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Baroclinic Critical Layers and Zombie Vortex Instability in Stratified Rotational Shear Flow

by

Meng Wang

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

 in

Engineering - Mechanical Engineering

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Philip S. Marcus, Chair Professor Ömer Savas Associate Professor Per-Olof Persson

Fall 2016

Baroclinic Critical Layers and Zombie Vortex Instability in Stratified Rotational Shear Flow

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Abstract

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Doctor of Philosophy in Engineering - Mechanical Engineering University of California, Berkeley

Professor Philip S. Marcus, Chair

Without instabilities, the gas in the protoplanetary disk around a forming protostar remains in orbit rather than falling onto the protostar and completing its formation into a star. Moreover without instabilities in the fluid flow of the gas, the dust grains within the disk's gas cannot accumulate, agglomerate, and form planets. Keplerian disks are linearly stable by Rayleighs theorem because the angular momentum of the disk increase s with increasing radius. This has led to the belief that there exists a large region in protoplanetary disks, known as the dead zone, which is stable to pure hydrodynamic disturbances. The dead zone is also believed to be stable against magneto-rotational instability (MRI) because the disks' cool temperatures inhibit ionization and therefore prevent the MRI. A recent study [15] shows the existence of a new hydrodynamic instability called the Zombie Vortex Instability (ZVI), where successive generations of self replicating vortices (zombie vortices) may fill the disk with turbulence and destabilize it. The instability is triggered by finite amplitude perturbations, including weak Kolmogorov noise, in stratified (with Brunt-Väisälä frequency N) flows, rotating with angular velocity Ω and shear σ . So far there are no observational evidences of the Zombie Vortex Instability and there are very few laboratory experiments of stratified plane Couette flow with background rotation in the literature. We perform systematic simulations exploring existence of Zombie Vortex Instability in terms of control parameters (Reynolds number $Re, \sigma/f$ and N/f). We present a parameter map showing two regimes where ZVI occurs, and interpret the physics that determines the boundaries of the two regimes. We also discuss the effects of viscosity and the existence of a threshold for *Re.* Our study on viscous effect, parameter map and its underlying physics provide guidance for designing practical laboratory experiments in which ZVI could be observed.

To my family

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Chapter 1 Introduction

Protoplanetary disk (PPD) is an astrophysical model, where there is a young star in the center and the dust and grains are rotating around the young star. Ninety percent of the mass in the PPD are concentrated in the young star. Under law of gravity, the dust and grains are rotating around the center star in orbits. The high pressure above and below the disk result into a pancake like structure of the disk, meaning most of the majority of the dust and grains are concentrated above and below a few pressure height of the middle plane. The accretion disk is a large scale astronomical system whose diameter is around 10 AU. AU represents the astronomical unit which equals the distance from the earth to the sun. The disk diameter is around 10^{12} meters.



Figure 1.1: Observation of accretion disk of HL Tauri

1.1 From the accretion disk

PPD is the model constructed in fifty years ago by astrophysicists to model the star formation process. It has been recently observed by an image of HL Tauri, a young star which is around 450 light years away from the earth. The central problem in the accretion disk is how they accrete and grow. In another words, what makes the dust and grains fall and collapse into the young star such that the star can aggregate and mature. It has been verified by the astrophysics community that it must be the turbulence existing the disk that drives the collapsing of the dusts. The reason is because, as we know, for the young star to grow, the dust and grains must fall into the star, which brings in a momentum from the boundary to the center of the disk. If we consider the accretion disk as an isolated system, by isolation I mean, the external forces from other star or galaxy is approximately zero due the distance between them is so large. From the conversation laws, we know that for isolated system, the mass, momentum and energy much be balanced. Without external forces, there must be some physics existing inside the disk that transfers the momentum from the center of the disk to the boundary, in order to balance the inward momentum flux by the falling of the dusts. After careful examination of the physics in the disk, people claim it is only the turbulence generated by the viscosity that is efficient enough to transport such amount of the momentum from the center to the boundary [3]. Thus this leads to the next question, what generates turbulence in the accretion disk?

Since then, people has been looking for the turbulence generating mechanism in the accretion disk. There are several theories have been come up based on different assumptions. One of them, which considers the magnetic filed of the accretion disk, called Magnetorotational Instability(MRI) has been quite popular recently. It arises when the angular velocity of a conducting fluid in a magnetic field decreases as the distance from the rotation center increases. It is also known as the Velikhov-Chandrasekhar instability or Balbus-Hawley instability in the literature [2]. However, MRI requires the existence of the magnetic field. The magnetic fields only exists in the area close to the young star. For the area where the distance from the young star is larger than 1 AU, the disk is so cool that the temperature of the disk is close to the absolute zero. The area is called the "dead zone". In dead zone, the temperature is too low for the particles to be ionized. Thus the magnetic effect in the dead zone is very small. MRI may not be a good candidate in this area. The existence of dead zone and the absence of the magnetic field in dead zone lead researchers to look for other kinds of turbulence generating mechanism which are purely driven by physics of the particles and plasma instead of magnetic effect, in another words, a purely hydrodynamic instability mechanism.

Previously, astrophysicists believe the pure hydrodynamic instability can not exist in the accretion disk, because they believe the accretion disk is hydrodynamically stable. Their argument is briefly phased as following. First, the dust and grains are rotating at the young star at certain orbits r, where is r is the distance from the center of the disk. The angular velocity of the dust and grains rotating around the star $\Omega(r)$ can be derived easily by the balance of the gravity and centrifugal forces. The result gives us the form of angular velocity

$$\Omega(r) \sim r^{-3/2} \tag{1.1}$$

In the area of the hydrodynamic instability, there is a famous law called the Rayleigh's Stability Criterion saying that, for the invsicid and non-stratified fluid, a necessary and sufficient condition for stability to axisymmetric disturbances is that the square of the circulation does not decrease anywhere ([8] [22]). If we check the accretion disk, we could easily see that the change of the square of the circulation on the radial direction increases

$$\frac{d(r^2\Omega)^2}{dr} \sim \frac{dr}{dr} \sim 1 \tag{1.2}$$

Thus by Rayleigh's criteria, the accretion disk is hydrodynamically stable! Unfortunately this argument has been proven to be wrong. The reason is because Rayleigh's criteria only works for the inviscid nonstratified fluid system, but the accretion disk is not such kind of system. Although small, viscosity does exit in the accretion disk. More importantly, the accretion disk is a stratified system, meaning the density of the disk is not a constant. In fact, the density of the disk varies in the vertical direction. Remember when we first introduce accretion disk, we mention that the disk has a pancake shape meaning the mess are concentrated around the mid-plane, thus brings in the stratification.

1.2 Instabilities in accretion disk

For the stratified fluid system, a new linear instability, called the StratoRotational Instability (SRI) [26] has been found analytically beyond the Rayleigh' criteria. The instability are found under the assumptions that wave-numbers are very small, the gap between the two rotating cylinders are small compared with the radius of the cylinder, and the assumption that the stratification in the system is very strong. Under those assumptions, it is reported analytically and numerically that there exists a linear growing mode in the area where Rayleigh's criteria predicts to be stable. They shows that the new boundary for linear stability or the modified Rayleigh's criteria for stratified fluid shall be as following

$$\frac{d(r\Omega)^2}{dr} \ge 0 \tag{1.3}$$

This work is further extended by [21] and [19]. The main extensions come from the remove of the constrains on SRI when firstly discovered. This include the small gap approximation and small wave numbers. Later on, people considered the viscous effect on SRI. SRI has been shown to be robust in those cases. Beyond the analytic work, SRI has been observed experimentally by [9]. The main discovery of the experimental work is that they verified the stability boundary predicted by SRI in equation 1.3. The observation of SRI is claimed to be the appearance of two traveling waves on the vertical direction. Those two traveling waves are traveling in opposite directions. From the experimental work, it seems the observations agrees very well with other on the stability boundary. Furthermore, they also verified that



(a)

Figure 1.2: SRI experiment by [9];Left:Experiment set up showin the apperance of SRI modes; Right:Comparison of experiments and analytic result for SRI stability regime; Red cross means unstable blue circle shows stable; Dashed line are the SRI stability boundary

SRI can only be observed when the wave number are small, saying the small wave number approximation has to be valid for the experimental observations. Numerical work for SRI with initial value simulation can be found in [10].

Beyond SRI, there are some other kind of linear instabilities have also been reported to be relevant to accretion disk. During the exploration of stratified plain Couette flow with background rotation, the gravity-wave-like instability has also been reported [25]. This kind of instability is claimed to be formed from the linear resonances between waves with oppositely singed wave momenta, for example, Kelvin wave and inertia-gravity wave existing in the system. The relationship between SRI and the gravity-wave-like instability is very clear yet. However, both of them claim the instability is excited due to the resonances of the linear traveling waves trapped by the boundary. Another instability called the Radiative Instability [11] has been reported. In radiative instability, the linear normal mode of column vortex in inviscid stratified flow becomes unstable. This is believed to be the emission of the internal waves from the vortex.

Linear instabilities discovered above seem like great candidates to excite turbulence in the accretion disks. They are linear instability, which shall exist with all sorts of initial conditions. They purely depend on the stratification and rotation of the system, thus they shall survive in the dead zone where there is no magnetic field. However, there is one problem that makes linear instabilities less appealing as they should. All those linear instabilities claimed to be excited by the coupling of the waves, requires the existence of solid boundaries, since it is the trapping of the waves on the boundary that leads to coupling of the waves that excite the instability. None of these instabilities have been reported to have the capacity to exist without any solid boundaries. However, in the accretion disk where the domain is so large, there is no well-posed boundaries existing in the accretion disk such that so waves can be easily trapped. An infinite boundary conditions sound more reliable to model the accretion disk, instead of one or two solid walls.

Despite the existence of the several instability in stratified rotational shear flow, a new finite-amplitude instability, called the self-replicating "Zombie" Vortex Instability(ZVI) has also been discovered [15]. ZVI is believed to exist in the system that contains horizontal shear, background rotation and vertical stratification. ZVI comes from the formation of a thin layer structure, called the critical layers. Critical layers are mathematical singularities in the fluid system. Those critical layers observed in ZVI have the capacity to spawn vortex on top of themselves. Those newly formed vortex created on the critical layers will excite their own critical layers, which we call it next generation of critical layers, in order to distinguish them from the original critical layers that spawn the vortex. The next generation critical layers will behave exactly like the original critical layers. They will spawn vortex on top of themselves. Those new created vortex will excite more critical layers and so on, until the whole field is occupied by those vortex and critical layers. Due to the self-replication property of those vortex, we call them "Zombie " vortex and this self-replication vortex instability, Zombie Vortex Instability.

There are several points which makes ZVI a great candidate for the turbulence generating mechanism in accretion disk. First, like the linear instabilities, it is excited purely by the hydrodynamic ingredients, which means they can be excited in the dead zone where there is no magnetic filed. Secondly, unlike other instabilities, ZVI does not require the existence of the physical boundaries. The infinite boundary condition works perfectly for ZVI. Third, ZVI seems more robust than other instabilities. The amplitude of ZVI is observed to be bigger than the amplitude of other linear instabilities.

Despite those nice properties of ZVI, what we know about such instability is very limited. There are lots of points remains unknown to us. For example, how does ZVI excite? Under what conditions ZVI will be excited and under what condition they will not? What are critical layers and what is the relationship between critical layers and the instability? Most importantly, can we observe ZVI in the laboratory experiments?

To answer those equestions above, a thorough exploration on both the analytic part of ZVI and numerical work of ZVI have been done. This is what I have been doing for the past a few years, to have a clear understanding of such instability, from how it is excited, under what conditions it will be excited, to how to analytically explain the relationship of critical layers and the instability, and finally how we could build up experiments to observe ZVI in the laboratory experiments.

1.3 Structure of the thesis

After years of work on this problem, those questions we have above have mostly answered and our experiments for ZVI is currently being conducted with our collaborators in Marseille France. In this thesis work, I will report and document all the finds I have on ZVI, from the theory analysis to numerical simulations, from mathematical deductions to physical explanations. The thesis structured as following,

Chapter 2 talks about the mathematical modeling of accretion disk. I will talk about how we build up the governing equations from the physics of the accretion disk. Then the linear normal mode analysis will be implemented to the system to have a understanding of the linear instability of the system.

Chapter 3 will focus on the inviscid regime. The key questions we want to answer in this chapter is, without considering of the viscous effect, under what condition, ZVI will be triggered. When they are triggered, what does it look like? A parameter map will be provided on this chapter, in terms of the physical ingredients in the system, denoting where to look for such instability.

Chapter 4 will focus on the numerical exploration on how viscous affects the instability, where a critical Reynolds number will be provided to guide the laboratory experiments.

Chapter 5 will focus on the numerical algorithm designed specifically for the stratified rotational shear flow, especially for the viscous case. A second order accurate algorithm with semi-analytic method is designed for such a problem. This work is a collaboration with my colleague Nelson Chen.

Chapter 6, I will summarize the work we have done on ZVI and provide my insights on the instability. Existing problems and what to continue on such instability will be discussed. A discussion on the instability in the general fluid system that contains horizontal shear, background rotation and vertical stratification will also be provided.

The analytic work on the critical layers are collaborated with Professor Patrick Huerre. The numerical work on the inviscid map and viscous effect are collaborated with my colleagues, Dr. Suyang Pei, Dr. Chung-Hsiang Jiang and Dr. Giulio Facchini. The numerical simulations on the titled vortex is collaborated with Dr. Pedram Hassanzadeh. The design and test of the numerical algorithm on the semi-analytic method is collaborated with Mr. Nelson Chen. Professor Philip Marcus provides insights and ideas in all of above.

Chapter 2

Mathematical Modeling and Normal Mode Analysis

In order to have a systematic investigation of the turbulence generating mechanism in accretion disk, we need to build up the mathematical models. If we focus on the large scale physics in the disk, i.e. not consider the quantum models, the governing equations shall come from the conservation laws, such as mass conservation, momentum conversation(Newton's second law) and the energy conservation.

2.1 Mathematical modeling of the system

Before we consider the mathematical equations, let us think about the physical ingredients existing in the accretion disk. There are four important physical ingredients, the horizontal shear, the background rotation, the stratification and the viscosity. Shear force or shear stress is the force whose direction is parallel to the surface. For example, the wind blowing on the surface of the lake can be regarded as a shear force, since the direction of the wind or wind force is parallel to the surface of the lake. Another famous example of the natural phenomena governed by the shear is the Great Red Spot(GRS) on Jupiter. GRS is a large scale rotation cloud on the Jupiter's atmosphere. The diameter of GRS is around three times the diameter of the earth. GRS has been existing for more than two hundred years, since human's first observation in 19th century. On the atmosphere of Jupiter, there exists several belts of clouds moving at different velocity. Those moving belts of the clouds are called the zonal winds are perfect example of shear. Since the force acting on the interfaces of the zonal winds are parallel to the interface. GRS is located on the interface of the zonal winds and has been believed to be strongly affected by the shear of the zonal winds.

Since the dust and grains are rotating around the young star, if we put ourselves in the rotating frame, there will be the centrifugal force and the Coriolis force in the rotating frame. The centrifugal force balances with the gravity between the dust and the young star, which results into the Kepler's law. We also need to consider the Coriolis force. The Coriolis force



Figure 2.1: Coordinates transformation from the (r, θ, z) to a local Cartesian box (x, y, z)

is due to the self-rotation of the system. For example, the earth is rotating with angular velocity one over 24 hours. The famous natural phenomena governed by the Coriolis force is the hurricanes. Hurricanes are large scale rotating atmospheric movement. It is totally governed by the pressure gradient from the center to the boundary of the hurricane and the Coriolis effects. The governing equations formed by these two terms are the famous thermal wind equations, which are usually used to estimate the radius and pressure at the center of the hurricanes.

As we have discussed before, the density in the vertical direction of the disk is not constant. We have to take into account the vertical stratification of the system. One example of stratification is the ocean water. As we know, due to the change of the temperature and the salinity, the density of the ocean is not constant. The deeper we go, the heavier it is. The stratification of the system brings in the buoyancy force in the system. The last physical ingredients we need to consider is the viscosity. Viscosity can be regarded as the friction of the particles in the system. The bigger the viscosity is, the stronger the interaction between the fluid particles will be, by which we call them more viscous.

For the system like accretion disk, the polar coordinates (r, θ, z) will come in handy, where r is the radius where the particles locates at, θ is the angle on the horizontal plane and z is the height of the particle from the mid-plane of the disk. However, since the length scale of the disk is too large to be directly modeled, typically people will build the mathematical models in a small Cartesian box at certain location (R_0, θ_0, z_0) . Inside the Cartesian box, the coordinates are (x, y, z). The x is the direction where the particles are moving at. We call this streamwise direction. y is radius direction and we call it cross-stream direction. z is the vertical direction. With the assumption that the size of the Cartesian box is very small compared with the radius R_0 , z_0 , i.e. $L_x \ll R_0$, $L_y \ll R_0$ and $L_z \ll z_0$, we could linearize the physical ingredients in the system.

For mass conservation, we need to think about whether the compressiblibility of the system. Since the Mach number is accretion disk is small $Ma \ 0.1$, we consider our system to be incompressible. For incompressible system, the mass conservation comes in form as the volume change of the particles are zero.

The second group of equations shall come from the momentum equations, which is the

Newton's second law. Newton's second law tells us the change of the momentum in the system is due to the external forces acting on the system. In order to construct the momentum equations, we need to consider what kinds of forces there are in the accretion disk. Typically, there are two kinds of forces, body force and surface force. As we have discussed, the forces existing in the system are, gravity from the center of the star, the pressure, buoyancy due to stratification, centrifugal force and Coriolis force by rotation, shear force and the viscous force. Among those, the centrifugal force can be rewritten into gradient of a scalar, thus it can be combined together with pressure and gravity into a total pressure term.

The last one comes from the conservation of energy. In our case, if we do not consider the energy source or sink. We could easily rewrite the energy equations in terms of the density terms.

As we mentioned, we linearzie all the physical ingredients in our local Cartesian box, such that the shear velocity is linear, the background rotation is rotating at a constant angular velocity Ω and the background stratification is linear in vertical direction. We could rewrite our total velocity \underline{u}_{total} as two parts, the background shear velocity and perturbation, i.e. $\underline{u}_{total} = \underline{U} + \underline{u}(x, y, z, t)$. The background shear velocity $\underline{U}(y) = -\sigma y \hat{x}$ with $-\sigma$ as the shear rate. We also separate our density into two parts, the background linear stratification and the density fluctuations, i.e. $\rho(x, y, z, t) = \overline{\rho}(z) + \tilde{\rho}(x, y, z, t)$, with $\overline{\rho}(z) = \rho_0(1 - \frac{N^2}{\rho_{0g}}z)$ For our stratification, we use the Boussinesq approximation that $\tilde{\rho}/\rho_0 \ll 1$, where g is the gravity and N is called the Brunt Väisälä frequency $N^2 = -\frac{g}{\rho_0} \frac{d\overline{\rho}(z)}{dz}$. Since the angular velocity of the system rotation is Ω , we use the Coriolis term $f = 2\Omega$ to represent the Coriolis effects. The kinematic viscosity of the fluid is ν and thermal diffusivity is κ , we could write our equations as

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
(2.1)

$$\frac{\partial u}{\partial t} = -(\underline{u} \cdot \nabla) u - U(y) \frac{\partial u}{\partial x} - \frac{\partial P}{\partial x} + (f + \sigma) v + \nu \nabla^2 u$$
(2.2)

$$\frac{\partial v}{\partial t} = -(\underline{u} \cdot \nabla) v - U(y) \frac{\partial v}{\partial x} - \frac{\partial P}{\partial y} - fu + \nu \nabla^2 v$$
(2.3)

$$\frac{\partial w}{\partial t} = -(\underline{u} \cdot \nabla) w - U(y) \frac{\partial w}{\partial x} - \frac{\partial P}{\partial z} - \frac{\tilde{\rho}}{\rho_0} g + \nu \nabla^2 w$$
(2.4)

$$\frac{\partial \tilde{\rho}}{\partial t} = -(\underline{u} \cdot \nabla) \,\tilde{\rho} - U(y) \frac{\partial \tilde{\rho}}{\partial x} + \frac{\rho_0 N^2}{g} w + \kappa \nabla^2 \tilde{\rho}$$
(2.5)

The mathematical models include five partial differential equations with five variables, velocity on three direction u, v, w, density $\tilde{\rho}$ and total pressure P. A close form solution seems quite difficult. We have to implement some simplifications if we would like to move on analytically. Then the linear normal mode analysis comes in handy.

2.2 Linear normal mode analysis

Linear normal mode analysis assumes that the physical quantities are linear in temporal space, linear in some but not all of the spacial space. For linear relations, we know that it can expressed as the exponential form with a normal mode. Thus if we would like to focus on the physics of the cross-stream direction y and assume that the physical quantities are linear in the temporal and spatial space. The physical quantities shall have the formula $q(t, x, y, z) = \hat{q}(y)e^{-i\omega t}e^{ik_x x + ik_z z}$, where k_x and k_z are the wave numbers for streamwise direction x and vertical direction z correspondingly. If we rewrite the temporal frequency $\omega = \omega_r + i\omega_i$. The imaginary part ω_i determines the linear stability state of the system. If ω_i is positive(negative), the system is linearly unstable(stable). For the cases where $\omega_i = 0$, the system is defined to be neutrally stable.

After making the linear normal mode assumptions, the partial differential equations all become ordinary differential equations, we could actually further simplify the group of equations to one ordinary differential equations. One example of this is in the two-dimensional plan Couette flow or pure shear flow, we could get the famous Rayleigh's equation for the inviscid case and the Orr-Sommerfeld equation for the viscous case. We would like to implement the exactly same technique and get the Rayleigh like equation for the inviscid case and the Orr-Sommerfeld-like equation for the viscous case.

Rayleigh like equations

First, let us further simplify the governing equations by assuming the flow field is inviscid and non-diffusive, i.e. $\nu = 0$ and $\kappa = 0$. Then the governing equations (2.1) to (2.5) are simplified as following,

$$ik_x\hat{u} + \frac{d\hat{v}}{dy} + ik_z\hat{w} = 0\tag{1}$$

$$i\Omega\hat{u} = -ik_x\hat{P} + (f+\sigma)\hat{v} \tag{2}$$

$$i\Omega\hat{v} = -f\hat{u} - \frac{d\hat{P}}{dy} \tag{3}$$

$$i\Omega\hat{w} = -\frac{\hat{\rho}}{\rho_0}g - ik_z\hat{P} \tag{4}$$

$$i\Omega\hat{\rho} = \frac{\rho_0 N^2}{g}\hat{w} \tag{5}$$

where $c = \omega/k_x$, Ω is a function of y and defined as $\Omega(y) = k_x(U-c)$, and $\frac{d\Omega}{dy} = -\sigma k_x$. N is the Brunt-Vaisala frequency which is a real constant number. For our convenience, the hat sign will be dropped from now on.

First Equation, relation between w and P

We get the first equation by eliminating ρ in equation (4) and (5). First, rewrite equation (5) as following,

$$\rho = \frac{\rho_0 N^2}{ig\Omega} w \tag{5}$$

and insert equation (5) into equation (4)

$$i\Omega\rho = -\frac{g}{\rho_0} \cdot \frac{\rho_0 N^2}{ig\Omega} w - ik_z P = -\frac{N^2}{i\Omega} w - ik_z P$$

rewrite the equation above as,

$$P = \frac{N^2 - \Omega^2}{\Omega k_z} w \tag{6}$$

Second Equation, relation between v and P

We get the second the equation by eliminating u in equation (2) and (3). We multiply equation (2) with f on each side, we have

$$i\Omega f u = -ik_x f P + f(f + \sigma)v \tag{7}$$

Multiply equation(3) by $i\Omega$ on each side, we have

$$i\Omega f u = \Omega^2 v - i\Omega \frac{dP}{dy} \tag{8}$$

Equation (7) minus equation (8), we have the second equation we want,

$$i\Omega \frac{dP}{dy} - ik_x fP = [\Omega^2 - f(f+\sigma)]v$$
(9)

Eliminate P, from the first and second equation

Before we plug equation (6) into equation (9), we need to compute the term $\frac{dP}{dy}$, let compute it first. We start by taking the derivative of y on equation (6)

$$\frac{dP}{dy} = \frac{N^2 - \Omega^2}{\Omega k_z} \frac{dw}{dy} + \left(\frac{d}{dy} \frac{N^2 - \Omega^2}{\Omega k_z}\right) w = \frac{N^2 - \Omega^2}{\Omega k_z} \frac{dw}{dy} + \frac{1}{k_z} \cdot \frac{-2\Omega \Omega' \Omega - \Omega' (N^2 - \Omega^2)}{\Omega^2} w$$

With $\Omega' = -\sigma k_x$, we have

$$\frac{dP}{dy} = \frac{N^2 - \Omega^2}{\Omega k_z} \frac{dw}{dy} + \frac{\sigma k_x}{k_z} \cdot \frac{N^2 + \Omega^2}{\Omega^2} w \tag{10}$$

Then plug equation(6) and equation(10) into equation(9), we will have

$$i\Omega(\frac{N^2 - \Omega^2}{\Omega k_z}\frac{dw}{dy} + \frac{\sigma k_x}{k_z} \cdot \frac{N^2 + \Omega^2}{\Omega^2}w) - ik_x f(\frac{N^2 - \Omega^2}{\Omega k_z}w) = [\Omega^2 - f(f + \sigma)]v$$

Simplify it, we get as following,

$$i\frac{N^{2}-\Omega^{2}}{k_{z}}\frac{dw}{dy} + i\frac{k_{x}[N^{2}(\sigma-f)+\Omega^{2}(\sigma+f)]}{\Omega k_{z}}w = [\Omega^{2}-f(f+\sigma)]v$$
(11)

We want to use equation (11) to eliminate the term $\frac{dv}{dy}$ in the continuity equation, so we have to compute the $\frac{dv}{dy}$ in terms of w. This is probably the most complicated term in this whole process. Let us start by taking the derivative of y on both sizes of equation (11)

$$i\frac{N^2 - \Omega^2}{k_z}\frac{d^2w}{dy^2} + i\left(\frac{d}{dy}\frac{N^2 - \Omega^2}{k_z}\right)\frac{dw}{dy} + i\frac{k_x[N^2(\sigma - f) + \Omega^2(\sigma + f)]}{\Omega k_z}\frac{dw}{dy} + \frac{ik_x}{k_z}\left(\frac{d}{dy}\frac{N^2(\sigma - f)}{\Omega} + \frac{d}{dy}(\sigma + f)\Omega\right)w = [\Omega^2 - f(f + \sigma)]\frac{v}{dy} + 2\Omega\Omega'v$$
(12)

Remember that $\Omega' = -\sigma k_x$, plug it in equation(12), we have

$$i\frac{N^{2} - \Omega^{2}}{k_{z}}\frac{d^{2}w}{dy^{2}} + i\frac{2\sigma k_{x}\Omega}{k_{z}}\frac{dw}{dy} + i\frac{k_{x}[N^{2}(\sigma - f) + \Omega^{2}(\sigma + f)]}{\Omega k_{z}}\frac{dw}{dy} + i\frac{\sigma k_{x}^{2}}{k_{z}}[(\sigma - f)\frac{N^{2}}{\Omega^{2}} - (\sigma + f)]w = [\Omega^{2} - f(f + \sigma)]\frac{dv}{dy} - 2\Omega\sigma k_{x}v$$
(12)

One more step of simplification, we have

$$i\frac{N^{2}-\Omega^{2}}{k_{z}}\frac{d^{2}w}{dy^{2}} + i\frac{k_{x}[N^{2}(\sigma-f)+\Omega^{2}(3\sigma+f)]}{\Omega k_{z}}\frac{dw}{dy} + i\frac{\sigma k_{x}^{2}}{k_{z}}[(\sigma-f)\frac{N^{2}}{\Omega^{2}} - (\sigma+f)]w = [\Omega^{2}-f(f+\sigma)]\frac{dv}{dy} - 2\Omega\sigma k_{x}v$$
(12)

Get the final equation

Our goal is to write all the terms in the continuity equation to w, we have the term $\frac{dv}{dy}$, we only have to worry about the term $ik_x u$. We will eliminate this term with equation (2). We multiply equation (1) by Ω and will get

$$i\Omega k_x u = -\Omega \frac{dv}{dy} - ik_z \Omega w \tag{13}$$

Next, we multiply k_x on both side of equation (2)

$$i\Omega k_x u = -ik_x^2 P + k_x(\sigma + f)v \tag{14}$$

From equation (13) and (14), we will have

$$ik_x^2 P = k_x(\sigma + f)v + \Omega \frac{dv}{dy} + ik_z \Omega w$$
(15)

Next step is to eliminate $\frac{dv}{dy}$ term from equation (12) and (15). We multiply equation (12) with Ω ,

$$i\Omega \frac{N^2 - \Omega^2}{k_z} \frac{d^2 w}{dy^2} + i \frac{k_x [N^2(\sigma - f) + \Omega^2(3\sigma + f)]}{k_z} \frac{dw}{dy} + i\Omega \frac{\sigma k_x^2}{k_z} [(\sigma - f) \frac{N^2}{\Omega^2} - (\sigma + f)] w = \Omega [\Omega^2 - f(f + \sigma)] \frac{dv}{dy} - 2\Omega^2 \sigma k_x v$$
(16)

Then we multiply equation (15) with $[\Omega^2 - f(f + \sigma)]$

$$ik_x^2[\Omega^2 - f(f+\sigma)]P = k_x(\sigma+f)[\Omega^2 - f(f+\sigma)]v + \Omega[\Omega^2 - f(f+\sigma)]\frac{dv}{dy} + ik_z\Omega[\Omega^2 - f(f+\sigma)]w$$
(17)

Eliminate $\frac{dv}{dy}$ from equation (16) and (17)

$$i\Omega \frac{N^2 - \Omega^2}{k_z} \frac{d^2 w}{dy^2} + i \frac{k_x [N^2(\sigma - f) + \Omega^2(3\sigma + f)]}{k_z} \frac{dw}{dy} + i\Omega \frac{\sigma k_x^2}{k_z} [(\sigma - f) \frac{N^2}{\Omega^2} - (\sigma + f)] w$$
$$-ik_x^2 [\Omega^2 - f(f + \sigma)] P = -k_x [(3\sigma + f)\Omega^2 - f(f + \sigma)^2] v - ik_z \Omega [\Omega^2 - f(f + \sigma)] w \quad (18)$$

Plug in equation (6) and (11), we have the final equation

$$\Omega^{2}(N^{2}-\Omega^{2})\frac{d^{2}w}{dy^{2}} + 4\Omega\sigma k_{x}N^{2}\frac{dw}{dy} + [N^{2}k_{x}^{2}(\sigma^{2}+f^{2}-\Omega^{2}) + \Omega^{2}k_{x}^{2}(\Omega^{2}-(\sigma+f)^{2}) + \Omega^{2}k_{z}^{2}(\Omega^{2}-f(f+\sigma))]w = 0$$
(19)

Equation for P

The goal is to replace every term in continuity equation to pressure P. We have the relation between w and P from equation (6). The rest two terms are $\frac{dv}{dy}$ and u

First, we get $\frac{dv}{dy}$ from equation (9). Take the derivative on both side of equation (9), we will have

$$[\Omega^2 - f(\sigma + f)]\frac{dv}{dy} = 2\sigma k_x \Omega v + i\Omega \frac{d^2 P}{dy^2} - ik_x (\sigma + f)\frac{dP}{dy}$$
(20)

Then replace v with equation (9), we will have

$$[\Omega^2 - f(\sigma + f)]^2 \frac{dv}{dy} = i\Omega[\Omega^2 - f(\sigma + f)] \frac{d^2P}{dy^2} + ik_x [\Omega^2(\sigma - f) + f(\sigma + f)^2] \frac{dP}{dy} - 2i\sigma f k_x^2 \Omega P \quad (21)$$

Secondly, we could get the relation between u and P by eliminating v in equation (2) and (3)

$$[\Omega^2 - f(\sigma + f)]u = (\sigma + f)\frac{dP}{dy} - k_x\Omega P$$
(22)

Plug equation (6), (21) and (22) into equation (1), we will have the **Rayleigh-like equation**

$$\Omega(N^2 - \Omega^2)[\Omega^2 - f(\sigma + f)]\frac{d^2P}{dy^2} + 2(N^2 - \Omega^2)\sigma k_x \Omega \frac{dP}{dy} + \Omega^4(k_x^2 + k_z^2) - \Omega^2[2k_z^2f(\sigma + f) + k_x^2N^2 + k_x^2f(f - \sigma)] + k_x^2N^2f(f - \sigma)P = 0$$
(2.6)

2.3 Singularities for Rayleigh-like equation

When the coefficients of the highest derivative terms goes to zero, it will generates singularities in the system. As we can see clearly in Rayleigh-like equation 2.6, there are three coefficients in front of the highest derivative terms. We could see clearly that these three points are all regular singular points. Thus, if any of them goes to zero, we will have a singular points.

$$\begin{split} \Omega &= 0 \Rightarrow Barotropic \ critical \ layer, & y^* = \frac{\omega}{\sigma k_x} \\ \Omega^2 - N^2 &= 0 \Rightarrow Baroclinic \ critical \ layer, & y^* = \frac{\omega \pm N}{\sigma k_x} \\ \Omega^2 - f(\sigma + f) &= 0 \Rightarrow Inertial \ critical \ layer, & y^* = \frac{\omega \pm \sqrt{f(f + \sigma)}}{\sigma k_x} \end{split}$$

The first singularity *barotropic* critical layer only depends on the background shear. It is exactly the same as the critical layer reported in 2d shear flow ([8]). We call it barotropic because it has nothing to do with the stratification.

