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Log-Concave Probability and Its Applications

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Abstract In many applications, assumptions about the log-concavity of a probability distribution allow just enough special structure to yield a workable theory. This paper catalogs a series of theorems relating log-concavity and/or log-convexity of probability density functions, distribution functions, reliability functions, and their integrals. We list a large number of commonly-used probability distributions and report the log-concavity or log-convexity of their density functions and their integrals. We also discuss a variety of applications of log-concavity that have appeared in the literature.

1 Introduction

A function f that maps a concave set into the positive real numbers is said to be *log-concave* if the function $\ln f$ is concave and *log-convex* if $\ln f$ is a convex function. The log-concavity or log-convexity of probability densities and their integrals has interesting qualitative implications in many areas of economics, in political science, in biology, and in industrial engineering.

This paper records and proves a series of related theorems on the log-concavity or log-convexity of univariate probability density functions, cumulative distribution functions, and their integrals. We examine the invariance of these properties under integration, truncations, and other transformations. We relate the properties of density functions to those of reliability functions, failure rates, and the monotonicity of the "mean-residual-lifetime function." We define the "mean-advantage-over-inferiors function" for truncated distributions and relate monotonicity of this function to log-concavity or log-convexity of the probability density function and its integral. We

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examine a large number of commonly-used probability distributions and record the log-concavity or log-convexity of density functions and their integrals. Finally, we discuss a variety of applications of log-concavity that have appeared in the literature.

Most of the results found in this paper have appeared somewhere in the literature of statistics, economics, and industrial engineering. The purpose of this paper is to offer a unified exposition of related results on the log-concavity and log-convexity of univariate probability distributions and to sample some applications of this theory. An earlier draft of this paper has been available on the web since 1989. The current version streamlines the exposition and proofs and makes note of several related papers that have appeared since 1989.

2 From Densities to Distribution Functions

2.1 Log-concavity begets Log-concavity

The results in this paper include a bag of tricks that can be used to identify log-concave distribution functions when more straightforward methods fail. Many familiar probability distributions lack closed-form cumulative distribution functions, but have *density functions* that are represented by simple algebraic expressions. Often, straightforward application of calculus determines whether the density function is log-concave or log-convex. Conveniently, it turns out that log-concavity of the density function implies log-concavity of the cumulative distribution function. Moreover, log-concavity of the c.d.f. is a sufficient condition for log-concavity of the integral of the c.d.f. We do not have to look far to find a useful application of this result. The cumulative normal distribution does not have a closed-form representation and direct verification of its log-concavity is difficult. But the normal density function is easily seen to be log-concave, since its natural logarithm is a concave quadratic function.

The fact that log-concavity is passed from functions to their integrals was proved by Prèkopa [32]. Prèkopa finds this result as a corollary of a general theorem that requires a great deal of mathematical apparatus. Theorem 1, which applies to the case of differentiable functions of a single real variable this result has a simple calculus proof which we present in the Appendix.[1]

Theorem 1 *Let f be a probability density function whose support is the interval (a, b), and let F be the corresponding cumulative distribution function:*

– *If f is continuously differentiable and log-concave on (a, b), then F is also log-concave on (a, b).*

[1] The proof used here is due to Dierker [15]. There is a useful extension of Theorem 1 to higher dimensions. Prèkopa shows that if f is a log-concave probability density function defined on R^n, then the "marginal density functions" will also be log-concave. See also An [4]

– *If F is log-concave on (a, b), then the left hand integral G, defined by $G(x) = \int_a^x F(x)$, is also a log-concave function on on (a, b).*

The following corollary of Theorem 1 is often useful for diagnosing log-concavity.

Corollary 1 *If the density function f is monotone decreasing, then F is log-concave and so is its left hand integral G.*

Proof: Since F is a c.d.f., it must be that F is monotone increasing. Therefore if f is monotone decreasing, it must be that $f(x)/F(x)$ is monotone decreasing. But $(\frac{f(x)}{F(x)})' = (\ln F(x))''$. Therefore if f is monotone decreasing, F must be log-concave. Log-concavity of G follows from Theorem 1 $\square$

2.2 Log-convexity (Sometimes) Begets Log-convexity

Log-convexity, unlike log-concavity, is not always inherited by the cumulative distribution function F from the density function f. Table 3 below lists examples of distribution functions that have strictly log-convex density functions and strictly *log-concave* distribution functions. But there is an easily diagnosed subset of log-convex density functions whose cdf's must also be log-convex. Let us define $f(a) = \lim_{x \to a} f(x)$. Then if $f(a) = 0$, the cdf F will inherit log-convexity from the density function.[2] Moreover, if F is log-convex, the left hand integral G, defined so that $G(x) = \int_a^x F(t)dt$, is also log-convex. A proof appears in the appendix.

Theorem 2 *Let f be a probability density function whose support is the interval (a, b), and let F be the corresponding cumulative distribution function:*

– *If f is continuously differentiable and log-convex on (a, b), and if $f(a) = 0$, then F is also log-convex on (a, b).*
– *If F is log-convex on (a, b), then the left hand integral G, defined by $G(x) = \int_a^x F(x)$, is also log-convex on (a, b).*

3 From Densities to Reliability Functions

3.1 Reliability Theory

Reliability theory is concerned with the time pattern of survival probability of a machine or an organism.[3] Let us consider a machine that will break

[2] Mark Yuying An [4] showed that F inherits log-convexity from f if $a = -\infty$. An's observation follows from our result, since for f to be a probability density function it must be that $f(-\infty) = 0$.

[3] A thorough and interesting treatment of reliability theory is found in Barlow and Proschan [7].

down and be discarded at some time in the interval (a, b). The survival density function f is defined so that $f(x)$ is the probability that a machine breaks down at age x. The probability that the machine breaks down before reaching age x is given by $F(x)$, where F is the cumulative distribution function defined by $F(x) = \int_a^x f(t)dt$. The *reliability function*, (also known as the *survival function*) $\bar{F}$, is defined so that $\bar{F}(x) = 1 - F(x)$ is the probability that the machine does *not* break down before reaching x. It follows from the definitions that $\bar{F}(x) = \int_x^b f(t)dt$. The conditional probability that a machine which has survived to time x will break down at time x is given by the *failure rate* (also known as the *hazard function*), which is defined by $r(x) = f(x)/\bar{F}(x)$. Let us also define a function H which is the right hand integral of the reliability function, so that $H(x) = \int_x^b \bar{F}(t)dt$.

3.2 Reliability Functions Inherit Log-concavity

Theorem 3 mirrors Theorem 1 by establishing that log-concavity is inherited by right-hand integrals as well as by left-hand integrals. According to Theorem 3, if the density function is log-concave, the reliability function, as well as the cumulative distribution function, will be log-concave. Furthermore, log-concavity of the reliability function is inherited by its right-hand integral.

Theorem 3 *Let f be a probability density function whose support is the interval (a, b), and let $\bar{F}$ be the corresponding reliability function:*

- *If the density function f is continuously differentiable and log-concave on (a, b), then $\bar{F}$ is also log-concave on (a, b).*
- *If $\bar{F}$ is log-concave on (a, b), then the right hand integral H of the reliability function, defined by $H(x) = \int_x^b \bar{F}(t)dt$, is also log-concave on (a, b).*

Corollaries 2 and 3 are useful consequences of Theorem 3.

