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VON NEUMANN ALGEBRAS OF SOFIC GROUPS WITH $\beta_1^{(2)} = 0$ ARE STRONGLY 1-BOUNDED.

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ABSTRACT. We show that if Γ is a finitely generated finitely presented sofic group with zero first L^2 Betti number and containing an element of infinite order, then the von Neumann algebra $L(\Gamma)$ is strongly 1-bounded in the sense of Jung. In particular, $L(\Gamma) \not\cong L(\Lambda)$ if Λ is any group with free entropy dimension > 1, for example a free group. The key technical result is a short proof of an estimate of Jung [9] using non-microstates entropy techniques.

1. INTRODUCTION.

In a series of papers [15, 16], Voiculescu introduced the notion of free entropy dimension, as an analog of Minkowski content in free probability theory. If $X_1, \ldots, X_n \in (M, \tau)$ are a self-adjoint n-tuple of elements in a tracial von Neumann algebra, $\delta_0(X_1, \ldots, X_n)$ is a measure of "how free they are". If $M = W^*(X_1, \ldots, X_n)$ satisfies the Connes embedding conjecture and is diffuse, $1 \leq \delta_0(X_1, \ldots, X_n) \leq n$. The value 1 is achieved e.g. when M is hyperfinite [7], while a free semicircular n-tuple generating the free group factor $L(\mathbb{F}_n)$ has free entropy dimension n [16].

The number $\delta_0(X_1, \ldots, X_n)$ is an invariant of the (non-closed) $*$-algebra generated by $X_1, \ldots, X_n$. In particular, if Γ is a finitely-generated discrete group, free entropy dimension of any generating set of the group algebra gives us a group invariant, $\delta_0(\mathbb{C}\Gamma)$ [18]. As it turns out, $\delta_0(\mathbb{C}\Gamma)$ has a close relationship with the first L^2 Betti number of Γ (we refer the reader to [11] for background on L^2-Betti numbers). Indeed, $\delta_0(\mathbb{C}\Gamma) \leq \beta_1^{(2)}(\Gamma) + 1$ for any finitely generated infinite group Γ [4]; and if δ_0 is replaced by its "non-microstates analog" δ^*, then equality holds [12].

Free entropy dimension has found a large variety of applications in von Neumann algebra theory; for example, it was used by Voiculescu to prove that free group factors $L(\mathbb{F}_n)$ have no Cartan subalgebras [16]. By now, there is a long list of various properties that imply that any generators of a von Neumann algebra M have free entropy dimension 1; we refer the reader to Voiculescu's survey [19] and to [8, 6].

In [8] Jung discovered a technical strengthening of the statement that $\delta_0(X_1, \ldots, X_n) = 1$, called *strong 1-boundedness*, which we will now briefly review.

The free entropy dimension δ_0 is defined by the formula [16]

$$\delta_0(X_1, \ldots, X_n) = n - \limsup_{\epsilon \to 0} \frac{\chi(X_1 + \epsilon^{\frac{1}{2}} S_1, \ldots, X_n + \epsilon^{\frac{1}{2}} S : S_1, \ldots, S_n)}{\log \epsilon^{\frac{1}{2}}}$$

where χ stands for free entropy and $S_1, \ldots, S_n$ is a free semicircular n-tuple, free from $X_1, \ldots, X_n$. The statement that $\delta_0(X_1, \ldots, X_n) \leq 1$ then translates into the estimate $\chi(X_1 +$

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$\epsilon^{\frac{1}{2}}S_1, \ldots, X_n + \epsilon^{\frac{1}{2}}S_n : S_1, \ldots, S_n) \leq (n-1)\log \epsilon^{\frac{1}{2}} + f(\epsilon)$ with $\limsup_\epsilon |f(\epsilon)/\log\epsilon| = 0$. Strong 1-boundedness is a strengthening of this inequality: it requires that

$$\chi(X_1 + \epsilon^{\frac{1}{2}}S_1, \ldots, X_n + \epsilon^{\frac{1}{2}}S_n : S_1, \ldots, S_n) \leq (n-1)\log \epsilon^{\frac{1}{2}} + \text{const}$$

for small ϵ and that at least one of the elements X_j has finite free entropy.

Jung proved the following amazing result: If $X_1, \ldots, X_n$ is a strongly 1-bounded n-tuple and $Y_1, \ldots, Y_m \in W^*(X_1, \ldots, X_n)$ is any other set of generators, then it is also strongly 1-bounded and in particular, $\delta_0(Y_1, \ldots, Y_m) = 1$ (the results of [6] allow one also to treat the case $m = \infty$).

