

UC Berkeley

UC Berkeley Electronic Theses and Dissertations

Title

Variations on Quantum Geometry

Permalink

<https://escholarship.org/uc/item/0x37w7dp>

Author

McCurdy, Shannon

Publication Date

2013

Peer reviewed|Thesis/dissertation

Variations on Quantum Geometry

by

Shannon Ries McCurdy

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Bruno Zumino, Chair

Professor Ori J. Ganor

Professor Marc A. Rieffel

Spring 2013

Variations on Quantum Geometry

Copyright 2013

by

Shannon Ries McCurdy

Abstract

Variations on Quantum Geometry

by

Shannon Ries McCurdy

Doctor of Philosophy in Physics

University of California, Berkeley

Professor Bruno Zumino, Chair

This dissertation explores various aspects of quantization and geometry. In particular, we analyze the ground states of a two-dimensional sigma-model whose target space is an elliptically fibered K_3 , with the sigma-model compactified on S^1 with boundary conditions twisted by a duality symmetry. We show that the Witten index receives contributions from two kinds of states: (i) those that can be mapped to cohomology with coefficients in a certain line bundle over the target space, and (ii) states whose wave-functions are localized at singular fibers. We also discuss the orbifold limit and possible connections with geometric quantization of the target space. We also provide a deformation quantization approach for differential forms on symplectic manifolds. After a description of the \mathbb{Z} -graded differential Poisson algebra, we introduce a covariant star product for exterior differential forms and give an explicit expression for it up to second order in the deformation parameter \hbar , in the case of symplectic manifolds. The graded differential Poisson algebra endows the manifold with a connection, not necessarily torsion-free, and places upon the connection various constraints.

To my friends and family.

Contents

I	Introduction	1
II	Ground States of Duality-twisted Sigma-Models with K_3 Target Space	10
1	Introduction	11
2	The σ-model and the mirror-symmetry twist	14
3	T^2 target space	17
3.1	Discrete symmetries	18
3.2	Mode expansion	19
3.3	Ground states	20
3.4	Fermion number	21
3.5	Other elements of $SL(2, \mathbb{Z})$	21
3.6	Relation to the noncommutative torus and the Landau problem	22
4	Elliptically fibered K_3 target spaces	23
4.1	Supersymmetry	25
4.2	Perturbative analysis	26
4.3	The Witten Index	32
4.4	The singular fibers	34
5	Other examples of mirror twists in the orbifold limit	37
5.1	K_3 as a T^4/\mathbb{Z}_2 orbifold	37
5.2	Mirror symmetry	39
6	Connection with geometric quantization	47
7	Discussion	50

III Covariant Star Product for Exterior Differential Forms on Symplectic Manifolds	52
8 Introduction	53
8.1 The Differential Poisson Calculus	54
9 A Covariant Star Product for Exterior Differential Forms	58
9.1 Hochschild Cohomology and Associativity	58
9.2 Chevalley Cohomology and Associativity	59
9.3 Star Product Equivalence	60
9.4 Explicit Form of $\alpha \star \beta$ to $\mathcal{O}(\hbar^2)$	61
10 Conclusion	63
Bibliography	64
A Appendix to Chapter III	70
A.1 The Contraction Operator	70
A.2 The connections $\nabla, \tilde{\nabla}$	71

Acknowledgments

Many thanks to my friends and family. My community has been a source strength and support; this endeavor would not have been successful without you.

I would also like to thank my committee, with special thanks to Bruno Zumino, for his engaging stories and his kindness. Special thanks also to Ori Ganor, for so generously sharing his brilliant insights and for his boundless positivity. I-Sheng Yang, Sharon Jue, Ruza Markov, and Omid Saremi: thank you for sharing the intellectual pleasure of puzzling over new physics and helping me learn the difficult skill we call research.

I would like to acknowledge the support all of my officemates over the years, most recently Kevin Grosvenor, Tom Griffin, Michael Girard, Ruza Markov, and Charles Melby-Thompson; thank you for chatting and smiling. I am grateful to Peter Shepard and Kevin Grosvenor for their valuable comments on this manuscript. I would also like to thank all of my students and fellow graduate student teachers for sharing the joy of teaching and learning physics. Anne Takizawa, Claudia Trujillo, and Donna Sakima provided essential support that helped make everything happen.

I am also indebted to the UC Berkeley Physics Department, the Berkeley Center for Theoretical Physics, the Theoretical Physics Group of Lawrence Berkeley National Laboratory, the UC Berkeley Graduate Division, the U.S. National Science Foundation (PHY-10-02399 and PHY-0457315), and the U.S. Department of Energy (ODE-AC02-05CH11231) for providing funding for this research. I would also like to thank Springer and the American Institute of Physics for the use of [43] and [66], respectively.

Part I

Introduction

String theory is the branch of theoretical physics that explores the highest energy scales that can be contemplated. At these high energies, we believe that all of the fundamental forces of nature are united. The two greatest achievements of 20th century physics, quantum field theory and general relativity, offer descriptions of sub-atomic distances and cosmological distances, respectively. However, for all their experimental successes, the two theories are not compatible, and a new theory is needed: a theory of quantum gravity. Physicists struggled for decades to find a unifying theory, most famously including Albert Einstein, but it was not until the development of string theory that physicists met with success, and the success of string theory far exceeded expectations. String theory postulates that at very short distances everything we think of as a particle is not actually point-like, but instead extended along a very small loop: a string. This simple idea leads to a theory that describes the fundamental interactions of nature, and in particular, automatically addresses the problem of quantum gravity.

General relativity describes the gravitational interactions among matter in terms of their effects on the geometry of spacetime. The familiar arc of a baseball's flight is actually a straight line through a spacetime curved by the mass of the Earth. Quantum field theory describes the behavior of the microscopic, with the quantum mechanical observables of the theory living at points in spacetime without interacting with the geometry of spacetime. Quantum field theory reveals that an electron does not move along a straight line from one place to another, but instead explores every possible path with a different probability. This non-deterministic nature and the associated wave-particle duality, makes quantum field theory so extraordinary, especially when compared to our everyday, classical intuition about how the world works.

The challenge of quantum gravity is to find a way to combine the non-deterministic nature of quantum field theory with the geometry of general relativity so that we can understand quantum geometry. However, simply combining general relativity and quantum field theory is an unsuccessful approach; the resulting theory's corrections diverge as one looks at smaller and smaller length scales. String theory spreads out interactions from points, zero dimensional plus time, to strings, one-dimensional plus time, and higher-dimensional membranes, softening the divergences in a consistent way. Not only does string theory provide a fascinating and consistent framework of quantum gravity, it also leads to a plethora of unexpected mathematical and physical features. It predicts extra dimensions of spacetime, supersymmetry, and new physical structures such as exotic states and branes which are only possible in quantum geometry. In this thesis, I explore several effects that arise in quantum gravity.

Quantization describes the process of extending a classical theory to a quantum theory. The starting point is the data of the classical theory, comprising of the classical phase space with canonical coordinates of position and momentum and the classical observables of real functions of the canonical coordinates. The ending point is

the quantum formulation of the theory, where the quantum data comprises quantum analogs of the classical data. The quantum phase space is the complex Hilbert space and the quantum observables are Hermitian operators acting on the Hilbert space.

There are several approaches to quantization, and the most familiar is canonical quantization, introduced by Dirac [28]. For a pedagogical review of approaches to quantization, see [2]. In the simple case of flat classical phase space, canonical quantization takes the space of square integrable complex functions of the position coordinates as the Hilbert space and the quantum operators $\hat{x}, \hat{p} = -i\hbar\partial_x$ as the quantum observables. The quantum operators satisfy the canonical commutation relation,

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \quad (0.1)$$

However, there are ambiguities in how to assign quantum operators to classical functions of higher order polynomials of the canonical coordinates.

Deformation quantization was originally introduced by Weyl [84], and later formalized by others [9, 10], as an attempt to provide an alternate theory of quantization for quantum mechanics. For a pedagogical review of deformation quantization, see [29, 2]. Deformation quantization approaches the question of quantization by deforming a commutative algebra of classical observables into a noncommutative, but still associative, algebra of quantum observables. In deformation quantization, classical functions on phase space are reached by the $\hbar \rightarrow 0$ limit of quantum operators. The noncommutative but associative products that appear in deformation quantization are called star products. Deformation quantization is relevant to string theory, as we will see below, and it is also a rich mathematical field in its own right.

The first star product discovered was the Weyl-Moyal star product for flat space [84, 48, 68]. It can be written in a compact form to all orders:

$$(f \star_M g)(x) = e^{\frac{i\hbar}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y=x}. \quad (0.2)$$

Here, f and g are functions of the coordinates x^i on flat space and θ^{ij} is an antisymmetric Poisson structure, in this case constant in the coordinates. The Poisson bracket, a bilinear map on functions, is given in terms of the Poisson structure in local coordinates by,

$$\{f, g\} = \theta^{ij}\partial_i f \partial_j g, \quad (0.3)$$

and it obeys the following properties,

$$\begin{aligned} \{f, g\} &= -\{g, f\}, & \{f, gh\} &= \{f, g\}h + g\{f, h\}, \\ \{f, \{g, h\}\} &+ \{g, \{h, f\}\} + \{h, \{f, g\}\} &= 0. \end{aligned} \quad (0.4)$$

The expansion to $\mathcal{O}(\hbar^2)$ of the Weyl-Moyal star product is,

$$(f \star_M g)(x) = fg + \frac{i\hbar}{2}\theta^{ij}\partial_i f \partial_j g + \frac{1}{2}\left(\frac{i\hbar}{2}\right)^2 \theta^{ij}\theta^{mn}\partial_i \partial_m f \partial_j \partial_n g + \mathcal{O}(\hbar^3).$$

The associativity property of the Weyl-Moyal star product follows easily from (0.2),

$$\begin{aligned}
((f \star_M g) \star_M h)(z) &= e^{\frac{i\hbar}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}}(f \star_M g)(x)h(z)|_{x=z} \\
&= e^{\frac{i\hbar}{2}\theta^{ij}\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial z^j}}e^{\frac{i\hbar}{2}\theta^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^l}}f(x)g(y)h(z)|_{x=y=z} \\
&= e^{\frac{i\hbar}{2}\theta^{ij}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}+\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}+\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right)}f(x)g(y)h(z)|_{x=y=z} \\
&= e^{\frac{i\hbar}{2}\theta^{ij}\frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial y^i}+\frac{\partial}{\partial z^j}\right)}e^{\frac{i\hbar}{2}\theta^{kl}\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^l}}f(x)g(y)h(z)|_{x=y=z} \\
&= (f \star_M (g \star_M h))(z).
\end{aligned}$$

On symplectic manifolds, where the Poisson bivector is invertible, it also has an equivalent integral form [84, 85, 13]:

$$\begin{aligned}
(f \star_{M,s} g)(x) &= \int W(x_1, x_2, x) f(x_1) g(x_2) d^n x_1 d^n x_2, \\
W(x_1, x_2, x) &= \frac{1}{\det(|\theta|)(\pi\hbar)^n} e^{\frac{2i}{\hbar}(x-x_1)^i \omega_{ij}(x-x_2)^j}.
\end{aligned} \tag{0.5}$$

Here, $\omega_{ij} = (\theta^{ij})^{-1}$. The argument of the exponential is $\left(\frac{4i}{\hbar}\right)$ times the symplectic area of a triangle with vertices (x_1, x_2, x) when $n = 2$.

From the work of [61, 18] and many others, the star product of functions on a general Poisson manifold is well known to all orders in the deformation parameter. The diagrammatic formula for the star product is [61],

$$f \star_K g = fg + \sum_{n=1}^{\infty} \hbar^n \sum_{\Gamma \in G_{n,2}} \omega_{\Gamma} B_{\Gamma}(f, g). \tag{0.6}$$

The $B_{\Gamma}(f, g)$ are bidifferential operators, homogeneous of degree n in the components of the Poisson structure, which here we denote by π^{ij} because it can depend on the coordinates x^i . The ω_{Γ} are weight coefficients. The Γ are admissible graphs, a subset of $G_{n,2}$, which is the set of graphs on $(n+2)$ vertices (see [61] for more details).

The result was subsequently elucidated by [18], who used a non-linear σ -model action and two real bosonic fields on a disk world sheet with a Poisson manifold target space to provide physical intuition for (0.6):

$$\begin{aligned}
(f \star_K g)(x) &= \int_{X(\infty)=x} \mathcal{D}X \mathcal{D}A f(X(1))g(X(0))e^{\frac{i}{\hbar}S[X,A]}, \\
S[X, A] &= \int_D \left(A_i(u) \wedge dX^i(u) + \frac{1}{2} \pi^{ij}(X(u)) A_i(u) \wedge A_j(u) \right).
\end{aligned} \tag{0.7}$$

The bosonic field X provides a map from the disk D with coordinates u to M , the Poisson manifold, and $0, 1$ and ∞ are three cyclically ordered points on the boundary of the disk. The bosonic field A is a one-form on D that takes values in the pull-back by X of the cotangent bundle of M .

In the symplectic case, when the Poisson structure is invertible, one can formally integrate over the one-form auxiliary field $A_i(u)$:

$$(f \star_{K,s} g)(x) = \int_{X(\infty)=x} \mathcal{D}X f(X(1))g(X(0))e^{\frac{i}{2\hbar} \int_D \omega_{ij}(X(u))dX^i(u) \wedge dX^j(u)} .$$

This can be further written as:

$$(f \star_{K,s} g)(x) = \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma f(\gamma(1))g(\gamma(0))e^{\frac{i}{\hbar} \int_\gamma d^{-1}\omega} .$$

The functional integral is over trajectories γ that map the boundary of the disk to the Poisson manifold M [18].

Star products were subsequently found to apply in many areas of physics, including string theory. In string theory, star products and noncommutative spacetime emerge naturally in the description of the quantization of open strings in the presence of non-zero but constant B -field. The noncommutativity emerges because the products of vertex operators of open strings often depends on the ordering along the boundary of the worldsheet [74].

As an example of a star product in string theory [78], take the open string in flat space with metric g , and with a constant background Neveu-Schwarz B -field, and Dp -branes. The action is a σ -model with constant background fields and a flat target space,

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma g_{ij} \partial_a x^i \partial^a x^j - \frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j . \quad (0.8)$$

For the world sheet $\Sigma = D$, a disk, the propagator on the boundary of the disk is,

$$\langle x^i(\tau) x^j(\tau') \rangle = -\alpha' G^{ij} \log(\tau - \tau')^2 + \frac{i}{2} \theta^{ij} \epsilon(\tau - \tau') ,$$

where,

$$G^{ij} = \left(\frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B} \right)^{ij} , \quad (0.9)$$

$$\theta^{ij} = -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{ij} . \quad (0.10)$$

The time-ordered commutator of the field x on the boundary is:

$$[x^i(\tau), x^j(\tau)] = T (x^i(\tau) x^j(\tau^-) - x^i(\tau) x^j(\tau^+)) = i\theta^{ij} .$$

One can think of the $x^i(\tau)$ as the coordinates of a noncommutative space with non-commutativity parameter of θ^{ij} . We see that noncommutative geometry has emerged, in this context from a worldsheet with boundary.

Furthermore, in the limit that $\alpha' \rightarrow 0$, the worldsheet action (0.8) reduces to a 0+1D action of the form,

$$S = -\frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j, \quad (0.11)$$

on the boundary of the world sheet. The propagator reduces to,

$$\langle x^i(\tau) x^j(\tau') \rangle = \frac{i}{2} \theta^{ij} \epsilon(\tau - \tau'), \quad \theta^{ij} = -\left(\frac{1}{B}\right)^{ij}.$$

The leading singularity in the product of two vertex operators $e^{ip \cdot x}(\tau)$ and $e^{iq \cdot x}(\tau')$, with $\tau > \tau'$ is,

$$: e^{ip \cdot x}(\tau) :: e^{iq \cdot x}(\tau') : \sim : e^{ip \cdot x} \star_M e^{iq \cdot x}(\tau') : .$$

This \star_M is the Weyl-Moyal star product from equation (0.2). Because \star_M is determined by associativity, translation invariance, and noncommutivity, it is the natural candidate for the leading terms of dimension-zero operator products [78].

Geometric quantization emerged from an attempt to systematize deformation quantization [62]. For a pedagogical review of geometric quantization, see [89, 34, 2]. In geometric quantization, one considers phase spaces with the additional structure of a Lie group. The geometric quantization procedure attempts to establish a correspondence between the Lie group acting on the classical phase space and classical observables and a group of unitary operators acting on the Hilbert space and quantum observables. The program of geometric quantization is as follows. First, one pre-quantizes the phase space by constructing the Hilbert space of the entire phase space. In the simple case of the one-dimensional harmonic oscillator, the pre-quantized dynamics coincide with the classical dynamics. Because the pre-quantum Hilbert space is too big, the pre-quantization procedure has failed to quantize the system. Next one tries to separate the degrees of freedom of the phase space into two groups using a polarization; for the harmonic oscillator, one can take a Kähler polarization. Now the system is quantizable, however, the energy spectrum of the theory is still not yet correct; for example, the energy spectrum of the harmonic oscillator requires a shift by $\frac{\hbar}{2}$ to match the known energy spectrum. More generally, this is a metaplectic correction, and after this correction one finally arrives at the true Hilbert space of the quantum theory. In Part II, we conjecture a relationship between geometric quantization and the ground states of a compactification with a mirror-symmetry twist, our model.

Mirror symmetry is a duality between two different geometries that is only possible at the quantum level. The discovery of the importance of quantum dualities such as mirror symmetry is one of the recent transformative discoveries in string theory. Seminal early works on mirror symmetry include [31, 64, 45]. Mirror symmetry was originally formulated as a map between two Calabi-Yau manifolds. A Calabi-Yau manifold of complex dimension d is a symplectic manifold with an integrable almost-complex structure that is compatible with the symplectic form (a Kähler manifold)

with $SU(d)$ holonomy. Mirror symmetry is a map between two Calabi-Yau manifolds, M and \tilde{M} , of complex dimension d such that their Hodge numbers are related by

$$h^{p,q}(M) = h^{d-p,q}(\tilde{M}). \quad (0.12)$$

A more general form of the symmetry, which does not require the mirror manifold to be Calabi-Yau, establishes an equivalence between a gauged linear σ -model and a Landau-Ginzburg theory. For certain cases, this map is an equivalence between two Calabi-Yau σ -models with the relation (0.12) above [56].

At the most intuitive level, one can understand mirror symmetry as an application of T -duality [79]. The simplest example of mirror symmetry occurs for the bosonic string compactified on the circle with radius R , and the mirror map is T -duality. One way to see that T -duality is a symmetry of the bosonic string compactified on the circle is to look at the spectrum of string excitations:

$$\begin{aligned} \alpha' M^2 &= \alpha' \left(\left(\frac{n}{R} \right)^2 + \left(\frac{wR}{\alpha'} \right)^2 \right) + 2(N_R + N_L) - 4, \\ nw &= N_R - N_L, \quad n, w \in \mathbb{Z}. \end{aligned} \quad (0.13)$$

Here $N_{R,L}$ are the left and right number operators, and n, w are integers that characterize the Kaluza-Klein momentum and winding number, respectively. By inspection, the spectrum is invariant under the interchange of $n \leftrightarrow w$, as long as the interchanged circle has a new radius $\tilde{R} = \frac{\alpha'}{R}$. This is the simplest example of T -duality and mirror symmetry. At the level of the bosonic fields $x_{R,L}$, the map is:

$$\tilde{x}_R = -x_R, \quad \tilde{x}_L = x_L, \quad \tilde{R} = \frac{\alpha'}{R}. \quad (0.14)$$

While this argument only shows the equivalence at the level of the spectrum, the equivalence also holds perturbatively for the full interacting string theory [11].

We would like to explore the consequences of a mirror-symmetry twist in our model. Twisted boundary conditions have been used variously for supersymmetry [73], gauge symmetries [17], Lorentz symmetries [81], and orbifolds [30]. Generic twisted boundary conditions on the world sheet with the identification $\sigma^1 + 2\pi \sim \sigma^1$ take the form:

$$x(\sigma^1 + 2\pi) = gx(\sigma^1), \quad (0.15)$$

where g is the twisting element of the symmetry in question. The case where the twisting element g is a duality of the theory has also been explored in various contexts (see, for example, [63, 65, 24, 39, 52, 25, 35, 58, 23, 60]). The mirror-symmetry twist we will consider in our model connects the fields just left of the cut at $\sigma^1 = 0$ (for all times σ^0) on the world sheet to the fields just right of the cut via the mirror-symmetry transformation.

The mirror-symmetry twist promotes mirror symmetry from a duality to a full symmetry of the theory. By this we mean the mirror \tilde{M} of M is M itself: the same manifold and at the same point in moduli-space. Since generically the mirror manifolds \tilde{M} and M do not have the same classical geometry, for example, in $d = 3$, the Euler characteristic changes sign,

$$\chi(\tilde{M}_{d=3}) = -\chi(M_{d=3}), \quad (0.16)$$

we are restricted to a rather special class of Calabi-Yau manifolds. In $d = 2$, there are two compact Calabi-Yau manifolds, K_3 and T^4 . K_3 manifolds have the following Hodge diamond, where $h^{p,q}(X)$ are the respective dimensions of the Dolbeault cohomology groups,

$$\begin{array}{ccccccc} & & h^{0,0} & & & & 1 \\ & & & & & & \\ & h^{1,0} & & h^{0,1} & & 0 & 0 \\ h^{2,0} & & h^{1,1} & & h^{0,2} & = 1 & 20 & 1 \\ & h^{2,1} & & h^{1,2} & & 0 & 0 & \\ & & h^{2,2} & & & & 1 & \end{array} \quad (0.17)$$

K_3 manifolds have many nice properties, such as the fact that any two K_3 manifolds are diffeomorphic to each other, though they may have differing complex structure. From the definition of mirror symmetry (0.12) and the Hodge diamond (0.17), we see that the mirror of a K_3 manifold is also a K_3 manifold. This makes K_3 a natural setting for the exploration of mirror-symmetry twists. In particular, we will focus on K_3 manifolds that can be represented as an elliptic fibration. Such spaces are described by fibering a torus over some base manifold, and the complex structure of the torus varies over the base.

In Part II, we consider a supersymmetric 1+1D σ -model with an elliptically fibered K_3 target space that is invariant under a suitable duality-symmetry transformation [43]. In this model, the duality-symmetry is a discrete nonperturbative symmetry: in some cases it is mirror symmetry realized through fiber-wise T -duality. We look at a compactification of this σ -model on S^1 with supersymmetric boundary conditions that include these mirror-symmetry twists. Our goal in Part II is to understand the low-energy description of this system. In particular, we study the ground states of this system and their connection to geometric quantization. The Witten index of the theory with the mirror-symmetry twist is given by,

$$I(\hat{\mathfrak{M}}) = \text{tr}\{(-1)^F \hat{\mathfrak{M}}\}, \quad (0.18)$$

where $\hat{\mathfrak{M}}$ is the mirror twist operator. The Witten index counts the difference between the number of zero-energy bosonic and fermionic states in the space with the mirror twist insertion. We show that the Witten index receives contributions from states that can be mapped to cohomology and states localized at singular fibers. We also discuss

the orbifold limit of K_3 and possible connections with geometric quantization of the target space. Through a doubling of the target space coordinates, we find evidence that the Hilbert space of ground states of a compactification of the duality-twisted theory and the Hilbert space of geometric quantization of the singular fiber are the same, and we give the operators in the IR limit. In this context, noncommutative geometry has also arisen, but this time on a closed worldsheet with modified boundary conditions.

Motivated by the appearance of noncommutative gauge theories in string theory, we also investigate a generalization of the star product for differential forms. A differential form of degree $|\alpha|$ is a covariant tensor field of degree $|\alpha|$ that is totally antisymmetric. In a local coordinate system $\{x^i, \dots, x^n\}$, a $|\alpha|$ -form α can be uniquely expressed as $\alpha = \frac{1}{|\alpha|!} \alpha_{i_1 \dots i_{|\alpha|}} dx^{i_1} \wedge \dots \wedge dx^{i_{|\alpha|}}$ where the coefficient $\alpha_{i_1 \dots i_{|\alpha|}}$ is completely antisymmetric. In Part III, we construct, to second order in the deformation parameter, a covariant star product for exterior differential forms on symplectic manifolds [66]. Since the $\mathcal{O}(\hbar)$ term of the star product on functions is the Poisson bracket, we use the differential Poisson algebra discussed in [21, 55, 12] as the $\mathcal{O}(\hbar)$ term of the star product on differentials. Noncommutative Poisson algebras are also discussed in [90]. As found in [21, 55, 12], endowing a symplectic manifold with a differential Poisson calculus is equivalent to endowing a symplectic manifold with a connection that has certain important restrictions. Modulo these restrictions, we find that it is possible to define a covariant star product to $\mathcal{O}(\hbar^2)$ on the space of differential forms on symplectic manifolds endowed with a differential Poisson calculus, and we give the explicit form. Our work was later extended and applied in [20, 83], for example.

Part II

Ground States of Duality-twisted Sigma-Models with K_3 Target Space

Chapter 1

Introduction

Consider a supersymmetric 1+1D σ -model with target space X (a Kähler manifold with metric G and 2-form field B) that is invariant under a suitable mirror-symmetry transformation, so that in this model mirror-symmetry is a discrete nonperturbative symmetry. We wish to study the Hilbert space \mathcal{H} of ground states of a compactification of this σ -model on S^1 with supersymmetric boundary conditions that include a *mirror-symmetry twist*. The mirror-symmetry twist is constructed as follows. Let (σ^0, σ^1) be the spacetime coordinates with $0 \leq \sigma^1 < 2\pi$ parameterizing S^1 , and $-\infty < \sigma^0 < \infty$ parameterizing time. To insert a mirror-symmetry twist at, say, $\sigma^1 = 0$, we first Wick rotate by setting $\sigma^0 = i\sigma^2$, then consider σ^1 as the (Euclidean) time direction and insert the operator $\hat{\mathcal{M}}$ that realizes mirror-symmetry at $\sigma^1 = 0$. Correlators of operators $\hat{\mathcal{O}}_1, \dots, \hat{\mathcal{O}}_n$ in this theory can be defined as $\text{tr}\{\hat{\mathcal{M}}(-1)^F e^{-2\pi\hat{H}} \hat{\mathcal{O}}_n \cdots \hat{\mathcal{O}}_1\}$, where \hat{H} is the Hamiltonian of the field theory, F is the fermion number, tr is the trace over the Hilbert space of the (untwisted) σ -model compactified on S^1 (which now corresponds to the space direction σ^2), and the operators are assumed to be in the Heisenberg picture and ordered according to increasing σ^1 .

Duality twists (sometimes referred to as “monodrofolds” or, in our context, as “mirrorfolds”) have been extensively studied in a variety of contexts (see for example [63, 65, 24, 39, 52, 25, 35, 58, 23, 60] for a sampling of the literature), and several interesting approaches have been developed to understand such backgrounds. For T^{2d} target spaces and their orbifolds the mirror-twist reduces to a T-duality twist, and the compactification is a sector of an asymmetric orbifold. For a K_3 target space, the spectrum of the theory at special limit points in moduli space (Gepner points) was computed in [60].