Corollary 2 *If the density function f is log-concave on (a, b), then the failure rate $r(x)$ is monotone increasing on (a, b).*

Proof: The failure rate is $r(x) = f(x)/\bar{F}(x) = -\bar{F}'(x)/\bar{F}(x)$. From Theorem 3, it follows that if f is log-concave, then $\bar{F}$ is also log-concave, and hence $\bar{F}'(x)/\bar{F}(x) = -r(x)$ is decreasing in x, so that $r(x)$ is increasing in x. □

Corollary 3 *If the density function f is monotone increasing, then the reliability function, $\bar{F}$, is log-concave and the failure rate is monotone increasing.*

Proof: Since $\bar{F}$ is a reliability function, it must be monotone decreasing. Therefore if f is monotone increasing, the failure rate $f/\bar{F}$ must be monotone increasing. But increasing failure rate is equivalent to a log-concave reliability function, which implies that the failure rate is monotone increasing and mean-residual-lifetime is monotone decreasing. □

Remark 1 The converse of Corollary 2 is not true. There exist probability distributions with monotone increasing failure rates but without log-concave density functions.

The "Mirror-image Pareto distribution," which is presented later in this paper, is an example of a distribution with monotone increasing failure rate, but with a density function that is log-convex rather than log-concave.

3.3 Reliability Functions (Sometimes) Inherit Log-convexity

Theorem 4 does for right hand integrals what Theorem 2 does for left hand integrals. The reliability function will inherit log-convexity from the density function if the density function approaches zero at the *upper* end of the interval (a, b).

Theorem 4 *Let f be a probability density function whose support is the interval (a, b), and let $\bar{F}$ be the corresponding reliability function:*

- *If f is continuously differentiable and log-convex on (a, b) and if $f(b) = 0$, then $\bar{F}$ is also log-convex on (a, b).*
- *If $\bar{F}$ is log-convex on (a, b), then the right hand integral H, defined by $H(x) = \int_x^b \bar{F}(t)dt$, is also log-convex on (a, b).*

4 Log-concavity Begets Monotonicity

4.1 The Mean-Residual-Lifetime Function

In the industrial engineering literature, the *mean-residual-lifetime function* MRL is defined so that $MRL(x)$ is the expected length of time before a machine that is currently of age x will break down. Suppose that the density function of length of life is given by a function f with support (a, b) and the corresponding reliability function is $\bar{F}$. Then the probability that a machine which has survived to age x will survive to age $t > x$ is $f(t)/\bar{F}(x)$. The mean residual lifetime function is therefore given by:

$$MRL(x) = \int_x^b t \frac{f(t)}{\bar{F}(x)} dt - x.$$

If $MRL(x)$ is a monotone decreasing function, then a machine will "age" with the passage of time, in the sense that it's expected remaining lifetime will diminish as it gets older. This property has been studied by Muth [27] and Swarz [37].

4.2 The Mean-Advantage-Over-Inferiors Function

The *mean-residual-lifetime function* has a mirror image, which we will call the *mean-advantage-over-inferiors function.*[4] In the case of length of life, the mean advantage over inferiors is the difference between the age x of a machine that has not broken down and the average age at breakdown of the machines that it has outlasted. Suppose that the survival density function f for machines has support (a, b). For any x and t, the conditional probability that a machine broke down at age t, given that it did not survive to age x, is $f(t)/F(x)$. The average age at breakdown of machines that broke down before age x is therefore $\int_a^x t(f(t)/F(x))dt$. The mean advantage over inferiors of a machine that survives to exactly age x is defined to be:

$$\delta(x) = x - \int_a^x t \frac{f(t)}{F(x)} dt.$$

We are particularly interested in the question of when the function $\delta(x)$ is monotone increasing in x. As we will demonstrate, this property has important implications in the economics of information and product quality. The application explored here is a variant of George Akerlof's "lemons" model, in which credible appraisal is possible but costly. [1]

4.3 Log-concavity and Monotonic Differences

One reason to be interested in log-concavity of the left hand integral of the cumulative distribution function $G(x) = \int_a^x F(t)dt$ and of the right hand integral of the reliability function $H(x) = \int_x^b \bar{F}(t)dt$ is that these properties are equivalent to monotonicity of the mean-advantage-over-inferiors and mean-residual-lifetime functions, respectively.

Lemma 1 *The mean-advantage-over-inferiors function $\delta(x)$ is monotone increasing if and only if $G(x)$ is log-concave.*[5]

Proof:

Integrating

$$\delta(x) = x - \int_a^x t \frac{f(t)}{F(x)} dt$$

by parts, we have

$$\delta(x) = x - \frac{xF(x) - \int_a^x F(t)dt}{F(x)} = \frac{\int_a^x F(t)dt}{F(x)} = \frac{G(x)}{G'(x)}.$$

[4] We find it a bit surprising that our invidious civilization has not created a common English word for this idea, but we haven't been able to find such a word.

[5] This result was previously reported and proved by Arthur Goldberger [19]. Goldberger attributes his proof to Gary Chamberlin.

Therefore $\delta(x)$ is monotone increasing if and only if $G'(x)/G(x)$ is monotone decreasing. The conclusion of Lemma 1 follows immediately from Remark 2. $\Box$

Combining the results of Lemma 1 and Theorem 1, we have the following.

Theorem 5 *The mean-advantage-over-inferiors function $\delta(x)$ is monotone increasing if either the density function f or the cumulative distribution function F is log-concave.*

Lemma 2 *The mean-residual-lifetime function $MRL(x)$ is monotone decreasing if and only if $H(x)$ is log-concave.*

Proof: Integrating

$$MRL(x) = \int_x^h \bar{f}(t)dt/\bar{F}(x) - x$$

by parts and noticing that $f(t) = -\bar{F}'(t)$, one finds that

$$MRL(x) = \frac{x\bar{F}(x) - \int_x^b \bar{F}(t)dt}{\bar{F}(x)} - x = \frac{-H(x)}{H'(x)}.$$

It follows that $MRL(x)$ is monotone increasing if and only if $H'(x)/H(x)$ is monotone decreasing. But $H'(x)/H(x)$ is monotone decreasing if and only if H is log-concave. $\Box$

Combining the results of Lemma 2 and Theorem 3, we have Theorem 6.

Theorem 6 *The mean residual lifetime function $MRL(x)$ will be monotone decreasing if the density function $f(x)$ is log-concave or if the reliability function $\bar{F}$ is log-concave.*

Since $\bar{F}$ is log-concave if and only if the failure rate is increasing, the following is an immediate consequence of Theorem 6.[6]

Corollary 4 *If the failure rate is monotone increasing, then the mean-residual lifetime function is monotone decreasing.*

4.4 Lemons with Costly Appraisals–An Application

Consider a population of used cars of varying quality all of which must be sold by their current owners. The current owner of each used car knows its quality, but buyers know only the probability density function f of quality in the population. At a cost of $\$c$, any used-car owner can have it credibly and accurately appraised, so that buyers will know its actual value. There is a large number of potential buyers, and a used car of quality x is worth $\$x$ to any of these buyers.

[6] This result is proved, in the industrial engineering literature, by Muth [27].

In equilibrium for this market, there will be a pivotal quality, x^*, such that the owners of used cars of quality $x > x^*$ choose to have their objects appraised, in which case they can sell their used cars for their actual values x and receive a net return of $\$x - c$. Owners of used cars worse than x^* will not have them appraised and will be able to sell them for the average value of unappraised used cars, which in this case is the average value of used cars that are no better than x^*. The owner of a used car of quality x^* will be indifferent between appraising and not appraising. If the owner of a used car of quality x^* has it appraised, she will get a net revenue of $x^* - c$. If she does not have her object appraised, she will be able to sell it for $\int_a^{x^*} tf(t)/F(x^*)dt$. Since this owner is indifferent between appraising and not appraising, it must be that

$$x^* - c = \int_a^{x^*} t\frac{f(t)}{F(x^*)}dt,$$

or equivalently that $\delta(x^*) = c$. If the function $\delta(\cdot)$ is monotone increasing, there will be a unique solution for the pivotal quality x^*. Moreover, if δ is not monotone increasing, there will be multiple equilibria for at least some values of c. [7]

5 Transformations, Truncations, and Mirror Images

5.1 Transformations

Some commonly-used distribution functions are defined by applying a simpler distribution to a transformed variable. For example, the lognormal distribution is defined on $(0, \infty)$ by the cumulative distribution function $F(x) = N(\ln(x))$ where N is the c.d.f. of the normal distribution. It happens that the normal distribution has a log-concave density function, and the transformation function $\ln(x)$ is a monotone increasing concave function. These two facts turn out to be sufficient to imply that the c.d.f. of the lognormal distribution is log-concave. On the other hand, the density function of the log-normal distribution is not log-concave.