His result means that strong 1-boundedness is a property of a generating set that has global consequences for the entire von Neumann algebra (one thus speaks of strongly 1-bounded von Neumann algebras). With few exceptions such as property (T) [10], all known implications stating that some property (presence of a Cartan subalgebra, tensor product decomposition, property Γ, etc.) entails $\delta_0 = 1$ for any generating set can be upgraded to state that the von Neumann algebra is actually strongly 1-bounded. We refer the reader to [8, 6] and references therein.

The main theorem of this paper states that if a finitely generated finitely presented sofic group Γ satisfies $\beta_1^{(2)}(\Gamma) = 0$ and has an element of infinite order, then the associated von Neumann algebra is strongly 1-bounded. In particular, any generating set of the von Neumann algebra $L(\Gamma)$ has free entropy dimension 1. This implies a number of free indecomposability properties of $L(\Gamma)$. For example, $L(\Gamma) \not\cong M * N$ with M, N diffuse von Neumann algebras; in particular, $L(\Gamma) \not\cong L(\mathbb{F}_n)$ for any $n \in [2, +\infty]$.

To obtain our result, we first give a new proof to a recent result of Jung [9] which allows one to deduce α-boundedness results for free entropy dimension under the assumption of existence of algebraic relations. Jung's estimate can be considered an improved version of the free entropy dimension estimates from [4, 3] that were previously obtained using non-microstates free entropy methods, but his proof relies on a number of highly technical microstates estimates. Our proof is considerably shorter than Jung's, avoiding much of the difficulty of dealing with matricial microstates. Indeed, we instead produce estimates for the non-microstates Fisher information and non-microstates free entropy and then use results of [2] to deduce the corresponding microstates free entropy inequality and strong boundedness of free entropy dimension. Finally, we apply our estimate to the case of sofic groups.

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2. Estimates on Fisher Information and Free Entropy.

2.1. **Preliminaries and notation.** Throughout this section we write $X = (X_1, \ldots, X_n)$ for an n-tuple of self-adjoint variables in a tracial von Neumann algebra (M, τ). Thus we write $W^*(X)$ for $W^*(X_1, \ldots, X_n)$, etc. If $S = (S_1, \ldots, S_n)$ is another n-tuple we write $X + \sqrt{\epsilon}S$ for the n-tuple $(X_1 + \sqrt{\epsilon}S_1, \ldots, X_n + \sqrt{\epsilon}S_n)$. We also write $\|X\|_2 = (\sum \|X_j\|_2^2)^{1/2}$.

We write $F = (F_1, \ldots, F_k)$ for a vector-valued non-commutative polynomial function of n variables. Here each $F_j \in \mathbb{C}[t_1, \ldots, t_n]$ is a non-commutative polynomial. We write ∂F for the $k \times n$ matrix $(\partial_j F_i)_{ij} \in M_{k\times n}(\mathbb{C}[t_1, \ldots, t_n] \otimes \mathbb{C}[t_1, \ldots, t_n])$, where ∂_j are the non-commutative difference quotient derivations determined by the Leibniz rule and $\partial_j t_i = \delta_{i=j} 1 \otimes 1$. We write

$F(X)$, $\partial F(X)$, etc. when we evaluate these functions by substituting $X = (X_1, \ldots, X_n)$ for $(t_1, \ldots, t_n)$.

We also view $\partial F(X)$ as a linear operator in $M_{k\times n}(W^*(X)\bar{\otimes}W^*(X)^o)$. It lies in the commutant of the von Neumann algebra $W^*(X)\bar{\otimes}W^*(X)^o$ acting on $L^2(W^*(X)\otimes W^*(X)^o)$ by $(a\otimes b)\cdot(\xi\otimes\eta) = \xi a\otimes b\eta$. The rank $\operatorname{rank}(\partial F(X))$ is the Murray-von Neumann dimension (over the algebra $W^*(X)\bar{\otimes}W^*(X)^o$ of the closure of the image of $\partial F(X)$.

Finally, for a tensor $Q = a\otimes b$ and an element x we write $Q\#x$ for axb. We use the same notation for the multiplication in the von Neumann algebra $W^*(X)\bar{\otimes}W^*(X)^o$. We also use the same notation for n-tuples and matrices: if $Q = (Q_{ij})$ is an $l\times k$ matrix and $Y = (Y_1, \ldots, Y_k)$ is a k-tuple, we write $Q\#Y$ for the l-tuple $(\sum_{j=1}^k Q_{ij}\#Y_j)_{i=1}^l$. With this notation we have the following perturbative formula: $F(X+\sqrt{\epsilon}S) = F(X)+\sqrt{\epsilon}\partial F\#S+O(\epsilon)$; it is sufficient to check its validity for monomial F, which is straightforward.

