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Tropical techniques in cluster theory and enumerative geometry

Author
Cheung, Man-Wai

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Tropical techniques in cluster theory and enumerative geometry

A dissertation submitted in partial satisfaction of the
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in

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by

Man Wai Cheung

Committee in charge:

Professor Dragos Oprea, Chair
Professor Benjamin Grinstein
Professor Mark Gross
Professor James McKernan
Professor Nolan Wallach

2016

The dissertation of Man Wai Cheung is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

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Chapter 2, in full, is a reprint of the paper "Faithful Realizability of Tropical Curves" as it appears in International Mathematics Research Notices 2015. Man-wai Cheung; Lorenzo Fantini, Jennifer Park, Martin Ulirsch.

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VITA

2008	B. S. in Mathematics *Hons*, Chinese University of Hong Kong
2008-2010	M. Sc. in Mathematics, Hong Kong University of Science and Technology
2010-20013	Graduate Teaching Assistant, University of California, San Diego
2016	Ph. D. in Mathematics, University of California, San Diego

PUBLICATIONS

Man Wai Cheung, Mark Gross, Greg Muller, Gregg Musiker, Dylan Rupel, Salvature Stella, Harold Williams, 'The Greedy Basis equals the Theta Basis", arXiv 1508.01404.

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ABSTRACT OF THE DISSERTATION

Tropical techniques in cluster theory and enumerative geometry

by

Man Wai Cheung

Doctor of Philosophy in Mathematics

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Professor Dragos Oprea, Chair

There are three parts in this thesis. First, we generalize the class of tropical curves from trivalent to 3-colorable which can be realized as the tropicalization of an algebraic curve whose non-archimedean skeleton is faithfully represented by Γ.

Second, we prove the equality of two canonical bases of a rank 2 cluster algebra, the greedy basis of Lee-Li-Zelevinsky and the theta basis of Gross-Hacking-Keel-Kontsevich.

Third, we link up scattering diagrams $\mathfrak{D}$ with quiver representations of corresponding quivers Q. We define a notion of good crossing of broken lines γ on $\mathfrak{D}$. Then we show if γ has good crossing over $\mathfrak{D}$, then it goes in the opposite direction of the Auslander-Reiten quiver of Q. Then we give a stratification of quiver representations by the bendings of γ.

Chapter 1

Introduction

There are two distinct directions in this thesis. The first one is about toric degenerations and tropical curves. We will introduce it in Section 1.1. The second one is about an attempt to make a connection between two areas, mirror symmetry and cluster algebras. Motivations and major results will be discussed in Section 1.2.

1.1 Faithful realizability of tropical curves

Curve counting has always been one of the major questions in algebraic geometry. Mikhalkin in [Mik05] gave a 'tropical' formula for the number of curves of genus g on a toric surface X passing through some number of points. The main idea is to provide a one-to-one correspondence between algebraic and tropical curves. Later, Nishinou and Siebert [NS06] developed such a correspondence theorem for rational curves in higher dimensional toric varieties. By using techniques from log geometry, they constructed a toric degeneration of the ambient toric variety controlled by the parameterized tropical curve.

Fantini, Park, Ulirsch and I [CFPU14] generalize the work of Nishinou-Siebert to get a larger class of tropical curves that can be realized as the tropicalization of algebraic curves whose non-archimedean skeleton is faithfully represented by the curves. One new thing about our approach is we consider tropical curves with valency greater than three. We have pushed the condition from trivalent to

3-colorable.

Let N a finitely generated free abelian group and set $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $M = Hom(N, \mathbb{Z})$. Let K be a nontrivially valued non-archimedean analytic field, and denote by R its valuation ring, K^{alg} a fixed algebraic closure of K. Let T be the split algebraic torus $\operatorname{Spec} K[M]$, and let T^{an} be the non-archimedean analytic space associated to T in the sense of [Ber90]. Following [Gub07], [Gub13], and [EKL06], one can define a continuous tropicalization map $\mathrm{trop} : T^{an} \longrightarrow N_{\mathbb{R}}$. Given a curve C in T, its *tropicalization* $\mathrm{Trop}(C)$ is the subset of $N_{\mathbb{R}}$ defined by $\mathrm{Trop}(C) := \mathrm{trop}(C^{an})$. Let C be a smooth, complete, and connected curve over K. Berkovich in [Ber90] associates to every semistable R-model $\mathcal{C}$ of C, i.e. a flat and proper R-scheme $\mathcal{C}$ with generic fiber C and nodal special fiber $\mathcal{C}_s$, a subset $\Sigma_{\mathcal{C}}$ of C^{an}, called a *skeleton*, and shows that it is a deformation retract of C^{an}. By [DM69], a semistable model $\mathcal{C}$ of C always exists. As an abstract graph, the skeleton Σ_C is the dual graph of the special fiber $\mathcal{C}_s$ of $\mathcal{C}$, and it can be naturally endowed with the structure of a metric graph. Given a finite set V of K^{alg}-points of C, we can associate an enlarged skeleton $\Sigma_{\mathcal{C},V}$ which contains $\Sigma_{\mathcal{C}}$ and has new edges of infinite length corresponding to the points of V

Starting with a non-superabundant, smooth and 3-colorable tropical curve $\Gamma \subset N_{\mathbb{R}}$, we construct a cone in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ by putting Γ in height one. Then we obtain a toric scheme $\mathfrak{X}$. Next, we define a suitable nodal curve C_0 in the special fiber of $\mathfrak{X}$ whose dual graph is the skeleton of Γ. By applying log smooth deformation theory, we lift the nodal curve C_0 to a proper, flat, semistable family of curves $\mathcal{C} \subset \mathfrak{X}$ with special fiber C_0. The generic fiber C of $\mathcal{C}$ will then be a smooth complete curve in the generic fiber X of $\mathfrak{X}$. We verify that our construction satisfies the properties we asserted. Our crucial insight is to give a combinatorial interpretation to the homomorphism of cohomology groups on C_0 controlling the logarithmic deformation theory. In particular, we found that for a non-superabundant tropical curve, this homomorphism is surjective and the deformation is unobstructed. More precisely, we prove

Theorem 1.1.1 (2.1.1 in Section 2)**.** *Let Γ be a non-superabundant, smooth and 3-colorable tropical curve with rational edge lengths. Then there exists a finite*

extension $K = \mathbb{C}((t^{1/\ell}))$ *of* $\mathbb{C}((t))$ *with valuation ring* R*, a toric scheme* $\mathfrak{X}$ *over* R *with big torus* T*, a complete smooth curve* C *over* K*, a semistable* R*-model* $\mathcal{C}$ *of* C *together with an embedding of* $\mathcal{C}$ *into* $\mathfrak{X}$*, and a finite set* $V \subseteq C(K^{alg})$ *of marked points, such that*

$$\mathrm{Trop}(C \cap T) = \Gamma$$

and the tropicalization is totally faithful with respect to $\Sigma_{\mathcal{C},V}$*.*

Combining the above theorem with a result of [CDMY14], we also prove

Theorem 1.1.2 (2.1.2 in Section 2)**.** *For every metric graph* G *with rational edge lengths there exists a tropical curve* Γ *in* $\mathbb{R}^n$*, where*

$$n = \max\left\{3, \max\{\deg v - 1 | v \in E(G)\}\right\} ,$$

a finite extension $K = \mathbb{C}((t^{1/\ell}))$ *of* $\mathbb{C}((t))$ *with valuation ring* R*, a toric scheme* $\mathfrak{X}$ *over* R *with big torus* $T = \mathbb{G}_m^n$*, a complete smooth curve* C *over* K*, a semistable* R*-model* $\mathcal{C}$ *of* C *together with an embedding of* $\mathcal{C}$ *into* $\mathfrak{X}$*, and a finite set* $V \subseteq C\big(K^{alg}\big)$ *of marked points, such that*

$$\mathrm{Trop}(C \cap T) = \Gamma ,$$

the skeleton $\Sigma_{\mathcal{C}}$ *is equal to* G*, and the tropicalization is totally faithful with respect to* $\Sigma_{\mathcal{C},V}$*.*

1.2 Cluster algebra and scattering diagrams

Fomin and Zelekinsky set up the theory of cluster algebras in 2000 in order to understand total positivity in algebraic groups and canonical bases in quantum groups. Roughly speaking, it is a subring of a field of rational functions. To define a cluster algebra, instead of knowing all the generators at the beginning, we start with initial data called an *initial seed* which includes some cluster variables. Then there is a procedure called mutation to generate more seeds. A cluster algebra would be defined to be the subring generated by the cluster variables in all the seeds.

In a few years, the theory developed rapidly and npw links with many other areas, e.g., Poisson geometry, integrable systems, higher Teichmüller spaces, algebraic geometry, and quiver representations. Later Fock and Gonchenov introduced a geometric point of view in [FG]. They introduce the $\mathcal{A}$ and $\mathcal{X}$ varieties which can be obtained by gluing 'seed tori' by birational map called cluster transformations.

From another world in mathematics, namely mirror symmetry, scattering diagram, theta functions and broken lines are introduced to understand if there exist certain duality between spaces. 2 dimensional scattering diagram were introduced by Kontsevich and Soibelman in [KS06] to study K3 surfaces. Leter Gross and Siebert [GS11] consider general scattering diagram to describe toric degenerations of Calabi-Yau varieties in order to construct mirror pairs. On the other hand, the notion of broken lines is developed by Gross in [Gro10] to understand Landau-Ginzburg mirror symmetry for P^2. Then Siebert, Carl and Pauperla [CPS10] made use of broken lines to describe regular functions in the context of [GPS10], and in particular to construct Landau-Ginzburg mirrors to varieties with effective anti-canonical bundle. In order to construct mirrors to log Calabi-Yau surfaces with maximal boundary with similar ideas, theta functions are introduced by Gross, Hacking and Keel in [GHK11]. Suggested by Abouzaid, a formula for multiplication of theta functions is given in [GHK11] in terms of trees of broken lines analogous to the formula using tropical Morse trees.

The discovery of [GHK15] and [GHKK14] reveals that there is a strong connection between cluster algebras and scattering diagrams. In particular, we can view each chamber in the scattering diagram as one of the seeds in the cluster algebras. The chambers can then be viewed as the 'seed tori' which 'glued' along the walls in the scattering diagrams. Then theta functions can be viewed as cluster variables. This can then united with the idea from Fock and Goncharov. The set of theta functions are proposed to be a canonical basis for cluster algebras. There are many other proposed bases as well, e.g. [GLS12], [Kel14], [DT11], [MSW13]. Together with Gross, Muller, Musiker, Rupel, Stella and Williams, we have proved that the theta basis in rank 2 agrees with the greedy basis. The result is contained in Chapter 5.

A natural question to ask is if there is any deep, intrinsic relation between the two subjects. With insight from Caldero and Chapoton in [CC06], theta functions are related to quiver Grassmannians. As broken lines are used to define theta functions, we propose each broken line corresponds to a family of subrepresentations of an ambient quiver representation. More specifically, the initial slope of the broken line tells us which ambient representation D we are working on. On the other hand the final slope of the broken line gives one of the subrepresentations E of D. So what does the bendings from the initial to the finial slopes of the broken lines mean?

By employing the machinery of motivic Hall algebras developed by Joyce [Joy07] and Hall algebra scatterings diagram introduced by Bridgeland [Bri15], we can convert the usual wall crossing to be conjugation by Hall algebra elements. In this way, each bending of the broken lines tells us stratas of quiver Grassmanians. Then the bendings of the broken lines actually describe the filtration of the quiver subrepresentation E. We will then be able to give a stratification of the quiver Grassmannian. Details of the stratification will be stated in Section 7.4.

Our calculation for each strata involves not only the tools from Hall algebra but also a closer understanding of quiver representations. More specifically, we need to understand the Hom and Ext groups between indecomposable quiver representations. This connects with the notion of Auslander-Reiten quivers which records maps between indecomposables. We have stated some useful properties of these concepts in Section 3.3.3. Furthermore, we have discovered scattering diagrams are Auslander-Reiten quivers in certain sense. The result is stated in Section 7.2. With all the machinery we have developed, we will be able to give an alternative proof of the Caldero-Chapoton formula and give a richer meaning to the equations.

1.3 Organization of this thesis

Chapter 2 contains the paper 'Faithful realizability of tropical curves'.

Chapter 3 gives the basic definition for cluster algebras and quiver representa-

tions. In the later chapter, we will relate scattering diagrams with Auslander-Reiten quivers. Thus, we also give definition of Auslander-Reiten (AR) quivers and related ideas in this chapter. Furthermore, in the computation of the statra of the quiver Grassmanian, we have used heavily several properties of AR quivers which are stated and proved in the last section of Chapter 3.

Chapter 4 gives the definitions for scattering diagrams, broken lines and theta functions. The association of scattering diagrams to cluster algebras by [GHKK14] is also indicated in this chapter. For the readability of the setup, we have simplified the definition of scattering diagram in the beginning. However, a more general set up for rank 2 cluster algebras is carefully computed in Section 4.2.2. This section also serves as an introduction to Chapter 5.

Chapter 5 states the paper 'The Greedy Basis is Theta Basis'. The major result of this paper is to show that greedy basis is the same as the theta basis in rank 2.

Chapter 6 gives an introduction to Hall algebra. We will also go over the definition for stability condition and then define the Hall algebra scattering diagram by [Bri15]. Then we will define the integration map to indicate how to pass from the Hall algebra scattering diagrams to ordinarily scattering diagrams. We further define a notion for broken lines in the Hall algebra context. At last, we state some properties in the Hall algebra scattering diagrams and broken lines for the use of the next chapter.

Chapter 7 contains the last results of this thesis. We will first introduce the relation between theta functions and quiver grassmannians. Then we will 'visualize' AR quiver on scattering diagram in Section 7.2. Next we will give an alternative proof of the Caldero-Chapoton formula by using the Hall algebra set up in Section 7.3. Afterward, we will describe how the bendings of broken lines give stratification of quiver representations. We will be able to tell the stratas explicitly. Finally in Section 7.4.4, we can tell the filtration of quiver representations is Harder-Narasimhan filtration in rank 2.

As indicated in the introduction, the main goal of the thesis is connecting several areas together. Therefore, there are several conflicts of notations between

different papers. We have tried to state the major confusion and the choices we have made in the Appendix. The reader are highly recommended to consult the Appendix while reading this thesis.

Chapter 2

Faithful realizability of tropical curves

2.1 Introduction

Let N be a finitely generated free abelian group, write $M = \mathrm{Hom}(N, \mathbb{Z})$ for its dual, and set $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Let K be a non-archimedean field, let T be the split algebraic torus $\operatorname{Spec} K[M]$, and let T^{an} be the non-archimedean analytic space associated to T in the sense of [Ber90]. Following [Gub07], [Gub13], and [EKL06], one can define a continuous tropicalization map $\mathrm{trop} : T^{an} \longrightarrow N_{\mathbb{R}}$. Given a curve C in T, its *tropicalization* $\mathrm{Trop}(C)$ is the subset of $N_{\mathbb{R}}$ defined by $\mathrm{Trop}(C) := \mathrm{trop}(C^{an})$. By the Bieri-Groves Theorem [BG84, Theorem A] and [EKL06, Theorem 2.2.3] the set $\mathrm{Trop}(C)$ can be canonically endowed with the structure of a 1-dimensional rational polyhedral complex. By using the lattice length on $N_{\mathbb{R}}$ with respect to N, $\mathrm{Trop}(C)$ can also be seen as a metric graph, with some non-compact edges which have infinite length. Moreover, we can associate weights to the edges of $\mathrm{Trop}(C)$ which satisfy a natural *balancing condition*.

Abstracting from these properties and following [Mik05], we define a *tropical curve* as a balanced weighted 1-dimensional rational polyhedral complex; see Definition 2.2.1 for a precise definition. It is then natural to ask which tropical curves can be realized as the tropicalization of an algebraic curve in T; this is

known as the problem of *realizability* of tropical curves. For the basics of tropical geometry we refer the reader to [Mik05], [Gub13] and [MS15].

Let us now assume that K is nontrivially valued, and denote by R its valuation ring. Let C be a smooth, complete, and connected curve over K. While the underlying topological space of the non-archimedean analytic curve C^{an} is an infinite graph, Berkovich shows in [Ber90, Section 4.3] that it has the homotopy type of a finite graph. More precisely, he associates to every semistable R-model $\mathcal{C}$ of C, i.e. a flat and proper R-scheme $\mathcal{C}$ with generic fiber C and nodal special fiber $\mathcal{C}_s$, a subset $\Sigma_{\mathcal{C}}$ of C^{an}, called a *skeleton*, and shows that it is a deformation retract of C^{an}. Note that a semistable model $\mathcal{C}$ of C always exists, possibly after a finite separable base change, by the semistable reduction theorem [DM69]. As an abstract graph, the skeleton Σ_C is the dual graph of the special fiber $\mathcal{C}_s$ of $\mathcal{C}$, and it can be naturally endowed with the structure of a metric graph.

Let K^{alg} denote a fixed algebraic closure of K. Given a finite set V of K^{alg}-points of C, we can associate an enlarged skeleton $\Sigma_{\mathcal{C},V}$ which contains $\Sigma_{\mathcal{C}}$ and has new edges of infinite length corresponding to the points of V. Again, there is an embedding of this enlarged skeleton as a deformation retract $\Sigma_{\mathcal{C},V}$ of C^{an}. We refer the reader to [BPR13, Section 3] for the details of this construction.

Given an embedding of C into a toric variety X with big torus T, we say that the corresponding tropicalization is *faithful* with respect to a skeleton $\Sigma_{\mathcal{C},V}$ if trop induces an isometric homeomorphism from $\Sigma_{\mathcal{C},V} \cap T^{an}$ onto its image in $\mathrm{Trop}(C \cap T)$. In addition, if $\Sigma_{\mathcal{C},V} \cap T^{an}$ surjects onto $\mathrm{Trop}(C \cap T)$, we say that the tropicalization is *totally faithful* with respect to $\Sigma_{\mathcal{C},V}$. Examples in [BPR11, Section 2.5] show that many tropicalizations are not faithful. In this article we study the following refinement of the question of realizability:

Question (Faithful realizability). Given a tropical curve $\Gamma \subseteq N_{\mathbb{R}}$, can we find a smooth, complete and connected curve C over K, a skeleton $\Sigma_{\mathcal{C},V}$ of C^{an}, and an embedding of C into a toric variety X with dense torus T such that

$$\mathrm{Trop}(C \cap T) = \Gamma$$

and the tropicalization is totally faithful with respect to $\Sigma_{\mathcal{C},V}$?

This question is also related to the problem of faithful tropicalization investigated in [BPR11] for curves and in [GRW14] in higher dimension.

We give a positive answer to both parts of the question of faithful realizability, working over a finite extension $\mathbb{C}((t^{1/\ell}))$ of $\mathbb{C}((t))$, when $\Gamma \subseteq N_{\mathbb{R}}$ is a tropical curve with rational edge lengths and fulfills the following three conditions:

- Γ is *non-superabundant* (Definition 2.4.1).
- Γ is *smooth* (Definition 2.2.2).
- Γ is *3-colorable* (Definition 2.2.3).

Non-superabundancy for tropical curves is a natural genericity condition, originally introduced in [Mik05], roughly saying that the deformation space of Γ as an embedded metric graph in $N_{\mathbb{R}}$ has the expected dimension (see the paragraph after Definition 2.4.1); in our situation we adopt the point of view of [Kat12]. The notion of 3-colorability is a mild combinatorial condition which is always satisfied by trivalent tropical curves and tropical curves of genus at most three. More precisely we prove the following result:

Theorem 2.1.1. Let Γ be a non-superabundant, smooth and 3-colorable tropical curve with rational edge lengths. Then there exists

- a finite extension $K = \mathbb{C}((t^{1/\ell}))$ of $\mathbb{C}((t))$ with valuation ring R,
- a toric scheme $\mathfrak{X}$ over R with big torus T,
- a complete smooth curve C over K,
- a semistable R-model $\mathcal{C}$ of C together with an embedding of $\mathcal{C}$ into $\mathfrak{X}$, and
- a finite set $V \subseteq C(K^{alg})$ of marked points,

such that

$$\mathrm{Trop}(C \cap T) = \Gamma$$

and the tropicalization is totally faithful with respect to $\Sigma_{\mathcal{C},V}$.

Now we can combine Theorem 2.1.1 with a result of [CDMY14], which states the following: Given a metric graph G with rational edge lengths, there is a non-superabundant, smooth and 3-colorable tropical curve Γ in $\mathbb{R}^n$ whose skeleton is G. We will recall their result and adjust it to our needs in Theorem 2.5.2. This leads to the following theorem:

Theorem 2.1.2. For every metric graph G with rational edge lengths there exists

- a tropical curve Γ in $\mathbb{R}^n$, where
$$n = \max\left\{3, \max\{\deg v - 1 | v \in E(G)\}\right\} ,$$
- a finite extension $K = \mathbb{C}((t^{1/\ell}))$ of $\mathbb{C}((t))$ with valuation ring R,
- a toric scheme $\mathfrak{X}$ over R with big torus $T = \mathbb{G}_m^n$,
- a complete smooth curve C over K,
- a semistable R-model $\mathcal{C}$ of C together with an embedding of $\mathcal{C}$ into $\mathfrak{X}$, and
- a finite set $V \subseteq C\big(K^{alg}\big)$ of marked points,

such that
$$\mathrm{Trop}(C \cap T) = \Gamma ,$$
the skeleton $\Sigma_{\mathcal{C}}$ is equal to G, and the tropicalization is totally faithful with respect to $\Sigma_{\mathcal{C},V}$.

In particular, if G has no univalent vertices then the skeleton $\Sigma_{\mathcal{C}}$ in Theorem 2.1.2 is the minimal skeleton of C^{an}, since each proper subgraph of G has homotopy type different than the homotopy type of C^{an}. The methods of [CDMY14] allow us to consider more general graphs G that may have infinite rays. Our proof applies to this situation, giving us the same data as in Theorem 2.1.2 together with an additional subset $V' \subseteq V$ of marked points such that $G = \Sigma_{\mathcal{C},V'}$.

Our proof of Theorem 2.1.1 generalizes the methods developed in [NS06] and extended in [Nis15a]. We now outline the steps of the proof: Let $\Gamma \subset N_{\mathbb{R}}$ be a non-superabundant, smooth and 3-colorable tropical curve with rational edge lengths. Choose a finite extension $K = \mathbb{C}((t^{1/\ell}))$ of $\mathbb{C}((t))$ such that all vertices of

Γ have coordinates in the value group of K. Define Δ to be the fan in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ obtained by putting a copy of Γ in the height one part $N_{\mathbb{R}} \times \{1\}$ of $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ and taking cones over all edges and vertices of Γ. The fan Δ defines a toric scheme $\mathfrak{X} = \mathfrak{X}_{\Delta}$ over $R = \mathbb{C}[[t^{\frac{1}{\ell}}]]$.

(I) In Section 2.2 we define a suitable nodal curve C_0 in the special fiber $\mathfrak{X}_s$ of $\mathfrak{X}$, whose dual graph is the skeleton of Γ.

(II) The special fiber $\mathfrak{X}_s$ is log smooth over the standard log point $O_0 = (\operatorname{Spec} \mathbb{C}, \mathbb{N})$; and the curve C_0, endowed with the log structure induced from $\mathfrak{X}_s$, is log smooth over O_0. These observations allow us to apply log smooth deformation theory in Section 2.3 and Section 2.4 in order to show that we can lift the nodal curve C_0 to a proper, flat semistable curve $\mathcal{C} \subseteq \mathfrak{X}$ over R with special fiber C_0. The generic fiber C of $\mathcal{C}$ is a smooth complete curve in the generic fiber X of $\mathfrak{X}$, which is a T-toric variety over K. We set $V = C \cap (X - T)$.

(III) Finally, in Section 2.5, we verify that this construction satisfies the properties asserted in Theorem 2.1.1: Lemma 2.5.1 shows that in this case $\operatorname{Trop}(C \cap T) = \Gamma$, and the results of [BPR11, Section 6] allow us to show that $\operatorname{trop}|_{\Sigma_{\mathcal{C},V}} : \Sigma_{\mathcal{C},V} \to \operatorname{Trop}(C)$ is totally faithful.

The crucial technical insight of this paper is contained in Section 2.4, where we give a combinatorial interpretation to the homomorphism of cohomology groups on C_0 controlling the logarithmic deformation theory used in Step (II) of our proof. In particular, we deduce that if Γ is non-superabundant, this homomorphism is surjective and the deformations are unobstructed.

Smoothness and 3-colorability of the tropical curve Γ are only used in Step (I) of our proof. The proof would work equally well with any other condition on Γ that allows us to construct a suitable nodal curve C_0 in $\mathfrak{X}_s$. The classical condition of trivalency used extensively in the literature implies 3-colorability; in particular, Theorem 5.1.7 holds for all smooth non-superabundant trivalent tropical curves.

The question of realizability of tropical curves has initially been studied by Mikhalkin [Mik05] in order to count algebraic curves in toric surfaces. Nishinou and

Siebert [NS06] generalize his results to loopless (hence non-superabundant) trivalent tropical curves in higher dimension, and Tyomkin [Tyo12] further generalizes this to non-superabundant trivalent tropical curves of higher genus. See [Gro11, Chapter 4] for an expository account of the methods of [NS06]. Special conditions ensuring the realizability of some superabundant tropical curves are studied by Speyer in [Spe05], and by Katz in [Kat12]. In the upcoming [Nis15a] Nishinou extends the approach of [NS06] to more general trivalent tropical curves (see also [Nis15b]). Another approach to the problem in the trivalent case has recently been developed by Lang in [Lan15]. Our Theorem 2.1.1 gives realizations for a wide class of tropical curves, including tropical curves of higher valence and higher genus.

2.2 Construction of the special fiber

We begin by recalling the definitions of a tropical curve and stating the combinatorial conditions our tropical curves will have to satisfy.

Definition 2.2.1. *Let Γ be the (non-compact) weighted graph obtained by removing all vertices of valence one from a finite connected weighted graph. We say that Γ is a* tropical curve *in $N_{\mathbb{R}}$ if it is endowed with a closed embedding $\Gamma \subset N_{\mathbb{R}}$ such that every vertex of Γ is contained in $N_{\mathbb{Q}}$, every edge of Γ is contained in a line of of rational slope of $N_{\mathbb{R}}$ and the* balancing condition *is satisfied: for every vertex v of Γ, if $e_1, \dots e_m$ are the edges of Γ adjacent to v, $w_1, \dots w_m \in \mathbb{N}$ are their weights and $\vec{e}_1, \dots, \vec{e}_m \in N$ are the primitive integer vectors from v in the direction of the edges e_j, then we have $\sum_{j=1}^{m} w_j \vec{e}_j = 0$.*

Moreover, a tropical curve $\Gamma \subset N_{\mathbb{R}}$ can be endowed with the structure of metric graph by using the lattice length on $N_{\mathbb{R}}$ with respect to N. That is, the length of an edge between the two vertices v_1 and v_2 is the largest positive real number l such that $v_2 - v_1 = lv$ for some $v \in N$, and the unbounded edges of Γ are then precisely those of infinite length. We then call *skeleton* of Γ the finite metric graph obtained by erasing from Γ all unbounded edges.

The following definition is the one-dimensional version of [MZ14, Definition 1.14]. For a planar tropical curve it also coincides with [Mik05, Definition 2.18].

Definition 2.2.2. *A tropical curve* $\Gamma \subset N_{\mathbb{R}}$ *is said to be* smooth *if all its weights are equal to 1 and, for every vertex* v *of* Γ*, there exists a basis* $\{b_1, \ldots, b_n\}$ *of* N *and a positive integer* $1 \leq d \leq n$ *such that the edges of* Γ *adjacent to* x *are in the directions of* $\left\{b_1, \ldots, b_d, -\sum_{i=1}^{d} b_i\right\}$.

We conclude our list of definitions with a mild combinatorial condition.

Definition 2.2.3. *A tropical curve* $\Gamma \subset N_{\mathbb{R}}$ *is said to be* 3-colorable *if there exists an ordering* $\{v_1, v_2, \ldots\}$ *of the vertices of* Γ *such that for every* i*, the vertex* v_i *is adjacent to less than three of the vertices* $\{v_j\}_{j<i}$.

Remark 2.2.1. A tropical curve Γ is 3-colorable if and only if its coloring number (in the sense of [EH66]) is at most three. It is simple to show that Γ is 3-colorable if and only if it is 2-degenerate in the sense of [LW70], i.e. if and only if every subgraph G of Γ contains a vertex that is adjacent to at most two other vertices of G.

Example 2.2.2. Every trivalent tropical curve is 3-colorable. Indeed, let Γ be a trivalent tropical curve and let m be the number of vertices of Γ. The balancing condition implies that Γ has at least one unbounded edge e; let v_m be the vertex adjacent to e. Then v_m has bounded valence at most two, and after removing it and and all the adjacent edges we can find one new vertex v_{m-1} of valence at most two. By repeating this argument until we have selected all the vertices of Γ we obtain an ordering which satisfies the condition of Definition 2.2.3, therefore Γ is 3-colorable. Observe that also any tropical curve of genus at most three is 3-colorable, since the smallest graph whose vertices have all bounded valence of three or more is a tetrahedron.

Let Γ be a tropical curve in $N_{\mathbb{R}}$. Let $K = \mathbb{C}((t^{1/\ell}))$ be a finite extension of $\mathbb{C}((t))$ such that all the vertices of the underlying graph of Γ lie in $N_{v(K^\times)} = N \otimes_{\mathbb{Z}} v(K^\times)$, where $v(K^\times) = \mathbb{Z}[\frac{1}{\ell}]$ is the value group of K, and let $R = \mathbb{C}[[t^{1/\ell}]]$ be the valuation ring of K. Define a set of cones in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ by putting a copy of Γ into $N_{\mathbb{R}} \times \{1\}$ and taking cones over all edges and vertices of Γ. More precisely, let Δ be the collection of the cones

$$c(F) = \overline{\mathbb{R}_{>0}(F \times \{1\})} \subseteq N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$$

and

$$c(F) \cap (N_{\mathbb{R}} \times \{0\}) \subseteq N_{\mathbb{R}} \times \mathbb{R}_{\geq 0},$$

where F is either an edge or a vertex of Γ. Then Δ is a fan. Indeed, as in the proof of [BGS11, Theorem 3.4] the only non-trivial part is to show that the set $\Delta_0 = \Delta \cap (N_{\mathbb{R}} \times \{0\})$ of the recession cones of Γ is a fan (see [BGS11, Examples 3.1 and 3.9(i)]). Since Γ is one-dimensional, the recession cones of Γ are either rays starting at the origin or the origin itself. Two such rays are either equal or their intersection is the origin and this observation suffices to show that the recession cones form a fan. Moreover, the fan Δ is $v(K^\times)$-admissible in the sense of [Gub13, Definition 7.5]. We set $\mathfrak{X}$ to be the toric scheme over R defined by Δ.

Remark 2.2.3. We could compactify $\mathfrak{X}$ by completing the cone Δ, but prefer not to do so, since we only want to keep the toric strata that are relevant to our construction.

Using the correspondence of [Gub13, 7.9] we can give an explicit description of the toric scheme $\mathfrak{X}$. Its generic fiber $\mathfrak{X}_\eta$ is the toric variety over K associated to the fan Δ_0. Its special fiber $\mathfrak{X}_s$ is reduced by [Gub13, Lemma 7.10], since the valuation v is discrete and the vertices of Γ are in $N_{v(K^\times)}$. The irreducible components of $\mathfrak{X}_s$ are toric varieties over the residue field $\mathbb{C}$, and they correspond bijectively to the vertices of Γ. Whenever two vertices v and w are connected by an edge e, the two components X_v and X_w are glued along the boundary divisor corresponding to e.

If Γ is smooth and v is a vertex of Γ with $\mathrm{val}_\Gamma(v) = d + 1$, from the explicit description of Γ around v of Definition 2.2.2, we deduce that the corresponding component X_v of $\mathfrak{X}_s$ is isomorphic to $P_d \times \mathbb{G}_m^{n-d}$, where $P_d := \mathbb{P}^d \setminus \{\text{orbits of } T \text{ of codimension 2 or higher}\}$, and the boundary divisors of P_d are all isomorphic to $\mathbb{G}_m^{d-1}$.

Example 2.2.4. Consider the admissible cone $\Delta \subseteq \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ obtained as described above by placing at height 1 the tropical curve $\Gamma \subseteq \mathbb{R}^2$ in Figure 2.1.

We obtain a toric scheme $\mathfrak{X}$ over $\mathbb{C}[[t]]$ whose generic fiber X is the toric variety associated to the recession fan $\Delta_0 = \Delta \cap (\mathbb{R}^2 \times \{0\})$ of Γ, which is the fan (Figure 2.2) in $\mathbb{R}^2$:

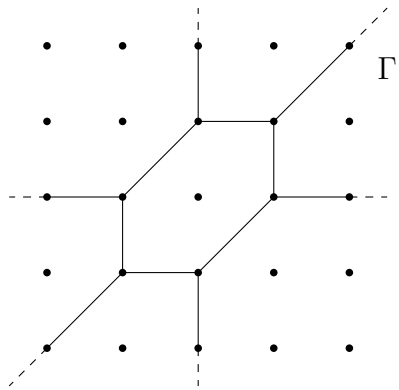

Figure 2.1: The tropical curve Γ

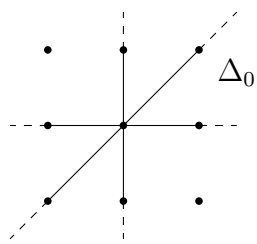

Figure 2.2: Recession fan $\Delta_0 = \Delta \cap (\mathbb{R}^2 \times \{0\})$

Therefore X is the toric surface obtained by blowing up $\mathbb{P}^2_{\mathbb{C}((t))}$ in three points, with the six closed points fixed by the torus removed. The special fiber $\mathfrak{X}_s$ of $\mathfrak{X}$ consists of six copies of $\mathbb{P}^2_{\mathbb{C}}$ without their closed torus invariant points, glued over the one-dimensional toric strata as indicated by their moment polytopes in Figure 2.3.

The following proposition is the main result of this section.

Proposition 2.2.4. *Suppose that Γ is a smooth and 3-colorable tropical curve. Then there is a complete and connected curve $C_0 \subseteq \mathfrak{X}_s$ fulfilling the following properties:*

1. *For each vertex $v \in \Gamma$, let $C_v := C_0 \cap X_v \subseteq \mathfrak{X}_s$; then $C_v \cong \mathbb{P}^1$.*

2. *Each C_v intersects every toric boundary stratum of X_v transversally.*

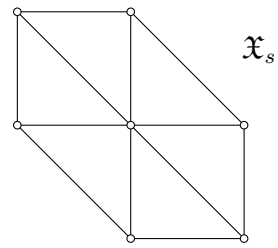

Figure 2.3: Moment polytopes

3. *If v and w are two vertices of Γ connected by an edge, then the components C_v and C_w intersect in a node.*

Proof. Let $\{v_i\}_i$ be an ordering of the vertices of Γ as in Definition 2.2.3. For every i, we inductively define a smooth rational curve C_{v_i} in $P_d \times \{1\} \subseteq X_{v_i} \cong P_d \times \mathbb{G}_m^{n-d}$ subject to the following condition: if v_j is a vertex of Γ adjacent to v_i, for $j < i$, the two curves C_{v_i} and C_{v_j} intersect in a point of $X_{v_i} \cap X_{v_j}$. Such a curve C_{v_i} exists since the condition that we are imposing is the passage through at most two given points. Moreover, we can choose the curves C_{v_i} meeting transversally each boundary stratum of $P_d \times \{1\}$, since a generic line in $\mathbb{P}^d$ intersects any coordinate hyperplane transversally away from the strata of codimension two or higher. Finally, let C_0 be the union of the curves C_{v_i}. $\square$

Example 2.2.5. Consider the tropical curve Γ and the associated toric scheme $\mathfrak{X}$ as in Example 2.2.4. Then the curve C_0 in $\mathfrak{X}_s$ constructed in Proposition 2.2.4 is a loop consisting of six copies of $\mathbb{P}^1_{\mathbb{C}}$, and it can be visualized using tropical lines in the moment polytopes as in Figure 2.4.