Theorem 7 establishes the inheritance of log-concavity and log-convexity under concave and convex transformations of variables.

Theorem 7 *Let F be a positive-valued, twice-differentiable function with support (a, b) and let t be a monotonic, twice-differentiable function from (a', b') to $(a, b) = (t(a'), t(b'))$ Define the function $\hat{F}$ with support (a', b') so that for all $x \in (a', b')$, $\hat{F}(x) = F(t(x))$.*

[7] The function $\delta(x)$ must be increasing over some range, since $\delta(a) = 0$ and $\delta(b) > 0$. Therefore if δ is not a monotone increasing function, it will be increasing over some range and decreasing over other ranges and hence for at least some values of c there will be multiple solutions.

– *If F is log-concave and t is a concave function, then $\hat{F}$ is log-concave.*
– *If F is log-convex and t is a convex function, then $\hat{F}$ is log-convex.*

Proof: Calculation shows that $(\ln F(x))''$ is of the same sign as $\frac{F''(x)}{F'(x)} - \frac{F'(x)}{F(x)}$, and $(\ln \hat{F}(x))''$ is of the same sign as $\frac{F''(x)}{F'(x)} + \frac{t''(x)}{t'(x)} - \frac{F'(x)}{F(x)}$.

If t is a concave function, then $\frac{t''(x)}{t'(x)} \leq 0$ and therefore if F is log-concave, it must be that $\frac{F''(x)}{F'(x)} + \frac{t''(x)}{t'(x)} - \frac{F'(x)}{F(x)} \leq 0$, which implies that $\hat{F}$ is log-concave.

If t is a convex function, then $\frac{t''(x)}{t'(x)} \geq 0$ and therefore if F is log-convex, it must be that $\frac{F''(x)}{F'(x)} + \frac{t''(x)}{t'(x)} - \frac{F'(x)}{F(x)} \geq 0$, which implies that $\hat{F}$ is log-convex. □

Linear transformations are both concave and convex. Therefore, as a corollary of Theorem 7, we can conclude that both log-concavity and log-convexity are preserved under linear transformations of variables, as described in Corollary 5. This result will be seen to have many useful applications.

Corollary 5
Let F be a function with support (a, b). Let t be a linear transformation from the real line to itself and define a function $\hat{F}$ with support $(t(a), t(b))$ so that $\hat{F}(x) = F(t(x))$.

– *If F is log-concave, then $\hat{F}$ is log-concave.*
– *If F is log-convex, then $\hat{F}$ is log-convex.*

5.2 Mirror-image Transformations

Consider a cumulative distribution function F and support (a, b). This distribution can be used to define another cumulative distribution function F^*, with support $(-b, -a)$, by setting $F^*(x) = \bar{F}(-x) = 1 - F(-x)$. The function F^*, defined in this way will be called the "mirror-image" of F, since the graphs of their density functions will be mirror-images, reflected around $x = 0$.

Theorem 8 *Let F and F^* be mirror-image cumulative distribution functions:.*

– *If the density function for either F or F^* is log-concave (log-convex), then so is the density function for the other.*
– *The c.d.f. for one of these functions is log-concave if and only if the reliability function of the other is log-concave.*
– *The mean-advantage-over-inferiors function for F^* is increasing (decreasing) if and only if the mean-residual-lifetime function for F is decreasing (increasing).*

Proof:

Since $F^*(x) = 1 - F(-x) = \bar{F}(-x)$, it must be that $F^{*\prime}(x) = F'(x)$. Therefore where f^* and f are the density functions for F^* and F, respectively, $f^*(x) = f(-x)$ for all x. Since $f^*(x) = f(-x)$, these two densities are related by a linear transformation of the variable x. It follows from Corollary 5 that f^* is log-concave (log-convex) if and only if f is log-concave (log-convex).

Since $F^*(x) = \bar{F}(-x)$, it also follows from Corollary 5 that F^* is log-concave (log-convex) if and only $\bar{F}$ is log-concave (log-convex).

The mean-advantage-over-inferiors function for F is monotone increasing (decreasing) in x if and only if G is a log-concave (log-convex) function of x, where $G(x) = \int_a^x F(t)dt$. The mean-residual-lifetime function for F^* is monotone decreasing (increasing) in x if and only if H^* is log-concave (log-convex), where $H^*(x) = \int_{-x}^{-a} \bar{F}^*(t)dt$. But $\bar{F}^*(x) = F(-x)$, so that $H^*(x) = \int_{-x}^{-a} F(-t)dt = \int_a^x F(t)dt = G(x)$. Since $H^*(x) = G(x)$, for all x, it must be that H^* is log-convex (log-concave) if and only if G is log-convex (log-concave). $\square$

If a probability distribution has a density function that is symmetric around zero, then this distribution will be its own mirror-image. In this case Theorem 8 has the following consequence.

Corollary 6 *If a probability distribution has a density function that is symmetric around zero, then*

- *The c.d.f. will be log-concave (log-convex) if and only if the reliability function is log-concave (log-convex).*
- *The mean-advantage over-inferiors function will be monotone increasing if and only if the mean-residual-lifetime is monotone decreasing.*

5.3 Truncations

Suppose that a probability distribution with support (a, b) is "truncated" to construct a new distribution function in which the probability mass is restricted to a subinterval, (a^*, b^*), of (a, b) while the relative probability density of any two points in this subinterval is unchanged. If F is the c.d.f. of the original distribution and F^* is the density function of the truncated distribution, then it must be that

$$F^*(x) = \frac{F(x) - F(a^*)}{F(b^*) - F(a^*)}.$$

But this means that the distribution function F^* is just a linear transformation of the F. It follows that the corresponding density functions are also linear transformations of each other, as are the left and right hand integrals of F and F^*. Applying Corollary 5, we can conclude the following.

Theorem 9 *If a probability distribution has a log-concave (log-convex) density function (cumulative distribution function), then any truncation of this probability distribution will also have a log-concave (log-convex) density function (cumulative distribution function).*

6 Log-concavity of Some Common Distributions

This section contains a catalog of information about the log-concavity and log-convexity of density functions, distribution functions, reliability functions, and of the integrals of the distribution functions and reliability functions. Descriptions and discussions of these distributions can be found in reference works by Patel, Kapadia, and Owen [30], Johnson and Kotz [22], and Patil, Boswell, and Ratnaparkhi [31], and Evans, Hastings, and Peacock [16]. None of these references deal extensively with log-concavity. Patel et. al. report results on the monotonicity of failure rates and mean residual lifetime functions for some of the distributions that are most commonly studied by reliability theorists.

Whatever we learn about log-concavity of distributions applies immediately to truncations of these distributions, since log-concavity of a density function or of its integrals is inherited under truncation. [8]

In the tables below, we usually describe distributions in a "standardized form," where the linear transformation that sets the scale and the "zero" of random variable is chosen for simplicity of the expression. Recall from Theorem 7 that log-concavity is preserved under linear transformations, so that the results listed here apply to the entire family of distributions defined by linear transformations of the random variable x in any of these distributions.