2.2. **The main estimate on Fisher information Φ^*.** In this paper we will denote by $\log_+(t)$ the function given by $\log_+(t) = t$ for $t > 0$ and $\log_+(0) = 0$.

Lemma 2.1. *Suppose that $F = (F_1, \ldots, F_k) \in \mathbb{C}[t_1, \ldots, t_n]^k$ is a vector-valued polynomial non-commutative function of n variables, and suppose that for some $X = (X_1, \ldots, X_n) \in (M, \tau)$, $F(X) = 0$. Assume that $\log_+[\partial F(X)^*\partial F(X)] \in L^1(W^*(X)\otimes W^*(X)^o)$.*

Let $S = (S_1, \ldots, S_n)$ be a free semicircular family, free from X. Then there exist $\epsilon_0 > 0$ and $f \in L^1[0, \epsilon_0]$ so that the free Fisher information satisfies the inequality

$$\Phi^*(X+\sqrt{\epsilon}S) \geq \frac{\operatorname{rank}\partial F(X)}{\epsilon} + f(\epsilon), \qquad 0 < \epsilon < \epsilon_0.$$

Proof. Denote by E_ϵ the conditional expectation from $W^*(X, S)$ to $W^*(X+\sqrt{\epsilon}S)$, and denote by the same letter the extension of E_ϵ to L^2. Then (see [17])

$$\Phi^*(X+\sqrt{\epsilon}S) = \|\xi_\epsilon\|_2^2$$

where

$$\xi_\epsilon = \epsilon^{-1/2}E_\epsilon(S).$$

Since $F(X) = 0$, using Taylor expansion we have that $F(X+\sqrt{\epsilon}S) = \sqrt{\epsilon}\partial F\#S + \zeta_1(\epsilon)$ with $\|\zeta_1(\epsilon)\|_2 = O(\epsilon)$. Thus, using contractivity of E_ϵ in L^2, we deduce:

$$\begin{aligned} E_\epsilon(\sqrt{\epsilon}\partial F(X)\#S) &= E_\epsilon(F(X+\sqrt{\epsilon}S)) + E_\epsilon(\zeta_1(\epsilon)) \\ &= F(X+\sqrt{\epsilon}S) + \zeta_2(\epsilon) \\ &= \sqrt{\epsilon}\partial F(X)\#S + \zeta_3(\epsilon) \end{aligned} \tag{1}$$

with $\|\zeta_j(\epsilon)\|_2 = O(\epsilon)$, $j = 2, 3$. Setting $X_\epsilon = X+\sqrt{\epsilon}S$, we also have:

$$\begin{aligned} E_\epsilon(\sqrt{\epsilon}\partial F(X)\#S) &= \sqrt{\epsilon}E_\epsilon(\partial F(X_\epsilon)\#S) + \zeta_4(\epsilon) \\ &= \sqrt{\epsilon}\partial F(X_\epsilon)\#E_\epsilon(S) + \zeta_4(\epsilon) \\ &= \sqrt{\epsilon}\partial F(X)\#E_\epsilon(S) + \zeta_5(\epsilon) \end{aligned}$$

where once again $\|\zeta_j(\epsilon)\|_2 = O(\epsilon)$, $j = 4, 5$. Combining this with (1) gives

$$\sqrt{\epsilon}\partial F(X)\#E_\epsilon(S) = \sqrt{\epsilon}\partial F(X)\#S + \zeta_6(\epsilon)$$

where $\|\zeta_6(\epsilon)\|_2 = O(\epsilon)$. Since $\xi_\epsilon = \epsilon^{-1/2}E_\epsilon(S)$, it follows that

$$\partial F(X)\#\xi_\epsilon = \epsilon^{-1/2}\partial F(X)\#S + \zeta'_\epsilon,$$

where $\|\zeta'_\epsilon\|_2 = O(1)$.

Let E' be the orthogonal projection onto the L^2 closure of $H = \mathrm{span}(W^*(X)SW^*(X)) \subset L^2(W^*(X,S))$. The map $Q \mapsto Q\#S$ is an isometric isomorphism of $L^2(W^*(X)\otimes W^*(X)^o)$ with H (as is easily verified by direct computation based on the freeness condition).