Our goal in this paper is relatively modest – to learn about the ground states of the theory, describe their wave-functions in more detail, and explore possible connections with geometric quantization of the target space. One of the motivations for this work is to develop tools to study compactifications with duality twists in gauge theory. For example, a compactification of 3+1D $\mathcal{N} = 4$ Super-Yang-Mills theory on S^1 with

an S-duality twist was studied in [40, 42, 80, 41], where a possible connection with Chern-Simons theory was also explored. This hints at a relation between our present σ -model problem and geometric quantization as follows. S-duality can be related to mirror-symmetry of a certain 1+1D σ -model by compactifying the 3+1D gauge theory on a Riemann surface [49, 15]. The σ -model's target space is the *Hitchin space* associated with the gauge group and the Riemann surface (see [59] for more details). On the other hand, the Hilbert space of Chern-Simons theory on a Riemann surface can be obtained by geometric quantization of the moduli space of flat connections on the surface [86], and the latter can naturally be embedded in the Hitchin space as the fixed locus of a certain global symmetry. Thus, the question arises whether the Hilbert space of the mirror-symmetry twisted σ -model compactification is related to the Hilbert space obtained by geometric quantization of the target space (or a suitable subspace of it).

Another motivation is to learn about the operator $\hat{\mathfrak{M}}$ that realizes mirror-symmetry. Mirror-symmetry is a nonperturbative phenomenon of 1+1D σ -models, and the Hilbert space of the mirror-twisted compactification carries information about the operator $\hat{\mathfrak{M}}$. Restricting to ground states is equivalent to taking the low-energy limit, where roughly speaking, correlators reduce to $\text{tr}\{\hat{\mathfrak{M}}(-1)^F \hat{\mathcal{O}}_n \cdots \hat{\mathcal{O}}_1\}$. Thus, understanding the Hilbert space of ground states (and the operators that act on it) is a way to probe the mirror symmetry operator $\hat{\mathfrak{M}}$.

The main focus of this paper is one particular case (with potential generalizations) — an elliptically fibered K_3 target space, with the mirror-symmetry that can be realized as T-duality on the fiber. While a lot of progress has been made in understanding mirror-symmetry (see e.g., [57, 56] and references therein) and certainly a lot is known about mirror-symmetry of K_3 , we are unaware of any explicit study of the ground states of the particular compactification that we are investigating in this paper. For the present analysis we begin with a careful study of a σ -model with T^2 target space, compactified with a T-duality twist. The Hilbert space of ground states naturally maps to the Hilbert space obtained by deformation quantization of T^2 at level 2. We then fiber this Hilbert space over the base of the elliptic fibration to construct ground states of the K_3 compactification. We then calculate the contribution of these states to the Witten index (which can be calculated by other means), and we find a discrepancy. We interpret this discrepancy as the contribution of ground states with wave-functions localized at singular fibers. We show that the two calculations of the Witten index are consistent if one ground state is associated with each singular fiber, and we bring supporting evidence for this conjecture from nonperturbative string theory.

The paper is organized as follows. In §2 we review some basic facts about supersymmetric nonlinear σ -models and define the mirror-twist. In §3 we discuss the simple case of a T^2 target space. In §4 we discuss the case of elliptically fibered K_3 surfaces with a mirror-symmetry twist that can be understood as local T-duality of the fiber. This section contains our main results for the ground states. In §5 we discuss duality

twists in the orbifold limit of the K_3 σ -model, and in §6 we further explore possible connections with geometric quantization. We conclude with a discussion in §7.

Chapter 2

The σ -model and the mirror-symmetry twist

Our spacetime is two-dimensional, and we work either in Minkowski signature with spacetime coordinates (σ^0, σ^1) or in Euclidean signature with complex (worldsheet) coordinate $z \equiv \sigma^1 + i\sigma^2$. The supersymmetric σ -models have an action of the form [91] (and we follow the conventions of [87]):

$$I_0 = \frac{1}{\pi} \int \left(\frac{1}{2}(G_{IJ} + iB_{IJ})\partial\phi^I \bar{\partial}\phi^J + \frac{i}{2}G_{IJ}\psi^I \partial\psi^J + \frac{i}{2}G_{IJ}\bar{\psi}^I \bar{\partial}\bar{\psi}^J + \frac{i}{2}G_{IJ}\psi^I \partial\phi^L \Gamma_{LK}^J \psi^K + \frac{i}{2}G_{IJ}\bar{\psi}^I \bar{\partial}\phi^L \Gamma_{LK}^J \bar{\psi}^K + \frac{1}{4}R_{IJKL}\psi^I \psi^J \bar{\psi}^K \bar{\psi}^L \right) d^2z \quad (2.1)$$

where $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$, ϕ^I ($I = 1, \dots, d$) are scalars, ψ^I and $\bar{\psi}^I$ are fermions, $G_{IJ}(\phi)$ are the components of the metric on target space X , Γ_{LK}^J are the Christoffel symbols derived from G_{IJ} , R_{IJKL} are the components of curvature, and $B_{IJ}(\phi)$ are the components of an antisymmetric B -field, assumed to be a closed 2-form. We will take X to be a Calabi-Yau manifold and label the complex coordinates by i, j, k, \dots and their complex conjugates by $\bar{i}, \bar{j}, \bar{k}, \dots$. The Kähler condition is equivalent to requiring that the metric G_{IJ} is of type $(1, 1)$ (i.e., $G_{i\bar{i}} = G_{\bar{i}i}$ are the only possible nonzero components) and further, the only allowed nonzero Christoffel symbols are Γ_{jk}^i and $\Gamma_{\bar{j}\bar{k}}^{\bar{i}}$. The Calabi-Yau condition is equivalent to having a coordinate system where $\sqrt{G} = \det G_{i\bar{i}} = 1$. We also assume that the 2-form B -field is of type $(1, 1)$ (i.e., only $B_{i\bar{i}} = -B_{\bar{i}i}$ can be nonzero).

Next, we introduce the mirror-symmetry-twist. We assume that X and the metric G and B -field are special so that “mirror-symmetry” is not only a duality but an actual symmetry. In other words, the mirror of X is the same manifold and at the same point in moduli-space. We then introduce a cut at $\sigma^1 = 0$ (for all times σ^0) and connect the fields just left of the cut to the fields just right of the cut via a mirror-symmetry transformation (which, of course, is not algebraic in the fields). Our goal is to understand the low-energy description of this system, i.e., the ground states.

The Lagrangian (2.1) is invariant under the supersymmetry transformations:

$$\begin{aligned}\delta\phi^I &= i\eta\psi^I + i\bar{\eta}\bar{\psi}^I, \\ \delta\psi^I &= -\eta\partial\phi^I - i\bar{\eta}\bar{\psi}^K\Gamma_{KM}^I\psi^M, \\ \delta\bar{\psi}^I &= -\bar{\eta}\bar{\partial}\phi^I - i\eta\psi^K\Gamma_{KM}^I\bar{\psi}^M,\end{aligned}\tag{2.2}$$

where η and $\bar{\eta}$ are left and right SUSY generators. Mirror-symmetry commutes with these transformations.

For a Kähler target space, we split the coordinates into holomorphic (labeled by i, j, k, \dots) and anti-holomorphic ($\bar{i}, \bar{j}, \bar{k}, \dots$). The nonzero metric components are of (1, 1)-type (i.e., $G_{i\bar{j}}$) and satisfy the Kähler condition ($\partial_{[i}g_{j]\bar{k}} = 0$). Defining $\bar{\psi}_i \equiv G_{i\bar{j}}\bar{\psi}^{\bar{j}}$ and $\psi_{\bar{i}} \equiv G_{j\bar{i}}\psi^j$, we can write the action as

$$\begin{aligned}I_0 &= \frac{1}{\pi} \int (G_{i\bar{j}}\partial\phi^i\bar{\partial}\phi^{\bar{j}} + i\psi_{\bar{j}}\bar{\partial}\psi^{\bar{j}} + i\bar{\psi}_i\partial\bar{\psi}^i \\ &\quad + i\bar{\partial}\phi^{\bar{i}}\Gamma_{\bar{i}k}^{\bar{j}}\psi_{\bar{j}}\psi^k + i\partial\phi^l\Gamma_{lk}^i\bar{\psi}_i\bar{\psi}^k + R_{\bar{j}}^k{}^i{}_{\bar{j}}\bar{\psi}_i\bar{\psi}^j\psi_{\bar{k}}\psi^{\bar{j}})d^2z\end{aligned}\tag{2.3}$$

For every complex structure under which the target space is Kähler there are additional SUSY transformations:

$$\begin{aligned}\delta\phi^i &= 0, \quad \delta\phi^{\bar{i}} = i\eta\psi^{\bar{i}} + i\bar{\eta}\bar{\psi}^{\bar{i}}, \\ \delta\psi^i &= -\eta\partial\phi^i, \quad \delta\psi^{\bar{i}} = -i\bar{\eta}\bar{\psi}^{\bar{j}}\Gamma_{\bar{j}k}^{\bar{i}}\psi^k, \\ \delta\bar{\psi}^i &= -\bar{\eta}\bar{\partial}\phi^i, \quad \delta\bar{\psi}^{\bar{i}} = -i\eta\psi^{\bar{j}}\Gamma_{\bar{j}k}^{\bar{i}}\bar{\psi}^k.\end{aligned}\tag{2.4}$$

Mirror-symmetry does not necessarily commute with these transformations. We will return to the question of how much SUSY is preserved in §4.1.

Later on, it will be convenient to have explicit expressions for the canonical duals of ϕ^i and $\phi^{\bar{i}}$. They are:

$$\Pi_i = \frac{1}{4}G_{i\bar{j}}\partial_0\phi^{\bar{j}} + \frac{1}{2}i\Gamma_{ij}^k\bar{\psi}_k\bar{\psi}^{\bar{j}}, \quad \Pi_{\bar{i}} = \frac{1}{4}G_{j\bar{i}}\partial_0\phi^j + \frac{1}{2}i\Gamma_{\bar{i}j}^{\bar{k}}\psi_{\bar{k}}\psi^{\bar{j}}.$$

and we have

$$[\Pi_i(\sigma_1), \phi^j(\sigma'_1)] = -i\delta_i^j\delta(\sigma_1 - \sigma'_1), \quad [\Pi_{\bar{i}}(\sigma_1), \phi^{\bar{j}}(\sigma'_1)] = -i\delta_{\bar{i}}^{\bar{j}}\delta(\sigma_1 - \sigma'_1).$$

and the fermions' commutation relations are

$$\{\psi^{\bar{i}}(\sigma_1), \psi_{\bar{j}}(\sigma'_1)\} = 4\pi\delta_{\bar{i}}^{\bar{j}}\delta(\sigma_1 - \sigma'_1), \quad \{\bar{\psi}^i(\sigma_1), \bar{\psi}_j(\sigma'_1)\} = 4\pi\delta_j^i\delta(\sigma_1 - \sigma'_1).$$

The supersymmetry generators are

$$Q = \int \psi^{\bar{i}}(\Pi_{\bar{i}} + G_{\bar{i}i}\partial_1\phi^i)d\sigma_1, \quad \bar{Q} = \int \bar{\psi}_i(G^{i\bar{i}}\Pi_{\bar{i}} + \partial_1\phi^i + \dots)d\sigma_1\tag{2.5}$$

[where (\dots) denotes higher order terms that are not important here], so that

$$\delta(\dots) = i[\eta Q + \bar{\eta} \bar{Q}, (\dots)].$$

Note that the definition of Q does not suffer from any normal ordering ambiguity, while the definition of \bar{Q} does require a normal ordering prescription (for $G_{i\bar{j}}(\phi)$ and $\Pi_{\bar{i}}$). We take this opportunity to recall that the sigma-model can be topologically twisted [87], and for our flat worldsheet $(S^1 \times \mathbb{R})$ the twisting has no effect on the spectrum. With ordinary supersymmetric (Ramond-Ramond) boundary conditions on S^1 the ground states correspond to Dolbeault cohomology, and Q is identified with $\bar{\partial}$ [87]. In the rest of this paper we will look for a geometric interpretation for the ground states of the model with boundary conditions that include a mirror-symmetry twist.

Chapter 3

T^2 target space

The simplest target space we can consider is $X = T^2$. We parameterize the metric and B -field in terms of two complex coordinates $\rho \equiv \rho_1 + i\rho_2$ and $\tau \equiv \tau_1 + i\tau_2$ as

$$ds^2 \equiv G_{IJ}dx^I dx^J = \frac{\rho^2}{\tau_2} |dx^1 + \tau dx^2|^2, \quad B_{IJ}dx^I \wedge dx^J = 2\rho_1 dx^1 \wedge dx^2,$$

where $0 \leq x^1, x^2 < 2\pi$ are periodic coordinates. The action is (and we now switch from coordinates x^I to fields ϕ^I):

$$I_0 = \frac{1}{2\pi} \int ((G_{IJ} + iB_{IJ})\partial\phi^I\bar{\partial}\phi^J + iG_{IJ}\psi^I\partial\psi^J + iG_{IJ}\bar{\psi}^I\bar{\partial}\bar{\psi}^J) d^2z. \quad (3.1)$$

T-duality acts as $\rho \rightarrow -1/\rho$, keeping τ fixed, and is a symmetry for $\rho = i$. From now on we assume that we are at the self-dual point $\rho = i$ of moduli space. Now, let us compactify direction σ^1 on S^1 so that $0 \leq \sigma^1 < 2\pi$. Usual (Ramond-Ramond) periodic boundary conditions require that the value of every fermionic field at $\sigma^1 = 0$ be equal to its value at $\sigma^1 = 2\pi$, and for bosonic fields both field and first derivative should be equal. In order to introduce a T-duality twist we first relax the condition of continuity and allow the fields at $\sigma^1 = 0$ to be independent of the fields at $\sigma^1 = 2\pi$. We denote by a subscript ($>$) or ($<$) the limiting values of the various fields as $\sigma^1 \rightarrow 0$ or $\sigma^1 \rightarrow 2\pi$:

$$\left. \begin{aligned} \phi_{(>)}^I &\equiv \phi^I(0, \sigma^2), & \phi_{(<)}^I &\equiv \phi^I(2\pi, \sigma^2), \\ \psi_{(>)}^I &\equiv \psi^I(0, \sigma^2), & \psi_{(<)}^I &\equiv \psi^I(2\pi, \sigma^2), \\ \bar{\psi}_{(>)}^I &\equiv \bar{\psi}^I(0, \sigma^2), & \bar{\psi}_{(<)}^I &\equiv \bar{\psi}^I(2\pi, \sigma^2). \end{aligned} \right\} \quad (3.2)$$

The T-duality twist is then accomplished by adding the following term to the action I_0 of (3.1):

$$I_1 = \frac{i}{2\pi} \int \sqrt{G}\epsilon_{IJ} \left[\phi_{(<)}^I \partial_2 \phi_{(>)}^J + 2\psi_{(<)}^I \psi_{(>)}^J + 2\bar{\psi}_{(<)}^I \bar{\psi}_{(>)}^J \right] d\sigma^2, \quad (3.3)$$

where $\partial_2 \equiv \partial/\partial\sigma^2$. The classical equations of motion that one gets by varying the action I_0+I_1 , where I_0 is given in (3.1) and I_1 in (3.3), are the free field equations away from the twist point $\sigma^1 = 0$, and at the twist point we get the boundary conditions:

$$\begin{aligned}\epsilon^I{}_J \partial_2 \phi^J_{(>)} &= i \partial_1 \phi^I_{(<)}, & \epsilon^I{}_J \partial_1 \phi^J_{(>)} &= -i \partial_2 \phi^I_{(<)}, \\ \epsilon^I{}_J \psi^J_{(>)} &= -\psi^I_{(<)}, & \epsilon^I{}_J \bar{\psi}^J_{(>)} &= -\bar{\psi}^I_{(<)}.\end{aligned}\tag{3.4}$$

These equations indeed realize T-duality. The term I_1 is also invariant under $\mathcal{N} = (2, 2)$ supersymmetry with parameters $\eta, \bar{\eta}$ (using the bulk Dirac equation):

$$\begin{aligned}\delta\phi^I &= i\bar{\eta}\psi^I + i\eta\bar{\psi}^I + i\bar{\eta}'\epsilon^I{}_J\psi^J + i\eta'\epsilon^I{}_J\bar{\psi}^J, \\ \delta\psi^I &= -\bar{\eta}\partial_z\phi^I - \bar{\eta}'\epsilon^I{}_J\partial_z\phi^J, \\ \delta\bar{\psi}^I &= -\eta\partial_{\bar{z}}\phi^I - \eta'\epsilon^I{}_J\partial_{\bar{z}}\phi^J.\end{aligned}$$

Beyond the classical level, the identification of (3.3) with the T-duality twist can be argued by taking direction σ^1 to be (Euclidean) time and realizing the operator of T-duality on wave-functionals of the fields at $\sigma^1 = 0$. The result is that a wave-functional of the field $\phi^I_{(<)}(\sigma^1)$ can be converted to the dual wave-functional by multiplying the wave-functional by the bosonic part of e^{-I_1} and path-integrating over $[\mathcal{D}\phi^I_{(<)}]$. (See [40] for more details.)

3.1 Discrete symmetries

Target space momentum and winding number are not conserved by I_1 [defined in (3.3)] but there is a combination of them that is conserved and gives rise to a useful discrete symmetry. To find this combination, consider first the theory without the T-duality twist. The momentum (p_1, p_2) and winding (w^1, w^2) quantum numbers (which take integer values) are conserved, but they do not commute with T-duality, which acts on them as:

$$\begin{pmatrix} p_1 \\ p_2 \\ w^1 \\ w^2 \end{pmatrix} \rightarrow \begin{pmatrix} w^2 \\ -w^1 \\ p_2 \\ -p_1 \end{pmatrix}.$$

However, the combinations

$$\mathfrak{U}_1 \equiv (-1)^{p_1+w^2}, \quad \mathfrak{U}_2 \equiv (-1)^{p_2+w^1},$$

which take values in \mathbb{Z}_2 , are conserved. We will argue below in §3.3 that when we insert the T-duality twist, \mathfrak{U}_1 and \mathfrak{U}_2 give rise to good \mathbb{Z}_2 -valued quantum numbers, but they acquire a nontrivial commutation relation:

$$\mathfrak{U}_1\mathfrak{U}_2 = -\mathfrak{U}_2\mathfrak{U}_1.\tag{3.5}$$

3.2 Mode expansion

T-duality acts as

$$\partial\phi^I \rightarrow G^{IJ} \epsilon_{JK} \partial\phi^K, \quad \bar{\partial}\phi^I \rightarrow -G^{IJ} \epsilon_{JK} \bar{\partial}\phi^K,$$

or explicitly,

$$\partial\phi^1 + \tau\partial\phi^2 \rightarrow i(\partial\phi^1 + \tau\partial\phi^2), \quad \partial\phi^1 + \bar{\tau}\partial\phi^2 \rightarrow -i(\partial\phi^1 + \bar{\tau}\partial\phi^2)$$

and similarly

$$\bar{\partial}\phi^1 + \tau\bar{\partial}\phi^2 \rightarrow -i(\bar{\partial}\phi^1 + \tau\bar{\partial}\phi^2), \quad \bar{\partial}\phi^1 + \bar{\tau}\bar{\partial}\phi^2 \rightarrow i(\bar{\partial}\phi^1 + \bar{\tau}\bar{\partial}\phi^2).$$

With the T-duality twist, the mode expansion is:

$$\begin{aligned} \psi^1 + (\tau_1 \pm i\tau_2)\psi^2 &= \sum_{n \in \mathbb{Z}} \mathfrak{b}_{n \pm \frac{1}{4}}^{\pm} e^{i(n \pm \frac{1}{4})(\sigma_1 + i\sigma_2)}, \\ \bar{\psi}^1 + (\tau_1 \pm i\tau_2)\bar{\psi}^2 &= \sum_{n \in \mathbb{Z}} \bar{\mathfrak{b}}_{n \mp \frac{1}{4}}^{\pm} e^{i(n \mp \frac{1}{4})(-\sigma_1 + i\sigma_2)}, \end{aligned} \tag{3.6}$$

and,

$$\begin{aligned} \partial\phi^1 + (\tau_1 \pm i\tau_2)\partial\phi^2 &= \sum_{n \in \mathbb{Z}} \alpha_{n \pm \frac{1}{4}}^{\pm} e^{i(n \pm \frac{1}{4})(\sigma_1 + i\sigma_2)}, \\ \bar{\partial}\phi^1 + (\tau_1 \pm i\tau_2)\bar{\partial}\phi^2 &= \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{n \mp \frac{1}{4}}^{\pm} e^{i(n \mp \frac{1}{4})(-\sigma_1 + i\sigma_2)}. \end{aligned} \tag{3.7}$$

Thus, the boundary conditions (3.4) create a spectral flow in the free field mode expansion and the modes are shifted to values in $\mathbb{Z} \pm \frac{1}{4}$.

As we shall see in §3.3, there are two ground states. We denote them by $|j\rangle$, with $j = 0, 1$. They satisfy

$$0 = \alpha_{n - \frac{1}{4}}^+ |j\rangle = \tilde{\alpha}_{n - \frac{3}{4}}^- |j\rangle = \alpha_{n - \frac{3}{4}}^- |j\rangle = \tilde{\alpha}_{n - \frac{1}{4}}^+ |j\rangle, \quad n = 1, 2, 3, \dots$$

and

$$0 = \mathfrak{b}_{n - \frac{1}{4}}^+ |j\rangle = \bar{\mathfrak{b}}_{n - \frac{3}{4}}^- |j\rangle = \mathfrak{b}_{n - \frac{3}{4}}^- |j\rangle = \bar{\mathfrak{b}}_{n - \frac{1}{4}}^+ |j\rangle, \quad n = 1, 2, 3, \dots$$

3.3 Ground states

The mode expansions (3.6)-(3.7) show that after the T-duality twist the worldsheet fields no longer have any zero modes. Nevertheless, we will now argue that the ground state is a doublet. One way to see this is by performing T-duality on direction x^1 . This duality interchanges the moduli ρ and τ and replaces the T-duality twist (which corresponded to $\rho \rightarrow -1/\rho$) with an ordinary geometrical twist ($\tau \rightarrow -1/\tau$) which for $\tau = i$ corresponds to a rotation by $\pi/2$ of the T^2 (around some fixed point, say the origin). We denote the scalar fields of this dual worldsheet theory by $\tilde{\Phi}^1$ and Φ^2 , so that $\tilde{\Phi}^1 + i\Phi^2$ is a coordinate on the dual torus. In this T-dual description the ground states correspond to fixed points of the rotation. These are the two allowed discrete values of the zero mode of the (dual) scalar worldsheet field, and there are two such fixed points: either the origin ($\tilde{\Phi}^1 = \Phi^2 = 0$) or the center of the torus ($\tilde{\Phi}^1 = \Phi^2 = \pi$). Furthermore, the operators \mathfrak{U}_1 and \mathfrak{U}_2 acquire a geometrical interpretation. \mathfrak{U}_1 is mapped to a translation (which commutes with the twist):

$$\mathfrak{U}_1 : \quad \tilde{\Phi}^1 + i\Phi^2 \mapsto (\tilde{\Phi}^1 + \pi) + i(\Phi^2 + \pi).$$

The operator \mathfrak{U}_2 is mapped to a topological charge which can be defined as follows (see [42] for a related discussion). Consider the 3-dimensional space M that we construct as a T^2 fibration over S^1 with the fiber being the target space and the base being the σ^1 circle. The homology of this space is $H_1(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$ where the \mathbb{Z} factor is generated by the homology class of a loop that wraps around the S^1 base at constant T^2 position $\tilde{\Phi}^1 = \Phi^2 = 0$, and the \mathbb{Z}_2 factor is generated by the homology class of a loop that wraps one of the short nontrivial cycles of the fiber T^2 at a constant σ^1 (which becomes a torsion class because of the identification induced by the twist). Now let $(\tilde{\Phi}^1(\sigma^1), \Phi^2(\sigma^1))$ be any smooth field configuration that respects the twist. We can associate a topological \mathbb{Z}_2 charge with it by interpreting it as a loop in M . The \mathbb{Z}_2 charge (with values ± 1) is then determined from the homology of the loop. This ± 1 charge is then identified with the eigenvalue of the operator \mathfrak{U}_2 . It is also not hard to check (see [42]) that the loop at the origin ($\tilde{\Phi}^1 = \Phi^2 = 0$) and the loop at the center ($\tilde{\Phi}^1 = \Phi^2 = \pi$) have opposite \mathfrak{U}_2 charge. If we denote by $|0\rangle$ the ground state that corresponds to the mode expansion around $\tilde{\Phi}^1 = \Phi^2 = 0$, and by $|1\rangle$ the ground state that corresponds to the mode expansion around $\tilde{\Phi}^1 = \Phi^2 = \pi$ then we can summarize the above observations as

$$\mathfrak{U}_1|0\rangle = |1\rangle, \quad \mathfrak{U}_1|1\rangle = |0\rangle, \quad \mathfrak{U}_2|0\rangle = |0\rangle, \quad \mathfrak{U}_2|1\rangle = -|1\rangle,$$

where we have arbitrarily assigned to $|0\rangle$ the \mathbb{Z}_2 charge $+1$. In particular, the commutation relation (3.5) follows. In section §3.6 we will present another way to verify that there are two ground states, in terms of deformation quantization of the target space.

3.4 Fermion number

The fermions have modes labeled by $n \pm \frac{1}{4}$, and all the zero modes have been eliminated by the T-duality twist. However, the ground states $|j\rangle$ have odd fermion number $(-1)^F = -1$, as we will now argue. To see this, we compactify the worldsheet on T^2 and calculate the Witten index in two ways. From the discussion above, the Witten index is $2(-1)^F$. On the other hand, by modular invariance, the Witten index is also

$$I = \text{tr}_{\mathcal{H}_0} [(-1)^F \mathcal{S}],$$

where the trace is taken over the Hilbert space of ground states of the 1+1D σ -model compactified on S^1 *without* a T-duality twist, and \mathcal{S} is the T-duality operator on that Hilbert space. The T-duality operator is part of the $\text{SL}(2, \mathbb{Z})_\rho \times \text{SL}(2, \mathbb{Z})_\tau$ group of dualities that act on the Hilbert space of ground states. Here $\text{SL}(2, \mathbb{Z})_\rho$ acts on the complexified area of T^2 , while $\text{SL}(2, \mathbb{Z})_\tau$ is the mapping class group which acts on the complex structure τ . The ground states of \mathcal{H}_0 correspond to the cohomology of T^2 and are in the $(2, 1) \oplus (1, 2)$ representation of the duality group. The two states corresponding to $H^1(T^2, \mathbb{Z})$ have odd fermion number and are singlets of $\text{SL}(2, \mathbb{Z})_\rho$, while the two states of $H^0(T^2, \mathbb{Z}) \oplus H^2(T^2, \mathbb{Z})$ are a doublet of $\text{SL}(2, \mathbb{Z})_\rho$ and a singlet of $\text{SL}(2, \mathbb{Z})_\tau$. The operator \mathcal{S} acts as the identity on $H^1(T^2, \mathbb{Z})$ and as the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $H^0(T^2, \mathbb{Z}) \oplus H^2(T^2, \mathbb{Z})$. The Witten index is therefore -2 and the states $|j\rangle$ therefore have odd fermion number, as stated.