6.1 Distributions with log-concave density functions

For distributions that have log-concave density functions, it is easy to determine the log-concavity of the distribution function and reliability function and the monotonicity of failure rates, of mean-advantage-over-inferiors, and of mean-residual-lifetime functions. If the density function f is log-concave, then we know from Theorem 1 that the cumulative distribution function F and the left-hand integral of the cumulative distribution function G are also

[8] Reliability theorists normally concern themselves only with distributions that are bounded from below by zero. It may therefore seem surprising that we apply the definitions of reliability theory to distributions whose support may be unbounded from below. For our purposes, this is justified, since log-concavity is preserved under truncations of random variables. If we find that a distribution has, for example, a log-concave reliability function with a support that is unbounded from below, then we know that any truncation of this distribution from below is log-concave and has a support with a lower bound.

log-concave. From Theorem 3 and its corollary, we know that the reliability function $\bar{F}$ and its right-hand integral H are log-concave, and that the failure rate (hazard function) $r(x)$ is monotone increasing. From Theorem 5 we know that the mean-advantage-over-inferiors function $\delta(x)$ is monotone increasing, and from Theorem 6, we know that the mean-residual-lifetime function $MRL(x)$is monotone decreasing.

Table 1 lists several commonly-used continuous, univariate probability distributions that have log-concave density functions. For all of these distributions except the Laplace distribution, we can verify log-concavity of the density function f by checking that $(\ln f(x))'' \leq 0$ for all x in the support of f. Some distributions, such as the Weibull distribution, the power function distribution, the beta function, and the gamma function have log-concave density functions only if their parameters fall into certain ranges. The parameter ranges where these distributions are log-concave are indicated in Table 1.

Table 1–Distributions with Log-concave Density Functions
(Distribution functions marked $*$ lack a closed-form representation.)

Name of Distribution	Support	Density Function f(x)	Cumulative Dist function F(x)	$(\ln f(x))''$
Uniform	$[0,1]$	1	x	0
Normal	$(-\infty,\infty)$	$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$	$*$	-1
Exponential	$(0,\infty)$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	0
Logistic	$(-\infty,\infty)$	$\frac{e^{-x}}{(1+e^{-x})^2}$	$\frac{1}{(1+e^{-x})^2}$	$-2f(x)$
Extreme Value	$(-\infty,\infty)$	$e^{-x}\exp\{-e^{-x}\}$	$\exp\{-e^{-x}\}$	$-e^{-x}$
Laplace (Double Exponential)	$(-\infty,\infty)$	$\frac{1}{2}e^{-\lvert x\rvert}$	$e^{\lambda x}$ if $x \leq 0$ $1-\frac{1}{2}e^{-x}$ if $x \geq 0$	0 for $x \neq 0$
Power Function ($c \geq 1$)	$(0,1]$	cx^{c-1}	x^c	$\frac{1-c}{x^2}$
Weibull ($c \geq 1$)	$[0,\infty)$	$cx^{c-1}e^{-x^c}$	$1-e^{-x^c}$	$\frac{1-c}{x^2}(1+cx^c)$
Gamma ($c \geq 1$)	$[0,\infty)$	$\frac{x^{c-1}e^{-x}}{\Gamma(c)}$	$*$	$\frac{1-c}{x^2}$
Chi-Squared ($c \geq 2$)	$[0,\infty)$	$\frac{x^{(c-2)/2}e^{-x/2}}{2^{c/2}\Gamma(c/2)}$	$*$	$\frac{2-c}{2x^2}$
Chi ($c \geq 1$)	$[0,\infty)$	$\frac{x^{c-1}e^{-x^2/2}}{2^{(c-2)/2}\Gamma(c)}$	$*$	$\frac{1-c}{x^2}-1$
Beta ($\nu \geq 1$, $\omega \geq 1$)	$[0,1]$	$\frac{x^{\nu-1}(1-x)^{\omega-1}}{B(\nu,\omega)}$	$*$	$\frac{1-\nu}{x^2}+\frac{1-\omega}{(1-x)^2}$
Maxwell	This is a Chi distribution with $c=3$			
Rayleigh	This is a Weibull distribution with $c=2$			

6.2 Distributions whose Density Functions are Not Log-Concave

Where the density function is not log-concave, determining the properties of the the cumulative distribution function F, the reliability function, $\bar{F}$, the mean-advantage-over inferiors function, and the mean-residual-lifetime function is a more complicated task.

One possible outcome is that f is log-convex. As is shown by the examples below, some distributions with log-convex density functions have log-concave c.d.f.'s, some have log-convex c.d.f.'s and some have c.d.f.'s which are neither log-concave nor log-convex.

For some probability distribution functions, f is neither log-concave nor log-convex but is log-concave over some interval of its support and log-convex over another interval.

Table 2 describes several distributions that do not have log-concave density functions.

Table 2–Distributions without Log-concave Density Functions

Name of Distribution	Support	Density Function f(x)	c.d.f. F(x)	$(\ln f(x))''$
Power ($c < 1$)	$(0, 1]$	cx^{c-1}	x^c	$\frac{1-c}{x^2}$
Weibull ($c < 1$)	$(0, \infty)$	$cx^{c-1}e^{-x^c}$	$1 - e^{-x^c}$	$\frac{1-c}{x^2}(1 + cx^c)$
Gamma ($c < 1$)	$(0, \infty)$	$\frac{x^{c-1}e^{-x}}{\Gamma(c)}$	*	$\frac{1-c}{x^2}$
Beta ($\nu > 1$ or $\omega > 1$)	$[0, 1]$	$\frac{x^{\nu-1}(1-x)^{\omega-1}}{B(\nu,\omega)}$	*	$\frac{1-\nu}{x^2} + \frac{1-\omega}{(1-x)^2}$
Arc-sine	$[0, 1]$	$\frac{1}{\pi\sqrt{x(1-x)}}$	$\frac{2}{\pi}\sin^{-1}(x)$	$\frac{1-2x}{2x^2(1-x^2)}$
Pareto	$[1, \infty)$	$\beta x^{-\beta-1}$	$1 - x^{-\beta}$	$(\frac{\beta+1}{x})^2$
Lognormal	$(0, \infty)$	$\frac{1}{x\sqrt{2\pi}}e^{-(\ln x)^2/2}$	*	$\frac{\ln x}{x^2}$
Student's t	$(-\infty, \infty)$	$\frac{(1+\frac{x^2}{n})^{-n+1/2}}{\sqrt{n}B(.5,n/2)}$	*	$(1-2n)\frac{n-x^2}{(n+x^2)^2}$
Cauchy	$(-\infty, \infty)$	$\frac{1}{\pi(1+x^2)}$	$\frac{1}{2} + \frac{tan^{-1}(x)}{\pi}$	$2\frac{x^2-1}{(x^2+1)^2}$
F distribution	$(0, \infty)$	See discussion of F distribution below		
Mirror-Image Pareto	$(-\infty, -1)$	$\beta x^{-\beta-1}$	$(-x)^\beta$	$(\frac{\beta+1}{x})^2$

Table 3 reports the log-convexity or log-concavity of density functions, distribution functions, and reliability functions, as well as the monotonicity of the mean-advantage-over-inferiors function $\delta(x)$ and the mean-residual-lifetime function $MRL(x)$.

