N(0, \lambda_1(1, m-1) + \omega_{m-1}, 0) \dim N(0, 0, {}'\lambda_1(m+1, n)) \quad (7.5.29)$$

is equal to

$$\dim N(\omega_m, 0, 0) \dim N(0, \lambda_1, 0). \quad (7.5.30)$$

By (7.2.3), (7.5.30) is equal to

$$\dim N(0, \lambda_1(1, m-1) + \omega_m, 0) \dim N(0, 0, \lambda_1(m+1, n))$$
$$+ \dim N(0, \lambda_1(1, m-1)', 0) \dim N(0, 0, \omega_{m+1} + \lambda_1(m+1, n)).$$

Then by removing common terms, we see it is sufficient to show that

$$\dim N(0, \lambda_1 + \omega_m, 0) + \dim N(0, \lambda_1(1, m-1) + \omega_{m-1}, 0) \dim N(0, 0, {}'\lambda_1(m+1, n))$$

is equal to

$$\dim N(0, \lambda_1(1, m-1) + \omega_m, 0) \dim N(0, 0, \lambda_1(m+1, n)).$$

But this follows from (7.2.1), finishing the proof.

# Bibliography


[1] R. Biswal, V. Chari, P. Shereen, and J. Wand, *Macdonald Polynomials and level two Demazure modules for affine* $\mathfrak{sl}_{n+1}$, J. Algebra **575** (2021), 159–191.

[2] M. Brito and V. Chari, *Tensor products and q-characters of HL-modules and monoidal categorifications*, Journal de l'École polytechnique—Mathématiques **6** (2019), 581–619.

[3] M. Brito, V. Chari, and A. Moura, *Demazure modules of level two and prime representations of quantum affine* $\mathfrak{sl}_{n+1}$, J. Inst. Math. Jussieu **17** (2018), no. 1, 75–105.

[4] V. Chari, *On the fermionic formula and the Kirillov-Reshetikhin conjecture*, Int. Math. Res. Notices **2001** (2001), no. 12, 629–654.

[5] V. Chari and S. Loktev, *Weyl, Demazure and fusion modules for the current algebra of* $sl_{r+1}$, Adv. Math. **207** (2006), no. 2, 928–960.

[6] V. Chari and A. Moura, *The restricted Kirillov–Reshetikhin modules for the current and twisted current algebras*, Comm. Math. Phys. **266** (2006), no. 2, 431–454.

[7] V. Chari and A. Pressley, *Weyl modules for classical and quantum affine algebras*, Representation Theory **211** (2001), 191–223.

[8] V. Chari, P. Shereen, R. Venkatesh, and J. Wand, *A Steinberg type decomposition theorem for higher level Demazure modules*, J. Algebra **455** (2016), 314–46.

[9] V. Chari and R. Venkatesh, *Demazure modules, fusion products, and Q-systems*, Comm. Math. Phys. **333** (2014), 799–830.

[10] M. Demazure, *Une nouvelle formule des caractères*, Bull. Sci. Math, 2e Série **98** (1974), no. 3, 163–172.

[11] V. G. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR **283** (1985), no. 5, 1060–1064.

[12] B. Feigin and S. Loktev, *On generalized Kostka polynomials and quantum Verlinde rule*, Differential Topology, Infinite-Dimensional Lie Algebras, and Applications, Amer. Math. Soc. Trans. Ser. 2 **194** (1999), 61–79.

[13] G. Fourier and P. Littelmann, *Weyl modules, Demazure modules, KR-modules, crystals, fusion products, and limit constructions*, Adv. Math. **211** (2007), no. 2, 566–93.

[14] H. Garland, *The arithmetic theory of loop algebras*, J. Algebra **53** (1978), 480–551.

[15] M. Jimbo, *A q-difference analogue of* $U(\mathfrak{g})$ *and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63–69.

[16] K. Naoi, *Weyl modules, Demazure modules and finite crystals for non-simply laced type*, Adv. Math. **229** (2012), no. 2, 875–934.

[17] J. Wand, *Demazure flags of local Weyl modules*, Ph.D. thesis, UC Riverside, 2015.