3.5 Other elements of $\text{SL}(2, \mathbb{Z})$

We can also consider twisting the boundary conditions by the $\text{SL}(2, \mathbb{Z})$ transformations

$$\pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

which keep the $\rho = e^{\pi i/3}$ value fixed and generate an abelian group of order $\mathbf{r} = 6$ (for the $+$ sign) and $\mathbf{r} = 3$ (for the $-$ sign). In addition, it is also useful to consider as a test case the twist by the central element $-1 \in \text{SL}(2, \mathbb{Z})$. This is equivalent to a geometrical rotation by π and keeps every value of ρ fixed. It has order $\mathbf{r} = 2$. The modified mode expansions have modes with values in $\mathbb{Z} \pm \frac{1}{\mathbf{r}}$. There are \mathbf{k} ground states $|j\rangle$ ($j = 0, \dots, \mathbf{k} - 1$) with $\mathbf{k} = 1$ for $\mathbf{r} = 6$, $\mathbf{k} = 3$ for $\mathbf{r} = 3$ and $\mathbf{k} = 4$ for $\mathbf{r} = 2$. Further explanation of the meaning of \mathbf{k} follows in section §3.6 and §6.

3.6 Relation to the noncommutative torus and the Landau problem

We are interested in the ground states of the theory described by the action $I_0 + I_1$ [see (3.1) and (3.3)]. The ground states of this system can also be established by a standard analysis of the relevant terms at low-energy. We have compactified the 1+1D theory on S^1 , and below the Kaluza-Klein scale (the inverse of the radius L_1 of the σ^1 circle) we can neglect the dependence of the fields on σ_1 . More precisely, the boundary conditions (3.4) imply that excited states have energy of the order of the σ^1 -compactification scale (at least $1/4L_1$). The only terms that survive the low-energy limit are those coming from I_1 , and since there is no longer any σ^1 dependence, we can set $\phi_{(>)}^I = \phi_{(<)}^I \equiv \phi^I(\sigma^2)$, and similarly for the fermionic fields. The fermionic fields become massive and irrelevant at low-energy, and we are left with the 0+1D action

$$I = -\frac{i}{2\pi} \int \epsilon_{IJ} \phi^I \partial_2 \phi^J d\sigma^2 = -\frac{i}{2\pi} \int \epsilon_{IJ} \phi^I d\phi^J. \quad (3.8)$$

Note that we have dropped the kinetic terms proportional to $(\partial_2 \phi^I)^2$ since they are IR irrelevant, as can be verified by simple dimensional analysis. It is nevertheless convenient to add the kinetic term again so as to get the Lagrangian of a Landau problem (a particle in a uniform magnetic field) on T^2 :

$$I_L = \frac{1}{2\pi} \int \left(\frac{1}{2} m G_{IJ} \dot{\mathbf{x}}^I \dot{\mathbf{x}}^J - \frac{i\mathbf{k}}{2} \epsilon_{IJ} \mathbf{x}^I \dot{\mathbf{x}}^J \right) dt, \quad (3.9)$$

where we set $t \equiv \sigma^2$, and $m = 2\pi L_1$ is the mass parameter (which is the circumference of the σ^1 direction). We have also introduced $\mathbf{k} \equiv 2$ in (3.9). Other values of \mathbf{k} appear in connection with the other elements of the $\text{SL}(2, \mathbb{Z})$ duality group discussed in §3.5. The ground states of (3.9) are Landau wave-functions (independent of m and the area of the T^2):

$$\begin{aligned} \varphi_{j,\mathbf{k}}(\mathbf{x}_1, \mathbf{x}_2) &= \frac{1}{2\pi} (2\mathbf{k}\tau_2)^{\frac{1}{4}} e^{\frac{i\mathbf{k}\mathbf{x}_1\mathbf{x}_2}{4\pi}} \sum_{n=-\infty}^{\infty} e^{i(\mathbf{k}n+j)\mathbf{x}_1 + \pi i\mathbf{k}\tau \left(\frac{\mathbf{x}_2}{2\pi} + n + \frac{j}{\mathbf{k}}\right)^2} \\ &= \frac{1}{2\pi} (2\mathbf{k}\tau_2)^{\frac{1}{4}} e^{\frac{i\mathbf{k}\mathbf{x}_1\mathbf{x}_2}{4\pi}} e^{\pi i\tau\mathbf{k}\left(\frac{\mathbf{x}_2}{2\pi}\right)^2} \Theta_{j,\mathbf{k}}\left(\frac{\mathbf{x}_1 + \tau\mathbf{x}_2}{2\pi}; \tau\right) \end{aligned} \quad (3.10)$$

where the Θ -function is defined as

$$\Theta_{j,\mathbf{k}}(u, \tau) \equiv \sum_{n=-\infty}^{\infty} e^{\pi i\mathbf{k}\tau(n+\frac{j}{\mathbf{k}})^2 + 2\pi i\mathbf{k}(n+\frac{j}{\mathbf{k}})u} \quad (3.11)$$

and is holomorphic in u and τ .

In the low-energy limit the kinetic term in (3.9) can be dropped, and as is well-known, we are left with the Lagrangian that describes a noncommutative T^2 with symplectic form $\frac{\mathbf{k}}{4\pi^2} d\mathbf{x}^1 \wedge d\mathbf{x}^2$. Thus, \mathbf{k} has the interpretation of the level of the deformation quantization problem of T^2 . We will return to this point in §6.

Chapter 4

Elliptically fibered K_3 target spaces

The analysis of §3 can be extended to curved target spaces X that can be represented as an *elliptic fibration*. Such spaces are described by fibering a torus with a complex structure that varies over some base \mathfrak{B} . More precisely, there exists a surjective holomorphic map π from the d dimensional space X to a complex space \mathfrak{B} of (complex) dimension $d - 1$ such that the inverse image $\pi^{-1}(p)$ of a generic point $p \in \mathfrak{B}$ is a torus. The complex structure τ of the fiber $\pi^{-1}(p)$ is allowed to vary (holomorphically) with the point p along the base \mathfrak{B} . An important example is $X = K_3$ with $\mathfrak{B} = \mathbb{C}\mathbb{P}^1$. This construction was developed in detail in [46] and is central to F-theory [82]. The complex structure τ is allowed to become ∞ (possibly after an $\text{SL}(2, \mathbb{Z})$ transformation) at a codimension-1 submanifold of \mathfrak{B} , which for a generic elliptically fibered K_3 turns out to be 24 isolated points $\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_{24}$. We will assume that sufficiently far from these points τ varies very slowly over distances of order 1 (the string-scale). The metric then takes the approximate form

$$ds^2 = \frac{\rho_2}{\tau_2} |d\mathbf{x}_1 + \tau d\mathbf{x}_2|^2 + g_{\mathfrak{z}\bar{\mathfrak{z}}} |d\mathfrak{z}|^2, \quad (4.1)$$

where \mathfrak{z} is a complex coordinate on \mathfrak{B} , $0 \leq \mathbf{x}_1, \mathbf{x}_2 < 2\pi$ are periodic coordinates on the fiber, ρ_2 is the constant area of the fiber, $\tau(\mathfrak{z})$ is a locally defined holomorphic function of the base, and τ_2 is the imaginary part of τ . The behavior of the complex structure τ as a function of \mathfrak{z} is conveniently encoded by the equation [46]:

$$y^2 = x^3 - f(\mathfrak{z})x - g(\mathfrak{z}), \quad (4.2)$$

where f and g are holomorphic functions of \mathfrak{z} , which for a K_3 are polynomials of degrees 8 and 12, respectively. The complex structure is given in terms of the j -function

$$j(\tau) = \frac{64f^3}{\frac{f^3}{27} - \frac{g^2}{4}}.$$

The hyper-Kähler metric $g_{\mathfrak{z}\bar{\mathfrak{z}}}$ is determined as follows. In the notation of [75], one defines a holomorphic function $a(\mathfrak{z})$ by solving $\frac{da}{d\mathfrak{z}} = \mathcal{f}_\alpha \frac{dx}{y}$, where α is a basic 1-cycle

of the T^2 fiber at \mathfrak{z} (the cycle that corresponds to a constant \mathbf{x}_2 , with \mathbf{x}_1 running from 0 to 2π). We then have $g_{\mathfrak{z}\bar{\mathfrak{z}}} = \tau_2 \left| \frac{da}{d\mathfrak{z}} \right|^2$. The form of the metric (4.1), where the components are independent of \mathbf{x}_1 and \mathbf{x}_2 , is approximate, with corrections that are exponentially small sufficiently far away from the points \mathfrak{z}_i of the singular fibers.¹

Now we take a sigma-model with a metric of the approximate form (4.1). We set $\rho = i$ (ρ is part of the Kähler moduli of the sigma-model). T-duality of the fibers now extends to mirror-symmetry of the whole space [79, 57]. We can then compactify on S^1 with a mirror-symmetry twist. The goal of this section is to understand the ground states of the twisted compactification in terms of a semi-classical description involving \mathfrak{B} alone. We specialize to the K_3 case for which, as mentioned above, there are 24 points $\mathfrak{z}_1, \dots, \mathfrak{z}_{24}$ where the fiber degenerates. We will refer to them as *special points*. We have chosen a metric on the base that is large enough so that (sufficiently far away from the special points) the complex structure $\tau(\mathfrak{z})$ varies sufficiently slowly, and we can then study the problem in a Born-Oppenheimer approximation whereby we treat the fields along the fiber as the “fast modes” and the fields along the base as the “slow-modes.” As we will describe below, we can then reduce the problem to what is essentially a quantum mechanical problem with \mathfrak{B} as the configuration space. We will develop this quantum mechanical description along the following lines:

1. Sufficiently far away from the special points \mathfrak{z}_i , we can derive a simple wave-equation for the wave-function.
2. The wave-functions have nontrivial monodromies around special points. In general when passing through a cut that emanates from a special point, the T^2 fiber undergoes a certain $SL(2, \mathbb{Z})$ (mapping class group) transformation, which acts nontrivially on the wave-function [46, 82]. These nontrivial monodromies around the \mathfrak{z}_i 's can be eliminated by encoding the wave-functions in terms of sections of certain holomorphic line bundles on the total space. The boundary conditions at the special points are determined by normalizability of the wave-function.
3. There are additional “bound states” whose wave-functions are localized at the special points and are out of reach of the classical analysis above. However, the Witten index allows us to glean some information about these bound states.

We now proceed to describe each part in detail.

¹ In Seiberg-Witten theory such spaces appear as moduli spaces of 3+1D $SU(2)$ gauge theories compactified on S^1 [77]. We are here considering the limit that the Kaluza-Klein energy scale is much smaller than the QCD scale, and the exponentially small corrections arise from BPS particles with Euclidean worldline along the circle.

4.1 Supersymmetry

The moduli space of the σ -model with K_3 target space is [76, 6]:

$$O(20, 4, \mathbb{Z}) \backslash O(20, 4, \mathbb{R}) / (O(20) \times O(4)) = O^+(20, 4, \mathbb{Z}) \backslash O^+(20, 4, \mathbb{R}) / (O(20) \times SO(4)), \quad (4.3)$$

where $O^+(20, 4, \mathbb{R})$ denotes an index-2 subgroup of $O(20, 4, \mathbb{R})$. The amount of world-sheet supersymmetry in the mirror-twisted setting is determined as follows. The σ -model has $\mathcal{N} = (4, 4)$ supersymmetry with an $SU(2)_L \otimes SU(2)_R$ R-symmetry (with $SU(2)_L$ acting on the left-moving sector of the CFT and $SU(2)_R$ acting on the right-moving sector). The supercharges are in the spinor representation $(2, 1) \oplus (1, 2)$. The R-symmetry $SU(2)_L \otimes SU(2)_R$ can be identified with a double-cover of the $SO(4)$ factor in (4.3) in the following sense [76, 6]. Set

$$K \equiv O(20) \times SO(4), \quad G \equiv O^+(20, 4, \mathbb{R}), \quad \Gamma \equiv O^+(20, 4, \mathbb{Z}).$$

Then, the supercharges of the theory live in a vector bundle over the moduli space $\Gamma \backslash G/K$ whose structure group is $SU(2)_L \times SU(2)_R$ which is a double cover of $SO(4)$. The principal bundle [i.e., the bundle one gets upon replacing the fiber of the vector bundle with the structure group $SO(4)$] is $\Gamma \backslash G/SO(20)$ (which is an $SO(4)$ bundle over the base $\Gamma \backslash G/K$). Now let $gK \in G/K$ be the point in moduli space, represented as a coset of groups. If $\hat{\mathfrak{M}} \in O^+(20, 4, \mathbb{Z})$ preserves gK then $g^{-1}\hat{\mathfrak{M}}g \in K$. Let $\rho \in SO(4)$ be the projection of $g^{-1}\hat{\mathfrak{M}}g$ to the $SO(4)$ factor of K . We will now argue that the left-moving supercharges transform as $(1, 1) \oplus (3, 1)$ under $\rho \in SO(4)$ and the right-moving supercharges transform as $(1, 1) \oplus (1, 3)$ [where representations of $SO(4)$ are written as representations of $SU(2)_L \otimes SU(2)_R$]. The singlets correspond to the preserved supersymmetries (2.2), while the triplets can be understood as follows. Schematically, the left-moving supercurrents are of the form $\psi \partial \phi$. The R-symmetries act on the ψ 's but not the ϕ 's, but ρ needs to act on $\partial \phi$ in a dual way to ψ in order to commute with the supersymmetries of (2.2). Assuming that ψ transforms in $(2, 1)$, we find that the SUSY generators are in $(2, 1) \otimes (2, 1) = (1, 1) \oplus (3, 1)$ as stated above.

As a special case, consider a point in moduli space that is invariant under an isometry. The isometry corresponds to an element $\hat{\mathfrak{M}}' \in O^+(19, 3, \mathbb{Z}) \subset O^+(20, 4, \mathbb{Z})$ and the projection ρ' of $g^{-1}\hat{\mathfrak{M}}'g$ to the $SO(3) \subset SO(4)$ factor of K determines the amount of preserved supersymmetry as follows. The isometry always preserves the supersymmetries of (2.2), and for every complex structure that the isometry preserves we get an extra supersymmetry of the type (2.4). By construction [6], the complex structures are in a triplet of $SO(3)$ (the group of which ρ' is an element), and so the number of (left and right) preserved supersymmetries of type (2.4) is the number of eigenvalues of 1 of ρ' . As an element of $SO(4)$, ρ' has an extra eigenvalue of 1, which corresponds to the preserved supersymmetry of type (2.2).

4.2 Perturbative analysis

We will now construct the wave equation that the ground-state wave-functions satisfy for a target space constructed by fibering a T^2 (of complex structure τ) over an open neighborhood of \mathbb{C} (parameterized by \mathfrak{z}), with a slowly varying $\tau(\mathfrak{z})$. We take the metric as in (4.1) and choose the fields that correspond to the complex coordinates as follows:

$$\mathfrak{z} \rightarrow \phi^{\mathfrak{z}}, \quad \bar{\mathfrak{z}} \rightarrow \phi^{\bar{\mathfrak{z}}}, \quad \mathfrak{w} \equiv \frac{1}{2\pi}(\mathbf{x}_1 + \tau(\mathfrak{z})\mathbf{x}_2) \rightarrow \phi^{\mathfrak{w}}, \quad \bar{\mathfrak{w}} \equiv \frac{1}{2\pi}(\mathbf{x}_1 + \bar{\tau}(\bar{\mathfrak{z}})\mathbf{x}_2) \rightarrow \phi^{\bar{\mathfrak{w}}}. \quad (4.4)$$

The periodicity of the holomorphic coordinate \mathfrak{w} is $\mathfrak{w} \sim \mathfrak{w} + 1 \sim \mathfrak{w} + \tau$, but the dependence of τ on \mathfrak{z} is inconvenient here. We will therefore work in a mixed convention where the fundamental fields are

$$\phi^{\mathfrak{z}}, \quad \phi^{\bar{\mathfrak{z}}}, \quad \phi^1, \quad \phi^2, \quad \psi^{\bar{\mathfrak{z}}}, \quad \psi_{\bar{\mathfrak{z}}}, \quad \bar{\psi}_{\mathfrak{z}}, \quad \bar{\psi}^{\mathfrak{z}}, \quad \psi^1, \quad \psi^2, \quad \bar{\psi}^1, \quad \bar{\psi}^2.$$

We denote the canonical dual momenta of the bosons in this formalism by

$$\Pi'_{\mathfrak{z}}, \quad \Pi'_{\bar{\mathfrak{z}}}, \quad \Pi'_1, \quad \Pi'_2.$$

They are related to the canonical momenta of $\phi^{\mathfrak{w}}$ and $\phi^{\bar{\mathfrak{z}}}$ by²

$$\Pi_{\mathfrak{w}} = \frac{i}{2\tau_2}(\bar{\tau}\Pi'_1 - \Pi'_2), \quad \Pi_{\bar{\mathfrak{z}}} = \Pi'_3 + \frac{i\tau'\phi^2}{2\tau_2}(\Pi'_2 - \bar{\tau}\Pi'_1),$$

and ψ^1 and ψ^2 are related to $\psi^{\bar{\mathfrak{w}}}$ and $\psi^{\bar{\mathfrak{z}}}$ by:

$$\psi^{\bar{\mathfrak{w}}} = \psi^1 + \bar{\tau}\psi^2 + \bar{\tau}'\phi^2\psi^{\bar{\mathfrak{z}}}.$$

We rewrite the left equation in (2.5) as

$$Q = \int \left[\psi^{\bar{\mathfrak{z}}}(\Pi'_{\bar{\mathfrak{z}}} + G_{i\bar{\mathfrak{z}}}\partial_1\phi^i) + \frac{i}{2\tau_2}(\psi^1 + \bar{\tau}\psi^2)(\Pi'_2 - \tau\Pi'_1 + G_{2I}\partial_1\phi^I - \tau G_{1I}\partial_1\phi^I) \right] d\sigma_1, \quad (4.5)$$

Now, we put the theory on S^1 with a mirror-symmetry twist. This is achieved by adding a term similar to (3.3) to the action. The mirror-symmetry operation acts as

$$(\psi^1 + \bar{\tau}\psi^2) \rightarrow i(\psi^1 + \bar{\tau}\psi^2),$$

and in order to preserve Q it must also act as,

$$(\Pi'_2 - \tau\Pi'_1 + G_{2I}\partial_1\phi^I - \tau G_{1I}\partial_1\phi^I) \rightarrow -i(\Pi'_2 - \tau\Pi'_1 + G_{2I}\partial_1\phi^I - \tau G_{1I}\partial_1\phi^I).$$

²The following relations are obtained by applying the change of variables from $(\mathbf{x}_1, \mathbf{x}_2, \mathfrak{z}, \bar{\mathfrak{z}})$ to $(\mathfrak{w}, \bar{\mathfrak{w}}, \mathfrak{z}, \bar{\mathfrak{z}})$ of (4.4). For example, $\Pi_{\mathfrak{w}}$ transforms as a component of a 1-form, and $\psi^{\bar{\mathfrak{w}}}$ as a component of a vector.

The mirror-symmetry twist eliminates the zero-modes of $(\psi^1 + \bar{\tau}\psi^2)$ and of the term it multiplies in (4.5), and their modes become nonintegral (in $\mathbb{Z} \pm \frac{1}{4}$). Since we are interested in the ground states, in the limit of slowly varying $\tau(\mathfrak{z})$ we can ignore the terms that contain these fields in (4.5). We are therefore left with

$$Q \rightarrow \int \psi^{\bar{3}}(\Pi'_{\bar{3}} + G_{i\bar{3}}\partial_1\phi^i)d\sigma_1 \rightarrow \int \psi^{\bar{3}}\Pi'_{\bar{3}}d\sigma_1. \quad (4.6)$$

We also dropped the term proportional to $\partial_1\phi^i$ because we are only interested in modes with zero momentum along S^1 . The ground states can now be constructed as follows. Let $\bar{\mathfrak{b}}_0^{\bar{3}}$ and $\mathfrak{b}_0^{\bar{3}}$ be the zero modes in the mode expansion of $\bar{\psi}^{\bar{3}}$ and $\psi^{\bar{3}}$, respectively. For constant $\tau(\mathfrak{z})$ (i.e., for $\mathbb{C} \times T^2$) we build a basis of ground state wavefunctions by combining a ground state of the T^2 compactification with a ground state of the \mathbb{C} compactification. The ground states of the T^2 compactification are labeled by $j = 0, \dots, \mathbf{k} - 1 = 1$, as in §3.3 (and we keep track of \mathbf{k} for possible generalizations as in §3.5). The ground states of the \mathbb{C} problem correspond to Dolbeault $\bar{\partial}$ -cohomology on \mathbb{C} (which are not normalizable, but that is not going to be a problem since we are not going to use all of \mathbb{C} but rather we are going to patch open subsets of \mathbb{C} to form a \mathbb{CP}^1). We denote by $|j\rangle$ the state that satisfies $\bar{\mathfrak{b}}_{3,0}|j\rangle = \mathfrak{b}_{3,0}|j\rangle = 0$ and corresponds to the j^{th} ground state of the T^2 problem. A generic ground state can then be written as

$$\omega_j(\mathfrak{z}, \bar{\mathfrak{z}})|j\rangle + \omega_{j,3}(\mathfrak{z}, \bar{\mathfrak{z}})\bar{\mathfrak{b}}_0^{\bar{3}}|j\rangle + \omega_{j,\bar{3}}(\mathfrak{z}, \bar{\mathfrak{z}})\mathfrak{b}_0^{\bar{3}}|j\rangle + \omega_{j,3\bar{3}}(\mathfrak{z}, \bar{\mathfrak{z}})\bar{\mathfrak{b}}_0^{\bar{3}}\mathfrak{b}_0^{\bar{3}}|j\rangle$$

where $(\mathfrak{z}, \bar{\mathfrak{z}})$ are the ‘‘center-of-mass’’ coordinates in the \mathfrak{z} (and $\bar{\mathfrak{z}}$) directions. To be annihilated by Q requires

$$\chi_j \equiv \omega_j + \omega_{j,3}d\mathfrak{z} + \omega_{j,\bar{3}}d\bar{\mathfrak{z}} + \omega_{j,3\bar{3}}d\mathfrak{z} \wedge d\bar{\mathfrak{z}}$$

to be a sum of $d\bar{\mathfrak{z}}\bar{\partial}_{\bar{\mathfrak{z}}}$ -closed forms, and in the topologically twisted theory (the A-model as in [87]) we also impose an equivalence up to $d\bar{\mathfrak{z}}\bar{\partial}_{\bar{\mathfrak{z}}}$ -exact forms. Since the theory is on a flat worldsheet, the topologically twisted theory is equivalent to the untwisted one, and so χ_j is in the $d\bar{\mathfrak{z}}\bar{\partial}_{\bar{\mathfrak{z}}}$ -cohomology. In particular, ω_j is a holomorphic 0-form and $\omega_{j,3}d\mathfrak{z}$ is a holomorphic 1-form. Given the structure of Q and the approximation (4.6), this statement is also true when $\tau(\mathfrak{z})$ varies slowly.

Next, we need to glue the $\{\chi_j\}_{j=0}^{\mathbf{k}-1}$ across cuts where the T^2 fiber undergoes an $\text{SL}(2, \mathbb{Z})$ transformation. We construct the expression

$$\chi \equiv \sum_{j=0}^{\mathbf{k}-1} \chi_j(\mathfrak{z}, \bar{\mathfrak{z}})\Theta_{j,\mathbf{k}}(\mathfrak{w}, \tau(\mathfrak{z})),$$

where the theta function $\Theta_{j,\mathbf{k}}(u, \tau)$ was defined in (3.11), and naturally corresponds to the ground state $|j\rangle$. We require the form χ to be single-valued across cuts, which follows from the fact that the collection of Θ -functions transforms in a dual way to χ_j ,

since the Θ -functions transform like the states $|j\rangle$. The Θ -functions are holomorphic and so χ is in the $d\bar{z}\bar{\partial}_z$ -cohomology. The $(0, 0)$ -form and $(1, 0)$ -form in χ are

$$\chi^{(0,0)} + \chi^{(1,0)} \equiv \sum_{j=0}^{\mathbf{k}-1} \omega_j \Theta_{j,\mathbf{k}}(\mathfrak{w}, \tau(\mathfrak{z})) + \sum_{j=0}^{\mathbf{k}-1} \omega_{j,\bar{z}} \Theta_{j,\mathbf{k}}(\mathfrak{w}, \tau(\mathfrak{z})) d\mathfrak{z},$$

which is a formal sum of a holomorphic section $\chi^{(0,0)}$ of a certain line-bundle \mathcal{L} (to be described in more detail below) and a holomorphic differential $\chi^{(1,0)}$ with coefficients in \mathcal{L} . The $(0, 1)$ and $(1, 1)$ components

$$\chi^{(0,1)} + \chi^{(1,1)} \equiv \sum_{j=0}^{\mathbf{k}-1} \omega_{j,\bar{z}} \Theta_{j,\mathbf{k}}(\mathfrak{w}, \tau(\mathfrak{z})) d\bar{z} + \sum_{j=0}^{\mathbf{k}-1} \omega_{j,\bar{z}\bar{z}} \Theta_{j,\mathbf{k}}(\mathfrak{w}, \tau(\mathfrak{z})) d\mathfrak{z} \wedge d\bar{z}$$

are of course not holomorphic. They are required to be in the $d\bar{z}\bar{\partial}_z$ cohomology, but this cohomology is not so convenient to work with. We will now show that it can be converted to the ordinary $\bar{\partial}$ cohomology. Consider for example the $(0, 1)$ Dolbeault cohomology with coefficients in some line-bundle \mathcal{L} over K_3 . Since $\bar{\partial} = d\bar{z}\bar{\partial}_z + d\bar{w}\bar{\partial}_{\bar{w}}$, we only need to show that every $\bar{\partial}$ -cohomology class has a representative that has no terms in it of the form $(\dots)d\bar{w}$ — in other words, its restriction to any elliptic fiber vanishes. Let φ be a $\bar{\partial}$ -closed $(0, 1)$ form with coefficients in \mathcal{L} over K_3 . Let $c_1(\mathcal{L})$ be the first Chern class. The restriction of $c_1(\mathcal{L})$ to any elliptic fiber is \mathbf{k} times the generator of $H^{(1,1)}(T^2, \mathbb{Z})$, and since \mathbf{k} is a positive integer, Serre duality implies that $H^1(T^2, \mathcal{L}|_{T^2}) = 0$. Now cover the base \mathfrak{B} with contractible open sets, such that each contains at most one special point \mathfrak{z}_j . We thus have $\mathfrak{B} = \bigcup_{\alpha} \mathcal{U}_{\alpha}$ written as a union of open patches. Let $\pi^{-1}\mathcal{U}_{\alpha}$ be the pre-image of the patch \mathcal{U}_{α} under the projection map $\pi : K_3 \rightarrow \mathfrak{B}$ (of the elliptic fibration). We have $H^1(\pi^{-1}\mathcal{U}_{\alpha}, \mathcal{L}) = 0$ and so the restriction of φ to $\pi^{-1}\mathcal{U}_{\alpha}$ can be written as $\bar{\partial}\psi_{\alpha}$ for some local function ψ_{α} . (This is also true if \mathcal{U}_{α} contains a \mathfrak{z}_j , by an explicit computation.) On $\mathcal{U}_{\alpha\beta} \equiv \mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta}$ we find that $\psi_{\alpha\beta} \equiv \psi_{\alpha} - \psi_{\beta}$ is holomorphic. Let $\{\rho_{\alpha}\}_{\alpha}$ be a partition of unity for the base $\mathbb{C}\mathbb{P}^1 \simeq \mathfrak{B}$ (i.e., $\rho_{\alpha}(\mathfrak{z}, \bar{\mathfrak{z}})$ is a function with support on \mathcal{U}_{α} and $\sum_{\alpha} \rho_{\alpha} = 1$ everywhere). Define $g_{\alpha} \equiv \sum_{\beta} \rho_{\beta} \psi_{\alpha\beta}$. Then the functions $f_{\alpha} \equiv \psi_{\alpha} - g_{\alpha}$ patch together to a global function on K_3 which we denote by f . (The last statement follows because, as is easy to see, $f_{\alpha} = f_{\beta}$ on $\mathcal{U}_{\alpha\beta}$.) It is then easy to check that the restriction of $\varphi - \bar{\partial}f$ to any elliptic fiber is zero. This completes the proof and demonstrates that we can work with the standard $\bar{\partial}$ differential operator instead of the cumbersome $d\bar{z}\bar{\partial}_z$.