Then E' commutes with $\partial F(X)\#\cdot$ and thus

$$\partial F(X)\#E'(\xi_\epsilon) = \epsilon^{-1/2}\partial F(X)\#S + \zeta''_\epsilon$$

with $\|\zeta''_\epsilon\|_2 = O(1)$ and $\zeta'' \in H$. Applying $(\partial F)^*\#$ to both sides and denoting $(\partial F)^*\#(\partial F)$ by Q finally gives us that there exists an $0 < \epsilon_0 < 1$ and a constant K for which

$$Q\#E'(\xi_\epsilon) = \epsilon^{-1/2}Q\#S + \zeta_\epsilon, \qquad 0 < \epsilon < \epsilon_0$$

with $\zeta_\epsilon \in H$, $\|\zeta_\epsilon\|_2 \leq K/2n^{1/2}$.

Let $P_\lambda = \chi_{[\lambda,+\infty)}(Q)$ and denote by R_λ the operator $f_\lambda(Q)$ where $f(x) = x^{-1}$ for $x \geq \lambda$ and $f(x) = 0$ for $0 \leq x < \lambda$. Then $R_\lambda Q = P_\lambda$.

Since projections are contractive on L^2, it follows that for any $\lambda > 0$, $\|P_\lambda\#S\|_2 \leq \|S\|_2 = n^{1/2}$ and furthermore

$$\begin{aligned}
\|\xi_\epsilon\|_2^2 &\geq \|P_\lambda E'(\xi_\epsilon)\|_2^2 \\
&= \|R_\lambda\#Q\#E_1(\xi_\epsilon)\|_2^2 \\
&= \|\epsilon^{-1/2}R_\lambda\#Q\#S + R_\lambda\#\zeta_\epsilon\|_2^2 \\
&\geq \epsilon^{-1}\tau(P_\lambda) - 2\epsilon^{-1/2}|\langle P_\lambda\#S, R_\lambda\#\zeta_\epsilon\rangle| \\
&\geq \epsilon^{-1}\tau(P_\lambda) - \epsilon^{-1/2}\lambda^{-1}K.
\end{aligned}$$

Let r be the rank of $\partial F(X)$ (i.e., $r = \lim_{\lambda\to 0}\tau(P_\lambda)$, where τ denotes the non-normalized trace on $M_{n\times n}(W^*(X)\bar{\otimes}W^*(X)^o)$), and set $\lambda = \epsilon^{1/4}$. Let $\phi(\lambda) = \tau(1-P_\lambda)$. Then

$$\begin{aligned}
\|\xi_\epsilon\|_2^2 &\geq \epsilon^{-1}(r - \phi(\lambda)) - \epsilon^{-1/2}\lambda^{-1}K \\
&= \frac{r}{\epsilon} - \epsilon^{-1}\phi(\epsilon^{1/4}) - \epsilon^{-3/4}K \\
&= \frac{r}{\epsilon} - \epsilon^{-1}\phi(\epsilon^{1/4}) + f_1(\epsilon)
\end{aligned}$$

for some $f_1 \in L^1[0,\epsilon_0]$. Thus to conclude the proof, it is enough to show that $\epsilon^{-1}\phi(\epsilon^{1/4})$ is integrable on $[0,\epsilon_0]$.

Let $d\mu(t)$ be the composition of the trace τ with the spectral measure of Q, so that $\phi(\lambda) = \int_{0^+}^{\lambda} d\mu(t)$. Let us substitute $u = \epsilon^{1/4}$ (so $du = (1/4)\epsilon^{-3/4}d\epsilon$, $u_0 = \epsilon_0^{1/4}$) into the integral expression for the L^1 norm of $\epsilon^{-1}\phi(\epsilon^{1/4})$:

$$\begin{aligned}
\int_{0^+}^{\epsilon_0} \frac{1}{\epsilon}\phi(\epsilon^{1/4})d\epsilon &= \frac{1}{4}\int_{0^+}^{u_0}\frac{1}{u}\phi(u)du \\
&= \frac{1}{4}\int_{0^+}^{u_0}(\partial_u \log u)\int_0^u d\mu(t)\ du - \left[\frac{1}{4}\phi(u)\log u\right]_{0^+}^{u_0} \\
&\leq \frac{1}{4}\int_0^{u_0}|\log_+ t|d\mu(t) + \frac{1}{4}\phi(u_0)|\log u_0| + \frac{1}{4}\lim_{\delta\to 0}|\log\delta|\phi(\delta).
\end{aligned}$$