The second kind comes from the background stratification and shear. We call it *baroclinic* critical layers to distinguish it from the barotropic ones.

The third kind, which depends on the background rotation and the shear. We can prove that actually this inertial critical layer is not a singular point at all. The way we prove it is by calculating the coefficients of other terms, instead of the pressure P. If the singularity exists, it should exist for all the physical quantities, not just for one quantity. Thus if it does not appear for any physical quantity, we know that it is not a singular point for the whole system.

Verification of singularities

This section describes the normal mode linear analysis on the critical layers in the stratified rotational shear flows. The barotropic and baroclinic critical layer have been described above. But we noticed the potential existence of a new kind of critical layer, the inertial critical layer as we call it above, which shows up in the coefficient of the second derivative of Pressure equation, along with the barotropic and baroclinic critical layers. This section will show that such kind of critical layer only shows up in the pressure equation, which make it spurious in the rotational stratified shear flow.

First Equation

Let us start wit the momentum equation

$$\frac{\partial \underline{u}}{\partial t} = -(\underline{u} \cdot \nabla)\underline{u} - \frac{\nabla P}{\rho_0} - \frac{\tilde{\rho}}{\rho_0}g\hat{z} + f\underline{u} \times \hat{z}$$
(1)

where $\tilde{\rho} = \rho - \overline{\rho}(z)$ and $\overline{\rho}(z) = \rho_0 (1 - \frac{N^2}{g}z)$ We take curl of the previous equation, and we will have the vorticity equation $\underline{\omega} = \nabla \times \underline{u}$.

$$\frac{\partial \underline{\omega}}{\partial t} = -(\underline{u} \cdot \nabla)\underline{\omega} + (\underline{\omega} \cdot \nabla)\underline{u} - \frac{g}{\rho_0}\nabla \times \tilde{\rho}\hat{z} + f\frac{\partial \underline{u}}{\partial z}$$
(2)

Linearize around the background shear flow $\underline{U}(y)$ or $\overline{\underline{\omega}} = -\frac{\partial U}{\partial y}\hat{z}$, i.e. $\underline{u} = U(y)\hat{x} + \epsilon \underline{u}'$ and $\underline{\omega} = \overline{\omega}\hat{z} + \epsilon\omega'$ and $\rho = \overline{\rho}(z) + \tilde{\rho} + \epsilon\rho'$, at the order of ϵ , we have the linearized equation,

$$\frac{\partial \underline{\omega}'}{\partial t} = -U\frac{\partial \underline{\omega}'}{\partial x} + v'\frac{\partial \sigma(y)}{\partial y}\hat{z} - u'\frac{\partial \underline{u}'}{\partial z} + \omega'_y u'(y)\hat{x} - \frac{g}{\rho_0}\nabla \times \tilde{\rho}'\hat{z} + f\frac{\partial \underline{u}'}{\partial z}$$
(3)

notice that $\underline{\omega}'$ and \underline{u}' are perturbations while $u'(y) = \frac{\partial U(y)}{\partial y}$. We rewrite $u'(y) = \sigma(y)$, the perturbed equation is written as following

$$\frac{\partial \underline{\omega}'}{\partial t} = -U\frac{\partial \underline{\omega}'}{\partial x} + v'\frac{d\sigma(y)}{dy}\hat{z} - \sigma(y)\frac{\partial \underline{u}'}{\partial z} + \omega_y'\sigma(y)\hat{x} - \frac{g}{\rho_0}\nabla \times \rho'\hat{z} + f\frac{\partial \underline{u}'}{\partial z}$$
(4)

Let us look at the normal modes of the perturbations $A'(x, y, z, t) = a(y)e^{ist+ik_xx+ik_zz}$, the equation is as following

$$i\underline{s}\underline{\omega} = -U(y)ik_{\underline{x}}\underline{\omega} + \frac{d\sigma(y)}{dy}v\hat{z} + ik_{\underline{z}}[f - \sigma(y)]\underline{u} + \sigma(y)\omega_{\underline{y}}\hat{x} - \frac{g}{\rho_{0}}\nabla \times \rho\hat{z}$$
(5)

Notice that $s = -\omega$ for our previous definition. Define $\Omega = s + U(y)k_x$, we look at Equ(5) on the cross-stream \hat{y} direction, here is what we have

$$i\Omega\omega_y = ik_z(f-\sigma)v + ik_x\frac{g}{\rho_0}\rho \tag{6}$$

Divide by i on each side of the equation, we end up the first equation between v, ω_y, ρ

$$\Omega\omega_y = k_z(f - \sigma)v + \frac{k_x g}{\rho_0}\rho \tag{7}$$

Second Equation

Take curl of Equ(4), we have

Notice that we have

$$\nabla \times \omega = \nabla \times \nabla \times \underline{u} = \nabla (\nabla \cdot \underline{u}) - \nabla^2 \underline{u} = -\nabla^2 \underline{u}$$
(9)

So we look at the \hat{y} component of Equ(8) with its normal modes, we shold check Equ(8) term by term

$$\hat{y} \cdot \frac{\partial \nabla \times \underline{\omega}'}{\partial t} = -is\nabla^2 v$$
$$\hat{y} \cdot -\frac{\partial}{\partial x} (\nabla \times U\underline{\omega}') = ik_x U(y)\nabla^2 v$$
$$\hat{y} \cdot \nabla \times v' \frac{d\sigma(y)}{dy} \hat{z} = -ik_x \frac{d\sigma}{dy} v$$
$$\hat{y} \cdot -\frac{\partial}{\partial z} \nabla \times \sigma(y)\underline{u}' = -ik_z \sigma(y)\omega_y$$
$$\hat{y} \cdot \nabla \times \omega_y' \sigma(y) \hat{x} = \sigma(y)ik_z \omega_y$$
$$\hat{y} \cdot -\frac{g}{\rho_0} \nabla \times \nabla \times \rho' \hat{z} = -\frac{g}{\rho_0}ik_z \frac{d\rho}{dy}$$
$$\hat{y} \cdot f \frac{\partial \nabla \times \underline{u}'}{\partial z} = ik_z f \omega_y$$

Sum the above terms up, we will have the second equation between v, ω_y, ρ

$$\Omega \nabla^2 v = k_x \frac{d\sigma(y)}{dy} v + \frac{g}{\rho_0} k_z \frac{d\rho}{dy} - k_z f \omega_y$$
(10)

Third Equation

Start with the density equation

$$\frac{\partial \tilde{\rho}}{\partial t} = -(\underline{u} \cdot \nabla)\tilde{\rho} + \rho_0 \frac{N^2}{g} w$$
(11)

The linearized normal mode equation is

$$is\rho = -ik_x U(y)\rho + \rho_0 \frac{N^2}{g}w \tag{12}$$

Now we need to replace w with some combination of v, ω_y , this can be easily done by toroidal and poloidal decomposition. If we rewrite the velocity \underline{u} in terms of poloidal and toroidal,

$$\underline{u} = \nabla \times \Psi \hat{y} + \nabla \times \nabla \times \Phi \hat{y} \tag{13}$$

Define $k^2 = k_x^2 + k_z^2$, We could easily see the following

$$v = k^{2}\Phi$$

$$\omega_{y} = k^{2}\Psi$$

$$w = ik_{x}\Psi + ik_{z}\frac{d\Phi}{dy}$$

We will have

$$w = ik_x \frac{\omega_y}{k^2} + ik_z \frac{dv/k^2}{dy} = \frac{i}{k^2} (k_x \omega_y + k_z \frac{dv}{dy})$$
(14)

Plug Equ(14) back to Equ(12), we will have the third equation with v, ω_y, ρ

$$\Omega \rho = \frac{\rho_0 N^2}{gk^2} (k_x \omega_y + k_z \frac{dv}{dy}) \tag{15}$$

Deduce to one equation

Equ(7), (10) and (15) are three equations with three components v, ω_y, ρ , from where we could eliminate w_y, ρ and deduce the second order ODE for v, let us put them together

$$\Omega\omega_y = k_z(f-\sigma)v + \frac{k_xg}{\rho_0}\rho$$
(16)

$$\Omega \nabla^2 v = k_x \frac{d\sigma(y)}{dy} v + \frac{g}{\rho_0} k_z \frac{d\rho}{dy} - k_z f \omega_y$$
(17)

$$\Omega \rho = \frac{\rho_0 N^2}{gk^2} (k_x \omega_y + k_z \frac{dv}{dy})$$
(18)

Eliminate ρ from Equ(16) and (18)

Times Ω on each side of Equ(16)

$$\Omega^2 \omega_y = \Omega k_z (f - \sigma) v + \frac{k_x g}{\rho_0} \Omega \rho$$
(19)

Plug in Equ(18)

$$(\Omega^2 - k_x^2 \frac{N^2}{k^2})\omega_y = \Omega k_z (f - \sigma)v + k_x k_z \frac{N^2}{k^2} \frac{dv}{dy}$$
(20)

Eliminate ρ in Equation (17)

From Equ(18), we calculate $\frac{d\rho}{dy}$

$$\frac{d\rho}{dy} = \frac{\rho_0 N^2}{gk^2} \frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right]$$
(21)

Plug Equ(21) to Equ(17), we have

$$\Omega \nabla^2 v = k_x \frac{d\sigma(y)}{dy} v - k_z f \omega_y + \frac{g}{\rho_0} k_z \cdot \frac{\rho_0 N^2}{gk^2} \frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right]$$
$$= k_x \frac{d\sigma(y)}{dy} v - k_z f \omega_y + k_z \frac{N^2}{k^2} \frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right]$$

$$\frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right] = -\frac{1}{\Omega^2} \frac{d\Omega}{dy} (k_x \omega_y + k_z \frac{dv}{dy}) + \frac{1}{\Omega} (k_x \frac{d\omega_y}{dy} + k_z \frac{d^2v}{dy^2})$$
$$= -\frac{\sigma k_x^2}{\Omega^2} \omega_y - \frac{\sigma k_x k_z}{\Omega^2} \frac{dv}{dy} + \frac{k_x}{\Omega} \frac{d\omega_y}{dy} + \frac{k_z}{\Omega} \frac{d^2v}{dy^2}$$

$$\begin{split} \Omega \nabla^2 v &= k_x \frac{d\sigma(y)}{dy} v - k_z f \omega_y + k_z \frac{N^2}{k^2} \frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right] \\ &= k_x \frac{d\sigma(y)}{dy} v - k_z f \omega_y - \frac{\sigma k_z k_x^2 N^2}{\Omega^2 k^2} \omega_y - \frac{\sigma k_x k_z^2 N^2}{k^2 \Omega^2} \frac{dv}{dy} + \frac{k_x k_z}{\Omega} \frac{N^2}{k^2} \frac{d\omega_y}{dy} + \frac{k_z^2 N^2}{\Omega k^2} \frac{d^2 v}{dy^2} \end{split}$$

Check the coefficients

Finally, we have

$$\Omega \nabla^2 v = k_x \frac{d\sigma(y)}{dy} v - k_z f \omega_y - \frac{\sigma k_z k_x^2 N^2}{\Omega^2 k^2} \omega_y - \frac{\sigma k_x k_z^2 N^2}{k^2 \Omega^2} \frac{dv}{dy} + \frac{k_x k_z}{\Omega} \frac{N^2}{k^2} \frac{d\omega_y}{dy} + \frac{k_z^2 N^2}{\Omega k^2} \frac{d^2 v}{dy^2}$$
(22)

with Equ(20)

$$\omega_y = \frac{\Omega k_z (f - \sigma) v + k_x k_z \frac{N^2}{k^2} \frac{dv}{dy}}{(\Omega^2 - k_x^2 \frac{N^2}{k^2})}$$
(23)

Let us check the second derivative coefficients first, the second derivative come from the terms $\Omega \nabla^2 v$, $\frac{k_x k_z}{\Omega} \frac{N^2}{k^2} \frac{d\omega_y}{dy}$, and $\frac{k_z^2 N^2}{\Omega k^2} \frac{d^2 v}{dy^2}$. So the coefficient for the second derivative is

$$\begin{array}{lll} Coefficient \ of \ \frac{d^2v}{dy^2} \\ = & \Omega - \frac{k_z^2 N^2}{\Omega k^2} - \frac{k_x k_z N^2}{\Omega k^2} \cdot \frac{\frac{k_x k_z N^2}{k^2}}{\Omega^2 - \frac{k_x^2 N^2}{k^2}} \\ = & \Omega - \frac{k_z^2 N^2}{\Omega k^2} - \frac{k_x^2 k_z^2 N^4}{\Omega k^2 (k^2 \Omega^2 - k_x^2 N^2)} \\ = & \frac{\Omega^2 k^2 (k^2 \Omega^2 - k_x^2 N^2) - k_z^2 N^2 (k^2 \Omega^2 - k_x^2 N^2) - k_x^2 k_z^2 N^4}{\Omega k^2 (k^2 \Omega^2 - k_x^2 N^2)} \\ = & \frac{\Omega^4 k^4 - \Omega^2 N^2 k^2 k_z^2 - \Omega^2 N^2 k^2 k_x^2}{\Omega k^2 (k^2 \Omega^2 - k_x^2 N^2)} \\ = & \frac{\Omega^4 k^4 - \Omega^2 k^4 N^2}{\Omega k^2 (k^2 \Omega^2 - k_x^2 N^2)} \\ = & \frac{\Omega (\Omega^2 - N^2)}{\Omega^2 - \frac{k_x^2}{k^2} N^2} \end{array}$$

As we could see, that $\Omega = 0$ correspond to the baroctropic critical layer and $\Omega^2 = N^2$ is the baroclinic critical layer. There is no the inertial critical layer which is $\Omega = f(f - \sigma)$ show up.

The terms that contribute to the first derivative of the v are $-k_z f \omega_y$, $\frac{\sigma k_z k_x^2 N^2}{\Omega^2 k^2} \omega_y$, $\frac{\sigma k_x k_z^2 N^2}{k^2 \Omega^2} \frac{dv}{dy}$

The terms that contribute to the v terms are, $\Omega \nabla^2 v$, $k_y \frac{d\sigma(y)}{dy} v$, $k_z f \omega_y$, $\frac{\sigma k_z k_x^2 N^2}{\Omega^2 k^2} \omega_y$, $\frac{k_x k_z}{\Omega} \frac{N^2}{k^2} \frac{d\omega_y}{dy}$. Then we have

$$Coefficient of v = -k^2 \Omega - k_x \frac{d\sigma(y)}{dy} + (k_z f + \frac{\sigma k_z k_x^2 N^2}{k^2 \Omega^2}) \cdot \frac{\Omega k_z (f - \sigma)}{\Omega^2 - k_x^2 \frac{N^2}{k^2}} - \frac{k_x k_z}{\Omega} \frac{N^2}{k^2} \left[\frac{d}{dy} \frac{\Omega k_z (f - \sigma)}{\Omega^2 - k_x^2 \frac{N^2}{k^2}}\right]$$

Notice we have

$$\Omega(y) = s + U(y)k_x, \quad \frac{d\Omega}{dy} = k_x\sigma$$

Now let us calculate the derivative term

$$\frac{d}{dy} \frac{\Omega k_z (f-\sigma)}{\Omega^2 - k_x^2 \frac{N^2}{k^2}} = \frac{k_z (f-\sigma) \frac{d\Omega}{dy} (\Omega^2 - k_x^2 \frac{N^2}{k^2}) - 2\Omega \frac{d\Omega}{dy} \Omega k_z (f-\sigma)}{(\Omega^2 - k_x^2 \frac{N^2}{k^2})^2}$$

$$= \frac{k_x k_z \sigma (f-\sigma) (\Omega^2 - k_x^2 \frac{N^2}{k^2}) - 2\Omega^2 k_x k_z \sigma (f-\sigma)}{(\Omega^2 - k_x^2 \frac{N^2}{k^2})^2}$$

$$= \frac{k_x k_z \sigma (f-\sigma) (-\Omega^2 - k_x^2 \frac{N^2}{k^2})}{(\Omega^2 - k_x^2 \frac{N^2}{k^2})^2}$$

$$= \frac{k^2 k_x k_z \sigma (\sigma-f) (\Omega^2 k^2 + k_x^2 N^2)}{(\Omega^2 k^2 - k_x^2 N^2)^2}$$

Plug the derivative back to the coefficient of v, we have

$$\begin{split} & Coefficient of \ v \\ = \ -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + (k_z f + \frac{\sigma k_z k_x^2 N^2}{k^2 \Omega^2}) \cdot \frac{\Omega k_z (f - \sigma)}{\Omega^2 - k_x^2 \frac{N^2}{k^2}} - \frac{k_x k_z}{\Omega} \frac{N^2}{k^2} [\frac{d}{dy} \frac{\Omega k_z (f - \sigma)}{\Omega^2 - k_x^2 \frac{N^2}{k^2}}] \\ = \ -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + (k_z f + \frac{\sigma k_z k_x^2 N^2}{k^2 \Omega^2}) \cdot \frac{\Omega k_z (f - \sigma)}{\Omega^2 - k_x^2 \frac{N^2}{k^2}} - \frac{k_x k_z}{\Omega} \frac{N^2}{k^2} \frac{k^2 k_x k_z \sigma (\sigma - f) (\Omega^2 k^2 + k_x^2 N^2)}{(\Omega^2 k^2 - k_x^2 N^2)^2} \\ = \ -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + \frac{\Omega^2 k^2 k_z f + \sigma k_z k_x^2 N^2}{k^2 \Omega^2} \cdot \frac{k^2 \Omega k_z (f - \sigma)}{\Omega^2 k^2 - k_x^2 N^2} - \frac{N^2 k_x^2 k_z^2 \sigma (\sigma - f) (\Omega^2 k^2 + k_x^2 N^2)}{\Omega (\Omega^2 k^2 - k_x^2 N^2)^2} \\ = \ -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + \frac{k_z^2 (f - \sigma) (\Omega^2 k^2 f + N^2 k_x^2 \sigma)}{\Omega (\Omega^2 k^2 - k_x^2 N^2)} + \frac{k_z^2 (f - \sigma)}{\Omega (\Omega^2 k^2 - k_x^2 N^2)} \frac{N^2 k_x^2 \sigma (\Omega^2 k^2 + k_x^2 N^2)}{\Omega^2 k^2 - k_x^2 N^2} \\ = \ -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + \frac{k_z^2 (f - \sigma)}{\Omega (\Omega^2 k^2 - k_x^2 N^2)^2} [(\Omega^2 k^2 f + N^2 k_x^2 \sigma) (\Omega^2 k^2 - k_x^2 N^2) + N^2 k_x^2 \sigma (\Omega^2 k^2 + k_x^2 N^2)] \\ = \ -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + \frac{k_z^2 (f - \sigma)}{\Omega (\Omega^2 k^2 - k_x^2 N^2)^2} [(\Omega^4 k^4 f + \Omega^2 N^2 k^2 k_x^2 \sigma - \Omega^2 N^2 k^2 k_x^2 f + N^2 \Omega^2 k^2 k_x^2 \sigma)] \\ = \ -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + \frac{k_z^2 (f - \sigma)}{\Omega (\Omega^2 k^2 - k_x^2 N^2)^2} [\Omega^4 k^4 f + \Omega^2 N^2 k^2 k_x^2 \sigma - \Omega^2 N^2 k^2 k_x^2 f + N^2 \Omega^2 k^2 k_x^2 \sigma)] \\ = \ -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + \frac{k_z^2 (f - \sigma)}{\Omega (\Omega^2 k^2 - k_x^2 N^2)^2} [\Omega^4 k^4 f + \Omega^2 N^2 k^2 k_x^2 \sigma - \Omega^2 N^2 k^2 k_x^2 f + N^2 \Omega^2 k^2 k_x^2 \sigma)] \\ = \ -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + \frac{k_z^2 (f - \sigma)}{\Omega (\Omega^2 k^2 - k_x^2 N^2)^2} [\Omega^4 k^4 f + \Omega^2 N^2 k^2 k_x^2 \sigma - \Omega^2 N^2 k^2 k_x^2 f + N^2 \Omega^2 k^2 k_x^2 \sigma)] \\ \end{cases}$$

Put the three terms together

$$\begin{aligned} Coefficient \ of \ \frac{d^2v}{dy^2} &= \frac{k^2\Omega(\Omega^2 - N^2)}{\Omega^2 k^2 - k_x^2 N^2} \\ Coefficient \ of \ \frac{dv}{dy} &= \frac{2N^2\Omega^2 k^2 k_z^2 k_x \sigma}{(\Omega^2 k^2 - N^2 k_x^2)^2} \\ Coefficient \ of \ v &= -k^2\Omega - k_x \frac{d\sigma(y)}{dy} + \frac{\Omega k^2 k_z^2 (f - \sigma)}{(\Omega^2 k^2 - k_x^2 N^2)^2} [\Omega^2 k^2 f + N^2 k_x^2 (2\sigma - f)] \end{aligned}$$

The coefficients of $\frac{d^2v}{dy^2}$ gives us the position of the critical layers. For linear shear $U(y)=\sigma(y)y=-\sigma y$

$$\begin{split} \Omega &= 0 \Rightarrow Barotropic \ critical \ layer, \quad y^* = \frac{\omega}{\sigma k_x} \\ \Omega^2 - N^2 &= 0 \Rightarrow Baroclinic \ critical \ layer, \quad y^* = \frac{\omega \pm N}{\sigma k_x} \\ Inertial \ critical \ layer \ does \ not \ exist \end{split}$$

It seems we have a new regular singular point at $\Omega = \pm \frac{k_x}{k}N$. But notice that this singularity only shows up in v, we never notice it in the vertical velocity equation, which is analogical to inertial one which only shows up in the Pressure equation.

From above, we have proved that there are only two kinds of critical layers, the batrotropic critical layer and the baroclinic critical layer.

2.4 Orr-Sommerfeld-like equations

Inspired by the technique with Toroidal and Poloidal to explore the existence of the inertial critical layer in inviscid rotational shear flow, we apply the toroidal and poloidal to deduce the Orr-Sommerfeld-like equation for viscous rotational shear flows. Our goal is not get the six order ODE for one parameter, instead, our goal is to reduce the governing equations to three equations contains the velocity and vorticity on the cross-stream direction, and the density perturbation, or we could further reduced them to two equations contain only the stream-wise velocity and vorticity, which is similar to the Orr-Sommerfeld-Squire equations in shear flow. We stick with our notation that x is the streamwise, y is the cross-stream and z is the vertical direction.

First Equation

Let us start with the momentum equation

$$\frac{\partial \underline{u}}{\partial t} = -(\underline{u} \cdot \nabla)\underline{u} - \frac{\nabla P}{\rho_0} - \frac{\tilde{\rho}}{\rho_0}g\hat{z} + f\underline{u} \times \hat{z} + \nu\nabla^2\underline{u}$$
(1)

where $\tilde{\rho} = \rho - \overline{\rho}(z)$ and $\overline{\rho}(z) = \rho_0(1 - \frac{N^2}{g}z)$ We take curl of the previous equation and

We take curl of the previous equation, and we will have the vorticity equation ω .

$$\frac{\partial \underline{\omega}}{\partial t} = -(\underline{u} \cdot \nabla)\underline{\omega} + (\underline{\omega} \cdot \nabla)\underline{u} - \frac{g}{\rho_0}\nabla \times \tilde{\rho}\hat{z} + f\frac{\partial \underline{u}}{\partial z} + \nu\nabla^2\underline{\omega}$$
(2)

Linearize around the background shear flow $\underline{U}(y)$ or $\overline{\underline{\omega}} = -\frac{\partial U}{\partial y}\hat{z}$, i.e. $\underline{u} = U(y)\hat{x} + \epsilon \underline{u}'$ and $\underline{\omega} = \overline{\omega}\hat{z} + \epsilon \omega'$ and $\rho = \overline{\rho}(z) + \tilde{\rho} + \epsilon \rho'$, at the order of ϵ , we have the linearized equation,

$$\frac{\partial \underline{\omega}'}{\partial t} = -U\frac{\partial \underline{\omega}'}{\partial x} + v'\frac{\partial \sigma(y)}{\partial y}\hat{z} - u'\frac{\partial \underline{u}'}{\partial z} + \omega'_y u'(y)\hat{x} - \frac{g}{\rho_0}\nabla \times \tilde{\rho}'\hat{z} + f\frac{\partial \underline{u}'}{\partial z} + \nu\nabla^2\underline{\omega}'$$
(3)

notice that $\underline{\omega}'$ and \underline{u}' are perturbations while $u'(y) = \frac{\partial U(y)}{\partial y}$. We rewrite $u'(y) = \sigma(y)$, the perturbed equation is written as following

$$\frac{\partial \underline{\omega}'}{\partial t} = -U\frac{\partial \underline{\omega}'}{\partial x} + v'\frac{d\sigma(y)}{dy}\hat{z} - \sigma(y)\frac{\partial \underline{u}'}{\partial z} + \omega_y'\sigma(y)\hat{x} - \frac{g}{\rho_0}\nabla \times \rho'\hat{z} + f\frac{\partial \underline{u}'}{\partial z} + \nu\nabla^2\underline{\omega}' \tag{4}$$

Let us look at the normal modes of the perturbations $A'(x, y, z, t) = a(y)e^{ist+ik_xx+ik_zz}$, the equation is as following

$$is\underline{\omega} = -U(y)ik_x\underline{\omega} + \frac{d\sigma(y)}{dy}v\hat{z} + ik_z[f - \sigma(y)]\underline{u} + \sigma(y)\omega_y\hat{x} - \frac{g}{\rho_0}\nabla \times \rho\hat{z} + \nu(\frac{d^2}{dy^2} - k^2)\underline{\omega} \quad (5)$$

Look at the cross-stream direction of the previous equation, we have

$$\frac{\partial \omega_y}{\partial t} = ik_z(f-\sigma)v + ik_x\frac{g}{\rho_0}\rho + \left[\nu\nabla^2 - ik_xU(y)\right]\omega_y \tag{6}$$

Define $\Omega = s + U(y)k_x$, we look at Equ(5) on the cross-stream \hat{y} direction, here is what we have

$$i\Omega\omega_y = ik_z(f-\sigma)v + ik_x\frac{g}{\rho_0}\rho + \nu(\frac{d^2}{dy^2} - k^2)\omega_y \tag{7}$$

Multipy *i* on each side of the equation and rewrite the equation above in order of the ω_y , we end up the first equation between v, ω_y, ρ

$$i\nu \frac{d^2 \omega_y}{dy^2} + (\Omega - i\nu k^2)\omega_y = k_z (f - \sigma)v + \frac{k_x g}{\rho_0}\rho$$
(8)

Second Equation

Take curl of Equ(4), we have

$$\frac{\partial \nabla \times \underline{\omega}'}{\partial t} = -\frac{\partial}{\partial x} (\nabla \times U \underline{\omega}') + \nabla \times v' \frac{d\sigma(y)}{dy} \hat{z} - \frac{\partial}{\partial z} \nabla \times \sigma(y) \underline{u}' + \nabla \times \omega_y' \sigma(y) \hat{x} - \frac{g}{\rho_0} \nabla \times \nabla \times \rho' \hat{z} + f \frac{\partial \nabla \times \underline{u}'}{\partial z} + \nu \nabla^2 \nabla \times \underline{\omega}'$$
(9)

Notice that we have

$$\nabla \times \omega = \nabla \times \nabla \times \underline{u} = \nabla (\nabla \cdot \underline{u}) - \nabla^2 \underline{u} = -\nabla^2 \underline{u}$$
(10)