Remark 2.2.6. Note that the curve C_0 constructed in Proposition 2.2.4 is a nodal curve, and its dual graph is equal to the graph underlying the skeleton of Γ. We remind the reader that the data of an n-dimensional toric scheme over R is essentially equivalent to the notion of a *toric degeneration*, a toric morphism from a complex toric variety of dimension $n+1$ to $\mathbb{A}^1_{\mathbb{C}}$ as in [NS06, Section 3], and the embedding $C_0 \subseteq \mathfrak{X}_s$ is a pre-log curve in the sense of [NS06, Definition 4.3].

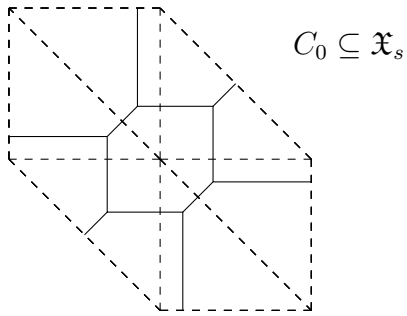

Figure 2.4: The curve C_0 visualized using tropical lines in the moment polytopes

2.3 Log smooth deformation theory

In this section, we explain the conditions under which we can use log smooth deformation theory to lift the nodal curve C_0 to a semistable curve $\mathcal{C}$ over R in $\mathfrak{X}$. Our approach is a generalization of the methods developed in [NS06].

We use logarithmic geometry in the sense of [Kat89], a theoretical framework that makes it possible to treat certain singularities, such as toric or normal crossings singularities, as if they were smooth. For the basics of this theory, see [Kat89] and [Gro11, Chapter 3].

In our setting, we endow the scheme $O = \operatorname{Spec} R$ with the divisorial log structure defined by its special fiber. Its generic fiber is then $\operatorname{Spec} K$ with the trivial log structure, while its special fiber is the standard log point $O_0 := (\operatorname{Spec} \mathbb{C}, \mathbb{C}^\times \times \mathbb{N})$. We endow a toric scheme $\mathfrak{X}$ with the divisorial log structure defined by its toric boundary; then $\mathfrak{X}$ is log smooth over O. If $Y \to X$ is a morphism of log schemes, we denote by $\Theta_{Y/X}$ the *log tangent sheaf* of Y over X. By [Oda88, Proposition 3.1], there is then a natural isomorphism $\Theta_{\mathfrak{X}/O} \cong \mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Z}} N$.

Finally, we endow the nodal curve $C_0 \subseteq \mathfrak{X}_s$ with the log structure inherited from the log structure of $\mathfrak{X}_s$ (see [Kat89, Example 1.5(3)]). Then C_0 is log smooth over the standard log point O_0. Note that the log structure of C_0 not only encodes information about the nodal points, but also marks the points of the intersection of C_0 with those toric boundary divisors of X_0 which lie in only one component of X_0. In the situation of Section 2.2, this means that C_0 has one marked point for each unbounded edge of Γ.

We now develop the log smooth deformation theory we need for our proof. Let $R_k = \mathbb{C}[[t^{\frac{1}{\ell}}]]/(t^{\frac{k+1}{\ell}})$ and endow $O_k = \operatorname{Spec} R_k$ with the log structure induced by $\mathbb{N} \to R_k : a \mapsto t^{\frac{a}{\ell}}$. Note that we have natural closed immersions $O_{k'} \hookrightarrow O_k$ for $0 \leq k' \leq k$.

Definition 2.3.1. *Let C_0 be a log smooth curve over O_0. A k-th order deformation of C_0 is a log smooth morphism $C_k \to O_k$ whose base change to O_0 is $C_0 \to O_0$.*

Suppose we are given a $(k-1)$-st order deformation $C_{k-1} \to O_{k-1}$ of $C_0 \to O_0$. By [Gro11, Proposition 3.40] there is an element $\mathrm{ob}(C_{k-1}/O_{k-1}) \in \mathrm{H}^2\left(C_0, \Theta_{C_0/O_0}\right)$ such that $C_{k-1} \to O_{k-1}$ lifts to a k-th order deformation $C_k \to O_k$ if and only if $\mathrm{ob}(C_{k-1}/O_{k-1}) = 0$. Since C_0 is of dimension one, we have $\mathrm{H}^2\left(C_0, \Theta_{C_0/O_0}\right) = 0$ and therefore such a lift always exists. Moreover, the set of such lifts is a torsor over $\mathrm{H}^1\left(C_0, \Theta_{C_0/O_0}\right)$. However, lifting a log smooth curve together with its embedding into $\mathfrak{X}$ is more complicated.

Let $f = f_0 : C_0 \hookrightarrow \mathfrak{X}_s$ be a strict closed immersion of C_0 into the special fiber $\mathfrak{X}_s$ of a toric scheme $\mathfrak{X}$, and consider the commutative diagram

$$\begin{tikzcd} C_0 \arrow[r, "f_0"] \arrow[d] & \mathfrak{X} \arrow[d] \\ O_0 \arrow[r] & O \end{tikzcd}$$

Definition 2.3.2. *A k-th lift of f_0 is a commutative diagram*

$$\begin{tikzcd} C_k \arrow[r, "f_k"] \arrow[d] & \mathfrak{X} \arrow[d] \\ O_k \arrow[r] & O \end{tikzcd}$$

where $f_k : C_k \to \mathfrak{X}$ is strict and $C_k \to O_k$ is a k-th lift of $C_0 \to O_0$. In this case Nakayama's lemma guarantees that f_k is a closed immersion.

The following proposition is the main result of this section; it expands on the argument in [Gro11, Theorem 3.41]. We refer the reader to [NS06, Lemma 7.2 and Proposition 7.3] for the original results.

Proposition 2.3.3. *Let f_{k-1} be a $(k-1)$-st lift of f_0 and suppose that the canonical homomorphism*

$$\mathrm{H}^1(C_0, \Theta_{C_0/O_0}) \longrightarrow \mathrm{H}^1(C_0, f_0^*\Theta_{\mathfrak{X}/O}) \tag{2.3.1}$$

is surjective. Then there exists a k-th lift of f_0 that extends f_{k-1}.

In [NS06] the authors assume that C_0 is rational. In this case

$$\mathrm{H}^1(C_0, f_0^*\Theta_{\mathfrak{X}/O}) = 0$$

and the homomorphism (2.3.1) is always surjective.

Proof of Proposition 2.3.3. Assume that we are given a $(k-1)$-st lift f_{k-1} and let $C_k \to O_k$ be a lift of $C_{k-1} \to O_{k-1}$.

Suppose that (2.3.1) is surjective. Choose an affine open cover (U_i) of C_0 and let $U_{ij} = U_i \cap U_j$. Since the U_i are affine, log smooth thickenings exist and are unique by [Gro11, Proposition 3.38]. Let U_i^k and U_i^{k-1} be log smooth thickenings of U_i over O_k and O_{k-1} respectively. In particular, we assume that U_i^k is a lifting of U_i^{k-1} to a log smooth scheme over O_k. We have gluing morphisms

$$\theta_{ij}^k : U_{ij}^k \longrightarrow U_{ji}^k$$

and

$$\theta_{ij}^{k-1} : U_{ij}^{k-1} \longrightarrow U_{ji}^{k-1}$$

that fulfill $\theta_{ij}^{k-1} = \theta_{ij}^k|_{U_{ij}^{k-1}}$ for all i and j. Moreover, we have lifts $f_i^{k-1} : U_i^{k-1} \to \mathfrak{X}$ that satisfy the compatibility condition $f_i^{k-1} = f_j^{k-1} \circ \theta_{ij}^{k-1}$ on U_{ij}^{k-1}.

Since $\mathfrak{X}$ is log smooth over O, we can find lifts $f_i^k : U_i^k \to \mathfrak{X}$ of the f_i^{k-1} to the thickening U_i^k of U_i^{k-1} for every i. By [Kat89, Proposition 3.9] the set of such lifts f_i^k on U_i^k forms a torsor over $\mathrm{H}^0\left(U_i, f^*\Theta_{\mathfrak{X}/O}|_{U_i}\right)$.

Now compare the two liftings f_i^k and $f_j^k \circ \theta_{ij}^{k-1}$ on U_{ij}^k. Note that they both lift $f_i^{k-1} = f_j^{k-1} \circ \theta_{ij}^{k-1}$ on U_{ij}^{k-1} and therefore differ by a section $\psi_{ij} \in \mathrm{H}^0(U_{ij}, f^*\Theta_{\mathfrak{X}/O}|_{U_{ij}})$, i.e. we have

$$f_i^k = f_j^k \circ \theta_{ij}^k + t^{\frac{k}{N}}\psi_{ij} \ .$$

The ψ_{ij} define a 2-cocycle of $f^*\Theta_{\mathfrak{X}/O}$ on C_0, since $\mathrm{H}^2(f^*\Theta_{\mathfrak{X}/O}) = 0$.

Since (2.3.1) is surjective, there is a 2-cocycle ϕ_{ij}^k for Θ_{C_0/O_0} such that $f \circ \phi_{ij} = \psi_{ij}$ for all i and j. We can now replace the lift C_k of C_{k-1} by the lift $\tilde{C}_k$ of C_{k-1} that is given by the gluing maps $\tilde{\theta}_{ij}^k = \theta_{ij}^k + t^{\frac{k}{N}}\phi_{ij}$. But then we have

$$\begin{aligned} f_i^k &= f_j^k \circ \theta_{ij}^k + t^{\frac{k}{N}}\psi_{ij} \\ &= f_j^k \circ (\tilde{\theta}_{ij}^k - t^{\frac{k}{N}}\phi_{ij}) + t^{\frac{k}{N}}\psi_{ij} \\ &= f_j^k \circ \tilde{\theta}_{ij}^k - t^{\frac{k}{N}}\psi_{ij} + t^{\frac{k}{N}}\psi_{ij} = f_j^k \circ \tilde{\theta}_{ij}^k \end{aligned}$$

and therefore we can glue the local lifts f_i^k to a global lift $f_k : \tilde{C}_k \to \mathfrak{X}$. □

In our setting, we can deduce the following result:

Proposition 2.3.4. *Let Γ, $\mathfrak{X}$, and C_0 be as in Section 2.2, and assume that the homomorphism* (2.3.1) *is surjective. Then there exists a semistable curve $\mathcal{C}$ over R with smooth generic fiber C and special fiber C_0, together with a closed immersion $\mathcal{C} \hookrightarrow \mathfrak{X}$ extending $C_0 \hookrightarrow \mathfrak{X}$.*

Proof. Taking the direct limit of all C_k, we obtain a formal scheme $\mathfrak{C}$ over O, with a closed immersion into the t-adic formal completion $\widehat{\mathfrak{X}}$ of $\mathfrak{X}$. Since C_0 is complete, all C_k are proper over O_k and therefore $\mathfrak{C}$ is proper over O. By Grothendieck's existence theorem [Gro61, Théorème 5.1.4], $\mathfrak{C}$ is then the t-adic formal completion of a closed subscheme $\mathcal{C}$ of $\mathfrak{X}$, proper over O. Note that by construction the special fiber of $\mathcal{C}$ is equal to C_0.

We want to show that with the log structure induced from $\mathfrak{X}$ the R-scheme $\mathcal{C}$ is log smooth over O. Since this can be checked in an étale neighborhood in $\mathcal{C}$ of a node p of C_0, without loss of generality we can assume that $C_0 = \operatorname{Spec}\big(\mathbb{C}[x,y]/(xy)\big)$. For $e > 0$, set $\mathcal{C}_e = \operatorname{Spec}\big(R[x,y]/(xy - t^{e/\ell})\big)$ with log smooth logarithmic structure as in [Gro11, Example 3.26]. By the description of log smooth curves of [Kat00, Proposition 1.1], there exists some $e > 0$ such that the special fiber of $\mathcal{C}_e$ is C_0. Therefore, for every $k > 0$ the restriction $\mathcal{C}_e \times_O O_k$ is the unique log smooth lifting of $C_0 \to O_0$ to O_k. This implies that $\mathcal{C} = \mathcal{C}_e$, so $\mathcal{C}$ is log smooth over O.

Then the generic fiber C of $\mathcal{C}$ is log smooth over K. Therefore C has only toric singularities, hence it is smooth, since it is one-dimensional. Since $\mathcal{C}$ is proper over O, the curve C is complete. □

Remark 2.3.1. Assume we are in the situation of Proposition 2.3.4. Then the log structure of $\mathcal{C}$, which is the one induced by the log structure of $\mathfrak{X}$, contains information not only about the nodes of C_0 but also about finitely many sections of $\mathcal{C} \to O$, which are disjoint since $\mathcal{C}$ is log smooth. In the special fiber, these sections cut out precisely the marked points of C_0, which correspond to the unbounded edges of Γ. On the other hand, in the generic fiber they cut out a finite set $V \subseteq C(K^{\mathrm{alg}})$ of marked points of C, which is precisely the intersection of C with the toric boundary of X. Therefore, this construction naturally gives rise to a generalized skeleton $\Sigma_{\mathcal{C},V}$ of C^{an}.

2.4 The abundancy map in cohomology

Let $\Gamma \subseteq N_{\mathbb{R}}$ be a tropical curve with skeleton G, and denote by E_G the set of edges of G.

Choose a direction for every edge $e \in E_G$ and write $\vec{e}$ for the vector in $N_{\mathbb{R}}$ connecting the two endpoints of e according to this direction.

We denote by $\mathrm{H}_1(\Gamma)$ the first simplicial homology group of Γ. An element of $\mathrm{H}_1(\Gamma)$ is a formal sum $\sum_{e \in E_G} a_e[e]$, with integer coefficients, forming a cycle in Γ. The next definition is due to Mikhalkin [Mik05, Section 2.6], but our formulation is essentially the one of [Kat12, Section 1].

Definition 2.4.1. *A tropical curve* Γ *is said to be* non-superabundant*, if the* abundancy map

$$\begin{aligned} \Phi_\Gamma \colon \mathbb{R}^{E_G} &\longrightarrow \mathrm{Hom}\big(\mathrm{H}_1(\Gamma), N_{\mathbb{R}}\big) \\ (\ell_e) &\longmapsto \left(\sum_{e \in E_G} a_e[e] \mapsto \sum_{e \in E_G} \ell_e a_e \vec{e} \right), \end{aligned}$$

is surjective.

We remark that the homomorphism $\Phi_\Gamma\big((l_e)_e\big)$ does not depend on the choice of a direction of the edges of Γ, as edge directions are specified in both the source and the target of $\mathrm{Hom}\big(\mathrm{H}_1(\Gamma), N_{\mathbb{R}}\big)$.

The intersection of the kernel of Φ_Γ with $\mathbb{R}^{E_G}_{>0}$ can be seen as the moduli space of metric graphs embedded in $N_\mathbb{R}$ which have the same combinatorial type as the skeleton of Γ, modulo translations. Then for Γ to be non-superabundant means that this moduli space has the expected dimension $\#E_G - b_1(\Gamma)n$, where $b_1(\Gamma)$ is the first Betti number of Γ and n is the rank of N. See [Mik05, Sections 2.4–2.6] for a thorough discussion of those dimension counts. For example, whenever Γ is trivalent, by [Mik05, Proposition 2.13] the expected dimension of the moduli space of metric graphs in $N_\mathbb{R}$ which have the same combinatorial type as Γ is $x + (n-3)(1 - b_1(\Gamma))$, where x is the number of unbounded edges of Γ.

Now let Γ be a tropical curve, let $\mathfrak{X}$ be the toric scheme as constructed in Section 2.2, and let C_0 be a nodal curve in $\mathfrak{X}_s$ fulfilling the conclusion of Proposition 2.2.4. The crucial result of this section is the following proposition, which gives a cohomological interpretation of the abundancy map.

Proposition 2.4.2. *There are a homomorphism* $\delta : \mathbb{C}^{E_G} \longrightarrow \mathrm{H}^1(C_0, \Theta_{C_0/O_0})$, *and an isomorphism* $\mathrm{H}^1(C_0, f^*\Theta_{\mathfrak{X}/O}) \cong \mathrm{Hom}\big(\mathrm{H}_1(\Gamma), N_\mathbb{C}\big)$ *such that the induced homomorphism*

$$\mathbb{C}^{E_G} \xrightarrow{\delta} \mathrm{H}^1(C_0, \Theta_{C_0/O_0}) \longrightarrow \mathrm{H}^1(C_0, f^*\Theta_{\mathfrak{X}/O}) \cong \mathrm{Hom}\big(\mathrm{H}_1(\Gamma), N_\mathbb{C}\big)$$

is equal to $\Phi_\Gamma \otimes \mathbb{C}$.

The proof of this statement will require several steps. We will begin by defining the homomorphism δ. In Lemma 2.4.1 we construct the isomorphism, and we conclude by explicitly computing the resulting composition.

Proof of Proposition 2.4.2. Note that there are compatible one-to-one correspondences between the vertices v of Γ and the components C_v of C_0 as well as between the edges e of G and the corresponding nodes, denoted $p(e)$, of C_0. We have two normalization exact sequences (see [Har77, Exercise IV.1.8]) of sheaves on C_0

$$\begin{tikzcd}
0 \arrow[r] & \Theta_{C_0/O_0} \arrow[r] \arrow[d] & \prod_v (\Theta_{C_0/O_0})|_{C_v} \arrow[r] \arrow[d] & \prod_e \underline{\mathbb{C}}_{p(e)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{C_0} \otimes N \arrow[r] & \prod_v (\mathcal{O}_{C_0/O_0} \otimes N)|_{C_v} \arrow[r] & \prod_e \underline{N_\mathbb{C}}_{p(e)} \arrow[r] & 0
\end{tikzcd} \tag{2.4.1}$$

where $\underline{\mathbb{C}}_{p(e)}$ and $\underline{N_{\mathbb{C}}}_{p(e)}$ denote the skyscraper sheaves at $p(e)$. These products of the skyscraper sheaves are identified with the cokernel of the maps on the left of the horizontal exact sequences.

The first and the second vertical maps are given by composing the natural map $\Theta_{C_0/O_0} \to f^*\Theta_{\mathfrak{X}/O}$ with the natural isomorphism $f^*\Theta_{\mathfrak{X}/O} \cong \mathcal{O}_{C_0} \otimes N$ of [Oda88, Proposition 3.1]. These maps induce the third vertical map. Taking the long exact cohomology sequences of these two short exact sequences, we obtain the following commutative square:

$$\begin{tikzcd} \prod_e \mathbb{C} \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^1(C_0, \Theta_{C_0/O_0}) \arrow[d] \\ \prod_e N_{\mathbb{C}} \arrow[r, "\delta"] & \mathrm{H}^1(C_0, \mathcal{O}_{C_0} \otimes N) \end{tikzcd} \tag{2.4.2}$$

Now we need the following lemma:

Lemma 2.4.1. There is an isomorphism

$$\alpha_0 : \mathrm{H}^1(C_0, \mathcal{O}_{C_0}) \xrightarrow{\sim} \mathrm{Hom}\big(\mathrm{H}_1(\Gamma), \mathbb{C}\big).$$

which induces the isomorphism

$$\alpha : \mathrm{H}^1(C_0, \mathcal{O}_{C_0} \otimes N) \xrightarrow{\sim} \mathrm{Hom}\big(\mathrm{H}_1(\Gamma), N_{\mathbb{C}}\big).$$

such that the composition $\alpha \circ \delta : \prod_e N_{\mathbb{C}} \to \mathrm{Hom}\big(\mathrm{H}_1(\Gamma), N_{\mathbb{C}}\big)$ is given by sending a family $(u_e)_e$ of elements $u_e \in N_{\mathbb{C}}$ to the homomorphism

$$\sum_e a_e[e] \longmapsto \sum_e a_e u_e$$

in $\mathrm{Hom}\big(\mathrm{H}_1(\Gamma), N_{\mathbb{C}}\big)$.

Proof. Consider the normalization short exact sequence

$$0 \to \mathcal{O}_{C_0} \to \prod_v \mathcal{O}_{C_v} \to \prod_e \underline{\mathbb{C}}_{p(e)} \to 0.$$

The associated long exact cohomology sequence is

$$0 \to \mathrm{H}^0(C_0, \mathcal{O}_{C_0}) \to \prod_v \mathrm{H}^0(C_v, \mathcal{O}_{C_v}) \to \prod_e \mathbb{C} \to \mathrm{H}^1(C_0, \mathcal{O}_{C_0}) \to 0, \tag{2.4.3}$$

since by the rationality of C_v we have $\mathrm{H}^1(C_v, \mathcal{O}_{C_v}) = 0$.

The first three terms of the sequence (2.4.3) are $\mathrm{Hom}(\cdot, \mathbb{C})$ of the reduced simplicial chain complex

$$\mathbb{Z}^{E_G} \to \mathbb{Z}^{V_G} \to \mathbb{Z} \to 0,$$

which defines $\mathrm{H}_1(\Gamma)$. Therefore we obtain an isomorphism $\mathrm{H}^1(C_0, \mathcal{O}_{C_0}) \cong \mathrm{H}^1(\Gamma, \mathbb{C})$, and the latter is isomorphic to $\mathrm{Hom}\big(\mathrm{H}_1(\Gamma), \mathbb{C}\big)$ by the universal coefficient theorem for Γ. Since N is a free abelian group, we may apply $- \otimes N$ to α_0, which induces the isomorphism α. In this case, the homomorphism $\prod_e N_{\mathbb{C}} \to \mathrm{Hom}\big(\mathrm{H}_1(\Gamma), N_{\mathbb{C}}\big)$ is given by sending a family $(u_e)_e$ of elements $u_e \in N_{\mathbb{C}}$ to the homomorphism

$$\sum_e a_e[e] \longmapsto \sum_e a_e u_e$$

in $\mathrm{Hom}\big(\mathrm{H}_1(\Gamma), N_{\mathbb{C}}\big)$. $\square$

Let us now finish the proof of Proposition 2.4.2. By Lemma 2.4.1, it is enough to show that the homomorphism

$$\prod_e \mathbb{C} \longrightarrow \prod_e N_{\mathbb{C}} \tag{2.4.4}$$

on the left of diagram (2.4.2) is given by sending $(l_e)_e$ to the family $(l_e\vec{e})_e$ in $\prod_e N_{\mathbb{C}}$.

Let e be an edge of Γ. Let v_1 and v_2 be the two vectors in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ pointing to the two ends of e, so that $v_2 - v_1 = \vec{e}$. We complete $\{v_1, v_2\}$ to a $\mathbb{Z}^n \oplus \frac{1}{\ell}\mathbb{Z}$-integral basis $\{v_1, \ldots, v_{n+1}\}$ of $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$, and we write $x_1, \ldots, x_{n+1}$ for the induced coordinates on the open affine torus-invariant subset $\mathfrak{U}$ corresponding to the cone in Δ containing e. Then $\Theta_{\mathfrak{U}/O}$ has generators $x_1\frac{\partial}{\partial x_1}, \ldots, x_{n+1}\frac{\partial}{\partial x_{n+1}}$ that fulfill the relation $T\frac{\partial}{\partial T} = 0$, where T denote the image in $\mathcal{O}_{\mathfrak{U}}$ of the coordinate $t^{\frac{1}{\ell}}$ on O. Since $f_0 : C_0 \hookrightarrow \mathfrak{X}$ is a strict closed immersion, the pullbacks $y_1 = f_0^* x_1$ and $y_2 = f_0^* x_2$ define coordinates on $\mathcal{O}_{C_0}$ such that the formal completion of $\mathcal{O}_{C_0,p(e)}$ is isomorphic to $\mathbb{C}[[y_1 y_2]]/(y_1 y_2)$. In these coordinates the stalk $(\Theta_{C_0/O_0})_{p(e)}$ is generated by the two elements $y_1\frac{\partial}{\partial y_1}$ and $y_2\frac{\partial}{\partial y_2}$, which fulfill the relation $y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} = 0$.

Therefore around $p(e)$ the natural homomorphism

$$\Theta_{C_0/O_0} \longrightarrow f^*\Theta_{\mathfrak{X}/O}$$

is given by the associations $y_1\frac{\partial}{\partial y_1} \mapsto x_1\frac{\partial}{\partial x_1}$ and $y_2\frac{\partial}{\partial y_2} \mapsto x_2\frac{\partial}{\partial x_2}$. Since $v_2 - v_1 = \vec{e}$ the map (2.4.4) sends $(0, \dots, 0, l_e, 0, \dots, 0)$ to $(0, \dots, 0, l_e\vec{e}, 0, \dots, 0)$, which concludes the proof of Proposition 2.4.2.

□

Remark 2.4.2. In particular, if Γ is trivalent, then the homomorphism (2.3.1) is precisely $\Phi_\Gamma \otimes \mathbb{C}$. Indeed, in this case, for every vertex v, we have $\Theta_{C_0/O_0}|_{C_v} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, so $\mathrm{H}^i(C_0, \Theta_{C_0/O_0}|_{C_v}) = 0$ for $i = 0, 1$, and therefore the homomorphism δ in Proposition 2.4.2 is an isomorphism by the long exact sequence associated to the first line of the diagram (2.4.1). This is the case considered in [Nis15a].

2.5 Proofs of Theorems 2.1.1 and 2.1.2

In this section we complete the proofs of Theorem 2.1.1 and Theorem 2.1.2. It is worthwhile to notice that 3-colorability and smoothness of the tropical curve $\Gamma \subseteq N_\mathbb{R}$ are only used to construct a suitable nodal curve $C_0 \subseteq \mathfrak{X}_s$ in Section 2.2. Whenever such a nodal curve exists, the non-superabundancy of Γ is sufficient for the conclusion of Theorem 2.1.1 to hold.

Let $\Gamma \subseteq N_\mathbb{R}$ be a tropical curve, let $\mathfrak{X}$ be the toric R-scheme defined as in Section 2.2, and denote by X the generic fiber of $\mathfrak{X}$.

Lemma 2.5.1. Let $\mathcal{C}$ be a flat R-curve in $\mathfrak{X}$ and denote by $C \subseteq X$ its generic fiber. If the special fiber $\mathcal{C}_s$ is proper over $\mathbb{C}$ and intersects every torus orbit contained in the special fiber of $\mathfrak{X}$, then $\mathrm{Trop}(C \cap T) = \Gamma$.

Proof. By [Gub13, Proposition 11.12] the properness of $\mathcal{C}_s$ implies that $\mathrm{Trop}(C) \subseteq \Gamma$, and therefore the vertices of $\mathrm{Trop}(C)$ are a subset of Γ. By Tevelev's Lemma [Gub13, Lemma 11.6] $\mathrm{Trop}(C)$ intersects the relative interior of every face in Γ. In particular, the vertices of $\mathrm{Trop}(C)$ coincide with the vertices of Γ. Again, since $\mathrm{Trop}(C)$ intersects the relative interior of every face of Γ, all one-dimensional faces of Γ already have to be contained in $\mathrm{Trop}(C)$, and therefore $\mathrm{Trop}(C) = \Gamma$. □

In the proof of Theorem 2.1.1 we use the *initial degeneration* $\mathrm{in}_P(C)$ of C along an open face P of $\mathrm{Trop}(C)$. In our situation $\mathrm{in}_P(C)$ can be defined as the

$\mathbb{C}$-scheme

$$\mathrm{in}_P(C) = \big(\mathcal{C}_s \cap \mathfrak{X}_P\big) \times T_P \ ,$$

where $\mathfrak{X}_P$ is the torus orbit in $\mathfrak{X}_s$ corresponding to P, and T_P is the subgroup of the reduction of the torus $\mathcal{T} = \operatorname{Spec} R[M]$ that stabilizes $\mathfrak{X}_P$. By [HK12, Lemma 3.6] this coincides with the usual definition of the initial degeneration of C at a point $p \in P$, as for example in [BPR11, Section 2.1], since $\mathcal{C}$ is proper over $O = \operatorname{Spec} R$ and the multiplication map $\mathcal{T} \times_O \mathcal{C} \to \mathfrak{X}$ is flat by the same argument as in [Hac08, Lemma 2.7] and surjective because $\mathcal{C}$ meets each torus orbit of $\mathfrak{X}$.

Proof of Theorem 2.1.1. Let $\Gamma \subset N_{\mathbb{R}}$ be a non-superabundant, smooth and 3-colorable tropical curve with rational edge lengths, let K and $\mathfrak{X}$ be be defined as in Section 2.2, and C_0 be a curve in $\mathfrak{X}_s$ as constructed in Proposition 2.2.4. Since Γ is non-superabundant, Proposition 2.4.2 and Proposition 2.3.4 imply that there is a semistable curve $\mathcal{C}$ over R with smooth generic fiber C and special fiber C_0, together with a closed immersion $\mathcal{C} \hookrightarrow \mathfrak{X}$. By Lemma 2.5.1 we have $\mathrm{Trop}(C \cap T) = \Gamma$. By construction of C_0, all the initial degenerations of C along open faces of Γ are smooth and irreducible, and so the tropical multiplicities are all one. Therefore, [BPR11, Corollary 6.11] implies that the tropicalization is faithful with respect to the skeleton $\Sigma_{\mathcal{C},V}$ of C^{an}, where $V = C \setminus (C \cap T)$ is the set of marked points as described in Remark 2.3.1. $\square$

Theorem 2.1.2 follows from Theorem 2.1.1 and the following result, which is a slight extension of a theorem of Cartwright–Dudzik–Manjunath–Yao.

Theorem 2.5.2 ([CDMY14]). Let G be a metric graph with rational edge lengths. Set

$$n = \max\big\{3, \max\{\deg v - 1 \mid v \in E_G\}\big\}$$

and let N be a free abelian group of rank n. Then there exists a non-superabundant, smooth and 3-colorable tropical curve $\Gamma \subseteq N_{\mathbb{R}}$ with rational edge lengths and whose skeleton is isomorphic to G.

Proof. Fix an integral basis $v_1, \ldots, v_n$ of $N_{\mathbb{R}}$. Let Γ be a tropical curve as given by [CDMY14, Theorem 1.1]. As noted in [CDMY14, Remark 2.8], the only part that

is not included there is the non-superabundance of Γ. Let G' be the minimal finite graph underlying the skeleton of Γ and consider a spanning tree $\mathbb{T}$ of G'. Then the set of $\epsilon \in E_{\tilde{G}\setminus\mathbb{T}}$ parametrizes a basis $\{c_\epsilon\}$ of $\mathrm{H}_1(\Gamma, \mathbb{Z})$, and c_ϵ is the only element of the basis which contains ϵ.

In order to show the surjectivity of the abundancy map it is enough to show that for all $\epsilon \in E_{\tilde{G}\setminus\mathbb{T}}$ and $1 \leq i \leq n$ the homomorphisms

$$\begin{aligned} f_{\epsilon,i} : \mathrm{H}_1(\Gamma) &\longrightarrow N_{\mathbb{R}} \\ c_{\epsilon'} &\longmapsto \begin{cases} v_i & \text{if } \epsilon' = \epsilon \\ 0 & \text{if } \epsilon' \neq \epsilon. \end{cases} \end{aligned}$$

are in the image of Φ_Γ. By the construction of [CDMY14], every ϵ will contain an edge e of G' such that $\vec{e}$ is parallel to v_i. Since c_ϵ is the only element of the basis which contains e, if we take the vector $\ell \in \mathbb{R}^{\#E_{G'}}$ with value $|v_i|$ in the e-th entry and 0 otherwise, we obtain $\Phi(\ell) = f_{\epsilon,i}$. $\square$

Remark 2.5.3. It is possible to extend Theorem 2.1.1, and therefore Theorem 2.1.2, to the equicharacteristic $p > 0$ case. Let $\Gamma \subset N_{\mathbb{R}}$ be a non-superabundant, smooth and 3-colorable tropical curve. Then for all but finitely many prime numbers p there exists a finite field extension K of $\mathbb{F}_p((t))$ such that we can construct a suitable curve C_0 in the special fiber of $\mathfrak{X}$ using the method of Section 2.2. The results of Section 2.3 remain valid over any discrete valuation ring of the form $k[[t]]$, where k is an arbitrary field, so in particular they hold over R. Observe that the abundancy map Φ_Γ defined in 2.4.1 is the base change $\Phi_{\Gamma,\mathbb{Z}} \otimes \mathbb{R}$ of the map $\Phi_{\Gamma,\mathbb{Z}} \colon \mathbb{Z}^{E_G} \to \mathrm{Hom}\big(\mathrm{H}_1(\Gamma, \mathbb{Z}), N\big)$ defined by

$$(\ell_e) \longmapsto \Big(\sum_{e \in E_G} a_e[e] \mapsto \sum_{e \in E_G} \ell_e a_e \vec{e} \Big)$$

as in Definition 2.4.1. For a field L we denote by $\Phi_{\Gamma,L}$ the base change of $\Phi_{\Gamma,\mathbb{Z}}$ to L. Given two fields L and L' of the same characteristic, the surjectivity of $\Phi_{\Gamma,L}$ is equivalent to the surjectivity of $\Phi_{\Gamma,L'}$, and in particular Γ is non-superabundant if and only if $\Phi_{\Gamma,\mathbb{Q}}$ is surjective. In this case the map $\Phi_{\Gamma,\mathbb{F}_p}$ is surjective for all but finitely many prime numbers p. Therefore, if Γ is non-superabundant, log

deformations of C_0 can be constructed for p big enough and it follows that, given a tropical curve Γ satisfying the hypotheses of Theorem 2.1.1, for all but finitely many prime numbers p we can find a finite field extension K of $\mathbb{F}_p((t))$ such that the conclusion of Theorem 2.1.1 holds over K.

This chapter, in full, is a reprint of the paper "Faithful Realizability of Tropical Curves" as it appears in International Mathematics Research Notices 2015. Man-wai Cheung; Lorenzo Fantini, Jennifer Park, Martin Ulirsch. The thesis author was the author of this paper.

Chapter 3

Introduction to cluster algebras and quiver representations

3.1 Cluster algebras

A cluster algebra $\mathcal{A}$ of rank n is a subring of the ring of rational functions in n variables over a field $\Bbbk$. Instead of knowing all the generators and relations in the beginning, we are only given some initial data which is called the initial seed. In this thesis, all cluster algebras are assumed without frozen variables.