Since by assumption $\|\log_+ Q\|_1 < \infty$, the intergral $\int_0^{u_0} |\log_+ t| d\mu(t)$ is finite. Since if $\delta < 1$, $|\log \delta| \leq |\log t|$ for $t \in (0, \delta]$,

$$|\log \delta| \phi(\delta) = \int_{0^+}^{\delta} |\log \delta| d\mu(t) \leq \int_0^{\delta} |\log_+ t| d\mu(t)$$

we conclude also that $\lim_{\delta \to 0} |\log \delta| \phi(\delta)$ is finite. Thus $\epsilon^{-1}\phi(\epsilon^{1/4}) \in L^1[0, \epsilon_0]$. □

Remark 2.2. If one drops the assumption that $\log_+[\partial F(X)^* \partial F(X)] \in L^1(W^*(X) \otimes W^*(X))$, the proof of Lemma 2.1 yields the estimate

$$\Phi^*(X + \sqrt{\epsilon} S) \geq \epsilon^{-1}\tau(P_\lambda) - \epsilon^{-1/2}\lambda^{-1}K,$$

where λ is arbitrary. This readily implies that $\delta^\star(X) \leq n - \tau(P_\lambda)$ for all λ, which shows that $\delta^*(X) \leq n - \operatorname{rank} \partial F(X)$. This estimate can be easily obtained from [4] by noting that for any finite-rank operator $T \in FR$,

$$0 = [F_j(X), T] = \sum_k [\partial_k F_j(X) \# T, X_k]$$

showing that the dimension of the L^2 closure of the space

$$\{(T_1, \dots, T_n) \in FR^n : \sum_j [T_j, X_j] = 0\}$$

is at least the rank of ∂F (compare [3]).

Remark 2.3. The proof of Lemma 2.1 suggests the following precise expression for the short-time asymptotics of the conjugate variable ξ_ϵ:

$$\xi_\epsilon = \epsilon^{-1/2} F \# S + \xi'_\epsilon + O(\epsilon^{1/2})$$

where $F = (F_{ij})_{i,j=1}^n$ is the projection onto the L^2 closure of the space $\{(T_1, \dots, T_n) \in FR^n : \sum_j [T_j, X_j] = 0\}$ and $\xi'_\epsilon = \partial^*_\epsilon(\delta_{ij} 1 \otimes 1 - F)$. Indeed, the intuition is that F is the projection onto the space of L^2 cycles, which is perpendicular to the space of L^2 derivations (which is the kernel of F). The kernel of F is a kind of "maximal domain of definition" of $\partial^*_{\epsilon=0}$; then $\xi'_\epsilon = E_\epsilon(\partial^*_{\epsilon=0}((\delta_{ij} 1 \otimes 1)_{ij} - F))$ is the "bounded part" of ξ_ϵ while $\epsilon^{-1/2} F \# S$ is the "unbounded part" (compare [13]). However, we were unable to prove this exact formula.

2.3. **Estimates on free entropy.**

Lemma 2.4. *Under hypothesis of Lemma 2.1, there exists $K < \infty$, $\epsilon_0 > 0$, so that for all $0 < \epsilon < \epsilon_0$, the non-microstates free entropy satisfies:*

$$\chi^*(X + \sqrt{\epsilon} S) \leq (\log \epsilon^{\frac{1}{2}})(\operatorname{rank} \partial F(X)) + K$$

In particular, the microstates entropy also satisfies

$$\chi(X + \sqrt{\epsilon} S : S) \leq \chi(X + \sqrt{\epsilon} S) \leq (\log \epsilon^{\frac{1}{2}})(\operatorname{rank} \partial F(X)) + K.$$

Proof. Let ϵ_0, f be as in Lemma 2.1; set $K' = \|f\|_{L^1[0,\epsilon_0]}$. By definition [17], up to a universal constant depending only on n,

$$\chi^*(X + \sqrt{\epsilon} S) = \frac{1}{2} \int_0^\infty -\Phi^*(X + \sqrt{\epsilon + t} S) + \frac{n}{1+t} dt$$

Thus up to a finite constant,

$$\begin{aligned}\chi^*(X+\sqrt{\epsilon}S) &= \frac{1}{2}\int_\epsilon^{\epsilon_0} -\Phi^*(X+\sqrt{t}S)dt\\ &\le \frac{1}{2}\int_\epsilon^{\epsilon_0} -\frac{\operatorname{rank}(\partial F(X))}{t}+K'\\ &= (\log\epsilon^{\frac{1}{2}})(\operatorname{rank}\partial F(X)) - \frac{1}{2}\log\epsilon_0 + K',\end{aligned}$$

where we used the inequality of Lemma 2.1 to pass from the first to the second line.