Table 3 Properties of Distributions without Log-concave Density

Name of Distribution	Density Function	c.d.f	$\delta(x)$	Reliability Function	$MRL(x)$
Power Function ($0 < c < 1$)	log-convex	log-concave	increasing	neither	nonmonotonic
Weibull ($0 < c < 1$)	log-convex	log-concave	increasing	log-convex	decreasing
Gamma ($0 < c < 1$)	log-convex	log-concave	increasing	log-convex	decreasing
Arc-Sine	log-convex	neither	nonmonotonic	neither	nonmonotonic
Pareto	log-convex	log-concave	increasing	log-convex	increasing
Lognormal	neither	log-concave	increasing	neither	nonmonotonic
Student's t	neither	neither	nonmonotonic	neither	nonmonotonic
Cauchy	neither	neither	undefined	neither	nonmonotonic
Mirror-Image Pareto	log-convex	log-convex	decreasing	log-concave	decreasing
Beta ($\nu > 1$ or $\omega > 1$)	See discussion of Beta distribution below				
F distribution	See discussion of F distribution below				

6.3 Remarks on Specific Distributions

The Uniform Distribution For the uniform distribution, there are simple algebraic expressions for all of the functions studied in this paper. The mean-advantage-over-inferiors function is $\delta(x) = \int_0^x F(t)dt/F(x) = x/2$, the failure rate (hazard function) is $r(x) = f(x)/\bar{F}(x) = \frac{1}{1-x}$, and the mean-residual-lifetime function is $MRL(x) = \int_x^1 \bar{F}(t)dt/\bar{F}(t) = (1-x)/2$.

The Normal Distribution The normal cumulative distribution function illustrates the usefulness of Theorems 1-4, since there do not exist closed-form expression for the c.d.f. or for the functions, $\delta(x)$, $r(x)$, and $MRL(x)$. Thus we are able to determine that the functions $\delta(x)$ and $r(x)$ are monotone increasing and that $MRL(x)$ is monotone decreasing, despite the fact that we can not write out these functions and calculate their derivatives.

The Extreme-Value Distribution The extreme value distribution arises as the limit as $n \to \infty$ of the greatest value among n independent random variables. This is sometimes known as the *Gumbel distribution*, or as a Type 1 Extreme Value distribution. In demography, this distribution is known as the *Gompertz distribution* and is frequently used to model the distribution of the length of human lives.

The Exponential Distribution Barlow and Proschan [7] point out that the exponential distribution is the only distribution for which the failure rate and the mean residual lifetime are constant. In most applications, the exponential distribution is written with the decay parameter λ. The failure rate is $f(x)/\bar{F}(x) = \lambda$. The mean residual lifetime function is $MRL(x) = \int_x^h \bar{F}(t)dt/\bar{F}(x) = \frac{1}{\lambda}$. If the lifetime of an object has an exponential distribution, then it does not "wear out" over time. That is to say, the probability of failure and the expected remaining lifetime remain constant so long as the object "survives".

The Laplace Distribution The Laplace density function is sometimes known as the *double exponential distribution*, since it is proportional to the exponential density for positive x and to the mirror-image of the exponential distribution for negative x. For the Laplace distribution, $\ln f(x) = -\lambda|x|$. The derivative of $\ln f(x)$ does not exist at $x = 0$, so that we can not verify log-concavity from the second derivative. However, concavity of the function $-\lambda|x|$ can be verified directly from the definition.

The Power Function Distribution The power function distribution has support $(a, b] = (0, 1]$, density function $f(x) = cX^{c-1}$, and c.d.f. $F(x) = x^c$. The mean-advantage-over-inferiors function is

$$\delta(x) = \frac{\int_a^x F(t)dt}{F(x)} = \frac{x}{1+c}.$$

Since $(\ln f(x))'' = \frac{c-1}{x^2}$, we see that f is strictly log-concave if $c > 1$, strictly log-convex if $0 < c < 1$, and log-linear (and hence both log-concave and log-convex) if $c = 1$.

If $0 < c < 1$, $f(a) = f(0) = \infty$ and $f(b) = f(1) = c$. Therefore neither Theorem 2 nor Theorem 4 applies, and we cannot use these theorems to conclude that either F or $\bar{F}$ inherits log-convexity from f. In fact, we can verify that F is log-concave by observing that $(\ln F(x))'' = \frac{-c}{x^2} < 0$. We also see by inspection that $\delta(x) = \frac{x}{1+c}$ is monotone increasing in x. Since $\bar{F}(x) = 1 - x^c$, calculation shows that

$$(\ln \bar{F}(x))'' = \frac{cx^{c-2}(1-c-x^c)}{(1-x^c)^2}.$$

Therefore $(\ln \bar{F}(x))''$ is negative for x close to 1 and positive for x close to 0, and hence $\bar{F}$ is neither log-concave nor log-convex. The right hand integral of the reliability function is $H(x) = \frac{c+x^{c+1}}{1+c} - x$ This function is found to be neither log-concave nor log-convex. Therefore the mean-residual-lifetime function is neither monotone decreasing nor monotone increasing.

The Weibull Distribution The Weibull distribution has support $(a, b) = (0, \infty)$ and density function,

$$f(x) = cx^{c-1}e^{-x^c}.$$

Calculation shows that $(\ln f(x))' = \frac{c-1}{x} - cx^{c-1}e^{-x^c}$, and $(\ln f(x))'' = \frac{1-c}{x^2}(1+cx^c)$. The sign of $(\ln f(x))''$ is negative, zero, or positive, respectively, as $c > 1$, $c = 1$, or $c < 1$. Therefore the Weibull distribution is log-concave if $c > 1$, log-linear if $c = 1$, and log-convex if $c < 1$.

If $0 < c < 1$, $f(a) = f(0) = \infty$ and $f(b) = f(\infty) = 0$. Since $f(b) = 0$, we can conclude from Theorem 4 that $\bar{F}$ is log-convex. Therefore the failure rate $r(x)$ is monotone decreasing and, by Theorem 6, mean residual lifetime is an *increasing* function of age.

Since $f(a) \neq 0$, we cannot conclude from Theorem 2 that F inherits log-convexity from f for $0 < c < 1$. In fact, we can establish by other means that in this case F is log-concave, rather than log-convex. If $0 < c < 1$, $(\ln f(x))' < 0$ for all $x > 0$. Therefore $f(x)$ is seen to be a monotone decreasing function, and by Corollary 1, it must be that F is log-concave, the left hand integral G is log-concave. From Theorem 5, it follows that $\delta(x)$ is monotone increasing.

The Gamma Distribution The Gamma distribution has support $(a, b) = (0, \infty)$ and density function $f(x) = \frac{x^{c-1}e^{-x}}{\Gamma(c)}$. Calculation shows that $(\ln f(x))' = \frac{(c-1)}{x} - 1$, and $(\ln f(x))'' = \frac{1-c}{x^2}$. The sign of $(\ln f(x))''$ is negative, zero, or positive, respectively, as $c > 1$, $c = 1$, or $c < 1$. Therefore the Gamma distribution is log-concave if $c > 1$, log-linear if $c = 1$, and log-convex if $c < 1$.

For the Gamma distribution with $c < 1$, we have $f(a) = f(0) = \infty$ and $f(b) = f(\infty) = 0$. Since $f(b) = 0$, it follows from Theorem 4 that if $c < 1$ the reliability function $\bar{F}$ and its right hand integral H both inherit log-convexity from f. Since $\bar{F}$ and H are log-convex, the failure rate must be decreasing in x, and the mean-residual-lifetime function must be increasing in x.

Since $f(a) \neq 0$, Theorem 2 does not establish log-convexity of the cumulative distribution function F. In fact, when $0 < c < 1$, we see that $(\ln f(x))' < 0$ for all $x > 0$, so that f is monotone decreasing on (a, b). It follows from Corollary 1 that the cumulative distribution function F is log-concave and from Theorem 1 it follows that G, the left hand integral of F is also log-concave. Theorem 5, therefore implies that $\delta(x)$ is monotone increasing.

The Chi-squared Distribution The Chi-square distribution with c degrees of freedom is a gamma distribution with parameter $c/2$. The most common application of the Chi-squared distribution comes from the fact that the sum of the squares of c independent standard normal random variables has a chi-square distribution with c degrees of freedom. Since the gamma distribution has a log-concave density function for $c \geq 1$, it must be that the sum of the squares of two or more independent standard normal random variables has a log-concave density function.