Let $\mathcal{K}_{\mathfrak{B}}$ be the canonical bundle of the base $\mathfrak{B} \simeq \mathbb{C}\mathbb{P}^1$ (whose transition functions are the Jacobians $d\mathfrak{z}_{\alpha}/d\mathfrak{z}_{\beta}$) and let $\mathcal{K}' \equiv \pi^*\mathcal{K}_{\mathfrak{B}}$ be the pullback of $\mathcal{K}_{\mathfrak{B}}$ under the projection map $\pi : K_3 \rightarrow \mathfrak{B}$. We can now summarize:

- Ground states of the form $\omega_j(\mathfrak{z}, \bar{\mathfrak{z}})|j\rangle$ correspond to elements of $H^0(K_3, \mathcal{L})$;
- Ground states of the form $\omega_{j,\bar{z}}(\mathfrak{z}, \bar{\mathfrak{z}})\bar{\mathfrak{b}}_0^{\bar{z}}|j\rangle$ correspond to elements of $H^0(K_3, \mathcal{L} \otimes \mathcal{K}')$;

- Ground states of the form $\omega_{j,\bar{3}}(\bar{3}, \bar{3})\bar{\mathbf{b}}_0^{\bar{3}}|j\rangle$ correspond to elements of $H^1(K_3, \mathcal{L})$;
- Ground states of the form $\omega_{j,\bar{3}\bar{3}}(\bar{3}, \bar{3})\bar{\mathbf{b}}_0^{\bar{3}}\bar{\mathbf{b}}_0^{\bar{3}}|j\rangle$ correspond to elements of $H^1(K_3, \mathcal{L} \otimes \mathcal{K}')$;

Let us now be more specific about \mathcal{L} . Let $\omega_{\mathfrak{B}}$ be the $(1, 1)$ -cohomology class that is Poincaré dual to the zero section \mathfrak{B}_0 ($x = y = \infty$) of the elliptic fibration, whose homology class we denote by $[\mathfrak{B}_0]$. Let $[\mathfrak{F}]$ denote the homology class of the fiber and let $\omega_{\mathfrak{F}}$ be the $(1, 1)$ -cohomology class that is Poincaré dual to $[\mathfrak{F}]$. The intersection numbers are

$$[\mathfrak{F}] \cdot [\mathfrak{F}] = 0, \quad [\mathfrak{B}_0] \cdot [\mathfrak{B}_0] = -2, \quad [\mathfrak{B}_0] \cdot [\mathfrak{F}] = 1. \quad (4.7)$$

We have

$$c_1(\mathcal{K}') = -2\omega_{\mathfrak{F}}, \quad c_1(\mathcal{L}) = \mathbf{k}\omega_{\mathfrak{B}} + n\omega_{\mathfrak{F}}, \quad (4.8)$$

where $n \in \mathbb{Z}$ still needs to be determined. We determined the coefficient \mathbf{k} of $\omega_{\mathfrak{B}}$ by the requirement that, when restricted to a fiber, $c_1(\mathcal{L})$ should be \mathbf{k} times the generator of the second cohomology of the fiber.

To proceed, we note that the Hirzebruch-Riemann-Roch theorem for K_3 -surfaces states that

$$h^0(K_3, \mathcal{L}) - h^1(K_3, \mathcal{L}) = h^0(K_3, \mathcal{L}) - h^1(K_3, \mathcal{L}) + h^2(K_3, \mathcal{L}) = 2 + \frac{1}{2}c_1(\mathcal{L})^2 = 2 + \mathbf{k}n - \mathbf{k}^2, \quad (4.9)$$

where we used the vanishing of $h^2(K_3, \mathcal{L})$, which follows from Serre duality [$h^2(K_3, \mathcal{L}) = h^0(K_3, \mathcal{L}^{-1})$] and the fact that \mathcal{L}^{-1} has no holomorphic sections since it restricts to a line bundle with a negative first Chern class on an elliptic fiber. Similarly,

$$h^0(K_3, \mathcal{L} \otimes \mathcal{K}') - h^1(K_3, \mathcal{L} \otimes \mathcal{K}') = 2 + \frac{1}{2}c_1(\mathcal{L} \otimes \mathcal{K}')^2 = 2 + \mathbf{k}(n - 2) - \mathbf{k}^2, \quad (4.10)$$

We can now calculate the total contribution of these “perturbative states” to the Witten index:

$$I_{\text{pert}} = - [h^0(K_3, \mathcal{L}) - h^1(K_3, \mathcal{L}) - h^0(K_3, \mathcal{L} \otimes \mathcal{K}') + h^1(K_3, \mathcal{L} \otimes \mathcal{K}')] = -2\mathbf{k} = -4. \quad (4.11)$$

The overall $(-)$ sign comes because, as we have seen in §3.4, the ground state $|j\rangle$ has odd fermion number $(-)^F = -1$.

The result (4.11) is not sensitive to the value of n , but for completeness let us determine n and the individual dimensions of the cohomologies. The dimension $h^0(K_3, \mathcal{L})$ of the space of holomorphic sections of a line bundle \mathcal{L} over K_3 can be determined as follows (see [37] for the relevant mathematical background). The question is equivalent to asking for the dimension of the space of meromorphic functions whose divisor of poles is no bigger than $n\mathfrak{F} + \mathbf{k}\mathfrak{B}_0$ (i.e., it has a pole of order no bigger than \mathbf{k} on \mathfrak{B}_0 and of order no bigger than n on \mathfrak{F}). For $\mathbf{k} = 2$, the general function with this property is

$$f(z, x) = P_n(z) + Q_{n-4}(z)x,$$

where P_n is a polynomial of degree n in z and Q_{n-4} is a polynomial of degree $(n-4)$. This is because in terms of \mathfrak{w} , x is proportional to the Weierstrass function $\wp(\mathfrak{w}; \tau)$, which has a pole of order 2 at $\mathfrak{w} = 0$. Moreover, near $z = \infty$ the elliptic fibration equation (4.2) makes sense in coordinates $z' = 1/z$ with $x' = x/z^4$ and $y' = y/z^6$. Therefore, near $z' = 0$ we find that $f(1/z', x'/z'^4)$ has a pole of order n in z' . We can therefore calculate the dimension of $h^0(K_3, \mathcal{L})$ as the number of independent coefficients in $f(z, x)$ and find:

$$(\mathbf{k} = 2) \quad h^0(K_3, \mathcal{L}) = \begin{cases} 2n - 2 & \text{for } n \geq 3 \\ n + 1 & \text{for } 0 \leq n \leq 2 \\ 0 & \text{for } n < 0 \end{cases} \quad (4.12)$$

For $\mathbf{k} = 3$ a similar argument shows that

$$P_n(z) + Q_{n-4}(z)x + R_{n-6}(z)y$$

is the general solution, so that

$$(\mathbf{k} = 3) \quad h^0(K_3, \mathcal{L}) = \begin{cases} 3n - 7 & \text{for } n \geq 5 \\ 2n - 2 & \text{for } 3 \leq n \leq 4 \\ n + 1 & \text{for } 0 \leq n \leq 2 \\ 0 & \text{for } n < 0 \end{cases} \quad (4.13)$$

while for $\mathbf{k} = 1$ we are left with only a polynomial $P_n(z)$ with

$$(\mathbf{k} = 1) \quad h^0(K_3, \mathcal{L}) = \begin{cases} n + 1 & \text{for } 0 \leq n \\ 0 & \text{for } n < 0 \end{cases} \quad (4.14)$$

We can now calculate $h^1(K_3, \mathcal{L})$ from (4.9), and in particular we find that

$$h^1(K_3, \mathcal{L}) = 0 \quad \text{for} \quad \begin{cases} n \geq 0, & \mathbf{k} = 1 \\ n \geq 3, & \mathbf{k} = 2 \\ n \geq 5, & \mathbf{k} = 3 \end{cases} \quad (4.15)$$

For $h^0(K_3, \mathcal{L} \otimes \mathcal{K}')$ and $h^1(K_3, \mathcal{L} \otimes \mathcal{K}')$ we get similar results after replacing $n \rightarrow n-2$ in (4.12)-(4.15).

Now let us determine n . The first Chern class of a line bundle can be calculated if a norm on the \mathbb{C} -fibers of the line bundle is given (with the assumption that the norm is independent of the patch used to calculate it). Thus, if s is a local section of \mathcal{L} then (see for instance [47] p148):

$$c_1(\mathcal{L}) = -\frac{i}{2\pi} [\partial\bar{\partial} \log(\|s\|^2)],$$

where $\|s\|$ is the norm of s (as a function of the point on K_3) and $[(\dots)]$ is the cohomology class of the 2-form (\dots) . In our case a natural norm is given as follows. Consider for example the 0-forms

$$\chi^{(0,0)} \equiv \sum_{j=0}^{\mathbf{k}-1} \chi_j^{(0,0)}(\mathfrak{z}, \bar{\mathfrak{z}}) \Theta_{j,\mathbf{k}}(\mathfrak{w}, \tau(\mathfrak{z})).$$

Since, we can write the normalized wave-function on T^2 as (3.10) with $\varphi_{j,\mathbf{k}}$ normalized so that $\int |\varphi_{j,\mathbf{k}}|^2 d\mathbf{x}_1 d\mathbf{x}_2 = 1$, we find the total norm

$$\|\chi^{(0,0)}\|^2 = \frac{1}{4\pi^2} \int (2\mathbf{k}\tau_2)^{1/2} e^{-\frac{\mathbf{k}\tau_2 \mathbf{x}_2^2}{2\pi}} |\chi^{(0,0)}|^2 |g_{\mathfrak{z}\bar{\mathfrak{z}}}| d^2 \mathfrak{z} d\mathbf{x}_1 d\mathbf{x}_2$$

and we can set

$$\|s\|^2 \rightarrow \frac{1}{4\pi^2} (2\mathbf{k}\tau_2)^{1/2} e^{-\frac{\mathbf{k}\tau_2 \mathbf{x}_2^2}{2\pi}} |\chi^{(0,0)}|^2.$$

Since $\chi^{(0,0)}$ is holomorphic, and setting $\mathbf{x}_2 = 2\pi(\mathfrak{w} - \bar{\mathfrak{w}})/(\tau - \bar{\tau})$, we find

$$c_1(\mathcal{L}) = -\frac{i}{2\pi} \left[\partial\bar{\partial} \left(\frac{-\mathbf{k}\tau_2 \mathbf{x}_2^2}{2\pi} \right) \right] - \frac{i}{4\pi} \left[\partial\bar{\partial} \log \tau_2 \right] = \frac{1}{4\pi^2} \mathbf{k} d\mathbf{x}_1 \wedge d\mathbf{x}_2 + \frac{i}{16\pi\tau_2^2} d\tau \wedge d\bar{\tau} \quad (4.16)$$

The integral of the second term over a fundamental domain \mathcal{F} of the $\text{SL}(2, \mathbb{Z})$ action on the upper-half τ -plane gives

$$\frac{i}{2\pi} \int_{\mathcal{F}} \frac{d\tau \wedge d\bar{\tau}}{8\tau_2^2} = \frac{1}{24}.$$

Since the elliptic fibration equation (4.2) describes the base \mathfrak{B} (given by the locus of $\mathbf{x}_1 = \mathbf{x}_2 = 0$) as a 24-fold cover of the fundamental domain \mathcal{F} , we find that the pullback of $c_1(\mathcal{L})$ to \mathfrak{B} is $24 \times (\frac{1}{24}) = 1$. Thus

$$c_1(\mathcal{L}) = \mathbf{k}\omega_{\mathfrak{B}} + (1 + 2\mathbf{k})\omega_{\mathfrak{F}}.$$

[Note that $\int_{\mathfrak{B}_0} \omega_{\mathfrak{B}} = [\mathfrak{B}_0] \cdot [\mathfrak{B}_0] = -2$, so $\int_{\mathfrak{B}_0} c_1(\mathcal{L}) = (1 + 2\mathbf{k} - 2\mathbf{k})$.] In particular, (4.15) implies that

$$h^1(K_3, \mathcal{L}) = h^1(K_3, \mathcal{L} \otimes \mathcal{K}') = 0, \quad \mathbf{k} = 1, 2, 3,$$

and all the states can be described by holomorphic sections of either \mathcal{L} or $\mathcal{L} \otimes \mathcal{K}'$. For $\mathbf{k} = 2$, for example, we find

$$h^0(K_3, \mathcal{L}) = 8, \quad h^0(K_3, \mathcal{L} \otimes \mathcal{K}') = 4.$$

4.3 The Witten Index

Mirror-symmetry of K_3 can formally be described [67, 8] by its action on what is known as a *Fourier-Mukai vector*. This is an element $v \equiv v_0 + v_2 + v_4$ in the lattice $H^0(K_3, \mathbb{Z}) \oplus H^2(K_3, \mathbb{Z}) \oplus H^4(K_3, \mathbb{Z})$, and we define $v^2 \equiv 2v_0 \cdot v_4 + v_2^2$ via the standard intersection form on cohomology. A mirror-symmetry element is described by a linear transformation on this vector v that preserves the lattice $H^0(K_3, \mathbb{Z}) \oplus H^2(K_3, \mathbb{Z}) \oplus H^4(K_3, \mathbb{Z})$ and preserves v^2 . Physically, in the context of type-IIA string theory compactification on $K_3 \times \mathbb{R}^{5,1}$, we understand v as a vector of D-brane charges (D0, D2, and D4). Mirror symmetry converts a combination of pointlike D0's, D2's that wrap (real) surfaces in K_3 , and D4's that wrap the entire K_3 into a similar combination but with different charges. The advantage of describing a duality in terms of a Fourier-Mukai vector is that the duality naturally corresponds to an $O(20, 4, \mathbb{Z})$ matrix.

We can now calculate the Witten index of the theory with the mirror-symmetry twist by requiring modular invariance of the torus partition function. Explicitly, we insert the mirror-symmetry operator $\hat{\mathfrak{M}}$ at $\sigma^1 = 0$, and we compactify (Euclidean) time σ^0 on S^1 with periodic boundary conditions for the fermions. We need to calculate the partition function, which we do by interchanging the roles of σ^0 and σ^1 . If σ^0 plays the role of “space”, the Hilbert space is the space of Ramond-Ramond ground states, which is naturally mapped to the Fourier-Mukai vector space.³ Letting $\hat{\mathfrak{M}}$ denote the $O(20, 4, \mathbb{Z})$ matrix that describes the action of $\hat{\mathfrak{M}}$ on the Fourier-Mukai vector, we calculate the Witten index as

$$I(\hat{\mathfrak{M}}) = \text{tr}\{(-1)^F \hat{\mathfrak{M}}\} = \text{tr} \hat{\mathfrak{M}}. \quad (4.17)$$

We have set $(-)^F = 1$, since we only have even-dimensional cohomology.

In the complex structure that is compatible with the elliptic fibration (i.e., $\pi : K_3 \rightarrow \mathfrak{B}$ is holomorphic), the nonzero Hodge numbers are $h^{0,0} = h^{2,2} = h^{2,0} = h^{0,2} = 1$ and $h^{1,1} = 20$. Let us now review some facts about the cohomology group $H^2(K_3, \mathbb{Z})$. We can construct a basis of the Poincaré dual homology group $H_2(K_3, \mathbb{Z})$ as follows. First, let $[\mathfrak{F}]$ and $[\mathfrak{B}_0]$ be the homology classes defined above equation (4.7), which correspond to the holomorphic submanifolds given by $\{\mathfrak{z} = 0\}$ and $\{x = y = \infty\}$, respectively. The remaining 18 independent generators of $H^2(K_3, \mathbb{Z})$ cannot be represented by analytic submanifolds but can be constructed as follows. Let γ be a smooth path on \mathfrak{B} from one of the special points \mathfrak{z}_i to another \mathfrak{z}_j ($i \neq j$), avoiding all other special points. While $\tau(\mathfrak{z})$ is a multivalued function, undergoing $\text{SL}(2, \mathbb{Z})$ transformations along cuts, we can find a small neighborhood of $\gamma \setminus \{\mathfrak{z}_i, \mathfrak{z}_j\}$ on which $\tau(\mathfrak{z})$ is represented by an analytic function without cuts. If in that representation

³In the string theory context each D-brane charge can be mapped to a (possibly fractional) set of Ramond-Ramond fluxes, as measured at infinity. The operators that correspond (under the CFT state-operator correspondence) to the Ramond-Ramond ground states of the σ -model are factors in the vertex operators that correspond to the type-II Ramond-Ramond fluxes.

$\tau(\mathfrak{z}_i) = \tau(\mathfrak{z}_j) = \infty$ (in general, the limits $\tau(\mathfrak{z}_i)$ and $\tau(\mathfrak{z}_j)$ are only guaranteed to be $\text{SL}(2, \mathbb{Z})$ equivalent) then we can construct a 2-cycle in K_3 by taking the basic α 1-cycle of \mathfrak{F} (from $\mathbf{x}_1 = 0$ to $\mathbf{x}_1 = 2\pi$ at a fixed \mathbf{x}_2), and dragging it along γ . This cycle will shrink to a point at both ends of γ and hence form a closed 2-cycle. It can be shown that in homology there are 18 linearly independent 2-cycles of this type, with intersection form that is equivalent to the $E_8 \oplus E_8 \oplus \mathbb{H}$ lattice, where \mathbb{H} is the 2-dimensional lattice \mathbb{Z}^2 with intersection form $(n_1, n_2)^2 = 2n_1n_2$ [for $(n_1, n_2) \in \mathbb{Z}^2$].

Let $\omega_{\mathfrak{F}}, \omega_{\mathfrak{B}}$ be the Poincaré dual $(1, 1)$ cohomology classes such that

$$\int_{[\mathfrak{F}]} \omega_{\mathfrak{F}} = 0, \quad \int_{[\mathfrak{B}]} \omega_{\mathfrak{F}} = \int_{[\mathfrak{F}]} \omega_{\mathfrak{B}} = 1, \quad \int_{[\mathfrak{B}]} \omega_{\mathfrak{B}} = -2.$$

For generic f_8 and g_{12} , all the integral $(1, 1)$ forms are linear combinations of $\omega_{\mathfrak{F}}$ and $\omega_{\mathfrak{B}}$ with integer coefficients. Let $\Upsilon \subset H^2(K_3, \mathbb{Z})$ be the sublattice of elements that are orthogonal (in terms of the intersection pairing) to both $\omega_{\mathfrak{F}}$ and $\omega_{\mathfrak{B}}$. It is generated by the cohomology classes of 2-cycles constructed from the paths between \mathfrak{z}_i and \mathfrak{z}_j as described above. We can now describe $\hat{\mathfrak{M}}$. Let x_0 and x_4 be generators of $H^0(K_3, \mathbb{Z})$ and $H^4(K_3, \mathbb{Z})$ so that $x_0 \cdot x_4 = 1$. Then $\hat{\mathfrak{M}}$ acts as $(+1)$ on Υ and acts as follows on the remaining generators:

$$\hat{\mathfrak{M}}(x_0) = \omega_{\mathfrak{F}}, \quad \hat{\mathfrak{M}}(x_4) = \omega_{\mathfrak{B}} + \omega_{\mathfrak{F}}, \quad \hat{\mathfrak{M}}(\omega_{\mathfrak{F}}) = -x_0, \quad \hat{\mathfrak{M}}(\omega_{\mathfrak{B}}) = x_0 - x_4.$$

According to the prescription of §4.1, this transformation preserves two pairs of left and right moving supersymmetries. We note that for $(2, 2)$ models, mirror symmetry always preserves all the left-moving supercharges and only one combination of the right-moving supercharges (see, e.g., [57]). Since we are dealing with a $(4, 4)$ model, there are two additional left and right moving supercharges that are not part of the $(2, 2)$ superconformal algebra, and one combination of the two additional right-moving supercharges is also preserved by $\hat{\mathfrak{M}}$. Calculation of the Witten index now gives:

$$I(\hat{\mathfrak{M}}) = \text{tr}\{(-1)^F \hat{\mathfrak{M}}\} = 20. \quad (4.18)$$

In §4.2 we calculated the contribution of $I_{\text{pert}} = -4$ to the Witten index from states with extended wave-functions along \mathfrak{B} . Comparing (4.11) to (4.18), we see that we need to account for an additional contribution of 24. Since there are exactly 24 special points of \mathfrak{B} with singular fibers, it stands to reason that there are extra “bound states” localized near these special points. We assume that the elliptic fibration is generic, so that the $\text{SL}(2, \mathbb{Z})$ monodromies (in τ) around the special points are all conjugate to $\tau \rightarrow \tau + 1$, and the behaviors of the fibration near all the special points are equivalent. Thus, if there is a contribution to the Witten index from localized ground states, it should be a multiple of 24. It is indeed pleasing that our analysis yielded a discrepancy of exactly 24. In the next subsection we will bring supporting evidence for the conjecture that each special point contributes one bound state.

This result can be extended to other values of \mathbf{k} (namely $\mathbf{k} = 1$ or $\mathbf{k} = 3$) as follows. We work at the value $\rho = e^{i\pi/3}$ (where ρ is the complexified area of the fiber). This value is fixed by $O(20, 4, \mathbb{Z})$ transformations that act as in §3.5 on the fiber. We can construct them by compounding $\hat{\mathfrak{M}}$ with an operator \mathcal{T} that acts as $(+1)$ on Υ and acts as follows on the remaining generators:

$$\mathcal{T}(x_0) = x_0, \quad \mathcal{T}(x_4) = x_4 + \omega_{\mathfrak{B}} + x_0, \quad \mathcal{T}(\omega_{\mathfrak{F}}) = \omega_{\mathfrak{F}} - x_0, \quad \mathcal{T}(\omega_{\mathfrak{B}}) = \omega_{\mathfrak{B}} + 2x_0.$$

This is the transformation that shifts the B -field by $\omega_{\mathfrak{B}}$. We find,

$$I(\hat{\mathfrak{M}}\mathcal{T}) = \text{tr}\{(-1)^F \hat{\mathfrak{M}}\mathcal{T}\} = 18, \quad I(\hat{\mathfrak{M}}\mathcal{T}^{-1}) = \text{tr}\{(-1)^F \hat{\mathfrak{M}}\mathcal{T}^{-1}\} = 22.$$

Locally on the base, $\hat{\mathfrak{M}}\mathcal{T}$ and $\hat{\mathfrak{M}}\mathcal{T}^{-1}$ act as $\text{SL}(2, \mathbb{Z})$ duality transformations of the fiber with $\mathbf{k} = 3$ and $\mathbf{k} = 1$ respectively. The index can therefore be written generally as $I = 24 - 2\mathbf{k}$ (for $\mathbf{k} = 1, 2, 3$) and the contribution of $-2\mathbf{k}$ is explained as in (4.11).

4.4 The singular fibers

We can gain more insight about the behavior near a singular fiber by embedding our problem into string theory. A natural string theory setting is a compactification of type-IIA string theory (it is more convenient to use type-IIA for reasons to become clear shortly) on a K_3 at a point in moduli space that is invariant under the mirror-symmetry element $\hat{\mathfrak{M}}$ considered in §4.3. We then compactify one of the remaining directions, say x_5 , on S^1 of radius R and insert an $\hat{\mathfrak{M}}$ -twist at $x_5 = 0$. In order to preserve some amount of target-space supersymmetry, we also insert at that point a suitable rotation $\gamma \in \text{Spin}(4)$ in directions $6, \dots, 9$.⁴ We then look for the ground states of a string state with winding number $w = 1$ along direction x_5 . The twist γ can be chosen so as to ensure that the ground states are normalizable and localized at $x_6 = x_7 = x_8 = x_9 = 0$. The worldsheet theory that solves this problem (at string coupling constant zero) is an asymmetric \mathbb{Z}_4 orbifold of an $S^1 \times K_3$ compactification with S^1 of radius $4R$. Since directions $5, \dots, 9$ are described by a flat target space their contribution is easy to analyze and we find that the Ramond-Ramond ground

⁴The amount of target space supersymmetry is determined as follows. Similarly to §4.1, set $K \equiv O(20) \times SO(4)$ and let $gK \in O^+(20, 4, \mathbb{R})/K$ be the point in moduli space, represented as a coset of groups. If $\hat{\mathfrak{M}} \in O^+(20, 4, \mathbb{Z})$ preserves gK then $g^{-1}\hat{\mathfrak{M}}g \in K$. Let $\rho \in SO(4)$ be the projection of $g^{-1}\hat{\mathfrak{M}}g$ to the $SO(4)$ factor of K . To construct the background we have to specify the action of $\hat{\mathfrak{M}}$ on target-space fermions, which requires a lift of ρ to the spin group. Let $e^{\pm i\varphi_1}$ and $e^{\pm i\varphi_2}$ be the eigenvalues of (the lift of) ρ in the spinor representation $(2, 1) \oplus (1, 2)$. Then the action of $\hat{\mathfrak{M}}$ on the supercharges has eigenvalues $e^{\pm i\varphi_j}$ ($j = 1, 2$), and the number of preserved SUSY generators is the number of eigenvalues that γ has from the set $\{e^{\mp i\varphi_j}\}_{j=1}^2$, when γ is expressed in the spinor representation $(2, 1) \oplus (1, 2)$. This can be contrasted with the worldsheet SUSY discussed in §4.1, where the scalar and tensor representations $2(1, 1) \oplus (3, 1) \oplus (1, 3)$ of $SO(4)$ were used instead of the spinor.

states of the string are in one-to-one correspondence with the states of the K_3 CFT compactified on S^1 with an $\hat{\mathcal{M}}$ -twist.