The inequality for the microstates entropy is due to the general inequalities $\chi(Z:Y)\le\chi(Z)\le\chi^*(Z)$; the first is trivial and the second is a deep result of Biane, Capitaine and Guionnet [2]. □

2.4. r**-boundedness.** The main consequence of our estimates is the following Theorem, originally due to Jung [9, Theorem 6.9]. We give a short alternative proof based on our estimates for non-microstates entropy.

Theorem 2.5. *Suppose that $F=(F_1,\dots,F_k)$ is a vector-valued polynomial non-commutative function of n variables, and suppose that for some $X=(X_1,\dots,X_n)\in(M,\tau)$, $F(X)=0$. Assume that $\log_+[\partial F(X)^*\partial F(X)]\in L^1(W^*(X)\otimes W^*(X))$. Then X is $n-\operatorname{rank}(\partial F(X))$ bounded.*

Proof. By Lemma 2.4,

$$\limsup_{\epsilon\to0}\chi(X+\sqrt{\epsilon}S:S)+\operatorname{rank}(\partial F(X))|\log\epsilon^{\frac{1}{2}}|\le K<\infty.$$

Thus by the last bullet point in [8, Corollary 1.4], X is $n-\operatorname{rank}(\partial F(X))$-bounded. □

Remark 2.6. Motivated by [8, Corollary 1.4] one can make the following definition: X is r-bounded for δ^* (resp. $\delta^\star$) if for some K and all $0<\epsilon<\epsilon_0$,

$$\chi^*(X+\sqrt{\epsilon}S)\le(n-r)\log\epsilon^{\frac{1}{2}}+K$$

(respectively, $\Phi^*(X+\sqrt{\epsilon}S)\ge\epsilon^{-1}(n-r)+\phi(\epsilon)$, $0<\epsilon<\epsilon_0$ with $\phi\in L^1[0,\epsilon_0]$). Then under the hypothesis of Theorem 2.5, X is $n-\operatorname{rank}(\partial F(X))$-bounded for δ^* and $\delta^\star$.

3. Applications.

Lemma 3.1. *Let Γ be a finitely generated finitely presented group. Then there exists an n-tuple $X=(X_1,\dots,X_n)$ of self-adjoint elements in $\mathbb{Q}\Gamma\subset L\Gamma$ and a vector-valued polynomial function $F=(F_1,\dots,F_k)$ with $F_j\in\mathbb{Q}[t_1,\dots,t_n]$ so that $F(X)=0$ and moreover*

$$\operatorname{rank}\partial F(X)=n-(\beta_1^{(2)}(\Gamma)+\beta_0^{(2)}(\Gamma)-1),$$

where $\beta_j^{(2)}(\Gamma)$ are the L^2-Betti numbers of Γ.

Proof. Let $g_1,\dots,g_m$ be generators of Γ and let $n=2m$. Consider the following generators $X_1,\dots,X_n$ for the group algebra $\mathbb{C}\Gamma$: $X_j=g_j+g_j^{-1}$ $X_{m+j}=i(g_j-g_j^{-1})$, $j=1,\dots,m$.

Denote by $h_1,\dots,h_m$ the generators of the free group $\mathbb{F}_m$, and let $Y_j=h_j+h_j^{-1}$, $Y_{n+j}=i(h_j-h_j^{-1})$, $j=1,\dots,m$ be generators of $\mathbb{C}\mathbb{F}_m$. Let $R_1,\dots,R_p\in\mathbb{F}_m$ be the relations satisfied by the generators $g_1,\dots,g_m$ of Γ.

Denote by $t_1,\dots,t_n$ the generators of the algebra $\mathscr{A}=\mathbb{C}[t_1,\dots,t_{2n}]$ of non-commutative polynomials in $2n$ variables. Then there is a canonical map from $\mathscr{A}\to\mathbb{C}\mathbb{F}_m$ given by $t_j\mapsto Y_j$. The algebra $\mathbb{C}\mathbb{F}_m$ is then the quotient of $\mathscr{A}$ by the ideal J_0 generated by the relations corresponding to the relations $h_j^{-1}h_j=h_jh_j^{-1}=1$ that hold in $\mathbb{C}\mathbb{F}_m$; written in terms of the generators t_j these relations take the form $F_j'=(t_j+t_{m+j})(t_j-t_{m+j})+4i$, $F_j''=(t_j-t_{m+j})(t_j+t_{m+j})+4i$, $j=1,\dots,m$.