The Chi Distribution Since $(\ln f(x))'' = -\frac{c-1}{x^2} - 1$, the chi distribution has a log-concave density function for $c \geq 1$.

The sample standard deviation from the sum of n independent standard normal variables has a chi distribution with $c = n/2$. Therefore the distribution of the sum of two or more independent standard normal variables is necessarily log-concave.

The chi distribution with $c = 2$ is sometimes known as the *Rayleigh distribution* amd the chi distribution with $c = 3$ is sometimes known as the *Maxwell distribution.*

The Beta Distribution The Beta distribution has support $(a, b) = (0, 1)$ and density function

$$f(x) = \frac{x^{\nu-1}(1-x)^{\omega-1}}{B(a,b)}.$$

Calculation shows that $(\ln f(x))'' = \frac{1-\nu}{x} + \frac{1-\omega}{x}$. Therefore if $\nu \geq 1$ and $\omega \geq 1$, then the density function is log-concave.

If $\nu < 1$ and $\omega < 1$, then the density function is log-convex. But in this case, Theorems 2 and 4 are of no assistance in determining log-convexity of F or $\bar{F}$, since $f(a) = f(b) = \infty$. More definite results apply for the special case of the Beta distribution where $\nu = \omega = .5$, which is known as the Arc-sine distribution and is discussed below.

If $\nu < 1$ and $\omega > 1$, the density function is neither log-convex nor log-concave on $(0, 1)$. In this case, however, the density function is monotone decreasing on $(0, 1)$, and therefore from Corollary 1 it follows that the distribution function F is log-concave and the mean-advantage-over-inferiors function δ is monotone decreasing.

If $\nu > 1$ and $\omega < 1$, the density function is again neither log-convex nor log-concave. In this case, the density function is monotone increasing on $(0, 1)$, and therefore by Corollary 3, the reliability function $\bar{F}$ is log-concave, the failure rate is monotone increasing, and mean-residual-lifetime is monotone decreasing.

The Arc-sine Distribution The Arc-sine distribution is the special case of the Beta distribution where $\nu = \omega = .5$. The cumulative distribution function has the closed-form expression, $F(x) = \frac{2}{\pi}\sin^{-1}(x)$. For this distribution,

$$(\ln f(x))'' = \frac{1-2x}{2x^2(1-x^2)},$$

which is positive for $x < 1/2$ and negative for $x > 1/2$. The Arc-sine distribution is therefore neither log-concave nor log-convex, but is log-convex on the interval, $(0, 1/2)$ and log-concave on the interval $(1/2, 0)$. It follows that on the interval $(1/2, 1)$, the cumulative distribution is log-concave and $\delta(x)$ is monotone decreasing.

The Arc-sine distribution has the property that $\bar{F}(x) = F(1-x)$. Since $1-x < 1/2$ when $x > 1/2$ and vice versa, it must be that on the interval $(0, 1/2)$ $\bar{F}$ is log-concave and $MRL(x)$ is monotone decreasing.

The Pareto Distribution For the Pareto distribution $(\ln(f(x))' = -\frac{\beta+1}{x}$ and $(\ln f(x))'' = \frac{\beta+1}{x^2} > 0$. Thus the density function is monotone decreasing and log-convex for all x. Although f is log-convex, the condition of theorem 2 does not apply (since $f(a) = \beta > 0$) and the c.d.f is not log-convex. In fact, since f is a decreasing function, it follows from Corollary 1 that the c.d.f, $F(x)$, is log-concave and therefore from Lemma 1 it must also be that δ is monotone increasing.

The reliability function for the Pareto distribution is $\bar{F}(x) = x^{-\beta}$. Therefore $(\ln \bar{F}(x))'' = \beta/x^2 > 0$. Therefore the reliability function is log-convex. The right hand integral, $H(x) = \int_x^\infty F(t)dt$, converges if and only if $\beta > 1$ and in this case, $H(x) = \frac{1}{\beta-1}x^{1-\beta}$. In this case, $(\ln H(x))'' = \frac{\beta-1}{x^2} > 0$. Therefore $H(x)$ is log-convex and the mean residual lifetime is a decreasing function of x.

The Lognormal Distribution The log-normal distribution has support $(0, \infty)$ and a cumulative distribution function $F(x) = N(\ln(x))$ where N is the c.d.f. of the normal distribution.

Since the normal distribution has a log-concave c.d.f., it follows from Theorem 7, that the lognormal distribution also has a concave c.d.f. From Theorem 5 it then follows that $\delta(x)$ is increasing.

Unlike the normal distribution, the lognormal distribution does *not* have a log-concave density function. The lognormal density function is

$$f(x) = \frac{1}{x\sqrt{2\pi}}e^{-(\ln x)^2/2}.$$

A bit of calculation shows that

$$(\ln f(x))'' = \frac{\ln x}{x^2}.$$

Since $\ln x$ is negative for $0 < x < 1$ and positive for $x > 1$, it must be that $f(x)$ is neither log-concave nor log-convex on its entire domain, but log-concave on the interval $(0, 1)$ and log-convex on the interval $(1, \infty)$. The failure rate of a log normally distributed random variable is neither monotone increasing nor monotone decreasing. (Patel, et.al. [30]). Furthermore the mean residual lifetime for the lognormal distribution is not monotonic, but is increasing for small values and decreasing for large values of x. (see Muth [27]). We have not found an analytic proof of either of these last two propositions. As far as we can tell, they have only been demonstrated by numerical calculation and computer graphics.

Student's t Distribution Student's t distribution is defined on the entire real line with density function

$$f(x) = \frac{(1 + \frac{x^2}{n})^{-n+1/2}}{\sqrt{n}B(.5, n/2)}$$

where $B(a, b)$ is the incomplete beta function and n is referred to as the number of degrees of freedom. For the t distribution $(\ln f(x))'' = -(n+1)\frac{n-x^2}{(n+x^2)^2}$. Therefore the density function of the t distribution is log-concave on the central interval $[-\sqrt{n}, \sqrt{n}]$ and log-convex on each of the outer intervals, $[-\infty, -\sqrt{n}]$ and $[\sqrt{n}, \infty]$. Although the t distribution itself is not log-concave, a truncated t distribution will be log-concave if the truncation is restricted to a subset of the interval $[-\sqrt{n}, \sqrt{n}]$.

We do not have a general, analytic proof of the concavity or non-concavity of the c.d.f. of the t distribution. But numerical calculations show that the c.d.f is neither log-concave nor log-convex for the cases of n =1, 2, 3, 4, and 24. Since the t distribution is symmetric, the reliability function is the mirror-image of the c.d.f. Therefore if the c.d.f. is neither log-concave nor log-convex, the reliability function must also be neither concave nor convex.

The Cauchy Distribution The Cauchy distribution is a Student's t distribution with 1 degree of freedom. It is equal to the distribution of the ratio of two independent standard normal random variables.

The Cauchy distribution has density function $f(x) = \frac{1}{\pi(1+x^2)}$ and c.d.f $F(x) = 1/2 + \frac{tan^{-1}(x)}{\pi}$. Then $(\ln f(x))'' = -2\frac{x^2-1}{(x^2+1)^2}$. This expression is negative if $|x| < 1$ and positive if $|x| > 1$. Like the rest of the family of t distributions, the density function of the Cauchy distribution is neither log-concave, nor log-convex.

The integral $\int_{-\infty}^{x} F(t)dt$ does not converge for the Cauchy distribution, and therefore the function G is not well-defined.