A *seed* is a pair $\Sigma = (\mathbf{a}, B)$, where

- $\mathbf{a}$ is a set of algebraically independent elements $\{A_1, \dots A_n\}$ lying in the field of rational functions of n independent variables;
- $B = (b_{ij})$ is a skew-symmetrizable integer matrix. A square integer matrix $B = (b_{ij})$ is called **skew-symmetrizable** if there exists a diagonal skew-symmetrizing matrix D with positive integer diagonal entries d_i such that DB is skew-symmetric, i.e. $d_i b_{ij} = -d_j b_{ji}$ for all i, j.

Elements in $\mathbf{a}$ are called **cluster variables** and B is called the **exchange matrix**. The seed we started with is called the *initial seed*. To generate new seeds, there is a procedure called *mutation* to obtain a new seed $\mu_k((\mathbf{a}, B)) =$

$\{\{A_1, \ldots, \hat{A}_k, \ldots, A_n\} \cup \{A'_k\}, \mu_k(B)\}$ for $k = 1, \ldots, n$, where

$$A_k A'_k = \prod_{b_{ik}>0} A_i^{b_{ik}} + \prod_{b_{ik}<0} A_i^{-b_{ik}},$$

and

$$(\mu_k(B))_{i,j} = \begin{cases} -b_{ij} & if\ k = i\ or\ k = j; \\ b_{ij} + \frac{|b_{ik}|b_{kj}+b_{ik}|b_{kj}|}{2} & otherwise. \end{cases}$$

Starting from an initial seed, one can apply all possible sequences of compositions of mutations (possibly infinite). This gives a set (possibly infinite) of all cluster variables generated under mutations. The **cluster algebra** $\mathcal{A} = \mathcal{A}(B, \mathbf{a})$ is the subalgebra of $\Bbbk(A_1, \ldots, A_n)$ generated by all cluster variables. We define the **exchange graph** of a cluster algebra to be the graph whose vertices are the seeds, and whose edges connect pairs of seeds which are connected by mutations.

Every cluster algebra belongs to a series $\mathcal{A}(B)$ consisting of all cluster algebras of the form $\mathcal{A}(B, \mathbf{a})$, where B is fixed and $\mathbf{a}$ is allowed to vary. Two series $\mathcal{A}(B)$ and $\mathcal{A}(B')$ are **strongly isomorphic** if B and B' can be obtained from each other by a sequence of mutation, modulo simultaneous relabeling of rows and columns.

Let us illustrated the construction by an example.

Example 3.1.1 (Rank 2 cluster algebra). In the rank 2 case, the initial seed can be represented as

$$\Sigma = (\{A_1, A_2\}, B = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}),$$

where b and c are positive integers. Mutating at 1 gives us

$$A_1 A'_1 = 1 + A_2^c.$$

Now we denote $x_3 = A'_1$ and put $x_1 = A_1$, $x_2 = A_2$. Then by repeating the mutation process and naming new cluster variables as x_{k+1}, we will get a recursive relation indexed by $k \in \mathbb{Z}$ as

$$x_{k-1}x_{k+1} = \begin{cases} x_k^b + 1 & \text{if } k \text{ is odd;} \\ x_k^c + 1 & \text{if } k \text{ is even.} \end{cases} \tag{3.1.1}$$

The *cluster algebra* $\mathcal{A}(b,c)$ is the $\mathbb{Z}$-subalgebra of $\mathbb{Q}(A_1, A_2)$ which they generate. Note that all the x_k, $k \neq 1, 2$ lies in $\mathbb{Q}(A_1, A_2)$. Each pair $\{x_k, x_{k+1}\}$ is called a *cluster* and a monomial in the variables of a cluster is called a *cluster monomial*.

In the above example, if we put $b = c = 1$, we will find out that there are only 5 cluster variables in $\mathcal{A}(1,1)$. In general, a cluster algebra with a finite number of cluster variables is called a **cluster algebra of finite type**.

Fomin-Zelevinsky [FZ03] showed that there is a bijection between the Cartan matrices C of finite type with cluster algebra $\mathcal{A}(B)$, where $C(B) = C$ is defined as

$$c_{ij} = \begin{cases} 2 & if\ i = j; \\ -|b_{ij}| & if\ i \neq j. \end{cases}$$

Another algebraic feature of cluster algebras is the Laurent phenomenon. [FZ02] shows that every cluster variable can be expressed as a Laurent polynomial of the initial variables with integral coefficients. Furthermore, Fomin and Zelevinsky further conjectured that all those coefficients are non-negative. There have been many attempt to this conjecture. For example, Caldero-Chapoton [CC06] on finite type, Lee-Schiffler [LS15] on skew-symmetric type, Gross-Hacking-Keel-Kontsevich, Maxim [GHKK14] for cluster algebras of geometric type. The GHKK construction will be discussed in Section 4.2.

We will give the definition of quiver representation in the next section. Let us first state the result from Caldero-Chapoton in this section.

In the finite classification, Fomin and Zelevinsky have shown [FZ03] the cluster algebras of finite type can be associated to a Dynkin quiver. At the same time, the set of indecomposable representations of a quiver of Dynkin type is in bijection with the set of positive roots by Gabriel's theorem. Using these correspondence, Caldero and Chapoton [CC06] then related cluster variables with indecomposable quiver representations. They found that the coefficients of the cluster variables are the Euler characteristics of the quiver Grassmannians. This proves that the coefficients are non-negative. Even though the result of Caldero-Chapoton is for finite type quivers, Dominguez-Geiss [DG14] generalized the result to generalized cluster categories.

Proposition 3.1.2. *Let Q be a finite quiver with vertices $1, \ldots, n$, and D a finite-dimensional representation of Q with dimension vector d. Denote $Gr(e, D) := \{E \in \mathrm{mod}(Q) | E \subseteq D, \dim(E) = e\}$ for $e \in \mathbb{N}^n$. Define the CC function as*

$$CC(D) = \frac{1}{A_1^{d_1} \cdots A_n^{d_n}} \sum_{0 \leq e \leq d} \chi(Gr(e, D)) \prod_{i=1}^{n} A_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)},$$

where the sum is taken over all vectors $e \in \mathbb{N}^n$ such that $0 \leq e_i \leq d_i$ for all i. Then we have $CC(D) = X_D$, the cluster variable obtained from D by composing Fomin-Zelevinsky's bijection with Gabriel's.

3.2 Quiver Representations

We will review some basic definitions and ideas of quiver representations in this section.

3.2.1 Quivers and quiver reprsentations

A **quiver** Q of rank n is a directed graph of n vertices. More precisely $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices of Q, Q_1 is the set of arrows of Q, $s, t : Q_1 \to Q_0$ two maps. For an arrow $\alpha \in Q_1$, $s(\alpha)$ is the starting point of α and $t(\alpha)$ is the end point of α. It can be written as $\alpha : s(\alpha) \to t(\alpha)$. Let us assume the vertices of Q are numbered $\{1, \ldots, n\}$. We will mainly talk about acyclic quivers Q. Thus we can further assume $i > j$ if there is an arrow goes from vertex j to i.

Definition 3.2.1. *A $\mathbb{C}$-linear representation $V = (V_a, V_\alpha)_{a \in Q_0, \alpha \in Q_1}$ of Q is defined by:*

- *To each point a in Q_0 is associated a $\mathbb{C}$ vector space V_a*
- *To each arrow $\alpha : a \to b$ in Q_1 is associated a $\mathbb{C}$-linear map $V_\alpha : V_a \to V_b$.*

*It is called **finite dimensional** if each vector space V_a is finite dimension. In this case, the **dimension vector** of M is defined to be the vector*

$$\dim(V) = (\dim V_a)_{a \in Q_0}.$$

A **morphism** of representations $\phi : V \to W$ is a collection $\phi = (\phi_i)_{i \in Q_0}$ of linear maps $\phi_i : V_i \to W_i$ for each vertex i such that the following commutative diagram commutes:

$$\begin{tikzcd} V_{s(\alpha)} \arrow[r, "\phi_{s(\alpha)}"] \arrow[d, "V_\alpha"] & W_{s(\alpha)} \arrow[d, "W_\alpha"] \\ V_{t(\alpha)} \arrow[r, "\phi_{t(\alpha)}"] & W_{t(\alpha)} \end{tikzcd}$$

Let $f : V \to V'$ and $g : V' \to V''$ be two morphisms of representations of Q. Then their composition is defined as $(gf)_a = g_a \circ f_a$ for $a \in Q_0$. Then $gf : V \to V''$ is also a morphism of representations. This defines a category $\mathrm{Rep}(Q)$ of representations of Q. We denote by $\mathrm{rep}(Q)$ the full subcategory of $\mathrm{Rep}(Q)$ consisting of the finite dimensional representations.

Let $V = (V_i, V_\alpha)$ and $W = (W_i, W_\alpha)$ be representation of Q. Then the **direct sum** $V \oplus W$ is defined as

$$(V \oplus W)_i = V_i \oplus W_i$$

$$(V \oplus W)_\alpha = \begin{pmatrix} V_\alpha & 0 \\ 0 & W_\alpha \end{pmatrix}$$

A representation V is called **indecomposable** if $V \neq 0$ and if $V = A \oplus B$, then $A = 0$ or $B = 0$.

Let $N = \mathbb{Z}Q_0$ and $M = \mathrm{Hom}(N, \mathbb{Z})$.Define a bilinear form, $\chi(\cdot, \cdot)$, the Euler form on N as

$$\chi(d, e) = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha : i \to j} d_i e_j \tag{3.2.1}$$

for $d, e \in N$. We further define a map $\mathcal{E} : N \to M$ by

$$\mathcal{E}(d) = \chi(\cdot, d). \tag{3.2.2}$$

Example 3.2.2. Let us take the Kronecker 2-quiver Q_2 as an example.

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & 2 \end{tikzcd}$$

Then the set of indecomposable representations would be of the form

$$\mathbb{C}^n \rightrightarrows \mathbb{C}^{n+1} \ , \ \mathbb{C}^k \rightrightarrows \mathbb{C}^k \ , \ \mathbb{C}^{n+1} \rightrightarrows \mathbb{C}^n \ ,$$

where $n \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 1}$.

Simple, projective and injective representations

Consider the path algebra $\mathbb{C}Q$. The **path algebra** $\mathbb{C}Q$ is the $\mathbb{C}$-algebra with basis the set of all the paths in the quiver Q and with multiplication defined on two basis elements α, α' by

$$\alpha\alpha' = \begin{cases} \alpha \cdot \alpha' & \text{if } s(\alpha') = t(\alpha) \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \cdot \alpha'$ means composing the paths α followed by α'. The product of two arbitrary elements $\sum_\alpha \lambda_\alpha \alpha$, $\sum_{\alpha'} \lambda_{\alpha'} \alpha'$, $\lambda_\alpha, \lambda_{\alpha'} \in \mathbb{C}$ is given by $\sum_\alpha \lambda_\alpha \lambda_{\alpha'} \alpha\alpha'$. Note that $\mathbb{C}Q$ is finite dimensional if and only if Q is a finite quiver without oriented cycles.

Example 3.2.3. Consider Q_2 in Example 3.2.2. Then $\mathbb{C}Q_2$ are generated by v_i, α_i, where v_i are constant paths on vertex i, $i = 1, 2$. The multiplications are $v_1\alpha_1 = \alpha_1$, $\alpha_2 v_2 = \alpha_2$, $v_i^2 = v_i$, $i = 1, 2$ and other path multiplications are zero in $\mathbb{C}Q_2$.

Denote $\operatorname{Mod} \mathbb{C}Q$ as the abelian category of all right $\mathbb{C}Q$-modules and $\operatorname{mod} \mathbb{C}Q$ the full subcategory of $\operatorname{Mod} \mathbb{C}Q$ whose objects are the finitely generated modules. Then there is an equivalence of categories $\operatorname{mod} \mathbb{C}Q \cong \operatorname{rep}(Q)$. Therefore, we can translate all the notions from module theory to the theory for quiver representations. In particular, a representation is projective (or injective) if it is equivalent to a projective (or injective) module in $\operatorname{mod} \mathbb{C}Q$. In this section, we can are to give explicit construction of simple, indecomposable projective, indecomposable injective representations.

Definition 3.2.4. *A **simple** representation is defined to be a non zero representation with no proper subrepresentations.*

Given a vertex i, define $S(i)$ to be the representation

$$S(i)_j = \begin{cases} \mathbb{C} & \text{if} \quad j = i \\ 0 & \text{if} \quad j \neq i \end{cases}$$

and $S(i)_\alpha = 0$. This is the simple representation associated to the vertex i.

Definition 3.2.5. *Let i be a vertex of Q.*

1. *Define the* ***projective representation*** *$P(i)$ at vertex i as*

$$P(i) = (P(i)_j, P(i)_\alpha)_{j \in Q_0, \alpha \in Q_1}$$

where $P(i)_j$ is the $\mathbb{C}$-vector space with basis the set of all paths from i to j in Q. For $\alpha : j \to l$, $P(i)_\alpha : P(i)_j \to P(i)_l$ is the linear map defined on the basis by composing the paths from i to j with the arrow $\alpha : j \to l$.

2. *Define the* ***injective representation*** *$I(i)$ at vertex i as*

$$I(i) = (I(i)_j, I(i)_\alpha)_{j \in Q_0, \alpha \in Q_1}$$

where $I(i)_j$ is the $\mathbb{C}$-vector space with basis the set of all paths from j to i in Q. For $\alpha : j \to Q$, $I(i)_\alpha : I(i)_j \to I(i)_l$ is the linear map defined on the basis by deleting the arrow α from those paths from j to i which start with α and sending to zero the path that do not start with α.

Example 3.2.6. For Q_2 in Example 3.2.2:

$$S(1) = \mathbb{C} \rightrightarrows 0 \ , S(2) = 0 \rightrightarrows \mathbb{C}$$

$$P(1) = \mathbb{C} \rightrightarrows \mathbb{C}^2 \ , P(2) = 0 \rightrightarrows \mathbb{C} \ = S(2)$$

$$I(1) = \mathbb{C} \rightrightarrows 0 \ = S(1), I(2) = \mathbb{C}^2 \rightrightarrows \mathbb{C}$$

There is one handy property about the indecomposable projectives for calculation.

Lemma 3.2.7. *Let V be a representation of Q. Then there are natural isomorphisms*

$$\mathrm{Hom}(P(i), V) \cong V_i,$$

where V_i is the vector space associated to vertex i.

The notion of projective resolution in module theory continues in $\operatorname{mod}\mathbb{C}Q$.

Definition 3.2.8. *Let V be a representation of Q. A* ***projective resolution*** *of V is an exact sequence*

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to V \to 0,$$

where each P_i is a projective representation.

In particular, as we are talking about representations of quivers without oriented cycles. The situation is even simpler

Theorem 3.2.9. *[Sch14, Theorem 2.15] Let V be a representation of Q. There exists a projective resolution of V of the form*

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0,$$

where $P_1 = \bigoplus_{\alpha \in Q_1} (\dim V_{s(\alpha)}) P(t(\alpha))$ and $P_0 = \bigoplus_{i \in Q_0} (\dim(V_i)) P(i)$.

Now consider the projective resolution of E

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E \longrightarrow 0.$$

Let F be any representation of Q. Applying $\operatorname{Hom}(\cdot, N)$ to the above projective resolution and using Lemma 3.2.7, we get the following exact sequence

$$0 \to \operatorname{Hom}(E,F) \to \operatorname{Hom}(P_0,F) \to \operatorname{Hom}(P_1,F) \to \operatorname{Ext}^1(E,F) \to 0.$$

By using Lemma 3.2.7, we have

$$\chi(E,F) = \dim \operatorname{Hom}(E,F) - \dim \operatorname{Ext}^1(E,F). \tag{3.2.3}$$

For the definition of Nakayama functor in the next section, let us define **minimal projective resolution**.

Definition 3.2.10. *Let $M \in \operatorname{rep} Q$. A* ***projective cover*** *of M is a projective representation P together with a surjective morphism $g : P \to M$ with the property that, whenever $g' : P' \to M$ is a surjective morphism with P' projective, then there*

exists a surjective morphism $h : P' \twoheadrightarrow P$ *such that the diagram commutes, i.e.* $gh = g'$.

$$\begin{tikzcd} & P' \arrow[dl, "h"'] \arrow[d, "g'"] & \\ P \arrow[r, "g"] & M \arrow[r] \arrow[d] & 0 \\ & 0 & \end{tikzcd}$$

Then

Definition 3.2.11. *A projective resolution*

$$\cdots \to P_3 \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} 0$$

is called ***minimal*** *if* $f_0 : P_0 \to M$ *is a projective cover and* $f_i : P_i \to \ker f_{i-1}$ *is a projective cover for every* $i > 0$.

3.3 Auslander-Reiten theory

3.3.1 Auslander-Reiten translation

Let Q^{op} be the quiver obtained from Q by reversing each arrow. Define

$$D = \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C}) : \mathrm{rep}\, Q \longrightarrow \mathrm{rep}\, Q^{op}$$

Let $P = \bigoplus_{i \in Q_0} P(i)$. We have that $\mathrm{Hom}(V, P)$ is a representation $(H_i, \phi_{\alpha^{op}})$ of Q^{op} for V a representation of Q, where $H_i = \mathrm{Hom}(V, P(i))$ for every $i \in Q_0$ and for $\alpha : i \to j$ in Q, define a morphism $\alpha : P(j) \to P(i)$ by $\alpha_l : P(j)_l \to P(i)_l$ by composing α with the paths from j to l. Then define $\phi_{\alpha^{op}} : H_j \to H_i$ as $\phi_{\alpha^{op}}(f) = \alpha \circ f$. That is we have the commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "\phi_{\alpha^{op}}(f)"'] & P(j) \arrow[d, "\alpha"] \\ & P(i) \end{tikzcd}$$

Then we define the **Nakayama functor** as

$$\nu = D\,\mathrm{Hom}(-, P) : \mathrm{rep}\, Q \longrightarrow \mathrm{rep}\, Q.$$

Example 3.3.1. Consider Q_2 again. We take $P(2) = 0 \rightrightarrows \mathbb{C}$. Then $\mathrm{Hom}(P(2), P)$ gives us $\mathbb{C}^2 \leftleftarrows \mathbb{C}$. Taking D will give us $\mathbb{C}^2 \rightrightarrows \mathbb{C}$, which is $I(1)$. Notice that this example also suggests that ν maps projectives to injectives which is true in general.

Now we are ready to define Auslander-Reiten translation.

Definition 3.3.2. *Let*

$$0 \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0 \to V \to 0$$

be a minimal projective resolution. Applying the Nakayama functor, we get an exact sequence

$$0 \to \tau V \to \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu V \to 0,$$

where $\tau V = \ker \nu p_1$ *is called the Auslander-Reiten translate of* M *and* τ *is the* ***Auslander-Reiten translation****.*

Example 3.3.3. Let us calculate $\tau(\mathbb{C}^2 \rightrightarrows \mathbb{C}^3)$ where $\mathbb{C}^2 \rightrightarrows \mathbb{C}^3$ is the indecomposable representation. First we have the minimal projective resolution

$$0 \xrightarrow{p_1} (0 \rightrightarrows \mathbb{C}) \xrightarrow{p_0} (\mathbb{C} \rightrightarrows \mathbb{C}^2)^2 \to (\mathbb{C}^2 \rightrightarrows \mathbb{C}^3) \to 0.$$

Applying the Nakayama functor will give us

$$0 \to \tau(\mathbb{C}^2 \rightrightarrows \mathbb{C}^3) \to (\mathbb{C}^2 \rightrightarrows \mathbb{C}) \xrightarrow{\nu p_1} (\mathbb{C} \rightrightarrows 0)^2 \xrightarrow{\nu p_0} 0 \to 0.$$

Thus $\tau(\mathbb{C}^2 \rightrightarrows \mathbb{C}^3) = 0 \rightrightarrows \mathbb{C}$.

There is a fundamental result called the Asulander-Reiten Formula to relate short exact sequences and morphisms in the module category. This result also give us a way to compute Ext^1 which is heavily used in this article.

Theorem 3.3.4. *[Sch14, Theorem 7.18, Auslander-Reiten formulas] Let* V, W *be* $\mathbb{C}Q$*-modules. We define*

$$\underline{\mathrm{Hom}}(V, W) = \mathrm{Hom}(V, W)/P(V, W),$$

$$\overline{\mathrm{Hom}}(V, W) = \mathrm{Hom}(V, W)/I(V, W),$$

where $P(V, W)$ *(*$I(V, W)$*) is the set of morphisms* $f \in \mathrm{Hom}(V, W)$ *such that* f *factors through a projective (injective)* $\mathbb{C}Q$*-module.*

Then there are isomorphisms

$$\mathrm{Ext}^1(V, W) \cong D\underline{\mathrm{Hom}}(\tau^{-1}W, V) \cong D\overline{\mathrm{Hom}}(W, \tau V).$$

Almost split sequence

There is a canonical extension between a module and its Auslander-Reiten translate. We have a notion of almost split sequence to describe the short exact sequence.

Definition 3.3.5. *A morphism $f : L \to E$ is called* ***left minimal almost split*** *if*

1. *f is not a section, i.e. there is no morphism $h : E \to L$ such that $hf = id_L$;*
2. *for each morphism $u : L \to U$ in $\operatorname{mod} \mathbb{C}Q$ which is not a section, there exists a morphism $u' : E \to U$ such that $u'f = u$;*
3. *if $h : E \to E$ is such that $hf = f$ then h is an automorphism of E.*

Similarly, a morphism $g : E \to F$ is called ***right minimal almost split*** *if*

1. *g is not a retraction, i.e. there is no morphism $h : F \to E$ such that $gh = id_F$;*
2. *for each morphism $v : V \to F$ in $\operatorname{mod} \mathbb{C}Q$ which is not a retraction, there exists a morphism $v' : V \to E$ such that $gv' = v$;*
3. *if $h : E \to E$ is such that $gh = g$ then h is an automorphism of E.*

Then we can define almost split sequence

Definition 3.3.6. *A short exact sequence in $\operatorname{mod} \mathbb{C}Q$*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is an ***almost split sequence*** *if f is a left minimal almost split morphism and g is a right minimal almost split morphism.*

Let $M = \oplus M_i$, where M_i are indecomposable. Then there exists M_i such that both $L \to M_i = M/(\oplus_{j \neq i} M_i)$ and $M_i \xrightarrow{g|_{M_i}} N$ are not zero. If not, we can decompose $M = A \oplus B$, where $A = \operatorname{im} f$ and $B = \ker g$. Then the sequence would be split which contradicts to the definition of almost split exact sequence.

The following theorem gives us the existence of almost split sequences.

Theorem 3.3.7. *[Sch14, Theorem 7.26] For any finite quiver Q without oriented cycle, there exists a bijection τ from the indecomposable non-projective representations to indecomposable non-injective representations such that for each non-projective indecomposable V, there exists an almost split sequence*

$$0 \to \tau V \to E \to V \to 0,$$

where τ is the Auslander-Reiten translation.

Example 3.3.8. Continuing our calculation in Example 3.3.3, we have the almost split sequence

$$0 \longrightarrow (0 \rightrightarrows \mathbb{C}) \longrightarrow (\mathbb{C} \rightrightarrows \mathbb{C}^2)^2 \longrightarrow (\mathbb{C}^2 \rightrightarrows \mathbb{C}^3) \longrightarrow 0.$$

3.3.2 Auslander-Reiten quiver

Let V and W be indecomposable modules in $\operatorname{mod} \mathbb{C}Q$. A homomorphism $f : V \to W$ is in $\operatorname{mod} \mathbb{C}Q$ is **irreducible** if f is neither a section nor a retraction; and if $f = gh$ for some morphism $h : V \to Z$, $g : Z \to W$, then either h is a section or g is a retraction.

Define $\operatorname{Irr}(V, W)$ as the set of irreducible morphisms from V to W. It is called the space of **irreducible morphisms**.

Definition 3.3.9. *[ASS06, Definition IV 4.6] Consider the path algebra $\mathbb{C}Q$, where Q is a finite acyclic quiver. The quiver $\Gamma(\operatorname{mod} \mathbb{C}Q)$ is defined as:*

- *The points of $\Gamma(\operatorname{mod} \mathbb{C}Q)$ are the isomorphism classes $[V]$ of indecomposable modules V in $\operatorname{mod} \mathbb{C}Q$.*
- *Let $[V]$, $[W]$ be the points in $\Gamma(\operatorname{mod} \mathbb{C}Q)$ corresponding to the indecomposable modules V, W in $\operatorname{mod} \mathbb{C}Q$. The arrows $[V] \to [W]$ are in bijective correspondence with the vectors of a basis of the K-vector space $Irr(V, W)$.*

*The quiver $\Gamma(\operatorname{mod} A)$ of the module category $\operatorname{mod} \mathbb{C}Q$ is called the **Auslander-Reiten quiver** of $\mathbb{C}Q$. We will write AR quiver for shorthanded notation.*

Proposition 3.3.10. *[ASS06, Proposition VIII 2.1] The Auslander-Reiten quiver* $\Gamma(\operatorname{mod}\mathbb{C}Q)$ *of* $\mathbb{C}Q$ *contains a connected component* $\mathcal{P}(\mathbb{C}Q)$ *where*

- *for every indecomposable* $\mathbb{C}Q$*-module* V *in* $\mathcal{P}(\mathbb{C}Q)$*, there exist a unique* $t \geq 0$ *and a unique* $a \in Q_0$ *such that* $V \cong \tau^{-t}P(a)$*.*
- $\mathcal{P}(\mathbb{C}Q)$ *contains a subquiver consisting of all the indecomposable projective* $\mathbb{C}Q$*-modules; and*
- $\mathcal{P}(\mathbb{C}Q)$ *is acyclic.*

$\Gamma(\operatorname{mod}\mathbb{C}Q)$ *also contains a connected component* $\mathcal{I}(\mathbb{C}Q)$*, where*

- *for every indecomposable* $\mathbb{C}Q$*-module* W *in* $\mathcal{I}(\mathbb{C}Q)$*, there exist a unique* $s \geq 0$ *and a unique* $a \in Q_0$ *such that* $W \cong \tau^{s}I(a)$*.*
- $\mathcal{I}(\mathbb{C}Q)$ *contains a subquiver consisting of all the indecomposable injective* $\mathbb{C}Q$*-modules; and*
- $\mathcal{I}(\mathbb{C}Q)$ *is acyclic.*

Furthermore, $\mathcal{P}(\mathbb{C}Q) = \mathcal{I}(\mathbb{C}Q)$ *if and only if* Q *is of Dynkin type.*

$\mathcal{P}(\mathbb{C}Q)$ is called the **pre-projecrive** component of $\Gamma(\operatorname{mod}\mathbb{C}Q)$ and $\mathcal{I}(\mathbb{C}Q)$ is called the **pre-injective** component of $\Gamma(\operatorname{mod}\mathbb{C}Q)$. An indecomposable $\mathbb{C}Q$-module is called **pre-projective** if it belongs to $\mathcal{P}(\mathbb{C}Q)$ and it is called **pre-injective** if it belongs to $\mathcal{I}(\mathbb{C}Q)$. From the above, if Q is of Dynkin type, then $\mathcal{P}(\mathbb{C}Q) = \mathcal{I}(\mathbb{C}Q)$ from the theorem above. Then $\Gamma(\operatorname{mod}\mathbb{C}Q)$ is connected and $\Gamma(\operatorname{mod}\mathbb{C}Q) = \mathcal{P}(\mathbb{C}Q) = \mathcal{I}(\mathbb{C}Q)$ from the properties of projectives and injectives. If Q is not of Dykin type, there are representations which is neither pre-projective or pre-injective. We will call the connected component $\mathcal{R}(\mathbb{C}Q)$ of $\Gamma(\operatorname{mod}\mathbb{C}Q)$ to be **regular** if $\mathcal{R}(\mathbb{C}Q)$ contains neither projective nor injective modules. An indecomposable representation is called **regular** if it belongs to a regular component of $\Gamma(\operatorname{mod}\mathbb{C}Q)$ and an arbitrary indecomposable representation is called **regular** if it is a direct sum of indecomposable regular representations.

Example 3.3.11. For Kronecker 2-quiver, we have the Auslander-Reiten quiver as follows

$$\begin{tikzcd}
 & \mathbb{C}\rightrightarrows\mathbb{C}^2 \arrow[dr, shift left] \arrow[dr, shift right] & & \cdots \arrow[dr, shift left] \arrow[dr, shift right] & & & & \mathbb{C}^2\rightrightarrows\mathbb{C} \arrow[dr, shift left] \arrow[dr, shift right] & \\
0\rightrightarrows\mathbb{C} \arrow[ur, shift left] \arrow[ur, shift right] & & \mathbb{C}^2\rightrightarrows\mathbb{C}^3 \arrow[ur, shift left] \arrow[ur, shift right] & & \mathcal{R}(Q) \arrow[u, dashed] & \cdots \arrow[ur, shift left] \arrow[ur, shift right] & & & \mathbb{C}\rightrightarrows 0
\end{tikzcd}$$

Note that the left component contains $P(1)$ and $P(2)$. Thus that is the pre-projective component. The right component contains $I(1)$ and $I(2)$ which means it is the pre-injective component. The middle component is the regular component which contains all the $\mathbb{C}^k\rightrightarrows\mathbb{C}^k$, where $k \geq 0$.

3.3.3 Computing Hom and Ext^1 group

We are going to state the properties of the Auslander-Reiten quiver which is useful for computing the Hom and Ext^1 group.

Let V and W be two indecomposable $\mathbb{C}Q$-module. A path in $\operatorname{mod} A$ from V to W is a sequence

$$V = V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} V_t = W$$

where all the V_i are indecomposable, and all the f_i are nonzero nonisomorphisms. In this case, V is called a **predecessor** of W and W is called a **successor** of V in $\operatorname{mod} A$.

Proposition 3.3.12. *[ASS06, VIII Lemma 2.5]*

- *Let $\mathcal{P}$ be a preprojective component of the quiver $\Gamma(\operatorname{mod}\mathbb{C}Q)$ and V be an indecomposable module in $\mathcal{P}$. Then the number of predecessors of V in $\mathcal{P}$ is finite and any indecomposable $\mathbb{C}Q$-module L such that $\mathrm{Hom}_{\mathbb{C}Q}(L, V) \neq 0$ is a predecessor of V in $\mathcal{P}$. In particular, $\mathrm{Hom}_{\mathbb{C}Q}(L, V) = 0$ for all but finitely many nonisomorphic indecomposable $\mathbb{C}Q$-modules L.*

- *Let $\mathcal{I}$ be a preinjective component of the quiver $\Gamma(\operatorname{mod}\mathbb{C}Q)$ and N be an indecomposable module in $\mathcal{I}$. Then the number of successors of W in $\mathcal{I}$ is finite and any indecomposable $\mathbb{C}Q$-module L such that $\mathrm{Hom}_{\mathbb{C}Q}(W, L) \neq 0$ is*

a predecessor of W in $\mathcal{I}$. In particular, $\mathrm{Hom}_{\mathbb{C}Q}(W, L) = 0$ *for all but finitely many nonisomorphic indecomposable $\mathbb{C}Q$-modules L.*

From the theorem above, we can tell immediately that if V is a predecessor of W, then $\mathrm{Hom}(W, V) = 0$, where V and $W \in \mathcal{P}$ or V and $W \in \mathcal{I}$. We can also understand the Hom group between different connected component in $\Gamma(\mathrm{mod}\,\mathbb{C}Q)$.

Proposition 3.3.13. *[ASS06, Corollary VIII 2.13] Let P, I and R be three indecomposable $\mathbb{C}Q$-module.*

1. *If P is preprojective and R is regular, then* $\mathrm{Hom}(R, P) = 0$.
2. *If P is preprojective and I is preinjective, then* $\mathrm{Hom}(I, P) = 0$.
3. *If R is regular and I is preinjective, then* $\mathrm{Hom}(I, R) = 0$.

We may write as $\mathrm{Hom}(\mathcal{R}, \mathcal{P}) = \mathrm{Hom}(\mathcal{I}, \mathcal{P}) = \mathrm{Hom}(\mathcal{I}, \mathcal{R}) = 0$.

Let us abuse the term 'predecessor' and 'sucessor' to say elements in $\mathcal{P}$ are predecessors of elements in $\mathcal{R}$, $\mathcal{I}$ and elements in $\mathcal{R}$ are predecessors of elements of $\mathcal{I}$. We will do the same for sucessor.

Important lemmas

From the properties of the AR quivers, we observe the following properties which will be useful for computations in later chapters.

Lemma 3.3.14.

$$\mathrm{Ext}^1(\mathcal{P}, \mathcal{R}) = \mathrm{Ext}^1(\mathcal{P}, \mathcal{I}) = \mathrm{Ext}^1(\mathcal{R}, \mathcal{I}) = 0.$$

Proof. Let us first show $\mathrm{Ext}^1(\mathcal{P}, \mathcal{R}) = 0$. If not, there exists a non-split exact sequence

$$0 \longrightarrow R \xrightarrow{f} V \xrightarrow{g} P \longrightarrow 0$$

where $R \in \mathcal{R}$, $P \in \mathcal{P}$ and for some $V \in \mathrm{rep}(Q)$. Decompose $V = \bigoplus V_i$ where V_i are indecomposable. Consider V_i such that $V_i \cap \mathrm{im}\, f \neq 0$ and $g(V_i) \neq 0$. If such a V_i does not exist, the exact sequence will split. Then if $V_i \in \mathcal{R}$, $g|_{V_i} : V_i \to P$

nonzero will contradict Theorem 3.3.13. Similarly, if $V_i \in \mathcal{P}$ or $\mathcal{I}$, it will contradict Theorem 3.3.13. Thus $\mathrm{Ext}^1(\mathcal{P}, \mathcal{R}) = 0$. Repeating the same argument will give us $\mathrm{Ext}^1(\mathcal{P}, \mathcal{I}) = \mathrm{Ext}^1(\mathcal{R}, \mathcal{I}) = 0$. □

Lemma 3.3.15. *If $V, W \in \mathcal{P}$ or $V, W \in \mathcal{I}$ and if $\mathrm{Hom}(V, W) \neq 0$, then*

$$\mathrm{Ext}^1(V, W) = 0.$$

Proof. Consider $V, W \in \mathcal{P}$. If V is projective, then $\mathrm{Ext}(V, W) = 0$ for all 2. Now assume V is not projective. Then by Auslander-Reiten Theorem 3.3.4 which tells $\mathrm{Hom}(W, \tau V) \neq 0$, i.e. we have a map

$$W \longrightarrow \tau V$$

By Theorem 3.3.7 and the remark above, there exists E for V non-projective such that there is an almost split exact sequence

$$0 \longrightarrow \tau V \xrightarrow{f} E \xrightarrow{g} V \longrightarrow 0.$$

We can again decompose $E = \bigoplus E_i$ where E_i are indecomposable. We can further assume $E_i \in \mathcal{P}$ for all i by Theorem 3.3.13. Pick E_i such that $E_i \cap \mathrm{im}(f) \neq 0$ and $g(E_i) \neq 0$. Such an E_i exists because the sequence is non-split. Then $\mathrm{Hom}(\tau V, E_i) \neq 0$ which implies τV is a predecessor of E_i. And $\mathrm{Hom}(E_i, V) \neq 0$ also implies E_i is a predecessor of V. Then there exists a path in AR quiver such that

$$W \to \tau V \to \cdots \to E_i \to \cdots \to V \to \cdots \to W.$$

The last arrow follows from the assumption $\mathrm{Hom}(V, W) \neq 0$. Then we obtain a cyclic in $\mathcal{P}$ which contradicts to Theorem 3.3.12. We can repeat a similar argument for $V, W \in \mathcal{I}$. □

Lemma 3.3.16. *Now if $V, W \in \mathcal{P}$ or $V, W \in \mathcal{I}$, assume $\mathrm{Ext}^1(W, V) \neq 0$, then V is a predecessor of W in the AR quiver. That is $\mathrm{Hom}(W, V) = 0$.*

Proof. Consider $V, W \in \mathcal{P}$. As $\mathrm{Ext}^1(W, V) \neq 0$, we have a non-split exact sequence

$$0 \longrightarrow V \longrightarrow E \longrightarrow W \longrightarrow 0,$$

for some $E \in \mathrm{rep}(Q)$. By repeating the arguments in the proofs of the two theorem above, we can decompose E as sums of indecomposable E_i with $E_i \in \mathcal{P}$. Furthermore, there exists E_i such that V is a predecessor of E_i and E_i is a predecessor of W in $\mathcal{P}$, then we have a path in $\mathcal{P}$:

$$V \to \ldots \to E_i \to \cdots \to W.$$

This shows that V is a predecessor of W. This holds similarly for $V, W \in \mathcal{I}$. $\square$

Combining all the lemmas above, we have if $V, W \in \mathcal{P}$ or $V, W \in \mathcal{I}$, if $\mathrm{Hom}(V, W) \neq 0$, then $\mathrm{Ext}^1(V, W) = 0$. However if $\mathrm{Ext}^1(W, V) \neq 0$, then $\mathrm{Hom}(W, V) = 0$. This is saying that

$$\chi(V, W) = \dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}^1(V, W)$$

actually equals to either $\dim \mathrm{Hom}(V, W)$ or $-\dim \mathrm{Ext}^1(V, W)$.