The relations R_j can be interpreted as polynomials (with rational coefficients) in $t_1,\dots,t_n$ by substituting $\frac{1}{2}(t_j+t_{m+j})$ for h_j and $\frac{1}{2i}(t_j-t_{m+j})$ for h_j^{-1}. Let $F_j'''=R_j-1$, $j=1,\dots,p$. Let J be the ideal in $\mathscr{A}$ generated by J_0 and F_j''', $j=1,\dots,p$. Then $\mathbb{C}\Gamma$ is precisely the quotient of $\mathscr{A}$ by J, the quotient map sending t_j to X_j. Let $F=(F_j''')_{j=1}^p\sqcup(F_j'')_{j=1}^n\sqcup(F_j')_{j=1}^n$, where $\sqcup$ refers to union of ordered tuples.

Let $\delta':\mathscr{A}\to L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o)$ be a derivation. If for all $j=1,\dots,p+2n$, $\delta'(F_j)=0$ then for any $x=aF_jb\in J$,

$$\delta'(x)=\delta'(a)(F_j)b+a(F_j)\delta'(b)+a\delta'(F_j)b=0$$

since F_j acts by zero on both the right and the left of $L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o)$ and $\delta'(F_j)=0$ by assumption; thus $\delta'(J)=0$. Conversely, if $\delta'(J)=0$ then $\delta(F_j)=0$ for all j. It follows that δ' descends to a derivation $\delta:\mathbb{C}\Gamma\to L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o)$ iff $\delta'(F_j)=0$ for all $j=1,\dots,p+2n$.

Let now $Q_l\in L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o)$, $l=1,\dots,n$. Then there exists a unique derivation $\delta':\mathscr{A}\to L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o)$ so that $\delta'(t_l)=Q_l$ for all l. This derivation descends to a derivation δ on $\mathbb{C}\Gamma$ (satisfying $\delta(X_l)=Q_l$) iff for all j,

$$0=\delta'(F_j)=\partial F_j(X)\#Q,$$

i.e., $Q\in\ker\partial F(X)$. It follows that the space of derivations $Z^1(\mathbb{C}\Gamma,L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o))$ is isomorphic to $\ker\partial F(X)\subset(L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o))^{\oplus n}$.

Let α denote the action of $L(\Gamma\times\Gamma^o)$ on $L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o)$ given by $(g\times h)\cdot(\xi\otimes\eta)=\rho(g)\xi\otimes\rho(h)\eta$, where ρ is the right regular representation. If $\delta:\mathbb{C}\Gamma\to L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o)$, then $\alpha_x\circ\delta$ is again a derivation, for any $x\in L(\Gamma\times\Gamma^o)$. It follows that $L(\Gamma\times\Gamma^o)$ acts on the space of these L^2 derivations; moreover, the isomorphism $Z^1(\mathbb{C}\Gamma,L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o))\cong\ker\partial F(X)$ is α-equivariant.

It is easily seen that

$$\dim_{L(\Gamma\times\Gamma^o)}Z^1(\mathbb{C}\Gamma,L^2(\mathbb{C}\Gamma\otimes\mathbb{C}\Gamma^o))=\beta_1^{(2)}(\Gamma)-\beta_0^{(2)}(\Gamma)+1$$

(see e.g. [4, 12]; note that $\beta_1^{(2)}(\Gamma)$ is the dimension of the space of derivations which are not inner, while $1-\beta_0^{(2)}(\Gamma)$ is the dimension of the space of inner derivations).

Putting things together, we obtain that

$$\beta_1^{(2)}(\Gamma)-\beta_0^{(2)}(\Gamma)+1=\dim_{L(\Gamma\times\Gamma^o)}\ker\partial F(X)=n-\operatorname{rank}\partial F(X)$$

as claimed. Note that by construction $F\in\mathbb{Q}[t_1,\dots,t_n]$. ☐

Theorem 3.2. *Let Γ be a finitely generated, finitely presented sofic group, and let $r=\beta_1^{(2)}(\Gamma)-\beta_1^{(2)}+1$. Then there exists a set of generators for $\mathbb{C}\Gamma$ which is r-bounded. In particular, if $\beta_1^{(2)}(\Gamma)=0$ and Γ contains an element of infinite order, then $L(\Gamma)$ is strongly* 1*-bounded.*

Proof. By Lemma 3.1 there exists an n-tuple $X = (X_1, \ldots, X_n)$ of self-adjoint elements in $\mathbb{Q}\Gamma \subset L\Gamma$ and a vector-valued polynomial function $F = (F_1, \ldots, F_k)$ with $F_j \in \mathbb{Q}[t_1, \ldots, t_n]$ so that $F(X) = 0$ and moreover $\operatorname{rank} \partial F(X) = n - (\beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1)$.