So we look at the \hat{y} component of Equ(8) with its normal modes, we shold check Equ(8) term by term

$$\begin{split} \hat{y} \cdot \frac{\partial \nabla \times \underline{\omega}'}{\partial t} &= -is \nabla^2 v \\ \hat{y} \cdot -\frac{\partial}{\partial x} (\nabla \times U \underline{\omega}') &= ik_x U(y) \nabla^2 v \\ \hat{y} \cdot \nabla \times v' \frac{d\sigma(y)}{dy} \hat{z} &= -ik_x \frac{d\sigma}{dy} v \\ \hat{y} \cdot -\frac{\partial}{\partial z} \nabla \times \sigma(y) \underline{u}' &= -ik_z \sigma(y) \omega_y \\ \hat{y} \cdot \nabla \times \omega_y' \sigma(y) \hat{x} &= \sigma(y) ik_z \omega_y \\ \hat{y} \cdot \nabla \times \omega_y' \sigma(y) \hat{x} &= -\frac{g}{\rho_0} ik_z \frac{d\rho}{dy} \\ \hat{y} \cdot f \frac{\partial \nabla \times \underline{u}'}{\partial z} &= ik_z f \omega_y \\ \hat{y} \cdot \nu \nabla^2 \nabla \times \underline{\omega}' &= -\nu \nabla^4 v = -\nu (\frac{d^4}{dy^4} - 2k^2 \frac{d^2}{dy^2} + k^4) v \end{split}$$

Sum the above terms up, we will have the second equation between v, ω_y, ρ

$$\frac{\partial \nabla^2 v}{\partial t} = ik_x \frac{d\sigma(y)}{dy} v + i\frac{g}{\rho_0} k_z \frac{d\rho}{dy} - ik_z f\omega_y + \left[\nu \nabla^4 - ik_x U(y) \nabla^2\right] v \tag{11}$$

or

$$\Omega \nabla^2 v = k_x \frac{d\sigma(y)}{dy} v + \frac{g}{\rho_0} k_z \frac{d\rho}{dy} - k_z f \omega_y - i\nu \nabla^4 v$$
(12)

Third Equation

Start with the density equation

$$\frac{\partial \tilde{\rho}}{\partial t} = -(\underline{u} \cdot \nabla) \tilde{\rho} + \rho_0 \frac{N^2}{g} w \tag{13}$$

The linearized normal mode equation is

$$is\rho = -ik_x U(y)\rho + \rho_0 \frac{N^2}{g}w \tag{14}$$

Now we need to replace w with some combination of v, ω_y , this can be easily done by toroidal and poloidal decomposition. If we rewrite the velocity \underline{u} in terms of poloidal and toroidal,

$$\underline{u} = \nabla \times \Psi \hat{y} + \nabla \times \nabla \times \Phi \hat{y} \tag{15}$$

Define $k^2 = k_x^2 + k_z^2$, We could easily see the following

$$v = k^{2}\Phi$$

$$\omega_{y} = k^{2}\Psi$$

$$w = ik_{x}\Psi + ik_{z}\frac{d\Phi}{dy}$$

We will have

$$w = ik_x \frac{\omega_y}{k^2} + ik_z \frac{dv/k^2}{dy} = \frac{i}{k^2} (k_x \omega_y + k_z \frac{dv}{dy})$$
(16)

Plug Equ(16) back to Equ(14), we will have the third equation with v, ω_y, ρ

$$\frac{\partial \rho}{\partial t} = i \frac{\rho_0 N^2}{gk^2} (k_x \omega_y + k_z \frac{dv}{dy}) - ik_x U(y)\rho \tag{17}$$

or

$$\Omega \rho = \frac{\rho_0 N^2}{gk^2} (k_x \omega_y + k_z \frac{dv}{dy})$$
(18)
Matrix form for three equations

There are several ways we could write as matrix form. First we put equation (6), (11), (17) together.

$$\begin{aligned} \frac{\partial \omega_y}{\partial t} &= ik_z(f-\sigma)v + ik_x \frac{g}{\rho_0}\rho + \left[\nu\nabla^2 - ik_x U(y)\right]\omega_y \\ \frac{\partial\nabla^2 v}{\partial t} &= ik_x \frac{d\sigma(y)}{dy}v + i\frac{g}{\rho_0}k_z \frac{d\rho}{dy} - ik_z f\omega_y + \left[\nu\nabla^4 - ik_x U(y)\nabla^2\right]v \\ \frac{\partial\rho}{\partial t} &= i\frac{\rho_0 N^2}{gk^2}(k_x\omega_y + k_z \frac{dv}{dy}) - ik_x U(y)\rho \end{aligned}$$

Write them as matrix form

$$\frac{\partial}{\partial t} \begin{pmatrix} \nabla^2 \\ I \\ I \end{pmatrix} \begin{pmatrix} v \\ \omega_y \\ \rho \end{pmatrix} = \begin{pmatrix} \nu \nabla^4 - ik_x U(y) \nabla^2 + ik_x \frac{d\sigma(y)}{dy} & -ik_z f & ik_z \frac{g}{\rho_0} \frac{d}{dy} \\ ik_z (f - \sigma) & \nu \nabla^2 - ik_x U(y) & ik_x \frac{g}{\rho_0} \\ ik_z \frac{\rho_0 N^2}{gk^2} \frac{d}{dy} & ik_x \frac{\rho_0 N^2}{gk^2} & -ik_x U(y) \end{pmatrix} \begin{pmatrix} v \\ \omega_y \\ \rho \end{pmatrix}$$

Notice this matrix form could be used to calculate the eigen-vectors, instead of using five prime parameters, we only have three here.

The other form come from equations (8), (12), (18)

$$\Omega\omega_y = k_z(f-\sigma)v + \frac{k_xg}{\rho_0}\rho - i\nu\nabla^2\omega_y$$
(19)

$$\Omega \nabla^2 v = k_x \frac{d\sigma(y)}{dy} v + \frac{g}{\rho_0} k_z \frac{d\rho}{dy} - k_z f \omega_y - i\nu \nabla^4 v$$
⁽²⁰⁾

$$\Omega \rho = \frac{\rho_0 N^2}{gk^2} (k_x \omega_y + k_z \frac{dv}{dy})$$
(21)

Deduce to two equation

Equ(8), (12) and (18) are three equations with three components v, ω_y, ρ , from where we could eliminate ρ and deduce the two equations for v, ω_y , let us put them together

$$\Omega\omega_y = k_z(f-\sigma)v + \frac{k_xg}{\rho_0}\rho - i\nu\nabla^2\omega_y$$
(22)

$$\Omega \nabla^2 v = k_x \frac{d\sigma(y)}{dy} v + \frac{g}{\rho_0} k_z \frac{d\rho}{dy} - k_z f \omega_y - i\nu \nabla^4 v$$
(23)

$$\Omega \rho = \frac{\rho_0 N^2}{gk^2} (k_x \omega_y + k_z \frac{dv}{dy})$$
(24)

Eliminate ρ from Equ(22) and (24)

Times Ω on each side of Equ(22)

$$\Omega^2 \omega_y = \Omega k_z (f - \sigma) v + \frac{k_x g}{\rho_0} \Omega \rho - i \nu \Omega \nabla^2 \omega_y$$
(25)

Plug in Equ(24)

$$(i\nu\Omega\nabla^2 + \Omega^2 - k_x^2 \frac{N^2}{k^2})\omega_y = \Omega k_z (f - \sigma)v + k_x k_z \frac{N^2}{k^2} \frac{dv}{dy}$$
(26)

Eliminate ρ in Equation (23)

From Equ(24), we calculate $\frac{d\rho}{dy}$

$$\frac{d\rho}{dy} = \frac{\rho_0 N^2}{gk^2} \frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right]$$
(27)

Plug Equ(27) to Equ(23), we have

$$\Omega \nabla^2 v = k_x \frac{d\sigma(y)}{dy} v - k_z f \omega_y + \frac{g}{\rho_0} k_z \cdot \frac{\rho_0 N^2}{gk^2} \frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right] - i\nu \nabla^4 v$$
$$= k_x \frac{d\sigma(y)}{dy} v - k_z f \omega_y + k_z \frac{N^2}{k^2} \frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right] - i\nu \nabla^4 v$$

$$\frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right] = -\frac{1}{\Omega^2} \frac{d\Omega}{dy} (k_x \omega_y + k_z \frac{dv}{dy}) + \frac{1}{\Omega} (k_x \frac{d\omega_y}{dy} + k_z \frac{d^2v}{dy^2})$$
$$= -\frac{\sigma k_x^2}{\Omega^2} \omega_y - \frac{\sigma k_x k_z}{\Omega^2} \frac{dv}{dy} + \frac{k_x}{\Omega} \frac{d\omega_y}{dy} + \frac{k_z}{\Omega} \frac{d^2v}{dy^2}$$

$$\begin{aligned} \Omega \nabla^2 v &= k_x \frac{d\sigma(y)}{dy} v - k_z f \omega_y + k_z \frac{N^2}{k^2} \frac{d}{dy} \left[\frac{1}{\Omega} (k_x \omega_y + k_z \frac{dv}{dy}) \right] - i\nu \nabla^4 v \\ &= k_x \frac{d\sigma(y)}{dy} v - k_z f \omega_y - \frac{\sigma k_z k_x^2 N^2}{\Omega^2 k^2} \omega_y - \frac{\sigma k_x k_z^2 N^2}{k^2 \Omega^2} \frac{dv}{dy} + \frac{k_x k_z}{\Omega} \frac{N^2}{k^2} \frac{d\omega_y}{dy} + \frac{k_z^2 N^2}{\Omega k^2} \frac{d^2 v}{dy^2} - i\nu \nabla^4 v \end{aligned}$$

Then we have

$$(i\nu\nabla^4 + \Omega\nabla^2)v = k_x \frac{d\sigma(y)}{dy}v - k_z f\omega_y - \frac{\sigma k_z k_x^2 N^2}{\Omega^2 k^2}\omega_y - \frac{\sigma k_x k_z^2 N^2}{k^2 \Omega^2} \frac{dv}{dy} + \frac{k_x k_z}{\Omega} \frac{N^2}{k^2} \frac{d\omega_y}{dy} + \frac{k_z^2 N^2}{\Omega k^2} \frac{d^2v}{dy^2}$$
(28)

with Equ(26)

$$(i\nu\Omega\nabla^2 + \Omega^2 - k_x^2\frac{N^2}{k^2})\omega_y = \Omega k_z(f-\sigma)v + k_xk_z\frac{N^2}{k^2}\frac{dv}{dy}$$
(29)

Write as matrix form

$$\begin{pmatrix} i\nu\nabla^4 + \Omega\nabla^2 \\ i\nu\Omega\nabla^2 + \Omega^2 \end{pmatrix} \begin{pmatrix} v \\ \omega_y \end{pmatrix} = \\ \begin{pmatrix} \frac{k_z^2 N^2}{\Omega k^2} \frac{d^2}{dy^2} - \frac{\sigma k_x k_z^2 N^2}{k^2 \Omega^2} \frac{d}{dy} + k_x \frac{d\sigma(y)}{dy} & \frac{k_x k_z}{\Omega} \frac{N^2}{k^2} \frac{d}{dy} - k_z f - \frac{\sigma k_z k_x^2 N^2}{\Omega^2 k^2} \\ \frac{k_x k_z N^2}{k^2} \frac{d}{dy} + \Omega k_z (f - \sigma) & \frac{k_x^2 N^2}{k^2} \end{pmatrix} \begin{pmatrix} v \\ \omega_y \end{pmatrix}$$

The equations above are the Orr-Sommerfeld equations for linear horizontal shear, linearly stratified flow with background rotation.

Chapter 3

Zombie Vortex Instability in Inviscid Regime

The exploration of the instability in stratified, rotational flow with horizontal shear has drawn great attention in recent years, due to the long lasting interests in the pursuit of turbulence generating mechanism in accretion disk. For the study of stratified Taylor-Couette system, a new linear instability, called the StratoRotational Instability (SRI) [26] has been found analytically beyond the Rayleigh' criteria, under the small wave-number approximation, thin-gap approximation and the strong stratification assumption. This work is further extended by [21] and [19]. SRI has been observed experimentally by [9] and numerically verified with initial value simulation [10]. During the exploration of stratified plain Couette flow with background rotation, the gravity-wave-like instability has also been reported [25]. This instability is believed to be relevant to SRI and it is form to the linear resonances between waves with oppositely singed wave momenta, for example, Kelvin wave and inertia-gravity wave.

Despite the existence of the several instability in stratified rotational shear flow, a new finite-amplitude instability, called the self-replicating "Zombie" Vortex Instability(ZVI) has also been discovered in such system[15]. Unlike other instabilities, ZVI exists in the parameter regime where all three physical parameters are on the same order. During the ZVI process, baroclinic critical layers are excited by initial vortex at certain locations, on which new generation of vortex are spawn. Those new generated vortex will excite their own critical layers and thus leads to the self-replicating mechanism that destabilize the system. In this paper, we provide a better understanding of how to detect ZVI numerically and under what condition that ZVI will be observed.

In this paper, we will systematically explore the existence of Zombie Vortex Instability under the physical parameters. In part 2, the key features such as critical layers, zombie vortex will be discussed in details. The whole progress of ZVI that determines whether it is stable or not will be shown as a guidance for our further exploration. Detailed in our numerical set-up will also be reported in this part. In part 3, all the parameters, including the physical parameters and numerical one are discussed. A parameter map is presented in part 4, as well as discussions on the regimes on the map. Physical explanation and interpretation will be provided. In the last part, we will discuss how to set up in the laboratory to observe such instability.

3.1 Problem Set-up

Governing equations, boundary conditions and numerical methods

Zombie Vortex Instability (ZVI) happens in vertically stratified, horizontal shear flow with background rotation. In order to fully understand the basic mechanism of ZVI, we need to figure out what the role of each of physical ingredients, i.e. shear, stratification and rotation, and how the instability will react if any of the these three parameter changes. In this paper, we define our background shear velocity $U(y) = -\sigma y \hat{x}$ on the horizontal direction, where σ is the shear rate, \hat{x} is the stream-wise direction and \hat{y} is the cross-stream direction. Notice that the vorticity associated with the background shear velocity field is σ . So if σ and f have the same (different) sign, the vorticity associated with the background shear will rotate in the same (different) direction of the background rotation. Thus we call our flow filed cyclonic(anti-cyclonic) based on the sign of σ/f . On the vertical direction \hat{z} , we assume our fluid is Boussinesq linearly stratified. The BruntVäisälä frequency $N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}(z)}{dz}$ is defined as a measurement of stratification, where g is the gravity, ρ_0 is the background density at the reference height and $\bar{\rho}(z)$ is the background stratification. Notice N is a constant for linear stratification. The background rotation on the vertical direction is represented by the Coriolis parameter $f = 2\Omega$ where Ω is the angular velocity of the system. If we separate the our total flow field as two parts, the background shear velocity and perturbation, i.e. $\underline{u}_{total} = \underline{U} + \underline{u}$. We also separate our density into two parts, the background linear stratification and the density fluctuations, i.e. $\rho(x, y, z, t) = \overline{\rho}(z) + \tilde{\rho}(x, y, z, t)$. After neglecting the viscous dissipation and thermal diffusion, the governing equations for the perturbations can be easily derived as following,

$$\begin{array}{lll} 0 & = & \displaystyle \frac{\partial u_x}{\partial x} + \displaystyle \frac{\partial u_y}{\partial y} + \displaystyle \frac{\partial u_z}{\partial z} \\ \\ \displaystyle \frac{\partial u_x}{\partial t} & = & \displaystyle -(\underline{u} \cdot \nabla) \, u_x - U\left(y\right) \displaystyle \frac{\partial u_x}{\partial x} - \displaystyle \frac{\partial P}{\partial x} + \left(f + \sigma\right) u_y \\ \\ \displaystyle \frac{\partial u_y}{\partial t} & = & \displaystyle -(\underline{u} \cdot \nabla) \, u_y - U\left(y\right) \displaystyle \frac{\partial u_y}{\partial x} - \displaystyle \frac{\partial P}{\partial y} - f u_x \\ \\ \displaystyle \frac{\partial u_z}{\partial t} & = & \displaystyle -(\underline{u} \cdot \nabla) \, u_z - U\left(y\right) \displaystyle \frac{\partial u_z}{\partial x} - \displaystyle \frac{\partial P}{\partial z} - \displaystyle \frac{\tilde{\rho}}{\rho_0} g \\ \\ \displaystyle \frac{\partial \tilde{\rho}}{\partial t} & = & \displaystyle -(\underline{u} \cdot \nabla) \, \tilde{\rho} - U\left(y\right) \displaystyle \frac{\partial \tilde{\rho}}{\partial x} + \displaystyle \frac{\rho_0 N^2}{g} u_z \end{array}$$

As you might have noticed, in the equations, the terms associated with linear shear velocity U(y) break the autonomy of the equations and thus triply periodic boundary conditions are not valid. In order to enforce the periodicity, without loss of generality, we could transfer our coordinates to the shearing sheet coordinates. The shearing sheet coordinate is a Galileo transformation of our frame to a moving frame where the observer(origin of the system) is moving with the background shear velocity. One of the advantage of applying such system is that, in the shearing sheet coordinates, the terms that break the autonomy of the equations will vanish and thus periodic boundary conditions are valid. Mathematically, the relationship between our shearing sheet system (x', y', z', t') and the original coordinates (x, y, z, t)is $(x', y', z', t') = (x + \sigma yt, y, z, t)$. After some simple algebra, the governing equations in shearing sheet coordinates can be expressed as

$$\begin{array}{lll} 0 &=& \displaystyle \frac{\partial u_x}{\partial x'} + (\frac{\partial}{\partial y'} + \sigma t' \frac{\partial}{\partial x'}) u_y + \frac{\partial u_z}{\partial z'} \\ \\ \displaystyle \frac{\partial u_x}{\partial t'} &=& \displaystyle -(\underline{u} \cdot \nabla') \, u_x + \sigma t' u_y \frac{\partial u_x}{\partial x'} - \frac{\partial P}{\partial x'} + (f + \sigma) \, u_y \\ \\ \displaystyle \frac{\partial u_y}{\partial t} &=& \displaystyle -(\underline{u} \cdot \nabla') \, u_y + \sigma t' u_y \frac{\partial u_y}{\partial x'} - (\frac{\partial}{\partial y'} + \sigma t' \frac{\partial}{\partial x'}) P - f u_x \\ \\ \displaystyle \frac{\partial u_z}{\partial t} &=& \displaystyle -(\underline{u} \cdot \nabla') \, u_z + \sigma t' u_y \frac{\partial u_z}{\partial x'} - \frac{\partial P}{\partial z'} - \frac{\tilde{\rho}}{\rho_0} g \\ \\ \quad \frac{\partial \tilde{\rho}}{\partial t} &=& \displaystyle -(\underline{u} \cdot \nabla') \, \tilde{\rho} + \sigma t' u_y \frac{\partial \rho}{\partial x'} + \frac{\rho_0 N^2}{q} u_z \end{array}$$

After the transformation, the triply periodic boundary conditions will be applied in the system.

We use fractional step methods to compute the different terms in the equations. The second-order accurate Adam-Bashforth method is used for nonlinear terms, Crank-Nicholson method for the pressure gradient and the semi-analytical method for the rest linear terms associated with shear, rotation and stratification. The idea for semi-analytic method is that we could compute the linear terms explicitly and precisely with its analytic formula. Details of such method and the shearing sheet transformation can be found in [4].

The process for ZVI

ZVI can be triggered by gaussian vortex and 3d random noise with Kolmogorov spectrum. The details on the initial conditions will be discussed in next section, here let us focus on the process we observed in ZVI, when triggered by a gaussian vortex. There are typically several stages we have observed during ZVI process, which are described in details as following. The corresponding figures are plotted in Fig.3.2.

a) the formation of baroclinic critical layers: Critical layers are mathematical singularities which are smoothed either by nonlinear effects or viscous/thermal dissipation. The

classical work on critical layers have been reported and extensively studied in the parallel shear flow [8]. We call the critical layers in plain-Couette flow barotropic critical layers since it is purely triggered by horizontal shear. In stratified rotational shear flow, we can find and derive the locations of critical layers by applying the linear normal mode analysis into the governing equations described above, by assuming that the amplitude of the perturbation are small compared with the background shear and stratification, and any prime variables q(x, y, z, t) in the system could be expressed in terms of the sum of normal modes $q(x, y, z, t) = \sum_{k_x} \sum_{k_z} \tilde{q}(y) exp(-\omega t + ik_x x + ik_z z)$, where k_x and k_z are the wave numbers on the stream-wise x and vertical direction z correspondingly (See [15] for details). There are two kinds of singularities that are both mathematically and physically valid: The first kind is the barotropic critical layers. It is the same kind as shown in parallel shear flow, which is only associated with shear rate σ and stream-wise wave number k_x ; The second kind is the baroclinic critical layer. It is linearly neutral mode which is excited not only by the horizontal shear, but also the vertical stratification. To distinguish from the classical barotropic critical layers, we call such critical layers *baroclinic* critical layer. Similar to barotropic critical layers who locations can be determined by the linear normal mode analysis, the location of baroclinic critical layers y^* could also be determined by the physical parameters that excited such signification is horizontal shear rate σ , vertical stratification N and the stream-wise wave number k_x . Since the excitement of barotropic critical layers requires the second derivative of the background shear velocity on the cross-stream direction to be valid, in our numerical simulations with linear shear velocity, they have never been excited nor observed. Only baroclinic critical layers have been excited and observed. With careful linear normal mode analysis, the locations of the baroclinic critical layers y^* are

$$y^* = \pm \frac{N}{\sigma k_x} \tag{3.1}$$

If we further define $k_x = \frac{2\pi m}{L_x}$ and m = 0, 1, 2... Equation above can be rewritten as

$$\frac{y^*}{L_x} = \pm \frac{N}{2\pi\sigma m} \tag{3.2}$$

As we can see from the equation and Fig.4.2, critical layers show up in groups, each of which is associated with one particular wave number. The lower the wave number is, the further away it is located from y = 0. In our simulations, the baroclinic critical layers are excited in the flow at very early stage(less than $12 \ 1/f$ time) and they are typically the first phenomenon we observe in the simulations. The baroclinic critical layers can be observed in the numerical simulation in terms of vertical velocity u_z and vertical vorticity ω_z clearly. As demonstrated in Fig.4.2(a), the critical layers have either shear-like or jet-like structures. Such structures will provide a mean-flow vorcity jump or a vortex sheet structure, which can be observed clearly in Fig.3.2(b). The vorticy jump or vortex sheet structure provided by the critical layers are very unstable and could result into a vortex by linear instability such as Kelvin-Helmoltz instability. The process of how vortex sheet would produce vortex under



(a) Vertical velocity u_z versus cross-stream y/L_x at (b) Vertical velocity u_z on x = 0 plane; Color red and $x = 0 \ z/L_x = 0.08$ blue mean positive and negative sign

Figure 3.1: Plot of vertical velocity u_z at 72 1/f time for Case $N/f = 1.0 \sigma/f = -0.75$; Dash line shows the critical layers location predicted by linear normal mode analysis

the effect of linear instability could be found in [14] and [24]. Indeed, we do observe the growing of vortex on the vortex sheet shown in Fig.3.2(c). Careful readers may question why the shape of the critical layers are not straight line along vertical z direction. The reason is due to the internal gravity waves which will be discussed in later session. Baroclinic critical layers have also been observed in the laboratory experiments during the research of instability of a vertically titled vortex in stratified fluid [5].

b) the formation of the vortex sheet: After the critical layers formed, the critical layers of the vertical vorticity get connected and the structure of vortex sheet is clearly formed. The vortex sheet contains two stripes next to each other, with one layer anti-cyclonic and the other layer cyclonic. The magnitude of anti-cyclone and cyclone are on the same order.

c) the formation of zombie vortex: Vortex sheet is unstable structure. Under the effect of linear instability, the anti-cyclonic(cyclonic) stripes of the vortex will roll up to vortex, if our flow is anti-cyclonic(cyclonic), while the cyclonic(anti-cyclonic) stripes remains.

d) self-replication of the zombie vortex: Process a) to c) are repeated based on the zombie vortex, i.e. the critical layers of the zombie vortex are generated and they are located at y^* from the location of the zombie vortex; vortex sheet formed on the critical layers; Due to the instability of the vortex sheet, a new generation of zombie vortex generated therefore.

e) the formation of zombie turbulence: Self-replication process continues until the whole flow field are filled up with critical layers and zombie vortex. The flow field reaches to a quasi-equilibrium fully developed turbulent state, where we could still see the stripes of critical layers separated the flow. We call this state zombie turbulence.

Those several stages of ZVI could also be clearly explained in terms of the vertical kinetic energy $E_{k,z}$ defined as





(a) Observation of critical layer (b) Observation of vortex sheet at $72 \ 1/f$ time at $213 \ 1/f$ time

(c) Formation of zombie vortex at $360 \ 1/f$ time



(d) Self-replication of zombie vortex at $1584 \ 1/f$ time

(e) Zombie turbulence at 5760 1/f time

Figure 3.2: Plot of vertical vorticity ω_z at $z/L_x = 0.25$ for different stages for Case N/f = 1.0 $\sigma/f = -0.75$; Color blue(red) represents anti-cyclonic(cyclonic); Green on the background represents zero;

$$E_{k,z} = \frac{1}{2} \int \rho_0 u_z^2 dV \tag{3.3}$$

Fig.3.3 shows a typical observation of the $E_{k,z}$ versus time. The vertical kinetic energy is a very good indication of ZVI process, since it is zero initially and becomes nonzero when the critical layers on the vertical velocity shows up. It can be viewed as measurement of how many critical layers and how strong they are. As we could see in Fig.3.3, the vertical kinetic energy is zero at the first beginning, this is due to the fact no critical layers have been excited at the first beginning and the vertical velocity of the flow field is zero. Then the baroclinic critical layers are excited and the $E_{k,z}$ starts to grow. During this period, we observe the critical layers on roll up to vortex layers and the next generation vortex start to form. Around $10001/\sigma$ time, due to the duplication of the zombie vortex and thus the duplication of the critical layers, $E_{k,z}$ has a sharp increases. This is corresponding to the stage(d) in ZVI process. Until 2000 $1/\sigma$ time, the flow field is totally destabilized and reaches a statistically turbulent equilibrium state.



Figure 3.3: Vertical kinetic energy for unstable case N/f = 1, $\sigma/f = -3/4$ shows clearly the several stages of ZVI process when initialized with vortex

Initial condition and criteria for ZVI

ZVI was first observed by initializing with Gaussian elliptical Vortex [15]. The reason we would like to use a vortex as initial condition are, first, the strong self-rotation property of vortex is famous for mass and momentum concentration. Secondly, the gaussian shape provides nonzero perturbations for all wave modes on three directions and mathematically, the fourier transform of a gaussin shape function remains gaussian shape for all wave numbers. The mathematical formula for gaussian vortex perturbation is

$$! = \omega_0 exp(-\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{H^2})$$
(3.4)

 ω_0 is the vorticity at the origin. *a* and *b* represents the horizontal length scale on streamwise *x* and cross-stream direction *y* while *H* represents the vertical length scale of the vortex.

Compared with other initial conditions such as random noise(which will be described later), the advantage of gaussian vortex is that the flow field are very clean such at we could



(a) Quiver plot showing the horizontal velocity field of the middle plan

(b) Vertical vorticity ω_z on the vertical direction

Figure 3.4: An anti-cyclonic Gaussian vortex as initial condition

see clearly the whole process and the structure of the ZVI, which has been demonstrated in the previous part. The disadvantage of the gaussian vortex is that the gaussian vortex is not the equilibrium state. Since it has been noticed and demonstrated that it takes roughly $1000 \ 1/\sigma$ time or equivalently 100 physical years for the critical layers, vortex sheet and self-replicating vortex to form and start to duplicate. It takes another roughly $1000 \ 1/\sigma$ time or equivalently 100 physical years to fully destabilize the system. During the vortex sheet and self-replicating vortex formation period, the vortex itself may become unstable or destroyed by the shear or stratification at the early stage. Indeed, we have observed the gaussian vortex was stretched and destroying due to strong shear or split vertically under strong stratification. In order to maintain our vortex stably existing long enough as a perturbation source to excite critical layers, we would like to initialize it as close to the equilibrium state. We already know that for two dimensional pure shear flow, the Moore-Saffman vortex introduced by [18] which satisfies the following equation, is the equilibrium vortex state

$$\frac{\omega_0}{\sigma} = \frac{\chi + 1}{\chi - 1} \frac{1}{\chi} \tag{3.5}$$

 $\chi = \frac{a}{b}$ is the horizontal aspect ratio of the elliptical vortex and ω_0 is the amplitude of the vortex. We also know that, for the stratified fluid with rotation, the equilibrium vortex satisfies a certain vertical aspect ratio [1]. The physical interpretation for the vortex in the stratified rotational flow is that, the background rotation would prefer a column vortex due to Taylor-Proudman theorem while the stratification effect would prefer a pancake like vortex shape. A balance between those two effects result into the vortex equilibrium solution. In conclusion, when initializing the vortex, we require our vortex satisfies the Moore-Saffman

vortex shape on the horizontal direction and also adjust the vertical aspect ration of the vortex when stratification and rotation are changed.

However, since the vortex is not an equilibrium. When simulations starts, there will be a period where the vortex self-adjust itself to the fluid system and the behavior of the vortex can not be predicted. For example, we have observed several cases that when the stratification is dominant, the vortex may be split to several vortex vertically. Or for strong shear fluid, the vortex might be stretched out directly by the strong horizontal shear. As we have demonstrated, ZVI require a certain amount of time for the critical layers becomes unstable. If the vortex gets destroyed at early stage, the lack of critical layers may lead to no zombiefication, in which case we can not determine that ZVI does not happen is due to it is a stable case or the destruction of the vortex. Thus, we shall use a more general initial condition which is the three dimensional Kolmogorov random noise.

The noise is initialized with isotropic random amplitude on three directions. The kinetic energy spectrum is proportional to $k^{-5/3}$ where $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$. The mathematical formula and deduction on the velocity can be found in [16]. The very important observations about ZVI initialized by random noise are, after the simulation starts, there will be small vortex generated and observed in the flow field. Those vortex could be regarded as the seeds to excite their own critical layers and replicate themselves to trigger the instability. The idea is instead of putting a non-equilibrium vortex (like gaussian vortex) as initial condition, we initialize the flow field with noise and let the flow filed to generate the vortex it preferred. Since those vortex are generated by the flow field, they could survive long enough in the flow field to excite critical layers and replicate themselves. Under such assumption, once the small vortex are generated in the flow field, the process shall has no differences with what we have observed when initializing with Gaussian vortex, i.e. the five stage process shall appear for each vortex seed. Further more, when the flow filed become fully destabilized, the turbulence shall reach to the same statistical state regardless of the initial condition. Indeed, we did have observe the convergence of the kinetic energy of the flow field when initialized with several initial conditions.