Chapter 4

Introduction to Scattering Diagrams and the Theta Basis

There are a lot of conjectures in mirror symmetry. Two major ones are Kontsevich's homological mirror symmetry conjecture (HMS) [Kon95] and the Strominger-Yau-Zaslow conjecture (SYZ) [SYZ96]. By HMS, the existence of a canonical basis of sections of ample line bundles is predicted when a particular choice of corresponding Lagrangian sections is given. Those elements in the canonical basis are called theta functions. The SYZ conjecure states that there are Lagrangian fibrations of X and its mirror $\check{X}$ to the same base space B such that the general fibres are dual tori. Both conjectures suggest the existence of theta functions. Morally, given a choice of Lagrangian sections corresponding to ample line bundles, elements of the canonical basis of sections are called theta functions.

With the idea of looking at fibrations from SYZ, Gross and Siebert constructed toric degenerations from affine manifolds with singularities. This construction introduced the idea of scattering diagrams. In [GHKK14], Gross, Hacking, Keel, and Kontsevich relate scattering diagrams with cluster algebras and propose that theta functions also formed the canonical basis for a given cluster algebra.

4.1 Scattering Diagram

In this section, we will go over the definition of scattering diagrams which will be associated to cluster algebras $\mathcal{A}(B)$. To simplify the discussion, we will assume that the skew-symmetrizable matrix B in the initial seed is skew-symmetric in this section.

Later in Chapter 5, we will discuss about rank 2 cluster algebras $\mathcal{A}(b,c)$ with B skew-symmetrizable. Thus the rank 2 scattering diagram associated to $\mathcal{A}(b,c)$ will be defined as an example in Section 4.2.2. This will give a hint to the construction for n-dimensonal scattering diagram as it is nothing more than replacing 2 by n.

Let N be a rank n lattice, $M = \mathrm{Hom}(N,\mathbb{Z})$. Write $M_{\mathbb{R}} = M \otimes \mathbb{R}$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Take $\Bbbk$ to be a field of characteristic 0. Fix a basis $f_1, \ldots, f_n$ of M. Given $m \in M$, we can write $A^m \in \Bbbk[M]$ as $A_1^{a_1} \cdots A_n^{a_n}$ if $m = a_1 f_1 + \cdots + a_n f_n$.

We further fix a skew-symmetric bilinear form on N

$$\{\cdot,\cdot\} : N \times N \to \mathbb{Z}.$$

Define the skew symmetric ϵ with $\epsilon_{ij} = \{e_i, e_j\}$. The skew-symmetric form induces a map

$$\begin{aligned} p^* : N &\longrightarrow M \\ n &\mapsto \{n,\cdot\} \end{aligned}$$

In this thesis, we will assume p^* to be injective. If p^* is not injective, we can replace initial data so that p^* is injective in the new setting. However, this replacement is tedious. Therefore, in the following set up, we will make the assumption to simplify the formulation. One may refer to Appendix B in [GHKK14] for the full details.

A **seed data** $s = (e_i)$ is a basis $e_1, \ldots, e_n$ of N. Note that s may or may not the standard basis of N. Denote

$$N^+ = \{\sum a_i e_i | a_i \geq 0, \sum a_i > 0\}$$

Definition 4.1.1. *A wall in $M_{\mathbb{R}} = M \otimes_Z \mathbb{R}$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where*

- $\mathfrak{d} \subseteq M_{\mathbb{R}}$, *support of the wall, is a convex rational polyhedral cone of codimension one, contained in* $n^{\perp}$ *for some* $n \in N^{+}$,
- $f_{\mathfrak{d}} \in \mathbb{C}[[A_1, \ldots, A_n]]$ *such that* $f_{\mathfrak{d}} = 1 + \sum_{k \geq 1} c_k A^{kp^*(n)}$ *for some* $c_k \in (A_1, \cdots A_n)$.

A wall $(\mathfrak{d}, f_{\mathfrak{d}})$ is called **incoming** if $p^*(n) \in \mathfrak{d}$. Otherwise it is called **outgoing**. Later we will subdivide the walls into small walls and the call the walls which contains $-p^*(n)$ as the **outgoing piece**.

Definition 4.1.2. *A scattering diagram* $\mathfrak{D}$ *is a collection of walls such that, for each* $k \geq 0$, *the set*

$$\{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} \mid f_{\mathfrak{d}} \neq 1 \bmod (A_1, \cdots, A_n)^k\}$$

is finite. The support of a scattering diagram, Supp$(\mathfrak{D})$, *is the union of the supports of its walls. We will write*

$$Sing(\mathfrak{D}) = \bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial \mathfrak{d} \cup \bigcup_{\mathfrak{d}_1, \mathfrak{d}_2 \in \mathfrak{D}, \dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = n-2} \mathfrak{d}_1 \cap \mathfrak{d}_2.$$

Now consider a smooth immersion

$$\gamma : [0, 1] \to M_{\mathbb{R}} \backslash \{0\}$$

with endpoints not in the support of $\mathfrak{D}$. Assume γ is transversal to each wall of $\mathfrak{D}$. For each power $k \geq 1$, we can find the sequence of numbers $0 < t_1 \leq t_2 \leq \cdots \leq t_s < 1$ with $\gamma(t_i) \in \mathfrak{d}_i$ for some i with $f_{\mathfrak{d}_i} \neq 1 \bmod \mathfrak{m}^k$ and $\mathfrak{d}_i \neq \mathfrak{d}_j$ whenever $t_i = t_j$. For each i, define $\mathfrak{p}_{\mathfrak{d}_i} \in \mathrm{Aut}_{\Bbbk-alg}\big(\Bbbk[[A_1, \ldots, A_n]]\big)$, the **path-ordered product** as

$$\mathfrak{p}_{\mathfrak{d}_i}(A^m) = A^m f_{\mathfrak{d}_i}^{\langle m, n_0 \rangle}, \tag{4.1.1}$$

where the lattice point $n_0 \in N$ is primitive, annihilates the tangent space to $\mathfrak{d}_i$, and is uniquely determined by the sign convention $\langle n_0, \gamma'(t_i) \rangle < 0$. Take $\mathfrak{p}^k_{\gamma, \mathfrak{D}} := \mathfrak{p}_{\mathfrak{d}_s} \circ \cdots \circ \mathfrak{p}_{\mathfrak{d}_1}$. We can then define the *path-ordered product* as

$$\mathfrak{p}_{\gamma, \mathfrak{D}} = \lim_{k \to \infty} \mathfrak{p}^k_{\gamma, \mathfrak{D}}. \tag{4.1.2}$$

With the path-ordered product, we can talk about equivalence among scattering diagrams.

Definition 4.1.3. *Two scattering diagram $\mathfrak{D}$, $\mathfrak{D}'$ are* ***equivalent*** *if $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma,\mathfrak{D}'}$ for all path γ for which both are defined.*

Note that the path-ordered product depends on the path γ on $\mathfrak{D}$. We will call $\mathfrak{D}$ consistent if the automorphism only depends on the endpoint of the path γ.

Definition 4.1.4. *A scattering diagram is* ***consistent*** *if $\mathfrak{p}_{\gamma,\mathfrak{D}}$ only depends on the endpoints of γ for any path γ for which $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is well defined.*

Not every scattering diagram $\mathfrak{D}$ is consistent; however, there always exists a consistent scattering diagram which contains $\mathfrak{D}$ as seen in the following theorem.

Theorem 4.1.5. *(Kontsevich-Soibelman [KS06], Gross-Siebert [GS11]) Given a scattering diagram $\mathfrak{D}$, there always exists a consistent scattering diagram $\mathfrak{D}'$ which contains $\mathfrak{D}$ such that $\mathfrak{D}' \setminus \mathfrak{D}$ only consists only of outgoing walls. And it is unique up to equivalent.*

The next result explains how to obtain Laurent polynomials out of scattering diagrams.

Theorem 4.1.6. *Let $\mathfrak{D} := \mathfrak{D}_s$ be as constructed above and consider a Laurent polynomial $f \in \mathbb{Z}[M]$. For any path γ which is regular with respect to $\mathfrak{D}$, $\mathfrak{p}_{\gamma,\mathfrak{D}}(f)$ can be viewed as an element of $\mathbb{Z}(M)$. If for any such γ in $M_{\mathbb{R}}$, with starting point in the first quadrant and endpoint in one of the chambers of $\mathfrak{D}$, we have that $\mathfrak{p}_{\gamma,\mathfrak{D}}(f)$ lies in $\mathbb{Z}[M]$, then f is a universal Laurent polynomial.*

Proof. This is [GHKK14, Theorem 4.4] applied to the case at hand. Specifically, let $\mathcal{A}$ be the cluster variety defined by the given choice of seed. By definition, $\mathcal{A}$ is obtained by gluing together a collection of tori via cluster transformations and thus a regular function on $\mathcal{A}$ is precisely a universal Laurent polynomial. On the other hand, in [GHKK14, Section 4] another variety $\mathcal{A}'$ is defined. This is done by associating a torus $\mathcal{A}'_\tau := \operatorname{Spec} \mathbb{Z}[M]$ to a chamber $\tau \subseteq M_{\mathbb{R}}$ of $\mathfrak{D}$. For any two chambers τ, τ' we can glue $\mathcal{A}'_\tau$ to $\mathcal{A}'_{\tau'}$ using the rational map defined on function fields by $\mathfrak{p}_{\gamma,\mathfrak{D}} : \mathbb{Z}(A_1, A_2) \to \mathbb{Z}(A_1, A_2)$, where γ is a path beginning in τ' and ending in τ. Performing these gluings gives $\mathcal{A}'$.

Now [GHKK14, Theorem 4.4] gives an explicit isomorphism between $\mathcal{A}$ and $\mathcal{A}'$, and thus the algebra of regular functions on $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic. Furthermore, this isomorphism restricts to the identity on the torus of $\mathcal{A}$ corresponding to the initial seed and the torus of $\mathcal{A}'$ corresponding to the positive chamber. In particular, a function f on this torus extends to a function on $\mathcal{A}'$ if $\mathfrak{p}_{\gamma,\mathfrak{D}}(f)$ lies in $\mathbb{Z}[M]$ for any path γ from the positive chamber to any other chamber. This shows the characterization of universal Laurent polynomials. $\square$

4.2 The association to cluster algebra

With a fixed seed data, we will now construct a scattering diagram. Set

$$\mathfrak{D}_{in,s} = \{(e_i^{\perp}, 1 + A^{p^*(e_i)}) | i = 1, \dots n\}. \tag{4.2.1}$$

Then by Theorem 4.1.5, there exists a consistent scattering diagram $\mathfrak{D}_s$, in the sense of Definition 4.1.2, containing $\mathfrak{D}_{in,s}$ and $\mathfrak{D}_s \setminus \mathfrak{D}_{in,s}$ contains outgoing walls only.

Similar to the definition of cluster algebra, we would also mutate the seed data s. For $k = 1, \dots n$, the mutated seed data $\mu_k(s)$ is a new basis

$$e_i' = \begin{cases} e_i + [\epsilon_{ik}]_+ e_k & i \neq k \\ -e_k & i = k \end{cases}$$

where $[\epsilon_{ik}]_+ = \max(0, \epsilon_{ik})$. The matrix $\epsilon' = \{e_i', e_j'\}$ is defined by the new set of (e_i).

By repeating what we did in Equation 4.2.1, we will obtain a new scattering diagram $\mathfrak{D}_{in,\mu_k(s)}$ and the corresponding consistent scattering diagram $\mathfrak{D}_{\mu_k(s)}$. As both scattering diagrams $\mathfrak{D}_s$, $\mathfrak{D}_{\mu_k(s)}$ correspond to the same cluster algebra, we are going to see they are the 'same' up to a transformation.

First, we let

$$\mathcal{H}_{k,+} = \{m \in M_{\mathbb{R}} | \langle e_k, m \rangle \geq 0\}, \qquad \mathcal{H}_{k,-} = \{m \in M_{\mathbb{R}} | \langle e_k, m \rangle \leq 0\}$$

Define the piecewise linear transformation $T_k : M \to M$ by

$$T_k(m) = \begin{cases} m + \langle e_k, m \rangle p^*(e_k) & m \in \mathcal{H}_{k,+} \\ m & m \in \mathcal{H}_{k,-} \end{cases} \tag{4.2.2}$$

for $m \in M$. We denote $T_{k,-} = T_k|_{\mathcal{H}_{k,-}}$ and $T_{k,+} = T_k|_{\mathcal{H}_{k,+}}$.

T_k acts on $\mathfrak{D}_s$ to give us a new scattering diagram $T_k(\mathfrak{D}_s)$. More precisely,

1. for each wall $(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}_s \setminus \{\mathfrak{d}_k\}$, $T_k(\mathfrak{d})$ leads us two possible walls, namely

$$(T_k(\mathfrak{d} \cap \mathcal{H}_{k,-}), T_{k,-}(f_\mathfrak{d})), (T_k(\mathfrak{d} \cap \mathcal{H}_{k,+}), T_{k,+}(f_\mathfrak{d})).$$

   If $\dim \mathfrak{d} \cap \mathcal{H}_{k,\pm} < \text{rank } M - 1$, we will take away $(T_k(\mathfrak{d} \cap \mathcal{H}_{k,\pm}), T_{k,\pm}(f_\mathfrak{d}))$.

   $T(f_\mathfrak{d})$ is the formal power series obtained by applying T to each exponent in $f_\mathfrak{d}$.

2. $T_k(\mathfrak{d}_s)$ contains the wall $\mathfrak{d}'_k = (e_k^\perp, 1 + A^{-p^*(e_k)})$.

Then we have

Theorem 4.2.1. *[GHKK14, Theorem 1.22] $T_k(\mathfrak{D}_s)$ is a consistent scattering diagram for $\mu_k(s)$ and $N^+_{\mu_k(s)}$, where $N^+_{\mu_k(s)} = \{\sum a_i e'_i | a_i \geq 0, \sum a_i > 0\}$, $\mu_k(s) = (e'_i)$. Furthermore, $\mathfrak{D}_{\mu_k(s)}$ and $T_k(\mathfrak{D}_s)$ are equivalent.*

4.2.1 Chamber structure

In this section, we will discuss the chamber structure of scattering diagram. Last section tells us that scattering diagrams after mutations are equivalent. This section will tell us that we can pull back the information into the initial scattering diagram.

Given a seed data $s = (e_1, \ldots, e_n)$, define

$$\mathcal{C}_s^+ := \mathcal{C}^+ = \{m \in M_\mathbb{R} | \langle e_i, m \geq 0, i = 1, \ldots n\}$$

$$\mathcal{C}_s^- := \mathcal{C}^- = \{m \in M_\mathbb{R} | \langle e_i, m \leq 0, i = 1, \ldots n\}$$

We will call $\mathcal{C}_s^+$ as the positive chamber. After mutation at vertex k, we will have another scattering diagram $\mathfrak{D}_{\mu_k(s)}$ and the corresponding positive chamber $\mathcal{C}^+_{\mu_k(s)}$. Note that the closures of $T_k^{-1}(\mathcal{C}^+_{\mu_k(s)})$ and $\mathcal{C}_s^+$ share a common codimension one face given by intersection with $e_k^\perp$. This is telling us that the scattering diagram is somehow a dual of the exchange graph of cluster algebra.

In general, take an arbitrary seed s_v and let v be the corresponding vertex in the exchange graph. Then consider the sequence of mutations $\mu_{k_1}, \ldots, \mu_{k_p}$ from the initial seed s to s_v. Denote $T_v = T_{k_p} \circ \cdots \circ T_{k_1}$ as the composition of the semilinear map T_{k_i}. Then we have $T_v(\mathfrak{D}_s) = \mathfrak{D}_{s_v}$ from Theorem 4.2.1. We pull back the positive chamber in $\mathfrak{D}_{s'}$ and hence define

$$\mathcal{C}_v^+ = T_v^{-1}(\mathcal{C}_{s_v}^+)$$

which is a chamber in $\mathfrak{D}_s$ corresponding to the vertex v. We further define Δ_s for the set of chambers $\mathcal{C}_v^+$ for v in the exchange graph. Note that the union of all the $\mathcal{C}_v^+$ may not equal to the whole $M_{\mathbb{R}}$. For example, in the next sections, we will see there is a region called the badlands which contained in $M_{\mathbb{R}} \setminus \cup_v \mathcal{C}_v^+$.

4.2.2 Example: Two-dimensional case

Let us illustrate what have just defined in 2-dimension explicitly. Now $M \cong \mathbb{Z}^2$. Then a two-dimensional scattering diagram is a diagram with walls that are rays or lines passing through the origin. The corresponding cluster algebra is $\mathcal{A}(b, c)$ defined in Example 3.1.1. As noted in the beginning of this chapter, the skew-symmetrizable matrix B in the initial seed of $\mathcal{A}(b, c)$ is $B = \begin{pmatrix} 0 & c \\ -b & 0 \end{pmatrix}$. which is not skew-symmetric.

First, we start with the fixed data: the lattice N, the dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$, the skew symmetric form $\{\cdot, \cdot\} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that this example also demonstrates the skew-symmetrizable matrix B in Section 3.1 and ϵ in Section 4.2 differ by a transpose.

In this case, we add a bit more information about the fixed data:

1. the integers (b, c) where the greatest common divisor of b, c is 1;
2. a sublattice $N^\circ = b\mathbb{Z} \times c\mathbb{Z}$ of N;
3. $M^\circ = \mathrm{Hom}(N^\circ, \mathbb{Z})$.

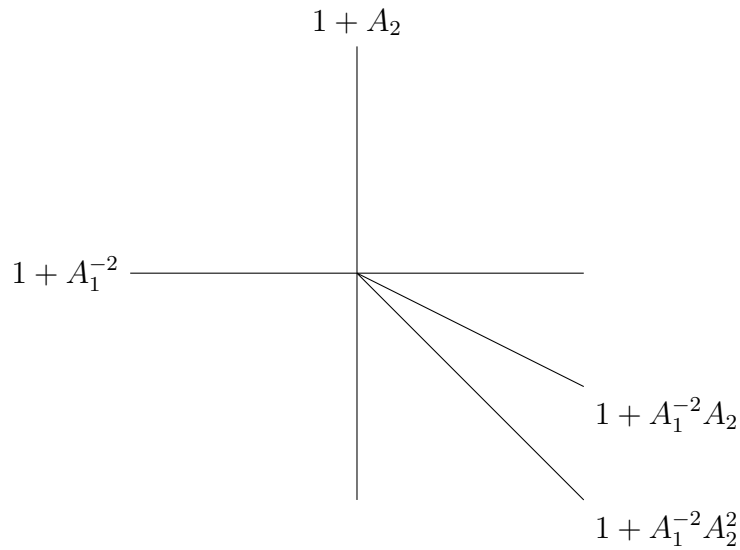

Figure 4.1: The scattering diagram $\mathfrak{D}_{(2,1)}$.

Then we consider the initial seed data s which is a standard basis ($e_1 = (1,0), e_2 = (0,1)$) of N. Then the dual basis of M would be (e_1^*, e_2^*) of M, and the corresponding basis for M° would be $f_1 = \frac{1}{b}e_1^*, f_2 = \frac{1}{c}e_2^*$. Then we continue to define the map

$$p^* : N \longrightarrow M^\circ,$$

with $n \mapsto \{n, \cdot\}$. Note that in this case $n = (n_1, n_2) \mapsto (-n_2, n_1) = -n_2 b f_1 + n_1 c f_2$. Thus we will follow what we did in last section, the constructed scattering diagram would be

$$\mathfrak{D}_{in,s} = \{(\mathbb{R}e_1, 1 + A^{(-b,0)}), (\mathbb{R}e_2, 1 + A^{(0,c)})\},$$

with components lie in M°. Then we obtain the consistent scattering diagram $\mathfrak{D}_{(b,c)}$ again by Theorem 4.1.5. The case of $\mathfrak{D}_{(2,1)}$ is illustrated in Figure 4.1. Note that since the initial diagram $\mathfrak{D}_{\text{in},(2,1)}$ consists of the walls $\big(\mathbb{R}(-1,0), 1 + A_1^{-2}\big)$ and $\big(\mathbb{R}(0,1), 1+A_2\big)$, the associated consistent scattering diagram $\mathfrak{D}_{(2,1)}$ contains $\mathfrak{D}_{\text{in},(2,1)}$ together with the two walls $\big(\mathbb{R}_{\leq 0}(-1,1), 1 + A_1^{-2}A_2^2\big)$ and $\big(\mathbb{R}_{\leq 0}(-2,1), 1 + A_1^{-2}A_2\big)$.

While this example portrays a scattering diagram with finitely many rays, the diagram $\mathfrak{D}_{(b,c)}$ will consist of an infinite number of rays precisely when $bc \geq 4$. Figure 4.2 illustrate the $\mathfrak{D}_{(2,2)}$ which is with infinitely rays of slope $(n, -(n+1))$, $(n+1, -n)$ and $(1,1)$. In general if $bc \geq 4$ then there are infinitely many discrete

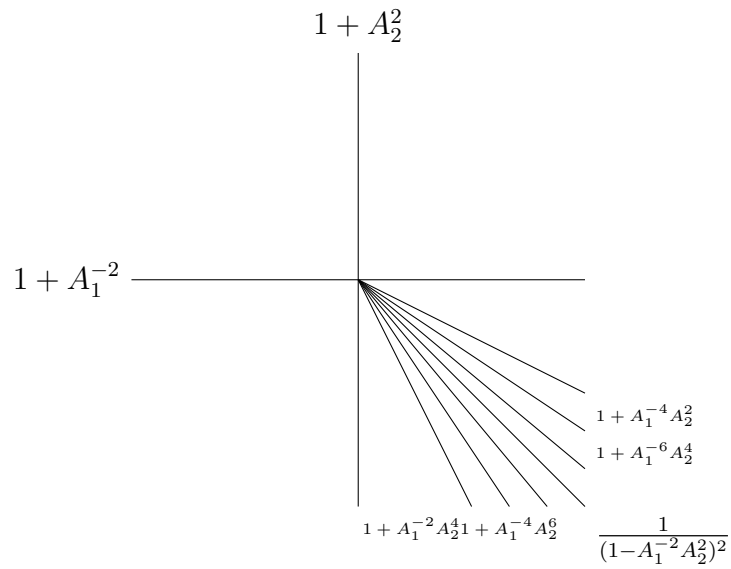

Figure 4.2: The scattering diagram $\mathfrak{D}_{(2,2)}$.

outgoing rays converge to the ray of slopes $-(bc \pm \sqrt{bc(bc-4)})/2b$. Then every ray of rational slope appears in the region within these two rational slopes. This region will be named as the **badlands**.

A detailed description of the diagram $\mathfrak{D}_{(b,c)}$ which appear for $bc \geq 4$ can be found in [GHKK14, Example 1.30].

4.3 Broken lines and theta functions

Broken lines were introduced in [Gro10] as a way of describing holomorphic disks which appear in mirror symmetry in a tropical manner. Their theory was further developed in [CPS10], and then used in [GHK11] and [GHKK14] to construct canonical bases in various circumstances.

Definition 4.3.1. *Let $\mathfrak{D}$ be a scattering diagram, $m \in M - \{0\}$ and $\mathcal{Q} \in M_{\mathbb{R}} - Supp(\mathfrak{D})$. A **broken line** for m with endpoint $\mathcal{Q}$ is a piecewise linear continuous proper path $\gamma : (-\infty, 0) \to M_{\mathbb{R}} \setminus Sing(\mathfrak{D})$ with a finite number of domains of linearity and a collection of monomials. A monomial $c_L A^{m_L} \in \Bbbk[M]$ is attached to each domain of linearity $L \subseteq (-\infty, 0)$ of γ. The path γ and the monomial $c_L A^{m_L}$ need to satisfy the following conditions:*

- $\gamma(0) = \mathcal{Q}$.
- *If L is the first (i.e., unbounded) domain of linearity of γ, then $c_L A^{m_L} = A^m$.*
- *For t in a domain of linearity, $\gamma'(t) = -m_L$.*
- *γ bends only when it crosses a wall. If γ bends from the domain of linearity L to L' when crossing $(\mathfrak{d}, f_\mathfrak{d})$, then $c_{L'} A^{m_{L'}}$ is a term in $\mathfrak{p}_{\gamma,\mathfrak{d}}(c_L A^{m_L})$.*

Definition 4.3.2. *Let $\mathfrak{D}, m, \mathcal{Q}$ as in Definition 4.3.1. For a broken line γ with initial slope m and endpoint $\mathcal{Q}$, denote $I(\gamma) = m$ and $b(\gamma) = \mathcal{Q}$. Define $Mono(\gamma) = c(\gamma)A^{F(\gamma)}$ to be the monomial $c_L A^{m_L}$ attached to the last domain of linearity L of γ. We further define*

$$\vartheta_{\mathcal{Q},m} = \sum_\gamma Mono(\gamma),$$

where the sum is over all broken lines for m with endpoint $\mathcal{Q}$.

The following summarizes the main properties of the theta functions as shown in [CPS10] and [GHKK14].

Theorem 4.3.3. *1. If $\mathfrak{D}$ is any consistent scattering diagram, $\mathcal{Q}$ and $\mathcal{Q}'$ are two general irrational points on $M_\mathbb{R} \setminus Supp(\mathfrak{D})$, and γ is a path joining $\mathcal{Q}$ to $\mathcal{Q}'$, then $\mathfrak{p}_{\gamma,\mathfrak{D}}(\vartheta_{\mathcal{Q},m}) = \vartheta_{\mathcal{Q}',m}$.*

2. (a) If $\mathcal{Q}$ and m lie in the interior of the same chamber of $\mathfrak{D}$, then $\vartheta_{\mathcal{Q},m} = x^m$.

(b) For $n = 2$, if $\mathcal{Q}$ lies in the interior of a chamber of $\mathfrak{D}$, then $\vartheta_{\mathcal{Q},m}$ is a Laurent polynomial for any m.

(c) For $n = 2$, if $\mathcal{Q}$ lies in the interior of the first quadrant, then $\vartheta_{\mathcal{Q},m}$ is a universal Laurent polynomial for any m.

Proof. (1) is a main result of [CPS10], see also [GHKK14, Theorem 3.5] for its application to scattering diagrams in the current context. (2a) is [GHKK14, Proposition 3.8] if $\mathcal{Q}$ and m are both in the positive quadrant of $M_\mathbb{R}$. If $\mathcal{Q}$ and m are in some other chamber, say σ, then by [GHKK14, Construction 1.38], there is a scattering diagram $\mathfrak{D}'$ obtained from a mutation of the initial seed defining $\mathfrak{D}$ and a piecewise linear map $T_v : M_\mathbb{R} \to M_\mathbb{R}$ which takes the support of $\mathfrak{D}$ to the support

of $\mathfrak{D}'$, and such that the positive chamber of $\mathfrak{D}'$ pulls back to σ. Furthermore, there is a one-to-one correspondence between broken lines for $\mathfrak{D}$ and $\mathfrak{D}'$ by [GHKK14, Proposition 3.6]. Thus the claim follows from [GHKK14, Proposition 3.8] applied to $\mathfrak{D}'$.

(2b) is [GHKK14, Example 7.18]. In slightly more detail, let $\Theta \subseteq M$ denote the set of $m \in M$ for which $\vartheta_{\mathcal{Q},m}$ is a Laurent polynomial for $\mathcal{Q}$ general in the first quadrant of $M_{\mathbb{R}}$. By [GHKK14, Theorem 7.16,(3)], Θ contains all points of M contained in chambers (i.e., the set of points denoted as $\Delta_V^+(\mathbb{Z})$ in [GHKK14, Theorem 7.16,(3)]). Thus in particular, Θ contains all integral points in the first three quadrants of $M_{\mathbb{R}}$. But by [GHKK14, Theorem 7.16,(4)], Θ is closed under addition, and hence consists of all points in M. It then follows that $\vartheta_{\mathcal{Q},m}$ is a Laurent polynomial for $\mathcal{Q}$ in any chamber by [GHKK14, Proposition 7.1].

Finally, (2c) follows from from (2b) and Theorem 4.1.6. □

Remark 4.3.4. If $\mathcal{Q}$ lies inside the positive chamber, and $m \in M$ lies in one of the chambers which is reachable from the positive chamber, i.e. $m \in \mathcal{C}^+ \in \Delta_s$ defined in Section 4.2.1. Then $\vartheta_{\mathcal{Q},m} = \mathfrak{p}_{\gamma,\mathfrak{D}}(A^m)$ for the path γ joining from chamber containing m to $\mathcal{Q}$. In this case, $\vartheta_{\mathcal{Q},m}$ is a cluster monomial. Further if $\mathcal{Q}$ is in the positive chamber, and $\vartheta_{\mathcal{Q},m}$ is a finite sum, then $\vartheta_{\mathcal{Q},m}$ is an element of the cluster algebra.

Remark 4.3.5. Theta functions are 'preserved' under mutation. More precisely, let $\mathfrak{D}_s$ be the scattering diagram associated to the seed data s, $\mathfrak{D}_{\mu(s)}$ be the one associated to $\mu(s)$ and T the semi-linear map defined in (4.2.2).

Then T defines a one-to-one correspondence from broken lines in $\mathfrak{D}_s$ with initial slope m and endpoint $\mathcal{Q}$ to broken lines in $\mathfrak{D}_s$ with initial slope $T(m)$ and endpoint $T(\mathcal{Q})$. Let $\vartheta^{\mu(s)}$ denote theta functions defined on $\mathfrak{D}_{\mu(s)}$. Then for $\mathcal{Q} \in \mathcal{H}_\pm$, we have

$$\vartheta^{\mu(s)}_{T(\mathcal{Q}),T(m)} = T_\pm(\vartheta_{\mathcal{Q},m}).$$

A proof of similar result is given in Theorem 5.3.3 so we are not going to repeat here.

Let us compute some broken lines and theta functions in the two dimensional scattering diagram shown in Section 4.2.2.

Example 4.3.6. Consider the scattering diagram $\mathfrak{D}_{(2,2)}$ in Figure 4.2 and let $\mathcal{Q}$ be a small irrational perturbation of the point $(1.5, 1)$. There are three broken lines with initial exponent $m = (1, -1)$ and endpoint $\mathcal{Q}$ as shown in Figure 4.3. First of all, we can have a broken line γ_1 which does not bend. Therefore

$$\mathrm{Mono}(\gamma_1) = A_1 A_2^{-1}.$$

There is the broken line γ_2 which bends only at the x-axis. Since

$$\mathfrak{p}_{(-1,1),\mathbb{R}(-1,0)}(A_1 A_2^{-1}) = A_1 A_2^{-1}(1 + A_1^{-2}) = A_1 A_2^{-1} + A_1^{-1} A_2^{-1},$$

to bend we need to choose the second term and obtain

$$\mathrm{Mono}(\gamma_2) = A_1^{-1} A_2^{-1}.$$

The last broken line γ_3 bends both at the x- and y-axes, the latter bend coming from

$$\mathfrak{p}_{(1,1),\mathbb{R}(0,1)}(A_1^{-1} A_2^{-1}) = A_1^{-1} A_2^{-1} + A_1^{-1} A_2.$$

This time we have

$$\mathrm{Mono}(\gamma_3) = A_1^{-1} A_2.$$

Thus the theta function associated to $m = (1, -1)$ with endpoint point $\mathcal{Q}$ is

$$\vartheta_{\mathcal{Q},(1,-1)} = A_1 A_2^{-1} + A_1^{-1} A_2^{-1} + A_1^{-1} A_2.$$

Example 4.3.7. Let us try one more calculation with broken lines. We take the same scattering diagram $\mathfrak{D}_{(2,2)}$ in Figure 4.2. Now take the initial exponent $m = (2, -2)$ with the same endpoint $\mathcal{Q}$. By similar calculations we get

$$\vartheta_{\mathcal{Q},(2,-2)} = A_1^2 A_2^{-2} + A_1^{-2} A_2^2 + A_1^{-2} A_2^{-2} + 2A_2^{-2} + 2A_1^{-2}.$$

Note that

$$\vartheta_{\mathcal{Q},(2,-2)} = \left(\vartheta_{\mathcal{Q},(1,-1)}\right)^2 - 2. \tag{4.3.1}$$

In the scattering diagram $\mathfrak{D}_{(2,2)}$ considered here, the ray with exponent $(1, -1)$ does not lie in the interior of any chamber. So neither $\vartheta_{\mathcal{Q},(1,-1)}$ nor $\vartheta_{\mathcal{Q},(2,-2)}$ is a cluster monomial. Later in Remark 7.1.2, we will give a reason from the quiver representation point of view about why we have $\vartheta_{\mathcal{Q},(2,-2)} = \left(\vartheta_{\mathcal{Q},(1,-1)}\right)^2 - 2$.