Furthermore, since $\partial F(X) \in \mathbb{Q}(\Gamma \times \Gamma^o)$ and Γ (thus also $\Gamma \times \Gamma^o$) is sofic, the determinant conjecture holds [5, 1, 11]. Thus $\log_+[\partial F(X)^* \partial F(X)] \in L^1(W^*(X) \otimes W^*(X))$.

We may thus apply Theorem 2.5 to conclude that $(X_1, \ldots, X_n)$ is $n - \operatorname{rank}(\partial F(X)) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma +) 1 = r$ bounded.

If Γ contains an element of infinite order, $\beta_0^{(2)}(\Gamma) = 0$. Thus if $\beta_1^{(2)}(\Gamma) = 0$, it follows that $r = 1$. Since Γ is sofic, its microstates spaces are non-empty. It follows that $L(\Gamma)$ is strongly 1-bounded since we can assume the generating set X contains an element with finite free entropy (e.g. the real part of the Haar unitary in $\mathbb{C}\Gamma$ coming from an element of infinite order in Γ). ☐

Remark 3.3. The proof of Theorem 3.2 still goes through if we assume that Γ satisfies the determinant conjecture (or, more precisely, that $\log_+[\partial F(X)^* \partial F(X)] \in L^1(W^*(X) \otimes W^*(X))$ for the specific function F we are dealing with) and that $L(\Gamma)$ satisfies the Connes embedding conjecture.

It is an open question whether property (T) von Neumann algebras are always strongly 1-bounded, even in the group case, although it is known that every generating set must have free entropy dimension at most 1 [10]. However, we have:

Corollary 3.4. *Let Γ be an finitely presented sofic property (T) group having an element of infinite order. Then $L(\Gamma)$ is strongly 1-bounded.*

Proof. Indeed, property (T) implies that $\beta_1^{(2)}(\Gamma) = 0$ and that the group is finitely generated (see e.g. [11]). ☐

As noted in [18, Corollary 4.6], the free entropy dimension δ_0 is an invariant of the group algebra $\mathbb{C}\Gamma$ of a discrete group.

Corollary 3.5. *Let M be a von Neumann algebra that is not strongly 1-bounded. (For example, $M = W^*(X)$ with $\delta_0(X) > 1$, e.g. $M = L(\Gamma)$ with $\delta_0(\mathbb{C}\Gamma) > 1$, e.g., $\Gamma = \mathbb{F}_n$, $n \geq 2$). If $M \cong L(\Lambda)$ with Λ finitely generated finitely presented sofic having an element of infinite order, then $\beta_1^{(2)}(\Lambda) \neq 0$.*

Proof. If $\beta_1^{(2)}(\Lambda) = 0$, Theorem 3.2 would imply that $L(\Lambda) \cong M$ is strongly 1-bounded. Contradiction. ☐

Conjecture 3.6. *Let Γ be a finitely presented finitely generated sofic group each having an element of infinite order with $\beta_1^{(2)} > 0$. Then*
(a) $L(\Gamma)$ is not strongly one bounded;
(b) $\delta_0(\mathbb{C}\Gamma) = \beta_1^{(2)}(\Gamma) + 1$.

Note that (b) $\Longrightarrow$ (a) since if $L(\Gamma)$ were strongly 1-bounded, then $\delta_0(\mathbb{C}\Gamma)$ would have to be 1. The inequality $\delta_0(\mathbb{C}\Gamma) \leq \beta_1^{(2)}(\Gamma) + 1$ is known (see [4]) and actually follows from the results of the present paper as well. However, the reverse inequality is only known in a few special cases, e.g. $\Gamma = \mathbb{F}_n$, or groups obtained from free groups by amalgamated free products over amenable subgroups [14].

Remark 3.7. Together with Corollary 3.5, Conjecture 3.6(a) would imply the following: if Γ and Λ are two finitely generated finitely presented sofic groups each having an element of infinite order and $L(\Gamma) \cong L(\Lambda)$ then $\beta_1^{(2)}(\Gamma)$, $\beta_1^{(2)}(\Lambda)$ are either simultaneously zero or simultaneously non-zero. In other words, the vanishing of $\beta_1^{(2)}$ is a W^*-equivalence invariant for such groups.

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