Now let us focus on the vicinity of a special point \mathfrak{z}_j . From the point \mathfrak{z}_j there emanates a cut that is accompanied by the $\text{SL}(2, \mathbb{Z})$ transformation $\tau \rightarrow \tau + 1$. We will now perform a series of dualities to relate our problem to a problem of quantizing an Ω -deformed [71, 70] instanton moduli space. Let $\mathbf{x}_1, \mathbf{x}_2$ be coordinates on the fiber \mathfrak{F} over a certain point \mathfrak{z} of the base such that direction \mathbf{x}_1 corresponds to the cycle that shrinks to zero at \mathfrak{z}_j . Now perform a fiberwise T-duality on direction \mathbf{x}_1 . This converts the fiber to a T^2 with complex structure $\tilde{\tau} = i$ and area τ_2 that becomes infinite at \mathfrak{z}_j . The monodromy $\tau \rightarrow \tau + 1$ around the cut shows that in this T-dual type-IIB background we have an NS5-brane at \mathfrak{z}_j that extends in directions $5, \dots, 9$, and is localized at the dual fiber location $(\tilde{\mathbf{x}}_1, \mathbf{x}_2) = (0, 0)$. The mirror-symmetry twist becomes a geometrical twist that acts as a rotation by $\pi/2$ in directions $\tilde{\mathbf{x}}_1, \mathbf{x}_2$. There are two fixed points

$$(\tilde{\mathbf{x}}_1, \mathbf{x}_2) = (0, 0) \text{ or } (\pi, \pi). \quad (4.19)$$

The (π, π) fixed point is far from the NS5-brane and so is not expected to have any additional bound states. Let us now analyze the $(0, 0)$ point by replacing the dual fiber with an \mathbb{R}^2 . We now have a fundamental string wrapped around direction x_5 with a geometrical twist in directions $\tilde{\mathbf{x}}_1, \mathbf{x}_2, x_6, x_7, x_8, x_9$ and in the vicinity of an NS5-brane spanning directions x_5, \dots, x_9 and localized at $\mathfrak{z} = \mathfrak{z}_j$ and $\tilde{\mathbf{x}}_1 = \mathbf{x}_2 = 0$. The question is whether we get additional states localized at $\mathfrak{z} = \mathfrak{z}_j$. Such states will be related to the existence of bound states of the string with the NS5-brane. We can generalize the question to one about bound states of a string with n NS5-branes, which corresponds to a monodromy of $\tau \rightarrow \tau + n$ in the elliptic fibration as would appear when n special points coincide (forming a singular fiber of type A_{n-1}). We can solve the question by performing S-duality, converting the string and NS5-branes into a D1 and n D5-branes. (The geometrical twists are similar to those used in the Ω -deformation context to regularize the moduli space of instantons [71, 70].) We can also regularize the moduli space by adding NSNS B -field components B_{67} and B_{89} to turn the moduli space of instantons into instantons on a noncommutative space [69, 14]. Let us now be more specific about the rotation γ that was introduced in order to preserve SUSY and eliminate zero modes. We can pick it to be a simultaneous rotation by $-\pi/4$ in two planes, the plane $x_6 - x_7$ and the plane $x_8 - x_9$, so that combined with the $\pi/2$ rotation in the plane of $\tilde{\mathbf{x}}_1 - \mathbf{x}_2$ the background will preserve $1/8$ (target space) supersymmetry. At weak string coupling-constant and large R , the D1-D5 dynamics is described locally [88, 32] by a point on the moduli space of $U(n)$ instantons (on a noncommutative \mathbb{R}^4 in directions $6, \dots, 9$) that varies as a function of x_5 , and with a twist at $x_5 = 0$ that corresponds to a rotation by $(-\pi/4, -\pi/4)$ in the planes $6 - 7$ and $8 - 9$, and a rotation by $\pi/2$ in the plane $1 - 2$ (which acts only on the fermionic degrees of freedom). The ground states are found by counting the number of instanton solutions that are invariant under the $(-\pi/4, -\pi/4)$ rotation.

Consider, for example, the case of an $SU(2)$ instanton and start with a commutative \mathbb{R}^4 . A rotation in \mathbb{R}^4 can be compensated by a global $SU(2)$ gauge transformation as long as the center of the instanton is at the origin. However, we are forbidding such transformations that modify the behavior at infinity (in directions 6, . . . , 9) and so we are left with only the zero-size instantons as a solution. These need to be regularized by turning \mathbb{R}^4 into a noncommutative \mathbb{R}^4 , as mentioned above.

Thus, in order to proceed we recall the extension [69, 14] of the ADHM construction for $U(n)$ instantons at level m (for us $m = 1$) on a noncommutative \mathbb{R}^4 . We take complex coordinates (z_0, z_1) on $\mathbb{R}^4 \simeq \mathbb{C}^2$ with noncommutativity given by

$$[z_0, z_0^\dagger] = [z_1, z_1^\dagger] = -\frac{\tilde{\zeta}}{2},$$

where $\tilde{\zeta}$ is a positive constant. One then picks $n \times n$ matrices B_0, B_1 , an $n \times m$ matrix X , and an $m \times n$ matrix Y (all matrices have constant complex entries) that satisfy the quadratic algebraic relations:

$$[B_0, B_1] + XY = 0, \quad [B_0, B_0^\dagger] + [B_1, B_1^\dagger] + XX^\dagger - Y^\dagger Y = \tilde{\zeta} \mathbb{I}_{n \times n}, \quad (4.20)$$

where $\mathbb{I}_{n \times n}$ is the identity matrix. The moduli space of solutions is then the solution to (4.20) subject to a $U(m)$ gauge equivalence defined to act on X and Y as

$$X \rightarrow X\Lambda^{-1}, \quad Y \rightarrow \Lambda Y, \quad \Lambda \in U(m). \quad (4.21)$$

One then looks for a $(2n + m) \times n$ matrix solution $\psi(z_0, z_1)$ for the linear equations

$$\begin{pmatrix} B_0 - z_0 & z_1 - B_1 & X \\ B_1^\dagger - z_1^\dagger & B_0^\dagger - z_0^\dagger & Y^\dagger \end{pmatrix} \psi = 0$$

where z_0, z_1 are short for $z_0 \mathbb{I}_{n \times n}, z_1 \mathbb{I}_{n \times n}$. One then constructs the gauge field as $A_\mu = \psi^\dagger \partial_\mu \psi$. We need an instanton solution that is invariant under the rotation

$$(z_0, z_1) \mapsto (e^{i\pi/4} z_0, e^{i\pi/4} z_1).$$

This rotation acts on the ADHM fields as

$$B_0 \rightarrow e^{\frac{i\pi}{4}} B_0, \quad B_1 \rightarrow e^{\frac{i\pi}{4}} B_1, \quad X \rightarrow e^{\frac{i\pi}{4}} X, \quad Y \rightarrow e^{\frac{i\pi}{4}} Y.$$

and it is easy to see that the only points in moduli space that are invariant under this rotation [up to a gauge transformation (4.21)] are those with $B_0 = B_1 = 0$ and $Y = 0$. For $m = 1$, it is easy to see that the surviving moduli space is $\mathbb{C}\mathbb{P}^{n-1}$. The Hilbert space of localized states coming from an A_{n-1} singular fiber is therefore equivalent to the cohomology of $\mathbb{C}\mathbb{P}^{n-1}$. In particular, a generic singular fiber (A_0 type) carries one extra state.

Chapter 5

Other examples of mirror twists in the orbifold limit

In the second part of this paper we will explore the ground states of theories with twists by other elements of the duality group $SO(20, 4, \mathbb{Z})$ and their possible relation to geometric quantization of K_3 . We will perform the computations in the orbifold limit T^4/\mathbb{Z}_2 . We start by reviewing some known properties of these orbifolds (see [5] for a comprehensive review).

5.1 K_3 as a T^4/\mathbb{Z}_2 orbifold

We take $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ as coordinates on T^4 , where each \mathbf{x}_i ($i = 1, \dots, 4$) has periodicity 2π . We also define the complex structure by

$$z = \frac{1}{2\pi}(\mathbf{x}_1 + \tau \mathbf{x}_2), \quad z' = \frac{1}{2\pi}(\mathbf{x}_3 + \tau' \mathbf{x}_4).$$

The \mathbb{Z}_2 orbifold is defined as

$$(z, z') \simeq (-z, -z'). \tag{5.1}$$

This orbifold has 16 fixed points

$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = (\epsilon_1\pi, \epsilon_2\pi, \epsilon_3\pi, \epsilon_4\pi); \quad (\epsilon_1, \dots, \epsilon_4 \in \{0, 1\}),$$

which we denote by $P(\epsilon_1, \dots, \epsilon_4)$. A smooth complex manifold can be constructed from this orbifold by “blowing-up” these fixed points and replacing each $P(\epsilon_1, \dots, \epsilon_4)$ with a $\mathbb{C}\mathbb{P}^1$ that we denote by $E(\epsilon_1, \dots, \epsilon_4)$. We recall that blowing up the origin of $\mathbb{C}^2/\mathbb{Z}_2$ is done by replacing the singular orbifold $(z, z') \simeq (-z, -z')$ with the subspace of $\mathbb{C}^2 \times \mathbb{C}\mathbb{P}^1$ described by coordinates $(z, z', [s, t])$ (where we use (z, z') as coordinates on \mathbb{C}^2 and $[s, t] \simeq [\lambda s, \lambda t]$ as homogeneous coordinates on $\mathbb{C}\mathbb{P}^1$) that satisfy $zt = z's$,

so that for $(z, z') \neq (0, 0)$ there is a unique solution $[s, t] = [z, z']$ but for $z = z' = 0$ the entire $\mathbb{C}\mathbb{P}^1$ survives. Blowing up an orbifold point in T^4/\mathbb{Z}_2 is done by replacing a small neighborhood of that point with a neighborhood of the blown-up origin of $\mathbb{C}^2/\mathbb{Z}_2$. So far we only discussed the complex structure. The σ -model requires also a complexified Kähler metric which is given by a Kähler 2-form k and NSNS B-field B . It was shown in [4] that the orbifold limit of the σ -model corresponds to the limit where the area of each $E(\epsilon_1, \dots, \epsilon_4)$ vanishes, while the flux of B on each $E(\epsilon_1, \dots, \epsilon_4)$ remains π . This orbifold has a \mathbb{Z}_2^4 group of isometries. The element $\Upsilon(\varpi_1, \dots, \varpi_4)$ that is labeled by $\varpi_1, \dots, \varpi_4 \in \mathbb{Z}_2$ (which we treat as numbers modulo 2) acts as

$$\Upsilon(\varpi_1, \dots, \varpi_4) : (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \mapsto (\mathbf{x}_1 + \pi\varpi_1, \mathbf{x}_2 + \pi\varpi_2, \mathbf{x}_3 + \pi\varpi_3, \mathbf{x}_4 + \pi\varpi_4),$$

$$\varpi_1, \dots, \varpi_4 \in \{0, 1\}. \quad (5.2)$$

Each of these isometries has fixed points. For example $\Upsilon(1, 1, 1, 1)$ has fixed points when each coordinate $\mathbf{x}_1, \dots, \mathbf{x}_4$ is $\pm\pi/4$, and with the identification (5.1), this gives 8 fixed points. Other than the identity, these isometries act nontrivially on the cycles $E(\epsilon_1, \dots, \epsilon_4)$:

$$E(\epsilon_1, \dots, \epsilon_4) \xrightarrow{\Upsilon(a_1, \dots, a_4)} E(\epsilon_1 + a_1, \dots, \epsilon_4 + a_4),$$

where $(\epsilon_i + a_i)$ is understood modulo 2.

We will need a convenient basis for the cohomology $H^2(K_3, \mathbb{Z})$. Let us start by defining the following elements of $H^2(K_3, \mathbb{Z})$:

- $[E(\epsilon_1, \dots, \epsilon_4)]$ (with $\epsilon_i \in \{0, 1\}$) denotes the Poincaré dual of one of the 16 $\mathbb{C}\mathbb{P}^1$ submanifolds $E(\epsilon_1, \dots, \epsilon_4)$ at the intersection of $z = \frac{1}{2}(\epsilon_1 + \tau\epsilon_2)$ and $z' = \frac{1}{2}(\epsilon_3 + \tau'\epsilon_4)$;
- $[M(i, j)] = -[M(j, i)]$ (with $1 \leq i < j \leq 4$) corresponds to the Poincaré dual of the submanifold given by the two equations $\mathbf{x}_i = \pm c_i$ and $\mathbf{x}_j = \pm c_j$ (with the \pm signs correlated), for some generic constants c_i, c_j (not equal to 0 or π). We take the orientation so that the dual class is represented by $\delta(\mathbf{x}_i - c_i)\delta(\mathbf{x}_j - c_j)d\mathbf{x}_i \wedge d\mathbf{x}_j$;

The intersection numbers are:

$$\begin{aligned} [E(\epsilon_1, \dots, \epsilon_4)] \cdot [E(\epsilon'_1, \dots, \epsilon'_4)] &= -2\delta_{\epsilon_1\epsilon'_1}\delta_{\epsilon_2\epsilon'_2}\delta_{\epsilon_3\epsilon'_3}\delta_{\epsilon_4\epsilon'_4}, \\ [E(\epsilon_1, \dots, \epsilon_4)] \cdot [M(i, j)] &= 0, \quad [M(i, j)] \cdot [M(k, l)] = 2\epsilon_{ijkl}, \end{aligned} \quad (5.3)$$

where ϵ_{ijkl} is the anti-symmetric Levi-Civita symbol. In addition to these cohomology classes, $H^2(K_3, \mathbb{Z})$ also contains some classes that are linear combinations of the $[E(\epsilon_1, \dots, \epsilon_4)]$'s and $[M(i, j)]$'s with fractional coefficients. We define

$$[W(i, j, \epsilon_i, \epsilon_j)] \equiv \frac{1}{2}[M(i, j)] - \frac{1}{2} \sum_{\epsilon'_1=0}^1 \cdots \sum_{\epsilon'_4=0}^1 \delta_{\epsilon_i\epsilon'_i}\delta_{\epsilon_j\epsilon'_j}[E(\epsilon'_1, \dots, \epsilon'_4)].$$

Then, one can show that $[W(i, j, \epsilon_i, \epsilon_j)] \in H^2(K_3, \mathbb{Z})$, and $H^2(K_3, \mathbb{Z})$ is spanned by the $[W(i, j, \epsilon_i, \epsilon_j)]$'s and the $[E(\epsilon_1, \dots, \epsilon_4)]$'s (which is an over-complete system). The $[W(i, j, \epsilon_i, \epsilon_j)]$'s are Poincaré dual to homology classes of submanifolds (not necessarily analytic) that can be constructed by setting c_i and c_j , in the definition of $[M(i, j)]$ above, to 0 or π . For example, $[W(3, 4, \epsilon_3, \epsilon_4)]$ is the Poincaré dual of the smooth analytic manifold defined by the holomorphic equation,

$$W(3, 4, \epsilon_3, \epsilon_4) = \{z' = \frac{1}{2}(\epsilon_3 + \epsilon_4\tau')\}. \quad (5.4)$$

which has genus 0. As another example, $[W(1, 3, \epsilon_1, \epsilon_3)]$ can be constructed by starting with the equations $\mathbf{x}_1 = \epsilon_1$ and $\mathbf{x}_3 = \epsilon_3$ for $\epsilon_1, \epsilon_3 \in \{0, 1\}$. This manifold is not analytic, of course, and it has a boundary. It intersects $E(\epsilon_1, \epsilon'_2, \epsilon_3, \epsilon'_4)$ along a circle which divides $E(\epsilon_1, \epsilon'_2, \epsilon_3, \epsilon'_4)$ into two hemispheres. If we attach one of these hemispheres to the manifold, for each of the four possible combinations of ϵ'_2 and ϵ'_4 , we can get a closed manifold whose Poincaré dual is given by $[W(1, 3, \epsilon_1, \epsilon_3)]$.

The equation $z = \pm z_0$, with generic z_0 , describes a $T^2 \subset T^4/\mathbb{Z}_2$. The cohomology class of this T^2 is given by

$$[\mathfrak{F}] \equiv [M(1, 2)],$$

and we can view it as the fiber of an elliptic fibration that becomes singular at the 4 points $z_0 = 0, \frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau$. As we take the limit $z_0 \rightarrow \frac{1}{2}(\epsilon_1 + \epsilon_2\tau)$, for example, the fiber T^2 turns into a union of the four exceptional divisors $\bigcup_{\epsilon_3, \epsilon_4} E(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ ($\epsilon_3, \epsilon_4 = 0, 1$) together with the sphere $W(1, 2, \epsilon_1, \epsilon_2)$ counted with multiplicity 2, so that the cohomology agrees:

$$2[M(1, 2)] = 2[W(1, 2, \epsilon_1, \epsilon_2)] + \sum_{\epsilon_3, \epsilon_4} [E(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)].$$

The fibration has a section which we take to be $W(3, 4, \epsilon_3 = 0, \epsilon_4 = 0)$. We define the cohomology class

$$[\mathfrak{B}] \equiv [W(3, 4, \epsilon_3 = 0, \epsilon_4 = 0)] = \frac{1}{2}[M(3, 4)] - \frac{1}{2} \sum_{\epsilon'_1, \epsilon'_2} [E(\epsilon'_1, \epsilon'_2, 0, 0)].$$

5.2 Mirror symmetry

In the orbifold limit the CFT is a free theory. Consider the space of states of the CFT on S^1 with coordinate $0 \leq \sigma^1 \leq 2\pi$. The Minkowski time coordinate is σ^0 . In the untwisted sector there are 4 scalar fields X^μ ($\mu = 1, \dots, 4$) with oscillator expansions [44, 72]:

$$X^\mu(\sigma^0, \sigma^1) = x^\mu + p^\mu \sigma^0 + \frac{i}{\sqrt{2}} \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{1}{m} \left(\alpha_m^\mu e^{-im(\sigma^0 - \sigma^1)} + \tilde{\alpha}_m^\mu e^{-im(\sigma^0 + \sigma^1)} \right)$$

as well as four left-moving fermionic ψ^μ fields and right-moving $\tilde{\psi}^\mu$ fields:

$$\psi^\mu(\sigma^0, \sigma^1) = \sum_{m \in \mathbb{Z}} \psi_m^\mu e^{-im(\sigma^0 - \sigma^1)}, \quad \tilde{\psi}^\mu(\sigma^0, \sigma^1) = \sum_{m \in \mathbb{Z}} \tilde{\psi}_m^\mu e^{-im(\sigma^0 + \sigma^1)}.$$

The twisted sector has similar expansions with $m \in \mathbb{Z} + \frac{1}{2}$ and with (x^1, \dots, x^4) taking values in one of the 16 fixed points of the \mathbb{Z}_2 action:

$$(x^1, \dots, x^4) = (\epsilon_1 \pi, \dots, \epsilon_4 \pi).$$

In what follows we will only be interested in the ground states. Thus we let all the oscillators with $m > 0$ be in their ground states and we set $p^\mu = 0$. For the untwisted sector this means that we only have to consider the fermionic zero modes. The ground states $|ab\rangle$ are labeled by two (Dirac) spinor indices $a = 1, \dots, 4$ and $b = 1, \dots, 4$ on which the zero modes act as:

$$\psi_0^\mu |ab\rangle = (\gamma^\mu)_a^{a'} |a'b\rangle, \quad \tilde{\psi}_0^\mu |ab\rangle = (\gamma^\mu)_b^{b'} |ab'\rangle,$$

where γ^μ are the Dirac matrices (in some arbitrary basis). The \mathbb{Z}_2 orbifold generator acts as $\psi_0^1 \cdots \psi_0^4 \tilde{\psi}_0^1 \cdots \tilde{\psi}_0^4$ and the surviving \mathbb{Z}_2 -even states can be written in the form

$$|ab\rangle = (\gamma^0)_{ab} |1234\rangle + \frac{1}{4} \epsilon_{\mu\nu\sigma\tau} (\gamma^0 \gamma^{\mu\nu})_{ab} |\sigma\tau\rangle + (\gamma^0 \gamma^{1234})_{ab} |\emptyset\rangle, \quad (5.5)$$

where we have defined a basis of $1 + 6 + 1$ \mathbb{Z}_2 -invariant states $|\emptyset\rangle$, $|\mu\nu\rangle = -|\nu\mu\rangle$, and $|1234\rangle$. The twisted sector has no fermionic zero modes, and so the ground states are only labeled by the (x^1, \dots, x^4) fixed point as $|\epsilon_1, \dots, \epsilon_4\rangle$.

In string theory these states appear as factors in the worldsheet description of Ramond-Ramond states for type-II compactification on $K_3 \times \mathbb{R}^{5,1}$. In type-IIA, for example, the states that correspond to modes of the RR 0-form, 2-form, and 4-form fields that are massless plane waves in the $\mathbb{R}^{5,1}$ directions and proportional to harmonic 0-forms, 2-forms, or 4-forms along K_3 contain $|ab\rangle$ as a factor. The states of the untwisted sector correspond to harmonic forms whose cohomology class is Poincaré dual to a \mathbb{Z}_2 invariant form on T^4 , and the twisted sector states correspond to harmonic forms that are Poincaré dual to $E(\epsilon_1, \dots, \epsilon_4)$.

From this description it is easy to derive the action of several dualities. We will consider several combinations of the basic operators defined as follows. We define \mathcal{S}_1 as the symmetry induced by T-duality on directions $\mathbf{x}_3, \mathbf{x}_4$ followed by a rotation $\mathbf{x}_3 \rightarrow \mathbf{x}_4$. It acts as

$$\begin{aligned} \mathcal{S}_1 |\emptyset\rangle &= |12\rangle, \quad \mathcal{S}_1 |12\rangle = -|\emptyset\rangle, \quad \mathcal{S}_1 |34\rangle = 2|\emptyset\rangle - 2|1234\rangle, \quad \mathcal{S}_1 |1234\rangle = |12\rangle + \frac{1}{2}|34\rangle, \\ \mathcal{S}_1 |\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle &= \frac{1}{2} \sum_{\epsilon'_3, \epsilon'_4} (-1)^{\epsilon_3 \epsilon'_4 + \epsilon_4 \epsilon'_3} |\epsilon_1, \epsilon_2, \epsilon'_3, \epsilon'_4\rangle, \quad \mathcal{S}_1 |ij\rangle = |ij\rangle, \quad (i = 1, 2, \quad j = 3, 4). \end{aligned} \quad (5.6)$$

We can understand the action of \mathcal{S}_1 on the twisted sector states as follows. First note that the orbifold CFT has a \mathbb{Z}_2^4 symmetry with elements that act as

$$\Omega(\varpi_1, \varpi_2, \varpi_3, \varpi_4)|\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle = (-1)^{\sum_{i=1}^4 \varpi_i \epsilon_i} |\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle.$$

The CFT also has another \mathbb{Z}_2^4 symmetry corresponding to the discrete isometries (5.2). The ground states of the untwisted sector are invariant under these symmetries, but the ground states of the twisted sector are not:

$$\Upsilon(\varpi_1, \varpi_2, \varpi_3, \varpi_4)|\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle = |\epsilon_1 + \varpi_1, \epsilon_2 + \varpi_2, \epsilon_3 + \varpi_3, \epsilon_4 + \varpi_4\rangle.$$

The operator \mathcal{S}_1 must convert discrete \mathbb{Z}_2 winding number in directions 3, 4 to discrete \mathbb{Z}_2 momentum in directions 4, 3, respectively, because \mathcal{S}_1 corresponds to T-duality. The symmetries $\Upsilon(\varpi_1, \dots, \varpi_4)$ play the role of discrete momentum while $\Omega(\varpi_1, \dots, \varpi_4)$ play the role of discrete winding number, thus

$$\mathcal{S}_1^{-1} \Upsilon(0, 0, \varpi_3, \varpi_4) \mathcal{S}_1 = \Omega(0, 0, \varpi_4, \varpi_3).$$

This explains the expression for $\mathcal{S}_1|\epsilon_1, \dots, \epsilon_4\rangle$ in (5.6). It is an eigenstate of the four $\Upsilon(0, 0, \varpi_3, \varpi_4)$'s with eigenvalues $(-1)^{\varpi_3 \epsilon_4 + \varpi_4 \epsilon_3}$.

The transformation \mathcal{S}_1 corresponds to the T-duality element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ for the T^2 in directions $\mathbf{x}_3, \mathbf{x}_4$, and we can also define \mathcal{T}_1 which corresponds to the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$.

$$\begin{aligned} \mathcal{T}_1|\emptyset\rangle &= |\emptyset\rangle, & \mathcal{T}_1|12\rangle &= |12\rangle + |\emptyset\rangle, & \mathcal{T}_1|34\rangle &= |34\rangle, & \mathcal{T}_1|1234\rangle &= |1234\rangle - \frac{1}{2}|34\rangle, \\ \mathcal{T}_1|\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle &= (-1)^{\epsilon_3 \epsilon_4} |\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle, & \mathcal{T}_1|ij\rangle &= |ij\rangle, & & & & (i = 1, 2, \quad j = 3, 4). \end{aligned} \tag{5.7}$$

Similarly to $\mathcal{S}_1, \mathcal{T}_1$ we define $\mathcal{S}_2, \mathcal{T}_2$ which act as T-duality on the T^2 in directions $\mathbf{x}_1, \mathbf{x}_2$. Thus,

$$\begin{aligned} \mathcal{S}_2|\emptyset\rangle &= |34\rangle, & \mathcal{S}_2|34\rangle &= -|\emptyset\rangle, & \mathcal{S}_2|12\rangle &= 2|\emptyset\rangle - 2|1234\rangle, & \mathcal{S}_2|1234\rangle &= |34\rangle + \frac{1}{2}|12\rangle, \\ \mathcal{S}_2|\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle &= \frac{1}{2} \sum_{\epsilon'_1, \epsilon'_2} (-1)^{\epsilon_1 \epsilon'_2 + \epsilon_2 \epsilon'_1} |\epsilon'_1, \epsilon'_2, \epsilon_3, \epsilon_4\rangle, & \mathcal{S}_2|ij\rangle &= |ij\rangle, & & & & (i = 1, 2, \quad j = 3, 4). \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} \mathcal{T}_2|\emptyset\rangle &= |\emptyset\rangle, & \mathcal{T}_2|34\rangle &= |34\rangle + |\emptyset\rangle, & \mathcal{T}_2|12\rangle &= |12\rangle, & \mathcal{T}_2|1234\rangle &= |1234\rangle - \frac{1}{2}|12\rangle, \\ \mathcal{T}_2|\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle &= (-1)^{\epsilon_1 \epsilon_2} |\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle, & \mathcal{T}_2|ij\rangle &= |ij\rangle, & & & & (i = 1, 2, \quad j = 3, 4). \end{aligned} \tag{5.9}$$

There is a map between the Fourier-Mukai vector space discussed in §4.3 and the vector space of ground states defined in (5.5),

$$\begin{aligned} x_0 &\mapsto |\emptyset\rangle, & x_4 &\mapsto |1234\rangle - \frac{1}{4} \sum_{\epsilon_1, \dots, \epsilon_4} |\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle, \\ [E(\epsilon_1, \dots, \epsilon_4)] &\mapsto |\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle - \frac{1}{2} |\emptyset\rangle, & [M(\mu, \nu)] &\mapsto |\mu\nu\rangle, \quad (1 \leq \mu < \nu \leq 4), \end{aligned} \quad (5.10)$$

We recall that (5.10) can be interpreted in terms of the phenomenon of “fractional branes” [26, 27]. the Fourier-Mukai vector can be thought of as a vector of Ramond-Ramond charges carried by some D-brane configuration. Due to the coupling, for example, between the NSNS 2-form field and the 1-form RR-field on the world volume of a D2-brane, a D2-brane that wraps an exceptional divisor also carries the charge equivalent of half a D0-brane.