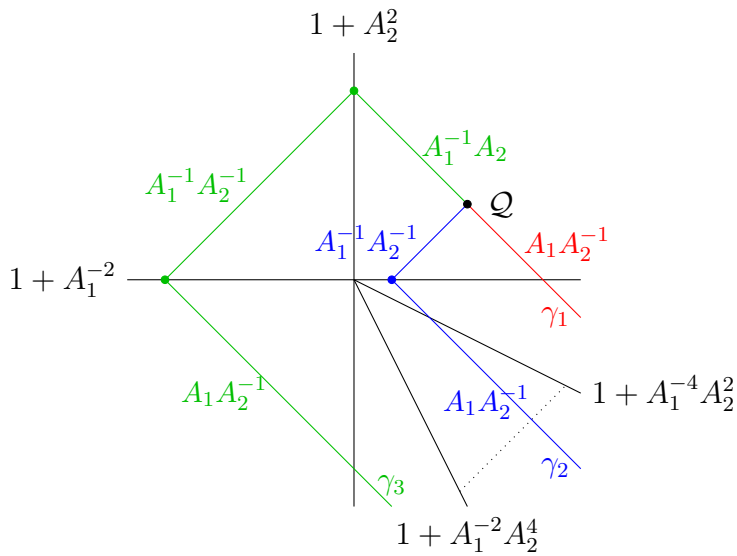

Figure 4.3: The scattering diagram $\mathfrak{D}_{(2,2)}$ and the broken lines described in Example 4.3.6.

This chapter overlaps with part of the paper "The Greedy Basis equals the Theta Basis". It is a joint work with Mark Gross, Greg Muller, Gregg Musiker, Dylan Rupel, Salvatore Stella, Harold Williams.

Chapter 5

The Greedy Basis is the Theta Basis

5.1 Introduction

Many bases for cluster algebras have been proposed, e.g. Generic Gases [GLS12], cluster monomials [Kel14], atomic bases [DT11], Bracelet [MSW13] and many others. In general, one expects any cluster algebra to admit several natural bases related in potentially subtle ways. A basic example of this is the relationship between the dual canonical and dual semicanonical bases of the coordinate ring of the positive unipotent subgroup of a simple algebraic group. Determining whether or not two constructions of canonical bases in a cluster algebra are the same is nontrivial.

One of the other constructions of a basis is the greedy basis [LLRZ14]. The greedy basis is defined for rank 2 cluster algebras so that all of its elements, not just cluster variables, have positive Laurent expansions. The resulting coefficients can be computed by enumerating combinatorial objects called compatible pairs which are related to maximal Dyck paths. By studying the behaviors of the bending of broken lines, we are able to show that the theta basis shares the same support with the greedy basis. This proves

Theorem 5.1.1. *[CGM$^+$15] The greedy basis and the theta basis of a rank 2 cluster*

algebra coincide.

5.2 The greedy bases

We will work on rank 2 cluster algebra defined in Example 3.1.1. From the result of Fomin-Zelevinsky, $\mathcal{A}(b,c)$ is actually a subalgebra of $\mathbb{Z}[A_1^{\pm 1}, A_2^{\pm 1}]$, rather than merely a subalgebra of $\mathbb{Q}(A_1, A_2)$.

Theorem 5.2.1. *[FZ02, Theorem 3.1] Given any cluster variable x_j, we have $x_j \in \mathbb{Z}[x_k^{\pm 1}, x_{k+1}^{\pm 1}]$ for every $k \in \mathbb{Z}$.*

We will denote by $\mathbb{Z}_{\geq 0}[x_k^{\pm 1}, x_{k+1}^{\pm 1}]$ the subspace of Laurent polynomials with positive coefficients. An element of $\mathbb{Q}(A_1, A_2)$ is a *universal Laurent polynomial* (resp. *positive universal Laurent polynomial*) if it is contained in $\mathbb{Z}[x_k^{\pm 1}, x_{k+1}^{\pm 1}]$ (resp. $\mathbb{Z}_{\geq 0}[x_k^{\pm 1}, x_{k+1}^{\pm 1}]$) for every $k \in \mathbb{Z}$. A primary result of [FZ07], specialized to the rank 2 setting, states that $\mathcal{A}(b,c)$ is precisely the set of univeral Laurent polynomials in $\mathbb{Q}(A_1, A_2)$.

Theorem 5.2.2. *[LS13, Rup12] Each cluster variable of $\mathcal{A}(b,c)$ is positive.*

An element of $\mathbb{Z}[A_1^{\pm 1}, A_2^{\pm 1}]$ is called *pointed at* $(a_1, a_2) \in \mathbb{Z}^2$ if it can be written in the form

$$A_1^{-a_1} A_2^{-a_2} \sum_{p_1, p_2 \geq 0} c(p_1, p_2) A_1^{bp_1} A_2^{cp_2},$$

where $c(p_1, p_2) \in \mathbb{Z}$ with $c(0,0) = 1$.

Proposition 5.2.3. *[LLZ14a, Proposition 1.5] Let z be pointed at $(a_1, a_2) \in \mathbb{Z}^2$ and suppose $z \in \mathbb{Z}_{\geq 0}[x_0^{\pm 1}, x_1^{\pm 1}] \cap \mathbb{Z}_{\geq 0}[x_1^{\pm 1}, x_2^{\pm 1}] \cap \mathbb{Z}_{\geq 0}[x_2^{\pm 1}, x_3^{\pm 1}]$. Then the pointed coefficients $c(p_1, p_2)$ satisfy the following recursive inequality:*

$$c(p_1, p_2) \geq \max\left(\sum_{k=1}^{p_1} (-1)^{k-1} c(p_1 - k, p_2) \binom{a_2 - cp_2 + k - 1}{k}, \right. \\ \left. \sum_{j=1}^{p_2} (-1)^{j-1} c(p_1, p_2 - j) \binom{a_1 - bp_1 + j - 1}{j} \right). \tag{5.2.1}$$

A positive element of $\mathcal{A}(b,c)$ is called *indecomposable* if it cannot be written as a sum of two positive elements. In the search for positive bases of $\mathcal{A}(b,c)$ one is naturally led to investigate the indecomposable positive elements. A sufficient condition for a positive pointed element to be indecomposable is the inequality (5.2.1) being an equality. It turns out that this requirement alone uniquely determines a collection of elements of $\mathcal{A}(b,c)$ with nice properties.

Theorem 5.2.4. *[LLZ14a, Theorem 1.7] For any $(a_1,a_2)\in\mathbb{Z}^2$ there exists a unique indecomposable positive element $x[a_1,a_2]\in\mathcal{A}(b,c)$ which is pointed at (a_1,a_2) and whose pointed coefficients satisfy the recursion*

$$c(p_1,p_2)=\max\left(\sum_{k=1}^{p_1}(-1)^{k-1}c(p_1-k,p_2)\binom{a_2-cp_2+k-1}{k},\right.$$
$$\left.\sum_{j=1}^{p_2}(-1)^{j-1}c(p_1,p_2-j)\binom{a_1-bp_1+j-1}{j}\right). \tag{5.2.2}$$

Moreover, the collection $\{x[a_1,a_2]:(a_1,a_2)\in\mathbb{Z}^2\}$ is a basis of $\mathcal{A}(b,c)$ which contains the cluster monomials and is independent of the choice of an initial cluster.

We will call $x[a_1,a_2]$ the *greedy element* pointed at (a_1,a_2) and call $\{x[a_1,a_2]:(a_1,a_2)\in\mathbb{Z}^2\}$ the *greedy basis* of $\mathcal{A}(b,c)$. In view of the definition of pointed elements, (a_1,a_2) is the $\mathbf{d}$-*vector* of $x[a_1,a_2]$; we refer to [FZ07] for the definitions and basic properties of $\mathbf{d}$-vectors. In order to better connect with the scattering diagram approach from Section 4.1, we now switch our point of view and consider ordinary support rather than pointed support. Given a Laurent polynomial $f=\sum_{m\in M}c_mA^m$ in $\mathbb{Z}[A_1^{\pm1},A_2^{\pm1}]$, the *support* of f is the set

$$\{m\in M\mid c_m\neq 0\}.$$

Theorem 5.2.5. *[LLZ14a, Proposition 4.1], [LLZ14b, Corollary 3.5] For $(a_1,a_2)\in\mathbb{Z}^2$, the smallest (possibly degenerate) lattice quadrilateral R_{a_1,a_2} containing the support of $x[a_1,a_2]$ is determined as follows.*

1. *If $a_1\le 0$ and $a_2\le 0$, then $R_{a_1,a_2}=\{(-a_1,-a_2)\}$.*

2. *If $a_1\le 0<a_2$, then $R_{a_1,a_2}=\{(p_1,-a_2):-a_1\le p_1\le -a_1+ba_2\}$.*

3. *If* $a_2 \leq 0 < a_1$, *then* $R_{a_1,a_2} = \{(-a_1, p_2) : -a_2 \leq p_2 \leq -a_2 + ca_1\}$.

4. *If* $0 < ba_2 \leq a_1$, *then*
$R_{a_1,a_2} = \{(p_1, p_2) : -a_1 \leq p_1 \leq -a_1 + ba_2,\ -a_2 \leq p_2 \leq -a_2 - cp_1\}$.

5. *If* $0 < ca_1 \leq a_2$, *then*
$R_{a_1,a_2} = \{(p_1, p_2) : -a_1 \leq p_1 \leq -a_1 - bp_2,\ -a_2 \leq p_2 \leq -a_2 + ca_1\}$.

6. *If* $0 < a_1 < ba_2$ *and* $0 < a_2 < ca_1$, *then*

$$\begin{aligned} R_{a_1,a_2} = &\left\{(p_1,p_2) \,\middle|\, -a_1 \leq p_1 < 0,\ -a_2 \leq p_2 < \left(\frac{a_2}{a_1} - c\right)p_1\right\} \\ &\bigcup \left\{(p_1,p_2) : -a_1 \leq p_1 < \left(\frac{a_1}{a_2} - b\right)p_2,\ -a_2 \leq p_2 < 0\right\} \\ &\bigcup \Big\{(-a_1 + ba_2, -a_2), (-a_1, -a_2 + ca_1)\Big\}. \end{aligned}$$

Moreover, if $z \in \mathcal{A}(b,c)$ *is pointed at* (a_1, a_2) *with support contained in* R_{a_1,a_2}, *then* $z = x[a_1, a_2]$.

Proof. The first claim is the content of [LLZ14a, Proposition 4.1] and [LLZ14b, Corollary 3.5].

Suppose $z \in \mathcal{A}(b,c)$ is pointed at (a_1, a_2) and that the support of z is contained in R_{a_1,a_2}. Suppose $z \neq x[a_1, a_2]$. Then there exists a monomial $A_1^{-a_1'} A_2^{-a_2'}$ appearing in z with a different coefficient than in $x[a_1, a_2]$. Our assumptions on z imply for any such monomial that we have $a_1' < a_1$ or $a_2' < a_2$. Choose a monomial with (a_1', a_2') minimal in lexicographic order. Then in the greedy basis expansion of z the element $x[a_1', a_2']$ must appear with nonzero coefficient.

Consider the points $O = (0,0)$, $A = (-a_1 + ba_2, -a_2)$, $B = (-a_1, -a_2)$, $C = (-a_1, -a_2 + ca_1)$, $D_1 = (-a_1 + ba_2, ca_1 - (bc+1)a_2)$, and $D_2 = (ba_2 - (bc+1)a_1, -a_2 + ca_1)$. Then the support region of $x[a_1, a_2]$ can be visualized as in Figure 5.1. To reach a contradiction, there are two cases to consider.

- If $(-a_1', -a_2')$ lies on or North of the line segment OB, i.e. $a_1 a_2' \leq a_1' a_2$, then we consider the point $C' = (-a_1', -a_2' + ca_2')$ at the Northern boundary of the

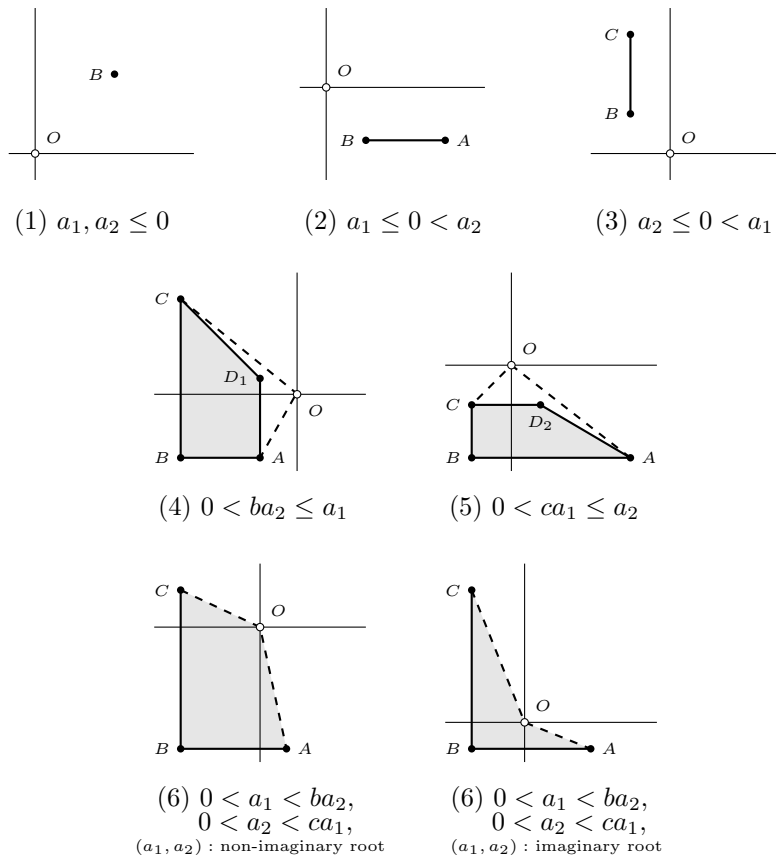

Figure 5.1: Support region of $x[a_1, a_2]$.

support region $R_{a_1',a_2'}$ of $x[a_1', a_2']$ and compare with the line segment OC. In this case, we have

$$-a_1(-a_2' + ca_1') = a_1a_2' - ca_1a_1' \leq a_1'a_2 - ca_1a_1' = -a_1'(-a_2 + ca_1)$$

and thus C' lies on or North of OC. If C' is North of OC or $C' \neq C$ is on OC, then it lies outside R_{a_1,a_2} which is impossible. Thus we must have $C' = C$, but this implies $(a_1', a_2') = (a_1, a_2)$ which clearly must be false.

- If $(-a_1', -a_2')$ lies on or East of the line segment OB, i.e. $a_1'a_2 \leq a_1a_2'$, then we consider the point $A' = (-a_1' + ba_2', -a_2')$ at the Eastern boundary of the support region $R_{a_1',a_2'}$ of $x[a_1', a_2']$ and compare with the line segment OA. In this case, we have

$$(-a_1' + ba_2')(-a_2) = a_1'a_2 - ba_2a_2' \leq a_1a_2' - ba_2a_2' = (-a_1 + ba_2)(-a_2')$$

and thus A' lies on or East of OA. If A' is East of OA or $A' \neq A$ is on OA, then it lies outside R_{a_1,a_2} which is impossible. Thus we must have $A' = A$, but this implies $(a_1', a_2') = (a_1, a_2)$ which is clearly false.

It follows that $z = x[a_1, a_2]$. □

The proof of Theorem 5.2.5 actually establishes the following stronger result, which never uses the special 'pointed' form, thus allowing for support anywhere in the region R_{a_1,a_2}.

Scholium 5.2.6. *If $z \in \mathcal{A}(b, c)$ is any element containing the monomial $A_1^{-a_1} A_2^{-a_2}$ with coefficient 1 and whose support is contained in the half-open quadrilateral $OABC$ from Figure 5.1 associated to (a_1, a_2), then $z = x[a_1, a_2]$.*

Remark 5.2.7. The existence of integers $c(p_1, p_2)$ satisfying the recursive equations (5.2.2), and thus the existence of the greedy basis itself, is quite non-trivial. The authors of [LLZ14a] characterize each $c(p_1, p_2)$ as the solution to an enumerative problem; specifically, the number of certain 'compatible pairs of edges' inside a type of lattice path called a 'maximal Dyck path'.

This enumerative description not only establishes the existence of the greedy basis, but shows that the coefficients $c(p_1, p_2)$ are manifestly non-negative. Finding naturally-defined bases for cluster algebras whose elements have positive coefficients has been one of the core goals of the theory since its inception.

5.3 From g-vectors to d-vectors

In this section, we will relate theta functions with greedy basis. As mentioned in Remark 4.3.4, if $\mathcal{Q}$ lies inside the positve chamber, and $m \in M$ is in $\mathcal{C} \in \Delta_s$, then $vartheta_{\mathcal{Q},m}$ is a cluster monomial. From [GHKK14, 7.5] that the **g**-vector of this cluster monomial is precisely m. On the other hand the description of greedy elements given in [LLZ14a] is in terms of their **d**-vectors (cf. Remark 1.9 ibid.). We refer to [FZ07] for the definition and basic properties of **g** and **d** vectors.

In order to compare the two we will leverage the observation that, in rank 2, these families of vectors are related by an easy piecewise-linear transformation

as explained in the paragraph following Conjecture 3.21 in [RS15]. We will do so via a scattering diagram $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ closely related to $\mathfrak{D}_{(b,c)}$.

Similar to the construction in Section 4.2, we define the piecewise-linear map $T : M_{\mathbb{R}} \to M_{\mathbb{R}}$ as

$$T(m) := \begin{cases} m & m_2 \geq 0 \\ m + (bm_2, 0), & m_2 \leq 0. \end{cases}$$

We will denote its domains of linearity by

$$\mathcal{H}_+ := \{m \in M_{\mathbb{R}} \,|\, m_2 \geq 0\} \quad \text{and} \quad \mathcal{H}_- := \{m \in M_{\mathbb{R}} \,|\, m_2 \leq 0\}.$$

Let T_+ and T_- be the linear extensions to $M_{\mathbb{R}}$ of $T|_{\mathcal{H}_+}$ and $T|_{\mathcal{H}_-}$ respectively (T_+ is just the identity map but it will be convenient to use this notation in what follows). Similar to the construction in Section 4.2, both T_+ and T_- act on pairs $(\mathfrak{d}, f_{\mathfrak{d}})$ so we can use them to define the image of such pairs under T.

Namely set

$$T(\mathfrak{d}, f_{\mathfrak{d}}) := \{T_+(\mathfrak{d} \cap \mathcal{H}_+, f_{\mathfrak{d}})), T_-(\mathfrak{d} \cap \mathcal{H}_-, f_{\mathfrak{d}}))\}.,$$

where $T_\pm(f_{\mathfrak{d}})$ is the formal power series obtained by applying T to each exponent in $f_{\mathfrak{d}}$. Having fixed the notation we are ready to introduce $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$. The set

$$T(\mathfrak{D}_{(b,c)}) := \bigcup_{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}_{(b,c)}} T(\mathfrak{d}, f_{\mathfrak{d}})$$

is not a scattering diagram according to Definition 4.1.2 (not all of its elements are walls for the same convex cone), but can be made into one by a few simple fixes.

First of all, $\big(\mathbb{R}(0,1), 1 + A_2^c\big)$ is the only wall of $\mathfrak{D}_{(b,c)}$ whose support is not totally contained in one of the domains of linearity of T; therefore, under T, it breaks into two parts:

$$\big(\mathbb{R}_{\geq 0}(0,1), 1 + A_2^c\big) \quad \text{and} \quad \big(\mathbb{R}_{\leq 0}(b,1), 1 + A_1^{bc} A_2^c\big).$$

Next note that, since $T(-1,0) = (-1,0)$ and $T(b,-1) = (0,-1)$, T maps all the walls of $\mathfrak{D}_{(b,c)} \smallsetminus \mathfrak{D}_{\text{in},(b,c)}$ to the third quadrant. Indeed, $\big(\mathbb{R}_{\leq 0}(-b,1), 1 + A_1^{-bc} A_2^c\big)$ is the wall with the biggest slope in $\mathfrak{D}_{(b,c)} \smallsetminus \mathfrak{D}_{\text{in},(b,c)}$ and its image is $\big(\mathbb{R}_{\leq 0}(0,1), 1 + A_2^c\big)$.

Definition 5.3.1. *$\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ is the scattering diagram obtained from $T\left(\mathfrak{D}_{(b,c)}\right)$ by replacing*

- $\left(\mathbb{R}(-1,0), 1+A_1^{-b}\right)$ *with* $\left(\mathbb{R}(1,0), 1+A_1^{b}\right)$,
- *both* $\left(\mathbb{R}_{\geq 0}(0,1), 1+A_2^{c}\right)$ *and* $\left(\mathbb{R}_{\leq 0}(0,1), 1+A_2^{c}\right)$ *with* $\left(\mathbb{R}(0,1), 1+A_2^{c}\right)$.

Its chamber is the cone $\sigma^{\mathbf{d}}$ generated by $(1,0)$ and $(0,1)$.

Remark 5.3.2. It is not too hard to see that the scattering diagram $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ is consistent. This fact, together with the uniqueness property implied by [GHKK14, Theorem 1.7], gives an alternative way to introduce it. Indeed, in analogy with the definition of $\mathfrak{D}_{(b,c)}$, one could consider the scattering diagram $\mathfrak{D}^{\mathbf{d}}_{\mathrm{in},(b,c)}$ given by

$$\mathfrak{D}^{\mathbf{d}}_{\mathrm{in},(b,c)} = \left\{\left(\mathbb{R}(1,0), 1+A_1^{b}\right), \left(\mathbb{R}(0,1), 1+A_2^{c}\right)\right\}$$

and obtain $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ using Theorem 4.1.5. The case of $\mathfrak{D}^{\mathbf{d}}_{(2,1)}$ is illustrated in Figure 5.2.

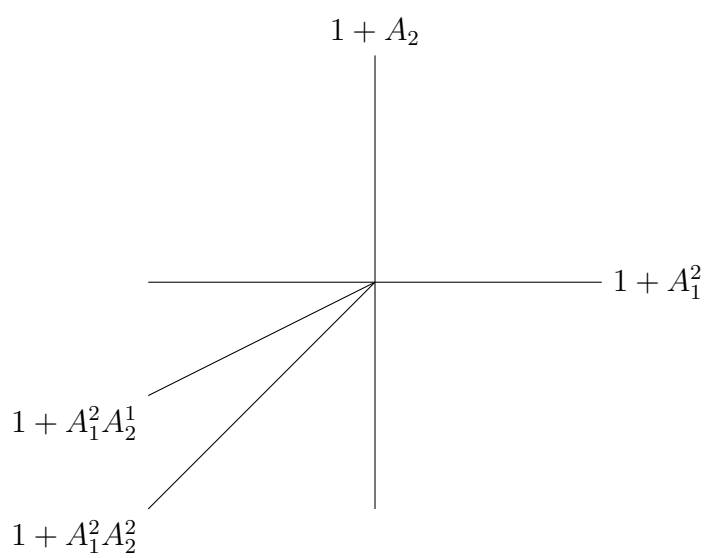

Figure 5.2: The scattering diagram $\mathfrak{D}^{\mathbf{d}}_{(2,1)}$.

For a broken line γ in $\mathfrak{D}_{(b,c)}$, we denote its image under T as $T(\gamma)$: this is the broken line in $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ whose underlying map is $T \circ \gamma$. Given any domain of linearity ℓ of γ, by subdividing it when necessary, we can always assume that either $\gamma(\ell) \subset \mathcal{H}_+$ or $\gamma(\ell) \subset \mathcal{H}_-$. The monomial attached to ℓ in $T(\gamma)$ is then obtained by

applying, accordingly, either T_+ or T_- to the exponent of the monomial attached to ℓ in γ.

Theorem 5.3.3. *The map T defines a one-to-one correspondence from broken lines in $\mathfrak{D}_{(b,c)}$ with exponent m and endpoint $\mathcal{Q}$ to broken lines in $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ with exponent $T(m)$ and endpoint $T(\mathcal{Q})$. In particular, for $\mathcal{Q} \in \mathcal{H}_+$ or $\mathcal{Q} \in \mathcal{H}_-$, we have*

$$\vartheta^{\mathbf{d}}_{T(\mathcal{Q}),T(m)} = T_+\left(\vartheta_{\mathcal{Q},m}\right) \quad \text{or} \quad \vartheta^{\mathbf{d}}_{T(\mathcal{Q}),T(m)} = T_-\left(\vartheta_{\mathcal{Q},m}\right)$$

respectively.

Proof. This is essentially the same as the argument of [GHKK14, Proposition 3.6]. To prove the statement, we only need to check the bending at the x-axis. Let ℓ, ℓ' be the domains of linearity of γ before and after bending along $\mathbb{R}(-1,0)$. So $c(\ell')A^{m(\ell')}$ is a term in

$$\mathfrak{p}_{-m(\ell),\mathbb{R}(-1,0)}\left(c(\ell)A^{m(\ell)}\right) = c(\ell)A^{m(\ell)}\left(1+A_1^{-b}\right)^{|m_2(\ell)|}.$$

First, assume γ passes from $\mathcal{H}_-$ to $\mathcal{H}_+$. In this case, we have $m_2(\ell) < 0$. Now in order for the monomial $c(\ell')A^{T_+(m(\ell'))} = c(\ell')A^{m(\ell')}$ attached to ℓ' in $T(\gamma)$ to satisfy the bending rule, it must be a term in

$$\mathfrak{p}_{-T_-(m(\ell)),\mathbb{R}(1,0)}\left(c(\ell)A^{T_-(m(\ell))}\right).$$

Since the second component of $T_-(m(\ell))$ is $m_2(\ell)$, we get

$$\begin{aligned}
\mathfrak{p}_{-T_-(m(\ell)),\mathbb{R}(1,0)}\left(c(\ell)A^{T_-(m(\ell))}\right) &= c(\ell)A^{T_-(m(\ell))}\left(1+A_1^{b}\right)^{-m_2(\ell)} \\
&= c(\ell)A^{m(\ell)}A_1^{bm_2(\ell)}\left(1+A_1^{b}\right)^{-m_2(\ell)} \\
&= c(\ell)A^{m(\ell)}\left(1+A_1^{-b}\right)^{-m_2(\ell)}.
\end{aligned}$$

This shows that $T(\gamma)$ satisfies the correct rule when bending along $(\mathbb{R}(1,0), 1+A_1^b)$ if γ passes from $\mathcal{H}_-$ to $\mathcal{H}_+$. By repeating similar calculations, we can see that this also holds when γ passes from $\mathcal{H}_+$ to $\mathcal{H}_-$. $\square$

The following demonstrates the utility of using $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$.

Proposition 5.3.4. *For any $m \in M$, if $\mathcal{Q}$ lies in the first quadrant, then*

$$\vartheta^{\mathbf{d}}_{q,m} = A^m \left(1 + f(A_1, A_2)\right)$$

where $f \in (A_1, A_2) \subseteq \Bbbk[A_1, A_2]$. In particular, m is the negative of the $\mathbf{d}$*-vector of* $\vartheta^{\mathbf{d}}_{\mathcal{Q},m}$.

Proof. For any $m \in M$ and any $\mathcal{Q}$ in the first quadrant, there is always a broken line γ for m and $\mathcal{Q}$ that does not bend at any wall. Therefore $\mathrm{Mono}(\gamma) = A^m$ always appears as a term in $\vartheta^{\mathbf{d}}_{\mathcal{Q},m}$.

However, because the functions attached to the walls of $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ are all of the form $1 + g(A_1, A_2)$ with $g(A_1, A_2) \in (A_1, A_2) \subseteq \Bbbk[[A_1, A_2]]$, it follows that any term coming from a broken line which bends must be of the form $cA^m A_1^{d_1} A_2^{d_2}$ with $d_1, d_2 \geq 0$, $d_1 + d_2 > 0$. This proves the result. □

Remark 5.3.5. Combining Theorem 5.3.3 with the above result, when $\mathcal{Q}$ is in the first quadrant we obtain the parametrization of theta functions we were after. Indeed, we get

$$\vartheta^{\mathbf{d}}_{\mathcal{Q},T(m)} = \vartheta_{\mathcal{Q},m}$$

with m being its $\mathbf{g}$-vector and $T(m)$ the negative of its $\mathbf{d}$-vector.

5.4 Proof that the bases coincide

We may now state the main theorem in our current notation.

Theorem 5.4.1. *For any integers $b, c > 0$, for each $m = (m_1, m_2) \in \mathbb{Z}^2$, and for each generic point $\mathcal{Q}$ in the first quadrant, we have that*

$$\vartheta^{\mathbf{d}}_{\mathcal{Q},m} = x[-m_1, -m_2]$$

as elements in the cluster algebra $\mathcal{A}(b, c)$. Hence, the greedy basis and the theta basis for $\mathcal{A}(b, c)$ coincide.

The proof will be to show that the support of $\vartheta^{\mathbf{d}}_{\mathcal{Q},m}$ is contained in the polygon R_{m_1,m_2} in Theorem 5.2.5. By Scholium 5.2.6, this is already enough to show that $\vartheta^{\mathbf{d}}_{\mathcal{Q},m} = x[-m_1, -m_2]$.

We begin our analysis by describing the "changes of direction" of a broken line γ in $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$. Let ℓ be a domain of linearity of γ. We say that γ *moves right* (resp. *up*) in ℓ if $m_1(\ell) < 0$ (resp. $m_2(\ell) < 0$). Conversely we will say that γ *moves left* or *down* in ℓ.

Lemma 5.4.2. *Let ℓ and ℓ' be two consecutive domains of linearity of a broken line γ in $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$. Then*

$$m_1(\ell) \leq m_1(\ell') \qquad \textit{and} \qquad m_2(\ell) \leq m_2(\ell').$$

Proof. Suppose γ bends along the wall $(\mathfrak{d}, f_{\mathfrak{d}})$ when passing from ℓ to ℓ' then $c(\ell')A^{m(\ell')}$ is a term in

$$\mathfrak{p}_{-m(\ell),\mathfrak{d}}\left(c(\ell)A^{m(\ell)}\right) = c(\ell)A^{m(\ell)}f_{\mathfrak{d}}^{m(\ell)\cdot n}$$

with $m(\ell) \cdot n > 0$. The desired property then follows immediately from the observation that, by how $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ has been constructed, all the exponents of the monomials of $f_{\mathfrak{d}}$ are non-negative. $\square$

An immediate consequence of this lemma is that, once a broken line begins to move left or down, it will continue to do so. In particular, if γ is a broken line ending in the first quadrant, it can move left (resp. down) only in the first and fourth (resp. second) quadrant.

At any point $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2) \in \gamma$ at which γ is linear with exponent $m = (m_1, m_2)$, define the *angular momentum* of γ at $\mathcal{Q}$ to be $\mathcal{Q}_2 m_1 - \mathcal{Q}_1 m_2$.

Lemma 5.4.3. *The angular momentum is constant on γ.*

Proof. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be two points on γ. First, assume that $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ and $\mathcal{Q}' = (\mathcal{Q}'_1, \mathcal{Q}'_2)$ are in the same linear region of γ, with exponent $m = (m_1, m_2)$. Since $\gamma' = -m$ at $\mathcal{Q}$, there is some t such that

$$(\mathcal{Q}'_1, \mathcal{Q}'_2) = (\mathcal{Q}_1 + tm_1, \mathcal{Q}_2 + tm_2)$$

Then the angular momentum at $\mathcal{Q}'$ is

$$(\mathcal{Q}_2 + tm_2)m_1 - (\mathcal{Q}_1 + tm_1)m_2 = \mathcal{Q}_2 m_1 - \mathcal{Q}_1 m_2$$

Next, assume that $\mathcal{Q}$ and $\mathcal{Q}'$ are points on γ on either side of a bend at a wall $(\mathfrak{d}, f_{\mathfrak{d}})$ at point $\mathcal{Q}'' = (\mathcal{Q}''_1, \mathcal{Q}''_2)$. If the exponent of γ at $\mathcal{Q}$ is $m = (m_1, m_2)$ and $f_{\mathfrak{d}}$ is a series in $A^{(w_1,w_2)}$, then the exponent of γ at $\mathcal{Q}'$ must be of the form $(m_1 + kw_1, m_2 + kw_2)$ for some positive integer k. By the argument of the previous paragraph, the angular momentum at $\mathcal{Q}$ is $\mathcal{Q}''_2 m_1 - \mathcal{Q}''_1 m_2$ and the angular momentum at $\mathcal{Q}'$ is

$$\mathcal{Q}''_2(m_1 + kw_1) - \mathcal{Q}''_1(m_2 + kw_2) = (\mathcal{Q}''_2 m_1 - \mathcal{Q}''_1 m_2) + k(\mathcal{Q}''_2 w_1 - \mathcal{Q}''_1 w_2)$$

Since the point $(\mathcal{Q}''_1, \mathcal{Q}''_2)$ lies on the ray through (w_1, w_2), the expression $\mathcal{Q}''_2 w_1 - \mathcal{Q}''_1 w_2$ is zero, and so the angular momenta at $\mathcal{Q}$ and $\mathcal{Q}'$ are the same. This equality extends transitively to any pair of points $\mathcal{Q}, \mathcal{Q}'$ on γ. $\square$

The sign of the angular momentum is a useful invariant for characterizing the qualitative behavior of a broken line. For a broken line ending in the first quadrant, the sign of the angular momentum characterizes whether that broken line could have passed through the fourth quadrant (positive) or the second quadrant (negative).

Lemma 5.4.4. *Let γ be a broken line $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ with endpoint $\mathcal{Q}$ in the first quadrant. If γ has positive (resp. negative) angular momentum, then the slope of the linear domains of γ decreases (resp. increases) at each bend, except possibly at the boundary of the first quadrant.*

Figure 5.3 depicts a broken line with positive angular momentum. The slopes of the linear domains decrease from $\frac{5}{4}$ to 1 to $\frac{1}{2}$ before increasing to $+\infty$.

Proof. The lemma is straightforward except for broken lines with initial exponent (m_1, m_2) with $m_1, m_2 < 0$. Consider a bend of γ at a point $(\mathcal{Q}_1, \mathcal{Q}_2)$ in a wall $(\mathfrak{d}, f_{\mathfrak{d}}(A^{(w_1,w_2)}))$. If the exponent immediately before the bend is (m_1, m_2), the exponent immediately after the bend is $(m_1 + kw_1, m_2 + kw_2)$ for some positive integer k.

Assume that $(\mathcal{Q}_1, \mathcal{Q}_2)$ is not in the boundary of the first quadrant, so that $(\mathcal{Q}_1, \mathcal{Q}_2)$ is a negative scalar multiple of the exponent (w_1, w_2). By this assumption,

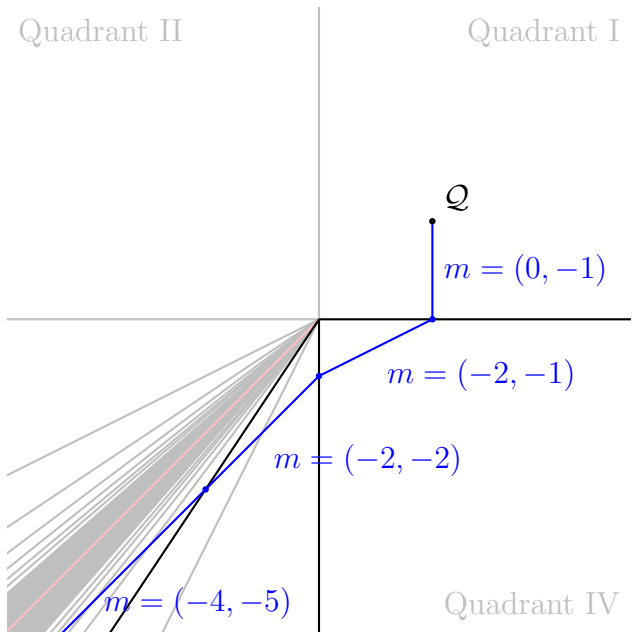

Figure 5.3: A broken line with positive angular momentum

in view of Lemma 5.4.2 and the fact that $\mathcal{Q}$ lies in the first quadrant, we have also $m_1 + kw_1, m_2 + kw_2 < 0$.