The advantage of the initial noise is that, our concerns that ZVI may be effected by the initial conditions has been eliminated. Even if the equilibrium state of vortex still remains unknown for our flow field, the right kind of vortex that could survive the flow field will be generated and ZVI shall be triggered under the right kind of parameters setup, which is crucial to our next discussion on how the physical parameters, instead of initial conditions affect the physics of ZVI. Compared with a clean gaussian vortex, the disadvantage of random noise is that, due to the fact the flow filed is all filled up with noise, we can not see the structure and the process of ZVI, which limits our capacity to determine which stage the flow filed is in. The criteria we use to determine if it is a stable case is that, the $E_{z,k}$ start to increase and saturate later on and the flow field has showed clearly the zombie turbulence. *Fig2.* shows a plot of a typical cases where we initialize with noise and it becomes unstable. At the very beginning we observe a sharp decrease of the vertical kinetic energy. This is due to the fact that the energy spectrum of the noise are at every wave number, and our hyperviscosity will damp out those energy at high wave numbers very quickly to avoid numerical





(b) Kinetic energy E_k spectrum against wave number

Figure 3.5: 3D random noise with Kolmogorov energy spectrum

blow-up problem. Then we will see the noise reaches to a stable period and then starts to increases due to ZVI.

Based on the discussion about the process of ZVI when initialized with Gaussian vortex and random noise, our criteria to determine whether ZVI was triggered or not is, first we initialized the flow field with Gaussian vortex, which allows us to quickly determine whether ZVI is triggered or not, by observing the flow field and the vertical kinetic energy. Once we observe ZVI is triggered, then we could conclude under such group of parameter, ZVI will be triggered without running simulations initialized noise. If ZVI was not observed with Gaussian vortex or Gaussian vortex is destroyed that it can not excite critical layers, simulations under the same group of parameters initialized with random noise will be run to check whether this is a stable case or not. If none of those simulations show no sign or patter of ZVI, for example, vertical kinetic energy decreases after long time simulations, we claim this is a stable case.

In summary, by implementing both Gaussian vortex and 3D random noise, the effect of initial conditions on ZVI has been eliminated. It is purely determined by the physical parameter whether ZVI will exit or not. In another words, it is the physical ingredients, the horizontal shear, the vertical stratification and the background that control the instability mechanism in the fluid system.

3.2 Parameters for ZVI

Physical parameters and dimensionless numbers

The goal of this paper is to present map of parameters, on which stable and unstable cases are marked. Before we talk about the parameter map, we need to consider how many control parameters in the physical system. There are viscosity, horizontal shear, stratification and rotation. We neglect the viscous effect by assuming the viscosity is too small that Regoes infinity. This leads us to a system with three physical parameters, horizontal shear σ , stratification N and background rotation f, all of which have the unit one over time. By dimensional analysis, three physical parameters lead to two dimensionless parameters, σ/f and N/f.

In the literature, σ/f is typically regarded as the Rossby number *Ro* since physically and mathematically it can be viewed as the ratio between the vorticity associated with the background shear velocity and the background rotation, which is the definition of Rossby number. Comparing with the instabilities in the Taylor-Couette system, it is analogous to the relative strain rate defined in Strato-Rotational Instability(SRI). It is the ratio between the horizontal shear and the background shear, which determines the horizontal shape of the zombie vortex by Equ(2.5).

On the other hand, N/f is the ratio between the stratification and rotation, which determines the vertical structure of the zombie vortex. It can be reviewed as the half of the inverse of the Froude number Fr = f/2N, which is the ratio between the angular velocity Ω and BruntVäisälä frequency N. As the readers may question, there are definitely other ways to represent the dimensionless number, for example, N/σ , f/σ or σ/N , f/N. The reasons we use rotation as the dimensionless parameter is, our original interest of this problem comes from the physics of the Protoplanetary Disk(PPD). Compared with shear and stratification, background rotation is considered to be more persistent in the system. Secondly, previous explorations of the instability in stratified shear flow with background rotation are also using the similar dimensionless numbers as we mentioned above.

Numerical parameters

After we determine the physical parameters existing in this problem, let us take a look at the numerical parameters, i.e. the computational domain size L_x, L_y, L_z , the use of hyper-viscosity and boundary damping.

Our numerical experiments are set up in the Shearing Sheet coordinates, which is a Galileo transformation of the inertial frame to the frame moving with the background shear. In the moving frame, the shear terms in the governing equations which breaks the autonomy of the governing equations disappears and we could use the periodic boundary conditions in the shearing frame which is computationally efficient comparing with one-direction non-periodic. One of the problem for using this coordinates is that by assuming periodicity, our computational domains may contaminated by its neighboring domain. For example, our critical layers are located on cross-stream y direction. Once ZVI is triggered, duplicated critical layers will be excited on the cross stream direction and keep expanding on that direction. If our domain size is too small, the critical layers excited by the neighboring domain may show up in our computational domains. As we have discussed by linear normal mode analysis, the position of critical layers are determined by Equ(2.2). If we rewrite the

equation with the replacement of stream-wise wave number $k_x = \frac{2\pi m}{L_x}$ where m is integer. We have the distance of the critical layers of wave number m is

$$y^* = \pm \frac{NL_x}{2\pi\sigma m} \tag{3.6}$$

As we can see, the critical layers are straight lines on the cross-stream direction \hat{y} and the critical layer with the smallest wave number m = 1 is the farthest from the center of the perturbation. We call this distance $\Delta = \frac{NL_x}{2\pi\sigma}$. Since zombie vortex will be generated on the critical layers, in order to observe the generation of zombie vortex, we need to make sure our domain size on the cross-size direction L_y are big enough to hold the critical layer of the new generations, i.e. $2\Delta \leq L_y/2$. This provides us a restriction on our domain size

$$\frac{L_y}{L_x} \ge \frac{2}{\pi} \frac{N}{\sigma} \tag{3.7}$$

When we are exploring the parameter map with different value of N/σ , we have to adjust our domain size ratio accordingly. The problem when we use too small L_y are, in shearing coordinates, it is periodic on the cross-stream direction. When L_y is too small, we may observe the critical layers of the neighboring computational box appearing in our computational domain, which may interact with the critical layers of the computational box we are looking at. For vertical domain size, zombie vortex will show up on certain height. Since the zombie vortex will not duplicate on the vertical direction, we typically use $L_z = L_y$ in our simulations.

Since we have no dissipation in our problem, we have to use hyper-viscosity to avoid numerical blow-up with limited computational allocation points. The hyper-viscosity is tuned such that kinetic energy spectrum is damped only at high wave numbers. As we have discussed in the previous part, the key point of why critical layers roll up to vortex is that, the nonlinearity of the baroclinic critical layers provide the vorticity jump(vortex sheet). Under the effect of linear instability such Kelvin-Helmholtz instability, the vortex sheet rolls up to vortex. The existence of any type viscosity, including hyperviscosity, will lead to the dissipation of the critical layes. Compared with purely inviscid case, the viscous dissipation of critical layers by hyperviscosity will make ZVI more difficult to form. Thus if ZVI is observed with hyperviscostiv under a group of parameters, it will be absolutely observed in a purely inviscid case with enough resolution and the same group of parameters! Secondly, for stable cases, we calculated the kinetic energy loss due to hyperviscosity. The amount of energy loss due to hyperviscosity is on the order of 0.1% compared with total kinetic energy in perturbations such as the vortex and critical layers. The reason it is so small is because the hyperviscosity parameter is tuned such that it only damps the high wave number phenomenons. However in our fluid system, the main physics such as the zombie vortex and the critical layers are around mid wave number range. Thus the energy contained in the high wave numbers are very small. Even if those energy in high wave numbers are mostly damped out by hyperviscosity, the total energy loss is neglectable. Therefore the using of hyperviscosity has no effect on the physics of our fluid system.



Figure 3.6: Stability diagram of the Zombie Vortex Instability(ZVI) in $(N/f, \sigma/f)$ parameter plane. Symbol \bigcirc represents for ZVI unstable and \times represents for ZVI stable; \triangle means the marginal stable

In order to avoid the reflection of the internal waves from the upper and lower boundary, boundary damping is implemented on the vertical direction. The damping conditions are simple Rayleigh damping.

3.3 Inviscid Parameter Map

We present a parameter map, in terms of dimensionless numbers σ/f and N/f. As we have discussed in previous session, the left half part of the map where σ/f is negative, is anti-cyclonic while the right half part of the map where σ/f is positive, we call it cyclonic regime, depending on whether the vorticity of the background shear velocity have the same sign with the background rotation or not.

The map is divided into four parts, a linear unstable regime where $\sigma/f < -1$, the ZVI unstable regime on both anti-cyclonic and cyclonic regime, the marginal stable regime and the stable regime. We will explain in details the observations in those regimes and provide our explanations.

Linear Unstable Regime $\sigma/f < -1$

During our exploration of ZVI on the map, we also found the existence of linear instability area on the map, which is $\sigma/f < -1$. Since we are mainly interested in ZVI which is a finite-amplitude instability, we shall avoid the linear unstable regime on the map. In the following session, we shall prove the existence of the linear instability both analytically and numerically. Since we are not interested in the linear stability of the flow, we will not give a fully linear analysis of the flow field. Instead we show the existence of a linear growing mode under certain conditions by linear normal mode analysis. Secondly, we will numerically verify the linear eigen-mode and eigen-functions.

Analytic analysis We will show a special cases where $\sigma + f < -1$ and there exist linear growing mode. The special case is when the perturbation is independent of the stream-wise direction, i.e. $k_x = 0$.

If we linearize the equations and look at the normal modes of primary variables, i.e. $q(t, x, y, z) = \tilde{q}(y)e^{-i\omega t}e^{ik_x x + ik_z z}$. For the special case $k_x = 0$ with the viscosity neglected. The linearized normal mode equations are

$$0 = \frac{d\tilde{u}_y}{dy} + ik_z\tilde{u}_z$$
$$-i\omega\tilde{u}_x = (f+\sigma)\tilde{u}_y$$
$$-i\omega\tilde{u}_y = -\frac{dP}{dy} - f\tilde{u}_x$$
$$-i\omega\tilde{u}_z = -ik_z\tilde{P} - \frac{\tilde{\rho}g}{\rho_0}$$
$$-i\omega\tilde{\rho} = \frac{N^2}{g}\tilde{u}_z$$

The boundary conditions are non-slip boundary on the cross-stream direction. One could get the second-order ordinary differential equation by simplifying the equations above,

$$(\omega^2 - N^2) \frac{d^2 \tilde{u}_y}{dy^2} = k_z^2 \left[\omega^2 - f(\sigma + f) \right] \tilde{u}(y)$$
(3.8)

Using the technique by [12], d/dy is approximately ik_y in the spectrum space, we will have the dispersion relation

$$\omega^2 = \frac{k_z^2}{k_y^2 + k_z^2} f(\sigma + f) + \frac{k_y^2}{k_y^2 + k_z^2} N^2$$
(3.9)

If we further assume that $k_z >> k_y$, the frequency $\omega \simeq f(\sigma + f)$. Since the rotation $f = 2\Omega$ is always positive, when $\sigma + f < 0$. there exists the linear growing mode $\omega = -i\sqrt{|f(\sigma + f)|}$



Figure 3.7: numerical eigenvalue ω plotting on complex space with $k_x = 0, k_z \gg k_y$

in figure 3.7, We have verified the analytic work using numerical eigen-value solvers to look for the growing modes for stratified rotational shear flow, with the assumption that $k_x = 0$ and $k_z >> k_y$. In order to be consistent with our numerical simulations for the parameter map, the eigenvalue calculation is computed in a box with Lx = Ly = Lz = 8, with periodic boundary conditions on stream-wise(\hat{x}) and vertical direction (\hat{z}). Chebyshev polynomials are applied on the cross-stream direction(\hat{y}) for the non-slip boundary conditions. The computational grid size is $128 \times 257 \times 128$. The plot of frequency ω is shown for $\sigma + f =$ -2, N = f = 2. Since this is the text for special case, $kx = 0, kz = 2\pi m/L_x$ and we use m = Nz/4 for the assumption of big k_z .

We could see clearly the existence of the linearly unstable modes in the regime $\sigma + f < 0$. A more rigorous argument is that $\sigma/f = -1$ separate the map from linearly stable and unstable, i.e. $\sigma/f < -1$ is linearly unstable while $\sigma/f > -1$ is linearly stable. The interpretation of $\sigma/f = -1$ is that it is the Rayleigh criteria [22] in corresponding Cartesian system. It can be easily proved by corresponding parameter transformations from the cylindrical coordinates to the Cartesian system in Appendix. B

ZVI unstable regime

For the area where $\sigma + f \ge 0$, it is all linearly stable. On our parameter map, linearly stable area is divided into three parts, ZVI unstable, marginal stable and stable. We will explain each area one by one. First let us focus on the unstable area.

ZVI in cyclonic regime

The first thing to notice for the unstable area is that, although ZVI was first discovered for the Keplerian shear case in the anti-cyclonic regime where $\sigma/f \leq 0$, it has also been found in the cyclonic regime when the background rotation and the vorticity associated with the background shear has the same sign. Similar to anti-cyclonic cases, they can be triggered by either Gaussian vortex, where the Moore-Saffman relation still satisfies, or by 3D random noise with Kolmogorov spectrum.

As we have shown in the previous session, the process for ZVI in the anti-cyclonic regime has clearly five stages, when triggered by vortex. When initialized with random noise, small anti-cyclonic vortex seeds will be generated by the flow field and thus trigger the ZVI process for each of the vortex seeds. Not surprisingly, similar patterns for cyclonic cases are observed. When initialized with cyclonic Gaussian vortex, we observed the same five stages: at first, the baroclinic critical layers are excited for the vertical velocity and vorticity, and their locations can still be predicted with Equ.(2.2). Secondly, vortex sheet structures are formed by the critical layers. Under the secondary linear stability such as the Kelvin-Helmholtz instability, vortex are spawned on the vortex sheet. Under the effect of horizontal shear and background rotation, the shape of the vortex is elliptical and turn out to be cyclonic instead of anti-cyclonic. On the cross-stream direction, those vortex are observed to be located at the critical layer positions since they are spawn in the vortex sheet which is formed by the critical layers. On the vertical direction, they have showed up at certain heights. The next stage is the self-replication of the cyclonic zombie vortex. New critical layers are excited by the zombie vortex and newly excited critical layers spawn next-generation zombie vortex until the whole flow field are filled up with zombie turbulence. Similar to the anti-cyclonic case, those five stages can be seen at the plot of vertical kinetic energy plot correspondingly.

When initialized with 3d random noise, cyclonic vortex seeds are generated by the flow field. Those cyclonic vortex seeds will excite their own critical layers until the zombie turbulence state is reached. Zombie turbulence for cyclonic cases share the similar patterns as the anti-cyclonic part, that there exist anti-cyclonic strips which separates the turbulence into regimes. The distance between the separations are around the critical layer positions.

The second observation of the ZVI unstable regime is that, there is a clear bound for the unstable regime for anti-cyclonic and cyclonic cases. For anti-cyclonic regime, the unstable area is constrained between $-1 \leq \sigma/f \leq -0.5$ horizontally and $0.5 \leq N/f \leq 2$ vertically. The unstable regime is roughly centered around the Keplerian case where $\sigma/f =$ -3/4, N/f = 1. On the other hand, for cyclonic regime, there is a clear lower bound on the horizontal direction that $\sigma/f \geq 0.5$. For the stratification on general, the cyclonic requires higher N/f compared with the anti-cyclonic cases. There also exists an universal lower bound for the stratification that $N/f \geq 1$. However, for cyclonic cases, we did not find a upper bound for shear and stratification. Instead, when the background shear and the stratification increases on the same order, i.e. $\sigma/f O(1)$, ZVI is still observed with strong shear and deeply stratified case such as $\sigma/f = 3$ and N/f = 4. The unstable area for cyclonic area can be roughly described to be between $1 \leq N/f \leq 4$. Despite the fact $1 \leq N/f \leq 4$ looks



Figure 3.8: Vertical vorticity of zombie turbulence on horizontal plan $N/f = 4, \sigma/f = 2$ observed above the middle plane $z/L_x = 0.5$; Red color represents cyclonic(positive sign) while blue color represents anti-cyclonic(negative sign)

like the Richardson number, we want to point out that our shear is horizontal shear rather than vertical shear, our ZVI unstable regime for cyclonic regime that N/σ is between 1 and 4 shares no similarity with the vertically stratified and vertical shear flow.

One interpretation of why ZVI is not observed for strong stratification and strong shear case for anti-cyclonic case is because the existence of the linear instability in the anti-cyclonic regime. One might argue the existence of a boundary for cyclonic cases when σ/f and N/fkeep increasing while maintaining $N/\sigma \sim O(1)$. Since the extreme case for such scenario is the vertically stratified fluid with horizontal shear, but no background rotation. ZVI can not be triggered without the background rotation. Due to limited resources, how far the ZVI cyclonic regime can go when increasing σ and N is beyond the scope of this paper.

moderate stratification effect

Although cyclonic and anti-cyclonic regime have different boundary for unstable regime, they do share some similarities, in respect of the effects of physical ingredients. In the following session, we will discuss about some of the trend we observed for stratification and shear separately.

First let us focus on the effect of the stratification, i.e. we fix the ration between shear and rotation. On the map, we could imagine we are moving upward and downward corre-



Figure 3.9: Log-plot of Vertical kinetic energy changes versus time for three cases, $\sigma/f = -0.75$, N/f = 0.5, 0.75, 1.0; All of which initialized as Ro = -1 Gaussian vortex

spondingly. When σ/f is fixed, ZVI is observed in a limited regime of $0.5 \leq N/f \leq 2$ for anti-cyclones and $1 \leq N/f \leq 4$ for cyclones. When stratification increases from the lower bound, it takes less time for the flow field to become unstable and the energy transported from the background shear to fluctuations seems to increases, when the background shear and the initial condition remains unchanged. Here we observe the trend by examining the vertical kinetic energy for various N/f. Notice that when BruntVäisälä frequency increases, the location of the baroclinic critical layers will get further and further from the center of the domain. We only report the cases where the domain sized unchanged the duplication of m = 1 critical layers stay inside the domain.

From 3.9, we could see that N/f increases, the vertical kinetic energy increases earlier and faster, indicating that the earlier formation of zombie vortex, and the faster duplication of the critical layers. For the case N/f = 0.75, the first zombie vortex is observed around $700 \ 1/f$ time and the flow filed becomes fully zombie turbulence around $2700 \ 1/f$. While for N/f = 1 case, the zmobie vortex is observed around $400 \ 1/f$ time and the zombie turbulence formed around $1800 \ 1/f$ time. The physical interpretation why moderate stratification will promote the instability comes from the energy analysis of ZVI. As we have discussed above, when the instability happens, the energy is transformed from the background shear to the perturbations, and later from the perturbations to the potential energy. As the BruntVäisälä frequency is associated with the potential energy, the bigger N is, the stronger momentum transportation there are on the vertical direction.

small shear effect

When we fix the ratio between stratification and rotation and decreases we could focus on the physics when shear is getting small. We set up several numerical experiments for various shears while maintaining the same stratification and background rotation. First, ZVI is still observed when $|\sigma/f|$ is as low as 0.5 for both cyclonic and anti-cyclonic cases. The zombiefication process is the same as Keplerian shear case $\sigma/f = -0.75$: The baroclinic critical layers will first appear, then we observe the spawn of zombie vortex within the critical layers. The magnitude of the first generation of zombie vortex are on the same order. The shape of the zombie vortex is elliptical with the long axis of the vortex lying on the streamwise direction \hat{x} . The zombie vortex tend to have smaller horizontal aspect ratio when shear is smaller, which is reasonable that it is the horizontal shear that stretches the vortex and results into the elliptical shape of the vortex and the decreased stretching effect leads to the bigger horizontal aspect ratio.

The second observation we have for small shear effect is the time evolution of ZVI. As we could see from the two plots of zombie vortex, despite of the similarity between the shape and magnitude of the zombie vortex, the time it takes for the appearance of the vortex are dramatically different, with the Keplearian shear case takes around 7201/f time while a slightly smaller shear case $\sigma/f = -0.65$ takes around 4 times longer. We have observed the dramatically slowing down of the ZVI process when shear is decreasing, until the case where $\sigma/f = -0.5$, ZVI stops. Figure 3.10 shows the kinetic energy E_k varies with σ/f . As we can see all of these cases saturates after a long time and E_k decreases for small shear, which indicates the ZVI getting weaker with all theses cases are initialized with Gaussian vortex of Ro = -1. During the period where ZVI happens, the growth rate of kinetic energy are positively correlated with the magnitude of shear, from where we could conclude that stronger shear will promote the ZVI process.

As for figure 3.11, the zombiefied cases all share the similar trend that the $E_{k,z}$ is zero at the first beginning because we have no initial vertical velocity. With the appearance of the baroclinic critical layers of the initial vortex, the vertical kinetic energy starts to increase gradually until the ZVI triggers. During the self-replication period of the zombie vortex, critical layers of several generations of zombie vortex are excited, and correspondingly, we see rapidly increases of the vertical kinetic energy. Until the late time that the whole flow field are filled up with zombie vortex, the flow becomes zombie turbulence, vertical kinetic energy start to saturate, as there is no more zombie vortex and critical layers created. Thus, we could define the time it takes to ZVI as the time from the beginning of the simulation where there is no vertical structure, to the time where the flow field is filled up with zombie turbulence ($E_{k,z}$ stop increasing and starts to saturates). Someone may argue why we don't use the total kinetic energy, instead of the vertical kinetic energy to measure the time ZVI takes, the reasons are as following, first, the initial E_k for different shear cases are different,



Figure 3.10: Log-plot of total kinetic energy E_k changes versus time for four cases, N/f = 1, $\sigma/f = -0.75, -0.7, -0.65, -0.5$, all of which initialized as Ro = -1 Gaussian vortex

due to the fact that we are using different aspect ratio of the gaussian vortex following Equ(3.1) such that the initial Ro = -1 for all the cases. Secondly, those vortex are not equilibrium solution of the flow field, we cannot predict the behavior of the vortex and thus we cannot conclude the early time phenomena we observe are due to ZVI or the self-adjusting of the vortex. However, the vertical kinetic energy represents the vertical structure of the flow field which are basicly the critical layers excited by the initial vortex and zombie vortex, which is independent of the initial vortex. Thus it is a pure indication of the self-adjusting of the vortex. Based on our argument, we summarize the time ZVI takes for each case in the table 3.1/

Marginal stable state

For the case where $|\sigma/f| = 0.5$, ZVI does not happen. We call the very small shear cases as marginal stable state, by which we mean, the whole physical status when shera is small is almost static. What we observe is that, the baroclinic critical layers will show up and so are the vortex layers. However, the vortex layers do not roll up to any zombie vortex, instead, the vortex layer saturate and the change of the magnitude of the vortex layers are



Figure 3.11: Log-plot of vertical kinetic energy $E_{k,z}$ changes versus time for four cases, $N/f = 1, \sigma/f = -0.75, -0.7, -0.65, -0.5$, all of which initialized as Ro = -1 Gaussian vortex

σ/f	N/f	time to ZVI $(1/f)$
-0.75	1.0	2000
-0.70	1.0	3360
-0.65	1.0	5520
-0.50	1.0	> 7920

Table 3.1: Zombiefication time versus shear

negelatable. We have run our simulation until 10000 1/f time, which is 5 time longer than the time Keplearian shear case will take to become unstable, we still did not observe the rolling up of vortex on the critical layers. As we could see clearly in 3.11 and 3.10, the total kinetic energy as well as the vertical kinetic energy saturate at early stage (around 500 1/f time). Both of the quantities remain unchanged. Secondly, the marginal stable state is observed for various values of N/f = 0.5, 1, 1.5, 2, 3. For small shear cases, the increasing of stratification does not seem to accelerate the ZVI process and we did not observe the rolling up of zombie vortex for all the cases. Thus we claim $\sigma/f = -0.5$ is the lower bound for ZVI in anti-cyclone cases.

The reason why the instability is getting slower is due to the fact that the energy resources of ZVI comes from the background shear. We have figured out that during the ZVI process, the energy is transported from the background shear to the fluctuations, and then from the fluctuations to the potential energy. In order to measure the amount of energy transported from the back ground shear, We define a new quantity called "shear energy" E_s , as a measurement of how much energy is drained from the background shear to the flow field.

$$E_s = \int \rho_0 \underline{U} \cdot \underline{u} dV \tag{3.10}$$

As we could see, for Keplearian Shear case $\sigma/f = -3/4$, the energy is transported from the background shear to fluctuations and when ZVI happens, the rate of shear energy transportation increases monotonically. For the cases of small shear, the energy transportation is quickly slowed down. For marginal case $\sigma/f = -0.5$, there is a constant energy transportation from the background to our flow field as a constant rate. This explains the saturation of the instability.

Internal wave and critical layers

Beyond the similarities that anti-cyclonic and cyclonic cases shared, we want to point out one differences we observed for cyclonic cases are the shape of critical layers. The critical layers for the anti-cyclonic cases, when plotted on the cross-stream and vertical direction, are appear to be "curved" in such a way that around the middle plan, it is not straight line. The reason is due to the interaction of the internal gravity waves and the critical layers around the middle plane. However for all the numerical simulations we have for cyclonic cases, regardless of stability, we did not observe the existence of the such waves. The shape of critical layers appear to be straight lines on the cross-stream and vertical plane.

Internal gravity waves in stratified rotational flow has been discovered and verified both analytically and numerically. With the effect of horizontal shear, the governing equations are not autonomous in cross-stream wise direction y. This problem can be solved using the technique by [12], that there exists a simple relationship between the absolute frequency of the internal waves ω and the relative frequency ω_0 ,



Figure 3.12: Shear Energy versus time, N/f = 1, $\sigma/f = -0.75, -0.7, -0.65, -0.5$, all of which initialized as Ro = -1 Gaussian vortex



(a) Vertical velocity at 64 1/f time, for $\sigma/f =$ (b) Vertical velocity at 64 1/f time, for $\sigma/f = -0.9, N/f = 1.5$, plot on the cross-stream and 1.5, N/f = 2, plot on the cross-stream and vertical(x = 0) plan vertical(x = 0) plan

Figure 3.13: Shape of critical layers in anti-cyclonic and cyclonic cases

$$\omega = \omega_0 + \underline{k} \cdot \underline{U} \tag{3.11}$$

 ω is the frequency occurring at fixed point in the space, which is the frequency we observe while ω_0 is the frequency of the inertial wave when we move with the background flow, which is the linear shear k is the wave number vector. We could apply the WKB approximation by assuming that the system rotation $f = 2\Omega$ and stratification are strong are strong compared with the mean flow (see [20]). We also assume that for the local wave number vector k, the stream-wise wave number k_x is one order of smaller than the wave number on cross-stream and vertical direction k_y, k_z . i.e. $k_x \ll k_y, k_z$. The intrinsic frequency ω_0 can be given by following, which is the same with the local frequency when $k_x = 0$.

$$\omega_0^2 = \frac{N^2 k_y^2}{k_y^2 + k_z^2} + \frac{f(f+\sigma)k_z^2}{k_y^2 + k_z^2}$$
(3.12)

Since k_y and k_z are local wave numbers, on a physical y - z plane where x is fixed, for any location y and z on the wave, we could say $\tan^2 \theta = \frac{k_y^2}{k_z^2}$, where θ represent the angle between the tangent line of the wave trace and the horizontal direction y.

$$\left(\frac{dz}{dy}\right)^2 \approx \tan^2\theta \approx \frac{\omega_0^2 - f(f+\sigma)}{N^2 - \omega_0^2} \tag{3.13}$$

Meanwhile, in our numerical simulations, we find that the local temporal frequency of the wave $\omega = 0$, by putting a numerical probe into the wave we observe. Using this numerical observation, we could easily rewrite the following differential equation

$$\left(\frac{dz}{dy}\right)^2 = \frac{(k_x \sigma y)^2 - f(f + \sigma)}{N^2 - (k_x \sigma y)^2}$$
(3.14)

Integrate the equation above will provide us a solution for the trace of the internal wave. It matches very well with our numerical observation, as shown below 3.14. Notice that this wave trace is valid under the assumption that the streamwise wave number is much smaller than the cross-stream and vertical direction. This assumption is cruel since the wave tracing without such assumption actually matches poorly with the numerical simulations.

The second observation we have for the internal wave is, although the temporal frequency of the waves are all zero, the spatial wave number of those waves are nonzero and it is the same as the critical layers it approaches. For example, in the Fig3.14, the red doted lines are critical layer positions determined by Equ(2.2). They are $k_x = 1$ and $k_x = 2$ critical layers from outside to the inside. We clearly see two wave approaching each critical layer. The spatial wave numbers of such wave are also $k_x = 1, 2$ correspondingly, which can be numerically verified. We can say that, for baroclinic critical layers, which is linearly neural stable, and associated with a certain wave number, there exists internal waves which has no temporal frequency and share the same spatial frequency with the critical layers. However, for the cyclonic cases, we have never observed the appearance of the interaction of the wave



 y/L_x

Figure 3.14: Vertical vorticity for the case $\sigma/f = -3/4 N/f = 1.5$; Red line represents the critical layer positions; Blue line represents the internal wave path; yellow circle represents the location where zombie vortex is observed;

and the critical layers. The structure of the critical layers are straight lines on cross-stream and vertical plane, as predicted by the linear normal mode analysis.

At last, the analysis of the wave also provides us some insight to the question we come up with earlier that why zombie vortex, spawn at different wave number critical layers, prefer different height. As shown in the Fig3.14, the yellow circles are the locations where zombie vorticies are observed. They are also the locations were the waves and critical layers meets. This coincidence are valid for critical layers with different wave numbers. Whether the zombie vortex is generated by the interaction of the critical layers and the internal waves or there is no physical interaction between these two remains an open question.