Combining (5.6) and (5.10) we get the action of \mathcal{S}_1 on the Fourier-Mukai vector v via its action on the basis:

$$\begin{aligned} \mathcal{S}_1(x_0) &= [M(1, 2)], & \mathcal{S}_1([M(1, 2)]) &= -x_0, \\ \mathcal{S}_1([M(i, j)]) &= [M(i, j)], & (i = 1, 2, \quad j = 3, 4), \\ \mathcal{S}_1([M(3, 4)]) &= -2x_4 - \frac{1}{2} \sum_{\epsilon_1, \dots, \epsilon_4} [E(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)] - 2x_0, \\ \mathcal{S}_1([E(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)]) &= -\frac{1}{2}[M(1, 2)] + \frac{1}{2} \sum_{\epsilon'_3, \epsilon'_4} (-1)^{\epsilon_3 \epsilon'_4 + \epsilon_4 \epsilon'_3} [E(\epsilon_1, \epsilon_2, \epsilon'_3, \epsilon'_4)] + \delta_{\epsilon_3 0} \delta_{\epsilon_4 0} x_0, \\ \mathcal{S}_1(x_4) &= \frac{1}{2}[M(3, 4)] + [M(1, 2)] - \frac{1}{2} \sum_{\epsilon_1, \epsilon_2} [E(\epsilon_1, \epsilon_2, 0, 0)] - x_0. \end{aligned} \quad (5.11)$$

The operation \mathcal{S}_1 is intuitively understood as T-duality on the $\mathbf{x}_3, \mathbf{x}_4$ directions.

For the special complex structure $\tau = i$ there is an additional \mathbb{Z}_4 symmetry generated by a $(\pi/2)$ -rotation of the $\mathbf{x}_1 - \mathbf{x}_2$ torus around the origin. It acts as

$$\begin{aligned} \mathcal{R}_2|\emptyset\rangle &= |\emptyset\rangle, & \mathcal{R}_2|34\rangle &= |34\rangle, & \mathcal{R}_2|12\rangle &= |12\rangle, & \mathcal{R}_2|1234\rangle &= |1234\rangle, \\ \mathcal{R}_2|\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\rangle &= |\epsilon_2, \epsilon_1, \epsilon_3, \epsilon_4\rangle, & \mathcal{R}_2|1j\rangle &= |2j\rangle, & \mathcal{R}_2|2j\rangle &= -|1j\rangle, & (j = 3, 4). \end{aligned} \quad (5.12)$$

We now compactify σ^1 on S^1 and insert a duality twist $\hat{\mathfrak{M}}$. We will start with the two cases with $\hat{\mathfrak{M}} = \mathcal{S}_1$ or $\hat{\mathfrak{M}} = \mathcal{S}_1 \mathcal{S}_2$. We wish to find the Hilbert space of ground states. Using (4.17) and the explicit expressions (5.6)-(5.7) and (5.8)-(5.9) we calculate the Witten indices:

$$I(\mathcal{S}_1) = 12, \quad I(\mathcal{S}_1 \mathcal{S}_2) = 8. \quad (5.13)$$

In §5.2-§5.2, we will describe the ground states directly by orbifolding the construction of §3. We can also consider other duality twists. Take for example the twists by $\mathcal{S}_1\mathcal{T}_1$ and $\mathcal{S}_1\mathcal{T}_1\mathcal{S}_2\mathcal{T}_2$ with calculated Witten indices:

$$I(\mathcal{S}_1\mathcal{T}_1) = 10, \quad I(\mathcal{S}_1\mathcal{T}_1\mathcal{S}_2\mathcal{T}_2) = 6. \quad (5.14)$$

These will be discussed in §5.2-§5.2. Lastly, in §5.2 we will study a combination of rotation on the base \mathfrak{B} and duality on the fiber \mathfrak{F} with the relevant Witten index

$$I(\mathcal{S}_1\mathcal{R}_2) = 4. \quad (5.15)$$

The \mathcal{S}_1 twist

We need to account for a Witten index of 12. The twist \mathcal{S}_1 acts as T-duality on directions $\mathbf{x}_3, \mathbf{x}_4$. Thus, we begin by adding

$$\frac{1}{2\pi} \int (\mathbf{x}'_3 d\mathbf{x}_4 - \mathbf{x}'_4 d\mathbf{x}_3), \quad \text{with } \mathbf{x}'_I \equiv \mathbf{x}_I(\sigma^1 = 2\pi), \quad \mathbf{x}_I \equiv \mathbf{x}_I(\sigma^1 = 0),$$

to the action of the CFT. The $\mathbf{x}_1, \mathbf{x}_2$ CFT is unchanged. In the untwisted sector we set $\mathbf{x}'_I = \mathbf{x}_I$ for $I = 1, 2, 3, 4$. At low-energy the $\mathbf{x}_3 - \mathbf{x}_4$ sector of the CFT reduces to geometric quantization of T^2 with symplectic form $2d\mathbf{x}_3 \wedge d\mathbf{x}_4$, resulting in a 2-dimensional Hilbert space that corresponds to the space of level-2 Θ -functions on T^2 . All states of this sector are bosonic. The $\mathbf{x}_1 - \mathbf{x}_2$ CFT reduces to supersymmetric quantum mechanics on T^2 with states corresponding to the cohomology $H^*(T^2, \mathbb{R})$. We need to keep only the states that are invariant under the orbifold action $\mathbf{x}_I \rightarrow -\mathbf{x}_I$. This acts on level- \mathbf{k} Θ -functions as $\Theta_{\mathbf{k},j} \rightarrow \Theta_{\mathbf{k},\mathbf{k}-j}$ and so for $\mathbf{k} = 2$ all Θ -functions are invariant. On the base \mathfrak{B} , the \mathbb{Z}_2 -invariant states are those corresponding to the even cohomology, i.e., 1 and $d\mathbf{x}_1 \wedge d\mathbf{x}_2$, and the \mathbb{Z}_2 -odd states are those that correspond to the odd cohomology, i.e., $d\mathbf{x}_1$ and $d\mathbf{x}_2$. Now, the missing piece of information that we need in order to combine the states of the $\mathbf{x}_3 - \mathbf{x}_4$ CFT with the states of the $\mathbf{x}_1 - \mathbf{x}_2$ CFT, to form \mathbb{Z}_2 -invariant states, is whether the single ground state of the fermionic sector of the $\mathbf{x}_3 - \mathbf{x}_4$ directions is \mathbb{Z}_2 -even or \mathbb{Z}_2 -odd. If this ground state is \mathbb{Z}_2 -even then we combine the 2 states of the $\mathbf{x}_3 - \mathbf{x}_4$ sector with the even dimensional cohomology states 1 and $d\mathbf{x}_1 \wedge d\mathbf{x}_2$. On the other hand, if the ground state is \mathbb{Z}_2 -odd then we combine the 2 states of the $\mathbf{x}_3 - \mathbf{x}_4$ sector with the odd dimensional cohomology states $d\mathbf{x}_1$ and $d\mathbf{x}_2$. In any case, the untwisted sector produces $2 \times 2 = 4$ states, but we need to know whether they contribute +4 or -4 to the Witten index. An argument as in §3.4 implies that the ground state of the fermionic $\psi^3, \psi^4, \bar{\psi}^3, \bar{\psi}^4$ system is actually \mathbb{Z}_2 -odd and has an odd fermion number. Altogether, combining it with the \mathbb{Z}_2 -odd and fermionic states $d\mathbf{x}_3$ and $d\mathbf{x}_4$ shows that the untwisted sector has $2 \times 2 = 4$ states that contribute +4 to the Witten index.

In the twisted sector we have to set $\mathbf{x}'_I = -\mathbf{x}_I$ for $I = 1, 2, 3, 4$. At low-energy the $\mathbf{x}_3 - \mathbf{x}_4$ sector reduces to geometric quantization of T^2 with symplectic form

$-2d\mathbf{x}_3 \wedge d\mathbf{x}_4$, again resulting in a 2-dimensional Hilbert space. These states can still be identified with level-2 Θ -functions provided we take the opposite complex structure where the holomorphic variable is $\bar{z}' \equiv \mathbf{x}_3 + \mathbf{x}_4\bar{\tau}'$ instead of $z' \equiv \mathbf{x}_3 + \mathbf{x}_4\tau$. For the $\mathbf{x}_1 - \mathbf{x}_2$ CFT the twisted boundary conditions leave us with only 4 states that are localized at the fixed points of \mathbb{Z}_2 , i.e., at

$$(\mathbf{x}_1, \mathbf{x}_2) = \text{one of } (0, 0), (0, \pi), (\pi, 0), (\pi, \pi). \quad (5.16)$$

They are all invariant under \mathbb{Z}_2 and the twisted sector therefore has $4 \times 2 = 8$ states. They must have even fermion number so as to contribute $+8$ to the Witten index, so that altogether we get a total of 12 bosonic states, in accordance with the Witten index calculated in (5.13). Note that the 4 wave-functions of the states of the untwisted sector are spread across the base \mathfrak{B} , while the 8 wave-functions of the twisted sector are localized at the 4 singular fibers, with 2 states for each singular fiber.

The $\mathcal{S}_1\mathcal{S}_2$ twist

The $\mathcal{S}_1\mathcal{S}_2$ twist acts as T-duality on both fiber and base. The untwisted sector has $2 \times 2 = 4$ states corresponding to geometric quantization of $T^2 \times T^2$ with symplectic form

$$\omega = 2d\mathbf{x}_1 \wedge d\mathbf{x}_2 + 2d\mathbf{x}_3 \wedge d\mathbf{x}_4.$$

This sector gives rise to 4 states which can be identified with states of geometric quantization of $(T^2 \times T^2)/\mathbb{Z}_2$ with symplectic form given by the same ω as above, which is Poincaré dual to $[M(1, 2)] + [M(3, 4)]$.

The twisted sector similarly has $2 \times 2 = 4$ states corresponding to geometric quantization of $(T^2 \times T^2)/\mathbb{Z}_2$ with symplectic form

$$\omega' = -2d\mathbf{x}_1 \wedge d\mathbf{x}_2 - 2d\mathbf{x}_3 \wedge d\mathbf{x}_4.$$

This sector gives rise to 4 states which can be identified with states of geometric quantization of $(T^2 \times T^2)/\mathbb{Z}_2$ with symplectic form given by the same ω as above, which is Poincaré dual to $-[M(1, 2)] - [M(3, 4)]$. Altogether in both sectors we get 8 bosonic states, in accordance with the Witten index calculated in (5.13).

The $\mathcal{S}_1\mathcal{T}_1$ twist

The twist \mathcal{S}_1 acts as T-duality on directions $\mathbf{x}_3, \mathbf{x}_4$ and the twist \mathcal{T}_1 acts as spectral flow on the B -field. Thus, we add

$$\frac{1}{2\pi} \int (\mathbf{x}'_3 d\mathbf{x}_4 - \mathbf{x}'_4 d\mathbf{x}_3 - \mathbf{x}_3 d\mathbf{x}_4), \quad \text{with } \mathbf{x}'_I \equiv \mathbf{x}_I(\sigma^1 = 2\pi), \quad \mathbf{x}_I \equiv \mathbf{x}_I(\sigma^1 = 0),$$

to the action of the CFT. The $\mathbf{x}_1, \mathbf{x}_2$ CFT is unchanged. As before, we set $\mathbf{x}'_I = \mathbf{x}_I$ for $I = 1, 2, 3, 4$ in the untwisted sector. Now at low-energy the $\mathbf{x}_3 - \mathbf{x}_4$ sector of

the CFT reduces to geometric quantization of T^2 with symplectic form $d\mathbf{x}_3 \wedge d\mathbf{x}_4$, with a 1-dimensional Hilbert space that corresponds to the level-1 Θ -function on T^2 , which is invariant under the orbifold action $\mathbf{x}_I \rightarrow -\mathbf{x}_I$. This state is bosonic. The $\mathbf{x}_1 - \mathbf{x}_2$ CFT is unchanged from 5.2, and since the ground state in the $\mathbf{x}_3 - \mathbf{x}_4$ sector is again \mathbb{Z}_2 -odd, we combine it with the odd cohomology states $d\mathbf{x}_1$ and $d\mathbf{x}_2$, and this contributes +2 to the Witten index.

In the twisted sector we have to set $\mathbf{x}'_I = -\mathbf{x}_I$ for $I = 1, 2, 3, 4$. At low-energy the $\mathbf{x}_3 - \mathbf{x}_4$ sector reduces to geometric quantization of T^2 with symplectic form $-3d\mathbf{x}_3 \wedge d\mathbf{x}_4$, with a 3-dimensional Hilbert space. However, the Hilbert space of level-3 Θ -functions that are invariant under the orbifold action $\mathbf{x}_I \rightarrow -\mathbf{x}_I$ is 2-dimensional; $\Theta_{3,0}$ and the linear combination $\Theta_{3,1} + \Theta_{3,2}$ are invariant. These states are bosonic.

The twisted sector of the $\mathbf{x}_1 - \mathbf{x}_2$ CFT is again the same as in 5.2, with 4 states corresponding to the fixed points (5.16), so altogether the twisted sector has $4 \times 2 = 8$ states. They must have even fermion number so as to contribute +8 to the Witten index, so that altogether we get a total of 10 bosonic states, in accordance with the Witten index calculated in (5.14).

The $\mathcal{S}_1\mathcal{T}_1\mathcal{S}_2\mathcal{T}_2$ twist

The $\mathcal{S}_1\mathcal{T}_1\mathcal{S}_2\mathcal{T}_2$ twist acts as T-duality and spectral flow on both fiber and base. The untwisted sector has $1 \times 1 = 1$ state corresponding to geometric quantization of $T^2 \times T^2$ with symplectic form

$$\omega = d\mathbf{x}_1 \wedge d\mathbf{x}_2 + d\mathbf{x}_3 \wedge d\mathbf{x}_4.$$

This state is \mathbb{Z}_2 even, and hence survives the orbifold projection of $(T^2 \times T^2)/\mathbb{Z}_2$. The twisted sector similarly has $2 \times 2 + 1 \times 1 = 5$ states which are the \mathbb{Z}_2 -even states of the Hilbert space that we get by geometric quantization of $T^2 \times T^2$ with symplectic form

$$\omega' = -3d\mathbf{x}_1 \wedge d\mathbf{x}_2 - 3d\mathbf{x}_3 \wedge d\mathbf{x}_4.$$

The $2 \times 2 = 4$ states come from the combination of \mathbb{Z}_2 invariant subspaces of the level-3 Θ -functions on each T^2 (given by $\Theta_{3,0}$ and $\Theta_{3,1} + \Theta_{3,2}$ on each T^2). The $1 \times 1 = 1$ state comes from the combination of 1-dimensional \mathbb{Z}_2 odd subspaces on each T^2 (given by $\Theta_{3,1} - \Theta_{3,2}$ on each T^2). Altogether, combining both sectors we get 6 bosonic states, which is in accordance with the Witten index that we calculated in (5.14).

The $\mathcal{S}_1\mathcal{R}_2$ twist

The \mathcal{S}_1 twist acts on directions $\mathbf{x}_3 - \mathbf{x}_4$ as in (5.2), while \mathcal{R}_2 acts as a rotation by $\pi/2$ of the $\mathbf{x}_1 - \mathbf{x}_2$ torus, which is assumed to have complex structure $\tau = i$. In the untwisted sector there are two ground states for the $\mathbf{x}_3 - \mathbf{x}_4$ system, corresponding

to $\Theta_{\mathbf{k},j}$ for $\mathbf{k} = 2$ ($j = 0, 1$), and there are two ground states of the $\mathbf{x}_1 - \mathbf{x}_2$ system corresponding to the two \mathcal{R}_2 fixed points $(\mathbf{x}_1, \mathbf{x}_2) = (0, 0)$ and (π, π) . In the twisted sector, we effectively add another \mathbb{Z}_2 twist, which is equivalent to changing the symplectic form from $2\mathbf{x}_3 \wedge \mathbf{x}_4$ to $-2\mathbf{x}_3 \wedge \mathbf{x}_4$ and at the same time replacing \mathcal{R}_2 with \mathcal{R}_2^{-1} . The generator of the \mathbb{Z}_2 action acts trivially on the bosonic parts of the ground states of both the untwisted and twisted sectors, but the fermionic part of the twisted ground state has opposite \mathbb{Z}_2 charge relative to its untwisted counterpart. (This can be seen, for example, using the state-operator correspondence after bosonization of the fermions.) Thus, only one sector, say the untwisted one, survives the orbifold projection on \mathbb{Z}_2 -invariant states. Altogether, we end up with four ground states, two for each fixed point $(\mathbf{x}_1, \mathbf{x}_2) = (0, 0)$ and (π, π) .

Chapter 6

Connection with geometric quantization

So far we have seen hints of a connection with geometric quantization of the target space. In particular, for the simple case of a T^2 target space, the relevant low-energy terms in the action reduce to the action of geometric quantization of T^2 at level $\mathbf{k} = 2$ [see (3.3)], and thus we have identified the Hilbert space of ground states with the Hilbert space $\mathcal{H}_{\text{gq}}(T^2, \mathbf{k})$ obtained by geometric quantization of T^2 with symplectic form $\frac{\mathbf{k}}{2\pi} d\mathbf{x}_1 \wedge d\mathbf{x}_2$. Geometric quantization defines a noncommutative geometry on T^2 , and the latter made a well-known appearance in string theory in [22, 33, 78]. As explained in [78], it appears when we take the limit of large B field for a sigma-model with T^2 target space formulated on a worldsheet with boundary. In this limit the worldsheet action reduces to a 0+1D action of the form (3.8) on the boundary (see also [18]). Noncommutative geometry also makes an appearance on D-branes probing asymmetric orbifolds [16]. In our context, noncommutative geometry arises on a closed worldsheet, but with modified boundary conditions along a nontrivial cycle (i.e., the T-duality twist along the S^1). It would be interesting to understand whether there is a more general connection between duality-twists and geometric quantization, and we will explore this a little bit further in this section.

A connection with geometric quantization would imply that there is a natural way to construct operators that act on $\mathcal{H}_{\text{gq}}(T^2, \mathbf{k})$. The ring of operators can be constructed from the generators $\exp(i\mathbf{x}_1)$ and $\exp(i\mathbf{x}_2)$, and therefore we would like to understand how to construct matrix elements of these operators naturally from the σ -model. Inserting $\exp(i\phi_j(\sigma_1))$ at some position σ_1 is not the right answer, because $\exp(i\mathbf{x}_j)$ should be dimensionless, whereas the normal ordered operators $:\exp(i\phi_j):$ have positive dimensions (which depend on the target space metric). As a matter of fact, even the operators $\exp(i\mathbf{x}_j)$ in the 0+1D Landau problem [given by the Lagrangian I_L of (3.9)] at low-energy do not flow directly to $\exp(i\mathbf{x}_j)$ of the geometric quantization system — there is a finite normalization constant in front.

A possible solution to this problem can be achieved as follows. Let us complexify

\mathbf{x}_j ($j = 1, 2$) by adding an imaginary part to define $\mathbf{z}_j \equiv \mathbf{x}_j + i\mathbf{y}_j$ with $\mathbf{y}_j \in \mathbb{R}$. We promote \mathbf{y}_j to 1+1D free bosonic fields and add fermionic superpartners. We now compactify as before with a T-duality twist on the $(\mathbf{x}_1, \mathbf{x}_2)$ -directions, and insert a geometrical twist corresponding to a rotation by $(\pi/2)$ on the $(\mathbf{y}_1, \mathbf{y}_2)$ -variables, namely:

$$\mathbf{y}_1(\sigma^1 = 0) = \mathbf{y}_2(\sigma^1 = 2\pi), \quad \mathbf{y}_2(\sigma^1 = 0) = -\mathbf{y}_1(\sigma^1 = 2\pi),$$

and similarly for the fermions, so as to preserve supersymmetry. This twist eliminates all zero modes, and leaves only one ground state in the $(\mathbf{y}_1, \mathbf{y}_2)$ system. In the combined $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)$ system, a matrix element of products of operators $\exp(i\mathbf{z}_j)$ between ground states is independent of the insertion points σ^1 and reduces to the matrix element of the corresponding product of $\exp(i\mathbf{x}_j)$ operators in \mathcal{H}_{gq} . The contribution of the higher modes of the \mathbf{y}_j 's cancels out the contribution of the higher modes of the \mathbf{x}_j 's. So we arrive at the identification of the low-energy limit (ground states) of a compactification of a σ -model with target space $T^2 \times \mathbb{R}^2$ on S^1 , with a combination of T-duality and geometrical twists, and geometric quantization of the T^2 , which is the fixed point set of the geometrical component of the twist.

Let us now extend these observations to geometric quantization on S^2 . We start with the orbifold $(T^2 \times \mathbb{R}^2)/\mathbb{Z}_2$ and add a twisted boundary condition that acts as T-duality on T^2 and as rotation by $\pi/2$ on \mathbb{R}^2 . The wave-functions of the ground states fall off fast far away from the origin of \mathbb{R}^2 . To count the number of ground states we recall the model of §5.2 which contained a similar twist, but with \mathbb{R}^2 replaced by T^2 . The $\pi/2$ rotation in that model had two fixed points, but the behavior near each fixed point is the same as for the $(T^2 \times \mathbb{R}^2)/\mathbb{Z}_2$ model. Thus, the number of ground states of the $(T^2 \times \mathbb{R}^2)/\mathbb{Z}_2$ model is half that found in §5.2, which is two ground states. These ground states should correspond to geometric quantization of T^2/\mathbb{Z}_2 , which is the space at the origin of \mathbb{R}^2 . The symplectic form is $\frac{\mathbf{k}}{4\pi^2} d\mathbf{x}_1 \wedge d\mathbf{x}_2$ (for $\mathbf{k} = 2$) whose integral over T^2/\mathbb{Z}_2 is 1. The space T^2/\mathbb{Z}_2 is equivalent as a complex manifold to \mathbb{CP}^1 , and so we expect that our system is equivalent to geometric quantization of $\mathbb{CP}^1 \simeq S^2$ at level 1. Indeed, the states of geometric quantization of \mathbb{CP}^1 at level 1 are constructed as sections of the line bundle $\mathcal{O}(1)$, and the Hilbert space is indeed two-dimensional. We can construct operators from \mathbb{Z}_2 -invariant operators on T^2 . Thus, we consider

$$\hat{\mathcal{O}}_{n_1, n_2} \equiv \cos(2\pi(n_1\mathbf{z}_1 + n_2\mathbf{z}_2)). \quad (6.1)$$

Let us now describe how these operators flow in the IR to operators that arise by geometric quantization of $T^2/\mathbb{Z}_2 \simeq \mathbb{CP}^1$. [That is to say, when projected to the Hilbert space of ground states the operators (6.1) can be identified with operators acting on the Hilbert space of geometrically quantized \mathbb{CP}^1 at level 1.] First let us describe how to get sections of $\mathcal{O}(1)$ by \mathbb{Z}_2 projection of sections of a line bundle of degree 2 on T^2 . We start with the Θ -functions defined in (3.11). These functions

satisfy

$$\Theta_{j,\mathbf{k}}(u+1, \tau) = \Theta_{j,\mathbf{k}}(u, \tau), \quad \Theta_{j,\mathbf{k}}(u+\tau, \tau) = e^{-2\pi i \mathbf{k}(u+\frac{1}{2}\tau)} \Theta_{j,\mathbf{k}}(u, \tau),$$

which makes them sections of a rank- \mathbf{k} line-bundle over T^2 . This line-bundle is invariant under \mathbb{Z}_2 , in the sense that if $f(u)$ is a section, i.e., satisfies

$$f(u+1) = f(u), \quad f(u+\tau) = e^{-2\pi i \mathbf{k}(u+\frac{1}{2}\tau)} f(u), \quad (6.2)$$

then $g(u) \equiv f(-u)$ is also a section of the same bundle. In particular, we have the relations

$$\Theta_{j,\mathbf{k}}(-u, \tau) = \Theta_{j,\mathbf{k}-j}(u, \tau).$$

Since $f(u)$ and $f(-u)$ are sections of the same bundle, we can mod out by \mathbb{Z}_2 by imposing the condition $f(u) = f(-u)$. For even \mathbf{k} , an even holomorphic function f that satisfies the boundary conditions (6.2) projects to a holomorphic section of the line bundle $\mathcal{O}(\mathbf{k}/2)$ on $T^2/\mathbb{Z}_2 \simeq \mathbb{CP}^1$. [For odd \mathbf{k} there is a problem at the point $u = \frac{1}{2}(1+\tau)$, since as can be easily seen from (6.2), the would-be section $f(\pm u)$ on T^2/\mathbb{Z}_2 would pick up a $(-)$ sign after traversing a small loop around $\pm \frac{1}{2}(1+\tau)$.] For $\mathbf{k} = 2$ we have the stronger statement that all holomorphic sections are \mathbb{Z}_2 -invariant. Thus, there is a natural map from the (\mathbb{Z}_2 -invariant) Hilbert space of ground states of geometric quantization of T^2 at level $\mathbf{k} = 2$ to the Hilbert space of ground states of geometric quantization of $T^2/\mathbb{Z}_2 \simeq \mathbb{CP}^1$ at level $\mathbf{k} = 1$. The Hilbert space of ground states of a compactification of the $(T^2 \times \mathbb{R}^2)/\mathbb{Z}_2$ theory with a T-duality and geometric rotation twist reduces to the Hilbert space of geometric quantization of the singular fiber T^2/\mathbb{Z}_2 at the origin of \mathbb{R}^2 . The \mathbb{Z}_2 invariant operators (6.1) reduce to the corresponding operators on the Hilbert space obtained by geometric quantization of T^2/\mathbb{Z}_2 . Explicitly, if we denote by $|j\rangle$ the state that corresponds to $\Theta_{j,\mathbf{k}}$ then the projection on the ground states is calculated by geometric quantization and we have the ‘‘clock and shift’’ matrices

$$e^{2\pi i \mathbf{x}_1} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^{2\pi i \mathbf{x}_2} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

from which the general $\hat{\mathcal{O}}_{n_1, n_2}$ can easily be calculated and we find

$$\cos(2\pi(n_1 \mathbf{z}_1 + n_2 \mathbf{z}_2)) \rightarrow \begin{cases} i^{n_1 n_2} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{n_1} \end{pmatrix} & \text{for even } n_2, \\ i^{n_1 n_2} \begin{pmatrix} 0 & 1 \\ (-1)^{n_1} & 0 \end{pmatrix} & \text{for odd } n_2. \end{cases}$$

Chapter 7

Discussion

This paper was devoted to a few case-studies of the low-energy limit of a compactification of a quantum field theory on S^1 with boundary conditions twisted by a nonperturbative symmetry. The field theory was a 1+1D supersymmetric σ -model with K_3 target space, and the nonperturbative symmetry was an element of the duality group $O(20, 4, \mathbb{Z})$. The “low-energy limit” problem becomes the question of identifying the ground states. We discussed several inequivalent elements of the duality group. Our main results refer to the element that can be interpreted as T-duality on the fiber, for an elliptically fibered K_3 , where the Kähler modulus of the fiber is set to the self-dual value. We found that there must exist ground states with wave-functions that are localized at the points of the base (of the elliptic fibration) where the T^2 fiber degenerates. In addition to these, there are also states that have a spread-out wave-function. We showed that these states are in one-to-one correspondence with sections of certain holomorphic line-bundles over the K_3 . We also supplemented the discussion with the analysis of other elements of $O(20, 4, \mathbb{Z})$ that we studied in the T^4/\mathbb{Z}_2 orbifold limit.