If the angular momentum $\mathcal{Q}_2 m_1 - \mathcal{Q}_1 m_2$ is positive, then the cross-product $w_2 m_1 - w_1 m_2$ is negative. But for positive k,

$$k(w_2 m_1 - w_1 m_2) = m_1(m_2 + kw_2) - m_2(m_1 + kw_1) < 0 \Rightarrow \frac{m_2 + kw_2}{m_1 + kw_1} < \frac{m_2}{m_1}$$

as desired. If the angular momentum is negative, the slope increases by an identical argument. □

We can now constrain the possible final exponent of a broken line, which will be used to bound the support of the corresponding theta function.

Lemma 5.4.5. *Let γ be a broken line in $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ which begins in the third quadrant, with endpoint $\mathcal{Q}$ in the first quadrant. Denote the initial exponent by $m = (m_1, m_2)$ and the final exponent by $m^{\mathcal{Q}} = (m_1^{\mathcal{Q}}, m_2^{\mathcal{Q}})$.*

1. *If γ has positive angular momentum, then $m_2 \leq m_2^{\mathcal{Q}} < 0$ and*

$$m_1 \leq m_1^{\mathcal{Q}} \leq \left(\frac{m_1}{m_2} - b\right) m_2^{\mathcal{Q}}$$

where the upper bound is equality only when $m^{\mathcal{Q}} = (m_1 - bm_2, m_2)$.

2. *If* γ *has negative angular momentum, then* $m_1 \leq m_1^{\mathcal{Q}} < 0$ *and*

$$m_2 \leq m_2^{\mathcal{Q}} \leq \left(\frac{m_2}{m_1} - c\right) m_1^{\mathcal{Q}}$$

where the upper bound is equality only when $m^{\mathcal{Q}} = (m_1, m_2 - cm_1)$.

Proof. Assume γ has positive angular momentum; consequently, γ passes through the fourth quadrant before entering the first quadrant. Let (m_1', m_2') be the exponent on γ in the fourth quadrant. By the preceding lemma, $\frac{m_2'}{m_1'} \leq \frac{m_2}{m_1}$ with equality only if γ doesn't bend before it reaches the fourth quadrant.

As the broken line passes into the first quadrant, it may bend at the wall $(\mathbb{R}(1,0), 1 + A^{(b,0)})$. By definition, the final exponent $m^{\mathcal{Q}}$ on γ must be an exponent that appears in $A^{(m_1', m_2')}(1 + A^{(b,0)})^{-m_2'}$. It follows that

$$(m_1^{\mathcal{Q}}, m_2^{\mathcal{Q}}) = (m_1' + kb, m_2')$$

for some $0 \leq k \leq -m_2'$. Consequently, $m_2^{\mathcal{Q}} = m_2' < 0$ and

$$m_1^{\mathcal{Q}} \leq m_1' - bm_2' = \left(\frac{m_1'}{m_2'} - b\right) m_2' = \left(\frac{m_1'}{m_2'} - b\right) m_2^{\mathcal{Q}} \leq \left(\frac{m_1}{m_2} - b\right) m_2^{\mathcal{Q}}$$

The second inequality is equality only if $(m_1', m_2') = (m_1, m_2)$, and so the composite inequality is equality only if $m^{\mathcal{Q}} = (m_1 - bm_2, m_2)$.

Analogous inequalities hold for negative angular momentum by the same argument. □

Proof of Theorem 5.4.1. If $m = (m_1, m_2)$ such that $m_1 \geq 0$ or $m_2 \geq 0$, then $\vartheta^{\mathbf{d}}_{\mathcal{Q},m}$ is the cluster monomial $x[-m_1, -m_2]$, as indicated in Remark 4.3.4. Next, assume that $m = (m_1, m_2)$ such that $m_1 \leq 0$ and $m_2 \leq 0$. The coefficient of $A^{(a_1,a_2)}$ in $\vartheta^{\mathbf{d}}_{\mathcal{Q},m}$ can have non-zero coefficient only if there is a broken line γ in $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ with initial exponent m and final exponent $a = (a_1, a_2)$. By the preceding lemma, this implies that

$$m_1 \leq a_1 \leq \left(\frac{m_1}{m_2} - b\right) a_2, \quad m_2 \leq a_2 \leq \left(\frac{m_2}{m_1} - c\right) a_1$$

Furthermore, the upper bounds are only satisfied in the specific cases when (a_1, a_2) is equal to $(m_1 - bm_2, m_2)$ or $(m_1, m_2 - cm_1)$. Since $\vartheta^{\mathbf{d}}_{\mathcal{Q},m} \in \mathcal{A}_{b,c}$, Scholium 5.2.6 implies that $\vartheta^{\mathbf{d}}_{\mathcal{Q},m}$ is a scalar multiple of $x[-m_1, -m_2]$.

To show they coincide, we consider the coefficient of $A^{(m_1,m_2)}$ in each element. The coefficient of $A^{(m_1,m_2)}$ in $x[-m_1, -m_2]$ is 1, by the definition of a pointed element. The coefficient of $A^{(m_1,m_2)}$ in $\vartheta^{\mathbf{d}}_{\mathcal{Q},m}$ is the sum of the coefficients of all broken lines in $\mathfrak{D}^{\mathbf{d}}_{(b,c)}$ with initial exponent (m_1, m_2) and final exponent (m_1, m_2). Since any bend in a broken line would increase one of the components of the exponent, this only happens for the unique broken line with initial exponent (m_1, m_2) that has no bends. Hence, the coefficient of $A^{(m_1,m_2)}$ in $\vartheta^{\mathbf{d}}_{\mathcal{Q},m}$ is 1, and so $\vartheta^{\mathbf{d}}_{\mathcal{Q},m} = x[-m_1, -m_2]$. $\square$

Remark 5.4.6. As mentioned in Remark 5.2.7, the coefficients of $x[-m_1, -m_2]$ may be interpreted as counting 'compatible pairs' in a lattice path called a 'maximal Dyck path'. One consequence of Theorem 5.4.1 is that the coefficients $c(p, q)$ are equal to a weighted sum of certain broken lines. An interesting open problem is to reprove the coincidence of the two bases by giving a combinatorial bijection between broken lines and compatible pairs which directly proves the equality of the respective coefficients.

This chapter contains part of the paper "The Greedy Basis equals the Theta Basis". It is a joint work with Mark Gross, Greg Muller, Gregg Musiker, Dylan Rupel, Salvatore Stella, Harold Williams.

Chapter 6

The Hall algebra scattering diagram

In this chapter, we will survey the link between stability conditions and scattering due to Bridgeland in [Bri15]. We will further interpret wall crossing in terms of Hall algebra multiplication defined in 6.1.1.

The definitions of the Hall algebra $H(Q)$, the groups $\hat{G}_{Hall}$, $\hat{G}_{reg}$, and the Lie algebras $\mathfrak{g}_{Hall}$, $\mathfrak{g}_{reg}$ are stated in Section 6.1. Naively speaking, elements of $\hat{G}_{Hall}$, $\mathfrak{g}_{Hall}$ are stacks while elements of $\hat{G}_{reg}$, $\mathfrak{g}_{reg}$ are varieties. Then the Hall algebra scattering diagram is defined to take values in $\mathfrak{g}_{Hall}$. The reason for the lengthy definition is that we are working with stacks. However, by a deep theorem of Joyce (Theorem 6.2.3), there exists an element in $\hat{G}_{reg}$ associated to semistable objects. Thus, we are actually working with $\mathfrak{g}_{reg}$.

To those who wish to skip the technical details, they can jump directly to Theorem 6.2.2 and Section 6.3.

6.1 Hall algebra

Let Q be a finite acyclic quiver. Denote the sets of vertices and arrows of Q as (Q_0, Q_1). Set $N = \mathbb{Z}^{Q_0}$, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Denote $N^{\oplus} = \{n \in N | n_i \geq 0 \ \forall i\}$.

Write $\mathbb{C}Q$ as the path algebra of Q. Let $\mathrm{rep}(Q) = \mathrm{mod}\mathbb{C}Q$ the abelian

category of finite dimensional representations of Q. Let $(e_i)_{i\in Q_0}$ be the canonical basis indexed by the vertices of Q.

In this section, we will introduce motivic Hall algebra developed by Joyce [Joy07].

Define the algebraic stack $\mathcal{M}$ parametrizing all objects of the category $\mathrm{rep}(Q)$ as a fibered category over the category of the schemes. The objects of $\mathcal{M}$ over a scheme S are pairs $(\mathcal{E}, \rho)$ where $\mathcal{E}$ is a locally free $\mathcal{O}_S$-module of finite rank, and $\rho : \mathbb{C}Q \to \mathrm{End}_S(\mathcal{E})$ is an algebra homomorphism. Let $\mathrm{St}\,/\mathcal{M}$ denote the full subcategory consisting of objects $f : X \to \mathcal{M}$ for which X is an algebraic stack of finite type over $\mathbb{C}$ and has affine stabilizers.

Definition 6.1.1. *The Grothendick group $K(\mathrm{St}\,/\mathcal{M})$ of stacks over $\mathcal{M}$ is the free abelian group with basis given by isomorphism classes of objects of $\mathrm{St}\,/\mathcal{M}$, modulo the subgroup spanned by the relations*

- *for every object $f : X \to \mathcal{M}$ in $\mathrm{St}\,/\mathcal{M}$, and every closed substack $Y \subset X$ with complementary open substack $U = X \setminus Y$, we have $[X \xrightarrow{f} \mathcal{M}] = [Y \xrightarrow{f|_Y} \mathcal{M}] + [U \xrightarrow{f|_U} \mathcal{M}]$;*

- *for every object in $\mathrm{St}\,/\mathcal{M}$, and every pair of morphisms $h_1 : Y_1 \to X$, $h_2 : Y_2 \to X$ which are locally trivial fibrations in the Zariski topology with the same fibres, we have $[Y_1 \xrightarrow{g\circ h_1} \mathcal{M}] = [Y_2 \xrightarrow{g\circ h_2} \mathcal{M}]$.*

$K(\mathrm{St}\,/\mathcal{M})$ has the structure of a $K(\mathrm{St}\,/\mathbb{C})$-module, defined by setting

$$[X] \cdot [Y \xrightarrow{f} \mathcal{M}] = [X \times Y \xrightarrow{f\circ\pi_2} \mathcal{M}]$$

and extending linearly. There is a unique ring homomorphism

$$\Upsilon : K(\mathrm{St}\,/\mathbb{C}) \to \mathbb{C}(q)$$

which takes the class of a smooth projective variety X over $\mathbb{C}$ to the Poincaré polynomial

$$\Upsilon([X]) = \sum_{i=0}^{2d} \dim_{\mathbb{C}} H^i(X_{an}, \mathbb{C}) \cdot q^{i/2} \in \mathbb{C}[q],$$

where X_{an} denotes X considered as a smooth projective variety, and $H^i(X_{an}, \mathbb{C})$ denotes singular cohomology. Note that we work with $\mathbb{C}(q)$, where $q = t^2$ in the setting of [Bri15].

Define the $\mathbb{C}(q)$ vector space

$$K_\Upsilon(\mathrm{St}\,/\mathcal{M}) = K(\mathrm{St}\,/\mathcal{M}) \otimes_{K(\mathrm{St}\,/\mathbb{C})} \mathbb{C}(q)$$

with the relation $[X \times Y \xrightarrow{f \circ \pi_2} \mathcal{M}] = \Upsilon([X]) \cdot [Y \xrightarrow{f} \mathcal{M}]$.

Next, let us equip $K(\mathrm{St}\,/\mathcal{M})$ with a ring structure. Denote $\mathcal{M}^{(2)}$ be the stack of short exact sequences in $\mathrm{rep}(Q)$. The objects of $\mathcal{M}^{(2)}$ over a scheme S consist of three pairs $(\mathcal{E}_i, \rho_i)$ of $\mathcal{M}(S)$ for $i = 1, 2, 3$, together with morphisms α and β of $\mathcal{O}_S$-modules which define a short exact sequence $0 \to \mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathcal{E}_3 \to 0$. There is a diagram

$$\begin{tikzcd} \mathcal{M}^{(2)} \arrow[r, "b"] \arrow[d, "(a_1,a_2)"] & \mathcal{M} \\ \mathcal{M} \times \mathcal{M} & \end{tikzcd}$$

where

$$b([0 \to A_1 \to B \to A_2 \to 0]) = B,$$

$$a_i([0 \to A_1 \to B \to A_2 \to 0]) = A_i, \text{ for } i = 1, 2.$$

Then the multiplication on $K(\mathrm{St}\,/\mathcal{M})$ is defined as

$$[X_1 \xrightarrow{f_1} \mathcal{M}] \star [X_2 \xrightarrow{f_2} \mathcal{M}] = [Z \xrightarrow{b \circ h} \mathcal{M}], \tag{6.1.1}$$

where Z and h are defined by

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d] & \mathcal{M}^{(2)} \arrow[r, "b"] \arrow[d, "(a_1,a_2)"] & \mathcal{M} \\ X_1 \times X_2 \arrow[r, "f_1 \times f_2"] & \mathcal{M} \times \mathcal{M} & \end{tikzcd}$$

where the square is Cartesian.

Then $K(\mathrm{St}\,/\mathcal{M})$ becomes a ring. As the multiplication operation is $K(\mathrm{St}\,/\mathbb{C})$-linear, it defines an algebra structure on the $\mathbb{C}(q)$-vector space $K_\Upsilon(\mathrm{St}\,/\mathcal{M})$. Then we get a $\mathbb{C}(q)$-algebra $H(Q)$.

The stack $\mathcal{M}$ can be decomposed as a disjoint union

$$\mathcal{M} = \coprod_{d \in N^{\oplus}} \mathcal{M}(d),$$

where $\mathcal{M}(d)$ parameterize representations of Q of the fixed dimension vector d. This induces a grading $H(Q) = \bigoplus_{d \in N^{\oplus}} H(Q)_d$, where $H(Q)_d = K_{\Upsilon}(\mathrm{St}\,/\mathcal{M}_d)$. Note that

$$H(Q)_0 = \mathbb{C}(q) \cdot 1 = [\mathcal{M}_0 \subset \mathcal{M}].$$

So we have a vector space decomposition $H(Q) = H(Q)_0 \oplus H(Q)_{>0}$, where $H(Q)_{>0} = \bigoplus_{d \in N^+} H(Q)_d$.

We can further consider $H(Q)_{>k} = \oplus_{\dim(n)>k} H(Q)_n$ for each $k \in \mathbb{N}$. Then we have the quotient $H(Q)_{\leq k} = H(Q)/H(Q)_{\geq k}$. By taking limit, we obtain the completed algebra

$$\hat{H}(Q) = \varprojlim H_{\leq k}(Q).$$

Define $\mathfrak{g}_{Hall} = H(Q)_{>0}$. Then we can obtain the corresponding completed Lie subalgebra

$$\hat{\mathfrak{g}}_{Hall} = \hat{H}(Q)_{>0} \subset \hat{H}(Q)$$

and the corresponding pro-unipotent group $\hat{G}_{Hall}$. By consider the embedding $\phi : \hat{G}_{Hall} \to \hat{H}(Q)$, we can identify

$$\hat{G}_{Hall} \cong 1 + \hat{H}(Q)_{>0} \subset \hat{H}(Q).$$

Through this identification, $\hat{G}_{Hall}$ can act on $\hat{H}(Q)$ by conjugation.

Objects in $H(Q)$ are of the form $f : X \to \mathcal{M}$ such that X is an algebraic stack of finite type over $\mathbb{C}$ and has affine stabilizers. Now we restrict X to be a variety and consider the $\mathbb{C}[q, q^{-1}]$-submodule $H_{reg}(Q)$ generated by those objects. By [Bri15, Theorem 5.2], $H_{reg}(Q)$ is closed under the Hall algebra product and is then a $\mathbb{C}[q, q^{-1}]$-algebra.

By repeating the similar grading decomposition as above, we obtain the Lie subalgebra $H_{reg}(Q)_{>0}$ of $H_{reg}(Q)$. Define

$$\mathfrak{g}_{reg} = (q-1)^{-1} H_{reg}(Q)_{>0} \subset \mathfrak{g}_{Hall}.$$

Then we take completion again and have the Lie algebra $\hat{\mathfrak{g}}_{reg} = (q-1)^{-1} \hat{H}_{reg}(Q)_{>0}$ and the corresponding pro-unipotent group $\hat{G}_{reg} \subset \hat{G}_{Hall}$.

6.2 Stability conditions and scattering diagrams

Now let us define the relevant notion of stability condition.

Definition 6.2.1. *Given $\omega \in M_{\mathbb{R}}$, an object $E \in \mathrm{rep}(Q)$ is said to be ω-**semistable** if*

- $\omega(E) = 0$,
- *every subobject $B \subset E$ satisfies $\omega(B) \leq 0$.*

By the definition of Grothendick group, we can identify elements in $K(\mathrm{rep}(Q))$ with its dimension vector, so we have $N \cong K(\mathrm{rep}(Q))$. Now given a fixed $\omega \in M_{\mathbb{R}}$, we can define a stability function $Z : K(\mathrm{rep}(Q)) \to \mathbb{C}$ as $Z(E) = -\omega(E) + i\delta(E)$. In this section, we take $\delta = (1, \ldots, 1)$, i.e. $\delta(E) = \dim(E)$. Note that if $E \neq 0$, $Z(E)$ lies on the upper half plane. So we can define the **phase** of E by

$$\phi(E) = \frac{1}{\pi} \arg Z(E).$$

Then an object $0 \neq E \in \mathrm{rep}(Q)$ is Z**-semistable** if every nonzero subobject $B \subset E$ satisfies $\phi(B) \leq \phi(E)$. Note that $0 \neq E \in \mathrm{rep}(Q)$ is ω-semistable precisely if it is Z-semistable with phase $1/2$.

For every $0 \neq E \in \mathrm{rep}(Q)$, there exists a Harder–Narasimhan filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{k-1} \subset E_k = E$ whose factors $F_i = E_i/E_{i-1}$ are Z-semistable with descending phase: $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_k)$.

Now for any interval $I \subset (0, 1)$ there is an open substack $\mathcal{M}_I = \mathcal{M}_I(\omega) \subset \mathcal{M}$ parameterising representations of Q lying in $\mathcal{P}(I) \subset \mathrm{rep}(Q)$, where $\mathcal{P}(I)$ is the full subcategory consisting of objects whose Harder Narasimhan factors have phases in I. This defines a corresponding element

$$1_I(\omega) = [\mathcal{M}_I(\omega) \subseteq \mathcal{M}] \in \hat{G}_{Hall} \subseteq \hat{H}(Q).$$

In particular, if $I = \{1/2\}$, by the remark above, $0 \neq E \in \mathrm{rep}(Q)$ is ω-semistable precisely if it is Z-semistable with phase $1/2$, then $1_{1/2}(\omega) = 1_{ss}(\omega)$.

After all these set up, we can finally state a result of Bridgeland about the linkage between stability conditions and scattering diagrams.

Theorem 6.2.2. *[Bri15, Theorem 6.5] There exists a consistent scattering diagram $\mathfrak{D}$ in $M_{\mathbb{R}}$ such that:*

1. *the support $\mathfrak{d}$ consists of maps $\omega \in M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{Z}) \otimes \mathbb{R}$ for which there exist ω-semistable objects in $\mathrm{rep}(Q)$;*

2. *the wall-crossing automorphism at a general point $\omega \in \mathfrak{d} \subset \mathrm{supp}(\mathfrak{D})$ is $\Phi_{\mathfrak{D}}(\mathfrak{d}) = 1_{ss}(\omega) \in \hat{G}_{Hall}$.*

This scattering diagram is unique up to equivalence. It is called the **Hall algebra scattering diagram**.

Bridgeland further [Bri15] shows that there is a bijection between equivalence classes of consistent $\hat{\mathfrak{g}}_{Hall}$-complexes and elements of the group $\hat{G}_{Hall}$. In this case, the Hall algebra scattering diagram $\mathfrak{D}$ corresponds to $\Phi_{\mathfrak{D}} = 1_{\mathrm{rep}(Q)} \in \hat{G}_{Hall}$.

Let us recall a theorem of Joyce

Theorem 6.2.3. *For any $\omega \in M_{\mathbb{R}}$ the element of $\hat{G} \subset \hat{H}(Q)$ defined by the inclusion of the open substack of ω-semistable objects $\mathcal{M}_{ss}(\omega) \subset \mathcal{M}$ corresponds to an element $1_{ss}(\omega) \in \hat{G}_{reg}$.*

6.3 Wall crossing and theta functions

In this section, we will have a closer look at the wall crossing automorphism. Before that, let us relate Hall algebra scattering diagram with torsion pair defined by ω in $\mathrm{rep}(Q)$.

Definition 6.3.1. *A **torsion pair** in $\mathrm{rep}(Q)$ is defined to be a pair of full additive subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathrm{rep}(Q)$ such that*

- *if $T \in \mathcal{T}$ and $F \in \mathcal{F}$, then $\mathrm{Hom}_{\mathrm{rep}(Q)}(T, F) = 0$*

- *for any $E \in \mathcal{A}$ there is a short exact sequence*

$$0 \to T \to E \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Then a torsion pair always exists.

Lemma 6.3.2. *[Bri15, Lemma 6.6] For each $\omega \in M_{\mathbb{R}}$ there is a torsion pair $(\mathcal{T}(\omega), \mathcal{F}(\omega)) \subset \mathrm{rep}(Q)$ defined by setting*

$$\begin{aligned}\mathcal{T}(\omega) &= \mathcal{P}(1/2, 1)\\ &= \{E \in \mathrm{rep}(Q) : \textit{any quotient object } E \to D \textit{ satisfies } \omega(D) > 0\}\\ \mathcal{F}(\omega) &= \mathcal{P}(0, 1/2]\\ &= \{E \in \mathrm{rep}(Q) : \textit{any subobject } F \subset E \textit{ satisfies } \omega(F) \leq 0\}\end{aligned}$$

We will denote $1_{\mathcal{T}}(\omega) := 1_{(1/2,1)}(\omega)$, $1_{\mathcal{F}}(\omega) := 1_{(0,1/2]}(\omega) \in \hat{G}_{Hall}$. Now let us set up to define theta function.

As $H(Q)$ is $N^{\oplus}$-graded, we can obtain an associated $N^{\oplus}$-graded algebra $H(Q) \otimes_{\mathbb{C}} \mathbb{C}[M]$ with relations

$$A^m \cdot H = q^{-m \cdot d} H \cdot A^m, \tag{6.3.1}$$

where $m \in M$, $H \in H(Q)_d$. By the same limiting process in Section 6.1, we obtain the completion $\widehat{H(Q) \otimes_{\mathbb{C}} \mathbb{C}[M]}$.

Next, we want to generalize the notions of broken line γ to the Hall algebra scattering diagram. We consider the projection $\chi : H(Q) \to \mathbb{C}$,

$$\chi([X]) = \Upsilon([X])|_{q^{1/2}=-1} = \sum (-1)^i \dim_{\mathbb{C}} H^i(X_{an}, \mathbb{C}), \tag{6.3.2}$$

where X_{an} denotes X as a smooth complex analytic variety.

To each linear piece of γ, the attaching monomials attaching are replaced by $[X \to \mathcal{M}] \otimes A^m$. Behavior at a bend at a wall $\mathfrak{d}$ is given by conjugation by $\Phi_{\mathfrak{D}}(\mathfrak{d})$. We can retain the ordinary attaching monomials by applying

$$\chi : H(Q) \otimes_{\mathbb{C}} \mathbb{C}[M] \to \mathbb{C}[M],$$

which is

$$\chi([X \to \mathcal{M}]) = \chi(X) A^{p^*(\dim(X))}. \tag{6.3.3}$$

Noted above that on crossing a generic point of a wall $\mathfrak{d} \in \mathfrak{D}$ in the positive direction, the wall crossing automorphisms are given by

$$A^m \mapsto \Phi_{\mathfrak{D}}(\mathfrak{d})(A^m).$$

This is the conjugation of $\hat{G}_{reg}$ acting on $H(Q)\widehat{\otimes_{\mathbb{C}}}\mathbb{C}[M]$. After taking χ to $\Phi_{\mathfrak{D}}(\mathfrak{d})(A^m)$, we will obtain the path-ordered products as in 4.1.2. Furthermore, if the quiver is acyclic, after applying the integration map to the Hall algebra scattering diagram we have the Hall algebra scattering diagram is equivalent to the cluster scattering by [Bri15, Lemma 11.4 and 11.5].

Then for each m lies in $\mathcal{C}$, where $\mathcal{C} \in \Delta$ (cluster complex), we can define a Hall algebra theta function as

$$\vartheta_m(\omega) = 1_{\mathcal{F}}(m)^{-1} A^m 1_{\mathcal{F}}(m) \in H(Q)\widehat{\otimes_{\mathbb{C}}}\mathbb{C}[M], \tag{6.3.4}$$

Note that here our direction goes from the negative chamber to the positive chamber which is the reverse of the set up in [Bri15].

Again after applying χ to this Hall algebra theta function, we will get back the theta function in usual sense. We will give more interpretation of theta function in Section 7.3.

6.4 Hall algebra broken lines

In this section, we will repeat similar ideas as in Section 4.3 to define Hall algebra broken lines.

Now we have $\mathfrak{D}$ a Hall algebra scattering diagram.

Definition 6.4.1. *Let $\mathfrak{D}$ be a Hall algebra scattering diagram, $m \in M \setminus \{0\}$ and $\mathcal{Q} \in M_{\mathbb{R}} \setminus Supp(\mathfrak{D})$.*

*A **Hall algebra broken line** for m with endpoint $\mathcal{Q}$ is a piecewise linear continuous proper path $\gamma : (-\infty, 0] \to M_{\mathbb{R}} \setminus Sing(\mathfrak{D})$ with a finite number of domains of linearity.*

An element $[\mathcal{N}\to\mathcal{M}]A^m \in H(Q) \otimes_{\mathbb{C}} \mathbb{C}[M]$, where $\mathcal{N}\to\mathcal{M}$ factors through $\mathcal{N}\to\mathcal{M}_d\to\mathcal{M}$, where d is the dimension vector for the representation, is attached to each domain of linearity $L \subseteq (-\infty, 0)$ of γ. The path γ and the $[\mathcal{N}\to\mathcal{M}]A^m$ need to satisfy the following conditions:

- $\gamma(0) = \mathcal{Q}$.

- *If L is the first (i.e., unbounded) domain of linearity of γ, then $[\mathcal{N}\to\mathcal{M}]A^m = A^m$. This is corresponding to the zero representation $[\cdot \mapsto 0]$.*

- *In each domain of linearity $L \subset (-\infty, 0]$, the attached Hall algebra monomial is $[\mathcal{N} \subset \mathcal{M}]A^m$. After applying integration map, by (6.3.3), we have*

$$\chi([\mathcal{N} \subset \mathcal{M}])A^m = \chi([\mathcal{N}])A^{p^*(\dim \mathcal{N})+m}.$$

  Then for $t \in L$, $\gamma'(t) = -(p^*(\dim \mathcal{N}) + m)$.

- *γ bends only when it crosses a wall. If γ bends from the domain of linearity L to L' when crossing $\mathfrak{d}$ defined in Theorem 6.2.2, then $[\mathcal{N} \subset \mathcal{M}]A^m$ is a term in $\Phi_{\mathfrak{D}}(\mathfrak{d}) \cdot ([\mathcal{N} \subset \mathcal{M}]A^m)$.*

6.5 Further properties in the Hall algebra scattering diagrams

In later sections, we will compute the Hall algebra wall crossing explicitly. Let us formulate some equations in the Hall algebra in this section.

In Section 6.2, for $\omega \in M_{\mathbb{R}}$, we define

$$1_{ss}(\omega) = [\mathcal{M}_{1/2}(\omega) \subseteq \mathcal{M}],$$

First note that if ω is a general point on some wall $\mathfrak{d}$ in the cluster complex, there is a representation D with $\dim D$ primitive which is a ω-semistable object. We are going to show D is indecomposable. If not, $D = D_1 \oplus D_2$ for some D_1, D_2 non zero. Then for all $\omega \in \mathfrak{d}_i$, $\omega(D_1) + \omega(D_2) = \omega(D) = 0$. As D_1 and D_2 are subojects of D, by definition of semistability, $\omega(D_1), \omega(D_2) \leq 0$. Therefore, $\omega(D_1) = \omega(D_2) = 0$. However, $\mathfrak{d}$ is of co-dimension 1 in N. Hence the normal space to $\mathfrak{d}$ is of dimension 1 only. This implies $\dim D_1$ and $\dim D_2$ are proportional to $\dim D$, contradicting to the condition that $\dim D$ is primitive. Thus D is indecomposable.

Hence $1_{ss}(\omega)$ is a formal sum

$$1_{ss}(\omega) = 1 + \sum_{k\geq 1} [BGL_k(D)\to\mathcal{M}], \tag{6.5.1}$$

where $BGL_k(D)$ are classifying space for GL_k. More precisely, $[BGL_k(D)\to\mathcal{M}]$ means we map a point to k copies of D, i.e. $\cdot \mapsto D^{\oplus k}$.

Theorem 6.2.2 tells us that the wall crossing automorphism at ω is conjugation by $1_{ss}(\omega)$. Therefore, we wish to understand $1_{ss}(\omega)^{-1}$ which is

$$1_{ss}(\omega)^{-1} = 1 + \sum_k (-1)^k \prod_{l=1}^{k} [BGL_{r_l}(D)\to\mathcal{M}], \tag{6.5.2}$$

where the product means multiplying k many $[BGL_{r_l}\to\mathcal{M}]$, for some $r_l \in N$, together. For example, up to degree 2, we have

$$\begin{aligned} 1_{ss}(\omega)^{-1} = 1 &- [BG_1(D)\to\mathcal{M}] - [BGL_2(D)\to\mathcal{M}] \\ &+[BG_1(D)\to\mathcal{M}]^2 + [BG_1(D)\to\mathcal{M}] \star [BGL_2(D)\to\mathcal{M}] \\ &+[BGL_2(D)\to\mathcal{M}] \star [BG_1(D)\to\mathcal{M}]. \end{aligned}$$

Let us also recall how to express GL_d as an element in $K(\mathrm{Var}/\mathbb{C})$:

Lemma 6.5.1. *[Bri12, Lemma 2.6]*

$$[\mathrm{GL}_d] = [\mathbb{A}^{d(d-1)/2}] \prod_{k=1}^{d} (\mathbb{A}^k - 1).$$

By similar techniques, we can also calculate the product $\prod_{l=1}^{k}[BGL_{r_l}(D)\to\mathcal{M}]$. As the product is associative, we can work on $[BGL_{r_1}(D)\to\mathcal{M}] * [BGL_{r_2}(D)\to\mathcal{M}]$ first. This consists of exact sequences

$$0 \longrightarrow D^{\oplus r_1} \longrightarrow Y \longrightarrow D^{\oplus r_2} \longrightarrow 0,$$

where the automorphisms of the exact sequence can be viewed as 'upper triangular' matrices of the form

$$\begin{pmatrix} \mathrm{GL}_{l_1} & * \\ 0 & \mathrm{GL}_{l_2} \end{pmatrix},$$

where $*$ denotes entries with arbitrary element in $\mathbb{C}$. Thus in general,

$$\prod_{l=1}^{k} [BGL_{r_l}(D)\to\mathcal{M}]$$

can be visualized as an $(r_1 + \cdots + r_k)$-square matrix:

$$\begin{pmatrix} \mathrm{GL}_{l_1} & * & * & * \\ 0 & \mathrm{GL}_{l_2} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & \mathrm{GL}_{l_k} \end{pmatrix}.$$

Hence after applying the integration map to $\prod_{l=1}^{k}[BGL_{r_l}(D)\to\mathcal{M}]$, we have

$$\begin{aligned} &\chi(\prod_{l=1}^{k}[BGL_{r_l}(D)\to\mathcal{M}]) \\ =&\frac{1}{\left(\prod_{l=1}^{k} q^{r_l(r_l-1)/2}\prod_{s=1}^{r_l}(q^s-1)\right)\prod_{u<v} q^{r_u r_v}}. \end{aligned}$$

Chapter 7

The relation between scattering diagrams and quiver representations

Let Q be an acyclic finite quiver with vertices $1, \ldots, n$. In this section, we will mostly denote representations of the quiver with upper case letters, e.g. D, and its dimension vector as lower case, say d.

7.1 Theta functions and the Caldero-Chapoton formula

Let D be a finite-dimensional representation of Q with dimension vector d. Recall the Caldero-Chapoton formula in Theorem 3.1.2.

$$CC(D) = \frac{1}{A_1^{d_1} \cdots A_n^{d_n}} \sum_{0 \leq e \leq d} \chi(Gr(e, D)) \prod_{i=1}^{n} A_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)},$$

where $Gr(e,D) := \{N \in \mathrm{mod}(Q) | N \subseteq D, \dim(N) = e\}$ for $e \in \mathbb{N}^n$, the sum is taken over all vectors $e \in \mathbb{N}^2$ such that $0 \leq e_i \leq d_i$ for all i. Then we have

$$\begin{aligned} CC(D) &= \frac{\prod A_i^{\sum_{i\to j} d_j}}{A_1^{d_1}\cdots A_n^{d_n}} \sum_{0\leq e\leq d} \chi(Gr(e,D)) \prod_{i=1}^{n} A_i^{\sum_{j\to i} e_j + \sum_{i\to j} -e_j} \\ &= A^{-\mathcal{E}(d)} \sum_{0\leq e\leq d} \chi(Gr(e,D)) A^{p^*(e)}, \end{aligned} \tag{7.1.1}$$

where $\mathcal{E}(d)$ is defined as in 3.2.2. Note that the suitable monomial corresponding to D should be $A^{-\mathcal{E}(d)}$ instead of $A^{\dim(D)}$. The term $A_i^{\sum_{i\to j} d_j}$ hints that we should look at the representations indexed by injective representations $I(i)$ instead of simple representations $S(i)$.