ZVI stable regime

In this session, we will discuss about the regime where ZVI is not observed. At the first beginning, it is easy to understand that when one or more of the three physical ingredients equals to zero, ZVI will not be triggered. When the horizontal shear vanished, our flow filed becomes stratified rotation flow. In this kind of flow, one of the common observations is Taylor column vortex predicted by the Taylor-Proundman theorem. Secondly, the excitement of baroclinic critical layers requires the existence of the horizontal shear. The same argument works for stratification too. Without density stratification, the flow becomes rotational shear flow. There is no baroclinic critical layers reported, let along the observation of Zombie Vortex Instability. As we have discussed in the previous session, lack of background rotation will lead to no ZVI also.

As you may have noticed, ZVI is only observed when all three of the physical ingredients, stratification, rotation and shear remains on the same order. The lack of any physical ingredients or one of the physical ingredients dominant will lead to stability on the parameter map. Such result can be interpreted from the perspective of critical layers and the long existence of the zombie vortex.

As we have observed, the continuing process of ZVI greatly replies on the self-replication of the zombie vortex. Even with 3d random noise as initial condition, small vorticies are still observed and considered as the seeds to trigger ZVI. Thus it is very important to make sure the zombie vortex can be generated and maintained for a while in order to excite new critical layers. If vortex is not generated in the flow field, or the vortex is generated but cannot last long enough to excite the critical layers, following generations will not be spawn. So we could say that the zombie vortex has to be quasi-equilibrium state such that it can last long enough to excite critical layers. Although the equilibrium state of vortex in such flow has not been found yet. We could still interpret the effect of the physical ingredients in simplified cases. As we know, if we neglect the vertical structure of the flow, the shape of the vortex on the horizontal direction can be determined by the ratio of shear and rotation in Equ(2.5). Thus the ratio between shear and rotation is to maintain the horizontal structure of the zombie vortex. When the shear becomes dominate, the vortex has to be very strong, in terms of Ro according to Equ(2.5) to survive such strong shear. Indeed, in our numerical simulations, for cases where $|\sigma| \gg N$, f, the vortex will be stretched and no barocilinic critical layers are excited.

Similar argument can be made on the vertical direction, when we consider the stratificaion and rotation. For the cases with strong stratification $N \gg \sigma$, f, when initializing with gaussian vortex, the initial Gaussian vortex could be easily split into smaller, shorter vortex vertically. The critical layers will be generated and the zombie vortex can also be observed. However, there are strong momentum dynamics on the background that the zombie vortex will be destroyed due to the strong vertical stratification effect. This numerical results can be easily interpreted by the asymptotic analysis in the paper [13], we could see that the magnitude of zombie vortex are in the order of $O((f/N)^2)$. If f/N is small, while σ/N remains order of unity, the vertical momentum equation is simply the hydrostatic balance at the leading order, which gives no change to generate the zombie vortex. While f/N gets big, the background stratification will be totally dominated by the dynamics of $O(f^2/N^2)$, which is far away from the vortex equilibrium on the vertical direction. Therefore the zombie vortex will be last very long, even if it is generated on the critical layers. The asymptotic results agrees with our numerical simulations.

Discussion and suggestions for experiments

Based on the parameter map, we would recommend the parameters for the experiments of observation of ZVI in the laboratory would be in the cyclonic regime. There are three reasons we suggests this. The first is because cyclonic regime is always linearly stable. Since our numerical simulations are using shearing sheet coordinates and we assume periodicity in the system, while in the real experiments with real boundary conditions, people claims the existence of StratoRotatinal Instability(SRI) in several literatures [26] in the anti-cyclonic regime $-1 \leq \sigma/f \leq -0.5$, although the theoretical analysis assumes the small wave number k_x and big stratification N/f. However, all the literatures have claimed SRI does not exist in cyclonic regime, either in Taylor-Couette or Cartesian system. Thus, if we observe any instability in anti-cyclonic regime, it shall be ZVI. Secondly, the parameters regime for cyclonic regime is much bigger than the anti-cyclonic regime. Since the parameter regime for anti-cyclonic regime is bounded on four directions. This will allow the experimentalists more freedom and space to build up the facilities and pick up the parameters for the experiments. Thirdly, we observed that for cyclonic cases, among the unstable cases, the bigger the shear and stratification is, the faster the ZVI is. Since on general, ZVI will show up in quite late time after we put in the initial condition, the faster to observe, the better it will be for lab experiments. The only problem that we may have for big shear and stratification cases are that it requires strong initial condition, i.e. Ro in our numerical simulations.

Some may argue that beyond SRI, there are other kinds of instabilities exists on our parameter map, such as the gravity-wave-like instabilities. There are mainly two differences with our work with them. The first is we don't have any boundaries on the cross-stream direction, thus all the instabilities excited by the coupling of gravity-like-waves are not applicable to our simulations and therefore not observed. This allows us to mainly focus on the ZVI without the interaction of any other mechanism. The second difference is, ZVI exist in moderate N/f and σ/f while the gravity-wave-like instabilities require rapid rotation and strong stratification σ/f small and N/f big.

Summary of chapter

In summary, ZVI can be triggered either by Gaussian vortex or 3d random noise. When triggered by vortex, there are clearly five stages for ZVI process, i.e. the excitement of critical layers and vortex sheet, the spawn of the zombie vortex on the vortex sheet due to linear instability, and self-replication of the zombie vortex. ZVI could be observed for both anti-cyclonic cases and cyclonic cases. The process for ZVI in both case are similar except that the anti-cyclonic(cyclonic) cases will excite anti-cyclonic(cyclonic) zombie vortex. A parameter map in terms of σ/f and N/f have been provided for the existence of ZVI. Linear instability has been observed and verified both analytically and numerically for the area where $\sigma + f \leq 0$. A linear growing mode has under the special case $k_x = 0$ has been provided to support our argument of linear instability. ZVI can be observed in the area where all three physical ingredients are on the same order. If one or more of the three ingredients is missing, or one of the physical ingredient becomes dominant, ZVI will be not triggered. An interpretation of such observation has been provided in terms of the vortex equilibrium in such flow. In the are of ZVI, the effects of stratification and shear has been discussed. A marginal stable state is found to be around $|\sigma/f| \sim 0.5$. In the marginal stable state, critical layers have been observed to be static, whose amplitude remains unchanged for more than five times longer than the typical time to observe ZVI. The reason of the fluctuations. The differences between the shape of the critical layers in the anti-cyclonic and cyclonic cases have been observed and explained.

One of the main reason for constructing the parameter map is to provide quantitative guidance for experimentalists to set up experiment and observe the Zombie Vortex Instability in the laboratory. The area we recommend for experiments are the cyclonic regime where sigma/f and N/f are rather large, since the cyclonic regime has bigger unstable area than the anti-cyclonic cases and there are no other kind of instabilities, such as SRI exist in the cyclonic regime.

3.4 Appendix: Taylor-Couette system and plan Couette system

For stratified rotational shear flow, we have three physical directions, stream-wise, crossstream and span-wise. They are $\hat{\theta}, \hat{r}, \hat{z}$ in Taylor-Couette system(TCS) and $\hat{x}, \hat{y}, \hat{z}$ in Plane Couette system(PCS) correspondingly. Notice that our stream-wise direction in PCS is \hat{x} which is different from the notation in SRI papers and our PRL paper on Zombie Vortex Instability(ZVI) where \hat{y} is the stream-wise direction.

In TCS, we have inner and outer cylinders with radius R_1, R_2 and angular velocity Ω_1, Ω_2 , and rotation ratio $\mu = \Omega_2/\Omega_1$ and the radius ratio $\eta = R_1/R_2$. The background flow is $\underline{u} = (0, r\Omega(r), 0)$ with angular velocity given by [23]

$$\Omega(r) = \Omega_1(\frac{A}{r^2} + B) \quad where \quad A = R_1^2(\frac{1-\mu}{1-\eta^2}) \quad B = \frac{\mu - \eta^2}{1-\eta^2}$$
(3.15)

In Yavneh[26] and Normand[19], they define a dimensionless number called relative strain rate S as

$$S = \frac{\bar{r}\Omega'(\bar{r})}{2\Omega(\bar{r})} = \frac{-A}{A + B\bar{r}^2}, \quad where \quad \bar{r} = (R_1 + R_2)/2 \tag{3.16}$$

The physical meaning of S is the strain rate on the cross-stream wise direction divided by the local rotational effects at the mean radius, which is the shear over rotation. Plug in the formula of A and B, we have

$$-\frac{1}{S} = 1 + \left(\frac{1+\eta}{2\eta}\right)^2 \left(\frac{\mu-\eta^2}{1-\mu}\right)$$
(3.17)

Also notice that, the cyclonicity of the flow is determined by the sign of $(\Omega^2)'$ at $r = \bar{r}$. We say the flow is cyclonic(anti-cyclonic) at $r = \bar{r}$, if the absolute value of the angular velocity Ω is increasing(decreasing) function of r at $r = \bar{r}$, i.e. $d\Omega/dr > 0 (< 0)$ is cyclonic(anti-cyclonic), with the assumption that $\Omega_1 > 0$.

In PCS, we have the background shear velocity as $\underline{u} = (-\sigma y, 0, 0)$ and the stratification represented by the Brunt-uäsälä frequency N and the background rotation $f = 2\Omega$. The dimensionless numbers in PCS are σ/f , N/f and Re for viscous flow.

Notice that the vorticity associated with the background shear is $! = \nabla \times \mathbf{u} = \sigma \hat{z}$. The flow is cyclonic(anti-cyclonic) when $\sigma > 0 (< 0)$.

Since relative strain rate S is the ratio between the local shear and rotation, and people define the Froude number as $Fr = \Omega(\bar{r})/N$, we have the following relation,

$$\frac{\sigma}{f} = S, \quad \frac{N}{f} = \frac{1}{2Fr} \tag{3.18}$$

In most of the reported cases in TCS, people are interested in Fr = 0.5 which is the Keplerian shear case. As we can see, it is the same as our PRL paper case that N/f = 1 in the accretion disk. We shall fix our Fr = 0.5, N/f = 1 in our following discussion and mainly focus on the σ/f .

The famous Rayleigh Criterion suggests that a sufficient and necessary condition for inviscid, axisymmetric perturbed circular Couette flow to be *unstable* is $d(r^2\Omega)^2/dr < 0$ which is equivalent to $\mu < \eta^2$ [22][8]. With equ(3), it is easy to see that

$$\mu < \eta^2 \Leftrightarrow -\frac{1}{S} < 1 \Leftrightarrow \frac{\sigma}{f} < -1 \tag{3.19}$$

Chapter 4

Viscous Effects on Zombie Vortex Instability

4.1 Introduction

In recent years, there have been a prosperity on the exploration of the instability in the protoplanetary disk. Several new instability mechanisms have been discovered and analyzed in theory, numerically as well as experimentally. More specifically, pure hydrodynamic instabilities such as StratoRotational Instability(SRI) [9] [26] and Radial instability [11] have been reported. Those instabilities are linear instabilities and believed to be formed by the resonate of the Kelvin waves trapped by the boundaries. Meanwhile, in the fluid system that includes horizontal shear, background rotation and vertical stratification, a new kind of finite amplitude instability, called the Zombie Vortex Instability (ZVI) has also been reported [15]. This finite amplitude instability involves the formation of baroclinic critical layers [5] and the discontinuity of the vertical vorticity, which will bring in a secondary linear instability and creates new generation of vortex. ZVI has been found to exist in a variety range of parameters. While for SRI which only exits in the cases where the vorticity associated with the background shear has the opposite sign of the background rotation. ZVI exits in a wider range than those linear instabilities. Also since SRI requires the existence of the boundary to trap the kelvin waves. It has never been found in systems that has no boundary [19]. ZVI has no dependency on the boundary and can be excited by a variety of initial conditions.

ZVI has been discovered in the plan couette flow with vertical stratification and background rotation. However, one important question remaining about the instability is that, how does the real viscosity effect on the instability? Among the questions about the viscous effects, the most interesting question we want to ask is the existence of a critical Reynolds number Re_{cr} such that, for strong viscous dissipation ZVI does not happen, and with small viscosity, ZVI can be observed. This is very important for laboratory experiments in the observation of ZVI. In summary, the questions we are going to answer with this paper are, does viscosity has effect on ZVI? If the answer is yes, how does the viscosity affect ZVI and is there a dimensionless number such as critical Reynolds number Re_{cr} that determines whether ZVI will be triggered or not? In this paper, we will explore the viscous effect on the ZVI and try to answer those question mentioned.

4.2 Problem set-up

ZVI is exited in the system involves horizontal shear, background rotation and vertical density stratification. We define our fluid system in Cartesian x, y, z system, where x is the streamwise direction and y is the cross-stream direction and z is the vertical direction. The shear velocity lies on the horizontal plan and has a linear form of $\mathbf{U}(y) = -\sigma y \hat{x}$, while the system is rotating on the vertical direction z at a constant angular velocity Ω . We use the Coriolis parameter $f = 2\Omega$ to denote the background rotation. The fluid is also linearly stratified. The BruntVäisälä frequency $N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}(z)}{dz}$ is defined as a measurement of stratification, where g is the gravity, ρ_0 is the background density at the reference height and $\bar{\rho}(z)$ is the background stratification. If we separate the our total flow field as two parts, the background shear velocity and perturbation, i.e. $\mathbf{u_{total}} = \mathbf{U} + \mathbf{u}$. We also separate our density into two parts, the background linear stratification and the density fluctuations, i.e. $\rho(x, y, z, t) = \bar{\rho}(z) + \tilde{\rho}(x, y, z, t)$. We use the half size of the cross-stream direction y as our characteristic length, the velocity of the wall $\sigma L_y/2$ as characteristic velocity and the density at the middle plan ρ_0 as our characteristic density. After neglecting the thermal diffusion, the dimensionless governing equations for the perturbations can be easily derived as following,

$$\begin{array}{lll} 0 &=& \displaystyle \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \\ \\ \displaystyle \frac{\partial u_x}{\partial t} &=& \displaystyle -(\underline{u}\cdot\nabla)\,u_x - U\left(y\right)\frac{\partial u_x}{\partial x} - \frac{\partial P}{\partial x} + (f/\sigma+1)\,u_y + \frac{1}{Re}\nabla^2 u_x \\ \\ \displaystyle \frac{\partial u_y}{\partial t} &=& \displaystyle -(\underline{u}\cdot\nabla)\,u_y - U\left(y\right)\frac{\partial u_y}{\partial x} - \frac{\partial P}{\partial y} - u_x f/\sigma + \frac{1}{Re}\nabla^2 u_y \\ \\ \displaystyle \frac{\partial u_z}{\partial t} &=& \displaystyle -(\underline{u}\cdot\nabla)\,u_z - U\left(y\right)\frac{\partial u_z}{\partial x} - \frac{\partial P}{\partial z} - \frac{2\tilde{\rho}g}{L_y\sigma^2} + \frac{1}{Re}\nabla^2 u_z \\ \\ \quad \frac{\partial \tilde{\rho}}{\partial t} &=& \displaystyle -(\underline{u}\cdot\nabla)\,\tilde{\rho} - U\left(y\right)\frac{\partial \tilde{\rho}}{\partial x} + \frac{N^2}{g\sigma}u_z \end{array}$$

Here the dimensionless background shear velocity is U(y) = -1 and we define our Re as $Re = \sigma L_y^2/4\nu$. Notice that we are using the length of the cross-stream direction in our Re. In order to have a clear understanding of the viscous effect on the instability, we want to control our parameters in our problem. The parameters existing in our problems are listed as following,

a. Physical parameters: As we know, there are three physical ingredients existing in our fluid system, the horizontal shear, which can be represented as σ , the background rotation on the vertical direction, which can be denoted by the Coriolis parameter $f = 2\Omega$ with Ω

as the angular velocity of the system. We use BruntVäisälä frequency $N^2 = -\frac{g}{\rho_0} \frac{d\bar{\rho}(z)}{dz}$ as a measurement of stratification. These three physical parameters will lead to two dimensionless numbers i.e. σ/f , N/f. Based on the parameter map in chapter 3, ZVI has been observed with different parameters. We need to take into account the parameters changes when looking at the viscous effect on the instability.

b. Initial conditions: There are two points we need to consider when talking about initial conditions, the first is what type of initial conditions we are talking about. ZVI can be triggered with both Gaussian vortex and 3D random noise with Kolmogorov scaling. The second point we need to take into account is, with a specific initial condition, will the amplitude of the initial condition has any effect on the critical Re.

c. Numerical parameters: Since in our simulations, we used a little bit hyper-viscosity since our numerical resolution is not as resolved as Kolmogorove length scale. The readers might question whether this artificial damping within the physical viscous dissipation will misguide us or not. We will show, with evidence that the use of the hyper-viscosity will not change the physics at all.

As listed above, we tackle those problems one by one. In the first part of the paper, we will look for the critical Re with a specific group of parameters, and a specific initial conditions. In the second part, we change the amplitude of the initial condition and then the type of the initial condition, and see how the critical Re changes with the initial conditions. In the third part, after we have a clear understanding of how the critical Re changes with initial condition, we will look for the critical Re with different physical parameters, i.e. what is $Re_{critical}(\sigma/f, N/f)$. At the last, we need prove that the use of hyper-viscosity has no effect on the physics. Based on our discovery, at the last, we will talk about why we need a local Re which is defined on the critical layer, to determine the ZVI.

The numerical method we use is similar to chapter 3, where we transform the system into Shearing Sheet coordinates. The benefits of doing so is to keep the periodicity of the system and the price we pay for it is the time dependency of derivative operator. In viscous case, we have viscous terms, which can be easily transformed into a time dependent wave numbers. The numerical algorithm we designed in the inviscid case also works with this extra viscous terms. For details of the algorithm, please see [4].

4.3 Critical *Re* for Keplerian shear case

In this session, we will pick up a specific parameter case N/f and σ/f , with a varying viscosity. With only viscosity as the varying parameter, we shall have a clear understanding of how the viscosity affect ZVI. Since one of our main motivation for ZVI, is to understand the turbulence generating mechanism in the Proto-Planetary Disk (PPD). A group of physical parameters that simulates the PPD will be a natural choice. Although there is no well-defined Gaussian vortex in the PPD, we still uses the 3D Gaussian vortex as initial condition since it provides us the capacity to observe the whole process of ZVI in chapter 3. In order to control the variables in our experiments, we fix the initial Ro = -1 in our case, and only



Figure 4.1: Magnitude of critical layers versus time for various Re

changes the viscosity. Based on our set-up, any phenomenon we observed shall be due to the change of viscosity.

When we add a little viscosity to our experiment $Re = 10^7$, ZVI is still observed, i.e. the appearance of viscous dissipation does not kill the instability immediately. Similar to the inviscid case, five processes of ZVI could still be clearly observed. The only differences are the amplitude of the critical layers are damped and the time it takes to become unstable is delayed.

As we have expected, if we keep increasing the viscosity, the viscous dissipation is strong enough that the instability is killed. That critical Re we found, for the group of simulations is between 2.5×10^6 and 5×10^6 . For cases $Re = 2.5 \times 10^6$ or slightly less, we did observe the formation of critical layers and the critical layers roll up to vortex, However, the viscous dissipation is so strong that the new zombie vortex did not have a chance to excite their own critic layers. If we keep increasing the viscosity, we do not even have a chance to observe the formation of the vortex sheet. What we observe is that the critical layers are excited at very early stage and then quickly died away.

The question now is, how does the viscosity kill the instability? Our answer is the viscosity kills the instability by killing the critical layers. As we have discovered, ZVI requires the long lasting of the critical layers (typically excited by a vortex), such that the zombie vortex can be generated on the critical layer locations due to local linear instability. If the critical
layers are quickly dissipated then there is no chance that the zombie vortex will be generated by linear instability. Second, we know that in order to trigger ZVI, there is a threshold that the *Ro* of the initial condition must satisfy [16]. The vertical vorticity on the critical layers is provided by the discontinuity of the critical layer structure on the horizontal direction. Thus, when there is a strong viscous damping on the critical layers, the vertical vorticity will also be damped, such that, even if the zombie vorticity is generated, their amplitude is not big enough to trigger ZVI.

From Fig.4.1, we are plotting the amplitude of $k_x = 1$ critical layers at the certain height against time. We could see clearly that, for those cases where ZVI is triggered, the amplitude of ZVI also plateaued out, saying that it is self-sustained. While for those stable cases where the viscosity is big, critical layers are all damped out. The stronger the viscosity is, the faster it damped the critical layers. Considering zombie vortex is spawn on the critical layer due to linear instability, we could conclude that the killing of ZVI is strongly correlated with the killing of critical layers. Second, we know that for the viscous dissipation term in the equation is $\nu \nabla^2 u$. In the Fourier space, the Laplacian operator can be represented with the wave number square, i.e. the bigger the wave number is, the stronger the viscous damping rate will be on the phenomena associated with that wave number. Since the wave number is inverse proportional to the length scale. Thus the smaller the length scale of the physical phenomena is, the stronger the viscous dissipation will be on that length scale. In the ZVI process, the phenomena with the smallest length scale is the critical layer. The critical layer by itself, is a mathematical singularity. Singularity shall have infinite amplitude and infinitesimal thickness. In our flow field, the critical layer has finite amplitude and finite thickness is due the smoothing of both viscous effect and the nonlinear effect. When there is no viscous effect in the inviscid case, it is the nonlinearility that smooths the critical layers. Since the critical layers are the smallest length scale phenomena in our case, the energy dissipation on the critical layer shall be the highest. Indeed, in Fig.4.2, we plot the spectral viscous dissipation versus the wave number. We could clearly see there is a peak on the critical layer wave number k_{cl} . $k_{cl} = 2\pi/\delta_{cl}$ and δ_{cl} is the thickness of the critical layers which can be measured numerically or predicted analytically.

In summary, in this session, we proved that, for the Keplerian shear case where $\sigma/f = -3/4$, N/f = 1, when initialized with Ro = -1 3d Gaussian vortex, the critical $Re_{critical}$ is between 2.5×10^6 and 5×10^6 . With solid argument and analysis of the energy dissipation spectral, we showed that it is through the viscous dissipation on the critical layers that the viscosity kills the instability. This conclusion can also in return, explains that why the critical Re for ZVI is so high. Since the critical layer are smoothed singularities, they are typically very thin. Thus a fair amount of viscosity can have a very big viscous dissipation because the viscous dissipation rate is proportional to the negative two power of the thickness.



(a) Vertical viscous dissipation spectral against wave (b) $k_x = 1$ critical layer from the initial value calculanumber tion

Figure 4.2: Viscous dissipation(negative) maximize at the critical layer length scale

4.4 Effect of initial conditions

In the previous session, we found the critical Re for a specific initial condition. In this session, we would like to change the initial condition and see whether the critical Re_{cr} will change or not. Again, we would like to change one variable at a time while maintaining the other unchanged. Here the variable we change is the initial condition. For the physical parameters, we still use $\sigma/f = -3/4$ and N/f = 1. The initial conditions include two parts, the amplitude of the initial condition, and the type of initial conditions. We will talk about these two parts one by one.

First, let us focus on the amplitude of the initial condition. For Gaussian vortex, it shall be the Rossby number $Ro = \frac{\omega_z}{f}$, where ω_z is the vorticity of the vortex at the center. Previously, we use Ro = -1. Here we would like to modify the Ro and see how the critical Re would change.

As we could see from Fig.4.3, it requires a certain value of Ro will zombiefy. However, when Ro is big enough that ZVI is triggered. Increasing Ro did not help on lower the critical Re. The explanation for such observation are as following. First, as we have show in chapter 3, once ZVI is triggered, it has no memory of its initial condition and the ZVI will reach the statistically same zombie turbulence state. Second, in our case with viscosity, the key whether ZVI will be triggered or not depends on how strong the viscous dissipation on the critical layers or not. The amplitude of the critical layers are determined by the fluid systems, such as the shear, the wave number. But it is independent of the initial conditions. The main function of the initial condition there is to excite and maintain the critical layers. Thus the amplitude of the initial condition shall be little influence on the critical Re for ZVI.



Figure 4.3: Initial amplitude of the vortex and the Re; × represent stable cases and \circ represents unstable cases

Second, we would like to check whether different types of initial conditions might change the critical Re or not. So far there are three kinds of initial conditions we have used. They are a) 3D Gaussian vortex b) 3D random noise and c) Zombie turbulence. The advantages and disadvantages of a) and b) as initial conditions have been discussed in details in chapter 3 and [16]. Zombie turbulence is the late time turbulence state of the corresponding inviscid case. In this session, we are focusing on the Keplerian shear case. We use the zombie turbulence as initial condition for viscous cases and increase the viscosity until the state where the self-sustained turbulent state stops.

Before we run our simulations with different type of initial conditions, we would like to discuss about how to measure the amplitude of initial condition. Here we define the measurement of the strength of the initial condition as the corresponding Rossby number \tilde{Ro} , which has different form for different type of initial conditions. For 3D Gausssian vortex, we use the Ro of the Gaussian vortex at its origin (x = 0, y = 0, z = 0) as its corresponding Rossby number, i.e. $\tilde{Ro} = \frac{\omega_{z,origin}}{f}$ where $\omega_{z,origin}$ is the vertical vorticity at the origin. For 3D random noise, we could plot the vertical vorticity ω_z in terms of the wave number k as use the Rossby number at the smallest scale (resolution scale) as the indication of the strength, i.e. $\tilde{Ro} = \frac{\omega_z(k_{resolution})}{f}$ where $\omega_z(k_{resolution})$ is the root-mean-square of the vertical vorticity at the resolution wave number $k_{resolution}$. The reason for such definition is because the vertical vorticity at the highest wave number determines whether ZVI will be triggered or not for iniviscid case when initialized with random noise [16]; At last, for zombie turbulence, the

Initial condition	$ \tilde{Ro} $	Re	Zombiefy
Vortex	0.3	4×10^7	YES
	0.3	1×10^7	YES
	0.3	5×10^6	YES
	0.3	$4.75 imes 10^6$	YES
	0.3	$2.5 imes 10^6$	NO
	1.5	$4.75 imes 10^6$	YES
	1.5	$2.5 imes 10^6$	NO
Turbulence	2.836	5×10^6	YES
	2.836	$2.5 imes 10^6$	YES
	2.836	1×10^6	NO
	2.836	5×10^5	NO
	2.836	$2.5 imes 10^5$	NO
Noise	0.05	1×10^7	NO
	0.05	5×10^6	NO
	0.05	$2.5 imes 10^6$	NO
	0.05	1×10^6	NO
	0.2	1×10^7	YES
	0.2	1×10^{6}	NO

Table 4.1: For Keplerian shear case $\sigma/f = -0.75$, N/f = 2, summary of part of the numerical simulations with different types of initial conditions in terms of corresponding Rossby number \tilde{Ro} and Re

corresponding Rossby number is defined as the ratio between the L1 norm of the vertical vorticity and the background rotation, $\tilde{Ro} = \frac{|\omega_z|_{\infty}}{f}$. Although the zombie turbulence has the similar energy spectrum as 3D random noise, we do not use the same definition as the random noise because the zombie turbulence is generated from our inviscid simulations with hyperviscosity. So the high wave number is damped by the hyperviscosity.

Table 4.1 shows some of the simulations with different initial conditions. We could see clearly from the table that for all three types of initial conditions, the critical Reynolds number $Re_{critical}$ fall between $1 \times 10^6 < Re_{critical} < 5 \times 10^6$. In another word, different types of initial conditions do not change the critical Re. When initialized with noise or zombie turbulence, if it is still unstable, the physical process and observations are exactly the same as the inviscid cases. The only differences are the amplitude are smaller than inviscid case.

In summary, in this session, we have proved that the critical Re is independent of the types of initial conditions and the amplitude of initial conditions. This conclusion agrees with our observation in the inviscid case that once triggered, ZVI e has no memory of the initial conditions. One intuitive way to explain this observation is that the critical Re is determined by the nonlinear effects of system that brings in the stability and the viscous effect that dissipates the energy. As we have shown in the first paper, once ZVI is triggered,



Figure 4.4: Four groups of parameters on the parameter map that we explored on the critical Re

it has no memory of the initial conditions and it reaches statistically the same turbulence state. Thus we could say that the nonlinear effect that creates and promotes the instability is independent of the initial condition. On the other hand, as we have discussed in the previous session, the dissipation is mainly focused on the critical layers. Critical layers can be excited by different initial conditions and the thickness of critical layers are determined by the parameters of the system, instead of the strength of the initial conditions. From this point of view, we could say that the viscous dissipation is also independent of the initial conditions. Thus the critical Re is determined by the balance of nonlinear and viscous effect, shall be independent of the initial condition naturally.

4.5 Viscous effect for Non-Keplerian case

After we have a clear understanding of how the viscous effect act on the instability and the Re_{cr} is independent of the initial condition, let us chance the physical parameters and see whether modifying the physical parameters will help in decreasing the Re_{cr} or not. Due to limited resources and time, we pick up three other case: another anti-cyclonic case where $\sigma/f = -3/4$, N/f = 2 and two cyclonic cases $\sigma/f = 2$, N/f = 4 and $\sigma/f = 3/4$, N/f = 2.

The reason why we pick up those cases are as following. Since the motivation for this research starts with the accretion disk. Thus a thorough exploration on the Keplerian shear case is necessary and natural. Beyond the Keplarian case, we have worked out the inviscid parameter map in part 1 of this paper indicating the stable and unstable regime of ZVI. For

σ/f	N/f	Initial condition	$ \tilde{Ro} $	Re	Zombiefy
-0.75	2	Vortex	0.3	5×10^6	YES
			0.3	1×10^{6}	NO
		Noise	0.2	5×10^6	YES
			0.2	1×10^6	NO
0.75	2	Vortex	2	5×10^7	YES
			2	$7.5 imes 10^6$	YES
			2	5×10^6	NO
			2	1×10^6	NO
		Turbulence	2.545	5×10^6	YES
				1×10^6	NO
2	4	Vortex	4	1×10^7	YES
			4	$5 imes 10^6$	YES
			4	1×10^6	NO

Table 4.2: Numerical simulations for non-Keplerian shear cases with different initial conditions

anti-cyclonic cases, the unstable regime is bounded in four directions. We also know that changing the shear in the anti-cyclonic case will make ZVI more difficult due to the fact that big shear in anti-cyclonic cases will lead to linear instability and small shear will greatly decrease the energy transported from the backgroud shear to the fluctuations that slows down the whole process. Thus we have to keep the shear for anti-cyclonic case the unchanged. For the stratification over rotation, we know that within certain range, the stronger the stratification is, the faster we will observe ZVI. Thus this parameter $\sigma/f = -0.75$, N/f = 2 will come in handy. As for the cyclonic cases, we have shown that the bigger the shear and the stratification are, the better it will promote ZVI. Thus we pick up two cases, one case that shear and stratification are as close to the anti-cyclonic case $\sigma/f = 0.75$, N/f = 2 and the other case that shear and stratification as big as possible, i.e. $\sigma/f = 2$, N/f = 4.