The association of a part of the Hilbert space of ground states with sections of a holomorphic line bundle \mathcal{L} over the K_3 target space suggests a relation with geometric quantization. As was explained in [7], the wave-functions of the Hilbert space obtained by geometric quantization of a Kähler manifold correspond to sections of a holomorphic line-bundle whose first Chern class is the class of the symplectic form (which is taken to be the Kähler class). It would be interesting to formulate a more precise connection between the ground states of a compactification with a mirror-symmetry twist and geometric quantization. Such a connection should also contain a dictionary for mapping operators on the σ -model side to operators on the geometric quantization side. We have begun to explore such a connection in §6, and we suggested that it might come about as follows. In a σ -model with a target space that is a T^{2n} fibration over a $2n$ -dimensional base, consider compactification on S^1 with a mirror-symmetry twist (realized locally as T-duality on the fiber) augmented by a suitable geometrical twist that is induced by an isometry of the base with isolated fixed points.

For a suitably chosen geometrical twist, the Hilbert space of ground states might be identified with the states of geometric quantization of the fibers (which can be singular) over the fixed points of the geometrical twist. We hope to explore this idea further in upcoming work.

Our results could have applications to the study of S-duality of 3+1D $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory. In [40, 42, 41] a compactification of $\mathcal{N} = 4$ SYM on S^1 with boundary conditions given by a combination of S-duality and R-symmetry twists was studied, and a possible connection between the low-energy limit and Chern-Simons theory was suggested, but is not very well understood or established. (See also [80] for another interesting approach.) The connection with our present setting appears if we compactify the 3+1D theory on a Riemann surface Σ_g (of genus g), together with a suitable topological twist. If Σ_g shrinks first, the SYM theory reduces to a certain 1+1D supersymmetric σ -model [49, 15] (and see [59] for a thorough review and new perspective). The target space is the Hitchin moduli space \mathcal{M}_H [54] associated with the gauge group G and Riemann surface Σ_g , and S-duality reduces to mirror-symmetry of \mathcal{M}_H . For $g > 1$ the Hitchin space has (complex) dimension $2(g - 1) \dim G$ (where $\dim G$ is the dimension of the gauge group) and can be represented as a fibration [54] of $T^{2(g-1)\dim G}$ over a $(g - 1) \dim G$ (complex) dimensional base \mathfrak{B} (and the fibers are allowed to degenerate over codimension-1 submanifolds of \mathfrak{B}). It is therefore interesting to study the ground states of a compactification of this σ -model on S^1 with a mirror-symmetry twist. We expect that there are two kinds of ground states: (i) those with wave-functions that can be expanded in terms of (holomorphic) Θ -functions of the fibers, with coefficients that are forms on the base (and it should be possible to map these to the cohomology with coefficients in a suitable line-bundle over \mathcal{M}_H); and (ii) states with wave-functions that are localized at the degenerate fibers. We hope to explore this subject and its connection to S-duality of SYM in more detail in future work.

Part III

Covariant Star Product for Exterior Differential Forms on Symplectic Manifolds

Chapter 8

Introduction

A star product is a deformation of an associative, commutative product of functions into a product that is still associative but no longer commutative. The original motivation behind star products was an alternative theory of quantization; Ref. [9] and Ref. [10] equated quantization with a deformation of the algebra of classical observables of functions on phase space, where the $\mathcal{O}(\hbar)$ term in the deformation can be taken to be the classical Poisson bracket (see Ref. [29] for a historical overview and references therein). Star products were subsequently found to apply in many areas of physics, including string theory.

From the work of Kontsevich [61], Cattaneo and Felder [18], and many others, the star product of functions on general Poisson manifolds is well known, in standard coordinates on \mathbb{R}^d , to all orders in the deformation parameter. Recently, Ref. [3] wrote the explicit form (to $\mathcal{O}(\hbar^3)$) of a covariant star product of functions on Poisson manifolds with torsion-free linear connections.

Motivated by the study of noncommutative gauge theories in string theory, we investigate the deformation of the \mathbb{Z} -graded exterior algebra of differential forms¹. Since the $\mathcal{O}(\hbar)$ term of the star product on functions is the Poisson bracket, we need a generalization of the Poisson bracket to differential forms. We use the differential Poisson algebra discussed in Ref. [21], Ref. [55] and, more recently, Ref. [12], Ref. [51], and Ref. [50] discusses similar material from the perspective of contravariant connections; the approach we use is more general (see Ref. [12]). As found in Ref. [21], Ref. [55], and Ref. [12], endowing a symplectic manifold with a differential Poisson calculus is equivalent to endowing a symplectic manifold with a connection Γ that has certain important restrictions, Equations (8.18)-(8.20). Although these results are already in the literature, we include a brief treatment of the differential Poisson calculus for completeness.

Modulo the important restrictions due to the differential Poisson calculus, we find that it is possible to define a covariant star product to $\mathcal{O}(\hbar^2)$ on the space of differential

¹The case of superspace is considered in Ref. [19].

forms on symplectic manifolds endowed with a differential Poisson calculus, and we give the explicit form in this note in Equations (8.14) and (9.5).

8.1 The Differential Poisson Calculus

For all exterior differential forms² α , β , and γ , a graded differential Poisson algebra on a Poisson manifold must satisfy the following axioms:

$$|\{\alpha, \beta\}| = |\alpha| + |\beta| \quad (8.1)$$

$$\{\alpha, \beta\} = (-1)^{|\alpha||\beta|+1}\{\beta, \alpha\} \quad (8.2)$$

$$\{\alpha, \beta + \gamma\} = \{\alpha, \beta\} + \{\alpha, \gamma\} \quad (8.3)$$

$$\{\alpha, \beta \wedge \gamma\} = \{\alpha, \beta\} \wedge \gamma + (-1)^{|\alpha||\beta|}\beta \wedge \{\alpha, \gamma\} \quad (8.4)$$

$$\{\alpha, \{\beta, \gamma\}\} + (-1)^{|\alpha|(|\beta|+|\gamma|)}\{\beta, \{\gamma, \alpha\}\} + (-1)^{|\gamma|(|\alpha|+|\beta|)}\{\gamma, \{\alpha, \beta\}\} = 0 \quad (8.5)$$

$$d\{\alpha, \beta\} = \{d\alpha, \beta\} + (-1)^{|\alpha|}\{\alpha, d\beta\}. \quad (8.6)$$

Note that $d^2 = 0$, and we take d to be undeformed. Axioms (8.1)-(8.5) give a \mathbb{Z} -graded Poisson bracket. The graded Jacobi identity (8.5) and the Leibniz rule (8.6) constrain the Poisson bracket on differential forms. In particular, on a symplectic manifold, the differential Poisson bracket between a function f and a 1-form α has all of the properties of a covariant differential operator in each argument, and thus defines a connection on the symplectic manifold,

$$\nabla_{X_f}\alpha \equiv \{f, \alpha\},$$

where X_f is the Hamiltonian vector field associated with f . The Leibniz rule (8.6) relates the coordinate variation of the Poisson bivector to the connection coefficients. Consider

$$d\{f, g\} = \{df, g\} + \{f, dg\}, \quad (8.7)$$

where f, g are both functions. Since this must be true for all functions f, g , the Leibniz rule (8.7) becomes, in local coordinates,

$$\partial_k \pi^{ij} = -\pi^{mj} \Gamma_{mk}^i - \pi^{im} \Gamma_{mk}^j, \quad (8.8)$$

where π^{ij} is the Poisson bivector (with inverse ω_{ij} on symplectic manifolds), and Γ_{mk}^i are the connection coefficients. In terms of the covariant derivative and the torsion, (8.8) can also be rewritten as $\nabla_k \pi^{ij} = -\pi^{ir} T_{rk}^j - \pi^{rj} T_{rk}^i$. From the same connection coefficients Γ_{mk}^i , one can define two connections $\nabla, \tilde{\nabla}$ with respective curvatures R, \tilde{R} . For example,

$$\nabla_i dx^k = -\Gamma_{ij}^k dx^j \quad \tilde{\nabla}_i dx^k = -\Gamma_{ji}^k dx^j. \quad (8.9)$$

²A differential form of degree $|\alpha|$ is a covariant tensor field of degree $|\alpha|$ that is totally anti-symmetric. In a local coordinate system $\{x^i, \dots, x^n\}$, a $|\alpha|$ -form α can be uniquely expressed as $\alpha = \frac{1}{|\alpha|!} \alpha_{i_1 \dots i_{|\alpha|}} dx^{i_1} \wedge \dots \wedge dx^{i_{|\alpha|}}$ where the coefficient $\alpha_{i_1 \dots i_{|\alpha|}}$ is completely antisymmetric.

These connections and curvatures differ when the torsion is non-zero. In this convenient notation, discussed further in the Appendix, the Leibniz rule (8.8) becomes

$$\tilde{\nabla}_k \pi^{ij} = 0. \quad (8.10)$$

Thus π^{ij} is covariantly constant under $\tilde{\nabla}$, and $\tilde{\nabla}$ is an almost symplectic connection³.

One can use the Leibniz condition (8.8) together with the Jacobi identity for the Poisson bivector to arrive at a cyclic relation for the torsion:

$$\sum_{(i,j,k)} \pi^{i\ell} \pi^{jm} T_{\ell m}^k = 0. \quad (8.11)$$

Note that a torsion-free connection is sufficient but not necessary to satisfy (8.11). Also note that (8.8) and the Jacobi identity for the Poisson bivector can also be combined to obtain the following cyclic relation:

$$\sum_{(i,j,k)} \pi^{i\ell} \nabla_\ell \pi^{jk} = 0. \quad (8.12)$$

As before, note that a symplectic connection is sufficient but not necessary to satisfy (8.12). If, in addition to $\tilde{\nabla}_k \pi^{ij} = 0$, one imposes $\nabla_k \pi^{ij} = 0$, then $T_{ij}^k = 0$ and there is only one covariant derivative $\nabla = \tilde{\nabla}$. Ref. [12] has a nice proof.

In this note, we do not require $\nabla_k \pi^{ij} = 0$.

To find the local coordinate expression for Poisson bracket between two 1-forms, consider the Leibniz rule again:

$$\{df, dg\} = d\{f, dg\} = d(\pi^{ij} \partial_i f \nabla_j dg) = \pi^{ij} \nabla_i df \wedge \nabla_j dg - \tilde{R}^{ij} \wedge i_i df \wedge i_j dg, \quad (8.13)$$

where, to get the right-most equality of (8.13), we have introduced a contraction operator⁴ i_k and defined $\tilde{R}^{ij} \equiv \frac{1}{2} \pi^{ik} \tilde{R}_{kab}^j dx^a \wedge dx^b$, where \tilde{R}_{kab}^j are the components of the curvature of $\tilde{\nabla}$. An important property \tilde{R}^{ij} is its symmetry in the upper two indices, $\tilde{R}^{ij} = \tilde{R}^{ji}$. This is easily shown from $[\tilde{\nabla}_m, \tilde{\nabla}_n] \pi^{ij} = 0$.

Using the graded product rule, one arrives at the general form of the Poisson bracket between differential forms:

$$\{\alpha, \beta\} = \pi^{ij} \nabla_i \alpha \wedge \nabla_j \beta + (-1)^{|\alpha|} \tilde{R}^{ij} \wedge i_i \alpha \wedge i_j \beta. \quad (8.14)$$

However, (8.14) does not satisfy the graded Jacobi identity a priori. Using (8.14) and the identities $\tilde{R}^{ij} = \tilde{R}^{ji}$, (A.2), and (A.4), as well as the following two identities⁵

$$\nabla_i \nabla_m \alpha = \frac{1}{2} \{\nabla_i, \nabla_m\} \alpha + \frac{1}{2} [\nabla_i, \nabla_m] \alpha \quad (8.15)$$

$$[\nabla_i, \nabla_m] \alpha = -R_{aim}^p dx^a \wedge i_p \alpha - T_{im}^p \nabla_p \alpha, \quad (8.16)$$

³In the literature, symplectic connections are additionally taken to be torsionless, but the connection defined here may have torsion.

⁴The contraction operator i_k is discussed fully in the Appendix. The interior product of a form α with a vector X has the following relation to the contraction operator: $i_X \alpha = X^k i_k \alpha$.

⁵The brackets $\{, \}$ in (8.15) are anticommutators.

we find:

$$\begin{aligned}
& \{\alpha, \{\beta, \gamma\}\} + (-1)^{|\alpha|(|\beta|+|\gamma|)} \{\beta, \{\gamma, \alpha\}\} - \{\{\alpha, \beta\}, \gamma\} = \\
& \quad (\pi^{mn} \partial_n \pi^{ij} + \pi^{in} \partial_n \pi^{jm} + \pi^{jn} \partial_n \pi^{mi}) \nabla_i \alpha \wedge \nabla_j \beta \wedge \nabla_m \gamma \\
& \quad - \pi^{ij} \pi^{mn} R_{aim}^p dx^a \wedge (i_p \alpha \wedge \nabla_j \beta \wedge \nabla_n \gamma + \nabla_n \alpha \wedge i_p \beta \wedge \nabla_j \gamma + \nabla_j \alpha \wedge \nabla_n \beta \wedge i_p \gamma) \\
& \quad + \pi^{mn} \nabla_n \tilde{R}^{ij} \wedge ((-1)^{|\alpha|} i_i \alpha \wedge i_j \beta \wedge \nabla_m \gamma + (-1)^{|\beta|} \nabla_m \alpha \wedge i_i \beta \wedge i_j \gamma \\
& \quad - (-1)^{|\alpha|+|\beta|} i_j \alpha \wedge \nabla_m \beta \wedge i_i \gamma) \\
& \quad - (-1)^{|\beta|} (\tilde{R}^{ij} \wedge i_i \tilde{R}^{mn} + \tilde{R}^{im} \wedge i_i \tilde{R}^{nj} + \tilde{R}^{in} \wedge i_i \tilde{R}^{jm}) \wedge i_m \alpha \wedge i_n \beta \wedge i_j \gamma.
\end{aligned} \tag{8.17}$$

For this to vanish for all α, β, γ , the connection coefficients Γ_{jk}^i must satisfy several additional conditions⁶:

$$R_{jkl}^i = 0 \tag{8.18}$$

$$\nabla_a \tilde{R}_{cd}^{mn} = 0 \tag{8.19}$$

$$\tilde{R}^{ab} \wedge i_b \tilde{R}^{mn} + \tilde{R}^{mb} \wedge i_b \tilde{R}^{na} + \tilde{R}^{nb} \wedge i_b \tilde{R}^{am} = 0. \tag{8.20}$$

Due to the Leibniz rule, these conditions are not independent. For example, consider three functions, f, g , and h . Following the argument in Ref. [12], let us define four functions:

$$\begin{aligned}
J_0(f, g, h) &= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \\
J_1(f, g, h) &= \{f, \{g, dh\}\} + \{g, \{dh, f\}\} + \{dh, \{f, g\}\} \\
J_2(f, g, h) &= \{f, \{dg, dh\}\} + \{dg, \{dh, f\}\} - \{dh, \{f, dg\}\} \\
J_3(f, g, h) &= \{df, \{dg, dh\}\} + \{dg, \{dh, df\}\} + \{dh, \{df, dg\}\}.
\end{aligned}$$

These functions $J_i(f, g, h)$ are obstructions to the graded Jacobi identity. $J_0(f, g, h) = 0$ gives the Jacobi identity for the Poisson bivector. Note that because the Leibniz rule holds,

$$dJ_0(f, g, h) = J_1(f, g, h) + J_1(g, h, f) + J_1(h, f, g).$$

So, $J_0(f, g, h) = 0$ implies that the cyclic permutation of $J_1(f, g, h)$ is identically zero. Similarly,

$$dJ_1(f, g, h) = J_2(f, g, h) - J_2(g, h, f),$$

and, due to the definition, $J_2(f, g, h) = J_2(f, h, g)$, we have,

$$dJ_1(f, g, h) = J_2(f, g, h) - J_2(g, f, h).$$

⁶The classical Yang-Baxter equation (CYBE) is given by $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in tensor product notation, where $[,]$ is the matrix commutator Ref. [38]. Using $\tilde{R}_{ab}^{ij} = \tilde{R}_{ab}^{ji} = -\tilde{R}_{ba}^{ij}$, one can verify that (8.20) implies that \tilde{R}_{ab}^{ij} satisfies the CYBE.

When $J_1(f, g, h) = 0$, this implies that $J_2(f, g, h)$ must be completely symmetric in its arguments. Finally,

$$dJ_2(f, g, h) = J_3(f, g, h),$$

so if $J_2(f, g, h) = 0$, then $J_3(f, g, h) = 0$. Therefore, (8.20) contains no new constraints; (8.20) is implied by the Leibniz rule, the Jacobi identity for the Poisson bivector, the vanishing of R , and the covariant constancy of \tilde{R}^{ij} .

If, in addition to obeying the Leibniz condition and (8.18)-(8.20), the connection happens to be torsionless, then \tilde{R} is identically zero. This is obvious from Equation (A.7) in the Appendix. In the torsionless case, the Poisson bracket reduces to

$$\{\alpha, \beta\}|_{T=0} = \pi^{mn} \nabla_m \alpha \wedge \nabla_n \beta.$$

Chapter 9

A Covariant Star Product for Exterior Differential Forms

On symplectic manifolds M , we deform $\Omega^*(M)$ to $\Omega^*(M)[[\hbar]]$. Elements of $\Omega^*(M)[[\hbar]]$ are formal power series in \hbar with coefficients in $\Omega^*(M)$: $\alpha = \alpha_0 + \sum_{n=1}^{\infty} \hbar^n \alpha_n$. A product of differential forms in $\Omega^*(M)[[\hbar]]$ is a “covariant star product of differential forms” if it satisfies the following properties for $\alpha, \beta, \gamma \in \Omega^*(M)[[\hbar]]$:

1. The product takes the form: $\alpha \star \beta = \alpha \wedge \beta + \sum_{n=1}^{\infty} \hbar^n C_n(\alpha, \beta)$ where the C_n are covariant bilinear differential operators of at most order n in each argument. The C_n are polynomials of order n in the Poisson bivector.
2. The product is associative: $(\alpha \star \beta) \star \gamma = \alpha \star (\beta \star \gamma)$
3. The order \hbar term is the differential Poisson bracket for forms (8.14).
4. The constant function, 1, is the identity: $1 \star \alpha = \alpha \star 1 = \alpha$
5. The C_n have degree zero: $|C_n(\alpha, \beta)| = |\alpha| + |\beta|$.

Condition 3 means that the symplectic manifold M is endowed with a connection, Γ , that obeys the requisite restrictions (8.10) and (8.18)-(8.20). Note that we require the Leibniz rule only at $\mathcal{O}(\hbar)$ of the star product.

9.1 Hochschild Cohomology and Associativity

The associativity condition at $\mathcal{O}(\hbar^n)$ can be expressed as a condition on the Hochschild coboundary of C_n ; this is well-known in the literature, but we review it briefly for completeness. For a \mathbb{Z} -graded associative algebra $A = \bigoplus_{j \in \mathbb{Z}} A^j$, where A^j is homogeneous of degree j , let $\alpha_i \in A^{|\alpha_i|}$. If C is a Hochschild p -cochain, it is p -linear in $\alpha_1, \dots, \alpha_p$ and homogeneous of degree $|C|$ such that $C(\alpha_1, \dots, \alpha_p) \in A^{|C|+|\alpha_1|+\dots+|\alpha_p|}$.

The Hochschild coboundary of C , $\delta_H C$, is a $(p+1)$ -cochain. For a p -cochain C of degree $|C|$, $\delta_H C$ is given by:

$$\begin{aligned} (\delta_H C)(\alpha_0, \dots, \alpha_p) &\equiv (-1)^{|C||\alpha_0|} \alpha_0 C(\alpha_1, \dots, \alpha_p) + \sum_{j=0}^{p-1} (-1)^{j+1} C(\alpha_0, \dots, \alpha_j \alpha_{j+1}, \dots, \alpha_p) \\ &\quad + (-1)^{p+1} C(\alpha_0, \dots, \alpha_{(p-1)}) \alpha_p \end{aligned}$$

A Hochschild p -cochain C is called a Hochschild p -cocycle if $(\delta_H C) = 0$. Like all coboundary operators, $\delta_H^2 = 0$. For $Z_H^p(A, A)$ the space of p -cocycles, and $B_H^p(A, A)$ the space of p -cocycles that are coboundaries of $(p-1)$ -cochains, the Hochschild cohomology space is $H_H^p(A, A) = Z_H^p(A, A)/B_H^p(A, A)$.

The C_n in the star product are Hochschild 2-cochains of degree zero. The coboundary of a Hochschild 2-cochain of degree zero is:

$$(\delta_H C_n)(\alpha, \beta, \gamma) = \alpha \wedge C_n(\beta, \gamma) - C_n(\alpha \wedge \beta, \gamma) + C_n(\alpha, \beta \wedge \gamma) - C_n(\alpha, \beta) \wedge \gamma.$$

The associativity condition at $\mathcal{O}(\hbar^n)$ can be written as:

$$(\delta_H C_n)(\alpha, \beta, \gamma) = \sum_{r+s=n; r, s > 0} (C_r(C_s(\alpha, \beta), \gamma) - C_r(\alpha, C_s(\beta, \gamma))) \quad \forall n \geq 1. \quad (9.1)$$

Note that the right-hand side of (9.1) is a 3-cocycle, since $\delta_H^2 C_n = 0$. Obstructions to extending the deformation are in $H_H^3(\Omega^*(M), \Omega^*(M))$, the space of Hochschild 3-cocycles that are not coboundaries.

9.2 Chevalley Cohomology and Associativity

Chevalley cohomology has implications for both associativity and equivalence; this is well-known in the literature, but we review it briefly for completeness. For a \mathbb{Z} -graded associative algebra $A = \bigoplus_{j \in \mathbb{Z}} A^j$, where A^j is homogeneous of degree j , let $\alpha_i \in A^{|\alpha_i|}$. Let σ denote a permutation of $\{1, \dots, p\}$, let $\epsilon(\sigma)$ denote the sign of the permutation, and let $\epsilon_{|\alpha|}(\sigma)$ denote the sign of σ acting on $\{\alpha_1, \dots, \alpha_p\}$ in the graded sense. If C is p -linear in $\alpha_1, \dots, \alpha_p$, one says that C is symmetric (respectively antisymmetric) in $\alpha_1, \dots, \alpha_p$ if,

$$\begin{aligned} C(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}) &= \epsilon_{|\alpha|}(\sigma) C(\alpha_1, \dots, \alpha_p) \\ \left(\text{respectively } C(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}) &= \epsilon(\sigma) \epsilon_{|\alpha|}(\sigma) C(\alpha_1, \dots, \alpha_p) \right). \end{aligned}$$

This is equivalent to, for all i ,

$$\begin{aligned} C(\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_p) &= (-1)^{|\alpha_i||\alpha_{i+1}|} C(\alpha_1, \dots, \alpha_p) \\ \left(\text{respectively } C(\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_p) &= -(-1)^{|\alpha_i||\alpha_{i+1}|} C(\alpha_1, \dots, \alpha_p) \right). \end{aligned}$$

Let the \mathbb{Z} -graded associative algebra A be equipped with a \mathbb{Z} -graded Poisson bracket $\{, \}$. If C is a Chevalley p -cochain, it is p -linear, antisymmetric, and homogeneous of degree $|C|$ such that $C(\alpha_1, \dots, \alpha_p) \in A^{|C|+|\alpha_1|+\dots+|\alpha_p|}$. The Chevalley coboundary $\delta_C C$ of C is $(p+1)$ -linear and antisymmetric:

$$\begin{aligned} (\delta_C C)(\alpha_0, \dots, \alpha_p) = & \\ & \sum_{i=0}^p (-1)^i \varepsilon_{|\alpha|}(i, 0 \dots \hat{i} \dots p) (-1)^{|C||\alpha_i|} \{\alpha_i, C(\alpha_0, \dots, \hat{\alpha}_i \dots, \alpha_p)\} \\ & + \sum_{0 \leq i < j \leq p} \varepsilon_{|\alpha|}(i, j, 0, \dots, \hat{i} \dots \hat{j} \dots p) (-1)^{i+j} C(\{\alpha_i, \alpha_j\}, \alpha_0, \dots, \hat{\alpha}_i \dots \hat{\alpha}_j \dots, \alpha_p), \end{aligned}$$

and $\delta_C^2 = 0$. A Chevalley p -cochain is called a Chevalley p -cocycle if $(\delta_C C) = 0$. For $Z_C^p(A, A)$ the space of p -cocycles, and $B_C^p(A, A)$ the space of p -cocycles that are coboundaries of $(p-1)$ -cochains, the Chevalley cohomology space is $H_C^p(A, A) = Z_C^p(A, A)/B_C^p(A, A)$.

The antisymmetric part of the associativity relation (9.1) at $\mathcal{O}(\hbar^3)$ requires that the antisymmetric part of $C_2(\alpha, \beta)$, $C_2^-(\alpha, \beta)$, be a Chevalley cocycle:

$$0 = \text{Skew}((\delta_H C_3)(\alpha, \beta, \gamma)) = -4(\delta_C C_2^-)(\alpha, \beta, \gamma). \quad (9.2)$$

Note that this argument could be extended to $\mathcal{O}(\hbar^k)$.

9.3 Star Product Equivalence

Two star products, $*$ and $\bar{*}$ with $C_i(\alpha, \beta)$ and $\bar{C}_i(\alpha, \beta)$ respectively, are formally equivalent if there exists a differential operator T of degree $|T| = 0$ of the form $T = id + \sum_{n=1}^{\infty} \hbar^n T_n$ such that

$$T(\alpha \bar{*} \beta) = T(\alpha) * T(\beta).$$

The equivalence condition at $\mathcal{O}(\hbar^n)$ is:

$$(\delta_H T_n)(\alpha, \beta) = \sum_{r+s=n; s < n} T_s(\bar{C}_r(\alpha, \beta)) - \sum_{r+s+t=n; s, t < n} C_r(T_s(\alpha), T_t(\beta)). \quad (9.3)$$

If two star products are equivalent to $\mathcal{O}(\hbar^n)$, then the condition for extending the equivalence is that a Hochschild 2-cocycle constructed from the T_k , $k \leq n$ must be a Hochschild coboundary of T_{n+1} . Note that the right-hand side of (9.3) is a Hochschild 2-cocycle. Obstructions to equivalence are in $H_H^2(\Omega^*(M), \Omega^*(M))$.