First the map CC takes a rigid indecomposable representation D with dimension vector d_i to the unique non-initial cluster variable $CC(D)$ ([Kel10, Corollary 5.6, Theorem 6.1]). A representation D is **rigid** if $\mathrm{Ext}^1(D,D) = 0$. By arguments similar to those in Section 3.3.3, a pre-projective or pre-injective representation is rigid. However a regular representation may not be rigid, e.g. for the Kronecker 2-quiver, we have non-split exact sequence

$$0 \longrightarrow (\mathbb{C} \rightrightarrows \mathbb{C}) \longrightarrow (\mathbb{C}^2 \rightrightarrows \mathbb{C}^2) \longrightarrow (\mathbb{C} \rightrightarrows \mathbb{C}) \longrightarrow 0.$$

Furthermore, if $E = E_1 \oplus \cdots \oplus E_s$ is a decomposition into indecomposables, then $CC(E) = \prod_{i=1}^{s} CC(E_i)$ ([Kel10, 5.6(b1)]). By Remark 4.3.4, cluster variables are theta functions. Thus a theta function associated to a cluster variable will give the same Laurent polynomial in initial variables as in the Caldero-Chapoton formula. We can see the relation between the Caldero-Chapoton formula and the theta function as

$$CC(D) = \vartheta_{\mathcal{Q},-\mathcal{E}(d)}, \tag{7.1.2}$$

where $\mathcal{Q}$ is a point in the positive chamber. By point 1 in Theorem 4.3.3, $\vartheta_{\mathcal{Q},-\mathcal{E}(d)} = \vartheta_{\mathcal{Q}',-\mathcal{E}(d)}$ if $\mathcal{Q}$ and $\mathcal{Q}'$ stays in the same chamber. We will further simplify the notation as

$$\vartheta_{-\mathcal{E}(d)} = \vartheta_{\mathcal{Q},-\mathcal{E}(d)}.$$

Furthermore, the Equality 7.1.2 also tells us that each monomial in the summand corresponds to a subrepresentation in D of dimension vector e. For example:

Example 7.1.1. Going back to Example 4.3.6, we have

$$\vartheta_{\mathcal{Q},(1,-1)} = A_1A_2^{-1} + A_1^{-1}A_2^{-1} + A_1^{-1}A_2.$$

We can write it as

$$\vartheta_{\mathcal{Q},(1,-1)} = \frac{A_1^2}{A_1^{-1}A_2^{-1}}(1 + A_1^2 + A_1^2A_2^2) = CC(\mathbb{C}\rightrightarrows\mathbb{C}).$$

The three terms corresponds to the three representations with dimension vectors $(0,0)$, $(0,1)$, $(1,1)$.

Remark 7.1.2. Let us recall (4.3.1),

$$\vartheta_{\mathcal{Q},(2,-2)} = \left(\vartheta_{\mathcal{Q},(1,-1)}\right)^2 - 2.$$

We learn that the theta functions does not satisfy the same equality satisfied by the CC formula, namely $CC(V \oplus W) = CC(V)CC(W)$ in the badlands region. The '-2' term in the equations means the theta $\vartheta_{\mathcal{Q},(2,-2)}$ fails to count the subrepresentations with dimension vector $(1,1)$. Furthermore, the number 2 comes from the euler characteristics of framed representations with dimension vector $(1,1)$ in the Kronecker 2-quiver. Therefore, the difference may suggest us to investigate the difference between moduli spaces of representations and moduli spaces of framed representations

In Section 5 [CGM+15], we have shown that rank 2 theta basis is the greedy basis. By using 5.1, we can find out the different between $\vartheta_{(s,-s)}$ and $(\vartheta_{(1,-1)})^s$. The missing terms correspond to representations with dimension vectors (t,t), where $t < s$. Thus all the difference comes from the badlands.

Chamber structure of scattering diagram

Consider a chamber $\mathcal{C} \in \Delta_s$ in the cluster complex in $\mathfrak{D}$. As noted in Section 4.2.1, $\mathcal{C}$ corresponds to a seed data s. For a cluster algebra of rank n, the chamber $\mathcal{C}$ will be a cone bounded by n walls. Thus, $\mathcal{C}$ is generated by n basis vectors $b_1, \ldots, b_n$ which lies in the intersection of the walls which bound the cone. If a b_i is not in the standard basis of $\mathbb{Z}^n$, then it corresponds to a cluster variable which is not initial. Thus by the identification between the Caldero-Chapoton

formula and theta functions, we can express $b_i = -\mathcal{E}(d_i)$ for some $d_i \in N^+$ which corresponds to indecomposable representation D_i with the dimension vector d_i. Further assume $\mathcal{C}$ corresponds to the seed consisting of non-initial variables only, and pick any point $c \in \mathcal{C}$, $c = \kappa_1 b_1 + \cdots + \kappa_n b_n$ for some non-negative integers $\kappa_1, \ldots, \kappa_n$. Then c would correspond to the representation $D = \kappa_1 D_1 + \cdots + \kappa_n D_n$. Note that $c = \mathcal{E}(D)$. Therefore, in the discussion in this chapter, we can always assume the representation is indecomposable.

For example, consider Figure 4.2, the scattering diagram associated to the Kronecker 2-quiver. Let us consider the chamber $\mathcal{C}$ spanned by $(2, -1)$ and $(3, -2)$. Then the corresponding cluster is $\vartheta_{(2,-1)}, \vartheta_{(3,-2)}$ which is associated to the indecomposable projectives $0 \rightrightarrows \mathbb{C}$, $\mathbb{C} \rightrightarrows \mathbb{C}^2$. For any point r in the chamber $\mathcal{C}$, we can write $r = \lambda_1(2, -1) + \lambda_2(3, -2)$. The corresponding quiver representation of r would be $(0 \rightrightarrows \mathbb{C})^{\lambda_1} + (\mathbb{C} \rightrightarrows \mathbb{C}^2)^{\lambda_2}$. Therefore, in the computations in this section, we can always assume the ambient representation D is indecomposable.

7.2 Good Crossing and the AR quiver

Consider a path γ in $\mathfrak{D}$. Assume γ crosses a wall $\mathfrak{d}$. Let $n \in N^+$ be the normal of $\mathfrak{d}$, i.e. $\mathfrak{d} \subset n^\perp$. Then the crossing is a **good crossing** if the path passes from the region where n is negative to the region it is positive. Otherwise, it is called a **bad crossing**.

Note that in the two-dimensional scattering diagram, if the path travels around the origin anti-clockwise from the lower half plane towards the positive chamber, the crossing is always good.

Now consider two walls $\mathfrak{d}_1$ and $\mathfrak{d}_2$ in the cluster complex. Let $f_i \in N^+$ be the normal of $\mathfrak{d}_i$ respectively. Then $\mathfrak{d}_1$ and $\mathfrak{d}_2$ may or may not intersect along a co-dimensional 2 polyhedral set which we call a **joint**. If $\mathfrak{d}_1$ intersects $\mathfrak{d}_2$ along a joint $\mathfrak{j}$, we decompose $\mathfrak{d}_1$ into three disjoint components $\mathfrak{d}_1^+$, $\mathfrak{j}$, $\mathfrak{d}_1^-$ where each of them are connected. Let us assume $-p^*(f_1)$ does not lie on the joint $\mathfrak{j}$. Then $\mathfrak{d}_1^+$ is defined to be the connected piece which contains the point $-p^*(f_1)$ which we will name the outgoing piece of $\mathfrak{d}_1$. Figure 7.1 illustrates the decomposition. If $\mathfrak{d}_1$, $\mathfrak{d}_2$

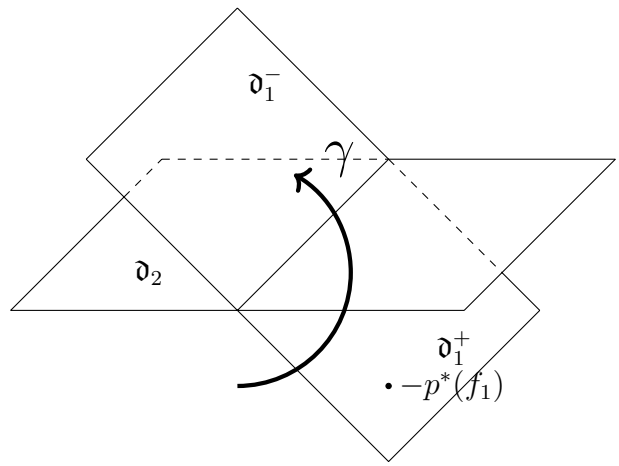

Figure 7.1: γ goes from $\mathfrak{d}_1^+$ to $\mathfrak{d}_2$

do not intersect along a joint, then we take $\mathfrak{d}_1^+ = \mathfrak{d}_1$. Note that $\mathfrak{d}_1^+$ still contains the point $-p^*(f_1)$ by construction.

Nest consider a path γ has good crossing from the outgoing piece of $\mathfrak{d}_1$ to the wall $\mathfrak{d}_2$. Please refer to Figure 7.1 about how γ goes.

From the discussion in Section 6.5, for $f_i \in N^+$, $i = 1, 2$, we can consider f_i as a dimension vector for some indecomposable semistable representation F_i. Then since γ having good crossing from outgoing piece of $\mathfrak{d}_1$ to $\mathfrak{d}_2$, $-p^*(f_1)$ points away from the positive direction of f_2, i.e.

$$\langle -p^*(f_1), f_2 \rangle < 0.$$

Here the inequality is strict as we have assume $-p^*(f_1)$ does not lie on $\mathfrak{d}_2$. By Section A.1, we have

$$\begin{aligned}
&\langle -p^*(f_1), f_2 \rangle = -\{f_1, f_2\} \\
=&\chi(f_1, f_2) - \chi(f_2, f_1) \\
=&\dim(\mathrm{Hom}(F_1, F_2)) - \dim \mathrm{Ext}^1(F_1, F_2) \\
&- \dim(\mathrm{Hom}(F_2, F_1)) + \dim(\mathrm{Ext}^1(F_2, F_1)).
\end{aligned} \tag{7.2.1}$$

We will now apply the lemmas in Section 3.3.3.

First assume Q is a connected acyclic quiver of finite type. Then from Theorem 3.3.10, we have the AR quiver is connected and it equals both the pre-projective component and the pre-injective component. At the same time the

regular component is empty. Note that if $\dim(\mathrm{Hom}(F_1, F_2)) \neq 0$, then F_1 is a predecessor of F_2 which implies $\dim(\mathrm{Hom}(F_2, F_1)) = 0$. From Section 3.3.3, we know that only one of the terms $\dim(\mathrm{Hom}(F_i, F_j))$, $\dim \mathrm{Ext}^1(F_i, F_j)$, $i \neq j$ is nonzero. In order to attain $\langle -p^*(f_1), f_2 \rangle < 0$, we will have

$$\langle -p^*(f_1), f_2 \rangle = -\dim \mathrm{Ext}^1(F_1, F_2) - \dim(\mathrm{Hom}(F_2, F_1)).$$

As the term above is negative, one of the $\dim \mathrm{Ext}^1(F_1, F_2)$ and / or $\dim(\mathrm{Hom}(F_2, F_1))$ is non-zero which implies F_2 is a predecessor of F_1 in the AR quiver by Lemma 3.3.16 and Theorem 3.3.12.

Now consider Q is a connected acyclic quiver with infinitely many indecomposables. Then the AR quiver consists of three components: pre-projective $\mathcal{P}$, regular $\mathcal{R}$ and pre-injective $\mathcal{I}$. Assume F_i lies in one of the $\mathcal{P}, \mathcal{R}, \mathcal{I}$, for $i = 1, 2$ and F_1, F_2 do not lie in the same component. We are going to see that γ travels a direction from $\mathcal{I}$ to $\mathcal{R}$, to $\mathcal{P}$. The reason is that from Theorem 3.3.13, we know that

$$\mathrm{Hom}(\mathcal{R}, \mathcal{P}) = \mathrm{Hom}(\mathcal{I}, \mathcal{P}) = \mathrm{Hom}(\mathcal{I}, \mathcal{R}) = 0.$$

And we also have

$$\mathrm{Ext}^1(\mathcal{P}, \mathcal{R}) = \mathrm{Ext}^1(\mathcal{P}, \mathcal{I}) = \mathrm{Ext}^1(\mathcal{R}, \mathcal{I}) = 0$$

from Lemma 3.3.14. Therefore by using the same argument as above, we know that

$$\langle -p^*(f_1), f_2 \rangle = -\dim \mathrm{Ext}^1(F_1, F_2) - \dim(\mathrm{Hom}(F_2, F_1)).$$

Hence we can only have three choices for F_1, F_2: (1). $F_2 \in \mathcal{I}$, $F_1 \in \mathcal{P}$; (2). $F_2 \in \mathcal{I}$, $F_1 \in \mathcal{R}$; (3). $F_2 \in \mathcal{R}$, $F_1 \in \mathcal{P}$. All three cases indicates that F_2 is the predecessor of F_1 in the general definition for predecessor. This is saying that if γ travels between walls corresponding to $\mathcal{P}, \mathcal{R}, \mathcal{I}$, it would have crossed from the pre-injective to the regular then to the pre-projective if the crossings are good.

We also have the cases $F_1, F_2 \in \mathcal{P}$ or $F_1, F_2 \in \mathcal{I}$. Then by Lemma 3.3.15, Lemma 3.3.16 together with Theorem 3.3.12, we have F_2 is a predecessor of F_1 from (7.2.1). Therefore, we can say if γ is having a good crossing in the scattering diagram, it is going in an opposite direction to the AR quiver. More precisely, we have

Proposition 7.2.1. *Given two walls $\mathfrak{d}_1$ and $\mathfrak{d}_2$ in the cluster complex in $\mathfrak{D}$. Let $f_i \in N^+$ be the normal of $\mathfrak{d}_i$ for $i = 1, 2$. If the two walls intersect along a joint, a co-dimensional 2 polyhedral set, we consider the outgoing piece $\mathfrak{d}_1^+$ of $\mathfrak{d}_1$ which contains $-p^*(f_1)$ instead of the entire $\mathfrak{d}_1$.*

Let γ be a path crossing the outgoing piece of $\mathfrak{d}_1 \in \mathfrak{D}$ and then $\mathfrak{d}_2 \in \mathfrak{D}$ with both crossings good. Let F_i be the corresponding indecomposable representation for $\mathfrak{d}_i$, $i = 1, 2$. Then we have F_2 is a predecessor of F_1 in the Auslander-Reiten quiver of Q.

7.3 Theta function

In this section, we are going to give an alternative proof of the Caldero-Chapoton formula by the machinery set up in Chapter 6. Notice that we will have our paths going towards the positive chamber of the scattering diagrams which differs from the direction in [Bri15].

Let $\mathcal{Q}$ be a point in the positive chamber. Let $m_0 \in \mathcal{C}$ for some $\mathcal{C} \in \Delta_s$, i.e. m_0 lies in the cluster complex. We further assume that $\mathcal{C}$ corresponds to a seed which does not contain any initial cluster variables. Then we can write $m_0 = -\mathcal{E}(d)$ for some $d \in (\mathbb{Z}_{\geq 0})^n$. Then from the discussion about chamber structure in Section 7.1, there is a representation D associated to m with dimension vector d. We can further assume D is indecomposable. Note that as m_0 is in the cluster chamber, ϑ_{m_0} corresponds to a cluster variable which implies D is not a regular representation.

Recall in Lemma 6.3.2:

$$\mathcal{F}(m_0) = \{E \in \mathrm{rep}(Q) : \text{any subobject } F \subset E \Rightarrow m_0(F) \leq 0\}.$$

For any $F \subset D$, we have the inclusion $F \to D$. Then $\mathrm{Hom}(F, D) \neq 0$. By decomposing F into sum of indecomposables F_i, then $\dim \mathrm{Hom}(F_i, D) > 0$. This implies F_i is a predecessor of D. By Lemma 3.3.15 and Lemma 3.3.14, we have $Ext^1(F_i, D) = 0$. Thus $\chi(f_i, d) = \dim \mathrm{Hom}(F_i, D) > 0$. That is $m_0(F) = -\mathcal{E}(d)(f) < 0$. Therefore we have $D \in \mathcal{F}(m_0)$.

Furthermore, for all $F \in \mathcal{F}(m_0)$ indecomposable, by definition of $\mathcal{F}(m_0)$, $\mathcal{E}(d)(F) = \chi(f, d) \geq 0$, where $f = \dim F$. By the discussion at the end of Section

3.3.3, we obtain $\chi(f,d) = \dim \mathrm{Hom}(F,D)$.

Recall the theta function we defined in (6.3.4)

$$1_{\mathcal{F}}(m_0)^{-1} A^{m_0} 1_{\mathcal{F}}(m_0).$$

Note that by the commutativity relation (6.3.1), we have

$$A^{m_0}[\mathcal{M}_e^{\mathcal{F}} \to \mathcal{M}] = q^{\mathcal{E}(D)(e)}[\mathcal{M}_e^{\mathcal{F}} \to \mathcal{M}] A^{m_0},$$

where objects in $[\mathcal{M}_e^{\mathcal{F}} \to \mathcal{M}]$ are representations $E \in \mathcal{F}(m_0)$ with dimension vector e. As noted above,

$$A^{m_0}[\mathcal{M}_e^{\mathcal{F}} \to \mathcal{M}] = q^{\dim \mathrm{Hom}(E,D)}[\mathcal{M}_e^{\mathcal{F}} \to \mathcal{M}] A^{m_0},$$

By viewing $q^{\dim \mathrm{Hom}(E,D)}$ as $[\mathbb{A}^{\dim \mathrm{Hom}(E,D)}]$ in the Hall algebra, we have

$$q^{\dim \mathrm{Hom}(E,D)}[\mathcal{M}_e^{\mathcal{F}} \to \mathcal{M}] = [\mathcal{M}_e^{\mathcal{F},d} \to \mathcal{M}],$$

where objects in $[\mathcal{M}_e^{\mathcal{F},d} \to \mathcal{M}]$ are pairs (E,ψ) with $E \in \mathcal{F}(m_0)$ having dimension vector e and $\psi : E \to D$ a map.

Now consider (E,ψ) as an object in $[\mathcal{M}_e^{\mathcal{F},d} \to \mathcal{M}]$. As $E \in \mathcal{F}(m_0)$, $R = \ker(\psi) \in \mathcal{F}(m_0)$. Consider the short exact sequence

$$0 \longrightarrow R \longrightarrow E \longrightarrow S \longrightarrow 0. \tag{7.3.1}$$

The map ψ induces an injective map $S \to D$. We wish to show $S \in \mathcal{F}(m_0)$. Note that by the properties of torsion pair in Definition 6.3.1, we have the following exact sequence

$$0 \longrightarrow T \longrightarrow S \longrightarrow F \longrightarrow 0,$$

where $T \in \mathcal{T}(m_0)$, $F \in \mathcal{F}(m_0)$. Since $D \in \mathcal{F}(m_0)$, $\mathrm{Hom}(T,D) = 0$ from the definition of torsion pair. So the exact sequence becomes

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[dr, "0"] & S \arrow[r] \arrow[d, hook] & F \arrow[r] & 0 \\
 & & D & &
\end{tikzcd}$$

This forces the map $T \to S$ to be the zero map in the exact sequence. Thus we have $S \cong F \in \mathcal{F}(m_0)$.

Going back to (7.3.1), we have

$$[\mathcal{M}_e^{\mathcal{F},d} \to \mathcal{M}] = 1_{\mathcal{F}}(m_0) * \mathcal{G}_{\mathcal{F}(m_0)}(D),$$

where objects in $\mathcal{G}_{\mathcal{F}(m_0)}(D)$ are representations E in $\mathcal{F}(m_0)$ equipped with an inclusion into D. Then

$$\begin{aligned}
\vartheta_{m_0} &= 1_{\mathcal{F}}(m_0)^{-1} A^{m_0} 1_{\mathcal{F}}(m_0) \\
&= 1_{\mathcal{F}}(m_0)^{-1} \sum_{e=\dim E} q^{\dim \operatorname{Hom}(E,D)} [\mathcal{M}_e^{\mathcal{F}} \to \mathcal{M}] A^{m_0} \\
&= 1_{\mathcal{F}}(m_0)^{-1} * 1_{\mathcal{F}}(m_0) * \mathcal{G}_{\mathcal{F}(m_0)}(D) A^{m_0} \\
&= \mathcal{G}_{\mathcal{F}(m_0)}(D) A^{m_0}
\end{aligned}$$

If Q is a quiver with finitely many indecomposables, by applying χ in 6.3.3 to $\mathcal{G}_{\mathcal{F}(m_0)}(D)$, we get

$$\vartheta_{m_0} = \chi(\mathcal{G}_{\mathcal{F}(m_0)}(D)) A^{m_0} = A^{-\mathcal{E}(D)} \sum_{0 \le e \le d} \chi(\operatorname{Gr}(e, D)) A^{p^*(e)}$$

which is exactly as in (7.1.1). Therefore, we obtain the Caldero-Chapoton formula.

7.4 Stratification of quiver grassmannian

We will repeat similar calculations as in the last section to give a stratification of quiver grassmannian.

Consider an indecomposable quiver representation D which is not regular. Then $m = -\mathcal{E}(d)$ lies in the cluster complex. Now consider a broken line $\gamma : (\infty, 0] \to M_{\mathbb{R}} \setminus \{0\}$ with endpoint $\mathcal{Q}$ in the positive chamber and initial slope $-\mathcal{E}(d)$. We further assume the final slope of the broken line is $-\mathcal{E}(d) + p^*(e)$ for some e. Let γ bend over the walls $\mathfrak{d}_1, \ldots, \mathfrak{d}_s$ in the cluster complex and assume all the bendings are good. Denote by $f^i \in N^+$ the normal vectors of $\mathfrak{d}_i$ for $i = 1, \ldots s$. From the remarks in last sections, the walls $\mathfrak{d}_i$ correspond to indecomposable representations F_i with dimension vector f^i.

For $i = 1, \ldots, s-1$, if $\mathfrak{d}_i$ intersect $\mathfrak{d}_{i+1}$ along a co-dimension 2 joint $\mathfrak{j}$ as in Section 7.2, we decompose $\mathfrak{d}_i$ into 3 connected components $\mathfrak{d}_i^+$, $\mathfrak{j}$ and $\mathfrak{d}_i^-$ where

$-p^*(f_i)$ lie on either $\mathfrak{d}_i^+$ or $\mathfrak{j}$. Let us further assume that γ crosses from $\mathfrak{d}_i^+$ to $\mathfrak{d}_{i+1}$. If $\mathfrak{d}_i$ does not intersect $\mathfrak{d}_{i+1}$ along a co-dimensional 2 joint, we can take $\mathfrak{d}_i^+ = \mathfrak{d}_i$. By repeating same argument as in Section 7.2, we will have the following 2 cases.

A If $-p^*(f_i)$ lies on $\mathfrak{j}$, then $\langle -p^*(f_i), f_{i+1}\rangle = -\{f_i, f_{i+1}\} = 0$. We then have

$$\begin{aligned}\dim \operatorname{Hom}(F_i, F_{i+1}) &= \dim \operatorname{Hom}(F_{i+1}, F_i)\\ = \dim \operatorname{Ext}^1(F_i, F_{i+1}) &= \dim \operatorname{Ext}^1(F_{i+1}, F_i) = 0.\end{aligned}$$

B If $-p^*(f_i)$ lies on $\mathfrak{d}_i^+$, and as γ crosses good from $\mathfrak{d}_i^+$ to $\mathfrak{d}_{i+1}$, then

$$\langle -p^*(f_i), f_{i+1}\rangle = -\{f_i, f_{i+1}\} < 0.$$

Therefore,

$$\langle -p^*(f_i), f_{i+1}\rangle = -\dim \operatorname{Ext}^1(F_i, F_{i+1}) - \dim \operatorname{Hom}(F_{i+1}, F_i).$$

and $\dim \operatorname{Hom}(F_i, F_{i+1}) = \dim \operatorname{Ext}^1(F_{i+1}, F_i) = 0$.

From the identification between theta functions and the Caldero-Chapoton formula by equation 7.1.1, the setup above is saying that we are considering the subrepresentations E of dimension e in D. By the bendings of γ, we have $e = \sum_i \lambda_i f_i$ for some $\lambda_i \in \mathbb{N}$.

7.4.1 First Bending

As γ has good bending over the first wall $\mathfrak{d}_1$, we have

$$\langle \mathcal{E}(d), f^1\rangle > 0.$$

As F_i and D are indecomposable, by lemmas in Section 3.3.3, we have

$$\langle \mathcal{E}(d), f^1\rangle = \chi(F_1, D) = \dim \operatorname{Hom}(F_1, D).$$

We can describe the first bending as follows.

Proposition 7.4.1. *Let γ be a broken line as defined above. Consider the first bending of γ over a general point ω_1 on the wall $\mathfrak{d}_1$ which corresponds to pre-projective/ pre-injective indecomposable representation F_1 of dimension vector f^1. Then the Hall algebra wall crossing automorphism is*

$$\Phi_{\mathfrak{D}}(\mathfrak{d}_1)A^{-\mathcal{E}(d)} = \sum_{\lambda} \mathcal{G}^1_{\lambda_1, f^1}(D)A^{-\mathcal{E}(d)},$$

where $\mathcal{G}^1_{\lambda, f^1}(D) = \{\psi : V \to D | V$ is a representation of dimension vector λf^1, and $\ker\psi$ contains no subrepresentation of dimension vector proportional to $f^1\}$.

Proof. By definition, the wall crossing at ω_1 is

$$\Phi_{\mathfrak{D}}(\mathfrak{d}_1)A^{-\mathcal{E}(d)} = 1_{ss}(\omega_1)^{-1}A^{-\mathcal{E}(d)}1_{ss}(\omega_1)$$

Then we employ (6.5.1)

$$\begin{aligned}
&= 1_{ss}(\omega_1)^{-1}A^{-\mathcal{E}(d)}(1 + \sum_{\ell \geq 1}[BGL_\ell(F_1) \to \mathcal{M}]) \\
&= 1_{ss}(\omega_1)^{-1}(1 + \sum_{\ell \geq 1} q^{\ell \cdot \chi(f^1, D)}[BGL_\ell(F_1) \to \mathcal{M}])A^{-\mathcal{E}(d)} \qquad (7.4.1) \\
&= 1_{ss}(\omega_1)^{-1}(1 + \sum_{\ell \geq 1} q^{\ell \dim \mathrm{Hom}(F^1, D)}[BGL_\ell(F_1) \to \mathcal{M}])A^{-\mathcal{E}(d)} \\
&= 1_{ss}(\omega_1)^{-1} \sum_{\ell \geq 0} 1^D_{ss}(\ell, \omega_1)A^{-\mathcal{E}(d)}, \qquad (7.4.2)
\end{aligned}$$

where $1^D_{F_1}(\ell, \omega_1)$ is the moduli space of semistable representations F_1' with a map $F_1' \to D$, where F_1' has dimension vector ℓf^1. (7.4.1) follows from the commutativity relation in (6.3.1). We obtain (7.4.2) by visualizing $q^{\ell \cdot \dim \mathrm{Hom}(F^1, D)}$ as a vector bundle of rank $= \ell \dim \mathrm{Hom}(F^1, D)$.

We now need to understand $1^D_{ss}(\ell, \omega_1)(\mathrm{Spec}\,\mathbb{C})$.

Given $\psi : F_1' \to D$, we consider $\ker\psi$. Take ω' to be a point very close to $\mathfrak{d}_1$ in the positive f_1 direction. By Definition 6.3.1 and Theorem 6.3.2, we have

$$0 \longrightarrow T \longrightarrow \ker\psi \longrightarrow F \longrightarrow 0$$

where $T \in \mathcal{T}(\omega')$, $F \in \mathcal{F}(\omega')$. We claim T has dimension vector proportional to $\dim F_1$. Firstly, as $T \subseteq \ker\psi \subseteq F_1'$, we have $\omega_1(T) \leq 0$ since F_1' is ω_1-semistable.

And by definition of $\mathcal{T}$, $\omega'(T) > 0$. As ω' can be arbitrary close to the wall $\mathfrak{d}_1$, we have $\omega_1(T) = 0$.

Consider the quotient $F_1 = F_1'/T$. As both F_1' and T have dimension vectors as multiplies of f^1, $F_1 = \lambda f^1$ for some λ. Furthermore, ψ induces a map $F_1 \to D$ with kernel in $\mathcal{F}(\omega')$. Thus we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] & F_1' \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[dl] & 0 \\ & & D & & \end{tikzcd}$$

where the row is an exact sequence.

From the exact sequence, we have

$$1_{ss}^D(\ell, \omega_1) = \sum_h [BGL_\ell(F_1) \to \mathcal{M}] * [\mathcal{G}^1_{\ell-h,f^1}(D) \to \mathcal{M}],$$

where $\mathcal{G}^1_{\ell-h,f^1}$ denotes denotes the stack of representations F_1 with dimension vector $(\ell - h)f^1$ and there exist a homomorphism $F_1 \to D$ with kernel in $\mathcal{F}(\omega')$. Then

$$\begin{aligned} \sum_\ell 1_{ss}^D(\ell, \omega_1) &= \sum_\ell \sum_h [BGL_h(F_1) \to \mathcal{M}] * [\mathcal{G}^1_{\ell-h,f^1}(D) \to \mathcal{M}] \\ &= \sum_h [BGL_h(F_1) \to \mathcal{M}] * \sum_\ell [\mathcal{G}^1_{\ell-h,f^1}(D) \to \mathcal{M}] \\ &= 1_{ss}(\omega_1) * \sum_\lambda [\mathcal{G}^1_{\lambda,f^1}(D) \to \mathcal{M}]. \end{aligned}$$

Therefore, we have

$$\Phi_{\mathfrak{D}}(\mathfrak{d}_1) A^{-\mathcal{E}(d)} = 1_{ss}(\omega_1)^{-1} \sum_{\ell \geq 0} 1_{ss}^D(\ell, \omega_1) A^{-\mathcal{E}(d)} = \sum_\lambda [\mathcal{G}^1_{\lambda,f^1}(D) \to \mathcal{M}] A^{-\mathcal{E}(d)}.$$

□

If F^1 is not regular, then by applying χ in 6.3.3, we have

$$\begin{aligned} \chi(\Phi_{\mathfrak{D}}(\mathfrak{d}_1) A^{-\mathcal{E}(d)}) &= \sum_\lambda \chi([\mathcal{G}^1_{\lambda,f^1}(D) \to \mathcal{M}]) A^{-\mathcal{E}(d)} \\ &= \sum_\lambda \chi\left(\mathrm{Gr}(\lambda, \mathrm{Hom}(F^1, D))\right) A^{\lambda p^*(f^1)} A^{-\mathcal{E}(d)} \end{aligned}$$

This is the path-ordered product defined in 4.1.1.

Now we are ready to move forward to the second bending. As $e = \sum \lambda_i f^i$, we will take the term

$$[\mathcal{G}^1_{\lambda_1, f^1}(D) \to \mathcal{M}] A^{-\mathcal{E}(D)}.$$

Applying the integration map, we have

$$\chi([\mathcal{G}^1_{\lambda_1}(D) \to \mathcal{M}]) A^{-\mathcal{E}(D)} = \chi\left(\mathrm{Gr}(\lambda_1, \mathrm{Hom}(F^1, D))\right) A^{-\mathcal{E}(D) + p^*(\lambda_1 f^1)}.$$

Let us consider the q-binomial coefficients $\binom{a}{b}_q = \frac{(q^a-1)\cdots(q^{a-b+1}-1)}{(q^b-1)\cdots(q-1)}$. Then the above equation is

$$\chi([\mathcal{G}^1_{\lambda_1}(D) \to \mathcal{M}]) A^{-\mathcal{E}(D)} = \binom{\dim \mathrm{Hom}(F^1, D)}{\lambda_1}_q A^{-\mathcal{E}(D) + p^*(\lambda_1 f^1)}.$$

Taking the limit $q \to 1$ will give us the usual broken line attaching monomial $\binom{\langle \mathcal{E}(d), f^1 \rangle}{\lambda_1} A^{-\mathcal{E}(D) + p^*(\lambda_1 f^1)}$.

Note that after the first bending, if we take $V_1 = F_1^{\oplus \lambda_1}$, we have

$$0 \subset V_1$$

as the first step of a filtration of $E \subset D$.

7.4.2 Second bending

In the next step we consider, γ crosses a second wall $\mathfrak{d}_2$. By Definition 6.4.1 of Hall algebra broken lines, $[\mathcal{G}^1_{\lambda_1, f^1}(D) \to \mathcal{M}] A^{-\mathcal{E}(D)}$ is attached to the linear piece of γ after the first bending. In the usual broken line, the attached monomial is $\chi\left(\mathrm{Gr}(\lambda_1, \mathrm{Hom}(F^1, D))\right) A^{-\mathcal{E}(D) + p^*(f^1)}$. As we are assuming the crossing of γ over $\mathfrak{d}_2$ is good, we have

$$\langle -\mathcal{E}(D) + p^*(f^1), -f_2 \rangle \geq 0,$$

i.e.

$$\mathcal{E}(d, f_2) - \{f_1, f_2\} \geq 0.$$

From our assumption that γ goes from $\mathfrak{d}_1^+$ to $\mathfrak{d}_2$ and the description in the beginning of Section 7.4, we have

$$-\{f_1, f_2\} \leq 0.$$

This implies $\mathcal{E}(d, f_2) \geq 0$. Therefore

$$\mathcal{E}(d, f_2) = \chi(f^2, d) = \dim \mathrm{Hom}(F^2, D).$$

We can then apply the wall crossing for the second bending at a general point ω_2 on $\mathfrak{d}_2$ on $[\mathcal{G}^1_{\lambda_1, f^1}(D) \to \mathcal{M}]A^{-\mathcal{E}(D)}$.

$$\begin{aligned}
&\Phi_{\mathfrak{D}}(\mathfrak{d}_2)\left([\mathcal{G}^1_{\lambda_1, f^1}(D) \to \mathcal{M}]A^{-\mathcal{E}(D)}\right) \\
=&1_{ss}(\omega_2)^{-1} \star [\mathcal{G}^1_{\lambda_1, f^1}(D) \to \mathcal{M}]A^{-\mathcal{E}(d)} \star 1_{ss}(\omega_2) \\
=&1_{ss}(\omega_2)^{-1} \star \sum_{\ell} [\mathcal{G}^1_{\lambda_1, f^1}(D) \to \mathcal{M}] \star q^{\dim(\mathrm{Hom}(F_2, D))} 1_{ss}(\ell, \omega_2) A^{-\mathcal{E}(d)} \\
=&1_{ss}(\omega_2)^{-1} \sum_{\ell} [\mathcal{G}^1_{\lambda_1, f^1}(D) \to \mathcal{M}] \star 1^D_{ss}(\ell, \omega_2) A^{-\mathcal{E}(d)}
\end{aligned}$$

where $1^D_{ss}(\ell, \omega_2)$ is the moduli space of semistable representation F_2' with a map $F_2' \to D$, where F_2' is a representation with dimension vector ℓf^2.

We will now put (6.5.2) into the above, i.e. we need to investigate

$$\sum_{\ell} \left(1 + \sum_{k} (-1)^k \prod_{l=1}^{k} [BGL_{r_l}(F_2) \to \mathcal{M}]\right) \star [\mathcal{G}^1_{\lambda_1, f^1}(D) \to \mathcal{M}] \star 1^D_{ss}(\ell\omega_2) \tag{7.4.3}$$

(7.4.3) looks complicated. Actually it is very pretty at each degree. Fix ℓ and k. For ease of reading, let us abuse notation to write $\mathcal{G}^1$ as $[\mathcal{G}^1_{\lambda_1, f^1}(D) \to \mathcal{M}]$. Let us the terms of dimension vector $s \cdot f_2$ in (7.4.3). We call this the multiplicities s contributed. In order to achieve multiplicities s, we can have two choices

1. $B_k \star \mathcal{G}^1 \star 1$
2. $B_{k-1} \star \mathcal{G}^1 \star B'$,

where B_k denote multiplying k of the classifying spaces $[BGL_{r_l} \to \mathcal{M}]$, for some $r_1, \ldots, r_k \in \mathbb{N}$, and B' is an object in $1^D_{ss}(\ell, \omega_2)$. Note that the inverse $1_{ss}(\omega_2)^{-1}$ is an alternating sum. So elements in the forms of (1) and (2) differ by a sign. Therefore, we can group the terms in the multiplicities s summand into pairs

$$\pm\left(B_k \star \mathcal{G}^1 - B_{k-1} \star \mathcal{G}^1 \star B'\right), \tag{7.4.4}$$

where $B_k = \prod_{l=1}^{k} [BGL_{r_l}(F_2) \to \mathcal{M}]$, $B_{k-1} = \prod_{l=1}^{k-1} [BGL_{r_l}(F_2) \to \mathcal{M}]$ and B' is the last multiple of B_k: $[BGL_{r_k}(F_2) \to \mathcal{M}]$.