As we have shown in the previous session, that the critical Re is independent of the type and amplitude of initial conditions. Thus with limited resources, there is no need for us to try all kinds of initial conditions and amplitudes. We only need to check that if the critical Re for those three cases will be on one order of smaller than the critical Re for Keplerian shear case which is on the order of 10^6 .

Similar to the Keplerian cases, when initialized with 3d Gaussian vortex, a small amount of viscosity will damp the amplitude of the critical layers as well as slowing down the ZVI process. When viscosity increases to a fair amount, the instability is not observed due to the high viscous dissipation of the critical layers. From our simulations, we conclude from the table that, $Re = 1 \times 10^6$ is a global lower bound for the cases we have explored.

The result that the critical Re_{cr} does not change for various initial conditions and the different parameters does not surprise us. As we have discovered on the Keplerian case, it

σ/f	N/f	$Re_{critical}$	
-0.75	1	(1×10^{6})	5×10^6]
-0.75	2	(1×10^{6})	5×10^{6}]
0.75	2	(5×10^{6})	7.5×10^6]
2	4	(1×10^{6})	5×10^{6}]

Table 4.3: Summary of critical Re for four cases

is due to the high viscous dissipation of the critical layers that viscosity is able to kill the instability. The viscous dissipation of the critical layers depends on the thickness of the critical layer and the wave number associated with it. The thickness of critical layers δ is proportional to $(k_x Re)^{-1/3}$, as demonstrated in [17] and [5]. Thus the change of stratification and background rotation N and f will have little effect on the dissipation rate. However, the thickness of critical layer does depend on the background shear σ , the smaller the shear is, the weaker the viscous dissipation will be. From our previous study, the region for ZVI to exist for various shear is not very large(see chapter 3). So there range of σ we could choose is quite limited. Second, considering the thickness is proportional to the -1/3 power of σ , a not dramatically different σ will not change that much of the thickness. As we have discussed, ZVI requires the long lasting existence of the critical layer and the whole stability also takes a fairly amount of time to become unstable. Together with the thickness of the critical layer, it is not surprising that we did not get much smaller Re for various physical parameters.

As we could show in this session, the critical Re for different parameters does not change either. Thus there is really no benefits we could get on the viscous part with different parameters. We will stick with our suggestions for experiments on the observation of ZVI on the cyclonic cases where σ/f and N/f are both big. As we have discussed in the last paper, there are several reasons why such area is the best for experiments. First the anti-cyclonic unstable regime is bounded while the anti-cyclonic regime is not. This will allow us a lots of freedom when designing the experiments. Second, in the anti-cyclonic regime, there exists other kinds of instabilities, such as SRI and linear instability. While for cyclonic regime, no any other kinds of instabilities in stratified rotational shear flow have been reported in such area. Third, we observe that in the cyclonic regime, the bigger the stratification and shear are, the faster the ZVI will be. Thus the cyclonic regime where σ/f and N/f are big is recommended for experiments.

4.6 Resolution of critical layers and the effect of hyper-viscosity

Since the first publication of ZVI, we have received feedback and questions, especially on the critical layers as well as the hyper-viscosity part. There are two main concerns on our work of

ZVI, the first concern is, since critical layers are singularities, how and could your numerical simulation handle the singularities? In another words, do you have enough resolutions for the critical layer. The second concern is, why do you use both hyper-viscosity and physical viscosity? Is it possible that ZVI is purely due to the effect of hyper-viscosity, thus it is an artificial phenomena, instead of real physical mechanism? We will address these two concerns one by one, with statements and evidence. Hopefully after you finish reading the following session, you will be convinced that we can resolve the critical layers and hyper-viscosity has no effect on the physics of ZVI.

First, let us talk about the critical layers. Indeed, critical layers are mathematically singularities. As we know, singularities, by definition, have infinite amplitude and infinitesimal thickness. The mathematical description of the singularity is the Dirac function, typically denoted as $\delta(x_0)$ where x_0 is the location of the singularity. However, living in the Newtonian mechanism world, the infinitely large and infinitesimally thin layer or point does not exist in our system. Thus there must exist some mechanism that smooth the singularity, by smoothing I mean, the amplitude is finite, and the singularity has a finite thickness. In the previous study on the critical layers, people have been able to show that it is either the nonlinearility or the viscosity that smoothed the critical layers. In another words, critical layers are singularities that are either smoothed by nonlinear effect, or viscous effect, or both, depending on the terms in the governing equations. When linear normal mode analysis is implemented, the nonlinear terms are ignored, thus we only have viscous effect. For inviscid initial value simulation where there are nonlinear terms, but no viscous terms, it is the nonlinear effect that smooths the critical layers. For our initial value simulation in this paper, we have both nonlinear effect and viscous effect, thus our critical layers are dual layer structure. In the following, we will discuss numerically for each scenario, which the critical layers shall look like and how we are able to resolve them. Notice that when we talk about critical layers in the following part, we mean the baroclinic critical layers we discovered in the stratified rotational flow with background shear. For barotropic critical layers excited purely by the background shear, please see [17].

Linear eigen-value calculation

As we have described previously, critical layers are singularities smoothed by nonlinear effect and viscous effect. We are able to identify them by using numerical tools, i.e. linear eigenvalue calculation or direct numerical simulations(DNS). In this session, we will focus on the linear eigen-value calculation. Without the nonlinear effect, the viscosity will be the factor to smooth the critical layers.

The equations for linear normal mode analysis are very straightforward. With the periodicity assumption on the streamwise direction \hat{x} and vertical direction \hat{y} , as well as the linear assumption that nonlinear effects are one order of small compared with the background flow, the mathematical formula for the fluid quantity can be expressed as $A(t, x, y, z) = \tilde{A}(y)e^{ik_x(x-ct)+ik_z z}$. We calculate the eigen-value c and its corresponding eigenvectors and check the vertical velocity \tilde{u}_z .



Figure 4.5: Inviscid eigen-vector showing the critical layer structure on vertical velocity; $c = -0.2216, k_x L_x = 64\pi, k_z L_x = 2\pi, \sigma/f = -1.5; N/f = 1.0;$ There are 256 collocation points on cross-stream \hat{y} direction.

Linearized inviscid case

In the purely inviscid case, a linear normal mode analysis will bring us all the linear eigenmodes with the nonlinear effect and the viscous effect eliminated. Thus, our critical layers will not be smoothed out by any physical effect, thus it shall be infinitesimally thin. Indeed, out linear eigen-value and eigen-vectors do show the existence of the critical layers at the location and the critical layers always shows three numerical grid points, no matter what numerical resolution we use and what parameters are for the system. In Fig.4.5, we could clearly see that the appearance of critical layers and their width are infinitesimally small, i.e. there are only three grid points inside the critical layers. Also since it is inviscid case, the eigenvalue c is a purely real number, which tells us the critical layers for inviscid case are neutrally stable.

Linearized viscous case

Secondly, let us add the viscosity to our linear system. Based on our previous discussion, we know that the bigger the bigger the viscosity, the thicker the critical layers will be. The concerns come from our critical Re for ZVI is on the order of 10^6 thus our critical layer might be too thin and we can not resolve it. In Fig.4.6, we could see clearly the critical layer structures, there are around 10 to 15 grind points inside the critical layers. Secondly, since there are viscosity in our flow field, the eigen-values are all decaying modes, i.e. the imaginary part of eigenvalue c are all negative. We pick up a specific eigenvalue to show the critical layer structure, since the critical layer locations are determined also by the eigenvalue c, so we may not be able to observe all the critical layers inside our domain.



Figure 4.6: Viscous eigen-vector showing the critical layer structure on vertical velocity; $c = 5.90 + 0.026i, Re = 5 \times 10^6, k_x L_x = 2\pi, k_z L_x = 2\pi, \sigma/f = -1.5; N/f = 1.0;$ There are 1024 collocation points on cross-stream \hat{y} direction.

Linearized viscous case with hyperviscosity

Thirdly, we could add the hyperviscosity term in our linear eigen-value calculation, since we know the analytic formula of such term in known to us. We would like to use exactly the same parameters as the viscous linear eigen-value calculation, with the hyperviscosity term. The parameters for hyperviscosity is exactly the same as those we use in our initial value calculation. There are mainly two questions we would like to investigate in such numerical experiment: first, could we found exactly the same eigenvalue and eigenvectors in the calculation with hyperviscosity; Second, by adding the hyperviscosity, does the eigenvector changes, i.e. does the thickness of the critical layer change. If the hyperviscosity will physically change the dynamics of the system, then we would expect the answers to the last two questions are both no. If there is no effect from the hyperviscosity term, we would expect to observe the same eigenvalues shows up and the thickness of critical layers does not change at all

Fig. 4.7 shows you the result of the calculation. We use the same parameters as in last session, the viscous linear eigenvalue calculation and the hyperviscosity exactly the same as it is in the initial value calculation. The eigen-values and eigenvectors are calculated. As shown in the figure, we are able to find the exactly the same eigenvalues with those has no hyperviscosity terms and it is easy to verify that the thickness of the critical layers do not change and the change of the magnitude is negelectable.



Figure 4.7: Viscous eigen-vector with hyperviscosity showing the critical layer structure on vertical velocity; c = 5.90 + 0.026i, $Re = 5 \times 10^6$, $k_x L_x = 2\pi$, $k_z L_x = 2\pi$, $\sigma/f = -1.5$; N/f = 1.0; There are 1024 collocation points on cross-stream \hat{y} direction.

the effect of hyper-viscosity in initial value simulation

For our initial value simulation, viscous or inviscid, there are nonlinear terms. The plots of critical layers can be found in Fig.4.2(b) for viscous case and chapter 3 for inviscid case. With 256 grid points on the cross-stream direction, there are around 10 points at least inside the critical layers, which tells us that are very well resolved. Now let us talk about hyperviscosity in the initial value simulations. The key question for hyperviscosity in our simulation is to make sure that hyperviscosity does no have physical effects on our simulations. A numerical simulation without any use of hyperviscosity shall be perfect for our case. However, The numerical resolution for Kolmogorov scale is too big to be computationally feasible in our case. With limited resources, hyperviscosity or some kind of modeling is essential in the simulations. We use hyperviscosity to damp the energy in the high wave numbers. The hyperviscosity has the formula $k_v \nabla^p \underline{u}$ and the hyperdiffusivity has the formula $k_d \nabla^p \tilde{\rho}$. k_v, k_d and the power p are parameters that we could tune for our calculation. Typically we use p = 8 or 16. We believe the hyperviscosity has no physical effect on our simulation and our argument and evidence are as following.

Hyperviscosity are damping on small scale (large wave number) phenomena

With the formula of our hyperviscosity $k_v \nabla^p \underline{u}$, define wave number $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$, it is easy to notice that the hyperdissipation term is proportional to k^p , i.e. the bigger the wave number is, the bigger the hyper-diffusion will be. We also know that the wave number k is proportional to 1/L where L is the length scale of the physical phenomena. Thus we could



Figure 4.8: Kinetic energy spectrum for case $Re = 1 \times 10^6$, $\sigma/f = -1.5$, N/f = 1 at 721/f time

conclude that the smaller the length scale associated with the physical phenomena is in our system, the bigger the hyperdiffusion will be.

Then the next question we shall answer is, what the wave number k_{hyper} that hyperviscosity has effect on? This question can be easily answered by plotting the kinetic energy spectrum against the wave number k. Fig. 4.8 shows such a plot, for the case where $\sigma/f - -1.5$, N/f = 1 and $Re = 10^6$. As we could see clearly, the kinetic energy spectrum has roughly Kolmogorov scale than wave number is smaller than the wave number hyperdiffusivity has effect on. In this case, it is $k < k_{hpyer}$. For wave number that are higher, the kinetic energy is quickly damped.

Thus we need to know that, in our numerical simulations, what are the small length scale phenomenon and whether that small scale phenomenon fall into the regime of hyperviscosity or not. As we have discussed clearly in the previous session, before Zombie turbulence formed, the smallest scale physical phenomena are the critical layer and the viscous dissipation peaks on the critical layer length scale. Then it is easy for us to check whether the length scale of critical layers fall into the hyperviscosity effective regime or not. If the length scale of critical layers are much bigger than the length scale that the hyperviscosity affects, it is a strong evidence that the hyperviscosity will not change the physics on the length scale of critical layers. This could also be easily verified by the kinetic energy spectrum. We have defined k_{cl} as $2\pi/\delta_{cl}$ where δ_{cl} is the thickness of the critical layers. In Fig.4.8, we could see clearly that the wave number associated with critical layers $k_{cl} = 22$ are much smaller than the wave number that hyperviscosity is acting on $k_{hyper} = 70$. Since critical layers are the physical phenomena of the smallest length scales and it is the viscous dissipation of critical layers that kills ZVI, the length scale that hyperviscosity is acting on is less than one third of the thickness of critical layers. Thus we could conclude that, the hyperviscosity is not heavily damping the critical layers.

Hyperviscosity has little effect on the critical layers

Some of the readers may still question that, even if we have proved hyperviscosity is not heavily damping the critical layers, it is still damping the critical layers, since the hyperviscosity is damping on all the wave numbers and particularly on the phenomenon whose wave number is bigger than k_{hyper} . In this part we will show evidence that the damping on critical layer phenomena is negletable.

The second experiment we would like to do is to compare the results between two numerical simulations, one with hypervicosity (Case 1) and the other one without any sort of artificial dissipation (Case 2). When designing the numerical experiments, we guarantee that the only difference between those two cases is the use of hyperviscosity, with all other conditions the same. Although for Case 2, there is no artificial dissipation, the simulation still produces valid results since at the first beginning of the simulations, the smallest length scale in our simulation is the critical layers and we have shown in previous session that our resolution can well resolve the critical layers. The numerical blow-up only happens later on, when the kinetic energy cascade into the small scales where numerical grid can not resolve. Thus at the beginning of the simulation in Case 2, we could still observe the form of critical layers and thus we could measure the thickness as well as the amplitude of the critical layers and compare them with Case 1. This comparison shall give us a clear observation how much hyperviscosity actually damps the critical layers.

Fig.4.9 shows the comparison. First, we notice that the critical layers show up on the vertical velocity u_z for both cases at the expected location. The amplitude for Case 2, which has the artificial damping, is slightly smaller than the Case 1 which has purely viscous dissipation. If we use the amplitude of the critical layer, $u_z(y = -1)$ as a measurement of the strength of critical layers. The numerical values for two cases are $u_{z,clean}(y = -1) = -5.01 \times 10^{-3}$ and $u_{z,hyper}(y = -1) = -4.84 \times 10^{-3}$. The relative differences between these two cases is roughly 3%, which is fairly small. Second, we could see clearly that the thickness of critical layers are numerically the same. Since the thickness are determined mathematically by the balancing of the leading physical terms in the governing equations. For the case in Fig.4.9, it is the nonlinear effect dominant case. The hyperviscosity shall have little effects on the thickness of nonlinear critical layers is not trivial. It is difficult for us to compare the thickness for this case with the analytical solution.



Figure 4.9: Vertical velocity u_z shows the critical layer structure for two cases; $\sigma/f = -1.5$, N/f = 1, $Re = 1 \times 10^7$ at time 36 1/f y is scaled such that m = 1 critical layer located at $y = \pm 1$ and u_z is scaled based on $\sigma L_y/2$.

The third evidence that hyperviscosity has no effects on the critical layers is the amount energy dissipated by hypervisocosity is much less than the amount of energy dissipated by real viscosity. Based on our numerical simulations, the energy dissipation due to hyperviscosity is less than one percent of the energy due to the real viscous dissipation. This is not surprising to us as we have discussed, most of the physical phenomenon happens in the length scales that are much larger than the hyperviscosity scale. We also have shown in the previous session, that the hyperviscosity effect on the large scale phenomenon is very small. Thus the amount of energy dissipated due to hyperviscosity is two orders of smaller than the energy dissipated by the real viscosity. This shall be another strong evidence that hyperviscosity is small enough that it does not affect the physics in our system.

Hyperviscosity has no effect on the system

The fourth numerical evidence we have done is to figure out without hyperviscosity, what the flow field would behave like. In this numerical experiment, we initialize the flow field with zombie turbulence, generated by the simulations where the parameters are observed to be



(a) Energy spectrum of zombie turbulence, with hy- (b) Energy spectrum after hyperviscosity is turned off, perviscosity

Figure 4.10: For parameters $N/f = 1, \sigma/f = -1.5, Re = 5 \times 10^6$, it is unstable case; After we observe the zombie turbulence with hyperviscosity; We turn off the hyperviscosity and the simulation blows up; As it shows clearly, there is an energy cascade to the smallest scale which is not resolved without hyperviscosity that leads to the numerical blow-up

unstable. Zombie turbulence is last time state of the flow field when ZVI happens. Detailed discussions can be found in [16]. In this numerical experiment, we set up the numerical experiments with initial vortex and with hyperviscosity, when the flow filed reaches the zombie turbulent state, we turn off the hyperviscosity while continuing the simulation. We will expect the our code to be numerically unstable. Thus the numerical results shall blow up and we could plot the energy spectrum as a function of wave number. If our resolution is not big enough to resolve the thickness of the critical layers, then the numerical blow up shall occur at the length scale of the thickness of the critical layer. On the other hand, if the critical layers are well-resolved, the numerical blow up shall be due to the cascade of the energy to the smallest scales and we shall expect a curl-up on the smallest length scale on the energy spectrum. Fig. 4.10 shows the energy spectrum of this numerical experiment. As we have expected, the energy spectrum curl up at the largest wave number and nothing unusual happens at the critical layer length scale.

The fifth evidence we would like to show that hyperviscosity has no effect on the physics is that, we could double the resolution in our numerical simulations and check if there is any change in our simulation results. Since the hyperviscosity is the only nonphysical term in the simulation, if the results with different resolutions show similar pattern, we could conclude that the hyperviscosity does not change the physics in the system. Since critical layer plays a very important role in ZVI and is deeply connected with the hyperviscosity, we would like to compare the critical layer structure of two runs with same parameters and different



Figure 4.11: $N/f = 1, \sigma/f = -1.5, Re = 4.75 \times 10^6$ at 160 1/f time; Vertical velocity u_z shows the critical layer structure with two different resolutions 256³ and 512³; y is rescaled such that m = 1 critical layer located at $y = \pm 1$; Notice that only half of the points of 512³ case are plotted in the figure

resolutions. If there is any effect caused by the hyperviscosity and the lack of resolution of the critical layers, we shall notice the differences on the results. Fig.4.11 shows the result.

Last but not the least, we would like to point out that the effects of hyperviscosity is damping the critical layers. Suppose we have enough resolutions that can resolve the small scales without any hyperviscosity, the total dissipation on the critical layers without any hyperviscosity shall be less than the cases where there exists hyperviscosity. Thus for cases with hyperviscosity, we observe it zombies. Then for the real cases where there is no hyperviscosity, it shall be more likely to zombiefy since the dissipation on critical layers are less.

In summary, in this session, we carefully examine from several aspects of the potential effect of hyperviscosity. We showed that first hyperviscosity heavily damps the physics with length scale that are much smaller than the critical layer length scale. The total amount of energy dissipated by the hyperviscosity is two order of smaller than the real viscous dissipation. The critical layers are well resolved by our numerical grid and without hyperviscosity, the code will blow up due to the energy cascade. Thus we conclude that

hyperviscosity causes no physical effect on our simulations.

4.7 Conclusions and future work

In this paper, we explored the viscous effects and how the viscous effect acting on ZVI. We first start with the Keplerian shear case where $\sigma/f = -1.5$ and N/f = 2. With a small amount of viscosity, the critical layers start to be damped due to the thin layer structure. Under one specific initial condition, we increase the viscosity while maintains other effects unchanged, a critical Re_d is found to be around 2.5×10^6 . We also find that it is due to the high viscous dissipation of the critical layers that the instability is killed. Secondly, we investigated whether the critical Re will vary with initial condition or not. There are two aspects we discussed, the amplitude of the initial condition and the different types of initial conditions. We find that as long as the amplitude of the initial conditions are big enough to excite the instability, increasing the amplitude of the initial condition does not help on decreasing the critical Re. Different type of initial conditions such as 3d random noise or Zombie turbulence have been explored and we found the critical Re remains at the order of 10^6 . Next, we verified the critical Re for cases with different physical parameters. We do not find any specific group of physical parameters that may significantly decrease the critical *Re.* The reason is because the thickness of critical layers barely varies with the change of parameters, when we have to maintain the order of shear, stratification and rotation to excite ZVI. At last, we discussed the effect of the hyperviscosity and the thickness of critical layers. We have shown that in both linear eigenvalue calculation and the initial value simulations, critical layers can be easily resolved. The effect of hyperviscosity is also discussed. The evidence why the hyperviscosity has not effect on the critical layers have been explained.

As we have discussed, the thickness of the critical layers determines the viscous dissipation rate on the thin layer structure, thus determined the critical Re where ZVI will happen or not. However, the definition we have on Re is a global one which depends on the length scale on the cross-stream direction \hat{y} , instead of the thickness of the critical layers. Meanwhile, it can be easily proved that the thickness of critical layers will not change with the cross-stream direction length L_y . This leads to a natural idea what we need a Re depending on the critical layers to better describe the viscous effects on ZVI. To be more precisely, we need a local Rewhich depends on the amplitude and width of baroclinic critical layers, instead of the global Re people typically use, to determine whether the viscous effects will kill ZVI or not. There are several different kinds of ways to determine such local Re. One good candidates are from the classical work on the baroclinic critical layers. To obtain the exact solution on the form of such local Re on baroclinic critical layers requires careful asymptotic analysis on both the nonlinear effect as well as the viscous effect, which falls beyond the scope of this paper.

Chapter 5

Numerical Algorithm for Stratified Rotational Shear Flow

In this chapter, I will discuss about the numerical algorithm I use for the initial value simulations for stratified rotational shear flow, for both inviscid and viscous cases. First I will discuss about the numerical difficulties for this problem. Second, I will briefly go through the numerical algorithm for inviscid and viscous case with triply periodic boundary conditions in shearing sheet coordinates. Third, this chapter will mainly focus on the case where non-triply periodic boundary conditions are implemented using Chebyshev polynomials. We invented a new algorithm for such case. This algorithm is invented in collaboration with Nelson Chen¹.

5.1 Governing equations and numerical difficulties

In this section, a brief introduction to the equations that this code attempts to solve, as well as an assumption about the flow field that would be generated, will be presented. Specifically, this would be the Boussinesq equations in shear flow. Then an overview of work previously done that serves as a foundation to this project will be presented. Finally, our goals and problems this project wishes to address based on the limitations of prior work will be explored.

Boussinesq Equations with background shear, stratification, rotation and viscosity

As we have discussed in details in Chapter 2, the equations of interest in this project are the Boussinesq equations with a constant rotation, stratification, and viscosity. This approximation filters out sound-waves is used because we only care larger scale physics, while still allowing for small density variations. This is due to the vertical stratification in the flow. These approximate equations are usually very accurate and simplifies the mathematics and

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physics, making our computations more efficient. A strong application of these equations is the study of ZVI. The Boussinesq equations are as follows

$$\frac{\partial U_x}{\partial t} = -(\underline{\mathbf{U}} \cdot \nabla)U_x + \nu \nabla^2 U_x - \frac{1}{\rho_0} \frac{\partial P}{\partial x} + U_y f$$
(5.1)

$$\frac{\partial U_y}{\partial t} = -(\underline{\mathbf{U}} \cdot \nabla)U_y + \nu \nabla^2 U_y - \frac{1}{\rho_0} \frac{\partial P}{\partial y} - U_x f$$
(5.2)

$$\frac{\partial U_z}{\partial t} = -(\underline{\mathbf{U}} \cdot \nabla)U_z + \nu \nabla^2 U_z - \frac{1}{\rho_0} \frac{\partial P}{\partial z} - \frac{\tilde{\rho}}{\rho_0} g$$
(5.3)

$$\frac{\partial \tilde{\rho}}{\partial t} = -(\underline{\mathbf{U}} \cdot \nabla) \tilde{\rho} + \rho_0 \frac{N^2}{g} U_z \tag{5.4}$$

$$0 = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}$$
(5.5)

Where $\underline{\mathbf{U}}$ is the total velocity field that dependent on both time and spatial coordinates, and $\tilde{\rho}$ is the density perturbation. ∇P is the pressure gradient. These are the variables that we are trying to solve. f is the Coriolis parameter that is equal to two times the constant background rotation angular velocity Ω_0 . In the Boussinesq approximation, the density is allowed to have a small perturbation such that $\rho(x, y, z, t) = \bar{\rho}(z) + \tilde{\rho}(x, y, z, t)$, where $\bar{\rho}$ is the background density and $\tilde{\rho}$ is the small perturbation. Additionally, ρ_0 is a reference density at a reference height that we choose. For simplicity we choose the reference density, $\rho_0 = 1$. N is called the Brunt Väisälä frequency given by the equation $N = -\frac{g}{\rho_0} \frac{d\bar{\rho}(z)}{dz}$. In our code, we have a fixed N. Lastly ν is the kinematic viscosity.

Additionally, we are going to assume that the flow can be decomposed to a background linear shear flow (with shear rate σ) plus a finite amplitude perturbation. This assumption is true in many astrophysical flows where there are intense shear in the system, like the zonal flows on Jupiter where the east-west winds create a strong shear background.

$$\underline{\mathbf{U}}(x, y, z, t) = \underline{\overline{\mathbf{U}}}(y) + \underline{\mathbf{u}}(x, y, z, t)$$
(5.6)

$$\underline{\overline{\mathbf{U}}}(y) = -\sigma y \hat{x} \tag{5.7}$$

It can be shown that a linear shear flow is at equilibrium and is steady with time. After substituting equation (4.6) into the Boussinesq equations, we are left with equations consisting mostly of the perturbation and some leftover shear terms from the nonlinear advection term.

$$\frac{\partial u_x}{\partial t} = -(\underline{\mathbf{u}} \cdot \nabla)u_x + \nu \nabla^2 u_x - \frac{\partial P}{\partial x} + u_y(f+\sigma) + \sigma y \frac{\partial u_x}{\partial x}$$
(5.8)

$$\frac{\partial u_y}{\partial t} = -(\underline{\mathbf{u}} \cdot \nabla)u_y + \nu \nabla^2 u_x - \frac{\partial P}{\partial y} - u_x f + \sigma y \frac{\partial u_y}{\partial x}$$
(5.9)

$$\frac{\partial u_z}{\partial t} = -(\underline{\mathbf{u}} \cdot \nabla)u_z + \nu \nabla^2 u_x - \frac{\partial P}{\partial z} - \tilde{\rho}g + \sigma y \frac{\partial u_z}{\partial x}$$
(5.10)

$$\frac{\partial \tilde{\rho}}{\partial t} = -(\underline{\mathbf{u}} \cdot \nabla) \tilde{\rho} + \frac{N^2}{g} u_z + \sigma y \frac{\partial \tilde{\rho}}{\partial x}$$
(5.11)

$$0 = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$
(5.12)

There are mainly two numerical difficulties in those equations. The first is if we use the traditional method, the time step we have to use is very very small which is not numerically efficient at all. The physical reason why we have to use very small numerical algorithm can be explained as following. For this problem, there are four physical ingredients involved, there are shear, rotation, stratification and viscous effect. For each of those physical ingredients, there is a physical time scale associated with it. For example, the time scale for rotation is basically the period of the self-rotation of the system. The time scale associated with the stratification is 1/N and the time scale for the shear is $1/\sigma$. As we know, in order to avoid the stiffness of the system, when implementing the numerical algorithm, we have to make sure that the time step for the algorithm is small than all of the physical times scales, such that all the physics can be very resolved and simulated. That means the time step has to be less than the smallest time scale in the system. In cases where one physical time scale is much smaller than other physical time scales, the time step is extremely small which makes the simulations unaffordable. One example of such case is the simulations for oceanic vortex. The simulations for oceanic vortex can be regarded as a simplified version of our problem, where there are only stratification and rotation. There is no shear. For the ocean water, f/N is roughly 0.01, which is exactly the case that we mentioned, one of the physical time scale is very small. Thus in order to produce long-enough simulations, it is quite popular for researchers to use f/N = 0.1, instead of 0.01 in their simulations. This problem has been perfectly solved by using the semi-analytical method proposed by [4]. In our system, we have eve more than two physical time scales, thus the semi-analytical method comes in handy to remove the time step restrictions.

The second numerical difficulty of our problem come from the shear term. The reason is because the shear term $U(y) = -\sigma y$ breaks the autonomy of our equations. We can not use the triply periodic boundary conditions any more. There are two ways to solve this problem. The first is to use the shearing sheet coordinates. In order to enforce the periodicity, without loss of generality, we could transfer our coordinates to the shearing sheet coordinates. The shearing sheet coordinate is a Galileo transformation of our frame to a moving frame where the observer(origin of the system) is moving with the background shear velocity. One of the advantage of applying such system is that, in the shearing sheet coordinates, the terms

that break the autonomy of the equations will vanish and thus periodic boundary conditions are valid. Mathematically, the relationship between our shearing sheet system (x', y', z', t')and the original coordinates (x, y, z, t) is $(x', y', z', t') = (x + \sigma yt, y, z, t)$. Shearing sheet coordinates works well for both inviscid or viscous cases, since the viscous terms are trivial to calculate within triply boundary conditions. The Laplatian operator ∇^2 is a constant and can be put into the linear terms with the semi-analytic algorithm. The price we pay for the shearing sheet coordinates is the spatial derivative terms (or the wave numbers) are time dependent. The combination of semi-analytic algorithm with shearing sheet coordinates works great for our system. Details for such algorithm can be found in [4].

The second way to solve the problem with the shear term is to live with the non-periodic boundary conditions. Instead of using the fourier basis functions, we could use the Chebyshev polynomials as basis functions on the non-periodic direction. The problem for using the Chebyshev polynomials is how to deal with the viscous terms. Since now the Laplatian operator can not be expressed as a constant, we can not use the semi-analytic method for this term any more. We invented a new algorithm to solve such problem, by using the classical Crank-Nicholson method for viscous terms. Before we show the step by step method, we want to talk about the prior work in order to have a though understand of how our algorithm works.

5.2 Prior work

Here, we will highlight two main body of works that lay the foundation for our algorithm. The first is an overview of spectral methods. The second is the usage of spectral methods, specifically, Fourier basis in two directions and Chebyshev polynomials in third direction, to solve the Navier-Stokes Equation. The third is a semi-analytic algorithm applied to Euler's equations.