The F Distribution The F distribution arises in statistical applications as the distribution of the ratio of two independent chi-square distributions with m_1 and m_2 degrees of freedom. The parameters m_1 and m_2, known as "degrees of freedom". The density function of an F distribution with m_1 and m_2 degrees of freedom is

$$f(x) = cx^{(m_1/2)-1}(1 + (m_1/m_2)x)^{-(m_1+m_2)/2}$$

where c is a constant that depends only on m_1 and m_2. The F distribution has support (a, b)=$(0, \infty)$.

For the F distribution,

$$(\ln f(x))'' = -(m_1/2-1)/x^2 + (m_1/m_2)^2(m_1+m_2)/2(1+m_1/m_2x)^{-2}.$$

If $m_1 > 2$, then $(\ln f(x))''$ is positive or negative depending on whether x is greater than or less than

$$\frac{m_2\sqrt{\frac{m_1-2}{m_1+m_2}}}{1-\sqrt{\frac{m_1-2}{m_1+m_2}}}.$$

Therefore the density function is neither log-concave nor log-convex when $m_1 > 2$.

If $m_1 \le 2$, then the density function is log-convex. Since $f(b) = f(\infty) = 0$, it follows from Theorem 4 that if $m_1 \le 2$, the reliability function $\bar{F}$ is log-convex and the mean-residual-lifetime function $MRL(x)$ is monotone increasing.

Mirror-image of the Pareto Distribution None of the examples listed so far has a monotone increasing mean-advantage-over-inferiors function, $\delta(x)$. Indeed, we have not come across a "named" distribution that has this property. But, according to Theorem 8, the mirror-image of a distribution that has monotone increasing mean-residual-lifetime must have monotone decreasing $\delta(x)$.

A simple probability distribution with increasing mean-residual-lifetime is the Pareto distribution. The mirror-image of the Pareto distribution has support $(-\infty, -1)$ and c.d.f. $F(x) = (-x)^{-\beta}$ where $\beta > 0$. For $\beta > 1$, $G(x) = \int_{-\infty}^{x} F(t)dt$ converges and $G(x) = (\beta-1)^{-1}(-x)^{1-\beta}$. Then $\delta(x) = G(x)/G'(x) = G(x)/F(x) = \frac{x}{1-\beta}$ and $\delta'(x) = \frac{1}{1-\beta} < 0$.

7 Notes on Related Literature

As far as we know, the earliest application of the assumption of log-concavity in the economics literature is due to Flinn and Heckman [18]. Economic applications can also be found in the industrial engineering literature in the context of *reliability theory*; see for example, Barlow and Proschan [7] and Muth, [27]. A pair of remarkable papers by Caplin and Nalebuff [11], [12] introduced Prèkopa's theorems on log-concave probability to the economics literature and applied them to voting theory and the theory of imperfect competition. Two useful theoretical papers by Mark Yuying An [3] and [4] discuss properties of log-concave and log-convex probability distributions. His paper contains several results not found here.[9] The main contributions of his paper are: 1) He shows that the standard results on inheritance of log-concavity can be established without the assumption that density functions are differentiable. 2) He pays more systematic attention to results concerning log-convexity than had been done previously. 3) He discusses the log-concavity of multivariate distributions.

Applications to labor economics and search theory Flinn and Heckman [18] consider a model of job search in which job offers arrive as a Poisson process and where the wage associated with a job offer is drawn from a random variable with distribution function F. They show that if the right hand integral of the reliability function, $H(x) = \int_{x}^{\infty}(1 - F(t))dt$ is log-concave, then with optimal search strategies, an increase in the rate of arrivals of job offers will increase the exit rate from unemployment.

Heckman and Honore [21] discuss a labor market in which workers have differing comparative advantage in each of two sectors of the economy. They show that if the distribution of differences of skills is log-concave, then incomes of workers who are able to choose occupations according to comparative advantage in a competitive market will be more equally distributed than they would be if workers were randomly assigned to sectors and paid their marginal products.

[9] An generously acknowledges an early draft of this paper, which predated his studies. In turn, our current paper has benefited from An's work. In particular, An's discussion motivated us to treat the inheritance theorems for log-convex distributions in a more systematic way. Our treatment of log-convexity is a slight generalization of that of An.

Applications to monopoly theory Consider a product whose consumers buy either one unit or none at all, and suppose that $F(\cdot)$ is the distribution function of consumers' reservation prices for this product. Then the quantity demanded at price p is proportional to $\bar{F}(p) = 1 - F(p)$ and a monopolistic seller's expected revenue $R(p)$ at price p is proportional to $p\bar{F}(p)$. Comparative statics is greatly simplified if the revenue function $R(\cdot)$ is quasi-concave. It is easy to show that log-concavity of the reliability function $\bar{F}(\cdot)$ implies quasi-concavity of $R(\cdot)$.[10] This fact finds frequent application in the economics literature. It is applied to the distribution of reservation demands for houses in Bagnoli and Khanna [6] and in a study of firm takeovers by Jegadeesh and Chowdry [13]. Segal [36] uses this assumption in his study of an optimal pricing mechanism for a monopolist who faces an unknown demand curve.

The assumption that willingness to pay is log-concavely distributed also plays a central part in the theory of price-competition with differentiated products. Dierker [15] develops foundations for a theory of price competition with differentiated products by showing that log-concavity of the distribution of certain preference parameters implies quasi-concavity of a firm's profits in its own price. Caplin and Nalebuff [11] are able to establish existence and uniqueness of equilibrium under assumptions that the density functions of the population distribution of certain preference parameters satisfy assumptions that are weaker than log-concavity. Further development of the relation between log-concavity and equlibrium in spatial markets can be found in Anderson, de Palma, and Thisse [2].

Fang and Norman [17] have discovered an important application of log-concave probability distributions to the theory of commodity bundling. They show that if a monopolist sells several goods and if each consumer's demand for any one of the bundled goods is uncorrelated with his demand for the others, then it will be more profitable for the seller to bundle these goods rather than sell them separately under the following conditions: a) the mean willingness to pay for each good exceeds marginal cost of that good b) the probability density of willingness to pay for each good is log-concave. It is well understood (see Armstrong [5]) that in the limit as bundles get large (and demands are independent), the distribution of average willingness to pay becomes highly concentrated about the mean willingness to pay and thus a bundling monopolist can capture almost all of consumers' surplus. Fang and Norman note that in order to ensure that bundling is profitable when only a small number of independently demanded commodities is available, one needs a stronger convergence result than the law of large numbers. The desired property is that the probability that the sample mean deviates from the population mean by a specified amount is *monotonically decreasing* in sample size. Not all probability distributions have this property, but

[10] In fact, as Caplin and Nalebuff [11] point out, quasi-concavity of the revenue function is implied by the condition that $1/\bar{F}(p)$ is a convex function of p, a condition which is weaker than log-concavity.

using a theorem of Proschan [33], Fang and Norman show that if the density function is log-concave, then the sample means converge monotonically as required.

Mechanism design theory With games of incomplete information, it is customary to convert the game into a game of imperfect, but complete, information by assuming that an opponent of unknown characteristics is drawn from a probability distribution over a set of possible "types" of player. For example, in the literature on contracts, it is assumed that the principal does not know a relevant characteristic of an agent. From the principal's point of view the agent's type is a random variable, with distribution function, F. It is standard to assume, as do Laffont and Tirole [23] or Corbett and de Groote [14] that F is log-concave. This assumption is required to make the optimal incentive contract invertible in the agent's type and thus to ensure a separating equilibrium. In the theory of regulation, the regulator does not know the firm's costs. Baron and Myerson [8] show that a sufficient condition for existence of a separating equilibrium is that the distribution function of types is log-concave. Rob [35] in a study of pollution claim settlements, Lewis and Sappington [24] in a study of regulatory theory, and Riordan and Sappington [34], in a study of government procurement, use essentially the same condition.