First, let us understand the term

$$B_k \star \mathcal{G}^1 = \prod_{l=1}^{k} [BGL_{r_l}(F_2) \to \mathcal{M}] \star \mathcal{G}^1.$$

The product in B_k means that we partition s into $(r_1, \ldots, r_k)$. Therefore, we are looking at the diagram

$$\begin{tikzcd} 0 \arrow[r] & F_2^{\oplus r_1} \oplus \cdots \oplus F_2^{\oplus r_k} \arrow[r] & Y \arrow[r] & F_1^{\lambda_1} \arrow[r] \arrow[d] & 0 \\ & & & D & \end{tikzcd} \tag{7.4.5}$$

where Y is such that the row is exact.

As the exact sequence may not split, the term Y is actually

$$\left[\frac{Ext^1(F_1^{\oplus \lambda_1}, F_2^{\oplus r_1} \oplus \cdots \oplus F_2^{\oplus r_k})}{G} \to \mathcal{M} \right] \tag{7.4.6}$$

where G is the group of matrices of the form $\begin{pmatrix} \mathrm{GL}_{r_1} & * & * & * \\ 0 & \mathrm{GL}_{r_2} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & \mathrm{GL}_{r_k} \end{pmatrix}$ as noted at the end of Section 6.5.

The second term $B_{k-1} \star \mathcal{G}^1 \star B'$ in (7.4.4) corresponds to two diagrams (7.4.7) and (7.4.9), where (7.4.7) refers to the product $\mathcal{G}^1 \star B'$ and (7.4.9) refers to $B_{k-1} \star (\mathcal{G}^1 \star B')$. First, $\mathcal{G}^1 \star B'$ gives

$$\begin{tikzcd} 0 \arrow[r] & F_1^{\lambda_1} \arrow[r] \arrow[d] & U_1 \arrow[r] & F_2^{r_k} \arrow[r] \arrow[d] & 0 \\ & D & & D & \end{tikzcd} \tag{7.4.7}$$

for some U_1. We are looking at fibres of projection to $\mathcal{G}^1$. Therefore we fix a map $F_1^{\lambda_1} \to D$ in $\mathcal{G}^1$.

From the discussion in the beginning of Section 7.4, we have $\mathrm{Ext}^1(F_2, F_1) = 0$. Thus the exact sequence above is split. Therefore the middle term is $U_1 = F_1^{\lambda} \oplus F_2^{r_k}$. Notice that there is an automorphism of the exact sequence, namely

$$\begin{tikzcd} 0 \arrow[r] & F_1^{\lambda} \arrow[r] \arrow[d, equal] & F_1^{\lambda} \oplus F_2^{r_k} \arrow[r, "g"] \arrow[d, "\pi"] & F_2^{r_k} \arrow[r] \arrow[d, equal] \arrow[dll, "i"] & 0 \\ 0 \arrow[r] & F_1^{\lambda} \arrow[r] & F_1^{\lambda} \oplus F_2^{r_k} \arrow[r] & F_2^{r_k} \arrow[r] & 0 \end{tikzcd}$$

with $g \circ \pi \circ i$ is an identity. We obtain the object

$$\left[\frac{\mathrm{Hom}(F_2^{r_k}, D)/\,\mathrm{Hom}(F_2^{r_k}, F_1^{\oplus\lambda_1})}{\mathrm{GL}_{r_k}} \to \mathcal{M}\right] = \left[\frac{\mathrm{Hom}(F_2^{r_k}, D/F_1^{\oplus\lambda_1})}{\mathrm{GL}_{r_k}} \to \mathcal{M}\right]. \quad (7.4.8)$$

The second multiplication $B_{k-1} \star (\mathcal{G}^1 \star B')$ gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_2^{\oplus r_1} \oplus \cdots \oplus F_2^{\oplus r_{k-1}} & \longrightarrow & U_2 & \longrightarrow & F_1^{\lambda} \oplus F_2^{r_k} & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & D & & \end{array} \quad (7.4.9)$$

These exact sequences are classified by $\mathrm{Ext}^1(F_1^{\lambda} \oplus F_2^{r_k}, F_2^{\oplus r_1} \oplus \cdots \oplus F_2^{\oplus r_{k-1}}) = \mathrm{Ext}^1(F_1^{\lambda}, F_2^{\oplus l_1} \oplus \cdots \oplus F_2^{\oplus l_{r-1}})$ as F_2 is non-regular. Combining with (7.4.8) the product gives us

$$\left[\frac{\mathrm{Hom}(F_2^{r_k}, D/F_1^{\oplus\lambda_1}) \times \mathrm{Ext}^1(F_1^{\lambda_1}, F_2^{\oplus r_1} \oplus \cdots \oplus F_2^{\oplus r_{k-1}})}{T} \to \mathcal{M}\right], \quad (7.4.10)$$

where T is as in (7.4.6).

The value of (7.4.4) would be the difference of (7.4.6) and (7.4.10)], i.e.

$$\left[\frac{\mathrm{Ext}^1(F_1^{\oplus\lambda_1}, F_2^{\oplus r_1} \oplus \cdots \oplus F_2^{\oplus r_k})}{T} \to \mathcal{M}\right] - \left[\frac{\mathrm{Hom}(F_2^{r_k}, D/F_1^{\oplus\lambda_1}) \times \mathrm{Ext}^1(F_1^{\lambda_1}, F_2^{\oplus r_1} \oplus \cdots \oplus F_2^{\oplus r_{k-1}})}{T} \to \mathcal{M}\right]$$

Let $h = \dim \mathrm{Hom}(F_2, D/F_1^{\oplus\lambda_1})$ and $e = \dim \mathrm{Ext}^1(F_1^{\oplus\lambda_1}, F_2)$. And $s = l_1 + \cdots + l_r$ as we are counting at degree s. Applying χ to the term above gives us

$$\begin{aligned} &\pm \frac{q^{se} - q^{(r_1+\cdots+r_{k-1})e + r_k h}}{\left(\prod_{l=1}^{k} q^{r_l(r_l-1)/2} \prod_{t=1}^{r_l}(q^t - 1)\right) \prod_{u<v} q^{r_u r_v}} \\ = &\pm \frac{q^{se}(q^{r_k h - r_k e} - 1)}{\left(\prod_{l=1}^{k} q^{r_l(r_l-1)/2} \prod_{t=1}^{r_l}(q^t - 1)\right) \prod_{u<v} q^{r_u r_v}}. \end{aligned}$$

One can show that after summing over all the partitions of s, the multiplities s term of (7.4.3) after applying χ would be

$$q^{se} \binom{h-e}{\lambda_2}_q.$$

Let S^2 be the space of the second bending with fixing $F_1^{\lambda_1}\to D$. Then S^2 would have the same Poincaré polynomial as the following space:

$$\sum_s [\mathbb{A}^{se} \times \mathrm{Gr}(s, \mathrm{Hom}(F_2, D/F_1^{\oplus\lambda_1}) - \mathrm{Ext}^1(F_1^{\oplus\lambda_1}, F_2))],$$

where if V, W are vector space, we use the relation $V - W$ to denote the vector space of dimension $\dim V - \dim W$. We now take the map $F_1^{\lambda_1}\to D$ back into account, and denote $[\mathcal{G}^2_{s,f^2}\to\mathcal{M}]$ as the degree s for the space we obtained after the second bending. Then $[\mathcal{G}^2_{s,f^2}\to\mathcal{M}]$ have the same Poincaré polynomial as

$$[\mathbb{A}^{s\dim \mathrm{Ext}^1(F_1^{\oplus\lambda_1})} \times \mathrm{Gr}(s, \mathrm{Hom}(F_2, D/F_1^{\oplus\lambda_1}) - \mathrm{Ext}^1(F_1^{\oplus\lambda_1})) \times \mathrm{Gr}(\lambda_1, \mathrm{Hom}(F_1, D))].$$

Therefore, the Hall algebra wall crossing at second bending would then be

$$\begin{aligned}&\Phi_{\mathfrak{D}}(\mathfrak{d}_2)\left([\mathcal{G}^1_{\lambda_1,f^1}(D)\to\mathcal{M}]A^{-\mathcal{E}(D)}\right)\\ =&\sum_s [\mathcal{G}^2_{s,f^2}(D)\to\mathcal{M}]A^{-\mathcal{E}(D)}.\end{aligned}$$

From the construction, we take λ_2 pieces of F_2. Thus the attaching Hall algebra monomial is

$$[\mathcal{G}^2_{\lambda_2,f^2}(D)\to\mathcal{M}]A^{-\mathcal{E}(D)}.$$

Applying integration map will give us

$$q^{\lambda_2 e}\binom{\chi(f^2, d-\lambda_1 f^1)}{\lambda_2}_q \binom{\chi(f^1, d)}{\lambda_1}_q A^{-\mathcal{E}(D)+p^*(\lambda_1 f^1+\lambda_2 f^2)}.$$

Taking the limit $q\to 1$, will give us the usual attaching monomial.

Moreover, in the Hall algebra multiplication, we build up the representation V_2 of dimension vector $\lambda_1 f^1 + \lambda_2 f^2$ from the exact sequence. Thus, we have a filtration

$$0 \subset V_1 \subset V_2$$

with the quotient $V_2/V_1 = F_1^{\oplus\lambda_1}$.

7.4.3 Further bending

We can repeat what we have done in the last section inductively.

Assume now γ is having the j-th bending from $\mathfrak{d}_{j-1}^{+}$ to $\mathfrak{d}_j$. From the previous $r-1$ bendings, we have the attaching Hall algebra monomial

$$[\mathcal{G}^{j-1}_{\lambda_1,\cdots,\lambda_{j-1},f^{j-1}} \to \mathcal{M}]A^{-\mathcal{E}(D)}.$$

And there is the filtration

$$0 \subset V_1 \subset \cdots \subset V_{j-1}.$$

Bending over $\mathfrak{d}_j$ and taking λ_j copoies of F_j will give us the space S^j having the same Poincaré polynomial as

$$[\mathbb{A}^{\lambda_j \operatorname{Ext}^1(V_{j-1},F_j)} \times \operatorname{Gr}(\lambda_j, \operatorname{Hom}(F_j, D/V_{j-1}) - \operatorname{Ext}^1(V_{j-1}, F_j))].$$

This also builds up one more piece in the filtration

$$0 \subset V_1 \subset \cdots \subset V_j,$$

where $V_j/V_{j-1} = F_j^{\oplus \lambda_j}$.

After the last bending of γ, notice that in our set up, we have

$$e = \sum_{j=1}^{s} \lambda_j f_j.$$

Thus the Hall algebra multiplication allows us to build up a filtration for a subrepresentation E of D with dimension vector e.

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_s = E, \tag{7.4.11}$$

where $V_j/V_{j-1} = F_j^{\oplus \lambda_j}$.

7.4.4 Harder-Narasimhan filtration

In two dimensions, a broken line γ can actually describe a Harder-Narasimhan filtration for a subrepresentation $E \subset D$.

Let $\mathcal{Q}$ be in the positive chamber. Now consider the stability function $Z : K(\operatorname{rep}(Q)) \to \mathbb{C}$ as

$$Z(F) = (\mathcal{E}(d) - p^*(e)) \cdot f + i\mathcal{Q} \cdot f.$$

Now consider the line $\beta : (0,1) \to \mathfrak{D}$ starting from $-\mathcal{E}(d) - p^*(e))$ and ending at $\mathcal{Q}$. β is parametrized by $\beta(t) = (1-t)(-\mathcal{E}(d) + p^*(e)) + t\mathcal{Q}$. When β cross the walls $\mathfrak{d}$ with normal vector $f \in N^+$ at t, we have

$$\langle (1-t)(-\mathcal{E}(d) + p^*(e)) + t\mathcal{Q}, f \rangle = 0.$$

Solving the above will give

$$t^{-1} - 1 = \frac{\mathcal{Q} \cdot f}{(\mathcal{E}(d) - p^*(e))(f)}.$$

This is the slope of Z.

Let γ be the broken line we considered in the beginning of Section 7.4. If γ has good bending over $\mathfrak{d}$, then $-(\mathcal{E}(d) - p^*(e))(-f) \geq 0$ from the properties of γ. Thus $(\mathcal{E}(d) - p^*(e))(f) \geq 0$. Therefore, $\frac{\mathcal{Q} \cdot f}{(\mathcal{E}(d) - p^*(e))(f)}$ has the same ordering as $\phi(F) = \frac{1}{\pi} \arg Z(F)$.

As $t^{-1} - 1$ is strictly decreasing, $\phi(F)$ is decreasing along β. Therefore slopes of the quotients in the filtration (7.4.11) are strictly decreasing. Hence the filtration (7.4.11) is a Harder-Narasimhan filtration.

7.4.5 Example

Let us illustrate what we have in this section by an example. We will consider the Kronecker 2-quiver Q_2 and the indecomposable representation $\mathbb{C}^5 \rightrightarrows \mathbb{C}^6$. Then we will consider the subrepresentation E with dimension vector $(2,4)$. By combining theta function and the Caldero-Chapoton formula, we learn that the Euler characteristic is $\chi(\mathbb{C}^2 \rightrightarrows \mathbb{C}^4, \mathbb{C}^5 \rightrightarrows \mathbb{C}^6) = 18$.

In terms of broken line language, we calculate

$$-\mathcal{E}(5,6) = (7,-6), \qquad -\mathcal{E}(5,6) + p^*(2,4) = (-1,-2).$$

Therefore, we are looking for broken line γ with initial slope $(7,-6)$, final slope $(-1,-2)$ and endpoint at some point $\mathcal{Q}$ in the positive chamber.

If we set $\mathcal{Q} = (2,1)$, we will obtain two broken lines γ_1 (blue) and γ_2 (red) as shown in Figure 7.2.

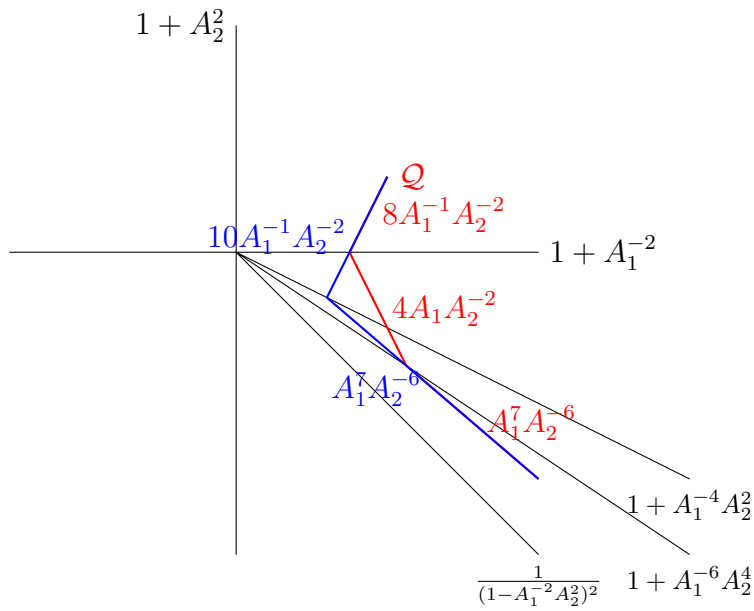

Figure 7.2: $\mathbb{C}^2 \rightrightarrows \mathbb{C}^4 \subset \mathbb{C}^5 \rightrightarrows \mathbb{C}^6$

Let us first start with γ_1 (the blue line). The broken line γ_1 bends over the wall $\mathfrak{d} = \{\mathbb{R}_{\geq 0}(2,-1), 1+A^{(-4,2)}\}$. From Section 7.4.1, we see that the bending contributes two copies of $\mathbb{C}\rightrightarrows\mathbb{C}^2$. Therefore the filtration for E_1 is

$$0 \subset (\mathbb{C}\rightrightarrows\mathbb{C}^2)^2 = E_1.$$

Furthermore the Hall algebra element after bending is $\mathcal{G}^1_{2,(1,2)}(\mathbb{C}^5\rightrightarrows\mathbb{C}^6)\to\mathcal{M}$, where the objects are $\mathbb{C}^2\rightrightarrows\mathbb{C}^4$ with a map $(\mathbb{C}^2\rightrightarrows\mathbb{C}^4) \xrightarrow{\psi} (\mathbb{C}^5\rightrightarrows\mathbb{C}^6)$ such that $\ker\psi$ does not contain any $\mathbb{C}\rightrightarrows\mathbb{C}^2$. Thus the space is

$$\mathrm{Gr}(2, \mathrm{Hom}\left((\mathbb{C}\rightrightarrows\mathbb{C}^2), (\mathbb{C}^5 \rightrightarrows \mathbb{C}^6)\right).$$

Next, we have γ_2. The broken line γ_2 has two bendings: first at the wall $\mathfrak{d}_1 = \{\mathbb{R}_{\geq 0}(3,-2), 1+A^{(-6,4)}\}$ then at the wall $\mathfrak{d}_2 = \{\mathbb{R}(1,0), 1+A^{(-2,0)}\}$. From the calculation of slopes, we know that both bendings take one copy of $\mathbb{C}^2\rightrightarrows\mathbb{C}^3$, $0\rightrightarrows\mathbb{C}$ respectively. This implies the filtration

$$0 \subset \mathbb{C}^2\rightrightarrows\mathbb{C}^3 \subset \mathbb{C}^2\rightrightarrows\mathbb{C}^4 =: E_2,$$

showing that $E_2 = (\mathbb{C}^2\rightrightarrows\mathbb{C}^3) \oplus (0\rightrightarrows\mathbb{C})$.

The first bending of γ_2 is similar to γ_1. Thus we obtain $[\mathcal{G}^1_{1,(2,3)}(\mathbb{C}^5 \rightrightarrows \mathbb{C}^6) \to \mathcal{M}]$ by Proposition 7.4.1. The second bending over a general point $\omega_2 \in \mathfrak{d}_2$ yields a term in

$$1_{ss}(\omega_2)^{-1} \star [\mathcal{G}^1_{1,(2,3)}(\mathbb{C}^5 \rightrightarrows \mathbb{C}^6) \to \mathcal{M}] A^{(7,-6)} \star 1_{ss}(\omega_2).$$

Write $\mathcal{G} = [\mathcal{G}^1_{1,(2,3)}(\mathbb{C}^5 \rightrightarrows \mathbb{C}^6) \to \mathcal{M}]$. As we only need degree 1 in this case, we only consider

$$\mathcal{G} A^{(7,-6)} \star [GL_1(0 \rightrightarrows \mathbb{C}) \to \mathcal{M}] - [GL_1(0 \rightrightarrows \mathbb{C}) \to \mathcal{M}] \star \mathcal{G} A^{(7,-6)}. \tag{7.4.12}$$

After commuting $A^{(7,-6)}$ and $BG_m(0 \rightrightarrows \mathbb{C})$, the first term becomes

$$\begin{aligned} &\mathcal{G} A^{(7,-6)} \star [BG_m(0 \rightrightarrows \mathbb{C}) \to \mathcal{M}] \\ =&\mathcal{G} q^{\dim \operatorname{Hom}(0 \rightrightarrows \mathbb{C}, \mathbb{C}^5 \rightrightarrows \mathbb{C}^6)} [BG_m(0 \rightrightarrows \mathbb{C}) \to \mathcal{M}] A^{(7,-6)} \\ =&\mathcal{G} \star 1^D_{ss}(1, \omega_2) A^{(7,-6)} \end{aligned}$$

The product consists of diagrams

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{C}^2 \rightrightarrows \mathbb{C}^3) \arrow[r] \arrow[d] & (\mathbb{C}^2 \rightrightarrows \mathbb{C}^4) \arrow[r] \arrow[dl] & (0 \rightrightarrows \mathbb{C}) \arrow[r] & 0 \\ & (\mathbb{C}^5 \rightrightarrows \mathbb{C}^6) & & & \end{tikzcd} \tag{7.4.13}$$

where the map $(\mathbb{C}^2 \rightrightarrows \mathbb{C}^4) \to (\mathbb{C}^5 \rightrightarrows \mathbb{C}^6)$ extends the map $(\mathbb{C}^2 \rightrightarrows \mathbb{C}^3) \to (\mathbb{C}^5 \rightrightarrows \mathbb{C}^6)$.

Note that $\operatorname{Ext}^1(0 \rightrightarrows \mathbb{C}, \mathbb{C}^2 \rightrightarrows \mathbb{C}^3) = 0$. Thus the term $\mathbb{C}^2 \rightrightarrows \mathbb{C}^4$ in the exact sequence above is actually $(0 \rightrightarrows \mathbb{C}) \oplus (\mathbb{C}^2 \rightrightarrows \mathbb{C}^3)$. In Section 7.4.2, we obtained the space

$$\left[\frac{\operatorname{Hom}(0 \rightrightarrows \mathbb{C}, \mathbb{C}^5 \rightrightarrows \mathbb{C}^6 / \mathbb{C}^2 \rightrightarrows \mathbb{C}^3)}{GL_1(0 \rightrightarrows \mathbb{C})}\right]$$

after fixing $(\mathbb{C}^2 \rightrightarrows \mathbb{C}^3) \to (\mathbb{C}^5 \rightrightarrows \mathbb{C}^6)$. Applying the integration map, we have

$$\chi\left(\left[\frac{\operatorname{Hom}(0 \rightrightarrows \mathbb{C}, \mathbb{C}^5 \rightrightarrows \mathbb{C}^6 / \mathbb{C}^2 \rightrightarrows \mathbb{C}^3)}{GL_1(0 \rightrightarrows \mathbb{C})}\right]\right) = \frac{q^h}{q-1},$$

where $h = \dim \operatorname{Hom}(0 \rightrightarrows \mathbb{C}, \mathbb{C}^5 \rightrightarrows \mathbb{C}^6 / \mathbb{C}^2 \rightrightarrows \mathbb{C}^3)$.

The second term $BGL_1(0 \rightrightarrows \mathbb{C}) \star \mathcal{G} A^{(7,-6)}$ gives

$$\begin{tikzcd} 0 \arrow[r] & (0 \rightrightarrows \mathbb{C}) \arrow[r] & (\mathbb{C}^2 \rightrightarrows \mathbb{C}^4) \arrow[r] & (\mathbb{C}^2 \rightrightarrows \mathbb{C}^3) \arrow[r] \arrow[d] & 0 \\ & & & (\mathbb{C}^5 \rightrightarrows \mathbb{C}^6) & \end{tikzcd} \tag{7.4.14}$$

which yields $\left[\frac{Ext^1(\mathbb{C}^2\rightrightarrows\mathbb{C}^3,0\rightrightarrows\mathbb{C})}{GL_1(0\rightrightarrows\mathbb{C})}\right]$ and

$$\chi(\left[\frac{Ext^1(\mathbb{C}^2\rightrightarrows\mathbb{C}^3,0\rightrightarrows\mathbb{C})}{BL_1(\mathbb{C}^5\rightrightarrows\mathbb{C}^6)}\right]) = \frac{q^e}{q-1},$$

where $e = \dim Ext^1(\mathbb{C}^2\rightrightarrows\mathbb{C}^3, 0\rightrightarrows\mathbb{C})$. Combining together, we have

$$\frac{q^h}{q-1} - \frac{q^e}{q-1} = q^e\frac{q^{h-e}-1}{q-1} = q^e\binom{h-e}{1}_q.$$

This is saying the moduli space S^2 has the same Poincaré polynomial as

$$\mathbb{A}^1 \times \left(\mathrm{Gr}\left(1, \mathrm{Hom}(0\rightrightarrows\mathbb{C}, \mathbb{C}^5\rightrightarrows\mathbb{C}^6/\mathbb{C}^2\rightrightarrows\mathbb{C}^3) - \mathrm{Ext}^1(\mathbb{C}^2\rightrightarrows\mathbb{C}^3, 0\rightrightarrows\mathbb{C})\right)\right).$$

Furthermore, note that our stability function is

$$Z(F) = -(-1,-2)\cdot F + i(2,1)\cdot F$$

By calculating, we have $Z(2,3) = 8+7i$ and $Z(0,1) = 2+i$. Thus $\phi(F) = \frac{1}{\pi}\arg Z(F)$ is decreasing. Hence the filtration

$$0 \subset \mathbb{C}^2\rightrightarrows\mathbb{C}^3 \subset \mathbb{C}^2\rightrightarrows\mathbb{C}^4$$

is Harder-Narasimhan.

Remark 7.4.2. At the end of this example, we would like to repeat the calculation about the terms in (7.4.12) so as to illustrate our future steps. The main difference between the two calculations is that we first fix the maps to the ambient space $\mathbb{C}^5\rightrightarrows\mathbb{C}^6$ in Section 7.4.2 then we went back to include the terms from first bending.

Now we would like to include the inclusion maps $\mathbb{C}^2\rightrightarrows\mathbb{C}^3$ in the Hall algebra multiplication. We again calculate the two terms in (7.4.12) separately. Following the same commutative relation, the first term is $\mathcal{G}\star 1^D_{ss}(1,\omega_2)A^{(7,-6)}$ and we have the same exact sequence (7.4.13)

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{C}^2\rightrightarrows\mathbb{C}^3) \arrow[r] \arrow[d] & (\mathbb{C}^2\rightrightarrows\mathbb{C}^4) \arrow[r] \arrow[dl, "\iota"] & (0\rightrightarrows\mathbb{C}) \arrow[r] & 0 \\
& (\mathbb{C}^5\rightrightarrows\mathbb{C}^6) & & &
\end{tikzcd}$$

We will now need to consider the map ι. Note that $\mathrm{Ext}(0\rightrightarrows\mathbb{C}, \mathbb{C}^2\rightrightarrows\mathbb{C}^3) = 0$. Thus the middle term $(\mathbb{C}^2\rightrightarrows\mathbb{C}^4)$ in the above exact sequence is $(0\rightrightarrows\mathbb{C}) \oplus (\mathbb{C}^2\rightrightarrows\mathbb{C}^3)$.

There are two case for ι: (1) injective; (2) non-injective. If ι is injective, then moduli space is $\mathrm{Gr}((0\rightrightarrows\mathbb{C})\oplus(\mathbb{C}^2\rightrightarrows\mathbb{C}^3),\mathbb{C}^5\rightrightarrows\mathbb{C}^6)$. Hence it is the Grassmannians which fibers over $\mathrm{Gr}(1,\mathrm{Hom}(\mathbb{C}^2\rightrightarrows\mathbb{C}^3,\mathbb{C}^5\rightrightarrows\mathbb{C}^6)$ with fiber

$$\mathrm{Gr}(1,\mathrm{Hom}(0\rightrightarrows\mathbb{C},(\mathbb{C}^5\rightrightarrows\mathbb{C}^6)/(\mathbb{C}^2\rightrightarrows\mathbb{C}^3)).$$

We are having 1 here as we take one copy of $0\rightrightarrows\mathbb{C}$ in this example.

If ι is not injective, let us write the space $\mathcal{S}$ as

$$[\text{maps } (\mathbb{C}^2\rightrightarrows\mathbb{C}^3)\oplus(0\rightrightarrows\mathbb{C})\to(\mathbb{C}^5\rightrightarrows\mathbb{C}^6) \text{ with kernel } 0\rightrightarrows\mathbb{C}].$$

The kernel of ι does not contain $\mathbb{C}^2\rightrightarrows\mathbb{C}^3$ since ι is an extension of the map $(\mathbb{C}^2\rightrightarrows\mathbb{C}^3)\to(\mathbb{C}^5\rightrightarrows\mathbb{C}^6)$. And by our calculation in the first bending, this map $(\mathbb{C}^2\rightrightarrows\mathbb{C}^3)\to(\mathbb{C}^5\rightrightarrows\mathbb{C}^6)$ would not contain any kernel proportional to $\mathbb{C}^2\rightrightarrows\mathbb{C}^3$.

Now we move to the second term $BGL_1(0\rightrightarrows\mathbb{C})\star\mathcal{G}A^{(7,-6)}$ which gives us the same exact sequence (7.4.14):

$$\begin{tikzcd}
0 \arrow[r] & (0\rightrightarrows\mathbb{C}) \arrow[r] & (\mathbb{C}^2\rightrightarrows\mathbb{C}^4) \arrow[r] \arrow[dr, "\nu"] & (\mathbb{C}^2\rightrightarrows\mathbb{C}^3) \arrow[r] \arrow[d] & 0 \\
 & & & (\mathbb{C}^5\rightrightarrows\mathbb{C}^6) &
\end{tikzcd}$$

Again as we do not fix the map $(\mathbb{C}^2\rightrightarrows\mathbb{C}^3)\to(\mathbb{C}^5\rightrightarrows\mathbb{C}^6)$, there is a composed map $\nu:(\mathbb{C}^2\rightrightarrows\mathbb{C}^4)\to(\mathbb{C}^5\rightrightarrows\mathbb{C}^6)$. We again have two cases: $(\mathbb{C}^2\rightrightarrows\mathbb{C}^4)$ is (a) split; (b) non-split.

If $\mathbb{C}^2\rightrightarrows\mathbb{C}^4$ split in the exact sequence (7.4.14), then the map ν represents the same space $\mathcal{S}$

$$[\text{maps } (\mathbb{C}^2\rightrightarrows\mathbb{C}^4)\to(\mathbb{C}^5\rightrightarrows\mathbb{C}^6) \text{ with kernel } 0\rightrightarrows\mathbb{C}].$$

as in the first term $\mathcal{G}\star 1^D_{ss}(1,\omega_2)A^{(7,-6)}$ in (7.4.12).

If $\mathbb{C}^2\rightrightarrows\mathbb{C}^4$ does not split, then the map ν is actually

$$(\mathbb{C}\rightrightarrows\mathbb{C}^2)^2\to(\mathbb{C}^5\rightrightarrows\mathbb{C}^6)$$

with image $\mathbb{C}^2\rightrightarrows\mathbb{C}^3$. Hence we are considering $[\mathrm{Gr}(1,\mathrm{Hom}(\mathbb{C}^2\rightrightarrows\mathbb{C}^3,\mathbb{C}^5\rightrightarrows\mathbb{C}^6)]$ with fiber $\mathrm{Gr}(1,\mathrm{Ext}^1(0\rightrightarrows\mathbb{C},\mathbb{C}^2\rightrightarrows\mathbb{C}^3))$.

Putting the two calculations together, we have the space $\mathcal{S}$ cancelled, and we still obtain the same space S^2 which has the same Poincaré polynomial as

$$[\mathrm{Gr}\left(1, \mathrm{Hom}(0 \rightrightarrows \mathbb{C}, (\mathbb{C}^5 \rightrightarrows \mathbb{C}^6)/(\mathbb{C}^2 \rightrightarrows \mathbb{C}^3)) - \mathrm{Ext}(0 \rightrightarrows \mathbb{C}, \mathbb{C}^2 \rightrightarrows \mathbb{C}^3)\right)].$$

fiber over $\mathrm{Gr}(1, \mathrm{Hom}(\mathbb{C}^2 \rightrightarrows \mathbb{C}^3, \mathbb{C}^5 \rightrightarrows \mathbb{C}^6))$.

In this calculation, we can see there is a 'secret' space $\mathcal{S}$ which got canceled out but it does appear in the Hall algebra calculation. This suggests that when we talked about Hall algebra broken lines as in Section 6.4, there are more Hall algebra broken lines then usual broken lines. We are working to understand the Hall algebra broken lines better.

Appendix A

Notations

In this thesis, we have adopted set up from numerous papers and so there is a confusion of notations. For clarity, let us clarify some confusing notations in this section.

First of all, our cluster variables are denoted as $A_1, \ldots, A_n$ instead of what Fomin-Zelevinsky did in [FZ02]. And we denote $\prod A_i^{m_i}$ as A^m instead of z^m in most mirror symmetry papers.

In most of the papers about scattering diagrams, e.g. [GHKK14], the path-ordered product (in 4.1.2) is denoted by θ. However, as too many different forms of θ is being used. We switch it to $\mathfrak{p}$.

Similay, in most papers, when broken line is mentioned, the endpoints are denoted as Q. While in [CGM$^+$15], we denote q as endpoints. To avoid confusion quiver Q, and q in Hall algebra, we set $\mathcal{Q}$ as the endpoints of the broken lines to express the lack of symbols.

As noted above, θ has been heavily used, we denote the general stability condition as ω instead of θ in [Bri15]. Continuing about the difference from [Bri15], our broken lines go from negative chamber to positive chamber. This is the opposite direction in the set up of [Bri15]. For theta function with initial slope m, we use ϑ_m instead of ϑ^m. And as $\mathcal{A}$ commonly stands for cluster algebra, we take out the use of $\mathcal{A} = \mathrm{rep}(Q)$. In this article, we will simply use $\mathrm{rep}(Q)$ as the abelian category of finite dimensional representations of a quiver Q without further shorthanded.

Our quiver is $1 \rightrightarrows 2$ instead of $1 \leftleftarrows 2$.

A.1 Skew-symmetric form

In this article, $\chi(\cdot,\cdot)$ denoted the Euler form defined in 3.2.1. We further define $\mathcal{E}(d) = \chi(\cdot, d)$ the functional. And it is defined in 3.2.2. In some papers, the Euler form is written as $\langle\cdot,\cdot\rangle$. We switch it here because we do not want to confuse the form with the usual inner product. $\{\cdot,\cdot\}$ here stands for the skew-symmetric form used to define scattering diagram. We define the matrix ϵ with entries $\epsilon_{ij} = \{e_i, e_j\}$. First note that the matrix ϵ is the transpose to the exchange matrix B in the definition of cluster algebras in Section 3.1.

To link up with the quiver theory, we can associate an arrow by setting $\epsilon_{i,j}$ to be the number of arrows from i to j (negative means opposite direction). Note that

$$\{e, f\} = \sum_{i,j} \epsilon_{i,j} e_i f_j = \sum_{i\to j} e_i f_j - \sum_{j\to i} e_i f_j$$

While in some other paper, e.g. [Kel10], there is a notion of skew-symmetric form $\langle\cdot,\cdot\rangle_a = \chi(e, f) - \chi(f, e)$. Note it is denoted as $\{,\}$ in some other papers, e.g. [Rei10],

$$\langle e, f\rangle_a = \chi(e, f) - \chi(f, e) = -\sum_{i\to j} e_i f_j + \sum_{i\to j} f_i e_j$$

Therefore we have

$$\{e, f\} = -\langle e, f\rangle_a = -(\chi(e, f) - \chi(f, e)) \tag{A.1.1}$$

If you consider it is not confusing enough, note that $\langle e, f\rangle_a$ is $\langle e, f\rangle$ in [Bri15].

At the same time, $\chi(X)$ denotes the Euler characteristic for the space X. A similar map on the Hall algebra defined in 6.3.2 is denoted as χ. It should be clear from the content to distinguish these two.

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