Spectral methods

The basic philosophy of spectral methods is that instead of discretizing differential equations by a set of collocation points, it is discretized to a summation of basis functions multiplied by their spectral coefficients. It can proven that the accuracy of finite-difference methods are proportional to $(1/N)^p$ where N is the number of grid points and p is a fixed constant. While spectral accuracy is proportional to $(1/N)^N$ [6]. Therefore, spectral methods are extremely accurate and needs much less degrees of freedom than finite-difference methods. It should noted that to compute nonlinear terms, many Fast-Fourier-transforms (FFTs) are needed to go back and forth between physical collocation space and spectral coefficient space, which is the bottleneck in parallel spectral simulations. However, the much fewer degrees of freedom make up for this efficiency difference. The velocities and pressure will be functions represented as a truncated summation of Fourier-Fourier-Chebyshev basis functions and their respective coefficients that are allowed to evolve in time.

$$\underline{\mathbf{u}}(x,y,z,t) = \sum_{l=-N_x}^{N_x} \sum_{m=-N_z}^{N_z} \sum_{n=0}^{N_y} \underline{\tilde{\mathbf{u}}}_{lmn}(t) e^{ik_x x} e^{ik_z z} T_n(y)$$

$$P(x, y, z, t) = \sum_{l=-N_x}^{N_x} \sum_{m=-N_z}^{N_z} \sum_{n=0}^{N_y} \tilde{P}_{lmn}(t) e^{ik_x x} e^{ik_z z} T_n(y)$$

Here l,m,n are integers that represents the index of $\underline{\tilde{\mathbf{u}}}_{lmn}(t)$ and $\tilde{P}_{lmn}(t)$, which are the spectral coefficients. $k_x = 2\pi l/L_x$ and $k_z = 2\pi m/L_z$ are the Fourier wave modes in the streamwise and vertical direction, where L_x and L_z are the lengths in the periodic directions. $T_n(y)$ are the Chebyshev polynomials in the cross-stream direction, where L_y is the length in this direction.

Furthermore, this algorithm is psuedo-spectral. Nonlinear terms are computed as products on collocation points instead of the convolution of spectral coefficients, which is computationally inefficient. FFTs are used to go between physical collocation space and spectral Fourier-Fourier-Chebyshev space. Additionally, it is also needed that some computations be done in Fourier-Fourier-Physical space, which will be referred to as mixed space.

It should be mentioned that under the Fourier-Fourier-Chebyshev transformation, the Fourier modes are decoupled, but the Chebyshev modes are all coupled. For this algorithm, the following formulation is solved for every Fourier mode separately. This also allows for high parallelization of the code for different processors to solve a different set of Fourier modes. One great advantage of using spectral methods is that taking derivatives become an algebraic process [6] and are usually much more accurate than using finite-difference or finite elements methods. For example,

$$\frac{\partial P}{\partial x} = \sum_{l=-N_x}^{N_x} \sum_{m=-N_z}^{N_z} \sum_{n=0}^{N_y} ik_x \tilde{P}_{lmn}(t) e^{ik_x x} e^{ik_z z} T_n(y)$$
$$\frac{\partial P}{\partial y} = \sum_{l=-N_x}^{N_x} \sum_{m=-N_z}^{N_z} e^{ik_x x} e^{ik_z z} \sum_{n=0}^{N_y} \tilde{P}_{lmn}(t) \frac{dT_n(y)}{dy}$$
$$\frac{\partial P}{\partial z} = \sum_{l=-N_x}^{N_x} \sum_{m=-N_z}^{N_z} \sum_{n=0}^{N_y} ik_z \tilde{P}_{lmn}(t) e^{ik_x x} e^{ik_z z} T_n(y)$$

 $\frac{dT_n}{dy}$ can be found in appendix B. The gradient, divergence, and laplacian spectral operators are built on these derivative definitions.

Solving Navier-Stokes equation with Chebyshev polynomials

The new algorithm was developed on based on an older algorithm to solve the incompressible Navier-Stokes equations (in cartesian coordinates) using Fourier basis in two directions, and Chebyshev polynomials in third direction [6]. Typically this is because the problems that are studied using this method are autonomous in two directions, but something would break this autonomy in the third direction. For example, in astrophysical flows, there may be a strong background shear that causes the flow to be non-periodic in one direction. Another example is in geophysical flows when there is strong stratification in the vertical direction.

It should also be noted that when using Chebyshev polynomials to solve boundary value problems, it is not possible to directly impose boundary conditions. Instead, the "tau" method is used, where small additional terms are added to the original differential equations to enforce constraints such as boundary conditions. These τ 's can be thought of as Green's functions that help enforce constraints in the problem. In this section, we will treat the x and z direction as the autonomous directions, and y as the Chebyshev direction.

$$\frac{\partial \mathbf{\underline{u}}}{\partial t} = -(\mathbf{\underline{u}} \cdot \nabla) \mathbf{\underline{u}} - \nabla P + \nu \nabla^2 \mathbf{\underline{u}} + (\tau_{x_1} \hat{x} + \tau_{y_1} \hat{y} + \tau_{z_1} \hat{z}) T_{N_{y-1}}(y)
+ (\tau_{x_2} \hat{x} + \tau_{y_2} \hat{y} + \tau_{z_2} \hat{z}) T_{N_y}(y)$$
(5.13)

Note that the τ 's only act on the highest two Chebyshev modes. The algorithm works almost entirely with the time-dependent spectral coefficients. The only time we compute in physical space is to compute the nonlinear advection term. However, we will keep the notation general to focus on the time-stepping algorithm.

This algorithm uses the traditional fractional method to treat different terms in the equation with different numerical time-stepping algorithms to optimize between efficiency and stability. Here, we treat the advection term with second order Adam-Bashforth, but we use Explicit-Euler as a starting step. In addition, we will add the current viscosity term in this step. This will serve as our first fractional step.

$$\underline{\mathbf{u}}^{n+\frac{1}{3}} = \underline{\mathbf{u}}^n - \left(\frac{3}{2}\Delta t(\underline{\mathbf{u}}^n \cdot \nabla)\underline{\mathbf{u}}^n - \frac{1}{2}\Delta t(\underline{\mathbf{u}}^{n-1} \cdot \nabla)\underline{\mathbf{u}}^{n-1}\right) + \frac{\nu\Delta t}{2}\underline{\mathbf{u}}^n \tag{5.14}$$

The pressure and τ 's are formally treated with Crank-Nicholson, but in this algorithm, by leveraging the fact that they are whatever they need to be to preserve the divergence-free condition and other constraints, they can be combined to terms that do not look like any specific time-stepping algorithms. As you will see, this is not the case in our semi-analytic algorithm, and we have to treat these terms with a Crank-Nicholson-like algorithm:

$$\underline{\mathbf{u}}^{n+\frac{2}{3}} = \underline{\mathbf{u}}^{n+\frac{1}{3}} + \frac{\Delta t}{2} (-\nabla P^{n+1} - \nabla P^{n} + ((\tau_{x_{1}}^{n} + \tau_{x_{1}}^{n+1})\hat{x} + (\tau_{y_{1}}^{n} + \tau_{y_{1}}^{n+1})\hat{y} + (\tau_{z_{1}}^{n} + \tau_{z_{1}}^{n+1})\hat{z})T_{N_{y-1}} + ((\tau_{x_{2}}^{n} + \tau_{x_{2}}^{n+1})\hat{x} + (\tau_{y_{2}}^{n} + \tau_{y_{2}}^{n+1})\hat{y} + (\tau_{z_{2}}^{n} + \tau_{z_{2}}^{n+1})\hat{z})T_{N_{y}})$$
(5.15)

Now combining the terms by defining the following quantities:

$$\Pi^{n+1} = \frac{\Delta t}{2} (P^{n+1} + P^n)$$
$$\hat{\tau}^{n+1} = \frac{\Delta t}{2} (\tau^{n+1} + \tau^n)$$

The second intermediate step equations are now:

$$\underline{\mathbf{u}}^{n+\frac{2}{3}} = \underline{\mathbf{u}}^{n+\frac{1}{3}} - \nabla \Pi^{n+1} + (\hat{\tau}_{x_1}^{n+1}\hat{x} + \hat{\tau}_{y_1}^{n+1}\hat{y} + \hat{\tau}_{z_1}^{n+1}\hat{z})T_{N_{y-1}} + (\hat{\tau}_{x_2}^{n+1}\hat{x} + \hat{\tau}_{y_2}^{n+1}\hat{y} + \hat{\tau}_{z_2}^{n+1}\hat{z})T_{N_y}$$
(5.16)

The last step is to calculate the new velocity from the implicit viscosity term. This is done by inverting the laplacian operator and applying boundary conditions. After inverting the laplacian operator, we place a bar on the new velocity field to denote that this velocity field is consistent except for the highest two Chebyshev modes because it was replaced by boundary conditions. A detailed explanation for this can be found in the appendix A. Using Taylor expansion, this algorithm is formally second-order. The last fractional step algorithm is

$$\underline{\mathbf{u}}^{n+1} = \underline{\mathbf{u}}^{n+\frac{2}{3}} + \frac{\nu \Delta t}{2} \nabla^2 \underline{\mathbf{u}}^{n+1}$$
(5.17)

$$\left(I - \frac{\nu \Delta t}{2} \nabla^2\right) \underline{\mathbf{u}}^{n+1} = \underline{\mathbf{u}}^{n+\frac{2}{3}} \tag{5.18}$$

$$\underline{\bar{\mathbf{u}}}^{n+1} = (I - \frac{\nu \Delta t}{2} \nabla^2)^{-1} \underline{\mathbf{u}}^{n+\frac{2}{3}}$$
(5.19)

To solve for the pressure and $\hat{\tau}^{n+1}$'s, we must use the divergence-free condition $\nabla \cdot \underline{\mathbf{u}}^{n+1} = 0$. Again, a detailed explanation is provided in the appendix A. A key aspect about this algorithm is the constraints used to compute the $\tau^{\hat{n}+1}$'s. These constraints are setting the highest two Chebyshev modes in the divergence to be zero and correcting the highest two Chebyshev modes of the velocity equation. τ_y 's and the pressure boundary conditions must be solved simultaneously, and then τ_x and τ_z can be solved after the implicit pressure gradient is computed.

$$0 = \nabla \cdot \underline{\mathbf{u}}^{n+1} \tag{5.20}$$

$$\underline{\mathbf{u}}^{n+1} = \underline{\bar{\mathbf{u}}}^{n+1} \tag{5.21}$$

There is a major problem with the constraints involving setting the highest two Chebyshev modes of the divergence equation to be zero. For a reasonable time-step size (typically

	N=32	N=64	N=128	N=256
$\nu \Delta t / L_y^2 = 2.5 * 10^{-4}$	10^{5}	10^{25}	10^{6}	10^{175}
$\nu\Delta t/L_y^2 = 2.5 * 10^{-5}$	10^{3}	10^{7}	10^{24}	10^{85}
$\nu\Delta t/L_y^2 = 2.5 * 10^{-6}$	10^{3}	10^{3}	10^{9}	10^{29}

Table 5.1: Condition Number for τ Matrix

 $\nu \Delta t/L_y^2 = 2.5 * 10^{-5}$), the algorithm cannot handle resolutions higher than 128 Chebyshev modes. When this algorithm was first developed, the state of the art computers could only handle up to 32 Chebyshev modes, much smaller than what we need now (greater than 256 modes). This problem occurs because the matrix equation to solve for $\hat{\tau}^{n+1}$'s become closer to singular as the resolution increases. This problem only occurs in the viscous version of the code, and is not present in the code for Euler's equation. It is the implicit viscous step that causes this singularity issue. One simple but inefficient way to fix this is to decrease the time-step size or increase the domain size. Another is to use a mixed-space algorithm. This issue can be seen in the condition number of the matrix as we vary the time-step size and resolution.

From table ??, it is clear that as one increases the spectral resolution (N number of modes), the condition number becomes exponentially worse. This causes the simulation to diverge very quickly. This problem is what needs to be solved in order to have a high-resolution Chebyshev viscous code.

Semi-analytic algorithm

In a paper published in the Journal of Computational Physics in 2006, J. Barranco presented a hydrodynamic code that uses spectral methods for the Anelastic Equations [4]. In particular, Barranco presented a semi-analytic method to treat the shear, Coriolis, and buoyancy terms. In this paper, the algorithm presented was:

$$\underline{\mathbf{u}}^{n+1} = e^{\underline{\pounds}\Delta t}\underline{\mathbf{u}}^n + \underline{\underline{\pounds}}^{-1} (e^{\underline{\pounds}\Delta t} - \underline{\underline{\mathbf{I}}}) (\frac{3}{2}\underline{\mathbf{N}}^n - \frac{1}{2}\underline{\mathbf{N}}^{n-1} - \nabla P^n) - \frac{\Delta t}{2}\nabla P^{n+1}$$
(5.22)

In this algorithm, \pounds is the linear operator that contains the coriolis and buoyancy term. $\underline{\mathbf{N}}$ is the nonlinear advection term. And ∇P is the pressure gradient. This algorithm is shown to be second order in the paper. This algorithm has also been applied to the inviscid Boussinesq equations to study ZVI in shear and stratified flows. The semi-analytic method will be discussed more in-depth later in the chapter.

5.3 Numerical method step by step

Based on the prior work presented in the previous subsections, several areas can use improvements. The biggest problem currently is the inability to use high resolution with any viscous Chebyshev code. We will present a method to stabilize the τ matrix in the following section. The second goal is to apply the semi-analytic algorithm to the viscous Boussinesq code to exactly solve the linear terms.

We have developed a three-dimensional hydrodynamic code using spectral methods to solve the Boussinesq equations, and to deal with the background shear, stratification, and rotation. This code is expanded using Fourier series as basis functions in the streamwise (x-direction) and vertical (z-direction) direction because these directions are assumed to be autonomous. The cross-stream direction (y-direction) has a background shear that breaks this autonomy, and thus a Chebyshev polynomial basis is used. The spectral coefficients associated with their respective spectral basis functions are allowed to evolve in time and are discretized using finite-difference methods [6]. The background forces are represented as linear terms in the differential equations and are treated with a semi-analytic method. This can be advantageous when any of these parameters become much larger than the others and ultimately become the bottleneck in how big of time-step we can take [7]. With the semianalytic treatment, these terms are exactly resolved so their bottlenecks have been removed. In addition, viscosity is included in this code and is treated with a semi-implicit method to avoid stability issues. However, the semi-implicit viscosity presents a different numerical instability when using high resolution with Chebyshev polynomials. The instability and a proposed solution will be presented in the body of this report. The nonlinear advection term is treated explicitly, and the pressure gradient is treated semi-implicitly.

In this section, a general overview of the numerical algorithm will be presented. Afterwards, the application of the numerical algorithm on our specific equations will be derived. As a reminder, When using a Fourier-Fourier-Chebyshev spectral discretization, we would compute the spectral coefficients for each Fourier mode separately because each Fourier mode is decoupled. The Chebyshev coefficients are all coupled so that we have to solve for them simultaneously. However, for this section, we will leave the equations in a general form to focus on conveying the time-stepping algorithm clearly. Majority of these equations would be applicable to both spectral methods or finite difference methods. The only exceptions are the way we discretized the shear term, we will talk about the divergence of velocity and the special treatment needed for a specific Fourier-Fourier mode in the following sections.

General numerical method

Any spatial and time dependent set of partial differential equation can be represented in the form:

$$\frac{\partial \underline{\mathbf{u}}}{\partial t} = \underline{\underline{\pounds}}(\underline{\mathbf{x}})\underline{\mathbf{u}} + \underline{\mathbf{N}}(\underline{\mathbf{x}}, t) + \underline{\mathbf{M}}(\underline{\mathbf{x}}, t)$$
(5.23)

Here, $\underline{\mathbf{u}}$ is the variable that needs to be solved. $\underline{\pounds}$ is the linear matrix operator in the differential equations that can be analytically solved simply. In this case, these would be the shear, Coriolis, and buoyancy terms. $\underline{\mathbf{N}}$ is the nonlinear term. In this case, this is the advection term. Finally, $\underline{\mathbf{M}}$ is the linear terms that cannot be analytically solved in a simple manner, and requires some numerical approximation. In this case, this would be the pressure gradient and viscosity terms.

Assuming that the linear operator $\underline{\pounds}$ is only a function of the spatial coordinates, the analytic solution to this differential equation would be [7]

$$\underline{\mathbf{u}}(\underline{\mathbf{x}},t) = e^{\underline{\pounds}t}\underline{\mathbf{u}}_0 + \int_{t_0}^t e^{\underline{\pounds}(t-s)}N(\underline{\mathbf{u}},s)ds + \int_{t_0}^t e^{\underline{\pounds}(t-s)}M(\underline{\mathbf{u}},s)ds$$
(5.24)

The discrete, but still analytic, version of this is:

$$\underline{\mathbf{u}}^{n+1} = e^{\underline{\pounds}\Delta t} \underline{\mathbf{u}}_n + \int_0^{\Delta t} e^{\underline{\pounds}(\Delta t - s)} N(\underline{\mathbf{u}}_n, s) ds + \int_0^{\Delta t} e^{\underline{\pounds}(\Delta t - s)} M(\underline{\mathbf{u}}_n, s) ds$$
(5.25)

As one can see, the linear terms in $\underline{\pounds}$ is solved exactly. The next step is to approximate the two integrals in the above equation. For the nonlinear terms $\underline{\mathbf{N}}$, an Adam-Bashforth-like method will be used, whereas the difficult linear terms $\underline{\mathbf{M}}$, a semi-implicit Crank-Nicholsonlike method will be used. The reason for the separate treatment is because in our case the nonlinear is most efficient with an explicit time-stepping method, whereas the difficult linear terms (pressure gradient and viscosity) are numerically unstable and requires a semi-implicit method to be stabilized. Additionally, for the sake of convenience and easy solvability, it is desired that implicit terms have no cross-coupling with other components. The numerical algorithm is now:

$$\underline{\mathbf{u}}^{n+1} = e^{\underline{\pounds}\Delta t} \underline{\mathbf{u}}_n + e^{\frac{\underline{\pounds}\Delta t}{2}} \Delta t (\frac{3}{2} \underline{\mathbf{N}}^n - \frac{1}{2} \underline{\mathbf{N}}^{n-1}) + e^{\underline{\pounds}\Delta t} \frac{\Delta t}{2} \underline{\mathbf{M}}^n + \frac{\Delta t}{2} \underline{\mathbf{M}}^{n+1}$$
(5.26)

$$0 = \nabla \cdot \underline{\mathbf{u}}^{n+1} \tag{5.27}$$

Another way of looking at this equation is as a fractional step method. Each step will consist of a dealing with a different term. Also, in this report, the velocity flow must also satisfy the additional divergence-free condition imposed by the incompressibility of the flow.

$$\underline{\mathbf{u}}^{n+\frac{1}{4}} = e^{\underline{\pounds}\,\Delta t}\underline{\mathbf{u}}_n \tag{5.28}$$

$$\underline{\mathbf{u}}^{n+\frac{2}{4}} = \underline{\mathbf{u}}^{n+\frac{1}{4}} + e^{\frac{\underline{\mathscr{E}}\Delta t}{2}} \Delta t (\frac{3}{2} \underline{\mathbf{N}}^n - \frac{1}{2} \underline{\mathbf{N}}^{n-1})$$
(5.29)

$$\underline{\mathbf{u}}^{n+\frac{3}{4}} = \underline{\mathbf{u}}^{n+\frac{1}{4}} + e^{\underline{\pounds}\Delta t} \frac{\Delta t}{2} \underline{\mathbf{M}}^{n}$$
(5.30)

$$\underline{\mathbf{u}}^{n+1} = \underline{\mathbf{u}}^{n+\frac{2}{3}} + \frac{\Delta t}{2} \underline{\mathbf{M}}^{n+1}$$
(5.31)

$$\nabla \cdot \underline{\mathbf{u}}^{n+1} = 0 \tag{5.32}$$

In the following sections, this algorithm will be applied directly to the Boussinesq equations. This algorithm is second-order accurate where a single time-step locally produces an error $\sim \Delta t^3$ and globally produces an error $\sim \Delta t^2$.

Step 1: Semi-analytic treatment of Coriolis, Buoyancy, and Shear forces

For the remainder of section 3's subsections, the equations will be written out in component form instead of vector form for cleanliness. The linear operator here $\underline{\pounds}(y)$ contains the shear, buoyancy, and Coriolis terms. Please note that the shear term is a function of the crossstream spatial coordinate. Since the linear operator is a function of y, the easiest way to compute the first intermediate step is to work in mixed-space, and then in subsequent steps work in fully spectral space.

$$\underline{\underline{\pounds}}(y) = \begin{bmatrix} \sigma y i k_x & f + \sigma & 0 & 0 \\ -f & \sigma y i k_x & 0 & 0 \\ 0 & 0 & \sigma y i k_x & -g \\ 0 & 0 & \frac{N^2}{g} & \sigma y i k_x \end{bmatrix}$$
(5.33)

An easy and straightforward way of computing the matrix exponential of the linear matrix is by doing an eigen-decomposition of the matrix and then exponentiating each eigenvalue of the diagonal matrix. Lets define the following constants:

$$\lambda_{1} \equiv f + \sigma$$

$$\lambda_{2} \equiv f$$

$$\lambda \equiv \sqrt{\lambda_{1}\lambda_{2}} = \sqrt{\frac{f + \sigma}{f}}$$

$$\zeta_{1} \equiv g$$

$$\zeta_{2} \equiv \frac{\bar{N}}{g}$$

$$\zeta \equiv \sqrt{\zeta_{1}\zeta_{2}} = \bar{N}$$

$$c \equiv i\sigma y k_{x}$$

In terms of these constants, the matrix exponential of the linear term is:

$$e^{\underline{\pounds}\Delta t} = e^{c\Delta t} \begin{bmatrix} \cos(\lambda\Delta t) & \frac{\lambda_1}{\lambda}\sin(\lambda\Delta t) & 0 & 0\\ -\frac{\lambda_2}{\lambda}\sin(\lambda\Delta t) & \cos(\lambda\Delta t) & 0 & 0\\ 0 & 0 & \cos(\zeta\Delta t) & -\frac{\zeta_1}{\zeta}\sin(\zeta\Delta t)\\ 0 & 0 & \frac{\zeta_2}{\zeta}\sin(\zeta\Delta t) & \cos(\zeta\Delta t) \end{bmatrix}$$
(5.34)

The velocity equations for the first quarter step are

$$u_x^{n+\frac{1}{4}} = e^{c\Delta t} (\cos(\lambda\Delta t)u_x^n + \lambda_1 \frac{\sin(\lambda\Delta t)}{\lambda\Delta t} u_y^n)$$
(5.35)

$$u_y^{n+\frac{1}{4}} = e^{c\Delta t} (\cos(\lambda\Delta t)u_y^n - \lambda_2 \frac{\sin(\lambda\Delta t)}{\lambda\Delta t}u_x^n)$$
(5.36)

$$u_z^{n+\frac{1}{4}} = e^{c\Delta t} (\cos(\zeta \Delta t) u_z^n - \zeta_1 \frac{\sin(\zeta \Delta t)}{\zeta} \tilde{\rho^n})$$
(5.37)

$$\tilde{\rho}^{n+\frac{1}{3}} = e^{c\Delta t} (\cos(\zeta \Delta t)\tilde{\rho}^n + \zeta_2 \frac{\sin(\zeta \Delta t)}{\zeta} u_z^n)$$
(5.38)

Step 2: Adam-Bashforth-like exponential propagation of Advection term

The advection term $(\underline{\mathbf{u}} \cdot \nabla)\underline{\mathbf{u}}$ here is a nonlinear term, so the common time-integration method used is the 2nd order Adam-Bashforth algorithm. This algorithm also uses Adam-Bashforth, but scaled by the linear matrix exponential to maintain 2nd order accuracy. Again to simplify the equations, lets define the quantities:

$$\begin{split} N_x^n &= -\frac{3}{2} \Delta t((\underline{\mathbf{u}}^n \cdot \nabla) u_x^n) + \frac{1}{2} \Delta t((\underline{\mathbf{u}}^{n-1} \cdot \nabla) u_x^{n-1}) \\ N_y^n &= -\frac{3}{2} \Delta t((\underline{\mathbf{u}}^n \cdot \nabla) u_y^n) + \frac{1}{2} \Delta t((\underline{\mathbf{u}}^{n-1} \cdot \nabla) u_y^{n-1}) \\ N_z^n &= -\frac{3}{2} \Delta t((\underline{\mathbf{u}}^n \cdot \nabla) u_z^n) + \frac{1}{2} \Delta t((\underline{\mathbf{u}}^{n-1} \cdot \nabla) u_z^{n-1}) \\ N_{\tilde{\rho}}^n &= -\frac{3}{2} \Delta t((\underline{\mathbf{u}}^n \cdot \nabla) \tilde{\rho}^n) + \frac{1}{2} \Delta t((\underline{\mathbf{u}}^{n-1} \cdot \nabla) \tilde{\rho}^{n-1}) \end{split}$$

With these nonlinear Adam-Bashforth terms defined, the second intermediate equations are

$$u_x^{n+\frac{2}{4}} = u_x^{n+\frac{1}{4}} + e^{\frac{c\Delta t}{2}} \left(\cos(\lambda \frac{\Delta t}{2}) N_x^n + \lambda_1 \frac{\sin(\lambda \frac{\Delta t}{2})}{\lambda} N_y^n \right)$$
(5.39)

$$u_y^{n+\frac{2}{4}} = u_y^{n+\frac{1}{4}} + e^{\frac{c\Delta t}{2}} \left(\cos\left(\lambda\frac{\Delta t}{2}\right)N_y^n - \lambda_2\frac{\sin\left(\lambda\frac{\Delta t}{2}\right)}{\lambda}N_x^n\right)$$
(5.40)

$$u_{z}^{n+\frac{2}{4}} = u_{z}^{n+\frac{1}{4}} + e^{\frac{c\Delta t}{2}} (\cos(\zeta \frac{\Delta t}{2}) N_{z}^{n} - \zeta_{1} \frac{\sin(\zeta \frac{\Delta t}{2})}{\zeta} N_{\tilde{\rho}}^{n})$$
(5.41)

$$\tilde{\rho}^{n+\frac{2}{4}} = \tilde{\rho}^{n+\frac{1}{4}} + e^{\frac{c\Delta t}{2}} \left(\cos\left(\zeta \frac{\Delta t}{2}\right) N_{\tilde{\rho}}^n + \zeta_2 \frac{\sin\left(\zeta \frac{\Delta t}{2}\right)}{\zeta} N_z^n \right)$$
(5.42)

Step 3: Crank-Nicholson-like treatment of Pressure, green's function, and Viscous terms

In the third intermediate step, the code has to partially deal with the difficult linear terms, which in this case are the pressure gradient, and viscous terms. In addition, since the algorithm uses Chebyshev polynomials as a basis, there are green's functions that are used to enforce boundary conditions and the divergence-free condition. The magnitude of the green's functions are represented by the unknown $\tau_{x_1}, \tau_{x_2}, \tau_{y_1}, \tau_{y_2}, \tau_{z_1}$ and τ_{z_2} associated with the green's functions (highest Chebyshev modes) T_{N_y-1} and T_{N_y} respectively. Here, the current step's pressure gradients, viscous terms, and τ 's are added into the equations. Once Again, defining the following quantities to simplify the equations:

$$\Pi^{n} = \frac{\Delta t}{2} P^{n}$$

$$\hat{\tau}^{n} = \frac{\Delta t}{2} \tau^{n}$$

$$M_{x}^{n} = \frac{\nu \Delta t}{2} \nabla^{2} u_{x}^{n} - \frac{\partial \Pi^{n}}{\partial x} + \hat{\tau}_{x_{1}}^{n} T_{N_{y-1}} + \hat{\tau}_{x_{2}}^{n} T_{N_{y}}$$

$$M_{y}^{n} = \frac{\nu \Delta t}{2} \nabla^{2} u_{y}^{n} - \frac{\partial \Pi^{n}}{\partial y} + \hat{\tau}_{y_{1}}^{n} T_{N_{y-1}} + \hat{\tau}_{y_{2}}^{n} T_{N_{y}}$$

$$M_{z}^{n} = \frac{\nu \Delta t}{2} \nabla^{2} u_{z}^{n} - \frac{\partial \Pi^{n}}{\partial z} + \hat{\tau}_{z_{1}}^{n} T_{N_{y-1}} + \hat{\tau}_{z_{2}}^{n} T_{N_{y}}$$

Please note that Π and $\hat{\tau}$ here are defined differently from section 2.2. Now the equations for the third intermediate step are

$$u_x^{n+\frac{3}{4}} = u_x^{n+\frac{2}{4}} + e^{c\Delta t} (\cos(\lambda\Delta t)M_x^n + \lambda_1 \frac{\sin(\lambda\Delta t)}{\lambda}M_y^n)$$
(5.43)

$$u_y^{n+\frac{3}{4}} = u_y^{n+\frac{2}{4}} + e^{c\Delta t} (\cos(\lambda\Delta t)M_y^n - \lambda_2 \frac{\sin(\lambda\Delta t)}{\lambda}M_x^n)$$
(5.44)

$$u_{z}^{n+\frac{3}{4}} = u_{z}^{n+\frac{2}{4}} + e^{c\Delta t} \cos(\zeta \Delta t) M_{z}^{n}$$
(5.45)

$$\tilde{\rho}^{n+1} = \tilde{\rho}^{n+\frac{2}{3}} + e^{c\Delta t} \zeta_2 \frac{\sin(\zeta \Delta t)}{\zeta} M_z^n$$
(5.46)

Step 4: Computing implicit Pressure and τ terms from divergence and velocity equations

The final step in the equations involve the future pressure gradient, τ 's, and viscosity terms.

$$u_x^{n+1} = u_x^{n+\frac{3}{4}} + \frac{\nu \Delta t}{2} \nabla^2 u_x^{n+1} - \frac{\partial \Pi^{n+1}}{\partial x} + \hat{\tau}_{x_1}^{n+1} T_{N_y-1} + \hat{\tau}_{x_2}^{n+1} T_{N_y}$$
(5.47)

$$u_y^{n+1} = u_y^{n+\frac{3}{4}} + \frac{\nu \Delta t}{2} \nabla^2 u_y^{n+1} - \frac{\partial \Pi^{n+1}}{\partial y} + \hat{\tau}_{y_1}^{n+1} T_{N_y-1} + \hat{\tau}_{y_2}^{n+1} T_{N_y}$$
(5.48)

$$u_z^{n+1} = u_z^{n+\frac{3}{4}} + \frac{\nu \Delta t}{2} \nabla^2 u_z^{n+1} - \frac{\partial \Pi^{n+1}}{\partial z} + \hat{\tau}_{z_1}^{n+1} T_{N_y-1} + \hat{\tau}_{z_2}^{n+1} T_{N_y}$$
(5.49)

This last step involves computing the implicit terms which will prove to be the most difficult part. This is because you have to solve a Helmholtz equation to compute the pressure-head and then another Helmholtz equation to compute the velocity from the implicit viscosity term. However, when one uses Chebyshev polynomials as a basis, it is actually not possible to directly impose boundary conditions. The only way to do this is to replace the last two equations for the highest two modes with boundary conditions. And you must do this once for the pressure equation and the second time when solving for the final velocity. Due to this, the flow will not be divergence free and the highest two modes will be inconsistent in the equations. Luckily, there are eight extra degrees of freedoms we can use, the six $\hat{\tau}$, and the two pressure boundary conditions that will represented by τ_{p_1} and τ_{p_2} . Please refer to appendix A for a detailed procedure on how to derive the pressure. The pressure head is now:

$$\Pi^{n+1} = \Pi_h + \hat{\tau}_{y_1}^{n+1} G_1 + \hat{\tau}_{y_2}^{n+1} G_2 + \hat{\tau}_{p_1}^{n+1} G_3 + \hat{\tau}_{p_2}^{n+1} G_4$$
(5.50)

where Π_h is the homogeneous part of pressure, and the four τ 's and G's are the green's functions. Using equation above, and solving another Helmholtz equation, we can arrive at the final velocity equations for the next time step. Once again, refer to appendix A for a full derivation of these equations.

$$\bar{u}_x^{n+1} = u_{\nu,x}^{n+1} + \hat{\tau}_{y_1}^{n+1} H_{1,x} + \hat{\tau}_{y_2}^{n+1} H_{2,x} + \hat{\tau}_{p_1}^{n+1} H_{3,x} + \hat{\tau}_{p_2}^{n+1} H_{4,x}$$
(5.51)

$$\bar{u}_{y}^{n+1} = u_{\nu,y}^{n+1} + \hat{\tau}_{y_{1}}^{n+1}H_{1,y} + \hat{\tau}_{y_{2}}^{n+1}H_{2,y} + \hat{\tau}_{p_{1}}^{n+1}H_{3,y} + \hat{\tau}_{p_{2}}^{n+1}H_{4,y}$$
(5.52)

$$\bar{u}_{z}^{n+1} = u_{\nu,z}^{n+1} + \hat{\tau}_{y_{1}}^{n+1}H_{1,z} + \hat{\tau}_{y_{2}}^{n+1}H_{2,z} + \hat{\tau}_{p_{1}}^{n+1}H_{3,z} + \hat{\tau}_{p_{2}}^{n+1}H_{4,z}$$
(5.53)

Here $\underline{\mathbf{u}}$ is the new velocity but with the incorrect highest two modes, and the τ 's and $\underline{\mathbf{H}}$'s are the new green's functions. $\underline{\mathbf{u}}_{\nu}^{n+1}$ and $\underline{\mathbf{H}}$'s are defined and derived in appendix A.