Log-concavity also arises in the analysis of auctions. Myerson and Satterthwaite [28], Matthews[26], and Maskin and Riley[25], impose conditions that are implied by log-concavity of the distribution function in order to characterize efficient auctions.

Applications to political science and law Many results from the theory of spatially differentiated markets have counterparts in the theory of voting and elections. An important paper by Nalebuff and Caplin [12] introduces powerful mathematical results that generalize the inheritance theorems for log-concave distributions and apply these concepts to voting theory and to the theory of income distribution. Weber [40] uses the assumption that individuals have single-peaked preferences and that the distribution of ideal points among individuals is log-concave to show the existence and uniqueness of equilibrium in a theory of "hierarchical" voting, where incumbents act as Stackelberg leaders with respect to potential entrants. primary elections are followed by general elections. Haimanko, LeBreton, and Weber [20] use similar assumptions to analyze equilibrium in a model where central governments use interregional redistribution to prevent succession of subgroups with divergent interests.

Cameron, Segal and Songer [10] study the transmission of information in a hierarchical court system. Their model has a lower court and a high court. The lower court hears the case, learns information that will not be directly available to the high court, and makes a decision. The high court's utility function differs from the lower court's and the high court tries to infer what the lower court learned from the decision it made. The high court must decide whether to incur the costs of reviewing the lower court's decision. There is a close parallel in logical structure to that found in the mechanism design literature.

Costly signalling As noted in Theorem 1, the mean-advantage-over-inferiors function $\delta(x)$ is increasing if and only if the left hand integral of the c.d.f. function is log-concave. The assumption that the distribution of quality has this property plays a critical role in theories of costly signalling and has found a variety of applications. Bergstrom and Bagnoli [9] develop a marriage market model in which there is asymmetric information about the quality of persons as potential marriage partners and where quality is revealed with the passage of time. In this model there is a unique equilibrium distribution of marriages by age and quality of the partners if $\delta(x)$ is increasing.

In Verrecchia [38], [39], a manager who wishes to maximize the market value of a firm must decide whether to incur a proprietary cost to disclose his information about the firm's prospects. Thus, the manager compares the market's expected value of the firm given his disclosure (less the cost of the disclosure) to the market's expected value of the firm given that the manager chooses to not disclose his private information. The resulting theory is essentially the same as that illustrated in section 4.4 of this paper.

Nöldeke and Samuelson [29] explore an evolutionary model in which males engage in costly signaling (as exemplified by the peacock's tail) to convince females that they are superior mates. The authors ask whether there can be a costly signaling equilibrium if females care about the net value of males after they have paid the cost of their signals. They assume that females choose from among n competing males. Where F is the cumulative distribution function of initial male quality, it turns out there exists an equilibrium with costly signaling if and only if the right hand integral of F^{n-1} is a log-concave function. A sufficient condition for this function to be log-concave is that the distribution function F is log-concave.

8 Appendix–Proofs of Inheritance Theorems

8.1 Proof of Theorems 1 and 2

We apply two Remarks based on elementary calculus to prove Lemma 3, from which Theorems 1 and 2 are almost immediate.

Remark 2 A continuously differentiable function $f : \Re \to \Re^+$ is log-concave (log-convex) if and only if $\frac{f'(x)}{f(x)}$ is a non-increasing (non-decreasing) function of x in (a, b).

Proof: The function $\ln f$ is concave (convex) if and only if

$$(\ln f(x))'' = \frac{d}{dx}\frac{f'(x)}{f(x)}$$

is non-positive (non-negative) for all x in (a, b). □

Remark 3 Where $F(x) = \int_a^x f(t)dt$, the function F is log-concave (log-convex) if and only if $f'(x)F(x) - f(x)^2$ is non-positive (non-negative) for all x in (a, b).

Proof: The function $\ln F$ is concave (convex) if and only if the expression

$$(\ln F(x))'' = \frac{d}{dx}\left(\frac{f(x)}{F(x)}\right) = \frac{f'(x)F(x) - f(x)^2}{F(x)^2}$$

is non-positive (non-negative) for all x in (a, b). □

Lemma 3 *Let f be a continuously-differentiable function, mapping the interval (a, b) into the positive real numbers, let $F(x) = \int_a^x f(t)dt$ for all x in (a, b), and define $f(a) = \lim_{x \to a} f(x)$. Then:*

- *If f is log-concave on (a, b), then F is also log concave on (a, b).*
- *If f is log-convex on (a, b) and if $f(a) = 0$, then F is also log convex on (a, b).*

Proof: If f is log-concave, then for all $x \in (a, b)$,

$$\frac{f'(x)}{f(x)}F(x) = \frac{f'(x)}{f(x)} \int_a^x f(t)dt \leq \int_a^x \frac{f'(t)}{f(t)} f(t)dt = \int_a^x f'(t)dt = f(x) - f(a),$$

where the inequality follows from Remark 2. Since $f(a) \geq 0$, it follows that

$$\frac{f'(x)}{f(x)}F(x) \leq f(x) - f(a) \leq f(x).$$

and therefore

$$f'(x)F(x) - f(x)^2 \leq 0.$$

From Remark 3 it follows that F is log-concave.

Reasoning similar to that of the previous paragraph leads to the conclusion that if f is log-convex and if $f(a) = 0$, then

$$\frac{f'(x)}{f(x)}F(x) \geq f(x) - f(a) \geq f(x).$$

It follows that $f'(x)F(x) - f(x)^2 \geq 0$, and then from Remark 3, it follows that F is log-convex. □

8.2 Proof of Theorems 3 and 4

We now apply Remarks 2 and 4 to prove Lemma 4, from which Theorems 3 and 4 are almost immediate.

Remark 4 Where $\bar{F}(x) = \int_x^b f(t)dt$, the function $\bar{F}$ is log-concave (log-convex) if and only if $f'(x)\bar{F}(x) + f(x)^2$ is non-negative (non-positive) for all x in (a, b).

Proof: The function $\ln \bar{F}$ is concave (convex) if and only if the expression

$$(\ln \bar{F}(x))'' = \frac{d}{dx}\left(\frac{-f(x)}{\bar{F}(x)}\right) = -\frac{f'(x)\bar{F}(x) + f(x)^2}{\bar{F}(x)^2}$$

is non-positive (non-negative) for all x in (a, b). □

Lemma 4 *Let f be a continuously-differentiable function, mapping the interval (a, b) into the positive real numbers, let $\bar{F}(x) = \int_x^b f(t)dt$ for all x in (a, b), and define $f(b) = \lim_{x \to b} f(x)$. Then:*

- *If f is log-concave on (a, b), then $\bar{F}$ is also log concave on (a, b).*
- *If f is log-convex on (a, b) and if $f(b) = 0$, then $\bar{F}$ is also log convex on (a, b).*

Proof: If f is log-concave, then for all $x \in (a, b)$,

$$\frac{f'(x)}{f(x)}\bar{F}(x) = \frac{f'(x)}{f(x)}\int_x^b f(t)dt \geq \int_x^b \frac{f'(t)}{f(t)} f(t)dt = \int_x^b f'(t)dt = f(b) - f(x),$$

where the inequality follows from Remark 2. Since $f(b) \geq 0$, it must be that

$$\frac{f'(x)}{f(x)}\bar{F}(x) \geq f(b) - f(x) \geq -f(x).$$

Therefore $f'(x)F(x) + f(x)^2 \leq 0$, and from Remark 3 it follows that $\bar{F}$ is log-concave.

Reasoning similar to that of the previous paragraph shows that if f is log-convex and if $f(b) = 0$, then

$$\frac{f'(x)}{f(x)}\bar{F}(x) \leq f(b) - f(x) = -f(x).$$

It follows that $f'(x)\bar{F}(x) + f(x)^2 \leq 0$, and from Remark 3, it then follows that $\bar{F}$ is log-convex. □

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