If two star products, $C_i(\alpha, \beta)$ and $\bar{C}_i(\alpha, \beta)$, are the same at $\mathcal{O}(\hbar)$ and equivalent to $\mathcal{O}(\hbar^2)$, then the associativity relation (9.1) at $\mathcal{O}(\hbar^2)$ requires that

$$(\delta_H \bar{C}_2)(\alpha, \beta, \gamma) = (\delta_H C_2)(\alpha, \beta, \gamma). \quad (9.4)$$

The difference between $\overline{C}_2(\alpha, \beta)$ and $C_2(\alpha, \beta)$ must be a Hochschild cocycle, which can be parameterized the following way:

$$\overline{C}_2(\alpha, \beta) = C_2(\alpha, \beta) + C_2^-(\alpha, \beta) + C_2^+(\alpha, \beta) + (\delta_H T_2)(\alpha, \beta),$$

where $C_2^-(\alpha, \beta)$ is an antisymmetric Hochschild 2-cocycle that is an derivation in each argument and $C_2^+(\alpha, \beta)$ is a symmetric Hochschild 2-cocycle that is not a Hochschild coboundary. The symmetric Hochschild coboundary part is given by $(\delta_H T_2)(\alpha, \beta)$. Also, $C_2^-(\alpha, \beta)$ must also be a Chevalley cocycle, due to (9.2).

If $\overline{C}_2(\alpha, \beta)$ and $C_2(\alpha, \beta)$ are to be equivalent star products at $\mathcal{O}(\hbar^2)$, then (9.3) requires that, for

$$T(\alpha) = \alpha + \hbar T_1(\alpha) + \hbar^2 \left(-T_2(\alpha) + \frac{1}{2} T_1(T_1(\alpha)) \right) + \dots,$$

$T_1(\alpha)$ must be a Hochschild cocycle, $C_2^+(\alpha, \beta) = 0$, and $C_2^-(\alpha, \beta)$ must be a Chevalley coboundary of T_1 :

$$C_2^-(\alpha, \beta) = -(\delta_C T_1)(\alpha, \beta).$$

Note that this argument can be extended to $\mathcal{O}(\hbar^k)$.

Thus, antisymmetric obstructions to equivalence at $\mathcal{O}(\hbar^2)$ are given by $H_C^2(\Omega^*(M), \Omega^*(M))$, and symmetric obstructions are given by symmetric terms in $H_H^2(\Omega^*(M), \Omega^*(M))$.

9.4 Explicit Form of $\alpha \star \beta$ to $\mathcal{O}(\hbar^2)$

A star product which satisfies Properties 1-5 to $\mathcal{O}(\hbar^2)$ for two arbitrary forms α, β has the following $C_2(\alpha, \beta)$:

$$\begin{aligned} C_2(\alpha, \beta) &\equiv \frac{1}{2} \pi^{ij} \pi^{mn} \nabla_i \nabla_m \alpha \wedge \nabla_j \nabla_n \beta + \frac{1}{3} (\pi^{mn} \nabla_n \pi^{ij}) \\ &+ \frac{1}{2} \pi^{in} \pi^{jp} T_{np}^m (\nabla_m \nabla_i \alpha \wedge \nabla_j \beta - \nabla_i \alpha \wedge \nabla_m \nabla_j \beta) + (-1)^{|\alpha|} \pi^{ij} \tilde{R}^{mn} \wedge \nabla_i i_m \alpha \wedge \nabla_j i_n \beta \\ &- \frac{1}{2} \tilde{R}^{ij} \wedge \tilde{R}^{mn} \wedge i_i i_m \alpha \wedge i_j i_n \beta - \frac{1}{3} \tilde{R}^{i\ell} \wedge i_\ell \tilde{R}^{mn} \wedge ((-1)^{|\alpha|} i_i i_m \alpha \wedge i_n \beta + i_m \alpha \wedge i_i i_n \beta). \end{aligned} \quad (9.5)$$

Properties 1 and 3-5 are manifestly satisfied by (8.14) and (9.5); a lengthy calculation verifies (8.14) and (9.5) also satisfy Property 2, given by Equation (9.1):

$$\begin{aligned} (\delta C_2)(\alpha, \beta, \gamma) &= C_1(C_1(\alpha, \beta), \gamma) - C_1(\alpha, C_1(\beta, \gamma)) \\ &= +\frac{1}{2} \pi^{ij} \pi^{mn} (\{\nabla_i, \nabla_m\} \alpha \wedge \nabla_n \beta \wedge \nabla_j \gamma - \nabla_m \alpha \wedge \nabla_i \beta \wedge \{\nabla_n, \nabla_j\} \gamma) \\ &\quad - (\pi^{mn} \nabla_n \pi^{ij} + \frac{1}{2} \pi^{in} \pi^{jp} T_{np}^m) \nabla_i \alpha \wedge \nabla_m \beta \wedge \nabla_j \gamma \\ &\quad + \pi^{mn} \tilde{R}^{ij} ((-1)^{|\alpha|} \nabla_m i_i \alpha \wedge i_j \beta \wedge \nabla_n \gamma + (-1)^{|\alpha|+|\beta|} i_i \nabla_m \alpha \wedge \nabla_n \beta \wedge i_j \gamma \\ &\quad - (-1)^{|\beta|} \nabla_m \alpha \wedge i_i \beta \wedge \nabla_n i_j \gamma - (-1)^{|\alpha|+|\beta|} i_i \alpha \wedge \nabla_m \beta \wedge i_j \nabla_n \gamma) \\ &\quad - (-1)^{|\beta|} \tilde{R}^{ij} \wedge i_i \tilde{R}^{mn} \wedge i_n \alpha \wedge i_j \beta \wedge i_m \gamma \\ &\quad + \tilde{R}^{ij} \wedge \tilde{R}^{mn} \wedge ((-1)^{|\beta|} i_i i_m \alpha \wedge i_n \beta \wedge i_j \gamma - (-1)^{|\alpha|+1} i_m \alpha \wedge i_i \beta \wedge i_n i_j \gamma), \end{aligned} \quad (9.6)$$

where the brackets $\{, \}$ in (9.6) are anticommutators. To arrive at this result, we have used Equations (8.11), (8.12), $\tilde{R}^{ij} = \tilde{R}^{ji}$, (8.18), (8.19) and (8.20), and the identities (A.2), (A.4), (8.15), and (8.16). Indeed, (9.5) was constructed by ansatz to satisfy (9.6).

Note that if the connection on M is torsionless, then the connection ∇ is a flat symplectic connection, and the star product reduces to

$$\alpha \star \beta|_{T=0} = \alpha\beta + \hbar\pi^{mn}\nabla_m\alpha \wedge \nabla_n\beta + \hbar^2\frac{1}{2}\pi^{ij}\pi^{mn}\nabla_i\nabla_m\alpha \wedge \nabla_j\nabla_n\beta + \mathcal{O}(\hbar^3).$$

In flat space with α, β restricted to functions, this reduces to the first three terms of the case considered in Theorem 5 of Ref. [9]. Indeed, if the connection is a flat symplectic connection, then the following product between differential forms will be associative to all orders:

$$\alpha \star \beta|_{T=0} = \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \pi^{i_1 j_1} \dots \pi^{i_r j_r} \nabla_{i_1} \dots \nabla_{i_r} \alpha \wedge \nabla_{j_1} \dots \nabla_{j_r} \beta.$$

Chapter 10

Conclusion

In this note, we present a noncommutative deformation (the star product) of the graded algebra of exterior differential forms to $\mathcal{O}(\hbar^2)$. We review and then use the graded differential Poisson bracket (8.14) at $\mathcal{O}(\hbar)$, and at $\mathcal{O}(\hbar^2)$ we give (9.5). We verify that this star product satisfies all the necessary properties, such as associativity, to $\mathcal{O}(\hbar^2)$.

The results of this paper may be generalized in at least four directions. One could examine the Chevalley and Hochschild cohomologies and determine if this star product is unique up to equivalence at $\mathcal{O}(\hbar^2)$ and if there are obstructions to extending the star product to $\mathcal{O}(\hbar^3)$. Barring possible obstructions, it should be possible (albeit laborious) to find the explicit star product for differential forms to $\mathcal{O}(\hbar^3)$ using the associativity relation (9.1). These calculations are in progress. Thirdly, one could look for a proof that this formal star product exists to all orders. Finally, one could apply this star product between differential forms to physics, such as gauge theories on noncommutative spaces, or generalizing the Seiberg-Witten map [78].

Bibliography

- [1] Ahmed Abouelsaood, Jr. Callan, Curtis G., C.R. Nappi, and S.A. Yost. Open Strings in Background Gauge Fields. *Nucl.Phys.*, B280:599, 1987.
- [2] S. T. Ali and M. Engliš. Quantization Methods:. *Reviews in Mathematical Physics*, 17:391–490, 2005.
- [3] M. Ammar, V. Chloup, and S. Gutt. Universal Star Products. *Letters in Mathematical Physics*, 84:199–215, June 2008.
- [4] Paul S. Aspinwall. Enhanced gauge symmetries and K3 surfaces. *Phys.Lett.*, B357:329–334, 1995.
- [5] Paul S. Aspinwall. K3 surfaces and string duality. pages 421–540, 1996.
- [6] Paul S. Aspinwall and David R. Morrison. String theory on K3 surfaces. 1994.
- [7] Scott Axelrod, Steve Della Pietra, and Edward Witten. GEOMETRIC QUANTIZATION OF CHERN-SIMONS GAUGE THEORY. *J.Diff.Geom.*, 33:787–902, 1991.
- [8] Claudio Bartocci, Ugo Bruzzo, Daniel Hernandez Ruiperez, and Jose M. Munoz Porras. Mirror symmetry on elliptic K3 surfaces via Fourier-Mukai transform. *Commun.Math.Phys.*, 195:79–93, 1998.
- [9] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation Theory and Quantization. 1. Deformations of Symplectic Structures. *Annals Phys.*, 111:61, 1978.
- [10] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation Theory and Quantization. 2. Physical Applications. *Annals Phys.*, 111:111, 1978.
- [11] K. Becker, M. Becker, and J.H. Schwarz. String theory and M-theory: A modern introduction. 2007.
- [12] E. J. Beggs and S. Majid. Semi-classical differential structures. *Pacific Journal of Mathematics*, 224(1):1–44, 2006.

- [13] F.A. Berezin, (Ed.) Kirillov, A.A., and (Ed.) Leites, D. INTRODUCTION TO SUPERANALYSIS. 1987.
- [14] Micha Berkooz. Nonlocal field theories and the noncommutative torus. *Phys.Lett.*, B430:237–241, 1998.
- [15] M. Bershadsky, A. Johansen, V. Sadov, and C. Vafa. Topological reduction of 4-d SYM to 2-d sigma models. *Nucl.Phys.*, B448:166–186, 1995.
- [16] Ralph Blumenhagen, Lars Gorlich, Boris Kors, and Dieter Lust. Asymmetric orbifolds, noncommutative geometry and type I string vacua. *Nucl.Phys.*, B582:44–64, 2000.
- [17] P. Candelas, Gary T. Horowitz, Andrew Strominger, and Edward Witten. Vacuum configurations for superstrings. *Nuclear Physics B*, 258(0):46 – 74, 1985.
- [18] Alberto S. Cattaneo and Giovanni Felder. A Path integral approach to the Kontsevich quantization formula. *Commun.Math.Phys.*, 212:591–611, 2000.
- [19] Alberto S. Cattaneo and Giovanni Felder. Relative formality theorem and quantisation of coisotropic submanifolds. *Advances in Mathematics*, 208(2):521 – 548, 2007.
- [20] M. Chaichian, M. Oksanen, A. Tureanu, and G. Zet. Noncommutative gauge theory using covariant star product defined between Lie valued differential forms. *Phys.Rev.*, D81:085026, 2010.
- [21] Chong-Sun Chu and Pei-Ming Ho. Poisson algebra of differential forms. *Int.J.Mod.Phys.*, 12:5573–5587, 1997.
- [22] Alain Connes, Michael R. Douglas, and Albert S. Schwarz. Noncommutative geometry and matrix theory: Compactification on tori. *JHEP*, 9802:003, 1998.
- [23] Mirjam Cvetič, Tao Liu, and Michael B. Schulz. Twisting $K3 \times T^{*2}$ orbifolds. *JHEP*, 0709:092, 2007.
- [24] Atish Dabholkar and Jeffrey A. Harvey. String islands. *JHEP*, 9902:006, 1999.
- [25] Atish Dabholkar and Chris Hull. Duality twists, orbifolds, and fluxes. *JHEP*, 0309:054, 2003.
- [26] Duiliu-Emanuel Diaconescu, Michael R. Douglas, and Jaume Gomis. Fractional branes and wrapped branes. *JHEP*, 9802:013, 1998.
- [27] Duiliu-Emanuel Diaconescu and Jaume Gomis. Fractional branes and boundary states in orbifold theories. *JHEP*, 0010:001, 2000.

- [28] Paul A.M. Dirac. The fundamental equations of quantum mechanics. *Proc.Roy.Soc.Lond.*, A109:642–653, 1925.
- [29] Giuseppe Dito and Daniel Sternheimer. Deformation quantization: Genesis, developments and metamorphoses. *To appear appear in the proceedings of Conference: C01-01-29.2*, 2002.
- [30] L. Dixon, J.A. Harvey, C. Vafa, and E. Witten. Strings on orbifolds. *Nuclear Physics B*, 261(0):678 – 686, 1985.
- [31] Lance J. Dixon. SOME WORLD SHEET PROPERTIES OF SUPERSTRING COMPACTIFICATIONS, ON ORBIFOLDS AND OTHERWISE. 1987.
- [32] Michael R. Douglas. Branes within branes. 1995.
- [33] Michael R. Douglas and Christopher M. Hull. D-branes and the noncommutative torus. *JHEP*, 9802:008, 1998.
- [34] Arturo Echeverria-Enriquez, Miguel C. Munoz-Lecanda, Narcisco Roman-Roy, and Carles Victoria-Monge. Mathematical foundations of geometric quantization. *Extracta Math.*, 13:135–238, 1998.
- [35] Alex Flournoy, Brian Wecht, and Brook Williams. Constructing nongeometric vacua in string theory. *Nucl.Phys.*, B706:127–149, 2005.
- [36] E.S. Fradkin and Arkady A. Tseytlin. Nonlinear Electrodynamics from Quantized Strings. *Phys.Lett.*, B163:123, 1985.
- [37] Robert Friedman, John Morgan, and Edward Witten. Vector bundles and F theory. *Commun.Math.Phys.*, 187:679–743, 1997.
- [38] J. Fuchs. *Affine Lie algebras and quantum groups: An Introduction, with applications in conformal field theory*. Cambridge monographs on mathematical physics. Univ. Pr., Cambridge, UK, 1992.
- [39] Ori J. Ganor. U duality twists and possible phase transitions in (2+1)-dimensions supergravity. *Nucl.Phys.*, B549:145–180, 1999.
- [40] Ori J. Ganor and Yoon Pyo Hong. Selfduality and Chern-Simons Theory. 2008.
- [41] Ori J. Ganor, Yoon Pyo Hong, Ruza Markov, and Hai Siong Tan. Static Charges in the Low-Energy Theory of the S-Duality Twist. *JHEP*, 1204:041, 2012.
- [42] Ori J. Ganor, Yoon Pyo Hong, and H.S. Tan. Ground States of S-duality Twisted N=4 Super Yang-Mills Theory. *JHEP*, 1103:099, 2011.

- [43] Ori J. Ganor, Sharon Jue, and Shannon McCurdy. Ground States of Duality-twisted Sigma-Models with K3 Target Space. *JHEP*, 1302:017, 2013.
- [44] Michael B. Green, J.H. Schwarz, and Edward Witten. SUPERSTRING THEORY. VOL. 1: INTRODUCTION. 1987.
- [45] Brian R. Greene and M.R. Plesser. DUALITY IN CALABI-YAU MODULI SPACE. *Nucl.Phys.*, B338:15–37, 1990.
- [46] Brian R. Greene, Alfred D. Shapere, Cumrun Vafa, and Shing-Tung Yau. Stringy Cosmic Strings and Noncompact Calabi-Yau Manifolds. *Nucl.Phys.*, B337:1, 1990.
- [47] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley-Interscience, 1978.
- [48] H.J. Groenewold. On the Principles of elementary quantum mechanics. *Physica*, 12:405–460, 1946.
- [49] Jeffrey A. Harvey, Gregory W. Moore, and Andrew Strominger. Reducing S duality to T duality. *Phys.Rev.*, D52:7161–7167, 1995.
- [50] E. Hawkins. The structure of noncommutative deformations. *J. Differential Geom.*, 77(3):385–424, 2007.
- [51] Eli Hawkins. Noncommutative rigidity. *Communications in Mathematical Physics*, 246:211–235, 2004.
- [52] Simeon Hellerman, John McGreevy, and Brook Williams. Geometric constructions of nongeometric string theories. *JHEP*, 0401:024, 2004.
- [53] A.C. Hirshfeld and P. Henselder. Deformation quantization in the teaching of quantum mechanics. *Am.J.Phys.*, 70:537–547, 2002.
- [54] Nigel J. Hitchin. The Selfduality equations on a Riemann surface. *Proc.Lond.Math.Soc.*, 55:59–131, 1987.
- [55] Pei-Ming Ho and Shun-Pei Miao. Noncommutative differential calculus for D-brane in nonconstant B field background. *Phys.Rev.*, D64:126002, 2001.
- [56] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, et al. Mirror symmetry. 2003.
- [57] Kentaro Hori and Cumrun Vafa. Mirror symmetry. 2000.
- [58] C.M. Hull. A Geometry for non-geometric string backgrounds. *JHEP*, 0510:065, 2005.

- [59] Anton Kapustin and Edward Witten. Electric-Magnetic Duality And The Geometric Langlands Program. *Commun.Num.Theor.Phys.*, 1:1–236, 2007.
- [60] Shinsuke Kawai and Yuji Sugawara. Mirrorfolds with K3 fibrations. *JHEP*, 0802:065, 2008.
- [61] Maxim Kontsevich. Deformation quantization of Poisson manifolds. 1. *Lett.Math.Phys.*, 66:157–216, 2003.
- [62] Bertram Kostant. Quantization and unitary representations. In C.T. Taam, editor, *Lectures in Modern Analysis and Applications III*, volume 170 of *Lecture Notes in Mathematics*, pages 87–208. Springer Berlin Heidelberg, 1970.
- [63] Alok Kumar and Cumrun Vafa. U manifolds. *Phys.Lett.*, B396:85–90, 1997.
- [64] Wolfgang Lerche, Cumrun Vafa, and Nicholas P. Warner. Chiral Rings in N=2 Superconformal Theories. *Nucl.Phys.*, B324:427, 1989.
- [65] James T. Liu and Ruben Minasian. U-branes and T**3 fibrations. *Nucl.Phys.*, B510:538–554, 1998.
- [66] Shannon McCurdy and Bruno Zumino. Covariant Star Product for Exterior Differential Forms on Symplectic Manifolds. *AIP Conf.Proc.*, 1200:204–214, 2010.
- [67] David R. Morrison. The Geometry underlying mirror symmetry. 1996.
- [68] J. E. Moyal. Quantum mechanics as a statistical theory. *Mathematical Proceedings of the Cambridge Philosophical Society*, 45:99–124, 1947.
- [69] Nikita Nekrasov and Albert S. Schwarz. Instantons on noncommutative R**4 and (2,0) superconformal six-dimensional theory. *Commun.Math.Phys.*, 198:689–703, 1998.
- [70] Nikita Nekrasov and Edward Witten. The Omega Deformation, Branes, Integrability, and Liouville Theory. *JHEP*, 1009:092, 2010.
- [71] Nikita A. Nekrasov. Seiberg-Witten prepotential from instanton counting. *Adv.Theor.Math.Phys.*, 7:831–864, 2004.
- [72] J. Polchinski. String theory. Vol. 1: An introduction to the bosonic string. 1998.
- [73] Ryan Rohm. Spontaneous supersymmetry breaking in supersymmetric string theories. *Nuclear Physics B*, 237(3):553 – 572, 1984.
- [74] Volker Schomerus. D-branes and deformation quantization. *JHEP*, 9906:030, 1999.

- [75] N. Seiberg and Edward Witten. Electric - magnetic duality, monopole condensation, and confinement in $N=2$ supersymmetric Yang-Mills theory. *Nucl.Phys.*, B426:19–52, 1994.
- [76] Nathan Seiberg. Observations on the Moduli Space of Superconformal Field Theories. *Nucl.Phys.*, B303:286, 1988.
- [77] Nathan Seiberg and Edward Witten. Gauge dynamics and compactification to three-dimensions. 1996.
- [78] Nathan Seiberg and Edward Witten. String theory and noncommutative geometry. *JHEP*, 9909:032, 1999.
- [79] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. Mirror symmetry is T duality. *Nucl.Phys.*, B479:243–259, 1996.
- [80] Yuji Terashima and Masahito Yamazaki. $SL(2, \mathbb{R})$ Chern-Simons, Liouville, and Gauge Theory on Duality Walls. *JHEP*, 1108:135, 2011.
- [81] C Vafa and Edward Witten. BOSONIC STRING ALGEBRAS. *Phys.Lett.*, B159:265, 1985.
- [82] Cumrun Vafa. Evidence for F theory. *Nucl.Phys.*, B469:403–418, 1996.
- [83] D.V. Vassilevich. Tensor calculus on noncommutative spaces. *Class.Quant.Grav.*, 27:095020, 2010.
- [84] H. Weyl. Quantenmechanik und gruppentheorie. *Zeitschrift fr Physik*, 46:1–46, 1927.
- [85] E. Wigner. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.*, 40:749–759, Jun 1932.
- [86] Edward Witten. Quantum Field Theory and the Jones Polynomial. *Commun.Math.Phys.*, 121:351, 1989.
- [87] Edward Witten. Mirror manifolds and topological field theory. 1991.
- [88] Edward Witten. Small instantons in string theory. *Nucl.Phys.*, B460:541–559, 1996.
- [89] N. Woodhouse. Geometric Quantization. 1980.
- [90] Ping Xu. Noncommutative poisson algebras. *American Journal of Mathematics*, 116(1):pp. 101–125, 1994.
- [91] B. Zumino. Supersymmetry and Kahler Manifolds. *Phys.Lett.*, B87:203, 1979.

Appendix A

Appendix to Chapter III

A.1 The Contraction Operator

The operation $i_{j_m}\alpha$, for α an $|\alpha|$ -form, denotes:

$$i_{j_m}\alpha \equiv \frac{1}{|\alpha|!} \sum_{m=1}^{|\alpha|} (-1)^{m+1} \alpha_{j_1 \dots j_m \dots j_{|\alpha|}} dx^{j_1} \wedge \dots \wedge \widehat{dx^{j_m}} \wedge \dots \wedge dx^{j_{|\alpha|}}, \quad (\text{A.1})$$

where $\widehat{dx^{j_m}}$ means to omit dx^{j_m} . Note that the interior product of a vector X and α an $|\alpha|$ -form can be denoted in coordinates using this contraction operator: $i_X\alpha = X^m i_m\alpha$. Similarly, $i_n i_m\alpha$, for α an $|\alpha|$ -form, denotes:

$$i_n i_m\alpha = \frac{1}{(|\alpha|-2)!} \alpha_{mnj_3 \dots j_{|\alpha|}} dx^{j_3} \wedge \dots \wedge dx^{j_{|\alpha|}} = -i_m i_n\alpha. \quad (\text{A.2})$$

And, applying (A.1) to a wedge product, we see that the contraction operator is an anti-derivation:

$$i_m(\alpha \wedge \beta) \equiv i_m\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge i_m\beta. \quad (\text{A.3})$$

Furthermore,

$$\nabla_m i_n\alpha \equiv \frac{1}{(|\alpha|-1)!} \nabla_m \alpha_{nj_2 \dots j_{|\alpha|}} dx^{j_2} \wedge \dots \wedge dx^{j_{|\alpha|}} = i_n \nabla_m \alpha. \quad (\text{A.4})$$

Note also that we take the wedge product to act on objects that are not strictly differential forms, like $\nabla_m\alpha$ and $i_m\alpha$. Objects such as $\nabla_m\alpha \wedge \nabla_n\beta$ mean:

$$\nabla_m\alpha \wedge \nabla_n\beta = \frac{1}{|\alpha|!|\beta|!} (\nabla_m\alpha)_{i_1 \dots i_{|\alpha|}} (\nabla_n\beta)_{j_1 \dots j_{|\beta|}} dx^{i_1} \wedge \dots \wedge dx^{i_{|\alpha|}} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{|\beta|}}$$

where the m, n indices are not antisymmetrized with the i, j indices.

A.2 The connections $\nabla, \tilde{\nabla}$

We define two connections from the same connection coefficients Γ_{ij}^k , as in (8.9). The curvature for these two connections are:

$$\begin{aligned} R_{mab}^i &= \partial_a \Gamma_{bm}^i - \partial_b \Gamma_{am}^i + \Gamma_{a\ell}^i \Gamma_{bm}^\ell - \Gamma_{b\ell}^i \Gamma_{am}^\ell \\ \tilde{R}_{mab}^i &= \partial_a \Gamma_{mb}^i - \partial_b \Gamma_{ma}^i + \Gamma_{\ell a}^i \Gamma_{mb}^\ell - \Gamma_{\ell b}^i \Gamma_{ma}^\ell. \end{aligned}$$

The torsion $T_{ij}^a = \Gamma_{ij}^a - \Gamma_{ji}^a$.

As noted in Ref. [21], [55], the difference between these two curvatures can be written as:

$$\tilde{R}_{kij}^a - R_{kij}^a = -\nabla_i T_{jk}^a - \nabla_j T_{ki}^a + T_{kb}^a T_{ij}^b + T_{ib}^a T_{jk}^b + T_{jb}^a T_{ki}^b. \quad (\text{A.5})$$

This makes it easy to see that with a torsion-free connection, R and \tilde{R} are equal, but when the torsion is non-zero, R and \tilde{R} can differ.

The first Bianchi identity is:

$$\sum_{(i,j,k)} R_{kij}^a = \sum_{(i,j,k)} (T_{bk}^a T_{ij}^b + \nabla_i T_{jk}^a). \quad (\text{A.6})$$

Since in our case the curvature R is zero, (A.5) and (A.6) give a nice relation between the curvature \tilde{R} and the torsion:

$$\tilde{R}_{kij}^a = \nabla_k T_{ij}^a \quad (R_{kij}^a = 0). \quad (\text{A.7})$$