Step 5: Computing τ 's and pressure boundary conditions

The τ 's and pressure boundary conditions are the last eight unknowns in our system of equations. The eight constraints that we need to solve for these unknowns will come from

restoring the last two modes of the velocities so that they satisfy our governing equations, and divergence-free condition.

Restoring the highest two Chebyshev modes of the velocity will involve substituting the \bar{u}^{n+1} into the u^{n+1} equations. Using the τ 's we will demand that the following equations be true.

For the highest two Chebyshev modes $N_y - 1$ and N_y

$$\bar{u}_x^{n+1} = u_x^{n+1} \tag{5.54}$$

$$\bar{u}_{y}^{n+1} = u_{y}^{n+1} \tag{5.55}$$

$$\bar{u}_z^{n+1} = u_z^{n+1} \tag{5.56}$$

As was explained in the prior work, the divergence-free conditions in the original algorithm has problems when increasing resolution in the Chebyshev direction. The new way to resolve this is use a different formulation to set all Chebyshev coefficients in the divergence to be zero. Instead of solving the τ s to directly set the last two modes of the divergence of velocity to zero, what we can do is to demand the divergence at the boundaries to be zero.

$$\nabla \cdot \underline{\mathbf{\bar{u}}}^{n+1} = 0 \Big|_{y=\pm L_y/2} \tag{5.57}$$

This method works because if you solved for the pressure so that the velocity is divergencefree without the τ 's, then the divergence of velocity's Chebyshev modes will all be zero except for the highest two modes.

$$let \ \nabla \cdot \underline{\mathbf{u}}^{n+1} \equiv \sum_{l=-N_x}^{N_x} \sum_{m=-N_z}^{N_z} \sum_{n=0}^{N_y} \tilde{D}_{lmn}(t) e^{ik_x x} e^{ik_z z} T_n$$

$$(5.58)$$

$$=\sum_{l=-N_x}^{N_x}\sum_{m=-N_z}^{N_z} (\tilde{D}_{lm(N_y-1)}T_{N_y-1} + \tilde{D}_{lmN_y}T_{N_y})e^{ik_xx}e^{ik_zz}$$
(5.59)

From this equation you can see that all the spectral coefficients are equal to zero except for the highest two Chebyshev modes. Now applying the new divergence-free constraints at the boundaries will yield the two linear equations

$$\tilde{D}_{lm(N_y-1)} + \tilde{D}_{lmN_y} = 0$$
(5.60)
$$\tilde{D}_{lm(N_y-1)} - \tilde{D}_{lmN_y} = 0$$
(5.61)

$$D_{lm(N_y-1)} - D_{lmN_y} = 0 (5.61)$$

The only solution to these two equations is $\tilde{D}_{lm(N_y-1)} = 0$ and $\tilde{D}_{lmN_y} = 0$. Furthermore, even when dealing with computer precision, the method still works where the solution will end up approximately $a = \mathcal{O}(10^{-16})$ and $b = \mathcal{O}(10^{-16})$. Even though this is a roundabout way

to force the highest two Chebyshev modes to the equal to zero (up to computer precision), it proves to be stable even with high resolution. Repeating, the same condition number test from section 2, the new condition numbers are close in order despite the increase in number of Chebyshev modes. This is in contrast to table **??**, where the condition number increases exponentially with number of Chebyshev modes.

	N=32	N=64	N=128	N=256
$\nu\Delta t/L_y^2 = 2.5 * 10^{-4}$	10^{5}	10^{6}	10^{6}	10^{7}
$\nu\Delta t/L_y^2 = 2.5 * 10^{-5}$	10^{6}	10^{6}	10^{7}	10^{7}
$\nu \Delta t / L_y^2 = 2.5 * 10^{-6}$	10^{7}	10^{7}	10^{7}	10^{8}

Table 5.2: Condition number for τ matrix with new constraints

Special treatment of the $(n_x = 0, n_z = 0)$ Fourier-Fourier mode

It is not possible to solve for the $(n_X = 0, n_z=0)$ Fourier mode's coefficients using the above method. This is because the matrix equation to solve for the pressure is singular and will provide no solution. Furthermore, the (0,0) mode of the pressure gradient in the y-direction $\partial \Pi^{n+1}/\partial y$ must be stored to included in the algorithm to achieve second-order accuracy. Therefore, a special treatment must be applied.

For the x and z directions, since they are discretized using a Fourier basis, the pressure gradients in these directions are always 0. So there is no need to solve the Helmholtz equation to compute the pressure. For the y-direction, it is even simpler. The (0,0) mode for the divergence demands that $\frac{\partial u_y}{\partial y} = 0$, when means that u_y does not change in the y direction (for the (0,0) mode) and so u_y must equal to the u_y at the boundaries. Hence the equations for the (0,0) mode velocities are:

$$u_x^{n+1} = u_{\nu,x}^{n+1} \tag{5.62}$$

$$u_y^{n+1} = 0 (5.63)$$

$$u_z^{n+1} = u_{\nu,z}^{n+1} \tag{5.64}$$

Again, the $\underline{\mathbf{u}}_{\nu}$ are defined in appendix A. Lastly, the (0,0) mode of $\partial \Pi^{n+1} / \partial y$ can computed when you substitute $u_y^{n+1} = 0$ into the y-direction velocity's equation.

$$\frac{\partial \Pi^{n+1}}{\partial y} = u_y^{n+\frac{3}{4}} \tag{5.65}$$

5.4 Results and preliminary tests

In this section, a series of tests are done to test the validity and usage of the developed code. The code is shown to be numerically second-order accurate. It is also stable under the

conditions researchers typically use to study ZVI. Lastly, we demonstrate that the code can produce critical layers to qualitatively validate the code.

The first series of tests that were done to this code was just to see if the code will be stable under different conditions. In these tests, σ , N, f, and Δt are varied to test the limits of the code. For a given parameter, the code either stayed stable or become unstable. The code was compared to the old, traditional fractional method to test the capabilities of the semi-analytic algorithm. All tests was initialized with a 3D Gaussian vortex.

The first test was to see if we can go up to 1024 Chebyshev modes and still be divergencefree using the new τ constraints. Here, $\sigma/f = 0.75$, N/f = 1, and $\Delta t = 0.01 \frac{1}{\sigma}$.

	New Method	Old Method
$N_y = 32$	Stable	Stable
$N_y = 64$	Stable	Stable
$N_y = 128$	Stable	Unstable
$N_y = 256$	Stable	Unstable
$N_y = 512$	Stable	Unstable
$N_y = 1024$	Stable	Unstable

Table 5.3: Stability of the new algorithm vs the traditional algorithm as a function of number of Chebyshev modes

Now varying the linear parameters, N, and f. Here, $\Delta t = 0.01 \frac{1}{\sigma}$, and $N_x = N_y = N_z = 64$

	Semi-Analytic	Fractional Step		Semi-Analytic	Fractional Step
N/f=1	Stable	Stable	f/N=1	Stable	Stable
N/f=50	Stable	Stable	f/N=50	Stable	Stable
N/f = 100	Stable	Stable	f/N=100	Stable	Stable
N/f = 120	Stable	Unstable	f/N = 120	Stable	Unstable
N/f = 150	Stable	Unstable	f/N=150	Stable	Unstable

(a) Varying N with $\sigma/f = 1$

	Semi-Analytic	Fractional Step
$\sigma/N = -30$	Stable	Unstable
$\sigma/N = -10$	Stable	Unstable
$\sigma/N = -2$	Stable	Stable
$\sigma/N = 1$	Stable	Stable
$\sigma/N = 2$	Stable	stable
$\sigma/N = 10$	Stable	Unstable
$\sigma/N = 30$	Stable	Unstable

(c) Varying σ with f/N = 1

Table 5.4: Varying N, f, and σ to see when the new method is better than the old method

(b) Varying f with $\sigma/N = 1$

Numerical accuracy

Even though analytically, the algorithm can be shown to be second order accurate using Taylor expansions, We also numerically prove that the algorithm is second order accurate. The initial condition we chose was a 3D Gaussian vortex,

$$\omega_z = \omega_0 exp(-x^2/a^2 - y^2/b^2 - z^2/c^2)$$
(5.66)

 ω_0 is the vortex strength, and a, b, and c are constants chosen so that the vorticity is close to zero near the boundaries.

The first simulation is run with a very small time-step size that we will call Δt^* and treated as the exact answer. Four more simulations is run with much bigger time-step sizes Δt_1 , Δt_2 , Δt_3 , and Δt_4 , where

$$\Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4, >> \Delta t^*$$

The simulations will be integrated to the same final time, T_f . The error is evaluated as the L2 norm of velocity at the time-step size of interest vs the "exact" solution.

$$\underline{\operatorname{error}} = ||\underline{\mathbf{u}}_i - \underline{\mathbf{u}}^*||_2$$

The log of the error vs. log of Δt are plotted. The slope of the plot is the order of accuracy. In figure 4.1, the errors for velocity and density shows that the code is second-order accurate. The equation shown in the graph are the fitted lines through the error points, and the slope of the linear equations are the order of accuracy.

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Figure 5.1: Loglog plots of L_2 Norm of error vs time-step size

Critical layer formation

As a demonstration of the code that was developed, a simulation was ran to show the formation of critical layers and the self-replication of critical layers. The code was again initialized with a 3D Gaussian vortex. The code setup was $\sigma/f = -0.75$, N/f = 1, $\Delta t = 0.015 \frac{1}{\sigma}$, $L_x/L_z = 1$ and $L_y/L_x = 1.5$ (this was chosen so that L_y is at least 6 times larger than the first generation critical layer location.), and $N_x = N_y = N_z = 256$. In addition, hyperviscosity and vertical boundary damping was applied. It should mentioned that at this resolution, hyperviscosity is needed. A simulation was ran with this set up but without hyperviscosity, and the simulation diverged.
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Figure 5.2: Vertical Vorticity for two different planes at t = 0



Figure 5.3: Two pairs of critical layers are formed at t = 720 $1/\sigma$

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Figure 5.4: The pair of critical layers has fully formed at = 1440 $1/\sigma$



Figure 5.5: Second generation of critical layers starting to form at t = 2160 $1/\sigma$

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Figure 5.6: Second generation fully formed at t = 2880 $1/\sigma$

5.5 Conclusion and future research

We have developed a three-dimensional spectral hydrodynamic code to study rapidly rotating, intensely sheared, and strongly stratified systems. This code is built using a number of different techniques to address the challenges associated with shear and stratification. Additionally, a new method to solve for the τ 's is presented to stabilize simulations with high resolution in Chebyshev direction and viscosity. The streamwise and vertical direction are discretized using a Fourier basis, while the cross-stream direction is discretized using a Chebyshev polynomial basis. The shear, coriolis, and buoyancy terms are treated with a semi-analytic method. The nonlinear advection is treated with an explicit Adam-Bashforthlike method. Lastly, the pressure gradient, viscous terms, and τ 's are treated with a semiimplicit Crank-Nicholson-like method. In summary, the code is numerically second-order accurate, stays stable even with high resolution, and is capable of simulating the formation of critical layers. In the future, this code will hopefully be used to study the viscous effects on ZVI and predicting ZVI in experiments.

Although this code can be applied to a wide variety of problems, some limitations exist. The code is built on solving equations that use the Boussinesq approximation, thus the flows are limited to subsonic flows where the density cannot deviate from the mean density too much. Also the viscosity is treated with a Crank-Nicholson-like method so at high wave numbers, the solution is highly oscillatory. A future code would include the viscosity in the linear operator and exponentiated exactly (up to spectral accuracy). However, that is a very challenging task. Also, a future version would allow the Brunt Väisälä frequency to vary as a function of spatial coordinates.

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$$\underline{\underline{\pounds}}(y) = \begin{bmatrix} \sigma y i k_x + \nu \nabla^2 & f + \sigma & 0 & 0\\ -f & \sigma y i k_x + \nu \nabla^2 & 0 & 0\\ 0 & 0 & \sigma y i k_x + \nu \nabla^2 & -\frac{1}{g}\\ 0 & 0 & \frac{N(x,y,z)^2}{g} & \sigma y i k_x \end{bmatrix}$$
(5.67)

5.6 Appendix A: Computing the Pressure and Implicit Viscosity term

For incompressible flows, the pressure is a "slave" to the flow. It is whatever it needs to be to maintain a divergence-free flow. Therefore, the equation to solve for the pressure-head is

$$\nabla \cdot \underline{\mathbf{u}}^{n+1} = 0 \tag{5.68}$$

$$\nabla \cdot \underline{\mathbf{u}}^{n+1} = \nabla \cdot (\underline{\mathbf{u}}^{n+\frac{3}{4}} - \nabla \Pi^{n+1} + \frac{\nu \Delta t}{2} \nabla^2 \underline{\mathbf{u}}^{n+1} + (\hat{\tau}_{x_1}^{n+1} \hat{x} + \hat{\tau}_{y_1}^{n+1} \hat{y} + \hat{\tau}_{z_1}^{n+1}) T_{N_{y-1}} + (\hat{\tau}_{x_2}^{n+1} \hat{x} + \hat{\tau}_{y_2}^{n+1} \hat{y} + \hat{\tau}_{z_2}^{n+1} \hat{z}) T_{N_y}$$
(5.69)

Since this code is in cartesian coordinates, it is assumed that the divergence operator and the Laplacian operator commute. Hence, the divergence of the future viscosity term $(\nabla \cdot \nabla^2 \underline{\mathbf{u}}^{n+1} = \nabla^2 \nabla \cdot \underline{\mathbf{u}}^{n+1} = 0)$ must also be equal to zero. Here, we are going to use a tilde to denote that we are working with an array of spectral coefficients.

 $\Pi(x_i, y_j, z_k, t^n) \leftarrow \text{FFT} \to \tilde{\Pi}_{l,m,n}(t^n)$

$$\nabla^{2} \tilde{\Pi}^{n+1} = \nabla \cdot \underline{\tilde{\mathbf{u}}}^{n+\frac{3}{4}} + i k_{x} (\hat{\tau}_{x_{1}}^{n+1} T_{N_{y-1}} + \hat{\tau}_{x_{2}}^{n+1} T_{N_{y}}) \hat{x} + \frac{\partial}{\partial y} (\hat{\tau}_{y_{1}}^{n+1} T_{N_{y-1}} + \hat{\tau}_{y_{2}}^{n+1} T_{N_{y}}) \hat{y} + i k_{z} (\hat{\tau}_{z_{1}}^{n+1} T_{N_{y-1}} + \hat{\tau}_{z_{2}}^{n+1} T_{N_{y}}) \hat{z})$$

$$(5.70)$$

After replacing the last two rows in the laplacian Chebyshev matrix with the boundary conditions (summation and alternating summation of the spectral coefficients):

$$\sum_{i=0}^{N_y-1} \tilde{\Pi}_i^{n+1} = \tau_{p_1}$$
$$\sum_{i=0}^{N_y-1} (-1)^i \tilde{\Pi}_i^{n+1} = \tau_{p_2}$$

When the last two rows are replaced, the $\hat{\tau}$'s in the x and z direction (Fourier directions) are discarded. The matrix equation then becomes the following:

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 $\forall nx, nz$ Fourier mode

$$\begin{bmatrix} \frac{\nabla_{spectral}^2}{1 & \dots & 1} \\ \hline 1^0 & \dots & -1^{n_z} \end{bmatrix} \begin{bmatrix} \tilde{\Pi} \end{bmatrix} = \begin{bmatrix} \nabla \cdot \underline{\tilde{\mathbf{u}}}^{n+3/4} \\ \hline 0 \\ \hline 0 \end{bmatrix} + \hat{\tau}_{y_1} \begin{bmatrix} \frac{\partial T_{N_y-1}}{\partial y} \\ \hline 0 \\ \hline 0 \end{bmatrix} + \hat{\tau}_{y_2} \begin{bmatrix} \frac{\partial T_{N_y}}{\partial y} \\ \hline 0 \\ \hline 0 \end{bmatrix}$$

$$+\tau_{p_1}\left[\frac{0}{1}\\0\right]+\tau_{p_2}\left[\frac{0}{1}\\1\right]$$

Now defining the following quantities:

$$\tilde{\Pi}_{h} = (\nabla^{2})^{-1} \nabla \cdot \underline{\tilde{\mathbf{u}}}^{n+\frac{3}{4}}$$
$$G_{1} = (\nabla^{2})^{-1} \frac{\partial T_{N_{y-1}}}{\partial y}$$
$$G_{2} = (\nabla^{2})^{-1} \frac{\partial T_{N_{y}}}{\partial y}$$
$$G_{3} = (\nabla^{2})^{-1} T_{N_{y-1}}$$
$$G_{4} = (\nabla^{2})^{-1} T_{N_{y}}$$

The pressure head is now:

$$\Pi^{n+1} = \Pi_h + \hat{\tau}_{y_1}^{n+1} G_1 + \hat{\tau}_{y_2}^{n+1} G_2 + \hat{\tau}_{p_1}^{n+1} G_3 + \hat{\tau}_{p_2}^{n+1} G_4$$
(5.71)

Substituting that into the $\underline{\mathbf{u}}^{n+1}$ equations

$$\underline{\tilde{\mathbf{u}}}^{n+1} = \underline{\tilde{\mathbf{u}}}^{n+\frac{3}{4}} + \frac{\nu\Delta t}{2} \nabla^2 \underline{\tilde{\mathbf{u}}}^{n+1} - \nabla (\tilde{\Pi}_h + \hat{\tau}_{y_1}^{n+1} G_1 + \hat{\tau}_{y_2}^{n+1} G_2 + \hat{\tau}_{p_1}^{n+1} G_3 + \hat{\tau}_{p_2}^{n+1} G_4)$$
(5.72)

$$+ (\hat{\tau}_{x_1}^{n+1} + \hat{\tau}_{x_2}^{n+1})\hat{x} + (\hat{\tau}_{y_1}^{n+1} + \hat{\tau}_{y_2}^{n+1})\hat{y} + (\hat{\tau}_{z_1}^{n+1} + \hat{\tau}_{z_2}^{n+1})\hat{z}$$
(5.73)

$$(I - \nabla^2)\underline{\tilde{\mathbf{u}}}^{n+1} = \underline{\tilde{\mathbf{u}}}^{n+\frac{3}{4}} - \nabla(\tilde{\Pi}_h + \hat{\tau}_{y_1}^{n+1}G_1 + \hat{\tau}_{y_2}^{n+1}G_2 + \hat{\tau}_{p_1}^{n+1}G_3 + \hat{\tau}_{p_2}^{n+1}G_4)$$
(5.74)

$$+ (\hat{\tau}_{x_1}^{n+1} + \hat{\tau}_{x_2}^{n+1})\hat{x} + (\hat{\tau}_{y_1}^{n+1} + \hat{\tau}_{y_2}^{n+1})\hat{y} + (\hat{\tau}_{z_1}^{n+1} + \hat{\tau}_{z_2}^{n+1})\hat{z}$$
(5.75)

Again, replacing the last two rows with the no-slip and no-penetration boundary conditions

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$$\begin{bmatrix} \underline{I - \nabla^2} \\ \underline{1 & \dots & 1} \\ 1^0 & \dots & -1^{n_z} \end{bmatrix} \begin{bmatrix} \underline{\tilde{\mathbf{u}}}^{n+1} \end{bmatrix} = \begin{bmatrix} \underline{\tilde{\mathbf{u}}}^{n+\frac{3}{4}} - \nabla \tilde{\Pi}_h \\ \underline{0} \\ 0 \end{bmatrix} + \hat{\tau}_{y_1}^{n+1} \begin{bmatrix} \underline{\nabla G1} \\ \underline{0} \\ 0 \end{bmatrix} + \hat{\tau}_{y_2}^{n+1} \begin{bmatrix} \underline{\nabla G2} \\ \underline{0} \\ 0 \end{bmatrix}$$

$$+\hat{\tau}_{p_1}^{n+1} \begin{bmatrix} \nabla G3\\ \hline 0\\ \hline 0 \end{bmatrix} + \hat{\tau}_{p_2}^{n+1} \begin{bmatrix} \nabla G4\\ \hline 0\\ \hline 0 \end{bmatrix}$$

Now defining the following quantities:

$$\begin{split} \tilde{\mathbf{u}}_{\nu}^{n+1} &= (I - \frac{\nu \Delta t}{2} \nabla^2)^{-1} \overline{(\tilde{\mathbf{u}}^{n+\frac{3}{4}} - \nabla \tilde{\Pi}_h)} \\ \overline{\mathbf{H}}_1 &= (I - \frac{\nu \Delta t}{2} \nabla^2)^{-1} \overline{\nabla G1} \\ \overline{\mathbf{H}}_2 &= (I - \frac{\nu \Delta t}{2} \frac{\nu \Delta t}{2} \nabla^2)^{-1} \overline{\nabla G2} \\ \overline{\mathbf{H}}_3 &= (I - \frac{\nu \Delta t}{2} \nabla^2)^{-1} \overline{\nabla G3} \\ \overline{\mathbf{H}}_4 &= (I - \frac{\nu \Delta t}{2} \nabla^2)^{-1} \overline{\nabla G4} \end{split}$$

Where the bar over the equation denotes that the last two modes are replaced with zeros. Finally the equations are (the bar denotes that the last two modes were replaced by the boundary conditions and will be corrected by the τ 's):

$$\underline{\bar{\mathbf{u}}}^{n+1} = \underline{\mathbf{u}}_{\nu}^{n+1} + \hat{\tau}_{y_1}^{n+1} \underline{\mathbf{H}}_{\underline{\mathbf{1}}} + \hat{\tau}_{y_2}^{n+1} \underline{\mathbf{H}}_{\underline{\mathbf{2}}} + \hat{\tau}_{p_1}^{n+1} \underline{\mathbf{H}}_{\underline{\mathbf{3}}} + \hat{\tau}_{p_2}^{n+1} \underline{\mathbf{H}}_{\underline{\mathbf{4}}}$$
(5.76)

5.7 Appendix B: Chebyshev Polynomials

Since the Fourier decomposition can only be applied to periodic domains, we need a way of representing systems that are not necessarily periodic. We do this with the Chebyshev polynomials [6]. The definition of the n^{th} Chebyshev polynomial $T_n(x)$ is:

$$T_n(x) := \cos(n \cos^{-1}(x))$$

They can also be equivalently defined by the recurrence relation:

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$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

They form a linearly independent, complete, orthogonal basis over the interval [1, -1]. Our goal is to represent a function $f(x); x \in [1, -1]$ as a sum of Chebyshev polynomials:

$$f(x) = \sum_{n=0}^{N} a_n T_n(x)$$

Chebyshev Derivatives

Taking the derivative of a Chebyshev sum is relatively easy, although not as straightforward as differentiating a Fourier series. We want to find $\frac{dT_n}{dx}$:

We know that

$$\frac{d}{dx} = \frac{d\theta}{dx}\frac{d}{d\theta}$$
$$\frac{dx}{d\theta} = \frac{d(\cos(\theta))}{d\theta}$$
$$= -\sin(\theta)$$

Which means that:

$$\frac{d}{dx}T_n(\cos(\theta)) = -\frac{1}{\sin(\theta)}\frac{d\cos(n\theta)}{d\theta}$$
$$= \frac{n\sin(n\theta)}{\sin(\theta)}$$

Which can be expressed as:

$$\frac{n\sin(n\theta)}{\sin(\theta)} = 2n\Big(\cos((n-1)\theta) + \cos((n-3)\theta) + \dots + \begin{cases} \cos(\theta) & \text{if n is even} \\ \cos(2\theta) + 1/2 & \text{if n is odd} \end{cases}\Big)$$
$$= 2n\Big(T_{n-1}(x) + T_{n-3}(x) + \dots + \begin{cases} T_1(x) & \text{if n is even} \\ T_2(x) + \frac{1}{2}T_0(x) & \text{if n is odd} \end{cases}\Big)$$

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Upon inspection, we can find the following matrix representation of the differentiation operator (assuming that N is odd):

$$\frac{d}{dx} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = 2 \times \begin{pmatrix} 0 & 1/2 & 0 & 3/2 & 0 & 5/2 & \cdots & N/2 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & N \\ 0 & 0 & 0 & 0 & 4 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$
$$=: D \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

Chapter 6

Summary and Future Work

Zombie Vortex Instability (ZVI) is a new finite amplitude instability driven by the interaction of horizontal shear, background rotation, vertical stratification. The instability comes from nonlinear baroclinic critical layers excited, the singularities excited by vortex. When the first generation of critical layers are excited, the logarithm singularity structure of the horizontal velocities will generate the discontinuity on vertical vorticity (vortex sheet). Vortex sheet are unstable structures and a secondary instability, such the Kelvin-Helmoltz instability or Rossby wave instability will appear on the vortex sheet, thus a new vortex is spawn on the location of critical layers. Those new born vortex will generate their own baroclinic critical layers. The newly form critical layers will spawn vortex. The self-replication process will continue until the whole system is fulled with critical layers and vortex, and thus reaches to a turbulent state.

We present a parameter map, in terms of the physical ingredients in the system. The parameter map shows clearly under what condition ZVI will be excited. ZVI has been found to exit not only for Keplerian case, but also for numerous parameter cases, with both anticyclonic and cyclonic regimes. ZVI is created and sustained under the condition that the stratification, rotation and shear are on the same order, which is believed to maintain the long lasting of the vortex that excite critical layers. When excited, ZVI shows clear patterns and can be observed in five stages, the formation of critical layers and vortex sheet, the roll up to zombie vortex, the self-replication of zombie vortex and the formation of zombie turbulence.

Two kinds of critical layers exits in stratified rotational shear flow, the barotropic critical layers and the baroclinic critical layers. Critical layers are mathematical singularities that are smoothed out by either nonlinear effects or viscous effects or both. The analytic solution of the barocolinic critical layers are provided with matched asymptotic expansions, showing the dual layer structure. The nonlinear dominate critical layers will bring in the instability and generate the zombie vortex. Viscous dominate critical layers will dissipate and stabilize the system. The critical layers are found to be logarithm singular on the horizontal velocity, which leads into a vertical vorticity jump conditions. The jump conditions bring in the secondary instability that generates vortex. The viscous effects dissipate the critical layer.

The critical Re for ZVI is at the order of 10⁶. ZVI is killed by the viscosity due the high dissipation on the critical layer length scale.

In order to simulate the dynamics of the stratified rotational shear flow, we have designed and developed several sets of numerical algorithm, including semi-analytical method in shearing sheet coordinates with inviscid and viscous cases, a brand-new semi-analytical numerical algorithm with one direction non-periodic with Chebyshev polynomials. The algorithm works correctly as second-order accurate and generates the zombie vortex.

The main questions for ZVI have been answered by this thesis. The left-overs are not very exciting but yet worth of exploring. The main goal remains is, can we actually build up the experiments in the laboratory to observe ZVI. The key questions in the experiment is how to observe the critical layers and maintain them in a long time. Once ZVI is verified in the experiment, the next step shall be, despite the fact that there are several hydrodynamic instabilities claimed to exist in the accretion disk, which one shall be the dominate? This can be explored by setting up the numerical experiments with parameters that will generate two or more instabilities in the system, and let them compete. The one lasts successfully suppress other instabilities will last to end, shall be the candidate for generating turbulence in the accretion disk.

However, it may take a long long time to verify whether our theory is right or wrong, with the real observation of the accretion disk in the universe which has last so long as close to eternity compared with human's life. I am pessimistic about the probability whether my work shall be approved or disapproved in my lifetime, or even in the lifetime of the next several generations, while yet I am optimistic about the existence of such a day, and so delighted to have such opportunity to contribute to the knowledge of human's understanding of our universe.

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